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Linearisation of tautological control systems

Andrew D. Lewis*

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Abstract

The framework of tautological control systems is one where "control" in the usual sense has been eliminated, with the intention of overcoming the issue of feedback-invariance. Here, the linearisation of tautological control systems is described. This linearisation retains the feedback-invariant character of the tautological control system framework and so permits, for example, a well-defined notion of linearisation of a system about an equilibrium point, something which has surprisingly been missing up to now. The linearisations described are of systems, first, and then about reference trajectories and reference flows.

Keywords. Geometric control theory, tautological control systems, linearisation

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1. Introduction

In recent work, Lewis [2014] introduced the notion of a "tautological control system." This class of control systems is developed to provide a framework for geometric control theory that is intrinsically feedback-invariant. As described at length in the introduction of [Lewis 2014], feedback-invariance is highly problematic for the standard "$\dot{x} = F(x, u)$" model in control theory. This is because, in such models, there is implicitly a fixed parameterisation of the control set and a fixed manner in which these controls appear in the dynamical model F. To account for the fact that the control parameterisation and the manner in which control appears in the equations may vary, while trajectories remain the same, there has been an extensive study of "feedback transformations." In the tautological control system framework, Lewis [2014, Proposition 5.39] shows that feedback transformations are comprised merely of diffeomorphisms of the state manifold i.e., there are no transformations involving the control set as are usually seen. This is a direct consequence of the elimination of control, in the usual sense, from the framework. This accounts, in part, for the use of the word "tautological" in the title for these systems.

To give a simple example of the sorts of problems that arise in the "ordinary" framework, and that the tautological control system framework is intended to overcome, let us consider the simple example of [Lewis 2014].

*Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/
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1.1 Example: (Linearisation is not well-defined) We consider two control-affine systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= x_3(t)u_1(t), & \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t),\\ \dot{x}_3(t) &= u_2(t), & \dot{x}_3(t) &= u_2(t), \end{aligned}$$

with $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(u_1, u_2) \in \mathbb{R}^2$. One can readily verify that these two systems have the same trajectories in the sense that the set of curves in the state space that arise from solving the two differential equations are the same (this corresponds to part (iii) of Definition 3.9). If we linearise these two systems about the equilibrium point at $(0, 0, 0)$—in the usual sense of taking Jacobians with respect to state and control [Isidori 1995, page 172], [Khalil 2001, §12.2], [Nijmeijer and van der Schaft 1990, Proposition 3.3], [Sastry 1999, page 236], and [Sontag 1998, Definition 2.7.14]—then we get the two linear systems

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 1 \end{bmatrix},$$

respectively. The linearisation on the left is not controllable, while that on the right is.

The example suggests that (1) classical linearisation is not independent of parameterisation of controls and/or (2) the classical notion of linear controllability is not independent of parameterisation of controls. In Section 4 we shall see that both things, in fact, are true: neither classical linearisation nor the classical linear controllability test are feedback-invariant. This may come as a surprise to some. •

This example is, by explicit design, very simple. A consequence of this is that an astute reader will note that by linearising the uncontrollable system about the control $(1, 0)$ rather than $(0, 0)$, one will obtain a controllable linearisation in both cases. This, however, is a kludge that leaves unanswered the question, "How should one define the notions of linearisation and linear controllability in such a way that any conclusions one draws from them are feedback-invariant?" The correct way to understand the example is to imagine giving the two different systems to two different people, telling neither that they are related, and then asking them to "linearise about the equilibrium point and determine whether the linearisation is controllable." They will assuredly come back with different conclusions, and it will be a rare occurrence (in the author's experience) that anyone will notice the potential ambiguity in this process.

It is also worth drawing attention to the fact that, while this paper focusses on linearisation, the lack of feedback-invariance arises in many other places in control theory. A survey of this with respect to (nonlinear) controllability and stabilisability is given by Lewis [Lewis 2012]. For our purposes here, suffice it to say that, once one is attuned to look for it, the problem of lack of feedback-invariance can be found in many areas of geometric control theory and manifested in a variety of ways.

While the tautological control system framework does address problems such as this, it does so at a cost: the theory is difficult to understand at a first glance, and it requires discipline not to fall back to the comforting world of "ordinary" control systems. The difficulty in the theory is that it relies in an unavoidable way on characterisations of time-varying vector fields using locally convex topologies for spaces of vector fields described by Jafarpour and Lewis [2014, Chapter 6]. Moreover, precious little guidance is provided by Lewis [2014] on how to work with tautological control systems; the emphasis in this original paper is to establish foundations. In this paper, therefore, we illustrate how one should work within the framework of tautological control system, while establishing a useful theory of linearisation for these systems. In subsequent papers we shall explore further how one works effectively in the world of tautological control systems.

Before we begin, it is worth pointing out that, apart from the problem revealed in Example 1.1, there are other difficulties with the very idea of classical Jacobian linearisation to which blind eyes seem to be routinely turned in practice. First of all, for models of the form "$F(x, u)$," one must assume that differentiation with respect to u can be done. For models of this sort, there is no reason to assume the control set to be a subset of $\mathbb{R}^m$, and so one runs into a problem right away. Even so, if one restricts to control-affine systems, where the notion of differentiation with respect to u seems not to be problematic, one must ignore the fact that the control set is generally not an open set—indeed, for control-affine systems there is no reason to have the control set be anything but an arbitrary subset of $\mathbb{R}^m$—and so these derivatives are not so easily made sense of. Therefore, even for the typical models one studies in control theory, there are good reasons to revisit the notion of linearisation.

The reader may be dismayed to learn that their nice simple theory of linearisation has now been replaced by a complicated mathematical construction requiring many pages to explain, and in a framework that takes many more pages to explain. A reader may well steadfastly use these facts to simply disregard the theory of linearisation presented here and the corresponding theory of tautological control systems. And it is very likely that, upon making this decision, such a reader will not regret it. However, in defence of the paper and of the theory of tautological control systems, it should be pointed out that there is some intellectual negligence in doing this, for Example 1.1 clearly points out a defect in things we understand (and teach![1]) about linearisation. It is our view that, until such defects are handled in a systematic way, it seems very unlikely that we will understand the very difficult fundamental structural problems of geometric control theory, problems such as controllability, stabilisability, and optimality.

We point out that geometric linearisation of control-affine systems, and a Linear Quadratic Regulator theory in this framework, has been carried out by Lewis and Tyner [2010]. But even the geometric approach in that work is refined and clarified by what we present here.

1.1. Outline of paper. Let us outline the contents of the paper.

As mentioned above, the framework of tautological control systems relies inextricably on locally convex topologies for spaces of vector fields. In Section 2 we review the theory for these as developed by Jafarpour and Lewis [2014]. One of the powerful facets of the theory of tautological control systems as expounded by Lewis [2014] is that it handles all common regularity classes—Lipschitz, finitely differentiable, smooth, and real analytic—in a unified manner. While these topologies are more or less well understood in the smooth and finitely differentiable case (and this understanding is fairly easily extended to the Lipschitz case, although this is rarely done), the description of the real analytic topology given by

[1] Note that this statement does not imply that the author advocates teaching undergraduates the theory of tautological control systems!

Jafarpour and Lewis [2014] is novel, and allows for the first time an understandable way of handling real analytic systems as easily as smooth systems. In Section 2.3 we describe the families of time-varying vector fields that are the backbone of the theory of tautological control systems, in that they permit the notion of trajectories. In Section 2.4 we introduce a class of "ordinary" control systems that will serve to provide a concrete reference point for tautological control system constructions as we go along.

In Section 3 we quickly go over the notation and definitions of [Lewis 2014] regarding tautological control systems. The presentation here is very sketchy indeed, and reader wishing for more context will want to review the material in [Lewis 2014].

In Section 4 we present our theory of linearisation. We develop linearisations of systems, as well as linearisation about trajectories and reference flows. We also give the special case of linearisation about equilibria, and with this we see how to rectify and explain the difficulties encountered in Example 1.1.

1.2. Notation. We shall use the slightly unconventional, but perfectly rational, notation of writing $A \subseteq B$ to denote set inclusion, and when we write $A \subset B$ we mean that $A \subseteq B$ and $A \neq B$. By id_A we denote the identity map on a set A. By $\mathbb{Z}$ we denote the set of integers, with $\mathbb{Z}_{\geq 0}$ denoting the set of nonnegative integers and $\mathbb{Z}_{>0}$ denoting the set of positive integers. We denote by $\mathbb{R}$ the set of real numbers. By $\mathbb{R}_{\geq 0}$ we denote the set of nonnegative real numbers and by $\mathbb{R}_{>0}$ the set of positive real numbers.

For a topological space $\mathcal{X}$ and $A \subseteq \mathcal{X}$, $\mathrm{int}(A)$ denotes the interior of A and $\mathrm{cl}(A)$ denotes the closure of A. Neighbourhoods will always be open sets.

Elements of $\mathbb{R}^n$ are typically denoted with a bold font, e.g., "$\boldsymbol{x}$." Similarly, matrices are written using a bold font, e.g., "$\boldsymbol{A}$."

If V is a $\mathbb{R}$-vector space and if $A \subseteq \mathsf{V}$, we denote by $\mathrm{conv}(A)$ the convex hull of A, by which we mean the set of all convex combinations of elements of A.

By λ we denote Lebesgue measure. If $I \subseteq \mathbb{R}$ is an interval and if $A \subseteq \mathbb{R}$, by $\mathrm{L}^1(I; A)$ we denote the set of Lebesgue integrable A-valued functions on I. By $\mathrm{L}^1_{\mathrm{loc}}(I; A)$ we denote the A-valued locally integrable functions on I, i.e., those functions whose restrictions to compact subintervals are integrable. In like manner, we denote by $\mathrm{L}^\infty(I; A)$ and $\mathrm{L}^\infty_{\mathrm{loc}}(I; A)$ the essentially bounded A-valued functions and the locally essentially bounded A-valued functions, respectively.

For an interval I and a topological space $\mathcal{X}$, a curve $\gamma\colon I \to \mathcal{X}$ is ***measurable*** if $\gamma^{-1}(\mathcal{B})$ is Lebesgue measurable for every Borel set $\mathcal{B} \subseteq \mathcal{X}$. By $\mathrm{L}^\infty(I; \mathcal{X})$ we denote the measurable curves $\gamma\colon I \to \mathcal{X}$ for which there exists a compact set $K \subseteq \mathcal{X}$ with

$$\lambda(\{t \in I \mid \gamma(t) \notin K\}) = 0,$$

i.e., $\mathrm{L}^\infty(I; \mathcal{X})$ is the set of ***essentially bounded*** curves. By $\mathrm{L}^\infty_{\mathrm{loc}}(I; \mathcal{X})$ we denote the ***locally essentially bounded*** curves, meaning those measurable curves whose restrictions to compact subintervals are essentially bounded.

Our differential geometric conventions mostly follow [Abraham, Marsden, and Ratiu 1988]. Whenever we write "manifold," we mean "second-countable Hausdorff manifold." This implies, in particular, that manifolds are assumed to be metrisable [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13]. If we use the letter "n" without mentioning what it is, it is the dimension of the connected component of the manifold M with which we are working at that time. The tangent bundle of a manifold M is denoted by $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}}\colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$. The derivative of a differentiable map $\Phi\colon \mathsf{M} \to \mathsf{N}$ is denoted by $T\Phi\colon \mathsf{TM} \to \mathsf{TN}$, with $T_x\Phi = T\Phi|\mathsf{T}_x\mathsf{M}$. If $I \subseteq \mathbb{R}$ is an interval and if $\xi\colon I \to \mathsf{M}$ is a curve that is differentiable at $t \in I$, we denote the tangent vector field to the curve at t by $\xi'(t) = T_t\xi(1)$. The flow of a vector field X is denoted by Φ_t^X, so $t \mapsto \Phi_t^X(x)$ is the integral curve of X passing through x at $t = 0$. We shall also use time-varying vector fields, but will develop the notation for the flows of these in the text.

We will work in both the smooth and real analytic categories. We will also work with finitely differentiable objects, i.e., objects of class C^r for $r \in \mathbb{Z}_{\geq 0}$. (We will also work with Lipschitz objects, but will develop the notation for these in the text.) An analytic manifold or mapping will be said to be of ***class*** $\mathbf{C}^{\boldsymbol{\omega}}$. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. The set of sections of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ of class C^r is denoted by $\Gamma^r(\mathsf{E})$. Thus, in particular, $\Gamma^r(\mathsf{TM})$ denotes the set of vector fields of class C^r. We shall think of $\Gamma^r(\mathsf{E})$ as a $\mathbb{R}$-vector space with the natural pointwise addition and scalar multiplication operations.

We shall make reference to elementary ideas from sheaf theory. It will not be necessary to understand this theory deeply, at least not in the present paper. A nice introduction to the use of sheaves in smooth differential geometry can be found in the book of Ramanan [2005]. More advanced and comprehensive treatments include [Bredon 1997, Kashiwara and Schapira 1990], and the classic [Godement 1958]. The discussion of sheaf theory in (Stacks 2013) is also useful.

We shall make use of locally convex topological vector spaces, and refer to [Rudin 1991] as a gentle introduction and to, e.g., [Jarchow 1981, Schaefer and Wolff 1999] for more advanced material. We denote by $\mathrm{L}(\mathsf{U};\mathsf{V})$ the set of continuous linear maps from U to V.

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2. Topologies for spaces of vector fields, time-varying vector fields, and "ordinary" control systems

In this section we review the definitions of the topologies we use for spaces of Lipschitz, finitely differentiable, smooth, and real analytic vector fields. We comment that all topologies we define are locally convex topologies, of which the normed topologies are a special case. However, few of the topologies we define, and none of the interesting ones, are normable. So a reader who is not familiar with locally convex topologies will have to do some reading; we recommend [Rudin 1991] as a nice introduction. In Section 2.3 we overview classes of time-varying vector fields following [Jafarpour and Lewis 2014], since these are an essential part of the theory of tautological control systems. In Section 2.4 we recall topological characterisations of "ordinary" control systems from [Jafarpour and Lewis 2016], as these provide for a useful point of departure for many notions for tautological control systems.

2.1. Fibre norms for jet bundles. The classes of vector fields we consider are all characterised by their derivatives in some manner. The appropriate device for considering derivatives of vector fields is the theory of jet bundles, for which we refer to [Saunders 1989] and [Kolář, Michor, and Slovák 1993, §12]. Local coordinate descriptions of these topologies are possible, [e.g., Hirsch 1976] for the smooth and finitely differentiable cases, but we prefer coordinate-free descriptions. The cost for this is the machinery presented in this and the next section. By $\mathsf{J}^m\mathsf{TM}$ we denote the vector bundle of m-jets of vector fields, with $\pi_{\mathsf{TM},m}\colon \mathsf{J}^m\mathsf{TM} \to \mathsf{M}$ denoting the projection. If X is a smooth vector field, we denote by j_mX the corresponding smooth section of $\mathsf{J}^m\mathsf{TM}$.

Sections of $\mathsf{J}^m\mathsf{TM}$ should be thought of as vector fields along with their first m derivatives. In a local trivialisation of TM, one has the local representatives of the derivatives, order-by-order. Such an order-by-order decomposition of derivatives is not possible globally, however. Nonetheless, following [Jafarpour and Lewis 2014, §2.1], we shall mimic this order-by-order decomposition globally using an affine connection ∇ on M; we refer to [Kolář, Michor, and Slovák 1993, §11, §17] or [Kobayashi and Nomizu 1963] for background on connections. Let $\mathrm{T}^m(\mathsf{T}^*\mathsf{M})$ denote the m-fold tensor product of $\mathsf{T}^*\mathsf{M}$ and let $\mathrm{S}(\mathsf{T}^*\mathsf{M})$ denote the symmetric tensor bundle. First note that ∇ defines a connection on $\mathsf{T}^*\mathsf{M}$ by duality. Then ∇ defines a connection ∇^m on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}$ by asking that the Leibniz Rule be satisfied for the tensor product. Then, for a smooth vector field X, we denote

$$\nabla^{(m)}X = \nabla^m \cdots \nabla^1\nabla X,$$

which is a smooth section of $\mathrm{T}^{m+1}(\mathsf{T}^*\mathsf{M} \otimes \mathsf{TM})$. By convention we take $\nabla^0 X = \nabla X$ and $\nabla^{(-1)}X = X$. (The funny numbering makes this agree with the constructions of Jafarpour and Lewis [2014].)

We then have a map

$$\begin{aligned} S^m_\nabla\colon \mathsf{J}^m\mathsf{TM} &\to \oplus_{j=0}^m(\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}) \\ j_mX(x) &\mapsto (X(x), \mathrm{Sym}_1 \otimes \mathrm{id}_{\mathsf{TM}}(\nabla X)(x), \dots, \mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(m-1)}X)(x)), \end{aligned} \tag{2.1}$$

which can be verified to be an isomorphism of vector bundles [Jafarpour and Lewis 2014, Lemma 2.1]. Here $\mathrm{Sym}_m\colon \mathrm{T}^m(\mathsf{V}) \to \mathrm{S}^m(\mathsf{V})$ is defined by

$$\mathrm{Sym}_m(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

Now we note that inner products on the components of a tensor product induce in a natural way inner products on the tensor product [Jafarpour and Lewis 2014, Lemma 2.3]. Thus, if we suppose that we have a Riemannian metric $\mathbb{G}$ on M, there is induced a natural fibre metric $\mathbb{G}_m$ on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}$ for each $m \in \mathbb{Z}_{\geq 0}$. We then define a fibre metric $\overline{\mathbb{G}}_m$ on $\mathsf{J}^m\mathsf{TM}$ by

$$\begin{aligned} &\overline{\mathbb{G}}_m(j_mX(x), j_mY(x)) \\ &\qquad = \sum_{j=0}^m \mathbb{G}_j\Big(\frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(j-1)}X)(x), \frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(j-1)}Y)(x)\Big). \end{aligned}$$

(The factorials are required to make things work out with the real analytic topology.) The corresponding fibre norm we denote by $\|\cdot\|_{\overline{\mathbb{G}}_m}$.

