

On the role of regularity in mathematical control theory¹

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Abstract

In this thesis, we develop a coherent framework for studying time-varying vector fields of different regularity classes and their flows. This setting has the benefit of unifying all classes of regularity. In particular, it includes the real analytic regularity and provides us with tools and techniques for studying holomorphic extensions of real analytic vector fields. We show that under suitable “integrability” conditions, a time-varying real analytic vector field on a manifold can be extended to a time-varying holomorphic vector field on a neighbourhood of that manifold. Moreover, in this setting, the “nonlinear” differential equation governing the flow of a time-varying vector field can be considered as a “linear” differential equation on an infinite dimensional locally convex vector space. We show that, in the real analytic case, the “integrability” of the time-varying vector field ensures convergence of the sequence of Picard iterations for this linear differential equation, giving us a series representation for the flow of a time-varying real analytic vector field.

Using the framework we develop in this thesis, we study a parametrization-independent model in control theory called tautological control system. In the tautological control system setting, instead of defining a control system as a parametrized family of vector fields on a manifold, it is considered as a subsheaf of the sheaf of vector fields on that manifold. This removes the explicit dependence of the systems on the control parameter and gives us a suitable framework for studying regularity of control systems. We also study the relationship between tautological control systems and classical control systems. Moreover, we introduce a suitable notion of trajectory for tautological control systems.

Finally, we generalize the orbit theorem of Sussmann and Stefan to the tautological framework. In particular, we show that orbits of a tautological control system are immersed submanifolds of the state manifold. It turns out that the presheaf structure on the family of vector fields of a system plays an important role in characterizing the tangent space to the orbits of the system. In particular, we prove that, for globally defined real analytic tautological control systems, every tangent space to the orbits of the system is generated by the Lie brackets of the vector fields of the system.

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Chapter 1

Introduction

1.1. Literature Review

In the mathematical theory of control, regularity of maps and functions plays a crucial role. Literature of control theory is replete with theories which work for a specific class of regularity but fail in other classes. The most well-known example is stabilizability of nonlinear control systems. While it is clear that for linear control systems controllability implies stabilizability, for nonlinear control systems this implication may or may not be true based on the regularity of the stabilizing feedback. In his 1979 paper, Sussmann proved that every real analytic system which satisfies some strong controllability condition is stabilizable by “piecewise analytic” controls [76]. However, some years later, Brockett showed that controllability of a nonlinear system does not necessarily imply stabilizability by “continuous” feedbacks [14]. Another example is the role that real analyticity plays in characterizing the fundamental properties of systems. One can show that the accessibility of real analytic systems at point x_0 can be completely characterized using the Lie bracket of their vector fields at point x_0 [79]. However, such a characterization is not generally possible for smooth systems [79]. While in the physical world there are very few, and possibly no, maps which are smooth but not real analytic, in mathematics the gap between these two classes of regularity is huge. This can be seen using the well-known fact that if a real analytic function is zero on an open set, then it is zero everywhere. However, using a partition of unity, one can construct many non-zero smooth functions that are zero on a given open set [49]. Moreover, the techniques and analysis for studying real analytic systems are sometimes completely different from their smooth counterparts.

Roughly speaking, a map f is real analytic on a domain D if the Taylor series of f around every point $x_0 \in D$ converges to f in a neighbourhood of x_0 . By definition, for the Taylor series of f on D to exist, derivatives of f of any order should exist and be continuous at every point $x_0 \in D$. This means that all real analytic maps are of class C^∞ . The converse implication is not true, since there are some examples of functions that are C^∞ but not real analytic. Although, nowadays, these examples of smooth but not real analytic functions are well-known, it is surprising to know

that at the beginning stages of theory of real analytic functions, mathematicians had difficulty understanding them. In the nineteenth century, mathematicians started to think more about the natural question of which functions can be expanded in a Taylor series around a point. Lagrange and Hankel believed that the existence of all derivatives of a function implies the convergence of its Taylor series [8]. Eventually, it was Cauchy who came up with the function

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

which is C^∞ everywhere not real analytic at $x = 0$ [16], [8].

Real analytic functions on \mathbb{R} have a close connection with the holomorphic functions defined on neighbourhoods of \mathbb{R} in \mathbb{C} . It is well-known that every real analytic function f on \mathbb{R} can be extended to a holomorphic function defined on an appropriate domain in \mathbb{C} . However, it may not be possible to extend the real analytic function f to a holomorphic function on the whole \mathbb{C} . This can be seen in the following example.

Example. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the real analytic function defined as

$$f(x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

If $\bar{f} : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic extension of f to the whole \mathbb{C} , then by the identity theorem, we should have

$$\bar{f}(z) = \frac{1}{1+z^2}, \quad \forall z \in \mathbb{C}.$$

However, the function $\frac{1}{1+z^2}$ is not defined at $z = i$ and $z = -i$ and this is a contradiction. So the holomorphic extension of f to the whole complex plane \mathbb{C} does not exist.

This observation suggests that one should consider a real analytic function as a germ of a holomorphic function. This perspective for real analytic functions motivates the definition of a natural topology on the space of real analytic functions. Unfortunately, there does not exist a single domain such that “every” real analytic function on \mathbb{R} can be extended to a holomorphic function on that domain. The following example shows this fact.

Example. For every $n \in \mathbb{N}$, consider the function $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{1}{1+n^2x^2}, \quad \forall x \in \mathbb{R}.$$

It is easy to see that, for every $n \in \mathbb{N}$, the function f_n is real analytic on \mathbb{R} . We show that there does not exist a neighbourhood Ω of \mathbb{R} in \mathbb{C} such that, for every $n \in \mathbb{N}$, the

real analytic function f_n can be extended to a holomorphic function on Ω . Suppose that such an Ω exists. Then there exists $r > 0$ such that

$$\{x \in \mathbb{C} \mid \|x\| \leq r\} \subseteq \Omega.$$

Now let $N \in \mathbb{N}$ be such that $\frac{1}{N} < r$ and suppose that \bar{f}_N be the holomorphic extension of f_N to Ω . Then, by the identity theorem, we have

$$\bar{f}_N(z) = \frac{1}{1 + N^2 z^2}, \quad \forall z \in \Omega.$$

By our choice of N , we have $\frac{i}{N} \in \Omega$, but \bar{f}_N is not defined at $z = \frac{i}{N}$. This is a contradiction and shows that such an Ω does not exist.

Thus the space of real analytic functions on \mathbb{R} , which we denote by $C^\omega(\mathbb{R})$, can be considered as the “union” of the spaces of holomorphic functions defined on neighbourhoods of \mathbb{R} in \mathbb{C} . This process of taking union can be made precise using the mathematical notion of “inductive limit”. The space of holomorphic functions on an open set $\Omega \subseteq \mathbb{C}$ has been studied in detail in the literature [51], [61]. One can show that the well-known “compact-open” topology on the space of holomorphic functions on Ω is generated by a family of seminorms [51]. The vector spaces equipped with a topology generated by a family of seminorms are called “locally convex topological vector spaces”. Therefore, we can present the space of real analytic functions on \mathbb{R} as an inductive limit of locally convex spaces. The “inductive limit topology” on $C^\omega(\mathbb{R})$ is defined as the finest topology which makes all the inclusions from the spaces of holomorphic functions to the space of real analytic functions continuous.

Locally convex topological vector spaces play a crucial role in the theory of topological vector spaces. Inductive limits of locally convex spaces arise in many fields, including partial differential equations, Fourier analysis, distribution theory, and holomorphic calculus. While there is little literature for inductive limit of arbitrary families of locally convex spaces, the countable inductive limit of locally convex spaces is rich in both theory and applications. Historically, locally convex inductive limits of locally convex spaces first appeared when mathematicians tried to define a suitable topology on the space of distributions. The importance of connecting maps in inductive limits of locally convex spaces was first realized by José Sebastião e Silva [71]. Motivated by studying the space of germs of holomorphic functions, Sebastião e Silva investigated inductive limits with compact connecting maps. Inductive limits with weakly compact connecting maps were studied later by Komatsu in [48], where he showed that weakly compact inductive limits share some nice properties with the compact inductive limits.

One can also characterize the space of real analytic functions on \mathbb{R} using the germs of holomorphic functions around compact subsets of \mathbb{R} . Let $\{K_i\}_{i \in \mathbb{N}}$ be a family of compact sets on \mathbb{R} such that $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}$ and

$$\text{cl}(K_i) \subseteq K_{i+1}, \quad \forall i \in \mathbb{N}.$$

Then the space of real analytic functions on \mathbb{R} is obtained by “gluing together” the vector spaces of germs of holomorphic functions on compact sets $\{K_i\}_{i \in I}$. The concept of “gluing together” mentioned above can be made precise using the notion of projective limit of vector spaces. The coarsest locally convex topology on $C^\omega(\mathbb{R})$ which makes all the gluing maps continuous is called the “projective limit topology” on $C^\omega(\mathbb{R})$. Having defined the “inductive limit topology” and “projective limit topology” on the space of real analytic functions on \mathbb{R} , it would be interesting to study the relation between these two topologies. To our knowledge, the first paper that studied the relation between these two topologies on the space of real analytic functions is [59], where it is shown that these two topologies are identical. There has been a recent interest in this topology due to its applications in the theory of partial differential equations [12], [52].

Since every locally convex topology on a vector space can be characterized using a family of generating seminorms, it is interesting to find a family of generating seminorms for C^ω -topology on $C^\omega(\mathbb{R})$. In [81] a family of generating seminorms for C^ω -topology on $C^\omega(\mathbb{R})$ has been introduced.

In chapter 3, by generalizing the above ideas, we study the C^ω -topology on the space of real analytic sections of a real analytic vector bundle (E, M, π) . For a real analytic vector bundle (E, M, π) , we denote the space of its real analytic sections by $\Gamma^\omega(E)$. Using the similar constructions as for the space of real analytic functions, we can define the “inductive limit” and “projective limit” topology on this space. It follows from [59] that the inductive limit topology and the projective limit topology on $\Gamma^\omega(E)$ are equivalent. This is the topology which we refer to as the C^ω -topology. In particular, using the results of [42], we define two families of generating seminorms for the C^ω -topology on $\Gamma^\omega(E)$.

In control theory, time-varying vector fields with measurable dependence on time arises naturally in studying open-loop systems. Properties of time-varying vector fields and their flows are essential in characterizing the fundamental properties of systems. However, the theory of time-varying vector fields and their flows has not been developed as much as its time-invariant counterpart. In order to study time-varying vector fields, it is convenient to adapt an operator approach to vector fields. The operator approach for studying time-varying vector fields and their flows in control theory started with the work of Agrachev and Gamkrelidze [3]. One can also find traces of this approach in the nilpotent Lie approximations for studying controllability of systems [77], [78]. In [3] a framework is proposed for studying complete time-varying vector fields and their flows. The cornerstone of this approach is the space $C^\infty(M)$, which is both an \mathbb{R} -algebra and a locally convex vector space. In this framework, a smooth vector field on M is considered as a derivation of $C^\infty(M)$ and a smooth diffeomorphism on M is considered as a unital \mathbb{R} -algebra isomorphism of $C^\infty(M)$. Using a family of seminorms on $C^\infty(M)$, weak topologies on the space of derivations of $C^\infty(M)$ and on the space of unital \mathbb{R} -algebra isomorphisms of $C^\infty(M)$ are defined [3]. Then a time-varying vector field is considered as a curve on the space of derivations of $C^\infty(M)$ and its flow is considered as a curve on the space of

\mathbb{R} -algebra isomorphisms of $C^\infty(M)$. While this framework seems to be designed for smooth vector fields and their flows, in [3] and [4] the same framework is used for studying time-varying “real analytic” vector fields and their flows.

In this thesis, we present a coherent framework for studying time-varying vector fields of different regularity classes. While we only focus on smooth and real analytic vector fields, this setting can be generalized to include a variety of regularity classes [42]. In order to include real analytic vector fields and their flows in our framework in a consistent way, we can generalize the operator approach of [3] by replacing the locally convex space $C^\infty(M)$ with $C^\nu(M)$, where $\nu \in \{\infty, \omega\}$. In particular, using the result of [26], we show that there is a one-to-one correspondence between real analytic vector fields on M and derivations of $C^\omega(M)$. Moreover, using the results of [60], we show that C^ω -maps are in one-to-one correspondence with unital \mathbb{R} -algebra homomorphisms on $C^\omega(M)$. This allows us to unify the smooth and real analytic classes of regularity in a setting where there is a one-to-one correspondence between C^ν -vector fields and derivations of $C^\nu(M)$ and there is a one-to-one correspondence between C^ν -maps on M and unital \mathbb{R} -algebra homomorphisms on $C^\nu(M)$. Using these characterizations, we can consider a time-varying C^ν -vector field on M as a curve on the vector space of derivations of $C^\nu(M)$. In order to study properties of this curve, we need to define a topology on the space of derivations of $C^\nu(M)$.

Using the C^ν -topology on $C^\nu(M)$, one can define the topology of pointwise-convergence on the space $L(C^\nu(M); C^\nu(M))$ of linear continuous maps between $C^\nu(M)$ and $C^\nu(M)$. One can show that $L(C^\nu(M); C^\nu(M))$ equipped with the topology of pointwise-convergence is a locally convex space with many nice properties. This topology also induces a locally convex topology on the space of derivations of $C^\nu(M)$ and enables us to study different properties of time-varying vector fields. In particular, we can use the framework in [6] to define and characterize the Bochner integrability of curves on $L(C^\nu(M); C^\nu(M))$.

As we mentioned above, one of the most fundamental and interesting properties of a real analytic vector field on \mathbb{R} is that it can be extended to a holomorphic vector field on a neighbourhood Ω of \mathbb{R} in \mathbb{C} . It would be interesting to ask whether such a holomorphic extension exists for a “time-varying” real analytic vector field. The following example shows that the answer to this question is negative in general.

Example. Define the time-varying vector field $X : \mathbb{R} \times \mathbb{R} \rightarrow T\mathbb{R}$ as

$$X(t, x) = \begin{cases} \frac{t^2}{t^2+x^2} \frac{\partial}{\partial x} & x \neq 0 \text{ or } t \neq 0, \\ 0 & x, t = 0. \end{cases}$$

One can easily see that X is measurable with respect to t and real analytic with respect to x . Now suppose that there exists $\mathbb{T} \subseteq \mathbb{R}$ a neighbourhood of $t = 0$, a connected neighbourhood \bar{U} of \mathbb{R} in \mathbb{C} , and a time-varying vector field $\bar{X} : \mathbb{T} \times \bar{U} \rightarrow T\mathbb{C}$ which is measurable in time and holomorphic in state such that

$$\bar{X}(t, x) = X(t, x) \quad \forall x \in \mathbb{R}, \forall t \in \mathbb{T}.$$

Since $0 \in \mathbb{T}$, there exists $t \in \mathbb{T}$ such that $\text{cl}(D_t(0)) \subseteq \bar{U}$. Let us fix this t and define the real analytic vector field $X_t : \mathbb{R} \rightarrow T\mathbb{R}$ as

$$X_t(x) = \frac{t^2}{t^2 + x^2} \frac{\partial}{\partial x}, \quad \forall x \in \mathbb{R},$$

and the holomorphic vector field $\bar{X}_t : \bar{U} \rightarrow T\mathbb{C}$ as

$$\bar{X}_t(z) = \bar{X}(t, z) \quad \forall z \in \bar{U},$$

Then it is clear that \bar{X}_t is a holomorphic extension of X_t . However, one can define another holomorphic vector field $Y : D_t(0) \rightarrow T\mathbb{C}$ by

$$Y(z) = \frac{t^2}{t^2 + z^2} \frac{\partial}{\partial z}, \quad \forall z \in D_t(0),$$

It is easy to observe that Y is also a holomorphic extension of X_t . Thus, by the identity theorem, we should have $Y(z) = \bar{X}_t(z)$, for all $z \in D_t(0)$. Moreover, we should have $\bar{U} \subseteq D_t(0)$. However, this is a contradiction with the fact that $\text{cl}(D_t(0)) \subseteq \bar{U}$.