2.2. Seminorms for spaces of vector fields. We shall describe topologies for spaces of smooth, finitely differentiable, Lipschitz, and real analytic vector fields by prescribing seminorms on these spaces, using the fibre norms from the preceding section. We shall not give any discussion of the nature of these topologies, although such considerations can, at times, be extremely important. Instead, we refer the reader to [Jafarpour and Lewis 2014].

For studying topologies on the space of real analytic vector fields, we work with a real analytic manifold M. For spaces of smooth, finitely differentiable, or Lipschitz vector fields, we suppose that M is smooth. In the real analytic case, we assume that the affine connection ∇ and the Riemannian metric $\mathbb{G}$ from the preceding section are real analytic. This is essential for the topology we define to make sense, and can always be done [Jafarpour and Lewis 2014, Lemma 2.4]. When we define seminorms for spaces of Lipschitz vector fields, we need to assume for the definitions to work that ∇ is the Levi-Civita connection associated with $\mathbb{G}$ [Jafarpour and Lewis 2014, §3.5].

To define our seminorms, we shall need some notation. We let $\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ denote the space of nonincreasing sequences in $\mathbb{R}_{>0}$, indexed by $\mathbb{Z}_{\geq 0}$, and converging to zero. We shall denote a typical element of $\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ by $\boldsymbol{a} = (a_j)_{j\in\mathbb{Z}_{\geq 0}}$. For $m \in \mathbb{Z}_{\geq 0}$, by $\Gamma^{m+\mathrm{lip}}(\mathsf{TM})$ we denote the set of vector fields that are m-times continuously differentiable and whose m-jet is Lipschitz continuous. (One can think of this in coordinates, but Jafarpour and Lewis [2014] provide geometric definitions, if the reader is interested.) If a vector field X is of class $\mathrm{C}^{m+\mathrm{lip}}$, then, by Rademacher's Theorem [Federer 1969, Theorem 3.1.6], its $(m+1)$st derivative exists almost everywhere. Thus we define

$$\begin{aligned}\operatorname{dil} j_m X(x) = \inf\{\sup\{\|\nabla^{[m]}_{v_y} j_m X\|_{\overline{\mathbb{G}}_m} \mid\ y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\\ j_m X \text{ differentiable at } y\} \mid\ \mathcal{U} \text{ is a relatively compact neighbourhood of } x\},\end{aligned}$$

which is the ***local sectional dilatation*** of X. Here $\nabla^{[m]}$ is the connection in $\mathsf{J}^m\mathsf{TM}$ induced by ∇ using the decomposition (2.1).

We may now define seminorms for our various classes of vector fields that define a locally convex topology in each case. We shall consider regularity $\nu \in \{m, m+\mathrm{lip}, \infty, \omega\}$ for $m \in \mathbb{Z}_{\geq 0}$.

1. $\nu = \infty$: We define the family of seminorms $p^{\infty}_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, by
$$p^{\infty}_{K,m}(X) = \sup\{\|j_m X(x)\|_{\overline{\mathbb{G}}_m} \mid\ x \in K\}.$$
2. $\nu = m$: We define the family of seminorms p^m_K, $K \subseteq \mathsf{M}$ compact, by
$$p^m_K(X) = \sup\{\|j_m X(x)\|_{\overline{\mathbb{G}}_m} \mid\ x \in K\}.$$
3. $\nu = m + \mathrm{lip}$: First, for $K \subseteq \mathsf{M}$ compact, we define
$$\lambda^m_K(X) = \sup\{\operatorname{dil} j_m X(x) \mid\ x \in K\}.$$
Then we define the family of seminorms $p^{m+\mathrm{lip}}_K$, $K \subseteq \mathsf{M}$ compact, by
$$p^{m+\mathrm{lip}}_K(X) = \max\{\lambda^m_K(X), p^m_K(X)\}.$$
4. $\nu = \omega$: We define the family of seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, by
$$p^{\omega}_{K,\boldsymbol{a}}(X) = \sup\{a_0 a_1 \cdots a_m \|j_m X(x)\|_{\overline{\mathbb{G}}_m} \mid\ x \in K,\ m \in \mathbb{Z}_{\geq 0}\}.$$

We call these topologies the $\mathbf{C}^{\nu}$***-topologies***, and refer to [Jafarpour and Lewis 2014] for discussion. Here we merely permit ourselves to say that it is the definition of the seminorms above for the C^{ω}-topology that make it possible to present the comprehensive theory that we give here. Without these useable seminorms, the theory would be much less satisfactory, probably not even worth presenting. However, because of the availability of these seminorms, what we can give is a unified presentation of a methodology across a wide class of regularity assumptions.

The degrees of regularity are ordered according to

$$\mathrm{C}^0 \supset \mathrm{C}^{\mathrm{lip}} \supset \mathrm{C}^1 \supset \cdots \supset \mathrm{C}^m \supset \mathrm{C}^{m+\mathrm{lip}} \supset \mathrm{C}^{m+1} \supset \cdots \supset \mathrm{C}^{\infty} \supset \mathrm{C}^{\omega}.$$

Where possible, we will state definitions and results for all regularity classes at once. To do this, we will let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and consider the regularity classes $\nu \in \{m+m', \infty, \omega\}$. In such cases we shall require that the underlying manifold be of class "C^r, $r \in \{\infty, \omega\}$, as required." This has the obvious meaning, namely that we consider class C^{ω} if $\nu = \omega$ and class C^{∞} otherwise. Proofs will typically break into the four cases $\nu = \infty$, $\nu = m$, $\nu = m + \mathrm{lip}$, and $\nu = \omega$. In some cases there is a structural similarity in the way arguments are carried out, so we will sometimes do all cases at once. In doing this, we will, for $K \subseteq \mathsf{M}$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^{\infty}_{K,k}, & \nu = \infty, \\ p^{m}_{K}, & \nu = m, \\ p^{m+\mathrm{lip}}_{K}, & \nu = m + \mathrm{lip}, \\ p^{\omega}_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases} \tag{2.2}$$

The convenience and brevity more than make up for the slight loss of preciseness in this approach.

2.3. Time-varying vector fields. The work of Jafarpour and Lewis [Jafarpour and Lewis 2014] is concerned with time-varying vector fields with measurable time dependence. In that work, a comprehensive and consistent theory for such vector fields, with varying regularity in state, is developed. The basic idea of the approach we give here is not new, and is also used by Agrachev and Gamkrelidze [Agrachev and Gamkrelidze 1978] in their presentation of "chronological calculus." A recent and nice exposition of this work, at least in the smooth case, can be found in the book [Agrachev and Sachkov 2004]. The essential idea is that one thinks of a time-varying vector field, not as a joint function of state and time, but as a vector field-valued function of time. In this case, it becomes important to provide properties of this vector field-valued function, typical properties being things like continuity, measurability, and integrability. The characterisations of these attributes rely essentially on a topology for the spaces of vector fields used, but this has been taken care of by virtue of the presentation in Section 2.2.

Thus, for $m \in \mathbb{Z}_{\geq 0}$, $m' \in \{0, \mathrm{lip}\}$, for $\nu \in \{m + m', \infty, \omega\}$, and for an interval $\mathbb{T} \subseteq \mathbb{R}$, characterisations are given for classes of time-varying vector fields denoted by $\mathrm{LI}\Gamma^{\nu}(\mathbb{T}; \mathsf{TM})$. There are two equivalent ways to present these classes of vector fields: (1) by directly prescribing the joint pointwise conditions on state and time in each regularity class; (2) using the C^{ν}-topologies. The former is probably the most understandable, but takes a few pages to write down. The latter is slick and elegant, but gives the impression of being too abstract to check. To save space, we shall give the topological characterisations and refer to [Jafarpour and Lewis 2014, Chapter 6] for the concrete descriptions, and the quite nontrivial proofs of the equivalence of the two descriptions.

The topological characterisation relies on notions of measurability, integrability, and boundedness in the locally convex spaces $\Gamma^\nu(\mathsf{TM})$. Let us review this quickly for an arbitrary locally convex space V, referring to references for details.

1. A function $\gamma\colon \mathbb{T} \to \mathsf{V}$ is ***measurable*** if $\gamma^{-1}(\mathcal{B})$ is Lebesgue measurable for every Borel set $\mathcal{B} \subseteq \mathsf{V}$.
2. It is possible to describe a notion of integral, called the ***Bochner integral***, for a function $\gamma\colon \mathbb{T} \to \mathsf{V}$ that closely resembles the usual construction of the Lebesgue integral. We refer to [Jafarpour and Lewis 2014] for a sketch of the construction, and to the references cited there for details; the note of Beckmann and Deitmar [2011] is particularly useful. A curve $\gamma\colon \mathbb{T} \to \mathsf{V}$ is ***Bochner integrable*** if its Bochner integral exists and is ***locally Bochner integrable*** if the Bochner integral of $\gamma|\mathbb{T}'$ exists for any compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$.
3. Finally, a subset $\mathcal{B} \subseteq \mathsf{V}$ is ***bounded*** if $p|\mathcal{B}$ is bounded any continuous seminorm p on V. A curve $\gamma\colon \mathbb{T} \to \mathsf{V}$ is ***essentially von Neumann bounded*** if there exists a bounded set $\mathcal{B}$ such that
$$\lambda(\{t \in \mathbb{T} \mid \gamma(t) \notin \mathcal{B}\}) = 0,$$
and is ***locally essentially von Neumann bounded*** if $\gamma|\mathbb{T}'$ is essentially von Neumann bounded for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$.

The following test for Bochner integrability and essential boundedness is one we shall use.

2.1 Lemma: (Test for Bochner integrability and von Neumann boundedness) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. For a manifold M of class C^r, an interval $\mathbb{T} \subseteq \mathbb{R}$, and a curve $\gamma\colon \mathbb{T} \to \Gamma^\nu(\mathsf{TM})$, the following two statements are equivalent:*

(i) γ is locally Bochner integrable;

(ii) for each of the seminorms p_K from (2.2), defined according to ν, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that $p_K \circ \gamma(t) \leq g(t)$ for every $t \in \mathbb{T}$,

and the following two statements are equivalent:

(iii) γ is locally essentially von Neumann bounded;

(iv) for each of the seminorms p_K from (2.2), defined according to ν, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that $p_K \circ \gamma(t) \leq g(t)$ for every $t \in \mathbb{T}$.

Proof: As indicated in [Jafarpour and Lewis 2014, Chapter 6], $\Gamma^\nu(\mathsf{TM})$ is complete and separable for all $\nu \in \{m+m', \infty, \omega\}$. By Theorems 3.1 and 3.2 of Beckmann and Deitmar [2011], we conclude that γ is locally Bochner integrable if and only if $p \circ \gamma$ is locally integrable for every continuous seminorm p for $\Gamma^\nu(\mathsf{TM})$. Since the seminorms from (2.2) define the locally convex topology of $\Gamma^\nu(\mathsf{TM})$, it suffices to check local integrability of $p_K \circ \gamma$. This proves the equivalence of the first two statements.

The equivalence of the second two statements follows from the definition of boundedness in a locally convex space. ■

With these notions, we can now define our classes of vector fields.

2.2 Definition: **(Classes of time-varying vector fields)** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. For a manifold M of class C^r and an interval $\mathbb{T} \subseteq \mathbb{R}$, let $X\colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ satisfy $X(t,x) \in \mathsf{T}_x\mathsf{M}$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$. Denote by X_t, $t \in \mathbb{T}$, the map $x \mapsto X(t,x)$ and suppose that $X_t \in \Gamma^\nu(\mathsf{TM})$ for every $t \in \mathbb{T}$. Then X is:

(i) a ***Carathéodory vector field of class*** $\mathbf{C}^{\boldsymbol{\nu}}$ if the curve $\mathbb{T} \ni t \mapsto X_t \in \Gamma^\nu(\mathsf{TM})$ is measurable;

(ii) ***locally integrally*** $\mathbf{C}^{\boldsymbol{\nu}}$***-bounded*** if the curve $\mathbb{T} \ni t \mapsto X_t \in \Gamma^\nu(\mathsf{TM})$ is locally Bochner integrable;

(iii) ***locally essentially*** $\mathbf{C}^{\boldsymbol{\nu}}$***-bounded*** if the curve $\mathbb{T} \ni t \mapsto X_t \in \Gamma^\nu(\mathsf{TM})$ is locally essentially von Neumann bounded.

We denote:

(iv) the set of Carathéodory vector fields of class C^ν by $\mathrm{CF}\Gamma^\nu(\mathbb{T};\mathsf{TM})$.

(v) the set of locally integrally C^ν-bounded vector fields by $\mathrm{LI}\Gamma^\nu(\mathbb{T};\mathsf{TM})$.

(vi) the set of locally essentially C^ν-bounded vector fields by $\mathrm{LB}\Gamma^\nu(\mathbb{T};\mathsf{TM})$. •

The classes of time-varying vector fields defined above have many excellent properties. Perhaps the most compelling of these is that the dependence of the flows of these vector fields on initial condition has regularity that matches ν. Let us be a little precise about this. We let $\nu \geq \mathrm{lip}$. The flow Φ^X of $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathsf{TM})$ is defined on an open subset D_X of the set $\mathbb{T} \times \mathbb{T} \times \mathsf{M}$ of final times, initial times, and initial states. For $(t, t_0, x_0) \in D_X$ there exists a neighbourhood $\mathcal{U}$ of x_0 such that the map

$$\mathcal{U} \ni x \mapsto \Phi^X(t, t_0, x) \in \mathsf{M}$$

is a C^ν-local diffeomorphism [Jafarpour and Lewis 2014]. The real analytic version of this result requires a deep understanding of the C^ω-topology.

2.4. Control systems. In this section we shall present a class of control systems of the "ordinary" sort. These systems, while of a standard form, are defined in such a way that the appropriate topology for the space of vector fields is carefully accounted for. The basic idea of the approach mirrors that of Section 2.3 for time-varying vector fields. Thus we consider a control system to be, not a joint function of state and control, but rather a vector field-valued function of control. As with time-varying vector fields, the topology on the space of vector fields is essential. This approach is explained in detail in [Jafarpour and Lewis 2016], so here we shall give an abbreviated presentation. We begin by describing "parameterised vector fields," then we turn to control systems.

We consider a parameter space $\mathcal{P}$, which is a topological space, and a mapping $X\colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ having the property $X(x,p) \in \mathsf{T}_x\mathsf{M}$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$. We denote by X^p the map $x \mapsto X(x,p)$, which is thus a vector field depending on the parameter p. We wish to give joint conditions on X so that the regularity is respected, not just for fixed p, but as p varies. As with time-varying vector fields, there are two ways to approach this: (1) by considering joint pointwise conditions on state and parameter; (2) by using the C^ν-topologies. As with time-varying vector fields, we choose the latter route since it saves space. We refer to [Jafarpour and Lewis 2016, §4] for the pointwise conditions and for the proofs of the equivalence of the two characterisations.

Unlike we saw for time-dependence, the situation for parameterised vector fields only relates to continuity, so requires no initial buildup.

2.3 Definition: (Classes of parameterised vector fields) Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. For a manifold M of class C^r and a topological space $\mathcal{P}$, let $X \colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ satisfy $X(x, p) \in \mathsf{T}_x\mathsf{M}$ for each $(x, p) \in \mathsf{M} \times \mathcal{P}$. Denote by X_x, $x \in \mathsf{M}$, the map $p \mapsto X(x, p)$ and by X^p, $p \in \mathcal{P}$, the map $x \mapsto X(x, p)$. Then X is a:

(i) ***separately parameterised vector field of class C^ν*** if X_x is continuous for every $x \in \mathsf{M}$ and if X^p is of class C^ν for every $p \in \mathcal{P}$;

(ii) ***jointly parameterised vector field of class C^ν*** if it is a separately parameterised vector field of class C^ν and if the map $\mathcal{P} \ni p \mapsto X^p \in \Gamma^\nu(\mathsf{TM})$ is continuous.

We denote the set of jointly parameterised vector fields of class C^ν by $\mathrm{JP}\Gamma^\nu(\mathcal{P}; \mathsf{TM})$. •

With these notions of parameterised sections, we readily define what we mean by a control system.

2.4 Definition: (Control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C^ν-control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a topological space called the ***control set***, and

(iii) $F \in \mathrm{JP}\Gamma^\nu(\mathcal{C}; \mathsf{TM})$. •

A special class of control systems is given in the next definition.

2.5 Definition: (Sublinear control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C^ν-sublinear control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a subset of a locally convex topological vector space V, $\mathcal{C}$ being called the ***control set***, and

(iii) F has the following property: for every continuous seminorm p for $\Gamma^\nu(\mathsf{TM})$, there exists a continuous seminorm q for V such that

$$p(F^{u_1} - F^{u_2}) \leq q(u_1 - u_2), \qquad u_1, u_2 \in \mathcal{C}. \qquad \bullet$$

Note that, by Definition 2.3, the sublinearity condition (iii) implies that a C^ν-sublinear control system is a C^ν-control system. A special sort of sublinear control system arises if there exists a continuous linear map $\Lambda \in \mathrm{L}(\mathsf{V}; \Gamma^\nu(\mathsf{TM}))$ for which $F(x, u) = \Lambda(u)(x)$. Such systems are called ***control-linear***, and include control-affine systems as a special case.

The governing equations for a control system are

$$\xi'(t) = F(\xi(t), \mu(t)),$$

for suitable functions $t \mapsto \mu(t) \in \mathcal{C}$ and $t \mapsto \xi(t) \in \mathsf{M}$. Of course, for these equations to be sensible, it should be the case that, upon substitution of a locally essentially bounded control, or a locally integrable control for sublinear systems, the resulting time-varying vector field is of the sort that possesses solutions. These matters are addressed by Jafarpour and Lewis [2016], where it is further shown that the flows of such time-varying vector fields depend on initial conditions in a manner consistent with the regularity of the control system, cf. the final paragraph of Section 2.3.

3. A review of tautological control systems

In this section we overview the theory of tautological control systems as introduced by Lewis [2014]. The theory is designed to deal with, among other things, the problem of feedback-invariance such as enunciated in Example 1.1. This is carried out by two main devices. The first is the replacement of the control set with an arbitrary family of vector fields, which becomes a topological space by virtue of the constructions of Section 2.2. This idea is not new, e.g., [Sussmann 1998], but we are able to extend it to new directions by virtue of our understanding of the real analytic topology. The second main feature of the approach of tautological control systems is the prescription of data locally, rather than globally, by virtue of presheaves and sheaves of sets of vector fields. This device allows for great generality in the sorts of things that are included as "control systems," and also provides a mechanism for systematically handling local constructions that abound in control theory, constructions such as linearisation (dealt with here) and local controllability, stabilisability, and optimality. Readers new to this approach can expect to take some time to understand the ideas behind the theory, and we hope that reading [Lewis 2014] is useful in this regard.

3.1. Presheaves and sheaves of sets of vector fields. One of the features of the tautological control system formulation is that it makes use of presheaves and sheaves of sets of vector fields in an essential manner. The motivation for this is explained by Lewis [2014], and we particularly refer to the formulation of the sub-Riemannian geodesic problem in that work as an illustration of the utility of the presheaf and sheaf formalism.

In any case, in this section we quickly review the essential material, which means that we merely give the definitions. A reader unfamiliar with these notions will want to look at the examples in [Lewis 2014].

3.1 Definition: (Presheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A ***presheaf of $\mathbf{C}^\nu$-vector fields*** is an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U})$ of $\Gamma^\nu(\mathsf{T}\mathcal{U})$ with the property that, for open sets $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$, the map

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}} \colon \mathscr{F}(\mathcal{U}) &\to \Gamma^\nu(\mathsf{T}\mathcal{V}) \\ X &\mapsto X|\mathcal{V} \end{aligned}$$

takes values in $\mathscr{F}(\mathcal{V})$. Elements of $\mathscr{F}(\mathcal{U})$ are called ***local sections*** of $\mathscr{F}$ over $\mathcal{U}$. •

The notion of a presheaf is natural, but the more restrictive notion of a sheaf is also sometimes useful, even in control theory.

3.2 Definition: (Sheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A presheaf $\mathscr{F}$ of sets of C^ν-vector fields is a ***sheaf of sets of*** $\mathbf{C}^\nu$***-vector fields*** if, for every open set $\mathcal{U} \subseteq \mathsf{M}$, for every open cover $(\mathcal{U}_a)_{a \in A}$ of $\mathcal{U}$, and for every choice of local sections $X_a \in \mathscr{F}(\mathcal{U}_a)$ satisfying $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X|\mathcal{U}_a = X_a$ for every $a \in A$. •

An important sort of presheaf is as illustrated by the following example.

3.3 Example: (Globally generated presheaf) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. If $\mathscr{X} \subseteq \Gamma^\nu(\mathsf{TM})$ is any family of vector fields on M, then we can define an associated presheaf $\mathscr{F}_{\mathscr{X}}$ of sets of vector fields by

$$\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \{X|\mathcal{U} \mid X \in \mathscr{X}\}.$$

Note that $\mathscr{F}(\mathsf{M})$ is necessarily equal to $\mathscr{X}$, and so we shall typically use $\mathscr{F}(\mathsf{M})$ to denote the set of globally defined vector fields giving rise to this presheaf. A presheaf of this sort will be called ***globally generated***. As shown in [Lewis 2014], globally generated presheaves are seldom sheaves. •

There is a process by which one naturally converts a presheaf into a sheaf. To make the construction, for a presheaf $\mathscr{F}$ of sets of vector fields, we denote by $\mathscr{F}_x$ the stalk at x, i.e., the set of germs of vector fields from $\mathscr{F}$. For a local section X defined in a neighbourhood of x, we denote by $[X]_x$ the germ of X at x. With this notation, we have the following definition.

3.4 Definition: (Sheafification of a presheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{F}$ be a presheaf of sets of C^ν-vector fields. The ***sheafification*** of $\mathscr{F}$ is the sheaf $\mathrm{Sh}(\mathscr{F})$ of sets of vector fields defined by

$$\mathrm{Sh}(\mathscr{F})(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid [X]_x \in \mathscr{F}_x \text{ for all } x \in \mathcal{U}\}.$$ •

The verification that $\mathrm{Sh}(\mathscr{F})$ is a sheaf is straightforward, and is given by Lewis [2014, Lemma 4.6].

3.2. Tautological control systems.

Equipped with the notion of a presheaf, it is easy to say what we mean by a tautological control system.

3.5 Definition: (Tautological control system and related notions) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required.

(i) A $\mathbf{C}^\nu$***-tautological control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where M is a manifold of class C^r whose elements are called ***states*** and where $\mathscr{F}$ is a presheaf of sets of C^ν-vector fields on M.

(ii) A tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***complete*** if $\mathscr{F}$ is a sheaf and is ***globally generated*** if $\mathscr{F}$ is globally generated.

(iii) The ***completion*** of $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is the tautological control system $\mathrm{Sh}(\mathfrak{G}) = (\mathsf{M}, \mathrm{Sh}(\mathscr{F}))$. •

Given a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$, there is a naturally associated tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ defined by taking

$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \mid u \in \mathcal{C}\}.$$

The presheaf $\mathscr{F}_\Sigma$ is obviously globally generated, and so generally is not a sheaf. It can, however, be sheafified if one wishes.

3.3. Open-loop systems, open-loop subfamilies, trajectories.

In order to define the notion of a trajectory for a tautological control system, there is a little buildup required. First we provide the tautological control system analogue of "plugging in control as a function of time" in the standard framework for control theory.

3.6 Definition: (Open-loop system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop system*** for $\mathfrak{G}$ is a triple $\mathfrak{G}_{\mathrm{ol}} = (X, \mathbb{T}, \mathcal{U})$ where

(i) $\mathbb{T} \subseteq \mathbb{R}$ is an interval called the ***time-domain***;

(ii) $\mathcal{U} \subseteq \mathsf{M}$ is open;

(iii) $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$. •

For a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$, there is a natural sort of open-loop system that arises by choosing an open-loop control $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ (or $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ for sublinear control systems). This is the open-loop system $\mathfrak{G}_{\Sigma,\mu} = (F^\mu, \mathbb{T}, \mathsf{M})$ for $\mathfrak{G}_\Sigma$ defined by

$$F^\mu(t)(x) = F(x, \mu(t)), \qquad t \in \mathbb{T},\ x \in \mathsf{M}.$$

It is shown in Propositions 6 and 7 of [Jafarpour and Lewis 2016] that F^μ is in $\mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ for control systems and in $\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ for sublinear control systems.

In "ordinary" control theory, one often wishes to restrict the class of controls from $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ or $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ (when this latter is defined) to a class with particular properties, e.g., piecewise constant. The following notion mimics this for tautological control systems.

3.7 Definition: (Open-loop subfamily) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop subfamily*** for $\mathfrak{G}$ is an assignment, to each interval $\mathbb{T} \subseteq \mathbb{R}$ and each open set $\mathcal{U} \subseteq \mathsf{M}$, a subset $\mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}) \subseteq \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ with the property that, if $(\mathbb{T}_1, \mathcal{U}_1)$ and $(\mathbb{T}_2, \mathcal{U}_2)$ are such that $\mathbb{T}_1 \subseteq \mathbb{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then

$$\{X|\mathbb{T}_1 \times \mathcal{U}_1 \mid X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_2, \mathcal{U}_2)\} \subseteq \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_1, \mathcal{U}_1). \qquad \bullet$$

Here are some examples of open-loop subfamilies, the last of which we shall need for our theory of linearisation.

3.8 Examples: (Open-loop subfamilies) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system.

1. The ***full subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\mathrm{full}}$ defined by

$$\mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T},\mathcal{U}) = \mathrm{LI}\Gamma^{\nu}(\mathbb{T};\mathscr{F}(\mathcal{U})).$$

   Thus the full subfamily contains all possible open-loop systems. Of course, every open-loop subfamily will be contained in this one.
2. The ***locally essentially bounded subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\infty}$ defined by asking that

$$\mathscr{O}_{\mathfrak{G},\infty}(\mathbb{T},\mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T},\mathcal{U}) \mid X \in \mathrm{LB}\Gamma^{\nu}(\mathbb{T};\mathsf{T}\mathcal{U})\}.$$

   Thus, for the locally essentially bounded subfamily, we require that the condition of being locally integrally C^{ν}-bounded be replaced with the stronger condition of being locally essentially C^{ν}-bounded.
3. The ***locally essentially compact subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\mathrm{cpt}}$ defined by asking that

$$\begin{aligned}\mathscr{O}_{\mathfrak{G},\mathrm{cpt}}(\mathbb{T},\mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T},\mathcal{U})| \ &\text{for every compact subinterval } \mathbb{T}' \subseteq \mathbb{T} \\ &\text{there exists a compact } K \subseteq \Gamma^{\nu}(\mathbb{T};\mathsf{T}\mathcal{U}) \\ &\text{such that } X(t) \subseteq K \text{ for almost every } t \in \mathbb{T}'\}.\end{aligned}$$

   Thus, for the locally essentially compact subfamily, we require that the condition of being locally essentially bounded in the von Neumann bornology (that defines the locally essentially bounded subfamily) be replaced with being locally essentially bounded in the compact bornology.[2]

   We comment that in cases when the compact and von Neumann bornologies agree, then of course we have $\mathscr{O}_{\mathfrak{G},\infty} = \mathscr{O}_{\mathfrak{G},\mathrm{cpt}}$. As pointed out by Jafarpour and Lewis [2014], this is the case when $\nu \in \{\infty,\omega\}$.
4. We can associate an open-loop subfamily to an open-loop system as follows. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0,\mathrm{lip}\}$, let $\nu \in \{m+m',\infty,\omega\}$, and let $r \in \{\infty,\omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M},\mathscr{F})$ be a C^{ν}-tautological control system, let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T},\mathcal{U})$. We denote by $\mathscr{O}_{\mathfrak{G},X}$ the open-loop subfamily defined as follows. If $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$, then we let

$$\mathscr{O}_{\mathfrak{G},X}(\mathbb{T}',\mathcal{U}') = \{X' \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}',\mathcal{U}') \mid X' = X|\mathbb{T}' \times \mathcal{U}'\}.$$

   If $\mathbb{T}' \not\subseteq \mathbb{T}$ and/or $\mathcal{U}' \not\subseteq \mathcal{U}$, then we take $\mathscr{O}_{\mathfrak{G},X} = \varnothing$. Thus $\mathscr{O}_{\mathfrak{G},X}$ is comprised of those vector fields from $\mathscr{O}_{\mathfrak{G}}$ that are merely restrictions of X to smaller domains. •

3.4. Trajectories. The preceding constructions allow one to define trajectories in a fairly straightforward manner. As in standard control theory, one should think of a trajectory as being an integral curve of some vector field arising after the substitution of control as a function of time.

[2] We shall not make essential use of bornologies in this paper. However, they are useful for understanding certain facets of the general theory, and we refer to [Lewis 2014] for discussion.

3.9 Definition: (Trajectory for tautological control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$.

(i) For a time-domain $\mathbb{T}$, an open set $\mathcal{U} \subseteq \mathsf{M}$, and for $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$, an ***$(X, \mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$.

(ii) For a time-domain $\mathbb{T}$ and an open set $\mathcal{U} \subseteq \mathsf{M}$, a ***$(\mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$ for some $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$.

(iii) A ***trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve that is a $(\mathbb{T}, \mathcal{U})$-trajectory for $\mathscr{O}_{\mathfrak{G}}$ for some time-domain $\mathbb{T}$ and some open set $\mathcal{U} \subseteq \mathsf{M}$.

We denote by:

(iv) $\mathrm{Traj}(X, \mathbb{T}; \mathcal{U})$ the set of $(X, \mathbb{T}, \mathcal{U})$-trajectories for $\mathscr{O}_{\mathfrak{G}}$;

(v) $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$ the set of $(\mathbb{T}, \mathcal{U})$-trajectories for $\mathscr{O}_{\mathfrak{G}}$;

(vi) $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G}})$ the set of trajectories for $\mathscr{O}_{\mathfrak{G}}$.

We shall abbreviate $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},\mathrm{full}}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathfrak{G})$ and $\mathrm{Traj}(\mathfrak{G}) = \mathrm{Traj}(\mathscr{O}_{\mathfrak{G},\mathrm{full}})$. •

Note that if we have a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$ with an open-loop control μ giving rise to the tautological control system $\mathfrak{G}_\Sigma$ and an open-loop system $\mathfrak{G}_{\Sigma,\mu}$, then trajectories are exactly curves ξ satisfying the open-loop control equations

$$\xi'(t) = F(\xi(t), \mu(t)),$$

as expected. However, this does not mean that, in this case, the trajectories of Σ and $\mathfrak{G}_\Sigma$ agree. This issue is discussed in detail in [Lewis 2014, §5.5], and we summarise these results in Section 3.5 below.

Given a trajectory, there is a naturally associated open-loop subfamily of which we shall make use.

3.10 Example: (The open-loop subfamily defined by a trajectory) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system, let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$, and let $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$. We denote by $\mathscr{O}_{\mathfrak{G},\xi}$ the open-loop subfamily defined as follows. If $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$ are such that $\xi(\mathbb{T}') \subseteq \mathcal{U}'$, then we let

$$\mathscr{O}_{\mathfrak{G},\xi}(\mathbb{T}', \mathcal{U}') = \{X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}', \mathcal{U}') \mid \xi'(t) = X(t, \xi(t)),\ \text{a.e. } t \in \mathbb{T}'\}.$$

If $\mathbb{T}' \not\subseteq \mathbb{T}$ or $\mathcal{U}' \not\subseteq \mathcal{U}$, or if $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$ but $\xi(\mathbb{T}') \not\subseteq \mathcal{U}'$, then we take $\mathscr{O}_{\mathfrak{G},\xi} = \varnothing$. Thus $\mathscr{O}_{\mathfrak{G},\xi}$ is comprised of those vector fields from $\mathscr{O}_{\mathfrak{G}}$ possessing ξ (restricted to the appropriate subinterval) as an integral curve. •

3.5. Tautological control systems and control systems. As part of our development above of tautological control systems, we indicated the natural way in which control systems give rise to tautological control systems. It turns out that one can also regard a globally generated tautological control system as an "ordinary" control system. Let us describe how to do this. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. To a globally generated C^ν -tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, we associate a C^ν-control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ by asking that $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ and $F_{\mathscr{F}}(x, X) = X(x)$.

Since, in the notation of Definition 2.3, $F^X = X$, it follows that $\Sigma_{\mathfrak{G}}$ is indeed a C^ν-control system as per Definition 2.4. Note that we now can make sense of the symbols $\Sigma_{\mathfrak{G}_\Sigma}$ and $\mathfrak{G}_{\Sigma_{\mathfrak{G}}}$, i.e., we can go back and forth between control systems and tautological control systems. Exactly the extent to which this returns you to where you started is stated in [Lewis 2014, Proposition 5.4]. In [Lewis 2014, §5.5] some theorems are provided which record the correspondence of trajectories for a control system and its corresponding tautological control system. We summarise these results as follows.

1. If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ is a C^ν-control system and if F is injective and open onto its image, then $\Sigma = \Sigma_{\mathfrak{G}_\Sigma}$.
2. If $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is a globally generated tautological control system, then $\mathfrak{G} = \mathfrak{G}_{\Sigma_{\mathfrak{G}}}$.
3. If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ is a C^ν-control system and if F is injective and proper, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$.
4. If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ is a C^ν-control system, if $\mathcal{C}$ is a Suslin topological space,[3] and if F is proper, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$. In particular, if $\mathcal{C}$ is compact, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$.
5. If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ is a C^ν-control-linear system, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$. In particular, if Σ is a control-affine system, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$.
6. For a globally generated tautological control system $\mathfrak{G} = (\mathsf{M}, \mathfrak{G})$, $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},\mathrm{full}}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma_{\mathfrak{G}})$.

4. Linearisation of tautological control systems

As an illustration of the fact that it is possible to do non-elementary things in the framework of tautological control systems, we present a fully developed theory for the linearisation of these systems. This theory is both satisfying and revealing. It is satisfying because it is very simple (if one knows a little tangent bundle geometry) and it is revealing because, for example, it clarifies and rectifies the hiccup with classical linearisation theory that was revealed in Example 1.1. It is, however, not trivial (as it shall occupy us for the remainder of the paper) and it does lead to results that make us uncomfortable (e.g., the linearisation of a time-invariant system about an equilibrium point may be time-varying). Our opinion is that this is a price one pays for a feedback-invariant theory; the decision on whether to pay this price is one that must be made by the reader.

In this section we work with systems of general regularity, only requiring that they be at least once differentiable so that we can easily define their linearisations. For dealing with Lipschitz systems, we will use the following result.

4.1 Lemma: $(\mathbf{C}^{(m+\mathrm{lip})-1} = \mathbf{C}^{(m-1)+\mathrm{lip}})$ *For a smooth manifold* M *and for* $m \in \mathbb{Z}_{>0}$*, if* $X \in \Gamma^{m+\mathrm{lip}}(\mathsf{TM})$*, then* $j_1 X \in \Gamma^{(m-1)+\mathrm{lip}}(\mathsf{J}^1\mathsf{TM})$*. Moreover,* $\mathrm{dil}\, j_{m-1}(j_1 X)(x) = \mathrm{dil}\, j_m X(x)$ *for every* $x \in \mathsf{M}$*.*

Proof: We need to show that $j_{m-1}(j_1 X)$ is locally Lipschitz. This, however, is clear since $j_{m-1} j_1 X$ is the image of $j_m X$ under the injection of $\mathsf{J}^m\mathsf{TM}$ in $\mathsf{J}^{m-1}\mathsf{J}^1\mathsf{TM}$ [Saunders 1989, Definition 6.2.25], and since $j_m X$ is Lipschitz by hypothesis.