As the above example suggests, without any joint condition on time and space, it is impossible to prove a holomorphic extension result similar to the time-invariant real analytic vector fields. It turns out that “local Bochner integrability” is the right joint condition for a time-varying real analytic vector field to have a holomorphic extension. Using the inductive limit characterization of the space of real analytic vector fields, we show that the “global” extension of “locally Bochner integrable” time-varying real analytic vector fields is possible. More specifically, we show that, for a locally Bochner integrable time-varying real analytic vector field X on M , there exists a locally Bochner integrable holomorphic extension on a neighbourhood of M . We call this result a “global” extension since it proves the existence of the holomorphic extension of a time-varying vector field to a neighbourhood of its “whole” state domain.

In order to study the holomorphic extension of a “single” locally Bochner integrable time-varying real analytic vector field, the global extension result is a perfect tool. However, this extension theorem is indecisive when it comes to questions about holomorphic extension of all elements of a family of locally Bochner integrable time-varying real analytic vector fields to a “single” domain. Using the projective limit characterization of space of real analytic vector fields, we show that one can “locally” extend every element of a bounded family of locally Bochner integrable time-varying real analytic vector fields to a time-varying holomorphic vector field defined on a single domain.

Another important question is the connection between time-varying vector fields and their flows. In [3], using the characterizations of vector fields as derivations and their flows as unital algebra isomorphism, the “nonlinear” differential equation on \mathbb{R}^n for flows of a complete time-varying vector field is transformed into a “linear” differential equation on the infinite-dimensional locally convex space $L(C^\infty(\mathbb{R}^n); C^\infty(\mathbb{R}^n))$.

While working with linear differential equations seems to be more desirable than working with their nonlinear counterparts, the fact that the underlying space of this linear differential equation is an infinite-dimensional locally convex spaces makes this study more complicated. In fact, the theory of linear ordinary differential equations on a locally convex spaces is completely different from the classical theory of linear differential equations on \mathbb{R}^n or Banach spaces [56]. While, for the Banach space case, every linear ordinary differential equations has a unique solution, the existence of solutions of an ordinary differential equation on a locally convex space heavily depends on the geometry of the underlying space [56]. Moreover, most of the theorems and techniques in classical differential equations are not applicable when dealing with locally convex spaces. For example, one can easily find counterexamples for Peano's existence theorem [56]. In [3] it has been shown that, if the vector field is integrable in time, real analytic in state, and has a bounded holomorphic extension to a neighbourhood of \mathbb{R}^n , the sequence of Picard iterations for the linear infinite-dimensional differential equation converges in $L(C^\infty(\mathbb{R}^n); C^\infty(\mathbb{R}^n))$. In this case, one can represent flows of a time-varying real analytic system as a series of iterated composition of the time-varying vector field.

The techniques that we developed in this thesis help us to study the differential equations governing the flows of time-varying C^ν -vector fields in a consistent way. The framework we developed for studying time-varying C^ν -vector fields plays a crucial role in this analysis. Using the fact that time-varying vector fields and their flows are curves on $L(C^\nu(M); C^\nu(M))$, we translate the nonlinear differential equation governing the flow a time-varying C^ν -vector field into a “linear” differential equation on $L(C^\nu(M); C^\nu(M))$. In the real analytic case, we show that a solution for the “linear” differential equation of a “locally Bochner integrable” time-varying real analytic vector field exists and is unique. In particular, using a family of generating seminorms on the space of real analytic functions, we show that the sequence of Picard iterations for our “linear” differential equation on the locally convex space $L(C^\omega(M); C^\omega(M))$ converges. This will generalize the result of [3, Proposition 2.1] to the case of locally Bochner integrable time-varying real analytic vector fields.

In chapter 4, we turn our attention to a new framework for modeling control systems. Since the advent of mathematical control theory, many different models have been used for studying control systems. In 1930's, starting with the works of Nyquist [63], Bode [11], and Black [10], the frequency response and transform methods became the dominant techniques for studying control systems. This methodology has the benefit of representing feedback in a nice way. In 1960's a paradigm shift has been made in control theory from the frequency based model to the state-space model. This change was partly motivated by the influential works of Kalman [45], [44] and Pontryagin and his collaborators [64]. In 1970's control theorists started to generalize the results of the state-space framework to nonlinear control systems. At the same time, a geometric approach to study classical mechanics, as in the work of Abraham and Marsden [1], started to gain prominence. Both of these developments inspired the use of differential geometry and Lie theory in control theory [31], [33], [35], [79],

[74], [75]. This resulted in a change of language and techniques in the mathematical theory of control. A geometric model for control theory considers a control system as a family of parametrized vector fields $\{F^u\}_{u \in \mathcal{U}}$, where \mathcal{U} is the set of all “inputs” or “controls”. Since this model is independent of coordinates, it is more convenient for a geometric study of control systems. However, this model has some shortfalls. While most of the fundamental properties of control systems depend on their trajectories, this model has dependence on the specific parametrization of the vector fields of the control system. This makes many techniques and theories that are presented in this model dependent on the specific choice of parameter for the vector fields of the system. The following example shows that even the simple “linear” test for controllability of systems is not parameterization-independent.

Example ([54]). Consider the control systems Σ_1 and Σ_2 on \mathbb{R}^3 , where Σ_1 is defined as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t),\end{aligned}$$

where $(u_1, u_2) \in \mathbb{R}^2$ and the control system Σ_2 is defined as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t) + x_3(t)v_1(t), \\ \dot{x}_3(t) &= v_2(t),\end{aligned}$$

where $(v_1, v_2) \in \mathbb{R}^2$. It is easy to note that under the bijective transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

the two control systems Σ_1 and Σ_2 are identical. This implies that Σ_1 and Σ_2 have the same trajectories and therefore have the same small-time local controllability. However, if we apply the linear test to Σ_1 and Σ_2 , one can observe that the linearization of Σ_2 is controllable at $(0, 0, 0)$ and this implies that Σ_2 (and therefore Σ_1) is small-time locally controllable at $(0, 0, 0)$. However, the linearization of Σ_1 is not controllable at $(0, 0, 0)$. This implies that the linear test is indecisive for small-time local controllability of Σ_1 at $(0, 0, 0)$.

This example motivates the construction of a setting for studying a control system which is independent of parametrization of vector fields of the system. In [55] a parameterization-independent methodology called “tautological control systems” has been developed using the notion of a sheaf of sets of vector fields. Considering control systems as sheaves of vector fields makes control systems at the level of definition independent of control parametrization. Moreover, it allows us to consider regularity

of the control systems in a consistent manner. In this thesis, following [55], we introduce tautological control systems and study their connection with the classical control systems. We also define the appropriate notion of trajectory for tautological control systems.

One of the most fundamental theorems in geometric control theory is the Chow–Rashevskii theorem. Let $S = \{X_1, X_2, \dots, X_n\}$ be a family of vector fields on a manifold M . Given a point $x_0 \in M$, the orbit of S passing through x_0 is the set of all points of M which can be reached from x_0 by traveling along trajectories of vector fields of S in both positive and negative times. The Chow–Rashevskii theorem connects some of the properties of orbits of S to the Lie brackets of vector fields in S . In order to state the Chow–Rashevskii theorem, we first need to introduce some terminology. A distribution on TM generated by S is the assignment D_S such that, for every $x \in M$, $D_S(x)$ is the vector subspace of T_xM defined by

$$D_S(x) = \text{span}\{X_1(x), X_2(x), \dots, X_n(x)\}.$$

We say that a vector field Y belongs to the distribution D_S (and we write $Y \in D_S$) if, for every $x \in M$, we have

$$Y(x) \in D_S(x).$$

A distribution D_S is involutive if, for every $Y, Z \in D_S$, $[Y, Z] \in D_S$. An integral manifold of D_S is a connected submanifold N of M such that $T_xN = D_S(x)$ for every $x \in N$. In other words, N is an integral manifold of D_S if D_S is the tangent space of N at every point. It is easy to check that not every distribution has an integral manifold. A distribution D_S on TM is called integrable if, for every $x \in M$, there exists a maximal integral manifold of D_S passing through x .

Given a family of vector fields $S = \{X_1, X_2, \dots, X_n\}$ on the manifold M , we denote the Lie algebra generated by this family of vector fields by $\text{Lie}(S)$. Chow [17] and Rashevskii [65] independently showed that if, for $x_0 \in M$, we have $\text{Lie}(S)(x_0) = T_{x_0}M$, then the orbit of S passing through x_0 contains a neighbourhood of x_0 . While the Chow–Rashevskii theorem only gives us information about orbit of S passing through x_0 when $\text{Lie}(S)(x_0) = T_{x_0}M$, it would be interesting to investigate the structure of this orbit when $\text{Lie}(S)(x_0) \neq T_{x_0}M$. As mentioned in [74], if the distribution $\text{Lie}(S)$ is integrable, it is still possible to apply the Chow–Rashevskii theorem for the maximal integral manifold of the distribution $\text{Lie}(S)$. Unfortunately, the distribution $\text{Lie}(S)$ is not generally integrable for a family of smooth vector fields. This can be seen using the following example.

Example ([74]). Consider the family of vector fields $S = \{X_1, X_2\}$ on \mathbb{R}^2 defined as

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \phi(x) \frac{\partial}{\partial y}. \end{aligned}$$

Where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\phi(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Let I be the maximal integral manifold of $\text{Lie}(\{X_1, X_2\})$ passing through $(0, 0) \in \mathbb{R}^2$. Note that we have

$$\dim(\text{Lie}(\{X_1, X_2\})(0, 0)) = 1.$$

Thus, the integral manifold I is a 1-dimensional manifold. However, we have

$$\text{Lie}(\{X_1, X_2\})(0, 0) = \text{span} \left\{ \frac{\partial}{\partial x}(0, 0) \right\}$$

This implies that, there exists $(x_0, y_0) \in I$ such that $x_0 > 0$. However, at (x_0, y_0) , we have

$$\dim(\text{Lie}(\{X_1, X_2\})(0, 0)) = 2,$$

which is a contradiction. Thus the integral manifold of $\text{Lie}(\{X_1, X_2\})$ passing through $(0, 0) \in \mathbb{R}^2$ does not exist.

As is shown in the example above, for a family $S = \{X_1, X_2, \dots, X_n\}$ of smooth but non-real-analytic vector fields, it is possible that the involutive distribution $\text{Lie}(S)$ does not have a maximal integral manifold. In 1974, Sussmann [74] and Stefan [72] independently proved a singular version of the Chow–Rashevskii theorem for smooth vector fields called the “orbit theorem”. Using the concatenation of flows of the vector fields $\{X_1, \dots, X_n\}$, Sussmann defined a distribution P_S . Instead of working with involutive distribution $\text{Lie}(S)$, Sussmann considered the distribution P_S . In particular, he showed that, given smooth vector fields $S = \{X_1, X_2, \dots, X_n\}$, the distribution P_S is integrable and its maximal integral manifolds are exactly the S -orbits [74]. This implies that S -orbits are smooth submanifolds of M and completely characterizes the tangent space to S -orbits.

Stefan introduced the notion of singular foliation in [72]. One can consider a singular foliation as a generalization of the notion of foliation where the leaves does not necessarily have the same dimension. Stefan proved that a family of smooth vector fields $S = \{X_1, X_2, \dots, X_n\}$ induces a “singular foliation” structure on M and using the same distribution P_S , he completely characterizes the tangent space to leaves of this foliation [72].

It would be natural to expect that if the distribution $\text{Lie}(S)$ is integrable, then it coincides with the distribution P_S . Both Sussmann and Stefan studied conditions under which the distribution $\text{Lie}(S)$ is integrable and $\text{Lie}(S)$ and P_S are identical. In the differential geometry literature, numerous conditions have been developed for the integrability of a distribution. The Frobenius theorem is one of the first and most well-known of these results. According to the Frobenius theorem, if the rank of the distribution $\text{Lie}(S)$ is locally constant at every point on its domain, then it

is integrable. In 1963, Hermann proposed two other conditions for integrability of $\text{Lie}(S)$ [32]. By considering the space of smooth vector fields as a module over the ring $C^\infty(M)$, Hermann showed that the module structure of the family of vector fields S plays a crucial role in the integrability of $\text{Lie}(S)$ [32]. He defined a $C^\infty(M)$ -module generated by vector fields in S and their Lie brackets and showed that if this module has some finiteness property, then the distribution $\text{Lie}(S)$ is integrable [32, 2.1(b)]. Hermann's second condition dealt with real analyticity of vector fields in S . He claimed that if the vector fields in S are real analytic, the distribution $\text{Lie}(S)$ is integrable [32, 2.1(c)]. However, he did not give any proof for this claim in his paper. Three years later, Nagano [62] showed that if the vector fields in S are real analytic, the distribution $\text{Lie}(S)$ is integrable. In 1970, Lobry introduced a weaker condition called "locally of finite type" and showed that (up to a minor error later discovered by Stefan) if the distribution $\text{Lie}(S)$ is locally of finite type, then it is integrable [57]. Using the Noetherian property of the space of germs of real analytic functions, Lobry claimed that an involutive family of real analytic vector fields is locally of finite type [57, Proposition 1.2.8]. However, Lobry did not give a complete proof of this claim.

In this thesis, following the approach of [72], we prove a similar orbit theorem for tautological control systems. While this result can be considered as a generalization of the orbit theorem, its proof is essentially the same as the proof of the orbit theorem for ordinary control systems. However, the power of the tautological control system approach becomes more clear in the real analytic case, where we show that the "locally finitely generated" property of the presheaf of Lie brackets of vector fields of a tautological control system can be used to characterize the orbits of the system using the Lie brackets of vector fields of the system.

1.2. Contribution of thesis

In this thesis, we develop a unifying framework for studying time-varying vector fields of smooth and real analytic regularity. The framework we developed can be generalized to include time-varying vector fields of a variety of regularity classes [42]. The contribution of this thesis is as follows.

1. As mentioned in the previous section, in [3] and [4], an operator framework has been used to study time-varying vector fields and their flows in \mathbb{R}^n . In the heart of the approach in [3] is the \mathbb{R} -algebra $C^\infty(\mathbb{R}^n)$ and the locally convex topology on it. In this approach, smooth vector fields are considered as derivations of \mathbb{R} -algebra $C^\infty(\mathbb{R}^n)$ and smooth maps are considered as unital \mathbb{R} -algebra homomorphisms on $C^\infty(\mathbb{R}^n)$. Then one can consider a time-varying vector field as a curve on the space of derivations of $C^\infty(\mathbb{R}^n)$. However, it seems that in both [3] and [4] there is no consistency between regularity of vector fields and their corresponding operators. For example, in [4], a real analytic vector field is also considered as derivation of \mathbb{R} -algebra $C^\infty(\mathbb{R}^n)$.