[3] Recall that this means that $\mathcal{C}$ is the continuous image of a complete, separable, metric space.

The last formula in the statement of the lemma requires us to make sense of $\operatorname{dil} j_{m-1}(j_1 X)$. This is made sense of using the fact that, by (2.1), one has $\mathsf{J}^1\mathsf{TM} \simeq \mathsf{T}^*\mathsf{M} \otimes \mathsf{TM}$, and so the Riemannian metric $\mathbb{G}$ on M and its Levi-Civita connection induce a fibre metric and linear connection in the vector bundle $\mathsf{J}^1\mathsf{TM}$ as in Sections 2.1 and 2.2 of [Jafarpour and Lewis 2014]. Now let us examine the inclusion of $\mathsf{J}^m\mathsf{TM}$ in $\mathsf{J}^{m-1}\mathsf{J}^1\mathsf{TM}$ to verify the final assertion of the lemma. We use (2.1) to write

$$\mathsf{J}^m\mathsf{TM} \simeq \oplus_{j=0}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}.$$

In this case, and using the generalisation of the isomorphism (2.1) for vector bundles as in [Jafarpour and Lewis 2014, Lemma 2.1], the inclusion of $\mathsf{J}^m\mathsf{TM}$ in $\mathsf{J}^1\mathsf{J}^{m-1}\mathsf{TM}$ becomes identified with the natural inclusions

$$\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM} \to \mathrm{S}^{j-1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{T}^*\mathsf{M} \otimes \mathsf{TM}, \qquad j \in \{0, 1, \dots, m-1\},$$

given by

$$\alpha^1 \odot \cdots \odot \alpha^j \otimes v \mapsto \sum_{k=1}^{j} \alpha^1 \odot \cdots \odot \alpha^{k-1} \odot \alpha^{k+1} \odot \cdots \odot \alpha^j \otimes \alpha^k \otimes v.$$

The fibre metric on $\mathrm{S}^j(\mathsf{T}^*\mathsf{M})$ is the restriction of that on $\mathrm{T}^j(\mathsf{T}^*\mathsf{M})$. Thus the preceding inclusion preserves the fibre metrics since these are defined componentwise on the tensor product. Similarly, since the connection in the symmetric and tensor products is defined so as to satisfy the Leibniz rule for the tensor product, the injection above commutes with parallel translation. It now follows from the characterisation of dilatation given in [Jafarpour and Lewis 2014, Lemma 3.12] that the final formula in the statement of the lemma holds. ■

4.1. Tangent bundle geometry. To make the constructions in this section, we recall a little tangent bundle geometry. Throughout this section, we let $m \in \mathbb{Z}_{>0}$, $m' \in \{0, \mathrm{lip}\}$, and let $\nu \in \{m + m', \infty, \omega\}$. We take $r \in \{\infty, \omega\}$, as required. The meaning of "$\nu - 1$" is obvious for all ν. But, to be clear, $\infty - 1 = \infty$, $\omega - 1 = \omega$, and, given Lemma 4.1, $(m + \mathrm{lip}) - 1 = (m-1) + \mathrm{lip}$.

Let $X \in \Gamma^\nu(\mathsf{TM})$. We will lift X to a vector field on TM in two ways. The first is the vertical lift, and is described first by a vector bundle map $\mathrm{vlft}\colon \pi_{\mathsf{TM}}^*\mathsf{TM} \to \mathsf{TTM}$ as follows. Let $x \in \mathsf{M}$ and let $v_x, w_x \in \mathsf{T}_x\mathsf{M}$. The **vertical lift** of u_x to v_x is given by

$$\mathrm{vlft}(v_x, u_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_x + t u_x).$$

Now, if $X \in \Gamma^\nu(\mathsf{TM})$, we define $X^V \in \Gamma^\nu(\mathsf{TTM})$ by $X^V(v_x) = \mathrm{vlft}(v_x, X(x))$. In coordinates $(x^1, \dots, x^n)$ for M with $((x^1, \dots, x^n), (v^1, \dots, v^n))$ the associated natural coordinates for TM, if $X = X^j \frac{\partial}{\partial x^j}$, then $X^V = X^j \frac{\partial}{\partial v^j}$. The vertical lift is a very simple vector field. It is tangent to the fibres of TM, and is in fact constant on each fibre.

The other lift of $X \in \Gamma^\nu(\mathsf{TM})$ that we shall use is the ***tangent lift***[4] which is the vector field X^T on TM of class $\mathrm{C}^{\nu-1}$ whose flow is given by $\Phi_t^{X^T}(v_x) = T_x\Phi_t^X(v_x)$. Therefore,

[4]This is also frequently called the ***complete lift***. However, "tangent lift" so much better captures the essence of the construction, that we prefer our terminology. Also, the dual of the tangent lift is used in the Maximum Principle, and this is then conveniently called the "cotangent lift."

explicitly,

$$X^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x\Phi^X_t(v_x).$$

In coordinates as above, if $X = X^j\frac{\partial}{\partial x^j}$, then

$$X^T = X^j\frac{\partial}{\partial x^j} + \frac{\partial X^j}{\partial x^k}v^k\frac{\partial}{\partial v^j}. \tag{4.1}$$

One recognises the "linearisation" of X in this expression, but one should understand that the second term in this coordinate expression typically has no meaning by itself. The flow for X^T is related to that for X according to the following commutative diagram:

$$\begin{tikzcd} \mathsf{TM} \arrow[r, "\Phi^{X^T}_t"] \arrow[d, "\pi_{\mathsf{TM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{M} \arrow[r, "\Phi^X_t"'] & \mathsf{M} \end{tikzcd} \tag{4.2}$$

Thus X^T projects to X in the sense that $T_{v_x}\pi_{\mathsf{TM}}(X^T(v_x)) = X(x)$. Moreover, X^T is a "linear" vector field (as befits its appearance in "linearisation" below), which means that the diagram

$$\begin{tikzcd} \mathsf{TM} \arrow[r, "X^T"] \arrow[d, "\pi_{\mathsf{TM}}"'] & \mathsf{TTM} \arrow[d, "T\pi_{\mathsf{TM}}"] \\ \mathsf{M} \arrow[r, "X"'] & \mathsf{TM} \end{tikzcd} \tag{4.3}$$

defines X^T as a vector bundle map over X.

We will be interested in the flow of the tangent lift in the time-varying case, and the next lemma indicates how this works.

4.2 Lemma: (Tangent lift for time-varying vector fields) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain. For $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathsf{TM})$ define $X^T\colon \mathbb{T}\times\mathsf{TM} \to \mathsf{TTM}$ by $X^T(t, v_x) = (X(t))^T(v_x)$. Then*

(i) $X^T \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T};\mathsf{TTM})$,

(ii) if $(t, t_0, x_0) \in D_X$, then $(t, t_0, v_{x_0}) \in D_{X^T}$ for every $v_{x_0} \in \mathsf{T}_{x_0}\mathsf{M}$, and

(iii) $X^T(t, v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} T_x\Phi^X_{t+\tau,t}(v_x)$.

Proof: (i) Since differentiation with respect to x preserves measurability in t,[5] and since the coordinate expression for X^T involves differentiating the coordinate expression for X, we conclude that X^T is a Carathéodory vector field. To show that $X^T \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T};\mathsf{TTM})$ requires, according to our definitions of Section 2.3, an affine connection on TM and a Riemannian metric on TM. We suppose, of course, that we have an affine connection ∇ and a Riemannian metric $\mathbb{G}$ on M. For simplicity of some of the computations below,

[5]Derivatives are limits of sequences of difference quotients, each of which is measurable, and limits of sequences of measurable functions are measurable [Cohn 2013, Proposition 2.1.5].

and without loss of generality, we shall suppose that ∇ is torsion-free. In case $r = \omega$, we suppose these are real analytic, according to [Jafarpour and Lewis 2014, Lemma 2.4]. In case $\nu = m + \mathrm{lip}$ for some $m \in \mathbb{Z}_{>0}$, we assume that ∇ is the Levi-Civita connection associated with $\mathbb{G}$.

Let us first describe the Riemannian metric on TM we shall use. The affine connection ∇ gives a splitting $\mathsf{TTM} \simeq \pi_{\mathsf{TM}}^*\mathsf{TM} \oplus \pi_{\mathsf{TM}}^*\mathsf{TM}$ [Kolář, Michor, and Slovák 1993, §11.11]. We adopt the convention that the second component of this decomposition is the vertical component so $T_{v_x}\pi_{\mathsf{TM}}$ restricted to the first component is an isomorphism onto $\mathsf{T}_x\mathsf{M}$, i.e., the first component is "horizontal." If $X \in \Gamma^\nu(\mathsf{TM})$ we denote by $X^H \in \Gamma^\nu(\mathsf{TTM})$ the unique horizontal vector field for which $T_{v_x}\pi_{\mathsf{TM}}(X^H(v_x)) = X(x)$ for every $v_x x \in \mathsf{TM}$, i.e., X^H is the "horizontal lift" of X. Let us denote by $\mathrm{hor}, \mathrm{ver}\colon \mathsf{TTM} \to \pi_{\mathsf{TM}}^*\mathsf{TM}$ the projections onto the first and second components of the direct sum decomposition. This then immediately gives a Riemannian metric $\mathbb{G}^T$ on TM by

$$\mathbb{G}^T(X_{v_x}, Y_{v_x}) = \mathbb{G}(\mathrm{hor}(X_{v_x}), \mathrm{hor}(Y_{v_x})) + \mathbb{G}(\mathrm{ver}(X_{v_x}), \mathrm{ver}(Y_{v_x})).$$

This is called the ***Sasaki metric*** [Sasaki 1958] in the case that ∇ is the Levi-Civita connection associated with $\mathbb{G}$.

Now let us determine how an affine connection on TM can be constructed from ∇. There are a number of ways to lift an affine connection from M to one on TM, many of these being described by Yano and Ishihara [1973]. We shall use the so-called "tangent lift" of ∇, which is the unique affine connection ∇^T on TM satisfying $\nabla^T_{X^T} Y^T = (\nabla_X Y)^T$ for $X, Y \in \Gamma^\nu(\mathsf{TM})$ [Yano and Kobayashi 1966, §7], [Yano and Ishihara 1973, page 30].

We have the following sublemma.

1 Sublemma: *If $X \in \Gamma^\nu(\mathsf{TM})$, if $v_x \in \mathsf{TM}$, if $k \in \mathbb{Z}_{\geq 0}$ satisfies $k \leq \nu$, if $X_1, \dots, X_k \in \Gamma^\infty(\mathsf{TM})$, and if $Z_a \in \{X_a^T, X_a^V\}$, $a \in \{1, \dots, k\}$, then the following formula holds:*

$$(\nabla^T)^{(k-1)} X^T(Z_1, \dots, Z_k) = \begin{cases} (\nabla^{(k)} X(X_1, \dots, X_k))^V, & Z_a \text{ vertical for some } a \in \{1, \dots, k\}, \\ (\nabla^{(k-1)} X(X_1, \dots, X_k))^T, & \text{otherwise.} \end{cases}$$

Proof: By [Yano and Kobayashi 1966, Proposition 7.2], we have

$$\nabla^T X^T(X_1^T) = (\nabla X(X_1))^T, \quad \nabla^T X^T(X_1^V) = (\nabla X(X_1))^V,$$

giving the result when $k = 0$. Suppose the result is true for $k \in \mathbb{Z}_{\geq 0}$, and let $Z_a \in \{X_a^T, X_a^V\}$, $a \in \{1, \dots, k+1\}$. First suppose that $Z_{k+1} = X_{k+1}^T(v_x)$. We then compute, using the fact that covariant differentiation commutes with contraction [Dodson and Poston 1991, Theorem 7.03(F)],

$$\begin{aligned}(\nabla^T)^{(k)} X^T(Z_1, \dots, Z_k, Z_{k+1}) = \nabla^T_{X_{k+1}^T}&((\nabla^T)^{(k-1)} X^T)(Z_1, \dots, Z_k) \\ &- \sum_{j=1}^k (\nabla^T)^{(k-1)} X^T(Z_1, \dots, \nabla^T_{X_{k+1}^T} Z_j, \dots, Z_k). \quad (4.4)\end{aligned}$$

We now consider two cases.

1. *None of* $Z_1, \dots, Z_k$ *are vertical:* In this case, by the induction hypothesis,
$$((\nabla^T)^{(k-1)} X^T)(Z_1, \dots, Z_k) = (\nabla^{(k-1)} X)(U_1, \dots, U_k))^T,$$
and [Yano and Kobayashi 1966, Proposition 7.2] gives
$$\nabla^T_{X^T_{k+1}}((\nabla^T)^{(k-1)} X^T)(Z_1, \dots, Z_k) = (\nabla_{X_{k+1}}(\nabla^{(k-1)} X)(U_1, \dots, U_k))^T.$$
Again using [Yano and Kobayashi 1966, Proposition 7.2] and also using the induction hypothesis, we have, for $j \in \{1, \dots, k\}$,
$$(\nabla^T)^{(k-1)} X^T(Z_1, \dots, \nabla^T_{X^T_{k+1}} Z_j, \dots, Z_k) = (\nabla^{(k-1)} X(U_1, \dots, \nabla_{X_{k+1}} U_j, \dots, U_k))^T.$$
Combining the preceding two formulae with (4.4) gives the desired conclusion for $k+1$ in this case.
2. *At least one of* $Z_1, \dots, Z_k$ *is vertical:* In this case, we have
$$((\nabla^T)^{(k-1)} X^T)(Z_1, \dots, Z_k) = (\nabla^{(k-1)} X)(U_1, \dots, U_k))^V$$
by the induction hypothesis. Applications of [Yano and Kobayashi 1966, Proposition 7.2] and the induction hypothesis give the formulae
$$\nabla^T_{X^T_{k+1}}((\nabla^T)^{(k-1)} X^T)(Z_1, \dots, Z_k) = (\nabla_{X_{k+1}}(\nabla^{(k-1)} X)(U_1, \dots, U_k))^V.$$
and, for $j \in \{1, \dots, k\}$,
$$(\nabla^T)^{(k-1)} X^T(Z_1, \dots, \nabla^T_{X^T_{k+1}} Z_j, \dots, Z_k) = (\nabla^{(k-1)} X(U_1, \dots, \nabla_{X_{k+1}} U_j, \dots, U_k))^V.$$
Combining the preceding two formulae with (4.4) again gives the desired conclusion for $k+1$ in this case.

If we take $Z_{k+1} = X^V_{k+1}$, an entirely similar argument gives the result for this case for $k+1$, and so completes the proof of the sublemma. ▼

To complete the proof of the lemma, let us for the moment simply regard X as a vector field of class C^ν, not depending on time. We will make use of the fact that, for every $v_x \in \mathsf{TM}$, $\mathsf{T}_{v_x}\mathsf{TM}$ is spanned by vector fields of the form $X^T_1 + Y^V_1$ since vertical lifts obviously span the vertical space and since tangent lifts of nonzero vector fields are complementary to the vertical space. Therefore, for a fixed v_x, we can choose $X_1, \dots, X_n, Y_1, \dots, Y_n \in \Gamma^\infty(\mathsf{M})$ so that $(X^T_1(v_x), \dots, X^T_n(v_x))$ and $(Y^V_1(v_x), \dots, Y^V_n(v_x))$ comprise $\mathbb{G}^T$-orthonormal bases for the horizontal and vertical subspaces, respectively, of $\mathsf{T}_{v_x}\mathsf{TM}$. Note that these vector fields depend on v_x, but for the moment we will fix v_x. We use the following formula given by Barbero-Liñán and Lewis [2012, Lemma 4.5] for any vector field W of class C^ν on M:
$$W^T(v_x) = W^H(v_x) + \mathrm{vlft}(v_x, \nabla_{v_x} W(x)), \tag{4.5}$$
keeping in mind that we are supposing ∇ to be torsion-free.

By the sublemma, if $Z_a = X^T_{j_a}$, $a \in \{1, \dots, k\}$, then we have
$$\begin{aligned}(\nabla^T)^{(k-1)} X^T(v_x)(Z_1(v_x), \dots, Z_k(v_x)) = (\nabla^{(k-1)} X(x)(X_{j_1}(x), \dots, X_{j_k}(x)))^H \\ + \mathrm{vlft}(v_x, \nabla_{v_x}(\nabla^{(k-1)} X(X_{j_1}, \dots, X_{j_k}))(x)),\end{aligned} \tag{4.6}$$

using (4.5) with $W = \nabla^{(k-1)}X(X_{j_1},\dots,X_{j_k})$. Again using (4.5), now with $W = X_{j_a}$, we have

$$X^T_{j_a}(v_x) = X^H_{j_a}(v_x) + \mathrm{vlft}(v_x, \nabla_{v_x}X_{j_a}(x)).$$

Since $X^T_{j_a}$ was specified so that it is horizontal at v_x, its vertical part must be zero, whence $\nabla_{v_x}X_{j_a}(x) = 0$. Therefore, expanding the second term on the right in (4.6), we get

$$\begin{aligned}(\nabla^T)^{(k-1)}X^T(v_x)(Z_1(v_x),\dots,Z_k(v_x)) = (\nabla^{(k-1)}&X(x)(X_{j_1}(x),\dots,X_{j_k}(x)))^H \\ &+ \mathrm{vlft}(v_x, \nabla^{(k)}X(x)(X_{j_1}(x),\dots,X_{j_k}(x),v_x)).\end{aligned} \tag{4.7}$$

Symmetrising this formula with respect to $\{1,\dots,k\}$ gives

$$\begin{aligned}P^k_{\nabla^T}(X^T)(v_x)(Z_1(v_x),\dots,Z_k(v_x)) = (P^k_{\nabla}&(X)(x)(X_{j_1}(x),\dots,X_{j_k}(x)))^H \\ &+ \mathrm{vlft}\Big(v_x, \nabla^k_{v_x}P^k_{\nabla}(X)(x)(X_{j_1},\dots,X_{j_k})\Big),\end{aligned} \tag{4.8}$$

where $P^k_{\nabla}(X) = \mathrm{Sym}_k \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(k-1)}X)$. Now consider $Z_a \in \{X^T_{j_a}, Y^V_{j_a}\}$, $a \in \{1,\dots,k\}$, and suppose that at least one of these vector fields is vertical. Then, by the sublemma, we immediately have the equality

$$P^k_{\nabla^T}(X^T)(v_x)(Z_1(v_x),\dots,Z_k(v_x)) = (P^k_{\nabla}(X_{j_1}(x),\dots,X_{j_k}(x)))^V, \tag{4.9}$$

where $\hat{X}_{j_1},\dots,\hat{X}_{j_k}$ are chosen from $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$, corresponding to the way that $Z_1,\dots,Z_k$ are defined.

Now let us use these formulae in the various regularity classes to obtain the lemma.

$\nu = \infty$: Let $K \subseteq \mathsf{TM}$ be compact and let $m \in \mathbb{Z}_{\geq 0}$. For the moment, suppose that X is time-independent. Combining (4.8) and (4.9), and noting that they hold as we evaluate $P^m_{\nabla^T}(X^T)(v_x)$ on a $\mathbb{G}^T$-orthonormal basis for $\mathsf{T}_{v_x}\mathsf{TM}$, we obtain the estimate

$$\|P^m_{\nabla^T}(X^T)(v_x)\|_{\mathbb{G}^T_m} \leq C(\|P^m_{\nabla}(X)(x)\|_{\mathbb{G}_m} + \|P^{m+1}_{\nabla}(X)(x)\|_{\mathbb{G}_{m+1}}\|v_x\|_{\mathbb{G}}), \qquad v_x \in K,$$

for some $C \in \mathbb{R}_{>0}$. Now, if we make use of the fibre norms induced on jet bundles as in Section 2.1, we have

$$\|j_mX^T(v_x)\|_{\overline{\mathbb{G}^T}_m} \leq C(\|j_mX(x)\|_{\overline{\mathbb{G}}_m} + \|j_{m+1}X(x)\|_{\overline{\mathbb{G}}_{m+1}}\|v_x\|_{\mathbb{G}}), \qquad v_x \in K,$$

for some possibly different $C \in \mathbb{R}_{>0}$. Since $v_x \mapsto \|v_x\|_{\mathbb{G}}$ is bounded on K, the previous estimate gives

$$\|j_mX^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m} \leq C\|j_{m+1}X_t(x)\|_{\overline{\mathbb{G}}_{m+1}}, \qquad v_x \in K,\ t \in \mathbb{T}, \tag{4.10}$$

for some appropriate $C \in \mathbb{R}_{>0}$.

Now we consider time-dependence, supposing that $X \in \mathrm{L}\Gamma\Gamma^\infty(\mathbb{T};\mathsf{TM})$. Then there exists $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_{m+1}X_t(x)\|_{\overline{\mathbb{G}}_{m+1}} \leq f(t), \qquad x \in K,\ t \in \mathbb{T}.$$

We then immediately have

$$\|j_mX^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m} \leq Cf(t), \qquad x \in K,\ t \in \mathbb{T},$$

showing that $X^T \in \mathrm{LI}\Gamma^\infty(\mathbb{T}; \mathsf{TTM})$, as desired.

$\nu = m$: This case follows directly from the computations in the smooth case.

$\nu = m + \mathrm{lip}$: Here we take $m = 1$ as the general situation follows by combining this with the previous case. We consider X to be time-independent for the moment. We let $K \subseteq \mathsf{TM}$ be compact. We have

$$\begin{aligned}\operatorname{dil} X^T(v_x) = \inf\{\sup\{\|\nabla^T_{Y_{v_y}} X^T\|_{\mathbb{G}^T} \mid\ v_y \in \mathrm{cl}(\mathcal{W}),\ \|Y_{v_y}\|_{\mathbb{G}^T} = 1,\ X^T \text{ differentiable at } v_y\}| \\ \mathcal{W} \text{ is a relatively compact neighbourhood of } v_x\}.\end{aligned}$$

Now we make use of (2.1), (4.10), and the fact that K is compact, to reduce this to an estimate

$$\begin{aligned}\operatorname{dil} X^T(v_x) \le C \inf\{\sup\{\|j_2X(y)\|_{\overline{\mathbb{G}}_1} \mid\ y \in \mathrm{cl}(\mathcal{U}),\ j_1X \text{ differentiable at } y\}| \\ \mathcal{U} \text{ a relatively compact neighbourhood of } x\}\end{aligned}$$

for some $C \in \mathbb{R}_{>0}$ and for every $x \in K$. This then gives $\operatorname{dil} X^T(v_x) \le C \operatorname{dil} j_1X(x)$ for $x \in K$. From this we obtain the estimate

$$\lambda^{\mathrm{lip}}_K(X^T) \le C p^{1+\mathrm{lip}}_{\pi_{\mathsf{TM}}(K)}(X).$$

From the proof above in the smooth case, we have

$$p^0_K(X^T) \le C' p^1_{\pi_{\mathsf{TM}}(K)}(X).$$

Combining these previous two estimates gives

$$p^{\mathrm{lip}}_K(X^T) \le C p^{1+\mathrm{lip}}_{\pi_{\mathsf{TM}}(K)}(X)$$

for some $C \in \mathbb{R}_{>0}$, and from this, this part of the result follows easily after adding the appropriate time-dependence.

$\nu = \omega$: For the moment, we take X to be time-independent. The following sublemma will allow us to estimate the last term in (4.8).

2 Sublemma: *Let* M *be a real analytic manifold, let* ∇ *be a real analytic affine connection on* M*, let* $\mathbb{G}$ *be a real analytic Riemannian metric on* M*, and let* $K \subseteq \mathsf{M}$ *be compact. Then there exist* $C, \sigma \in \mathbb{R}_{>0}$ *such that*

$$\|\nabla^k P^k_\nabla(X)(x)\|_{\overline{\mathbb{G}}_{k+1}} \le 2\|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}}$$

for every $x \in K$ *and* $k \in \mathbb{Z}_{\ge 0}$*.*

Proof: We use (2.1) to represent elements of $\mathsf{J}^k\mathsf{TM}$. Following [Kolář, Michor, and Slovák 1993, §17.1], we think of a connection $\tilde{\nabla}^k$ on $\mathsf{J}^k\mathsf{TM}$ as being defined by a vector bundle mapping

$$\begin{tikzcd} \mathsf{J}^k\mathsf{TM} \arrow[r, "\tilde{S}_k"] \arrow[d] & \mathsf{J}^1\mathsf{J}^k\mathsf{TM} \arrow[d] \\ \mathsf{M} \arrow[r] & \mathsf{M} \end{tikzcd}$$

The connection $\nabla^{[k]}$, thought of in this way and using the decomposition of (2.1), gives the associated vector bundle mapping as zero. Now, with our identifications, we see that $P^k_\nabla(X) = j_kX - j_{k-1}X$, noting that $\mathsf{J}^{k-1}\mathsf{T}\mathsf{M}$ is a subbundle of $\mathsf{J}^k\mathsf{T}\mathsf{M}$ with our identification. Therefore, by definition of $\nabla^{[k]}$,

$$\nabla^k(P^k_\nabla(X)) = \nabla^{[k]}(j_kX - j_{k-1}X) = j_1(j_kX - j_{k-1}X).$$

As we pointed out in the proof of Lemma 4.1 above, the inclusion of $\mathsf{J}^{k+1}\mathsf{T}\mathsf{M}$ in $\mathsf{J}^1\mathsf{J}^k\mathsf{T}\mathsf{M}$ preserves the fibre metric. Therefore,

$$\|\nabla^k(P^k_\nabla(X))(x)\|_{\mathbb{G}_k} \le \|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}} + \|j_kX(x)\|_{\overline{\mathbb{G}}_k} \le 2\|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}},$$

as desired. ▼

Let $K \subseteq \mathsf{T}\mathsf{M}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$. As in the smooth case, but now using the preceding sublemma, we obtain an estimate

$$\|j_mX^T(v_x)\|_{\overline{\mathbb{G}^T}_m} \le C\|j_{m+1}X(x)\|_{\overline{\mathbb{G}}_{m+1}}, \qquad x \in K,\ m \in \mathbb{Z}_{\ge 0},$$

for some suitable $C \in \mathbb{R}_{>0}$.

Now, taking $X \in \mathrm{L}\mathrm{I}\Gamma^\omega(\mathbb{T}; \mathsf{T}\mathsf{M})$, there exists $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\ge 0})$ such that

$$a'_0a'_1 \cdots a'_{m+1}\|j_{m+1}X_t(x)\|_{\overline{\mathbb{G}}_{m+1}} \le f(t), \qquad x \in K,\ t \in \mathbb{T},\ m \in \mathbb{Z}_{\ge 0},$$

where $a'_{j+1} = a_j$, $j \in \{1, \dots, m\}$, and $a'_0 = C$. We then immediately have

$$a_0a_1 \cdots a_m\|j_mX^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m} \le f(t), \qquad x \in K,\ t \in \mathbb{T},\ m \in \mathbb{Z}_{\ge 0},$$

showing that $X^T \in \mathrm{L}\mathrm{I}\Gamma^\omega(\mathbb{T}; \mathsf{T}\mathsf{T}\mathsf{M})$, as desired.

(iii) We now prove the third assertion. It is local, so we work in a chart. Thus we assume that we are working in an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. We let $\boldsymbol{X} \colon \mathbb{T} \times \mathcal{U} \to \mathbb{R}^n$ be the principal part of the vector field so that a trajectory for $\boldsymbol{X}$ is a curve $\boldsymbol{\xi} \colon \mathbb{T} \to \mathcal{U}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(t) = \boldsymbol{X}(t, \boldsymbol{\xi}(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

The solution with initial condition $\boldsymbol{x}_0$ and t_0 we denote by $t \mapsto \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)$. For fixed $(t_0, \boldsymbol{x}_0) \in \mathbb{T} \times \mathcal{U}$ and for t sufficiently close to t_0, let us define a linear map $\boldsymbol{\Psi}(t) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n; \mathbb{R}^n)$ by

$$\boldsymbol{\Psi}(t) \cdot \boldsymbol{w} = \boldsymbol{D}_3\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) \cdot \boldsymbol{w}.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) = \boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)), \qquad \text{a.e. } t,$$

for t sufficiently close to t_0. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) &= \boldsymbol{D}_3(\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \\ &= \boldsymbol{D}_2\boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \boldsymbol{D}_3\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0).\end{aligned}$$

In the preceding expression, we have used [Schuricht and von der Mosel 2000, Corollary 2.2] to swap the time and spatial derivatives. This shows that $t \mapsto \mathbf{\Psi}(t)$ satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}(t) = \boldsymbol{D}_2\boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \mathbf{\Psi}(t), \qquad \mathbf{\Psi}(t_0) = \boldsymbol{I}_n.$$

By [Sontag 1998, Proposition C.3.8], $t \mapsto \mathbf{\Psi}(t)$ can be defined for all t such that $(t, t_0, \boldsymbol{x}_0) \in D_{\boldsymbol{X}}$. Moreover, for $\boldsymbol{v}_0 \in \mathbb{R}^n$ (which we think of as being the tangent space at $\boldsymbol{x}_0$), the curve $t \mapsto \boldsymbol{v}(t) \triangleq \mathbf{\Psi}(t) \cdot \boldsymbol{v}_0$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t) = \boldsymbol{D}_2\boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \boldsymbol{v}(t).$$

Returning now to geometric notation, the preceding chart computations, after sifting through the notation, show that

$$\Phi^{X^T}(t, t_0, v_{x_0}) = T_x\Phi^X(t, t_0, x_0)(v_{x_0}),$$

and differentiation with respect to t at t_0 gives this part of the lemma.

(ii) This was proved along the way to proving (iii). ■

We will also use some features of the geometry of the double tangent bundle, i.e., TTM. This is an example of what is known as a "double vector bundle," and we refer to [Mackenzie 2005, Chapter 9] as a comprehensive reference. A review of the structure we describe here can be found [Barbero-Liñán and Lewis 2012], along with an interesting application of this structure. We begin by noting that the double tangent bundle possesses two natural vector bundle structures over $\pi_{\mathsf{TM}} \colon \mathsf{TM} \to \mathsf{M}$:

$$\begin{tikzcd} \mathsf{TTM} \arrow[r, "\pi_{\mathsf{TTM}}"] \arrow[d, "T\pi_{\mathsf{TM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{TM} \arrow[r, "\pi_{\mathsf{TM}}"'] & \mathsf{M} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{TTM} \arrow[r, "T\pi_{\mathsf{TM}}"] \arrow[d, "\pi_{\mathsf{TTM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{TM} \arrow[r, "\pi_{\mathsf{TM}}"'] & \mathsf{M} \end{tikzcd}$$

The left vector bundle structure is called the ***primary vector bundle*** and the right the ***secondary vector bundle***. We shall denote vector addition in the vector bundles as follows. If $u, v \in \mathsf{TTM}$ satisfy $\pi_{\mathsf{TTM}}(u) = \pi_{\mathsf{TTM}}(v)$, then the sum of u and v in the primary vector bundle is denoted by $u +_1 v$. If $u, v \in \mathsf{TTM}$ satisfy $T\pi_{\mathsf{TM}}(u) = T\pi_{\mathsf{TM}}(v)$, then the sum of u and v in the secondary vector bundle is denoted by $u +_2 v$.

The two vector bundle structures admit a naturally defined isomorphism between them, described as follows. Let ρ be a smooth map from a neighbourhood of $(0, 0) \in \mathbb{R}^2$ to M. We shall use coordinates (s, t) for $\mathbb{R}^2$. For fixed s and t define $\rho_s(t) = \rho^t(s) = \rho(s, t)$. We then denote

$$\frac{\partial}{\partial t}\rho(s, t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho_s(t) \in \mathsf{T}_{\rho(s,t)}\mathsf{M}, \quad \frac{\partial}{\partial s}\rho(s, t) = \frac{\mathrm{d}}{\mathrm{d}s}\rho^t(s) \in \mathsf{T}_{\rho(s,t)}\mathsf{M}.$$

Note that $s \mapsto \frac{\partial}{\partial t}\rho(s, t)$ is a curve in TM for fixed t. The tangent vector field to this curve we denote by

$$s \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s, t) \in \mathsf{T}_{\frac{\partial}{\partial t}\rho(s,t)}\mathsf{TM}.$$

We belabour the development of the notation somewhat since these partial derivatives are not the usual partial derivatives from calculus, although the notation might make one think they are. For example, we do not generally have equality of mixed partials, i.e., generally we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s,t) \neq \frac{\partial}{\partial t}\frac{\partial}{\partial s}\rho(s,t).$$

Now let ρ_1 and ρ_2 be smooth maps from a neighbourhood of $(0,0) \in \mathbb{R}^2$ to M. We say two such maps are **equivalent** if

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_1(0,0) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_2(0,0).$$

To the equivalence classes of this equivalence relation, we associate points in TTM by

$$[\rho] \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(0,0).$$

The set of equivalence classes is easily seen to be exactly the double tangent bundle TTM. We easily verify that

$$\pi_{\mathsf{TTM}}([\rho]) = \frac{\partial}{\partial t}\rho(0,0), \quad T\pi_{\mathsf{TM}}([\rho]) = \frac{\partial}{\partial s}\rho(0,0). \tag{4.11}$$

Next, using the preceding representation of points in TTM, we relate the two vector bundle structures for TTM by defining a canonical involution of TTM. If ρ is a smooth map from a neighbourhood of $(0,0) \in \mathbb{R}^2$ into M, define another such map by $\bar{\rho}(s,t) = \rho(t,s)$. We then define the ***canonical tangent bundle involution*** as the map $I_{\mathsf{M}} \colon \mathsf{TTM} \to \mathsf{TTM}$ given by $I_{\mathsf{M}}([\rho]) = [\bar{\rho}]$. Clearly $I_{\mathsf{M}} \circ I_{\mathsf{M}} = \mathrm{id}_{\mathsf{TTM}}$. In a natural coordinate chart for TTM associated to a natural coordinate chart for TM, the local representative of I_{M} is

$$((\boldsymbol{x},\boldsymbol{v}),(\boldsymbol{u},\boldsymbol{w})) \mapsto ((\boldsymbol{x},\boldsymbol{u}),(\boldsymbol{v},\boldsymbol{w})).$$

One readily verifies that I_{M} is a vector bundle isomorphism from TTM with the primary (resp. secondary) vector bundle structure to TTM with the secondary (resp. primary) vector bundle structure [Barbero-Liñán and Lewis 2012, Lemma A.4].

The following technical lemma is Lemma A.5 from [Barbero-Liñán and Lewis 2012].

4.3 Lemma: (A property of vertical lifts) *If $w \in \mathsf{TTM}$ satisfies $\pi_{\mathsf{TTM}}(w) = v$ and $T\pi_{\mathsf{TM}} = u$ and if $z \in \mathsf{T}_x\mathsf{M}$, then*

$$w +_2 I_{\mathsf{M}} \circ \mathrm{vlft}(u,z) = w +_1 \mathrm{vlft}(v,z).$$

The final piece of tangent bundle geometry we will consider concerns presheaves and sheaves of sets of vector fields on tangent bundles. We shall need the following natural notion of such a presheaf.

4.4 Definition: (Projectable presheaf) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{G}$ be a presheaf of sets of vector fields of class C^ν on TM. The presheaf $\mathscr{G}$ is ***projectable*** if

$$\mathscr{G}(\mathcal{W}) = \{Z|\mathcal{W} \mid \ Z \in \mathscr{G}(\pi_{\mathsf{TM}}^{-1}(\pi_{\mathsf{TM}}(\mathcal{W})))\}. \qquad \bullet$$

The idea is that a projectable sheaf is determined by the local sections over the open sets $\pi_{\mathsf{TM}}^{-1}(\mathcal{U})$ for $\mathcal{U} \subseteq \mathsf{M}$ open.