In this thesis, we develop a unified setting for studying smooth and real analytic time-varying vector field in a consistent manner. In order to do this, we generalize the frameworks in [3] and [4] in two ways. First, instead of working with the \mathbb{R} -algebra $C^\infty(M)$ for all regularity classes, we consider the \mathbb{R} -algebra $C^\nu(M)$, where $\nu \in \{\infty, \omega\}$, consistent with the regularity of the problem. Using the results of Grabowski [26] and Michael [60], we show that there is a one-to-one correspondence between C^ω -vector fields on M and derivations on $C^\omega(M)$ and there is a one-to-one correspondence between C^ω -maps on M and unital \mathbb{R} -algebra homomorphisms of $C^\omega(M)$. These results together with that proved in [4] allow us to develop a setting for studying time-varying C^ν -vector fields and their flows which is consistent with regularity of the vector fields. While, we only include the space of smooth and real analytic functions in this thesis, the extension of this setting to other classes of regularity is straightforward and has been done in [42]. Secondly, instead of focusing only on the space of functions on M , we study the space of sections of a vector bundle (E, M, π) . One can easily see that functions and vector fields can be obtained from this constructions by considering the sections of the trivial line bundle and tangent bundle on M respectively. Moreover, we define suitable topology on the space of C^ν -sections, which makes it into a complete locally convex space. This topology in the smooth case is the well-known “smooth compact-open” topology. However, for the real analytic sections, following [59], we define the topology using the germ of holomorphic sections. This topology, which turns out to be finer than the restriction of smooth compact-open topology, plays a crucial role in studying the local and global holomorphic extension of time-varying real analytic vector fields.

2. It is well-known that every real analytic function on \mathbb{R} can be extended to a holomorphic function on a neighbourhood $\Omega \subseteq \mathbb{C}$ of \mathbb{R} . Similarly, one can extend a real analytic function (vector field) on a real analytic manifold M to a holomorphic function (vector field) on a complexification of M [18, Lemma 5.40]. In control theory, it is sometimes crucial to work with time-varying vector fields. Since the class of real analytic regularity is of considerable importance in control theory, it is interesting to study time-varying real analytic vector fields. As mentioned in the introduction, the class of real analytic maps has a close connection with the class of holomorphic maps. Therefore, one would like to see if it is possible to extend a time-varying real analytic vector field to a time-varying holomorphic vector field. In the previous section we showed that generally such an extension is not possible. One would then try to impose some appropriate joint regularity condition on time and state to ensure the existence of such an extension. In chapter 3, we develop tools and techniques for studying time-varying vector fields as curves on $\Gamma^\nu(TM)$. In particular, following the approach in [6] and using the properties of the locally convex space $\Gamma^\nu(TM)$, we study and characterize the “Bochner integrability” of curves on $\Gamma^\nu(TM)$.

We then use the inductive limit representation of the space $\Gamma^\omega(TM)$ to show that one can extend a “locally Bochner integrable” time-varying real analytic vector field on M to a locally Bochner integrable holomorphic vector field on a complexification of M (Theorem 3.7.4). We call this result the “global” extension result, since it proves the existence of a holomorphic extension on the “whole” domain M . While this theorem is the right tool for studying extension of a “single” time-varying real analytic vector field, it is indecisive for studying the holomorphic extension of a family of time-varying real analytic vector fields. In Theorem 3.7.8, using the projective limit representation of the space $\Gamma^\omega(TM)$, we show that all members of a “bounded” family of locally Bochner integrable real analytic vector fields have holomorphic extensions to a common domain of states.

3. Following the chronological calculus of Agrachev and Gamkrelidze [3], one can consider the “nonlinear” differential equation governing the flow of a time-varying vector field as a “linear” differential equation on the infinite-dimensional locally convex space $L(C^\infty(M); C^\infty(M))$. In [3] and [4] this “linear” differential equation has been studied on $L(C^\infty(M); C^\infty(M))$ for the real analytic vector fields. The approach used in both [3] and [4] is to construct the so-called sequence of Picard iterations for the “linear” differential equation and show that this sequence converges. In [3] it has been shown that, if a locally integrable time-varying real analytic vector field on \mathbb{R}^n has a bounded extension to a locally integrable time-varying holomorphic vector field on \mathbb{C}^n , then the sequence of Picard iterations for the extended vector field converges in $L(C^{\text{hol}}(M); C^{\text{hol}}(M))$. In [4], it has been shown that, for every locally integrable time-varying real analytic vector field on a real analytic manifold M , the sequence of Picard iterations converges in $L(C^\infty(M); C^\infty(M))$. In this thesis, we show that for a time-varying C^ν -vector fields, the associated “linear” differential equation is a differential equation on the locally convex space $L(C^\nu(M); C^\nu(M))$. In Theorem 3.8.1 we study this “linear” differential equation for the holomorphic and real analytic case in a consistent manner. As mentioned in the previous section, we show that a locally Bochner integrable time-varying real analytic vector field X can be considered as a locally Bochner integrable curve on $L(C^\omega(M); C^\omega(M))$. Therefore, the “linear” differential equation for the flows of X is a “linear” differential equation on the locally convex space $L(C^\omega(M); C^\omega(M))$. Using the extension results developed in Theorem 3.7.4 and appropriate estimates for the seminorms on the space of real analytic functions (equation (3.3.2)), we show that the sequence of Picard iterations for this “linear” differential equation converges in $L(C^\omega(M); C^\omega(M))$. This will give us the convergence of the sequence of Picard iterations for the locally Bochner integrable time-varying real analytic vector field in the real analytic topology.
4. We finish chapter 3 by studying the connection between locally Bochner integrable time-varying C^ν -vector fields and their flows. We consider locally

Bochner integrable time-varying C^ν -vector fields as locally Bochner integrable curves on the space $L(C^\nu(M); C^\nu(M))$. Moreover, by Theorem 3.8.1, we know that the flow of a locally Bochner integrable time-varying C^ν -vector field can also be considered as a locally absolutely continuous curve on the space $L(C^\nu(M); C^\nu(U))$, for some open set U . In Example 3.9.1, it has been clarified that any map which assigns to a locally Bochner integrable time-varying vector field its flow can only be defined into the space of “germs” of flows. We then proceed to define the “exponential map” which assigns to the germ of a locally Bochner integrable curve around $(t_0, x_0) \in \mathbb{R} \times M$, the germ of its flow around (x_0, t_0) . Using suitable topologies on the space of locally Bochner integrable curves on $L(C^\nu(M); C^\nu(M))$ and the space of locally absolutely continuous curves on $L(C^\nu(M); C^\nu(U))$, one can induce topologies on domain and codomain of the exponential map. In particular, we show that, using these topologies, the exponential map is sequentially continuous.

5. One can use the framework developed in chapter 3 to study regularity of control systems. Using the C^ν -topologies developed in chapter 3, we define a new class of control systems called C^ν -control systems. This control system is a family of parametrized vector fields $\{F^u\}_{u \in \mathcal{U}}$, where the control set \mathcal{U} is a general “topological space”. We study regularity of C^ν -control systems and we show that, by imposing appropriate conditions on $\{F^u\}_{u \in \mathcal{U}}$ coming from C^ν -topology on $\Gamma^\nu(M)$, we could ensure that the regularity of the flows of a C^ν -control system is consistent with the regularity of the system itself. In the real analytic case, the class of C^ω -control systems is new and deep. Moreover, using the C^ω -topology on the space of real analytic vector fields, we study the relationship between real analytic control systems and holomorphic control systems. In particular, we show that when \mathcal{U} is locally compact, every real analytic control system can be extended “locally” to a holomorphic control system on an appropriate manifold.
6. In chapter 4, following [55], we present a model for studying control system called “tautological control system”. Instead of considering control systems as a “parametrized” family of vector fields, a tautological control system is defined as a presheaf of vector fields. Therefore, the tautological framework removes the dependence of the definition of control systems on “control parametrization”. However, this arises some difficulties in defining trajectories of a system. We define étalé trajectories of a tautological control systems and we show that this notion of the trajectory is the right one for studying orbits of the system and its properties.
7. In chapter 5, we study the orbits of C^ν -tautological control systems. We generalize proof of the orbit theorem given in [72] to the case of C^ν -tautological control systems. In particular, for every C^ν -tautological control system $\Sigma = (M, \mathcal{F})$, we define a subpresheaf $\overline{\mathcal{F}}$ of the sheaf of C^ν -vector fields on M , which is called

the homogeneous presheaf associated to \mathcal{F} . We show that, for every point x on the manifold M , the tangent space to the orbit of Σ passing through x is $\overline{\mathcal{F}}(x)$. Similar to the classical orbit theorem, it would be interesting to see whether one can characterize $\overline{\mathcal{F}}(x)$ using the Lie brackets of vector fields of the system. In particular, one would like to find conditions under which we have

$$\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x).$$

Given a family of vector fields S , Hermann observed that the module structure of the vector fields of the system plays an essential role in the integrability of the distribution $\text{Lie}(S)$. More specifically, Hermann showed that if $\text{Lie}(S)$ is a locally finitely generated module, then the distributions $\text{Lie}(S)$ and P_S coincides [31, 2.1(c)]. Unfortunately, one cannot generalize Hermann's condition for tautological control systems. In particular, Example 5.4.13 gives a real analytic tautological control system (M, \mathcal{F}) such that, for every open neighbourhood U of x , $\text{Lie}(\mathcal{F})(U)$ is a locally finitely generated $C^\omega(M)$ -module but $\overline{\mathcal{F}}(x) \neq \text{Lie}(\mathcal{F})(x)$.

Thus, for a tautological control system (M, \mathcal{F}) , the fact that $\text{Lie}(\mathcal{F})$ is a locally finitely generated module does not generally imply that $\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x)$. In this thesis, we consider $\text{Lie}(\mathcal{F})$ as a subpresheaf of the sheaf of C^ν -vector fields on M . Then we show that if $\text{Lie}(\mathcal{F})$ is a “locally finitely generated presheaf”, then, for every $x \in M$, we have

$$\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x).$$

Thus the difference between $\text{Lie}(\mathcal{F})$ being a locally finitely generated module and a locally finitely generated presheaf is crucial here. It should be noted that the first condition is a condition on the local section of $\text{Lie}(\mathcal{F})$, while the second one is a condition on germs of sections in $\text{Lie}(\mathcal{F})$. In particular, for a globally generated real analytic tautological control system $\Sigma = (M, \mathcal{F})$, using the Weierstrass Preparation theorem and the Noetherian property of analytic sheaves, one can show that the presheaf $\text{Lie}(\mathcal{F})$ is locally finitely generated. Therefore, for globally generated real analytic tautological control systems, we have

$$\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x).$$

Chapter 2

Mathematical notation and background

In this chapter, we present the mathematical background and notation used in this thesis. Our treatment of every subject in this section is not comprehensive and we refer to the references for a detailed study of each topic.

2.1. Manifolds and mappings

Definition 2.1.1. A **multi-index of order m** is an element $(r) = (r_1, r_2, \dots, r_m) \in (\mathbb{Z}_{\geq 0})^m$. For all multindices (r) and (s) of order m , every $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, and $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, we define

$$\begin{aligned} |r| &= r_1 + r_2 + \dots + r_m, \\ (r) + (s) &= (r_1 + s_1, r_2 + s_2, \dots, r_m + s_m), \\ (r)! &= r_1! r_2! \dots r_m!, \\ x^{(r)} &= x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}, \\ D^{(r)} f(x) &= \frac{\partial^{|r|} f}{\partial x_1^{r_1} \partial x_2^{r_2} \dots \partial x_m^{r_m}}, \\ \binom{(r)}{(s)} &= \binom{r_1}{s_1} \binom{r_2}{s_2} \dots \binom{r_m}{s_m}. \end{aligned}$$

We denote the multi-index $(0, 0, \dots, 1, \dots, 0) \in (\mathbb{Z}_{\geq 0})^m$, where 1 is in the i -th place, by $\widehat{(i)}$. One can compare multindices $(r), (s) \in (\mathbb{Z}_{\geq 0})^m$. We say that $(s) \leq (r)$ if, for every $i \in \{1, 2, \dots, m\}$, we have $s_i \leq r_i$.

Definition 2.1.2. The space of all decreasing sequences $\{a_i\}_{i \in \mathbb{N}}$ such that $a_i \in \mathbb{R}_{>0}$ and $\lim_{n \rightarrow \infty} a_n = 0$ is denoted by $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$.

Definition 2.1.3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $x_0 \in \Omega$. A mapping $f : \Omega \rightarrow \mathbb{R}^l$ is **smooth**, or **of class C^∞** at point x_0 , if there exists a neighbourhood $U \subseteq \Omega$ of x_0

such that, for every $m \in \mathbb{N}$ and every $(r) \in (\mathbb{Z}_{\geq 0})^m$, the mapping $D^{(r)}f : U \rightarrow \mathbb{R}^{m^{|r|}}$ is continuous.

A mapping $f : \Omega \rightarrow \mathbb{R}^l$ is **smooth** on Ω if, for every $x_0 \in \Omega$, f is smooth at x_0 .

Definition 2.1.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $x_0 \in \Omega$, and $f : \Omega \rightarrow \mathbb{R}^l$ be a C^∞ -mapping at x_0 . Then the **Taylor series** of f at x_0 is the power series

$$\sum_{(r) \in (\mathbb{Z}_{\geq 0})^m} \frac{1}{(r)!} [D^{(r)}f(x_0)] (x - x_0)^{(r)}. \quad (2.1.1)$$

A C^∞ -mapping $f : \Omega \rightarrow \mathbb{R}^l$ is **real analytic** or **of class C^ω** if, for every $x_0 \in \Omega$, there exists $\rho > 0$ such that the Taylor series (2.1.1) of f at x_0 converges to $f(x)$ for all $\|x - x_0\| < \rho$.

A mapping $f : \Omega \rightarrow \mathbb{R}^l$ is **real analytic** on Ω if, for every $x_0 \in \Omega$, it is real analytic at x_0 .

The class of C^ω -maps are strictly contained in class of C^∞ -maps. The following theorem characterizes real analytic functions as a subset of smooth functions [49, Proposition 2.2.10].

Theorem 2.1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \rightarrow \mathbb{R}^l$ be smooth on Ω . Then the following are equivalent:*

1. f is real analytic on Ω ;
2. there exists open set $\bar{U} \subseteq \mathbb{C}^n$ such that $\Omega \subset \bar{U}$ and a holomorphic function $\bar{f} : \bar{U} \rightarrow \mathbb{C}^l$ such that

$$\bar{f}(x) = f(x), \quad \forall x \in \Omega;$$

3. for every $x \in \Omega$, there exists $V \subseteq \Omega$ containing x and $R, C > 0$ such that

$$\left\| \frac{\partial^{|r|} f}{\partial x^{(r)}}(x) \right\| \leq \frac{C(r)!}{R^{|r|}}, \quad \forall (r) \in (\mathbb{Z}_{\geq 0})^m.$$

Definition 2.1.6. Let $\nu \in \{\infty, \omega, \text{hol}\}$, $n \in \mathbb{N}$, and $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, where $\mathbb{F} = \mathbb{C}$ if $\nu = \text{hol}$. Then a C^ν -**manifold** is a Hausdorff, second countable topological space M equipped with a family \mathcal{A} of maps such that

1. for every $\phi \in \mathcal{A}$, $\phi : D_\phi \rightarrow R_\phi$ is a homeomorphism from an open subset $D_\phi \subseteq M$ onto an open set $R_\phi \subseteq \mathbb{F}^n$,
2. $\bigcup_{\phi \in \mathcal{A}} D_\phi = M$,
3. for every $\phi, \psi \in \mathcal{A}$, the map $\phi \circ \psi^{-1} : \psi(D_\phi \cap D_\psi) \rightarrow \phi(D_\phi \cap D_\psi)$ is of class C^ν .