4.2. Linearisation of systems. Throughout this section, unless stated otherwise, we let $m \in \mathbb{Z}_{>0}$, $m' \in \{0, \mathrm{lip}\}$, and let $\nu \in \{m+m', \infty, \omega\}$. We take $r \in \{\infty, \omega\}$, as required.

When linearising, one typically does so about a trajectory. We will do this also. But before we do so, let us provide the notion of the linearisation of a *system*. The result, gratifyingly, is a system on the tangent bundle. Before we produce the definition, let us make a motivating computation. We let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a globally generated tautological control system of class C^ν. As in Section 3.5, we have the corresponding C^ν-control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ with $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ and $F_{\mathscr{F}}(x, X) = X(x)$. This is a control system whose control set is a vector space, and so is a candidate for classical Jacobian linearisation, provided one is prepared to overlook technicalities of differentiation in locally convex spaces... and we are for the purposes of this motivational computation. In Jacobian linearisation one considers perturbations of state and control. In our framework, we linearise about a state/control (x, X). We perturb the state by considering a C^1-curve $\gamma\colon J \to \mathsf{M}$ defined on an interval J for which $0 \in \mathrm{int}(J)$ and with $\gamma'(0) = v_x$. Thus we perturb the state in the direction of v_x. We perturb the control from X in the direction of $Y \in \mathscr{F}(\mathsf{M})$ by considering a curve of controls $s \mapsto X + sY$. Let us then define $\sigma\colon \mathcal{N} \to \mathsf{M}$ on a neighbourhood $\mathcal{N}$ of $(0,0) \in \mathbb{R}^2$ by

$$\sigma(t, s) = \Phi_t^{X+sY}(\gamma(s));$$

thus $\sigma(t, s)$ gives the flow at time t corresponding to the perturbation at parameter s. Now we compute

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s}\sigma(t,s) &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X+sY}(\gamma(s)) \\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X}(\gamma(s)) + \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X+sY}(x) \\
&= \frac{\partial}{\partial t}T_x\Phi_t^X(\gamma'(s)) + I_{\mathsf{M}}\Big(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\Phi_t^{X+sY}(x)\Big) \\
&= \frac{\partial}{\partial t}T_x\Phi_t^X(\gamma'(s)) + I_{\mathsf{M}}\Big(\frac{\partial}{\partial s}(X+sY)(\Phi_t^{X+sY}(x))\Big),
\end{aligned}$$

from which we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\sigma(0,0) = X^T(v_x) + I_{\mathsf{M}}(\mathrm{vlft}(X(x), Y(x))) = X^T(v_x) + Y^V(v_x), \tag{4.12}$$

using Lemma 4.3.

The formula clearly suggests what the linearisation of a tautological control system should be. However, we need the following lemma to make a sensible definition in our sheaf framework.

4.5 Lemma: (Presheaves for linearisation) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathscr{F}$ be a presheaf of sets of C^ν-vector fields on a C^r-manifold M. Then there exist unique projectable presheaves $\mathscr{F}^T$ and $\mathscr{F}^V$ of $\mathrm{C}^{\nu-1}$-vector fields and C^ν-vector fields on TM with the property that*

$$\mathscr{F}^T(\pi_{\mathsf{TM}}^{-1}(\mathcal{U})) = \{X^T \mid X \in \mathscr{F}(\mathcal{U})\}$$

and

$$\mathscr{F}^V(\pi_{\mathsf{TM}}^{-1}(\mathcal{U})) = \{X^V \mid X \in \mathscr{F}(\mathcal{U})\}$$

for every open set $\mathcal{U} \subseteq \mathsf{M}$. Moreover,

(i) $\mathscr{F}^T$ *is a sheaf if and only if* $\mathscr{F}$ *is a sheaf,*
(ii) $\mathscr{F}^V$ *is a sheaf if and only if* $\mathscr{F}$ *is a sheaf,*
(iii) $\mathrm{Sh}(\mathscr{F}^T) = \mathrm{Sh}(\mathscr{F})^T$, *and*
(iv) $\mathrm{Sh}(\mathscr{F}^V) = \mathrm{Sh}(\mathscr{F})^V$.

Proof: Let $\mathcal{W} \subseteq \mathsf{TM}$ be open and note that $\mathcal{U}_{\mathcal{W}} = \pi_{\mathsf{TM}}(\mathcal{W})$ is open. For $\mathcal{W} \subseteq \mathsf{TM}$ open we define

$$\mathscr{F}^T(\mathcal{W}) = \{X^T|\mathcal{W} \mid\ X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})\}$$

and

$$\mathscr{F}^V(\mathcal{W}) = \{X^V|\mathcal{W} \mid\ X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})\}.$$

If $\mathcal{W}, \mathcal{W}' \subseteq \mathsf{TM}$ are open with $\mathcal{W}' \subseteq \mathcal{W}$ and if $X^T|\mathcal{W} \in \mathscr{F}^T(\mathcal{W})$, then, for $v_x \in \mathcal{W}'$, we have

$$(X^T(v_x)|\mathcal{W}')(v_x) = ((X|\mathcal{U}_{\mathcal{W}'})^T)(v_x),$$

this making sense since $X^T(v_x)$ depends only on the values of X in a neighbourhood of x, and since $\mathcal{U}_{\mathcal{W}'}$ contains a neighbourhood of x if $v_x \in \mathcal{W}'$. In any case, we have that

$$X^T|\mathcal{W}' \in \mathscr{F}^T(\mathcal{W}'),$$

which shows that $\mathscr{F}^T$ is a presheaf. A similar argument, of course, works for $\mathscr{F}^V$. This gives the existence assertion of the lemma. Uniqueness follows immediately from the requirement that $\mathscr{F}^T$ and $\mathscr{F}^V$ be projectable.

(i) Suppose that $\mathscr{F}$ is a sheaf. We shall first show that $\mathscr{F}^T$ is a sheaf. Let $\mathcal{W} \subseteq \mathsf{TM}$ be open, and let $(\mathcal{W}_a)_{a\in A}$ be an open cover of $\mathcal{W}$. Let $Z_a \in \mathscr{F}^T(\mathcal{W}_a)$, supposing that

$$Z_a|\mathcal{W}_a \cap \mathcal{W}_b = Z_b|\mathcal{W}_a \cap \mathcal{W}_b$$

for $a, b \in A$. For each $a \in A$, we have, by our definition of $\mathscr{F}^T$ above, $Z_a = X_a^T|\mathcal{W}_a$ for $X_a \in \mathscr{F}(\mathcal{U}_{\mathcal{W}_a})$. Using the fact that $\Gamma^{\nu-1}(\mathsf{TTM})$ is a sheaf, we infer that there exists $Z \in \Gamma^{\nu-1}(\mathsf{TTM})$ such that $Z|\mathcal{W}_a = X_a^T|\mathcal{W}_a$ for each $a \in A$. Now, for each $x \in \mathcal{U}_{\mathcal{W}}$, let us fix $a_x \in A$ such that $x \in \pi_{\mathsf{TM}}(\mathcal{W}_a)$. Note that $Z|\mathcal{W}_{a_x} = X_{a_x}^T|\mathcal{W}_{a_x}$ and so there is a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}_{\mathcal{W}_{a_x}}$ of x and $X_x \in \Gamma^{\nu-1}(\mathsf{T}\mathcal{U}_x)$ such that $X_x = X_{a_x}|\mathcal{U}_x$. In particular, $X_x \in \mathscr{F}(\mathcal{U}_x)$. Moreover, since $\mathscr{F}^T$ is projectable, we can easily see that $[X_x]_x$ is independent of the rule for choosing a_x. Now let $x_1, x_2 \in \mathsf{M}$ and let $x \in \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$. By projectability of $\mathscr{F}^T$, there exist a neighbourhood $\mathcal{V}_x \subseteq \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$ and $X'_x \in \mathscr{F}(\mathcal{V}_x)$ such that

$$X_{a_{x_j}}^T|\mathcal{W}_{a_{x_j}} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{V}_x) = (X'_x)^T|\mathcal{W}_{a_{x_j}}, \qquad j \in \{1, 2\}.$$

We conclude, therefore, that $X_{x_1}(x) = X_{x_2}(x)$. Thus we have an open covering $(\mathcal{U}_x)_{x\in\mathcal{U}_{\mathcal{W}}}$ of $\mathcal{U}_{\mathcal{W}}$ and local sections $X_x \in \mathscr{F}(\mathcal{U}_x)$ pairwise agreeing on intersections. Since $\mathscr{F}$ is a sheaf, there exists $X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})$ such that $X|\mathcal{U}_x = X_x$ for each $x \in \mathcal{U}_{\mathcal{W}}$. Since

$$X^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_x^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_{a_x}^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x),$$

projectability of $\mathscr{F}^T$ allows us to conclude that $Z = X^T|\mathcal{W}$.

Now suppose that $\mathscr{F}^T$ is a sheaf and let $\mathcal{U} \subseteq \mathsf{M}$ be open, let $(\mathcal{U}_a)_{a\in A}$ be an open covering of $\mathcal{U}$, and let $X_a \in \mathscr{F}(\mathcal{U}_a)$, $a \in A$ be such that $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$. This implies that

$$X_a^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a \cap \mathcal{U}_b) = X_b^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a \cap \mathcal{U}_b).$$

Therefore, by hypothesis, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a) = X_a^T$ for each $a \in A$. Projecting to M gives $X|\mathcal{U}_a = X_a$ for each $a \in A$, showing that $\mathscr{F}$ is a sheaf.

(ii) To show that $\mathscr{F}^V$ is a sheaf can be made with an identically styled argument as above in showing that $\mathscr{F}^T$ is a sheaf. The argument, indeed, is even easier since vertical lifts do not depend on the value of their projections in a neighbourhood of a point in TM, only on the projection at the point.

(iii) Let $\mathcal{W} \subseteq \mathsf{TM}$ be open and let $Z \in \mathrm{Sh}(\mathscr{F}^T)(\mathcal{W})$. This means that, for each $v_x \in \mathcal{W}$, $[Z]_{v_x} \in \mathscr{F}^T_{0,v}$. Therefore, there exist a neighbourhood $\mathcal{W}_{v_x}$ of v_x and $X_x \in \mathscr{F}(\mathcal{U}_{\mathcal{W}_{v_x}})$ such that $Z|\mathcal{W}_{v_x} = X_x^T|\mathcal{W}_{v_x}$. We now proceed as in the preceding part of the proof. Thus, for each $x \in \mathcal{U}_{\mathcal{W}}$ let us fix $v_x \in \mathcal{W}$. Note that $Z|\mathcal{W}_{v_x} = X_{v_x}^T|\mathcal{W}_{v_x}$ and so there is a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}_{\mathcal{W}_{v_x}}$ of x and $X_x \in \Gamma^{\nu-1}(\mathsf{T}\mathcal{U}_x)$ such that $X_x = X_{v_x}|\mathcal{U}_x$. In particular, $X_x \in \mathscr{F}(\mathcal{U}_x)$. Moreover, since $\mathscr{F}^T$ is projectable, we can easily see that $[X_x]_x$ is independent of the rule for choosing $v_x \in \mathcal{W}$. Now let $x_1, x_2 \in \mathsf{M}$ and let $x \in \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$. By projectability of $\mathscr{F}^T$, there exist a neighbourhood $\mathcal{V}_x \subseteq \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$ and $X'_x \in \mathscr{F}(\mathcal{V}_x)$ such that

$$X^T_{v_{x_j}}|\mathcal{W}_{v_{x_j}} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{V}_x) = (X'_x)^T|\mathcal{W}_{v_{x_j}}, \qquad j \in \{1,2\}.$$

We conclude, therefore, that $X_{x_1}(x) = X_{x_2}(x)$. Thus we have an open covering $(\mathcal{U}_x)_{x\in\mathcal{U}_{\mathcal{W}}}$ and local sections $X_x \in \mathscr{F}(\mathcal{U}_x)$ pairwise agreeing on intersections. Thus there exists $X \in \mathrm{Sh}(\mathscr{F}(\mathcal{U}_{\mathcal{W}}))$ such that $X|\mathcal{U}_x = X_x$ for each $x \in \mathcal{U}_{\mathcal{W}}$. Since

$$X^T|\mathcal{W}_{v_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_x^T|\mathcal{W}_{v_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_{v_x}^T|\mathcal{W}_{v_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x),$$

projectability of $\mathrm{Sh}(\mathscr{F}^T)$ allows us to conclude that $Z = X^T|\mathcal{W}$, i.e., $Z \in \mathrm{Sh}(\mathscr{F})^T(\mathcal{W})$.

(iv) A similar argument as in the preceding part of the proof works to give this part of the proof as well. ■

With the preceding computations and sheaf lemma as motivation, we make the following definition.

4.6 Definition: (Linearisation of a tautological control system) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. The ***linearisation*** of $\mathfrak{G}$ is the $\mathrm{C}^{\nu-1}$-tautological control system $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$, where the projectable presheaf of sets of vector fields $T\mathscr{F}$ is characterised uniquely by the requirement that, for every open subset $\mathcal{U} \subseteq \mathsf{M}$,

$$T\mathscr{F}(\pi_{\mathsf{TM}}^{-1}(\mathcal{U})) = \{X^T + Y^V \mid X, Y \in \mathscr{F}(\mathcal{U})\}. \qquad \bullet$$

This definition may look a little strange at a first glance. However, as we go along, we shall use the definition in more commonplace settings, and we will see then that it connects to more familiar constructions.

4.3. Trajectories for linearisations. As a tautological control system, $T\mathfrak{G}$ provides a forum for all of the constructions of Section 3. In particular, the linearisation has trajectories, so let us look at these.

Let us first think about open-loop systems. By definition, an open-loop system for $T\mathfrak{G}$ is a triple $(Z, \mathbb{T}, \mathcal{W})$ with $\mathbb{T} \subseteq \mathbb{R}$ an interval, $\mathcal{W} \subseteq \mathsf{TM}$ an open set, and $Z \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; T\mathscr{F}(\mathcal{W}))$. Thus $Z(t) = X(t)^T + Y(t)^V$ for $X, Y\colon \mathbb{T} \to \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W}))$. We will write $Z = X^T + Y^V$ with the understanding that this means precisely what we have just written. We should, however, verify that X and Y have useful properties.

4.7 Lemma: (Property of open-loop systems for linearisation) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. Let $\mathbb{T}$ be a time-domain and let $\mathcal{W} \subseteq \mathsf{TM}$ be open. If $Z \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; T\mathscr{F}(\mathcal{W}))$ is given by*

$$Z(t, v_x) = X^T(t, v_x) + Y^V(t, v_x)$$

for maps $X, Y\colon \mathbb{T} \times \pi_{\mathsf{TM}}(\mathcal{W}) \to \mathsf{TM}$ for which $X_t, Y_t \in \Gamma^\nu(\mathbb{T}; \pi_{\mathsf{TM}}(\mathcal{W}))$ for every $t \in \mathbb{T}$, then $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W})))$ and $Y \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W})))$.

Proof: It is possible to make oneself believe the lemma by a coordinate computation. However, we shall give a coordinate-free proof. To do this, we will use the Riemannian metric $\mathbb{G}^T$ and the affine connection ∇^T on TM defined by a Riemannian metric $\mathbb{G}$ and affine connection ∇ on M, as described in the proof of Lemma 4.2. For simplicity, and since we will make use of some formulae derived in the proof of Lemma 4.2 where this assumption was made, we suppose that ∇ is torsion-free.

Since we will be calculating iterated covariant differentials as in Section 2.1, only now using the affine connection ∇^T on TM, we should also think about the character of $\mathrm{T}^k(\mathsf{T}^*\mathsf{TM})$. For $v_x \in \mathsf{T}_x\mathsf{M}$, $T_{v_x}\pi_{\mathsf{TM}}$ is a surjective linear mapping from $\mathsf{T}_{v_x}\mathsf{TM}$ to $\mathsf{T}_x\mathsf{M}$. Thus its dual, $(T_{v_x}\pi_{\mathsf{TM}})^*$, is an injective linear mapping from $\mathsf{T}^*_x\mathsf{M}$ to $\mathsf{T}^*_{v_x}\mathsf{TM}$. It induces, therefore, an injective linear mapping from $\mathrm{T}^k(\mathsf{T}^*_x\mathsf{M})$ to $\mathrm{T}^k(\mathsf{T}^*_{v_x}\mathsf{TM})$ [Bourbaki 1989, Proposition III.5.2.2]. Yano and Kobayashi [1966] call this the ***vertical lift*** of $\mathrm{T}^k(\mathsf{T}^*\mathsf{M})$ into $\mathrm{T}^k(\mathsf{T}^*\mathsf{TM})$. Note that vertically lifted tensors, thought of as multilinear maps, vanish if they are given a vertical vector as one of their arguments, i.e., they are "semi-basic" (in fact, they are even "basic"). Note that $\mathsf{T}^*_{v_x}\mathsf{TM} \simeq \mathsf{T}^*_x\mathsf{M} \oplus \mathsf{T}^*_x\mathsf{M}$ by dualising the splitting of the tangent bundle. So as to notationally distinguish between the two components of the direct sum, let us denote the first component by $(\mathsf{T}^*_x\mathsf{M})_1$ and the second component by $(\mathsf{T}^*_x\mathsf{M})_2$, noting that the first component is defined to be the image of the canonical injection from $\mathsf{T}^*_x\mathsf{M}$ to $\mathsf{T}^*_{v_x}\mathsf{TM}$. We then have

$$\mathrm{T}^k((\mathsf{T}^*_x\mathsf{M})_1 \oplus (\mathsf{T}^*_x\mathsf{M})_2) \simeq \bigoplus_{a_1,\dots,a_k \in \{1,2\}} (\mathsf{T}^*_x\mathsf{M})_{a_1} \otimes \cdots \otimes (\mathsf{T}^*_x\mathsf{M})_{a_k}$$

by [Bourbaki 1989, §III.5.5]. Let

$$\pi_k\colon \mathrm{T}^k(\mathsf{T}^*_{v_x}\mathsf{TM}) \to (\mathsf{T}^*_x\mathsf{M})_1 \otimes \cdots \otimes (\mathsf{T}^*_x\mathsf{M})_1$$

be the projection onto the component of the direct sum decomposition.