The family \mathcal{A} is called an **atlas** on M . Members of \mathcal{A} are called **coordinate charts** on M . We usually refer to a coordinate chart as a pair (D_ϕ, ϕ) . The integer n is called the **dimension** of the manifold and we write $\dim_{\mathbb{F}} M = n$.

Definition 2.1.7. Let M and N be two C^ν -manifolds and $x_0 \in M$. A mapping $F : M \rightarrow N$ is of **class** C^ν at x_0 if there exists a coordinate chart (ϕ, D_ϕ) on M containing x_0 and a coordinate chart (ψ, D_ψ) on N containing $f(x_0)$ such that $F(D_\phi) \subseteq D_\psi$ and the mapping

$$\psi \circ F \circ \phi^{-1} : \phi(D_\phi) \rightarrow \psi(D_\psi),$$

is of class C^ν at $\phi(x_0)$.

A mapping $f : M \rightarrow N$ is of **class** C^ν if, for every $x \in M$, it is of class C^ν at x .

If $\nu = \infty$, we consider the manifold M to be of class C^∞ . Manifolds of class C^ν are also called **smooth manifolds**.

If $\nu = \omega$, the manifold M is considered to be of class C^ω . Manifolds of class C^ω are usually called **real analytic manifolds**.

Finally, if $\nu = \text{hol}$, we consider the manifold M to be a C^{hol} -manifold. Manifolds of class C^{hol} are usually called **holomorphic manifolds**.

Definition 2.1.8. The set of all C^ν -functions $f : M \rightarrow \mathbb{F}$ is denoted by $C^\nu(M)$.

The set $C^\nu(M)$ is clearly an \mathbb{F} -vector space under addition and scalar multiplication defined as

$$\begin{aligned} (f + g)(x) &= f(x) + g(x), & \forall f, g \in C^\nu(M), \\ (\lambda f)(x) &= \lambda f(x), & \forall f \in C^\nu(M), \lambda \in \mathbb{F}. \end{aligned}$$

Since every C^ν -manifold of dimension n is locally diffeomorphic with \mathbb{F}^n , one can see that locally C^ν -functions do exist. However, in the case of holomorphic manifolds, it is possible that the set of globally defined holomorphic functions are very restricted (for example, when M is a compact holomorphic manifold, then $C^{\text{hol}}(M)$ consists only of locally constant functions) [28, Chapter V, Theorem 2.4]. We restrict our analysis to a class of holomorphic manifolds which turns out to have good supply of globally defined holomorphic functions.

Definition 2.1.9. A C^{hol} -manifold is called a **Stein manifold** if,

1. for every compact set $K \subseteq M$, the set

$$\widehat{K} = \{z \in M \mid \|f(z)\| \leq \sup_{z \in K} \|f(z)\|, \text{ for every } f \in C^{\text{hol}}(M)\},$$

is a compact subset of M .

2. the set $C^{\text{hol}}(M)$ separate points on M (i.e., for every $z_1, z_2 \in M$ such that $z_1 \neq z_2$, there exists $f \in C^{\text{hol}}(M)$ such that $f(z_1) \neq f(z_2)$).

One can show that Stein manifolds have “enough” global functions to construct local charts.

Theorem 2.1.10. *Let M be a Stein manifold. Then, for every $z_0 \in M$, there exist $f_1, f_2, \dots, f_n \in C^{\text{hol}}(M)$ and a neighbourhood U around z_0 such that (U, ϕ) , where $\phi : U \rightarrow \mathbb{R}^n$ given by*

$$\phi(z) = (f_1(z), f_2(z), \dots, f_n(z)), \quad \forall z \in U,$$

is a coordinate chart around z_0 .

For a C^ν -manifold M , one can attach a vector space to every point on a manifold and glue these vector spaces in such a way that the whole space “locally” looks like $M \times \mathbb{R}^k$. These structures arises frequently in differential geometry and are called vector bundles.

Definition 2.1.11. Let M be a C^ν -manifold. A C^ν -**vector bundle** over M is a triple (E, M, π) where E is a C^ν -manifold and $\pi : E \rightarrow M$ is a C^ν -map such that

1. for every $x \in M$, the set $E_x = \pi^{-1}(x)$ (which is usually called the **fiber of E at x**) is a k -dimensional \mathbb{F} -vector space;
2. for every $x \in M$, there exists a neighbourhood U of x in M and a C^ν -diffeomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{F}^k$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{F}^k \\ & \searrow \pi & \downarrow \text{pr}_1 \\ & & U \end{array}$$

where $\text{pr}_1 : U \times \mathbb{F}^k \rightarrow U$ is the projection into the first factor;

3. for every $x \in M$, the restriction of Φ to E_x is a linear isomorphism from E_x to \mathbb{F}^k .

Definition 2.1.12. Let (E, M, π) be a C^ν -vector bundle. A C^ν -**section** of (E, M, π) is a C^ν -map $\sigma : E \rightarrow M$ such that

$$\pi \circ \sigma = \text{id}_M.$$

The set of all C^ν -sections of a C^ν -vector bundle (E, M, π) is denoted by $\Gamma^\nu(E)$. One can show that $\Gamma^\nu(E)$ is an \mathbb{F} -vector space.

Let M be a C^ν -manifold. One can show that $(M \times \mathbb{R}, M, \text{pr}_1)$ is a C^ν -vector bundle. Moreover, C^ν -sections of $(M \times \mathbb{R}, M, \text{pr}_1)$ are exactly C^ν -functions on M .

Now we can define maps between vector bundle and subbundles.

Definition 2.1.13. Let (E, M, π) and (E', M', π') be two C^ν -vector bundles. A C^ν **vector bundle map** between (E, M, π) and (E', M', π') is a pair of maps (F, f) such that

1. The maps $F : E \rightarrow E'$ and $f : M \rightarrow M'$ are C^ν and the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & M' \end{array}$$

2. for every $x \in M$, the map $F|_{E_x} : E_x \rightarrow E'_{f(x)}$ is linear.

Vector bundle (E, M, π) is a **generalized vector subbundle** of (E', M', π') if there exists a C^ν -vector bundle map (F, f) such that

1. The maps $F : E \rightarrow E'$ and $f : M \rightarrow M'$ are C^ν -embeddings, and
2. for every $x \in M$, the map $F|_{E_x} : E_x \rightarrow E'_{f(x)}$ is injective.

Let (E, M, π) be a C^ν -vector bundle and $U \subseteq M$ be an open subset of M . We set $E|_U = \pi^{-1}(U)$. Then one can show that $(E|_U, U, \pi|_U)$ is a generalized C^ν -vector subbundle of (E, M, π) .

Definition 2.1.14. For the space \mathbb{R}^n , we define the Euclidian norm $\|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\|\mathbf{v}\|_{\mathbb{R}^n} = (v_1^2 + v_2^2 + \dots + v_n^2)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbb{R}^n.$$

For the space \mathbb{C}^n , we define the norm $\|\cdot\|_{\mathbb{C}^n} : \mathbb{C}^n \rightarrow \mathbb{R}$ as

$$\|\mathbf{v}\|_{\mathbb{C}^n} = (v_1\bar{v}_1 + v_2\bar{v}_2 + \dots + v_n\bar{v}_n)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbb{C}^n.$$

Let M be an n -dimensional C^ν -manifold and (U, ϕ) be a coordinate chart on M . Then we define $\|\cdot\|_{(U, \phi)} : U \rightarrow \mathbb{R}$ as

$$\|x\|_{(U, \phi)} = \|\phi(x)\|_{\mathbb{F}^n}, \quad \forall x \in U.$$

Let M be an n -dimensional C^ν -manifold and (U, ϕ) be a coordinate chart on M . Let f be a C^ν -function on M . Then, for every multi-index (r) , we define $\|D^{(r)}f(x)\|_{(U, \phi)}$ as

$$\|D^{(r)}f(x)\|_{(U, \phi)} = \|D^{(r)}(f \circ \phi)(\phi^{-1}(x))\|_{\mathbb{F}}, \quad \forall x \in U.$$

When the coordinate chart on M is understood, we usually omit the subscript (U, ϕ) in the norm.

Let (E, M, π) be a C^ν -vector bundle, M be an n -dimensional C^ν -manifold, (U, ϕ) be a coordinate chart on M , and $\eta : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ be a local trivialization for (E, M, π) . Let X be a C^ν -section of (E, M, π) . Then, for every multi-index (r) , we define $\|D^{(r)}X(x)\|_{(U, \phi, \eta)}$ as

$$\|D^{(r)}X(x)\|_{(U, \phi, \eta)} = \|D^{(r)}(\eta \circ X \circ \phi^{-1})(\phi(x))\|_{\mathbb{F}}, \quad \forall x \in U.$$

When the coordinate chart on M and the local trivialization on (E, M, π) is understood, we usually omit the subscript (U, ϕ, η) in the norm.

2.2. Complex analysis

In this section we review some well-known theorems from the theory of several complex variables.

Definition 2.2.1. Let $\omega \in \mathbb{C}^n$. An **open polydisc** $D_{(r)}(\omega)$ for $(r) = (r_1, r_2, \dots, r_n) \in (\mathbb{R}_{>0})^n$ is a subset of \mathbb{C}^n defined as

$$D_{(r)}(\omega) = \{z \in \mathbb{C}^n \mid \|z_i - \omega_i\| < r_i, \quad \forall i \in \{1, 2, \dots, n\}\}.$$

The Cauchy integral formula is one of the fundamental results in theory of functions with one complex variable. Repeated application of Cauchy's integral formula for one variable will give us the following generalization of Cauchy's integral formula for several complex variables [37, Theorem 2.2.1].

Theorem 2.2.2. Let $D_{(r)}(z_0)$ be an open polydisc in \mathbb{C}^n and $f \in C^0(\text{cl}(D_{(r)}(z_0)))$, such that f is analytic in each z_i on $D_{(r)}(z_0)$, when other variables are fixed. Then we have

$$f(z) = \frac{1}{(2\pi)^n} \int_{\partial_0 D_{(r)}(z_0)} \frac{f(z_1, z_2, \dots, z_n)}{(\omega_1 - z_1)(\omega_2 - z_2) \dots (\omega_n - z_n)} dw_1 dw_2 \dots dw_n,$$

where $\partial_0 D_{(r)}(z_0)$ is not the boundary of $D_{(r)}(z_0)$, but it is a set defined as

$$\partial_0 D_{(r)}(z_0) = \prod_{i=1}^n \{z \in \mathbb{C} \mid \|z_i - (z_0)_i\| = r_i\}.$$

Using the Cauchy's integral formula for several variables, one can get Cauchy's estimate for derivatives of holomorphic functions [37, Theorem 2.2.1].

Theorem 2.2.3. Let $f \in C^{\text{hol}}(D_{(r)}(z_0))$. Then we have

$$\|D^{(s)} f(z_0)\| \leq \frac{(s)!}{(r)^{(s)}} \sup\{\|f(z)\| \mid z \in \text{cl}(D_{(r)}(z_0))\}.$$

2.3. Topological vector spaces

2.3.1. Basic definitions. In this section we consider vector spaces over the field \mathbb{F} , where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 2.3.1. Let V be a vector space over field \mathbb{F} with addition $+$: $V \times V \rightarrow \mathbb{F}$ and scalar multiplication \cdot : $\mathbb{F} \times V \rightarrow V$. Then a topology τ on V is called a **linear topology** if, with respect to τ , both addition and scalar multiplication are continuous. The pair (V, τ) is called a **topological vector space**.

Definition 2.3.2. Let (V, τ) be a topological vector space.

A subset $B \subseteq V$ is **bounded** if, for every neighbourhood U of 0 in V , there exists $\alpha \in \mathbb{F}$ such that $B \subseteq \alpha U$.

A subset $C \subseteq V$ is **circled** if, for every $\alpha \in \mathbb{F}$ such that $\|\alpha\| \leq 1$, we have

$$\alpha C \subseteq C.$$

A subset $R \subseteq V$ is **radial** if, for every $v \in V$, there exists $\lambda_v \in \mathbb{F}$ such that $v \in \lambda R$ for every λ such that $\|\lambda\| \leq \|\lambda_v\|$.

A subset $A \subseteq V$ is **convex** if, for every $\alpha \in [0, 1]$ and every $x, y \in A$, we have

$$\alpha x + (1 - \alpha)y \in A$$

A subset $A \subseteq V$ is **absolutely convex** if it is convex and circled.

A family \mathcal{U} of open neighbourhoods of 0 in V is a **local base** for V if, for every neighbourhood W of 0 in V , there exists $U \in \mathcal{U}$ such that $U \subseteq W$.

A family \mathcal{S} of open neighbourhoods of 0 in V is a **local subbase** for V if, for every neighbourhood W of 0 in V , there exists $n \in \mathbb{N}$ and $S_1, S_2, \dots, S_n \in \mathcal{S}$ such that $\bigcap_{i=1}^n S_i \subseteq W$.

Definition 2.3.3. A topological vector space V satisfies the **Heine–Borel property** if every closed and bounded subset of V is compact.

2.3.2. Locally convex topological vector spaces.

Definition 2.3.4. A topological vector space (V, τ) is **locally convex** if there exists a local base \mathcal{U} for V consisting of convex sets.

Definition 2.3.5. Let V be a vector space. A **seminorm** on V is a map $p : V \rightarrow \mathbb{R}$ such that

1. $p(v) \geq 0, \quad \forall v \in V,$
2. $p(\alpha v) = \|\alpha\|p(v), \quad \forall \alpha \in \mathbb{F}, \forall v \in V,$
3. $p(v + w) \leq p(v) + p(w), \quad \forall v, w \in V.$

For every seminorm p on V and every $\epsilon > 0$, one can define a subset $U_{p,\epsilon}$ as

$$U_{p,\epsilon} = \{v \in V \mid p(v) < \epsilon\}.$$

Then the **topology generated by** p is the linear topology on V for which the family $\{U_{p,\epsilon}\}_{\epsilon \in \mathbb{R}_{>0}}$ is a local subbase.

One can also associate to every convex, circled, and radial subset of V a seminorm.

Definition 2.3.6. For every convex, circled, and radial set $U \subseteq V$, the **Minkowski functional** of U is the map $p_U : V \rightarrow \mathbb{R}$ defined as

$$p_U(v) = \inf\{\alpha \in \mathbb{R}_{\geq 0} \mid \alpha^{-1}v \in U\}, \quad \forall v \in V.$$

Since U is radial, we have

$$p_U(v) < \infty, \quad \forall v \in V.$$

Since U is circled, we have

$$p_U^{-1}([0, 1]) = U.$$

Finally, since U is convex, the Minkowski functional p_U is a seminorm [68, Theorem 1.35].

One can show that locally convex topological vector spaces can be characterized using a family of seminorms.

Theorem 2.3.7. *A topological vector space (V, τ) is locally convex, if and only if there exists a family of seminorms $\{p_i\}_{i \in \Lambda}$ on V which generates the topology τ .*

Similarly, one can characterize boundedness in a locally convex topological vector space using a family of generating seminorms [68, Theorem 1.37].

Theorem 2.3.8. *Let V be a locally convex space with a family of generating seminorms $\{p_i\}_{i \in \Lambda}$. Then $B \subseteq V$ is bounded if and only if, for every $i \in \Lambda$, there exists $M_i > 0$ such that*

$$p_i(v) < M_i, \quad \forall v \in B.$$

2.3.3. Uniformity on topological vector spaces. In a metric space there are many concepts which are not topological but depend on the specific metric we use on the space. One of these properties is being a Cauchy sequence. The following example shows that, for a sequence in a topological space, the property of being Cauchy is not topological.

Example 2.3.9 ([46]). Let $X = (0, \infty)$ and consider two metrics $d_1 : X \times X \rightarrow \mathbb{R}$ and $d_2 : X \times X \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} d_1(x, y) &= |x - y|, & \forall x, y \in X \\ d_2(x, y) &= \left| \frac{1}{x} - \frac{1}{y} \right|, & \forall x, y \in X. \end{aligned}$$

Since the map $f : (0, \infty) \rightarrow (0, \infty)$ defined by

$$f(x) = \frac{1}{x}$$

is a homeomorphism, d_1 and d_2 induce the same topology on X (which is exactly the subspace topology from \mathbb{R}). However, the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ on X is Cauchy in (X, d_1) and it is not Cauchy in (X, d_2) .

In order to study Cauchy sequences on topological spaces, topology does not suffice and one needs some extra structure. This structure can be obtained by generalizing the notion of metric.

Definition 2.3.10. Let X be a set and let $U, V \subseteq X \times X$. Then we define $U^{-1} \subseteq X \times X$ as

$$U^{-1} = \{(x, y) \mid (y, x) \in U\}.$$

Also we define $U \circ V \subseteq X \times X$ as

$$U \circ V = \{(x, y) \in X \times X \mid \exists z \in X \text{ s.t. } (x, z) \in U \text{ and } (z, y) \in V\}.$$

Definition 2.3.11. Let X be a set. A **uniformity** on X is a family \mathfrak{U} of subsets of $X \times X$ such that

1. every member of \mathfrak{U} contains $\{(x, x) \mid x \in X\}$,
2. if $U \in \mathfrak{U}$, then $U^{-1} \in \mathfrak{U}$,
3. if $U \in \mathfrak{U}$, then there exists $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$.
4. for every $U, V \in \mathfrak{U}$, we have $U \cap V \in \mathfrak{U}$,
5. if $U \in \mathfrak{U}$, then, for every V such that $U \subseteq V$, we have $V \in \mathfrak{U}$.

By a **uniform space**, we mean a pair (X, \mathfrak{U}) , where X is a topological space and \mathfrak{U} is a uniformity on X .

Definition 2.3.12. Let \mathfrak{U} be a uniformity on a topological space X . Then a subfamily \mathfrak{N} of \mathfrak{U} is a **base** for \mathfrak{U} if, for every $U \in \mathfrak{U}$, there exists $N \in \mathfrak{N}$ such that $N \subseteq U$.

Using a uniformity on a topological space, one can define the notion of completeness. We first define nets on topological spaces.

Definition 2.3.13. Let Λ be a set. A binary relation \succeq **directs** Λ if

1. for every $i, j, k \in \Lambda$, $i \succeq j$ and $j \succeq k$ implies $i \succeq k$,
2. for every $i \in \Lambda$, we have $i \succeq i$,
3. for every $i, j \in \Lambda$, there exists $m \in \Lambda$ such that $m \succeq i$ and $m \succeq j$.

A **directed set** is a pair (Λ, \succeq) such that \succeq directs Λ .

Definition 2.3.14. Let (Λ, \succeq) be a directed set. A **net** from (Λ, \succeq) to a space Y is a function $s : \Lambda \rightarrow Y$. We usually denote a net s by $\{s_\alpha\}_{\alpha \in \Lambda}$.

Definition 2.3.15. Let (Λ, \succeq) be a directed set and (X, \mathfrak{U}) be a uniform space. Let $\{s_\alpha\}_{\alpha \in \Lambda}$ be a net in the uniform space (X, \mathfrak{U}) . Then $\{s_\alpha\}_{\alpha \in \Lambda}$ is a **Cauchy net** in (X, \mathfrak{U}) if, for every $U \in \mathfrak{U}$, there exists $\beta \in \Lambda$ such that, for every $m, n \in \Lambda$ with the property that $m \succeq \beta$ and $n \succeq \beta$, we have

$$(s_m, s_n) \in U.$$

Definition 2.3.16. A uniform space (X, \mathfrak{U}) is **complete** if every Cauchy net in (X, \mathfrak{U}) converges to a point in X .

Since we are studying topological vector spaces, we would like to see if it is possible to define uniformity on a topological vector space using its topological structure. We first define translation-invariant uniformities on vector spaces.

Definition 2.3.17. Let V be a vector space. A uniformity \mathfrak{U} on V is **translation-invariant** if there exists a base \mathfrak{N} for \mathfrak{U} such that, for every $N \in \mathfrak{N}$ and every $x, y, z \in V$, $(x, y) \in N$ if and only if $(x + z, y + z) \in N$.

The following theorem is of considerable importance in the theory of topological vector spaces. It allows us to study the notions of completeness and Cauchy convergence in topological vector spaces without ambiguity [69, Chapter 1, §1.4].

Theorem 2.3.18. *Let V be a topological vector space. The topology of V induces a unique translation-invariant uniformity \mathfrak{U} . In particular, if \mathfrak{B} is a local neighbourhood base for topology on V , then \mathfrak{N} defined as*

$$N = \{(x, y) \in V \times V \mid x - y \in V\}$$

is a base for the uniformity \mathfrak{U} .

Therefore, given a topological vector space V , one can define Cauchy nets and Cauchy sequences on V using this unique translation-invariant uniformity. In general a topological vector space equipped with this translation-invariant uniformity may not be complete. However, one can show that V is contained as a dense subset in a complete topological vector space \widehat{V} [69, Chapter 1, §1.5].

Theorem 2.3.19. *Let V be a Hausdorff, locally convex vector space. Then there exists a unique complete, locally convex vector space \widehat{V} which contains V as a dense subset.*

We usually call the unique locally convex vector space \widehat{V} associated with V the **completion of V** .

2.3.4. Dual space and dual topologies.

Definition 2.3.20. Let V be a topological vector space. Then the **algebraic dual** of V , which is denoted by V^* , is the set

$$V^* = \{f : V \rightarrow \mathbb{F} \mid f \text{ is linear}\}$$

It is easy to see that V^* is a vector space.

The **topological dual** of V , which is denoted by V' , is the set

$$V' = \{f : V \rightarrow \mathbb{F} \mid f \text{ is continuous and linear}\}$$

The set V' is clearly a vector space.

Definition 2.3.21. Let (V, τ) be a locally convex space and let V' be the topological dual of V . Then the coarsest topology on V , which makes all members of V' continuous is called the **weak topology** V .

It can be shown that the weak topology on V is a locally convex topology. In particular, one can define a family of generating seminorms for the weak topology on V .

Theorem 2.3.22. *Let V be a locally convex space and V' be the topological dual of V . Then, for every $f \in V'$, we define the seminorm $p_f : V \rightarrow \mathbb{F}$ as*

$$p_f(v) = |f(v)|, \quad \forall v \in V.$$

The family of seminorms $\{p_f\}_{f \in V'}$ generates the weak topology on V .

To distinguish between topologies on a locally convex space V , we denote the weak topology on V by $\sigma(V, V')$. Since $(V, (\sigma(V, V')))$ is a locally convex space, one can define appropriate notions of compactness, boundedness and, convergence on it.

Definition 2.3.23. Let V be a locally convex space.

1. A subset $K \subseteq V$ is **weakly compact** if it is compact in $(V, \sigma(V, V'))$,
2. a subset $S \subseteq V$ is **relatively weakly compact** if the $\sigma(V, V')$ -closure of S is compact in $(V, (\sigma(V, V')))$,
3. a sequence $\{v_n\}_{n \in \mathbb{N}}$ in V is **weakly convergent** if there exists $v \in V$ such that $\{v_n\}_{n \in \mathbb{N}}$ converges to v in $(V, \sigma(V, V'))$,
4. a sequence $\{v_n\}_{n \in \mathbb{N}}$ in V is **weakly Cauchy** if $\{v_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(V, \sigma(V, V'))$.

For a metrizable topological space X it is well-known that a set $K \subseteq X$ is compact if and only if every sequence in K has a convergent subsequence. Let V be a Banach space. Then it is clear that V is a metrizable space and therefore one can characterize the compact subsets of V using the above fact. However, the weak topology on V is not metrizable. Thus it would be interesting to see if the same characterization holds for weakly compact subsets of V . Eberlein–Smulian Theorem answers this question affirmatively [69, Chapter IV, Corollary 2]. Eberlein–Smulian Theorem is usually considered as one of the deepest theorems in studying the weak topology on Banach spaces.

Theorem 2.3.24. *Let V be a Banach space and $A \subseteq V$. Then the following statements are equivalent:*

- (i) *The weak closure of A is weakly compact,*
- (ii) *each sequence of elements of A has a subsequence that is weakly convergent.*

One can get a partial generalization of the Eberlein–Smulian Theorem for complete locally convex spaces [69, Chapter IV, Theorem 11.2].

Theorem 2.3.25. *Let V be a complete locally convex space and $A \subseteq V$. If every sequence of elements of A has a subsequence that is weakly convergent, then the weak closure of A is weakly compact.*

When V is a locally convex space, one can define many different linear topology on V' which makes it into a locally convex space.

Definition 2.3.26. Let V be a locally convex vector space and \mathfrak{S} be a family of subsets of V directed under the inclusion such that

$$\bigcup_{S \in \mathfrak{S}} S = V.$$

For every $S \in \mathfrak{S}$ and every open neighbourhood $U \subseteq \mathbb{F}$ containing 0, we define $B(S, U)$ by

$$B(S, U) = \{\alpha \in V' \mid \alpha(S) \subseteq U\}.$$

The family $\{B(S, U)\}$ form the neighbourhood base for a topology on V' called **\mathfrak{S} -topology**.

In general, the vector space V' endowed with \mathfrak{S} -topology may not be a Hausdorff locally convex topological vector space. However, it can be shown that by imposing some restrictions on \mathfrak{S} , one can ensure that \mathfrak{S} -topology on V' is Hausdorff [69, Chapter III, Theorem 3.2].

Theorem 2.3.27. *Let \mathfrak{S} be a family of subsets of V such that $\bigcup_{S \in \mathfrak{S}} S = V$ and, for every $S \in \mathfrak{S}$ and every $\alpha \in V'$, $\alpha(S)$ is bounded in \mathbb{F} . Then the \mathfrak{S} -topology on V' is a Hausdorff locally convex topology.*

In particular, when \mathfrak{S} is family of finite sets of V , it is clear that, for every $\alpha \in V'$, $\alpha(S)$ is bounded in \mathbb{R} .

Definition 2.3.28. 1. If \mathfrak{S} is the family of finite subsets of V , then the \mathfrak{S} -topology on V' is called the **weak-* topology** on V' . We denote the weak-* topology on V' by $\sigma(V', V)$.

2. If \mathfrak{S} is the family of bounded subsets of V , then the \mathfrak{S} -topology on V' is called the **strong topology** on V' . We denote the strong topology on V' by $\beta(V', V)$.

One can find a family of generating seminorms for the \mathfrak{S} -topology on V' [69, Chapter III, §3].

Theorem 2.3.29. *Let $\{p_i\}_{i \in I}$ be a family of generating seminorms for the locally convex topology on V . Then, for every set $S \in \mathfrak{S}$, we define the seminorm $p_{S,i} : V' \rightarrow \mathbb{R}$ by*

$$p_{S,i}(\alpha) = \sup_{x \in S} \|\alpha(x)\|.$$

The family $\{p_{S,i}\}_{\{S \in \mathfrak{S}, i \in I\}}$ is a generating family of seminorms on for the \mathfrak{S} -topology on V' .

Corollary 2.3.30. *Let V be a locally convex topological vector space. For every $v \in V$, we define the seminorm $p_v : V' \rightarrow \mathbb{F}$ as*

$$p_v(f) = |f(v)|, \quad \forall f \in V'$$

The family of seminorm $\{p_v\}_{v \in V}$ generates the weak- topology on V' .*

It is easy to show that, for a locally convex space V , the strong topology on V' is finer than the weak-* topology on V' .

Definition 2.3.31. Let V and W be two topological vector spaces.

1. A linear map $L : V \rightarrow W$ is **continuous** if, for every open set $U \subseteq W$, the set $L^{-1}(U)$ is open in V . The set of all linear continuous maps $L : V \rightarrow W$ is denoted by $L(V; W)$.
2. A linear map $L : V \rightarrow W$ is **bounded** if, for every bounded set $B \subseteq V$, the set $L(B)$ is bounded in W .
3. A linear map $L : V \rightarrow W$ is **compact** if there exists a neighbourhood U of 0 in V such that $L(U)$ is relatively compact in W .
4. Let V and W be locally convex topological vector spaces. A linear map $L : V \rightarrow W$ is **weakly compact** if there exists a neighbourhood U of 0 in V such that $L(U)$ is relatively compact in $(W, \sigma(W, W'))$ (or equivalently $L(U)$ is relatively compact in the weak topology on W).

5. A subset $S \subseteq L(V; W)$ is **pointwise bounded** if, for every $v \in V$, the set $S(v)$ is bounded in W .
6. Let $S \subseteq L(V; W)$ be a subset and $v_0 \in V$. Then S is **equicontinuous at v_0** if, for every open neighbourhood U of 0 in W , there exists an open neighbourhood O of 0 in V such that

$$\alpha(v - v_0) \in U, \quad \forall \alpha \in S, \forall v - v_0 \in O.$$

7. A subset $S \subseteq L(V; W)$ is **equicontinuous** if it is equicontinuous at v for every $v \in V$.

2.3.5. Curves on locally convex topological vector spaces. In this section, we denote the Lebesgue measure on \mathbb{R} by \mathbf{m} .

Definition 2.3.32. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval and let $\mathcal{M}(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \rightarrow \mathbb{F}$. A function $f : \mathbb{T} \rightarrow \mathbb{F}$ is **integrable** on \mathbb{T} if

$$\int_{\mathbb{T}} \|f(\tau)\| d\mathbf{m} < \infty.$$

We denote the set of all integrable members of $\mathcal{M}(\mathbb{T})$ by $L^1(\mathbb{T})$. A function $f : \mathbb{T} \rightarrow \mathbb{F}$ is **locally integrable** if, for every compact set $K \subseteq \mathbb{T}$, we have

$$\int_K \|f(\tau)\| d\mathbf{m} < \infty.$$

The set of all locally integrable functions on \mathbb{T} is denoted by $L^1_{\text{loc}}(\mathbb{T})$.

A function $f : \mathbb{T} \rightarrow \mathbb{F}$ in $\mathcal{M}(\mathbb{T})$ is **essentially bounded** if there exists a compact set $K \subseteq \mathbb{F}$ such that

$$\mathbf{m}\{x \mid f(x) \notin K\} = 0.$$

We denote the set of all essentially bounded members of $\mathcal{M}(\mathbb{T})$ by $L^\infty(\mathbb{T})$.

A function $f : \mathbb{T} \rightarrow \mathbb{F}$ in $\mathcal{M}(\mathbb{T})$ is **locally essentially bounded** if, for every compact set $K \subseteq \mathbb{T}$, there exists a bounded set $B_K \subseteq \mathbb{F}$ such that

$$\mathbf{m}\{x \in K \mid f(x) \notin B_K\} = 0.$$

We denote the set of all locally essentially bounded members of $\mathcal{M}(\mathbb{T})$ by $L^\infty_{\text{loc}}(\mathbb{T})$.

Definition 2.3.33. Let V be a locally convex space with a family of generating seminorms $\{p_i\}_{i \in \Lambda}$ and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A curve $f : \mathbb{T} \rightarrow V$ is **integrally bounded** if, for every $i \in \Lambda$, we have

$$\int_{\mathbb{T}} p_i(f(\tau)) d\mathbf{m} < \infty.$$

A function $s : \mathbb{T} \rightarrow V$ is a **simple function** if there exist $n \in \mathbb{N}$, measurable sets $A_1, A_2, \dots, A_n \subseteq \mathbb{T}$, and $v_1, v_2, \dots, v_n \in V$ such that $\mathbf{m}(A_i) < \infty$ for every $i \in \{1, 2, \dots, n\}$ and

$$s = \sum_{i=1}^n \chi_{A_i} v_i.$$

The set of all simple functions from the interval \mathbb{T} to the vector space V is denoted by $S(\mathbb{T}; V)$.