With all of the preceding, we can now make sense of the following sublemma.

1 Sublemma: *If, for $X, Y \in \Gamma^\nu(\mathsf{TM})$, we have $Z = X^T + Y^V$, then we have*

$$\pi_k \otimes \mathrm{id}_{\mathsf{TTM}}((\nabla^T)^{(k)} Z(0_x)) = \nabla^{(k)} X(x) \oplus (\nabla^{(k)} Y(x))$$

for $k \in \mathbb{Z}_{\geq 0}$ satisfying $k \leq \nu$.

Proof: Obviously we can consider two special cases, the first where $Y = 0$ and the second where $X = 0$. When $Y = 0$, the result follows from Sublemma 1 from the proof of Lemma 4.2, especially the formula (4.7) we derived from the sublemma. When $X = 0$ the result immediately follows from the same sublemma. ▼

By the preceding sublemma, $Z(t, 0_x) = X(t, x) \oplus Y(t, x)$. Since the projections onto the first and second component of the direct sum decomposition of TTM are continuous, we immediately conclude that $X, Y \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; \mathsf{T}(\pi_{\mathsf{TM}}(\mathcal{W})))$.

The remainder of the proof breaks into the various cases of regularity.

$\nu = \infty$: Let $K \subseteq \mathsf{M}$ be compact and let $m \in \mathbb{Z}_{\geq 0}$. Since K is also a compact subset of TM, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|j_m Z(t, 0_x)\|_{\overline{\mathbb{G}^T}_m} \leq g(t), \qquad t \in \mathbb{T},\ x \in K.$$

Let $\overline{\pi}_m \colon \mathsf{J}^m\mathsf{TTM} \to \oplus_{j=0}^m \mathrm{T}^j(\pi^*_{\mathsf{TM}}\mathsf{TM}) \otimes \mathsf{TTM}$ be defined by

$$\overline{\pi}_m(j_m Z'(v_x)) = \sum_{j=0}^m \pi_j \otimes \mathrm{id}_{\mathsf{TTM}}((\nabla^T)^{(j-1)} Z'(v_x)),$$

this making sense by virtue of (2.1). By the sublemma, by the definition of $\mathbb{G}^T$, and by the definition of the fibre metrics on $\mathsf{J}^m\mathsf{TM}$ and $\mathsf{J}^m\mathsf{TTM}$ induced by the decomposition of (2.1), we have

$$\|\overline{\pi}_m(j_m Z(t, 0_x))\|^2_{\overline{\mathbb{G}^T}_m} = \|j_m X(t, x)\|^2_{\overline{\mathbb{G}}_m} + \|j_m Y(t, x)\|^2_{\overline{\mathbb{G}}_m}.$$

This gives

$$\|j_m X(t, x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \ \ \|j_m Y(t, x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad t \in \mathbb{T},\ x \in K,$$

which gives the lemma in this case.

$\nu = m$: From the computations above in the smooth case we have that X and Y are locally integrally C^{m-1}-bounded. To show X is, in fact, locally integrally C^m-bounded, we will use the computations from the proof of Lemma 4.2. Let $K \subseteq \mathsf{M}$ and let

$$K_1 = \{v_x \in \mathsf{TM} \mid \ x \in K,\ \|v_x\|_{\mathbb{G}} \leq 1\}$$

so K_1 is a compact subset of TM. For the moment, let us fix $t \in \mathbb{T}$. We now recall equation (4.8) which gives a formula for $P^m_{\nabla^T}(X^T_t)$ when all arguments are horizontal. Since, in the expression (4.8), v_x is arbitrary, by letting it vary over vectors of unit length we get an estimate

$$\|P^m_{\nabla}(X_t)(x)\|_{\mathbb{G}_m} \leq C(p^{m-1}_K(X_t) + p^{m-1}_{K_1}(X^T_t))$$

for some $C \in \mathbb{R}_{>0}$. Since $X, Y \in \mathrm{LI}\Gamma^{m-1}(\mathbb{T}; \mathsf{M})$ and since $X^T = Z - Y^V \in \mathrm{LI}\Gamma^{m-1}(\mathbb{T}; \mathsf{TM})$, by (2.1) there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|j_m X_t(x)\| \leq g(t), \qquad (t, x) \in \mathbb{T} \times K,$$

which gives $X \in \mathrm{LI}\Gamma^m(\mathsf{TM})$.

$\nu = m + \mathrm{lip}$: This follows from the computations above, cf. the proof of the Lipschitz part of the proof of Lemma 4.2.

$\nu = \omega$: Let $K \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Since K is also a compact subset of TM, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m Z(t, 0_x)\|_{\overline{\mathbb{G}^T}_m} \leq g(t), \qquad t \in \mathbb{T},\ x \in K,\ m \in \mathbb{Z}_{\geq 0}.$$

As in the smooth case we have

$$\|\overline{\pi}_m(j_m Z(t, 0_x))\|^2_{\overline{\mathbb{G}^T}_m} = \|j_m X(t, x)\|^2_{\overline{\mathbb{G}}_m} + \|j_m Y(t, x)\|^2_{\overline{\mathbb{G}}_m}.$$

This gives

$$a_0 a_1 \cdots a_m \|j_m X(t, x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \quad a_0 a_1 \cdots a_m \|j_m Y(t, x)\|_{\overline{\mathbb{G}}_m} \leq g(t),$$

for $t \in \mathbb{T}$, $x \in K$, and $m \in \mathbb{Z}_{\geq 0}$, which gives the lemma. ■

Next let us think about open-loop subfamilies for linearisations. Generally speaking, one may wish to consider different classes of open-loop systems for the "tangent lift part" and the "vertical lift part" of a linearised system. The open-loop systems for the tangent lift part will be those giving rise to reference trajectories and reference flows. On the other hand, the open-loop systems for the vertical lift part will be those that we will allow as perturbing the reference flow. There is no reason that these should be the same. While this proliferation of open-loop subfamilies will lead to some notational complexity, the freedom to carefully account for these possibilities is one of the strengths of our theory. Indeed, in standard Jacobian linearisation, it is difficult to keep track of how the controls—constraints on them and attributes of them—are carried over to the linearisation. In our theory, this is natural.

We first make tangent and vertical lift constructions for open-loop subfamilies.

4.8 Definition: (Tangent and vertical lifts of open-loop subfamilies) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$, and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$.

(i) The ***tangent lift*** of $\mathscr{O}_{\mathfrak{G}}$ is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G}}$ for $(\mathsf{TM}, \mathscr{F}^T)$ defined by

$$\mathscr{O}^T_{\mathfrak{G}}(\mathbb{T}, \mathcal{W}) = \{X^T|\mathcal{W} \mid\ X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))\}$$

for a time-domain $\mathbb{T}$ and for $\mathcal{W} \subseteq \mathsf{TM}$ open.

(ii) The ***vertical lift*** of $\mathscr{O}_{\mathfrak{G}}$ is the open-loop subfamily $\mathscr{O}^V_{\mathfrak{G}}$ for $(\mathsf{TM}, \mathscr{F}^V)$ defined by

$$\mathscr{O}^V_{\mathfrak{G}}(\mathbb{T}, \mathcal{W}) = \{Y^V|\mathcal{W} \mid\ Y \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))\}$$

for a time-domain $\mathbb{T}$ and for $\mathcal{W} \subseteq \mathsf{TM}$ open. •

4.9 Definition: (Open-loop subfamily for linearisation) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. An ***open-loop subfamily*** for $T\mathfrak{G}$ defined by a pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ of open-loop subfamilies for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0} + \mathscr{O}^V_{\mathfrak{G},1}$ defined by:

$$X^T + Y^V \in (\mathscr{O}^T_{\mathfrak{G},0} + \mathscr{O}^V_{\mathfrak{G},1})(\mathbb{T}, \mathcal{W}) \iff X^T \in \mathscr{O}^T_{\mathfrak{G},0}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W})),\ Y^V \in \mathscr{O}^V_{\mathfrak{G},1}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W})).$$

•

Note that the restriction properties of open-loop subfamilies as per Definition 3.7 are satisfied by our construction above, so the result is indeed an open-loop subfamily for $T\mathfrak{G}$.

Next we can define what we mean by trajectories for the linearisation in the more or less obvious way.

4.10 Definition: (Trajectory for linearisation of tautological control system) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$.

(i) For a time-domain $\mathbb{T}$, an open set $\mathcal{W} \subseteq \mathsf{TM}$, and for $X \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \mathcal{U})$ and $Y \in \mathscr{O}_{\mathfrak{G},1}$, an ***$(X, Y, \mathbb{T}, \mathcal{W})$-trajectory*** for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a curve $\Upsilon\colon \mathbb{T} \to \mathcal{W}$ such that $\Upsilon'(t) = X^T(t, \Upsilon(t)) + Y^V(t, \Upsilon(t))$.

(ii) For a time-domain $\mathbb{T}$ and an open set $\mathcal{W} \subseteq \mathsf{TM}$, a ***$(\mathbb{T}, \mathcal{W})$-trajectory*** for the pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a $(\mathbb{T}, \mathcal{W})$-trajectory for $\mathscr{O}^T_{\mathfrak{G},0} + \mathscr{O}^V_{\mathfrak{G},1}$.

(iii) A plain ***trajectory*** for the pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a curve that is a $(\mathbb{T}, \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ for some time-domain $\mathbb{T}$ and some open $\mathcal{W} \subseteq \mathsf{TM}$.

We denote by:

(iv) $\mathrm{Traj}(X, Y, \mathbb{T}; \mathcal{W})$ the set of $(X, Y, \mathbb{T}, \mathcal{U})$-trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$;

(v) $\mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1}))$ the set of $(\mathbb{T}, \mathcal{U})$-trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$;

(vi) $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ the set of trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$.

We shall abbreviate

$$\mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},\mathrm{full}}, \mathscr{O}_{\mathfrak{G},\mathrm{full}})) = \mathrm{Traj}(\mathbb{T}, \mathcal{W}, T\mathfrak{G})$$

and $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G},\mathrm{full}}, \mathscr{O}_{\mathfrak{G},\mathrm{full}}) = \mathrm{Traj}(T\mathfrak{G})$. •

Now that we have been clear about what we mean by the trajectory of a linearised system, let us say some things about these trajectories.

4.11 Proposition: (Trajectories for the linearisation of a tautological control system) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$, and let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain and let $\mathcal{W} \subseteq \mathsf{TM}$ be open. If $\xi^T \in \mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1}))$ then the following statements hold:*

(i) there exist $X \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))$ and $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))$ such that

$$(\xi^T)'(t) = X^T(t, \xi^T(t)) + Y^V(t, \xi^T(t));$$

(ii) there exists $\xi \in \mathrm{Traj}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}), \mathscr{O}_{\mathfrak{G},0})$ *such that the diagram*

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\xi^T"] \arrow[dr, "\xi"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{T}\mathcal{U}}"] \\ & \mathsf{M} \end{tikzcd}$$

commutes, i.e., ξ^T *is a vector field along* ξ*.*

Proof: The first assertion follows from Lemma 4.7. The second assertion follows by taking $\xi = \pi_{\mathsf{TM}} \circ \xi^T$, and noting that

$$\xi'(t) = T_{\xi^T(t)}\pi_{\mathsf{TM}}((\xi^T)'(t)) = T_{\xi^T(t)}\pi_{\mathsf{TM}}(X^T(t, \xi^T(t)) + Y^V(t, \xi^T(t))) = X(t, \xi(t))$$

and X is an open-loop system for $\mathscr{O}_{\mathfrak{G},0}$. ■

4.4. Linearisation about reference trajectories and reference flows. Let us now slowly begin to pull back our general notion of linearisation to something more familiar. In this section we will linearise about two sorts of things, trajectories and flows. We will see in the next section that it is the distinction between these two things that accounts for the problems observed in Example 1.1.

But for now, we proceed in general. We let $\mathfrak{G}$ be a tautological control system and $\mathscr{O}_{\mathfrak{G}}$ an open-loop subfamily. We recall from Example 3.10 that, if $\mathbb{T}$ is a time-domain, if $\mathcal{U} \subseteq \mathsf{M}$ is open, and if $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$, then $\mathscr{O}_{\mathfrak{G},\xi}$ is the open-loop subfamily associated to the trajectory ξ, i.e., all open-loop systems from $\mathscr{O}_{\mathfrak{G}}$ possessing ξ as a trajectory. Having made this recollection, we make the following definition.

4.12 Definition: (Linearisation of a tautological control system about a trajectory) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $\xi_{\mathrm{ref}} \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},0})$. The ***$(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$-linearisation of $\mathfrak{G}$ about ξ_{ref}*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,\xi_{\mathrm{ref}}} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory Υ for $(\mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$ satisfying $\pi_{\mathsf{TM}} \circ \Upsilon = \xi_{\mathrm{ref}}$, and where $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. •

By definition, a trajectory for the linearisation about the reference trajectory ξ_{ref} is a curve $\Upsilon \colon \mathbb{T}' \to \mathcal{W}$ satisfying

$$\Upsilon'(t) = X^T(t, \Upsilon(t)) + Y^V(t, \Upsilon(t)),$$

for $X \in \mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$ and for $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$, and where Υ is a tangent vector field along ξ_{ref}. Note that there may well be trajectories for $(\mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$ that are *not* vector fields along ξ_{ref}; we just do not call these trajectories for the linearisation *about* ξ_{ref}.

Let us now talk about linearisation, not about a trajectory, but about a flow. Here we recall the notion of the open-loop subfamily associated to an open-loop system in Example 3.8–4.

4.13 Definition: (Linearisation of a tautological control system about a flow) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $X_{\mathrm{ref}} \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \mathcal{U})$. The ***$\mathscr{O}_{\mathfrak{G},1}$-linearisation of $\mathfrak{G}$ about X_{ref}*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,X_{\mathrm{ref}}} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$, where $\mathbb{T}' \subseteq \mathbb{T}$ and where $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. •

By definition, a trajectory for the linearisation about the reference flow X_{ref} is a curve $\Upsilon\colon \mathbb{T}' \to \mathcal{W}$ satisfying

$$\Upsilon'(t) = X^T_{\mathrm{ref}}(t, \Upsilon(t)) + Y^V(t, \Upsilon(t)),$$

for $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$. Note that the definition of $\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}$ necessarily implies that $\pi_{\mathsf{TM}} \circ \Upsilon$ is an integral curve for X_{ref}. Unlike the case of linearisation about a reference trajectory, we do not specify that the trajectories for the linearisation about a reference flow follow a specific trajectory for $\mathfrak{G}$, although one can certainly do this as well.

4.5. Linearisation about an equilibrium point. Continuing to make things concrete, let us consider linearising about trivial reference trajectories and reference flows. We begin by considering what an equilibrium point is in our framework.

4.14 Definition: (Tautological control system associated to an equilibrium point) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $x_0 \in \mathsf{M}$.

(i) The ***tautological control system for $\mathfrak{G}$ at x_0*** is the C^ν-tautological control system $\mathfrak{G}_{x_0} = (\mathsf{M}, \mathrm{Eq}_{\mathscr{F},x_0})$, where

$$\mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}) = \{X \in \mathscr{F}(\mathcal{U}) \mid X(x_0) = 0_{x_0}\}.$$

(ii) If there exists an open set $\mathcal{U} \subseteq \mathsf{M}$ for which $\mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}) \neq \varnothing$, then x_0 is an ***equilibrium point*** for $\mathfrak{G}$. •

Of course, by properties of presheaves, if $X \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U})$, then $X|\mathcal{V} \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{V})$ for every open set $\mathcal{V} \subseteq \mathcal{U}$. Thus $\mathfrak{G}_{x_0}$ is indeed a tautological control system.

Let us examine the nature of tautological control systems at x_0. This amounts to understanding any particular structure that one can associate to vector fields that vanish at a point. This is the content of the following lemma.

4.15 Lemma: (Properties of vector fields vanishing at a point) *Let M be a smooth manifold, let $x_0 \in \mathsf{M}$, and let $X \in \Gamma^1(\mathsf{M})$. If $X(x_0) = 0_{x_0}$, then there exists a unique $A_{X,x_0} \in \mathrm{End}_{\mathbb{R}}(\mathsf{T}_{x_0}\mathsf{M})$ satisfying either of the following equivalent characterisations:*

(i) noting that $X^T|\mathsf{T}_{x_0}\mathsf{M}\colon \mathsf{T}_{x_0}\mathsf{M} \to \mathsf{V}_{0_{x_0}}\mathsf{TM} \simeq \mathsf{T}_{x_0}\mathsf{M}$, $A_{X,x_0} = X^T|\mathsf{T}_{x_0}\mathsf{M}$;

(ii) $A_{X,x_0}(v_{x_0}) = [V, X](x_0)$ where $V \in \Gamma^\infty(\mathsf{M})$ satisfies $V(x_0) = v_{x_0}$.

Proof: We will show that the characterisation from part (i) makes sense, and that it agrees with the second characterisation.

First, note that, since $X(x_0) = 0_{x_0}$, $T_{v_{x_0}}\pi_{\mathsf{TM}}(X^T(v_{x_0})) = 0_{x_0}$ for every $v_{x_0} \in \mathsf{T}_{x_0}\mathsf{M}$. Thus $X^T(v_{x_0}) \in \mathsf{V}_{0_{x_0}}\mathsf{TM}$, as claimed. That $X^T|\mathsf{T}_{x_0}\mathsf{M}$ is linear is a consequence of the fact that X^T is a linear vector field, i.e., that the diagram (4.3) commutes. In the particular case that $X(x_0) = 0_{x_0}$, the diagram implies that X^T is a linear map from $\mathsf{T}_{x_0}\mathsf{M}$ to $\mathsf{T}_{0_{x_0}}\mathsf{TM}$. As we already know that $X^T|\mathsf{T}_{x_0}\mathsf{M}$ is $\mathsf{V}_{0_{x_0}}\mathsf{TM}$-valued, the characterisation from part (i) does indeed uniquely define an endomorphism of $\mathsf{T}_{x_0}\mathsf{M}$.