One can define **Bochner integral** of a simple function $s = \sum_{i=1}^n \chi_{A_i} v_i$ as

$$\int_{\mathbb{T}} s(\tau) d\mathbf{m} = \sum_{i=1}^n \mathbf{m}(A_i) v_i.$$

It is easy to show that the above expression does not depend on choice of $A_1, A_2, \dots, A_n \subseteq \mathbb{T}$.

A curve $f : \mathbb{T} \rightarrow V$ is **Bochner approximable** if there exists a net $\{f_\alpha\}_{\alpha \in \Lambda}$ of simple functions on V such that, for every seminorm p_i , we have

$$\lim_{\alpha} \int_{\mathbb{T}} p_i(f_\alpha(\tau) - f(\tau)) d\mathbf{m} = 0.$$

The net of simple functions $\{f_\alpha\}_{\alpha \in \Lambda}$ is an **approximating net** for the mapping f .

Theorem 2.3.34 ([6]). *Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be an approximating net for the mapping $f : \mathbb{T} \rightarrow V$. Then $\{\int_{\mathbb{T}} f_\alpha(\tau) d\mathbf{m}\}_{\alpha \in \Lambda}$ is a Cauchy net.*

Let $f : \mathbb{T} \rightarrow V$ be a mapping and let $\{f_\alpha\}_{\alpha \in \Lambda}$ be an approximating net of simple functions for f . If the net $\{\int_{\mathbb{T}} f_\alpha(\tau) d\mathbf{m}\}_{\alpha \in \Lambda}$ converges, then we say that f is **Bochner integrable**. One can show that the limit of $\{\int_{\mathbb{T}} f_\alpha(\tau) d\mathbf{m}\}_{\alpha \in \Lambda}$ doesn't depend on the choice of approximating net and is called **Bochner integral of f** . The set of all Bochner integrable curves from \mathbb{T} to V is denoted by $L^1(\mathbb{T}; V)$.

A curve $f : \mathbb{T} \rightarrow V$ is **locally Bochner integrable** if for every compact set $J \subseteq \mathbb{T}$, the map $f|_J$ is Bochner integrable. The set of all locally Bochner integrable curves from \mathbb{T} to V is denoted by $L^1_{\text{loc}}(\mathbb{T}; V)$.

Definition 2.3.35. Let V be a locally convex vector space, $\{p_i\}_{i \in \Lambda}$ a family of generating seminorms on V , and $\mathbb{T} \subseteq \mathbb{R}$ an interval. Then, for every $i \in \Lambda$, we define a seminorm $p_{i, \mathbb{T}}$ as

$$p_{i, \mathbb{T}}(f) = \int_{\mathbb{T}} p_i(f(\tau)) d\mathbf{m}, \quad \forall f \in L^1(\mathbb{T}; V).$$

The family of seminorms $\{p_{i, \mathbb{T}}\}$ generates a topology on $L^1(\mathbb{T}; V)$, which is called the **L^1 -topology**.

Since locally convex spaces generally do not satisfy Heine–Borel property, one can define two notion of bounded curves on locally convex spaces. The first notion which we call von Neumann bounded deals with the bounded sets in a locally convex spaces, while the second notion which is called bounded in compact bornology deals with compact sets in the locally convex space.

Definition 2.3.36. A curve $f : \mathbb{T} \rightarrow V$ is **essentially von Neumann bounded** if there exist a bounded set $B \in V$ such that

$$\mathbf{m}\{x \mid f(x) \notin B\} = 0.$$

The set of all essentially von Neumann bounded curves from \mathbb{T} to V is denoted by $L^\infty(\mathbb{T}; V)$.

A curve $f : \mathbb{T} \rightarrow V$ is **locally essentially von Neumann bounded** if, for every compact interval $\mathbb{I} \subseteq \mathbb{T}$, the curve $f|_{\mathbb{I}}$ is essentially von Neumann bounded. The set of all locally essentially von Neumann bounded curves from \mathbb{T} to V is denoted by $L_{\text{loc}}^\infty(\mathbb{T}; V)$.

A curve $f : \mathbb{T} \rightarrow V$ is **essentially bounded in compact bornology** if there exist a compact set $K \in V$ such that

$$\mathbf{m}\{x \mid f(x) \notin K\} = 0.$$

The set of all essentially bounded in compact bornology curves from \mathbb{T} to V is denoted by $L^{\text{cpt}}(\mathbb{T}; V)$.

A curve $f : \mathbb{T} \rightarrow V$ is **locally essentially bounded in compact bornology** if, for every compact interval $\mathbb{I} \subseteq \mathbb{T}$, the curve $f|_{\mathbb{I}}$ is essentially bounded in compact bornology. The set of all locally essentially bounded in compact bornology curves from \mathbb{T} to V is denoted by $L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; V)$.

One can easily see that for locally convex spaces that satisfy the Heine–Borel property, bounded curves in von Neumann bornology and bounded curves in compact bornology coincide.

Definition 2.3.37. Let V be a locally convex vector space, $\{p_i\}_{i \in \Lambda}$ a family of generating seminorms on V , and $\mathbb{T} \subseteq \mathbb{R}$ an interval. Then, for every $i \in \Lambda$, we define a seminorm $q_{i, \mathbb{T}}$ as

$$q_{i, \mathbb{T}}(f) = \inf\{M > 0 \mid p_i(f(t)) \leq M, \text{ a.e. on } \mathbb{T}\}.$$

The family of seminorms $\{q_{i, \mathbb{T}}\}$ generates a topology on $L^\infty(\mathbb{T}; V)$ which is called the **L^∞ -topology**.

Definition 2.3.38. A curve $f : \mathbb{T} \rightarrow V$ is **absolutely continuous** if there exists a Bochner integrable curve $g : \mathbb{T} \rightarrow V$ such that, for every $t_0 \in \mathbb{T}$, we have

$$f(t) = f(t_0) + \int_{t_0}^t g(\tau) d\mathbf{m}, \quad \forall t \in \mathbb{T}.$$

The set of all absolutely continuous curves on V on the interval \mathbb{T} is denoted by $\text{AC}(\mathbb{T}; V)$.

Definition 2.3.39. A curve $f : \mathbb{T} \rightarrow V$ is **locally absolutely continuous** if, for every compact interval $\mathbb{T}' \subseteq \mathbb{T}$, the curve $f|_{\mathbb{T}'} : \mathbb{T}' \rightarrow V$ is absolutely continuous. The set of all locally absolutely continuous curves on V on the interval \mathbb{T} is denoted by $\text{AC}_{\text{loc}}(\mathbb{T}; V)$.

Theorem 2.3.40. *Let V be a complete, separable locally convex space, $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and $f : \mathbb{T} \rightarrow V$ be a curve on V . Then f is locally integrally bounded if and only if it is locally Bochner integrable.*

Proof. Our proof here is restatement of the proof of Theorem 3.2 and Theorem 3.3 in [6]. Without loss of generality, assume that \mathbb{T} is compact. Let $\{p_i\}_{i \in \Lambda}$ be a family of generating seminorms for V . If f is locally Bochner integrable then, by definition, if $i \in \Lambda$ we have

$$\int_{\mathbb{T}} p_i(f(\tau)) d\mathbf{m} < \infty.$$

This means that f is integrally bounded.

To prove the converse, since V is separable there exists a sequence $\{v_j\}_{j \in \mathbb{N}}$ such that $\{v_1, v_2, \dots\}$ is dense in V . For every $i, j, n \in \mathbb{N}$, we set

$$A_i^{n,j} = \left\{ t \in \mathbb{T} \mid p_i(f(t)) > \frac{1}{n}, p_i(f(t) - v_j) < \frac{1}{n} \right\}.$$

We define the simple functions $\{s_i^n\}_{i,n \in \mathbb{N}}$ as

$$s_i^n = \sum_{j=1}^n \chi_{A_i^{n,j}} v_j.$$

Note that, by construction, we have

$$p_i(f(t) - s_i^n(t)) \leq \frac{1}{n}, \quad \forall t \in \mathbb{T}$$

So by Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} p_i(f(\tau) - s_i^n(\tau)) d\mathbf{m} = 0.$$

This means that there exists $N \in \mathbb{N}$ such that

$$\int_{\mathbb{T}} p_i(f(\tau) - s_i^N(\tau)) d\mathbf{m} < 1, \quad \forall i \in \mathbb{N}.$$

This implies that f is Bochner-approximable. Since V is complete, every Bochner approximable function is Bochner integrable. So f is Bochner integrable. \square

Theorem 2.3.41. *Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For every $g \in L^\infty(\mathbb{T})$, we define the bounded functional $L_g : L^1(\mathbb{T}) \rightarrow \mathbb{R}$ as*

$$L_g(f) = \int_{\mathbb{T}} f(\tau) g(\tau) d\mathbf{m}, \quad \forall f \in L^1(\mathbb{T}).$$

Then for every bounded linear functional L on $L^1(\mathbb{T})$, there exists $g \in L^\infty(\mathbb{T})$ such that $L = L_g$.

Proof. The proof is given in [67, Chapter 8]. \square

2.3.6. Inductive limit and projective limit of topological vector spaces.

Definition 2.3.42. Let V be a vector space, $\{V_i\}_{i \in \Lambda}$ be a family of topological vector spaces, and $\{L_i\}_{i \in \Lambda}$ be a family of continuous linear maps such that $L_i : V \rightarrow V_i$. Then we define the **projective topology** on V with respect to $\{V_i, L_i\}_{i \in \Lambda}$ as the coarsest topology on V such that all the maps L_i are continuous.

One can easily show that the projective topology turns V into a topological vector space. Moreover if, for every $i \in \Lambda$, the topological vector space V_i is locally convex, then the projective topology on V is also locally convex.

Theorem 2.3.43 ([69]). *Let $\{V_i\}_{i \in \Lambda}$ be a family of locally convex topological vector spaces and $\{L_i\}_{i \in \Lambda}$ is a family of continuous linear maps such that $L_i : V \rightarrow V_i$. Then the projective topology on V with respect to $\{V_i, L_i\}_{i \in \Lambda}$, makes V into a locally convex topological vector space.*

Definition 2.3.44. Let V be a vector space, $\{V_i\}_{i \in \Lambda}$ be a family of topological vector spaces, and $\{L_i\}_{i \in \Lambda}$ be a family of continuous linear maps such that $L_i : V_i \rightarrow V$. Then the **inductive linear topology** on V with respect to $\{V_i, L_i\}_{i \in \Lambda}$ is the finest linear topology on V which makes all the maps L_i continuous.

One may note that, contrary to the case of the projective topology, the inductive topology may be different in different categories.

Definition 2.3.45. Let V be a vector space, $\{V_i\}_{i \in \Lambda}$ be a family of locally convex topological vector spaces, and $\{L_i\}_{i \in \Lambda}$ be a family of continuous linear maps such that $L_i : V_i \rightarrow V$. Then the **inductive locally convex topology** on V with respect to $\{V_i, L_i\}_{i \in \Lambda}$ is the finest locally convex topology on V which makes all the maps L_i continuous.

Here we present definitions of the inductive limit and the projective limit of a directed family of objects in a general category. However, in this thesis we only study limit in the categories of sets, vector spaces, topological vector spaces, and locally convex spaces.

Definition 2.3.46. Let (Λ, \succeq) be a directed set and $\{X_\alpha\}_{\alpha \in \Lambda}$ be a family of objects and let $\{f_{\alpha\beta}\}_{\beta \succeq \alpha}$ be a family of morphisms such that $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$. Then the **inductive limit** of $\{X_\alpha, f_{\alpha\beta}\}_{\beta \succeq \alpha}$ is a pair $(X, \{g_\alpha\}_{\alpha \in \Lambda})$, where

1. X is an object and $\{g_\alpha\}_{\alpha \in \Lambda}$ is a family of morphisms such that $g_\alpha : X_\alpha \rightarrow X$ and, for every $\alpha, \beta \in \Lambda$ with $\beta \succeq \alpha$, the following diagram commutes:

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \\
 & \searrow g_\alpha & \downarrow g_\beta \\
 & & X
 \end{array}$$

2. $(X, \{g_\alpha\}_{\alpha \in \Lambda})$ is universal with respect to $\{X_\alpha, f_{\alpha\beta}\}_{\beta \succeq \alpha}$ in the sense that, for every other pair $(Y, \{h_\alpha\}_{\alpha \in \Lambda})$ such that, for every $\beta \succeq \alpha$, the following diagram commutes:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \\ & \searrow h_\alpha & \downarrow h_\beta \\ & & Y \end{array}$$

there exists a unique morphism $i : X \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccccc} X_\alpha & & \xrightarrow{f_{\alpha\beta}} & & X_\beta \\ & \searrow g_\alpha & & g_\beta & \swarrow \\ & & X & & \\ & \searrow h_\alpha & \downarrow i & & \swarrow h_\beta \\ & & Y & & \end{array}$$

We usually denote the inductive limit of the inductive system $\{X_\alpha, f_{\alpha\beta}\}_{\beta \succeq \alpha}$ by $\varinjlim X_\alpha$.

Definition 2.3.47. Let (Λ, \succeq) be a directed set, $\{X_\alpha\}_{\alpha \in \Lambda}$ be a family of objects, and $\{f_{\alpha\beta}\}_{\beta \succeq \alpha}$ be a family of morphisms such that $f_{\alpha\beta} : X_\beta \rightarrow X_\alpha$. Then the **projective limit** of $\{X_\alpha, f_{\alpha\beta}\}_{\alpha, \beta \in \Lambda}$ is a pair $(X, \{g_\alpha\}_{\alpha \in \Lambda})$, where

1. X is an object and $\{g_\alpha\}_{\alpha \in \Lambda}$ is a family of morphisms such that $g_\alpha : X \rightarrow X_\alpha$ and, for every $\beta \succeq \alpha$, the following diagram commutes:

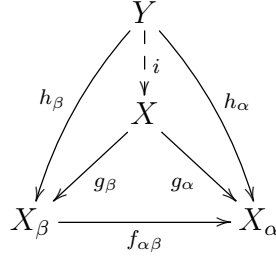
$$\begin{array}{ccc} X & & \\ g_\beta \downarrow & \searrow g_\alpha & \\ X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \end{array}$$

2. $(X, \{g_\alpha\}_{\alpha \in \Lambda})$ is universal with respect to $\{X_\alpha, f_{\alpha\beta}\}_{\beta \succeq \alpha}$ in the sense that, for every other pair $(Y, \{h_\alpha\}_{\alpha \in \Lambda})$ such that, for every $\beta \succeq \alpha$, the following diagram commutes:

$$\begin{array}{ccc} Y & & \\ h_\beta \downarrow & \searrow h_\alpha & \\ X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \end{array}$$

there exists a unique morphism $i : Y \rightarrow X$ such that the following diagram

commutes:



We denote the projective limit of the inductive system $\{X_\alpha, f_{\alpha\beta}\}_{\beta \geq \alpha}$ by $\varprojlim X_\alpha$.

Let $\{V_i\}_{i \in \mathbb{N}}$ be a countable family of locally convex topological vector spaces and $\{f_i\}_{i \in \mathbb{N}}$ be a family of linear continuous maps such that $f_i : V_i \rightarrow V_{i+1}$. Then, for every $j > i$, one can define $f_{ij} : V_i \rightarrow V_j$ as

$$f_{ij} = f_j \circ f_{j-1} \circ \dots \circ f_i.$$

So we get a directed system $(V_i, \{f_{ij}\}_{j>i})$ of locally convex spaces and continuous linear maps.

In the case of countable inductive limits, we sometimes refer to $\{V_i, f_i\}_{i \in \mathbb{N}}$ as a directed system. By this we mean the directed system $(V_i, \{f_{ij}\}_{j>i})$ which is generated from $\{V_i, f_i\}_{i \in \mathbb{N}}$ as above.

Assume that, according to Definition 2.3.46, the pair $(V, \{g_i\}_{i \in \mathbb{N}})$ is the locally convex inductive limit of a directed system $\{V_i, f_i\}_{i \in \mathbb{N}}$. So V is a locally convex space and $g_i : V_i \rightarrow V$ are continuous linear maps. This completely characterizes open sets in V in terms of open sets in V_i . In particular, one has the following theorem [69, Chapter 2, §6].

Theorem 2.3.48. *Let $\{V_i, f_i\}_{i \in \mathbb{N}}$ be a directed system of locally convex spaces and the pair $(V, \{g_i\}_{i \in \mathbb{N}})$ be the locally convex inductive limit of $\{V_i, f_i\}_{i \in \mathbb{N}}$. Then a local base for V consists of all radial, convex, and circled subset U of V such for every $i \in \mathbb{N}$, we have $g_i^{-1}(U)$ is an open set in V_i .*

In many problems in functional analysis, one would like to study bounded sets of $V = \varinjlim V_i$ in terms of bounded sets of the locally convex spaces V_i . The following definition classifies inductive limits of locally convex spaces based on their bounded sets.

Definition 2.3.49. Let $\{V_i, f_i\}_{i \in \mathbb{N}}$ be a directed system of locally convex spaces and the pair $(V, \{g_i\}_{i \in \mathbb{N}})$ be the locally convex inductive limit of $\{V_i, f_i\}_{i \in \mathbb{N}}$. The inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is **regular** if, for every bounded set $B \subset V$, there exists $m \in \mathbb{N}$ and a bounded set $B_m \subset V_m$ such that the restriction map $g_m|_{B_m} : B_m \rightarrow V$ is a bijection onto B .

The inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is **boundedly retractive** if, for every bounded set $B \subset V$, there exists $m \in \mathbb{N}$ and a bounded set $B_m \subset V_m$ such that the restriction map $g_m|_{B_m} : B_m \rightarrow V$ is a homeomorphism onto B .

Definition 2.3.50. Let $\{V_i\}_{i \in \mathbb{N}}$ be a family of locally convex topological vector spaces and let $\{f_i\}_{i \in \mathbb{N}}$ be a family of continuous linear maps such that $f_i : V_i \rightarrow V_{i+1}$.

1. The inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is **compact** if, for every $i \in \mathbb{N}$, the map $f_i : V_i \rightarrow V_{i+1}$ is compact.
2. The inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is **weakly compact** if, for every $i \in \mathbb{N}$, the map $f_i : V_i \rightarrow V_{i+1}$ is weakly compact.

Theorem 2.3.51. Let $\{V_i\}_{i \in \mathbb{N}}$ be a family of locally convex topological vector spaces and let $\{f_i\}_{i \in \mathbb{N}}$ be a family of linear continuous maps such that $f_i : V_i \rightarrow V_{i+1}$. Then

1. if the inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is weakly compact, then it is regular, and
2. if the inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is compact, then it is boundedly retractive.

Proof. The first part of this theorem has been proved in [48, Theorem 6] and the second part in [48, Theorem 6']. \square

However, one can find boundedly retractive inductive families which are not compact [7]. In [66], Retakh studied an important condition on inductive families of locally convex spaces called condition (M).

Definition 2.3.52. Let $\{V_i\}_{i \in \mathbb{N}}$ be a family of locally convex topological vector spaces and let $\{f_i\}_{i \in \mathbb{N}}$ be a family of linear continuous maps such that $f_i : V_i \rightarrow V_{i+1}$. The inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ satisfies **condition (M)** if there exists a sequence of absolutely convex neighbourhoods $\{U_i\}_{i \in \mathbb{N}}$ of 0 such that, for every $i \in \mathbb{N}$, we have $U_i \subseteq V_i$ and,

1. for every $i \in \mathbb{N}$, we have $U_i \subseteq f_i^{-1}(U_{i+1})$, and
2. for every $i \in \mathbb{N}$, there exists $M_i > 0$ such that, for every $j > M_i$, the topologies induced from V_j on U_i are all the same.

Theorem 2.3.53. Let $\{V_i\}_{i \in \mathbb{N}}$ be a family of normed vector spaces and let $\{f_i\}_{i \in \mathbb{N}}$ be a family of continuous linear maps such that $f_i : V_i \rightarrow V_{i+1}$. Suppose that the inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is regular. Then inductive system $\{V_i, f_i\}_{i \in \mathbb{N}}$ is boundedly retractive if and only if it satisfies condition (M).

Proof. This theorem is proved in [7, Proposition 9(d)]. \square

2.3.7. Tensor product of topological vector spaces.

Definition 2.3.54. If $M \subseteq V$ and $N \subseteq W$ be subsets, then we define $M \otimes N \subseteq V \otimes W$ as

$$M \otimes N = \left\{ \sum_{i=1}^k \lambda_i (m_i \otimes n_i) \mid \lambda_i \in \mathbb{F}, m_i \in M, n_i \in N, k \in \mathbb{N} \right\}.$$

If V and W are locally convex topological vector spaces, in general there is no canonical way of defining a topology on $V \otimes W$. In this section we recall two common topologies on the tensor product $V \otimes W$.

Definition 2.3.55. The finest locally convex topology on $V \otimes W$ which makes the canonical map $\mu : V \times W \rightarrow V \otimes W$ continuous is called the **projective tensor product topology**. The vector space $V \otimes W$ equipped with this topology is denoted by $V \otimes_\pi W$.

Definition 2.3.56. The finest locally convex topology on $V \otimes W$ which makes the map $\mu : V \times W \rightarrow V \otimes W$ separately continuous is called the **injective tensor product topology**. The vector space $V \otimes W$ equipped with this topology is denoted by $V \otimes_\epsilon W$.

Even if the locally convex spaces V and W are complete, it is always the case that $V \otimes_\pi W$ and $V \otimes_\epsilon W$ are not complete. We denote the completions of $V \otimes_\pi W$ and $V \otimes_\epsilon W$ by $V \widehat{\otimes}_\pi W$ and $V \widehat{\otimes}_\epsilon W$ respectively.

Theorem 2.3.57 ([43]). *Let $E \subset V$ be a dense subset of V and $F \subset W$ be a dense subset of W . Then $E \otimes_\pi F$ is a dense subset of $V \widehat{\otimes}_\pi W$.*

Definition 2.3.58. Let V and W be locally convex spaces and p and q be seminorms on V and W respectively. Then we define the map $p \otimes q : V \otimes W \rightarrow \mathbb{F}$ as

$$p \otimes q(u) = \inf \left\{ \sum_{i=1}^n p(x_i)q(y_i) \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

It is easy to check that $p \otimes q$ is a seminorm on $V \otimes W$.

Theorem 2.3.59 ([69]). *Let V and W be locally convex spaces and let $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ be families of generating seminorms for V and W respectively. Then the family $\{p_i \otimes q_j\}_{(i,j) \in I \times J}$ is a generating family of seminorms for $V \otimes_\pi W$.*

Theorem 2.3.60 ([43]). *Let $\mathbb{T} \subseteq \mathbb{R}$ and E be a complete locally convex space. Then there exists a linear homeomorphism between $L^1(\mathbb{T}; E)$ and $L^1(\mathbb{T}) \widehat{\otimes}_\pi E$.*

Proof. We first show that the map $\iota : S(\mathbb{T}) \otimes_\pi E \rightarrow S(\mathbb{T}; E)$ defined as

$$\iota(\phi(t) \otimes v) = \phi(t)v, \quad \forall v \in E, \forall \phi \in S(\mathbb{T}),$$

is a linear homeomorphism. It is clear that ι is surjective. Let $\beta = \sum_{i=1}^n \beta_i(t) \otimes v_i \in S(\mathbb{T}) \otimes E$ be such that $\iota(\beta) = 0$. Without loss of generality, one can assume that the set $\{v_1, v_2, \dots, v_n\}$ is linearly independent. By assumption, we have

$$\sum_{i=1}^n \beta_i(t)v_i = 0, \quad \text{a.e. } t \in \mathbb{T}.$$

Now if we choose $\{v_1^*, v_2^*, \dots, v_n^*\}$ in E' such that

$$\langle v_i, v_j^* \rangle = \delta_{ij}, \quad \forall i, j \in \{1, 2, \dots, n\},$$

then we get

$$\beta_j(t) = \sum_{i=1}^n \beta_i(t) \langle v_i, v_j^* \rangle = 0, \quad \text{a.e. } t \in \mathbb{T}.$$

So we have $\beta = 0$. This implies that the map ι is a bijection.

Now we show that ι is continuous. Let q be the norm on $L^1(\mathbb{T})$ defined as

$$q(\phi) = \int_{\mathbb{T}} |\phi(\tau)| d\tau.$$

Let p be a seminorm on E and $s = \sum_{i=1}^n \chi_{A_i} \otimes v_i$. Then we have

$$\int_{\mathbb{T}} p \circ \iota(s(\tau)) d\tau = \int_{\mathbb{T}} p\left(\sum_{i=1}^n \chi_{A_i}(\tau) v_i\right) d\tau \leq \int_{\mathbb{T}} \sum_{i=1}^n \chi_{A_i}(\tau) p(v_i) d\tau = \sum_{i=1}^n \left(\int_{\mathbb{T}} \chi_{A_i}(\tau) d\tau \right) p(v_i).$$

Since the above relation holds for every representation of the simple function s , we get

$$\int_{\mathbb{T}} p \circ \iota(s(\tau)) d\tau \leq q \otimes p(s).$$

To show that ι is an open map, we can choose a representation $s = \sum_{i=1}^n \chi_{A_i} \otimes v_i$ such that $\{A_i\}_{i=1}^n$ are mutually disjoint. Then we have

$$q \otimes p(s) \leq \sum_{i=1}^n \mathbf{m}(A_i) p(v_i) = \int_{\mathbb{T}} p \circ \iota(s(\tau)) d\tau.$$

Thus ι^{-1} is continuous and so ι is open. This implies that ι is a linear homeomorphism. Now we extend ι to the completion of $S(\mathbb{T}) \otimes_{\pi} E$. By Theorem 2.3.57, the completion of $S(\mathbb{T}) \otimes_{\pi} E$ is $L^1(\mathbb{T}) \widehat{\otimes}_{\pi} E$. Since E is complete, the completion of $S(\mathbb{T}; E)$ is $L^1(\mathbb{T}; E)$. So ι extends to a linear homeomorphism $\widehat{\iota} : S(\mathbb{T}) \widehat{\otimes}_{\pi} E \rightarrow L^1(\mathbb{T}; E)$. This completes the proof. \square

Theorem 2.3.61 ([43]). *Let $\mathbb{T} \subseteq \mathbb{R}$ be a compact interval and E be a complete locally convex space. Then there exists a linear homeomorphism between $C^0(\mathbb{T}; E)$ and $C^0(\mathbb{T}) \widehat{\otimes}_{\epsilon} E$.*

2.3.8. Nuclear spaces.

Definition 2.3.62. Let E and F be two Banach spaces. An operator $L : E \rightarrow F$ is **nuclear** if there exists sequences $\{f_n\}_{n \in \mathbb{N}}$ in E' and $\{g_n\}_{n \in \mathbb{N}}$ in F and a sequence of complex numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, and we can write

$$L(v) = \sum_{n=1}^{\infty} \lambda_n f_n(v) g_n, \quad \forall v \in E.$$

One can also generalize the notion of nuclear operators for operators between locally convex vector spaces.

Let E be a locally convex space and U be a convex, circled, and radial subset of E containing 0. Then, for every $n \in \mathbb{N}$, one can define U_n as

$$U_n = \left\{ \frac{1}{n}x \mid x \in U \right\}.$$

The family $\{U_n\}_{n \in \mathbb{N}}$ form a local base for a locally convex topology τ_U on E . The space $(E, \tau_U)/p_U^{-1}(0)$ is denoted by E_U . We denote the completion of E_U by \widehat{E}_U . It is easy to see that E_U is a normable space with norm p_U . One can define the quotient map $i : E \rightarrow \widehat{E}_U$ as

$$i(v) = [v], \quad \forall v \in E.$$

Similarly, if B is a convex, circled, and bounded subset of E , then $E_1 = \bigcup_{n=1}^{\infty} nB$ is a subspace of E . The gauge seminorm p_B can be seen to be a norm on E_1 . We denote the normed space (E_1, p_B) by E_B .

Definition 2.3.63. Let E and F be two locally convex vector spaces. A continuous map $L : E \rightarrow F$ is called **nuclear** if there exists an equicontinuous sequence $\{f_n\}_{n \in \mathbb{N}}$ in E' , a sequence $\{g_n\}_{n \in \mathbb{N}}$ contained in a convex, circled, and bounded subset $B \subseteq F$ such that F_B is complete, and a sequence of complex numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, and we can write

$$L(v) = \sum_{n=1}^{\infty} \lambda_n f_n(v) g_n, \quad \forall v \in E.$$

Theorem 2.3.64 ([69]). *Every nuclear mapping is compact.*

Definition 2.3.65. A locally convex vector space E is **nuclear** if, for every convex circled neighbourhood of 0, the inclusion $i : E \rightarrow \widehat{E}_U$ is nuclear.

One can see that nuclear spaces satisfies the Heine–Borel property. The following theorem has been proved in [69, Chapter III, §7].

Theorem 2.3.66. *Let E be a nuclear space. Then every closed and bounded subset of E is compact.*

One important property of nuclear spaces is that, for nuclear spaces, the topological tensor product is uniquely defined. This property can be used to characterize nuclear spaces [43, §21.2, Theorem 1].

Theorem 2.3.67. *Let E and F be locally convex spaces. Then F is nuclear if and only if $E \otimes_{\pi} F = E \otimes_{\epsilon} F$.*

One can also identify the strong dual of the tensor product of metrizable locally convex spaces with tensor product of their strong duals when one of the locally convex spaces are nuclear [69, Chapter IV, Theorem 9.9].

Theorem 2.3.68. *Let E and F be metrizable locally convex space and F be nuclear. Then the strong dual of $E \widehat{\otimes} F$ can be identified with $(E', \beta(E', E)) \widehat{\otimes} (F', \beta(F', F))$*

2.4. Sheaves and presheaves

In this section, we define the notions of presheaf and sheaf on a topological space X . In this thesis, we mostly use presheaves and sheaves of sets, presheaves and sheaves of rings, and presheaves and sheaves of modules. Thus it is more natural to start with definition of presheaves and sheaves of arbitrary “objects” in this section.

Definition 2.4.1. Let X be a topological space. A **presheaf** of objects \mathcal{F} on X assigns to every open set $U \subseteq X$ an object $\mathcal{F}(U)$ and to every pair of open sets (U, V) with $U \subseteq V$ a morphism

$$r_{U,V} : \mathcal{F}(V) \rightarrow \mathcal{F}(U),$$

which is called the **restriction map** such that

1. for every open set $U \subseteq X$, we have $r_{U,U} = \text{id}$;
2. for all open sets $U, V, W \subseteq X$ such that $U \subseteq V \subseteq W$, we have

$$r_{U,V} \circ r_{V,W} = r_{U,W}.$$

For an open set $U \subseteq X$, an element $s \in \mathcal{F}(U)$ is called a **local section** of \mathcal{F} over U . If $s \in \mathcal{F}(V)$ and $W \subseteq V$ be an open set, then we denote $r_{W,V}(s)$ by $s|_W$ and we call it “restriction of s to W ”.