Let us now show that the characterisation of part (ii) agrees with that of part (i). By [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19], we have

$$\begin{aligned}\operatorname{vlft}(0_{x_0}, [V, X](x_0)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{\Phi^X_{-t}(x_0)}\Phi^X_t \circ V \circ \Phi^X_{-t}(x_0) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{x_0}\Phi^X_t \circ V(x_0) = X^T(V(x_0)),\end{aligned}$$

as desired. ■

According to the lemma, we can make the following definitions.

4.16 Definition: (Data associated with linearisation about an equilibrium point) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. For an equilibrium point $x_0 \in \mathsf{M}$ for $\mathfrak{G}$, we define

$$\mathscr{L}_{\mathscr{F},x_0} = \{A_{X,x_0} \mid [X]_{x_0} \in (\mathrm{Eq}_{\mathscr{F},x_0})_{x_0}\}$$

(where $(\mathrm{Eq}_{\mathscr{F},x_0})_{x_0}$ denotes the stalk of the presheaf $\mathrm{Eq}_{\mathscr{F},x_0}$ at x_0) and

$$\mathscr{F}(x_0) = \{X(x_0) \mid [X]_{x_0} \in \mathscr{F}_{x_0}\}. \qquad \bullet$$

Associated to an equilibrium point are natural notions of open-loop systems that preserve the equilibrium point.

4.17 Definition: (Open-loop subfamilies and equilibrium points) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. If $x_0 \in \mathsf{M}$ and if $\mathscr{O}_{\mathfrak{G}}$ is an open-loop subfamily for $\mathfrak{G}$, the open-loop subfamily $\mathscr{O}_{\mathfrak{G},x_0}$ is defined by specifying that, for a time-domain $\mathbb{T}$ and an open set $\mathcal{U} \subseteq \mathsf{M}$,

$$\mathscr{O}_{\mathfrak{G},x_0}(\mathbb{T}, \mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}; \mathcal{U}) \mid X(t) \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}),\ t \in \mathbb{T}\}. \qquad \bullet$$

Note that the only trajectory of $\mathscr{O}_{\mathfrak{G},x_0}$ passing through x_0 is the constant trajectory $t \mapsto x_0$, as it should be.

It is now more or less obvious how one should define linearisations about an equilibrium point. This can be done for trajectories and flows. We start with trajectories.

4.18 Definition: (Linearisation of a tautological control system about an equilibrium trajectory) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$ and let $x_0 \in \mathsf{M}$. The ***$(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$-linearisation of $\mathfrak{G}$ about x_0*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,x_0} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}, \mathcal{W})$-trajectory for the $(\mathscr{O}_{\mathfrak{G},0,x_0}, \mathscr{O}_{\mathfrak{G},1})$-linearisation about the trivial reference trajectory $t \mapsto x_0$, where $\mathbb{T}$ is a time-domain and where $\mathcal{W}$ is a neighbourhood of $\mathsf{T}_{x_0}\mathsf{M}$. •

By definition and by the characterisation of X^T at equilibria, a trajectory for the linearisation about x_0 will be a curve $\Upsilon\colon \mathbb{T} \to \mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\Upsilon'(t) = A_{X(t),x_0}(\Upsilon(t)) + b(t),$$

where $t \mapsto X(t)$ is a curve in $\mathscr{L}_{\mathscr{F},x_0}$ whose nature is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},0}$, e.g., it may be locally integrable, locally essentially bounded, piecewise constant, etc., and where $t \mapsto b(t)$ is a curve in $\mathscr{F}(x_0) \subseteq \mathsf{T}_{x_0}\mathsf{M}$, again whose nature is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. Note that the linearisation about x_0 will, therefore, generally be a family of *time-dependent* linear systems on $\mathsf{T}_{x_0}\mathsf{M}$. This may come as a surprise to those used to Jacobian linearisation, but we will see in Example 4.25 below how this arises in practice.

Let us now talk about linearisation about an equilibrium point, not about a trajectory, but about a flow.

4.19 Definition: (Linearisation of a tautological control system about an equilibrium flow) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $x_0 \in \mathsf{M}$, let $\mathcal{U} \subseteq \mathsf{M}$ be a neighbourhood of x_0, and let $X_{\mathrm{ref}} \in \mathscr{O}_{\mathfrak{G},0,x_0}(\mathbb{T}, \mathcal{U})$. The ***$\mathscr{O}_{\mathfrak{G},1}$-linearisation of $\mathfrak{G}$ about (X_{ref}, x_0)*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,X_{\mathrm{ref}}} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$, where $\mathbb{T}' \subseteq \mathbb{T}$ and where $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. •

In this case, we have a prescribed curve $t \mapsto X_{\mathrm{ref}}(t)$ such that $X_{\mathrm{ref}}(t, x_0) = 0_{x_0}$ for every t. Thus this defines a curve $A_{X_{\mathrm{ref}}(t),x_0}$ in $\mathscr{L}_{\mathscr{F},x_0}$. By definition, a trajectory for the linearisation about the pair (X_{ref}, x_0) is a curve $\Upsilon\colon \mathbb{T}' \to \mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\Upsilon'(t) = A_{X_{\mathrm{ref}}(t),x_0}(\Upsilon(t)) + b(t),$$

where $t \mapsto b(t)$ is a curve in $\mathscr{F}(x_0)$ having properties determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. Note that this linearisation will still generally be time-dependent, but it is now a *single* time-dependent linear system, not a family of them, as with linearisation about a trajectory. Moreover, if X_{ref} is chosen to be time-independent, then the linearisation will also be time-invariant. But there is no reason in the general theory to do this.

The above comments about the possibility of time-varying linearisations notwithstanding, there is one special case where we can be sure that linearisations will be time-independent, and this is when $\mathscr{L}_{\mathscr{F},x_0}$ consists of a single vector field. The following result gives a common case where this happens. Indeed, the ubiquity of this situation perhaps explains the neglect of the general situation that has led to the seeming contradictions in the standard treatments, such as are seen in Example 1.1.

4.20 Proposition: (Time-independent linearisations for certain control-affine systems) *Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^1-control-affine system with $\mathcal{C} \subseteq \mathbb{R}^k$ and*

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x).$$

For $x_0 \in \mathsf{M}$, suppose that *(i) there exists* $\boldsymbol{u}_0 \in \mathcal{C}$ *such that*

$$f_0(x_0) = \sum_{a=1}^{k} u_0^a f_a(x_0)$$

and

(ii) $(f_1(x_0), \dots, f_k(x_0))$ *is linearly independent.*

Then x_0 *is an equilibrium point for* $\mathfrak{G}_\Sigma$ *and* $\mathscr{L}_{\mathscr{F}_\Sigma, x_0}$ *consists of a single linear map.*

Proof: Let us define

$$f_0' = f_0 - \sum_{a=1}^{k} u_0^a f_a,$$

noting that $f_0' \in \mathscr{F}_\Sigma$. Since $f_0'(x_0) = 0_{x_0}$, we conclude that x_0 is an equilibrium point. Now suppose that $F(x_0, \boldsymbol{u}) = 0_{x_0}$. Thus

$$f_0(x_0) + \sum_{a=1}^{k} u^a f_a(x_0) = 0_{x_0} \quad \Longrightarrow \quad f_0(x_0) = -\sum_{a=1}^{k} u^a f_a(x_0).$$

This last equation has a solution for $\boldsymbol{u}$, namely $\boldsymbol{u} = -\boldsymbol{u}_0$, and since $(f_1(x_0), \dots, f_m(x_0))$ is linearly independent, this solution is unique. Thus, for any neighbourhood $\mathcal{U}$ of x_0,

$$\mathrm{Eq}_{\mathscr{F}_\Sigma, x_0}(\mathcal{U}) = \Big\{ f_0 - \sum_{a=1}^{k} u_0^a f_a(x_0) \Big\} = \{ f_0'(x_0) \}.$$

This shows that $\mathscr{L}_{\mathscr{F}_\Sigma, x_0} = \{ A_{f_0', x_0} \}$, as claimed. ■

While we are definitely not giving a comprehensive account of controllability in this paper, in order to "close the loop" on Example 1.1, let us consider here how one talks about linear controllability in our framework. First we introduce some general notation.

4.21 Definition: (Subspaces invariant under families of linear maps) Let F be a field, let V be an F-vector space, let $\mathscr{L} \subseteq \mathrm{End}_{\mathsf{F}}(\mathsf{V})$, and let $S \subseteq \mathsf{V}$. By $\langle \mathscr{L}, S \rangle$ we denote the smallest subspace of V that (i) contains S and (ii) is invariant under L for every $L \in \mathscr{L}$. •

We can give a simple description of this subspace.

4.22 Lemma: (Characterisation of smallest invariant subspace) *If* F *is a field, if* V *is an* F*-vector space, if* $\mathscr{L} \subseteq \mathrm{End}_{\mathsf{F}}(\mathsf{V})$*, and if* $S \subseteq \mathsf{V}$*, then* $\langle \mathscr{L}, S \rangle$ *is spanned by elements of* V *of the form*

$$L_1 \circ \cdots \circ L_k(v), \qquad k \in \mathbb{Z}_{\geq 0},\ L_1, \dots, L_k \in \mathscr{L},\ v \in S. \tag{4.13}$$

Proof: Let $\mathsf{U}_{\mathscr{L},S}$ be the subspace spanned by elements of the form (4.13). Clearly $S \subseteq \mathsf{U}_{\mathscr{L},S}$ (taking the convention that $L_1 \circ \cdots \circ L_k(v) = v$ if $k = 0$) and, if $L \in \mathscr{L}$, then $L(\mathsf{U}_{\mathscr{L},S}) \subseteq \mathsf{U}_{\mathscr{L},S}$ since an endomorphism from $\mathscr{L}$ maps a generator of the form (4.13) to another generator of this form. Therefore, $\langle \mathscr{L}, S \rangle \subseteq \mathsf{U}_{\mathscr{L},S}$. Now, if $v \in S$, then clearly $v \in \langle \mathscr{L}, S \rangle$. Since $\langle \mathscr{L}, S \rangle$ is invariant under endomorphisms from $\mathscr{L}$, $L(v) \in \langle \mathscr{L}, S \rangle$ for every $v \in S$ and $L \in \mathscr{L}$. Recursively, we see that all generators of the form (4.13) are in $\langle \mathscr{L}, S \rangle$, whence $\mathsf{U}_{\mathscr{L},S} \subseteq \langle \mathscr{L}, S \rangle$ since $\mathsf{U}_{\mathscr{L},S}$ is a subspace. ■

With the preceding as setup, let us make the following definition.

4.23 Definition: (Linear controllability) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$, and let $x_0 \in \mathsf{M}$ be an equilibrium point for $\mathfrak{G}$. The system $\mathfrak{G}$ is ***linearly controllable*** at x_0 if there exists $S \subseteq \mathscr{F}(x_0)$ such that (i) $0_{x_0} \in \mathrm{conv}(S)$ and (ii) $\langle \mathscr{L}_{\mathscr{F}, x_0}, S \rangle = \mathsf{T}_{x_0}\mathsf{M}$. •

4.24 Remark: (Relationship to rank test) For readers who may not recognise the relationship between our definition of linear controllability and the classical Kalman rank test [Brockett 1970, Theorem 13.3], we make the following comments. Consider the linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

with $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and for appropriately sized matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Using Lemma 4.22 and the Cayley–Hamilton Theorem, it is easy to check that the smallest $\boldsymbol{A}$-invariant subspace containing image($\boldsymbol{B}$) is exactly the columnspace of the Kalman controllability matrix,

$$\left[\ \boldsymbol{B} \mid \boldsymbol{A}\boldsymbol{B} \mid \cdots \mid \boldsymbol{A}^{n-1}\boldsymbol{B}\ \right].$$

For the more geometric approach to topics in linear system theory, we refer to the excellent book of Wonham [1985]. •

We state linear controllability as a definition, not a theorem, because we do not want to develop the definitions required to state a theorem. However, it *is* true that a system that is linearly controllable according to our definition is small-time locally controllable in the usual sense of the word. This is proved by Aguilar [2010, Theorem 5.14]. The setting of Aguilar is not exactly that of our paper. However, it is easy to see that this part of Aguilar's development easily translates to what we are doing here.

Let us close this section, and the technical part of the paper, by revisiting Example 1.1 where we saw that the classical picture of Jacobian linearisation presents some problems.

4.25 Example: (Revisiting Example 1.1) We work with the system

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t)u_1(t), \\
\dot{x}_3(t) &= u_2(t).
\end{aligned}$$

We could as well work with the other representation for the system from Example 1.1, but since the family of vector fields is the same (what changes between the two representations is the parameterisation of the set of vector fields!), we will get the same conclusions; this, after all, is the point of our feedback-invariant approach.

This, of course, is a control-affine system, and the resulting tautological control system is $\mathfrak{G} = (\mathbb{R}^3, \mathscr{F})$ where $\mathscr{F}$ is the globally generated presheaf with

$$\mathscr{F}(\mathbb{R}^3) = \{f_0 + u^1 f_1 + u^2 f_2 \mid (u^1, u^2) \in \mathbb{R}^2\},$$

with

$$f_0 = x_2 \frac{\partial}{\partial x_1}, \quad f_1 = x_3 \frac{\partial}{\partial x_2}, \quad f_2 = \frac{\partial}{\partial x_3}.$$

We have an equilibrium point at $(0, 0, 0)$.

1 Lemma: $\mathrm{Eq}_{\mathscr{F},(0,0,0)}(\mathbb{R}^3) = f_0 + \mathrm{span}_{\mathbb{R}}(f_1)$.

Proof: It is clear that $f_0(0,0,0) = f_1(0,0,0) = 0$, and, therefore, any linear combination of f_0 and f_1 will also vanish at $(0,0,0)$, and particularly those from the affine subspace $f_0 + \mathrm{span}_{\mathbb{R}}(f_1)$. Conversely, if

$$f_0(0,0,0) + u^1 f_1(0,0,0) + u^2 f_2(0,0,0) = 0,$$

then $u^2 = 0$ and so the resulting vector field is in the asserted affine subspace. ▼

We, therefore, have

$$\mathscr{L}_{\mathscr{F},(0,0,0)} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}.$$

We also have

$$\mathscr{F}((0,0,0)) = \{ b f_2(0,0,0) \mid b \in \mathbb{R} \} = \left\{ \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}.$$

Thus a curve in $\mathscr{L}_{\mathscr{F},(0,0,0)}$ has the form

$$t \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a(t) \\ 0 & 0 & 0 \end{bmatrix}$$

for a function a having whatever properties might be induced from the open-loop subfamily $\mathscr{O}_{\mathfrak{G},0}$ one is using, e.g., locally integrable, locally essentially bounded. A curve in $\mathscr{F}((0,0,0))$ has the form

$$t \mapsto \begin{bmatrix} 0 \\ 0 \\ b(t) \end{bmatrix}$$

for a function b having whatever properties might be induced from the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$ one is using. Trajectories for the linearisation about $(0,0,0)$ then satisfy

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b(t) \end{bmatrix}.$$

Note that this is not a fixed time-varying linear system, but a family of these, since the function a is not *a priori* specified, but is variable.

Next let us look at two instances of linearisation about a reference flow by choosing the two reference flows $X_1 = f_0$ and $X_2 = f_0 + f_1$. We use coordinates $((x_1, x_2, x_3), (v_1, v_2, v_3))$ for $\mathsf{T}\mathbb{R}^3$ and we compute

$$X_1^T = x_2 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial v_1}, \qquad X_2^T = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + v_2 \frac{\partial}{\partial v_1} + v_3 \frac{\partial}{\partial v_2}.$$

If $t \mapsto Y(t)$ is a time-dependent vector field with values in $\mathscr{F}(\mathbb{R}^3)$, then

$$Y_t = f_0 + \nu_1(t) f_1 + \nu_2(t) f_2 = x_2 \frac{\partial}{\partial x_1} + \nu_1(t) x_3 \frac{\partial}{\partial x_2} + \nu_2(t) \frac{\partial}{\partial x_3},$$

for functions ν_1 and ν_2 whose character is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. The linearisation about the two reference flows are described by the differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= 0; & \dot{x}_2(t) &= x_3(t); \\ \dot{x}_3(t) &= 0, & \dot{x}_3(t) &= 0, \\ \dot{v}_1(t) &= v_2(t) + x_2(t), & \dot{v}_1(t) &= v_2(t) + x_2(t), \\ \dot{v}_2(t) &= \nu_1(t) x_3(t), & \dot{v}_2(t) &= v_3(t) + \nu_1(t) x_3(t), \\ \dot{v}_3(t) &= \nu_2(t), & \dot{v}_3(t) &= \nu_2(t), \end{aligned}$$

respectively. The linearisations about $(X_1, (0,0,0))$ and $(X_2, (0,0,0))$ will be time-independent since the vector fields X_1 and X_2 are time-independent, and we easily determine that these linearisations are given by

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \nu_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \nu_2(t) \end{bmatrix},$$

respectively. These are exactly the two distinct linearisations we encountered in Example 1.1. Thus we can see here what was going on in Example 1.1: we were linearising about two different reference flows. This also highlights the dangers of explicit and fixed parameterisations by control: one can unknowingly make choices that affect conclusions.

We comment that the reason this example does not meet the conditions of Proposition 4.20 is that the vector fields f_1 and f_2 are not linearly independent at $(0,0,0)$. The distribution generated by these vector fields has $(0,0,0)$ as a singular point. These sorts of matters will doubtless be interesting in subsequent studies of geometric control systems in our framework.

Finally, using Lemma 4.22, we can easily conclude that this system is linearly controllable. •

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