One of the most well-known examples of presheaves is the presheaf of locally defined functions.

Example 2.4.2. Let M be a C^ν -manifold. We define C_M^ν which assigns to every open set $U \subseteq M$ the ring of C^ν -functions on U and to every pair of open sets $U, V \subseteq M$ with $U \subseteq V$ the ring homomorphism $r_{U,V} : C_M^\nu(V) \rightarrow C_M^\nu(U)$ defined as

$$r_{U,V}(f) = f|_U.$$

Then one can easily check that C_M^ν is a presheaf of rings on M .

Given a presheaf of rings \mathcal{R} over X , one can also define presheaf of \mathcal{R} -modules over X .

Definition 2.4.3. Let X be a topological space and \mathcal{R} be a presheaf of rings on X . Then \mathcal{F} is a **presheaf of \mathcal{R} -modules** over X if it assigns to every open set $U \subseteq X$ the $\mathcal{R}(U)$ -module $\mathcal{F}(U)$ and to every pair of open sets $U \subseteq V$ a homomorphism $r_{U,V}^{\mathcal{F}}$ such that

1. for every open set $U \subseteq X$, we have $r_{U,U}^{\mathcal{F}} = \text{id}$;
2. for all open sets $U, V, W \subseteq X$ such that $U \subseteq V \subseteq W$, we have

$$r_{U,V}^{\mathcal{F}} \circ r_{V,W}^{\mathcal{F}} = r_{U,W}^{\mathcal{F}};$$

3. for every $f, g \in \mathcal{F}(U)$ and every $\alpha \in \mathcal{R}(U)$, we have

$$r_{U,V}^{\mathcal{F}}(f + \alpha g) = r_{U,V}^{\mathcal{F}}(f) + r_{U,V}^{\mathcal{R}}(\alpha)r_{U,V}^{\mathcal{F}}(g),$$

where $r^{\mathcal{R}}$ is the restriction map for the presheaf \mathcal{R} .

The most important example of presheaf of modules is the presheaf of locally defined vector fields on a manifold.

Example 2.4.4. Let M be a C^ν -manifold. In Example 2.4.2, we defined C_M^ν as a presheaf of ring of locally defined C^ν -functions. In this example, we define the presheaf Γ_M^ν of C_M^ν -modules over M . Let us define Γ_M^ν which assigns to every open set $U \subseteq M$ the $C^\nu(U)$ -module of locally defined C^ν -vector fields on U and assigns to every pair of open sets (U, V) with $U \subseteq V \subseteq M$ a restriction map $r_{U,V} : \Gamma_M^\nu(V) \rightarrow \Gamma_M^\nu(U)$ defined as

$$r_{U,V}(X) = X|_U, \quad \forall X \in \Gamma_M^\nu(V).$$

Therefore, the set $\Gamma_M^\nu(U)$ is defined as

$$\Gamma_M^\nu(U) = \{X : U \rightarrow TM \mid X \text{ is of class } C^\nu \text{ and } X(x) \in T_x M, \quad \forall x \in U\},$$

One can easily check that Γ_M^ν is a presheaf of C_M^ν -modules over M .

Definition 2.4.5. Let \mathcal{F} be a presheaf of objects on X . Then a **subpresheaf** \mathcal{H} of \mathcal{F} is an assignment to every open set $U \subseteq X$, an object $\mathcal{H}(U) \subseteq \mathcal{F}(U)$ such that, for every pair of open sets $U \subseteq V \subseteq X$, the restriction map

$$r_{U,V}|_{\mathcal{H}(V)} : \mathcal{H}(V) \rightarrow \mathcal{F}(U),$$

takes its values in $\mathcal{H}(U)$.

It is easy to see that a subpresheaf is itself a presheaf.

Definition 2.4.6. Let $\{s_\alpha\}_{\alpha \in \Lambda}$ be a family of sections of the sheaf \mathcal{F} of \mathcal{R} -modules on X , i.e., for every $\alpha \in \Lambda$, we have $s_\alpha \in \mathcal{F}(X)$. Then, one can define a subpresheaf \mathcal{H} of \mathcal{R} -modules as

$$\mathcal{H}(U) = \text{span}_{\mathcal{R}(U)} \{s_\alpha|_U \mid \alpha \in \Lambda\}, \quad \forall \text{ open sets } U \subseteq X.$$

The subpresheaf \mathcal{H} is called **the subpresheaf of \mathcal{R} -modules generated by $\{s_\alpha\}_{\alpha \in \Lambda}$** .

One can easily notice that presheaves lack many nice local to global properties. One can enhance the local to global properties of presheaves by adding some extra conditions on their sections.

Definition 2.4.7. A **sheaf** of objects \mathcal{G} on X is a presheaf of sets on X such that, for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} U_i$, the following properties hold.

1. **Locality property** : Two elements $s, t \in \mathcal{F}(U)$ are the same if, for every $i \in I$, we have $r_{U_i, U}(t) = r_{U_i, U}(s)$.
2. **Gluing property** : If $\{s_i\}_{i \in I}$ is a family of sections of \mathcal{F} such that, for every $i, j \in I$, we have

$$\begin{aligned} s_i &\in \mathcal{F}(U_i), \\ r_{U_i \cap U_j, U_i}(s_i) &= r_{U_i \cap U_j, U_j}(s_j), \end{aligned}$$

then there exists $s \in \mathcal{F}(U)$ such that, for every $i \in I$, we have $r_{U_i, U}(s) = s_i$.

Similar to presheaves, one can define a subsheaf of a sheaf.

Definition 2.4.8. Let \mathcal{G} be a sheaf of objects on X . Then a **subsheaf** \mathcal{H} of \mathcal{G} is a subpresheaf which is itself a sheaf.

Example 2.4.9. One can easily show that the presheaf of rings C_M^ν defined in Example 2.4.2 and the presheaf Γ_M^ν of C_M^ν -modules defined in Example 2.4.4 are sheaves.

While the above example shows that both of the presheaves we define in Example 2.4.2 and Example 2.4.4 are sheaves, it is not generally true that every presheaf is a sheaf. The following example shows that not every presheaf is a sheaf.

Example 2.4.10 ([55]). Let M be a C^ν -manifold and let \mathbb{G} be a Riemannian metric on M . Then, for every open set $U \subseteq M$, we define

$$\mathcal{F}_{\text{bdd}}(U) = \{X \in \mathcal{G}^\nu(U) \mid \sup\{\|X(x)\|_{\mathbb{G}} \mid x \in U\} < \infty\}.$$

Also, we assign to every pair of open sets (U, V) with $U \subseteq V \subseteq M$, a restriction map $r_{U, V} : \mathcal{F}_{\text{bdd}}(V) \rightarrow \mathcal{F}_{\text{bdd}}(U)$ defined as

$$r_{U, V}(X) = X|_U, \quad \forall X \in \mathcal{F}_{\text{bdd}}(V).$$

One can easily check that \mathcal{F}_{bdd} is a presheaf of sets. In fact, \mathcal{F}_{bdd} is a presheaf of vector spaces on M . However, \mathcal{F}_{bdd} is not a sheaf of sets. This can be easily seen in the case $M = \mathbb{R}$, by showing that this presheaf doesn't satisfy the gluing property. Consider the covering $\{U_i\}_{i \in \mathbb{N}}$ of \mathbb{R} , where

$$U_i = (-i, i), \quad \forall i \in \mathbb{N},$$

and the family of locally defined vector field $\{X_i\}_{i \in \mathbb{N}}$, where $X_i : U_i \rightarrow TR \simeq \mathbb{R} \times \mathbb{R}$ is defined as

$$X_i(x) = (x, x), \quad \forall x \in U_i.$$

For every $i, j \in \mathbb{N}$ such that $i > j$, we have $U_j \subseteq U_i$. Thus we get

$$r_{U_i \cap U_j, U_i}(X_i) = r_{U_j, U_i}(X_i) = X_j = r_{U_j, U_j}(X_j) = r_{U_i \cap U_j, U_j}(X_j).$$

It is also easy to see that, for every $i \in \mathbb{N}$, $\|X_i\| < i$. Therefore, for every $i \in \mathbb{N}$, $X_i \in \mathcal{F}_{\text{bdd}}(U_i)$ and, for every $i, j \in \mathbb{N}$, we have

$$r_{U_i \cap U_j, U_i}(X_i) = r_{U_i \cap U_j, U_j}(X_j).$$

However, there does not exist a bounded vector field $X \in \mathcal{F}_{\text{bdd}}(\mathbb{R})$ such that $r_{U_i, \mathbb{R}}(X) = X_i$, for all $i \in \mathbb{N}$.

One can capture the local properties of a presheaf around a set A in a structure called stalk of the presheaf over A .

Definition 2.4.11. Let \mathcal{F} be a presheaf on X and $A \subseteq X$ be a set. The set of all open neighbourhoods of A in X is denoted by \mathcal{N}_A . We define an equivalence relation \simeq_A between local sections of \mathcal{F} . Let $U, V \in \mathcal{N}_A$ and $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ be two local sections of \mathcal{F} . Then we say that $s \simeq_A t$ if there exists an open set $W \in \mathcal{N}_A$ such that $W \subseteq U \cap V$ and

$$r_{W, U}(s) = r_{W, V}(t).$$

For every local section $s \in \mathcal{F}(U)$, where $U \in \mathcal{N}_A$, we define the **germ of s over A** as the equivalence class of s under the equivalence relation \simeq_A . We denote the germ of s over A by $[s]_A$.

The **stalk of \mathcal{F} over A** is the inductive limit

$$\varinjlim_{U \in \mathcal{N}_A} \mathcal{F}(U).$$

We usually use the symbol \mathcal{F}_A for the stalk of \mathcal{F} over A . One can easily show that the stalk of \mathcal{F} over A consists of all equivalence classes under equivalent relation \simeq_A .

If \mathcal{R} is a presheaf of rings, then, for every $x \in X$, one can define addition and multiplication on \mathcal{R}_x as follows. Let $[s]_x, [t]_x \in \mathcal{R}_x$. Then, there exists open sets $U, V \subseteq X$ such that

$$\begin{aligned} s &\in \mathcal{R}(U), \\ t &\in \mathcal{R}(V). \end{aligned}$$

Let $W \subseteq U \cap V$ be an open set in X . Then, we define

$$\begin{aligned} [s]_x \cdot [t]_x &= [(s|_W)(t|_W)]_x, \\ [s]_x + [t]_x &= [(s|_W) + (t|_W)]_x. \end{aligned}$$

One can easily check that the multiplication and the addition defined above are well-defined. Thus, \mathcal{R}_x is a ring. Similarly, for a presheaf of \mathcal{R} -modules \mathcal{F} over X one can define an \mathcal{R}_x -module structure on \mathcal{F}_x .

Definition 2.4.12. A sheaf \mathcal{F} of \mathcal{R} -modules on the topological space X is called **locally finitely generated** if, for every $x \in X$, there exists a neighbourhood $U \subseteq X$ of x and a family of finite sections $\{s_1, s_2, \dots, s_n\}$ of \mathcal{F} on U , such that \mathcal{F}_y is generated by the set $\{[s_1]_y, [s_2]_y, \dots, [s_n]_y\}$ as \mathcal{R}_y -modules, for every $y \in U$.

Theorem 2.4.13. Let $\nu \in \{\omega, \text{hol}\}$, M be a C^ν -manifold, and $\{s_\alpha\}_{\alpha \in \Lambda}$ be a family of sections of the sheaf C^ν defined on M , i.e., for every $\alpha \in \Lambda$, we have $s_\alpha \in C^\nu(M)$. Then the subpresheaf of C^ν generated by $\{s_\alpha\}_{\alpha \in \Lambda}$ is locally finitely generated.

Proof. The proof in the holomorphic case is given in [30, Theorem H.8 & Corollary H.9]. For the real analytic case, the proof is similar. \square

Definition 2.4.14. Let \mathcal{F} be a presheaf on X . Then the **étalé space** of \mathcal{F} is defined as

$$\text{Et}(\mathcal{F}) = \bigcup_{x \in X} \mathcal{F}_x.$$

One can equip this space with a topology using a basis consisting of elements of the form

$$B(U, X) = \{[X]_x \mid x \in U\}, \quad \forall U \text{ open in } X, \forall X \in \mathcal{F}(U).$$

This topology on $\text{Et}(\mathcal{F})$ is usually called the **étalé topology**.

As Example 2.4.10 shows, presheaves are not necessarily sheaves. However, it can be shown that every presheaf can be converted to a sheaf in a natural way. This procedure is called sheafification.

Definition 2.4.15. Let \mathcal{F} be a presheaf of objects on X . Then, for every open set $U \subseteq X$, we define

$$\text{Sh}(\mathcal{F})(U) = \{s : U \rightarrow \text{Et}(\mathcal{F}) \mid \forall x \in U, s(x) \in \mathcal{F}_x \text{ and } s \text{ is continuous on } U\}.$$

One can show that the assignment $\text{Sh}(\mathcal{F})$ is a sheaf [80, Theorem 7.1.8]

Theorem 2.4.16. Let \mathcal{F} be a presheaf of objects on X . Then $\text{Sh}(\mathcal{F})$ is a sheaf on X .

2.5. Groupoids and pseudogroups

In this section, we review definitions and elementary properties of groupoids and pseudogroups. In particular, we show that one can always associate a groupoid to a family of local diffeomorphism. This groupoid plays an important role in our study of the orbits of a C^ν -tautological system in chapter 5

Definition 2.5.1. A **groupoid** \mathcal{G} is a pair (Ω, B) and five structural maps $(t, s, \text{id}, (\cdot)^{-1}, *)$, where $t : \Omega \rightarrow B$ is called **target map**, $s : \Omega \rightarrow B$ is called **source map**, $\text{id} : B \rightarrow \Omega$ is called **object inclusion map**, $(\cdot)^{-1} : \Omega \rightarrow \Omega$ is called **inversion map** and $* : \Omega \otimes \Omega \rightarrow \Omega$ is called **partial multiplication map**, where $\Omega \otimes \Omega = \{(\xi, \eta) \in \Omega \times \Omega \mid t(\xi) = s(\eta)\}$. These maps have the following properties:

1. $t(\xi * \eta) = t(\xi)$ and $s(\xi * \eta) = s(\eta)$, for all $(\xi, \eta) \in \Omega \otimes \Omega$;
2. $\xi * (\eta * \chi) = (\xi * \eta) * \chi$;
3. $s(\text{id}(x)) = t(\text{id}(x)) = x$, for all $x \in B$;
4. $\xi * \text{id}(t(\xi)) = \xi$ and $\text{id}(s(\xi)) * \xi = \xi$, for all $\xi \in \Omega$;
5. $\xi * \xi^{-1} = \text{id}(t(\xi))$ and $\xi^{-1} * \xi = \text{id}(s(\xi))$, for all $\xi \in \Omega$.

Definition 2.5.2. If \mathcal{G} is a groupoid, then the set Ω^x defined as

$$\Omega^x = \{\xi \mid \xi \in t^{-1}(x)\},$$

is called the t -**fiber** at $x \in B$, and the set Ω_x defined as

$$\Omega_x = \{\xi \mid \xi \in s^{-1}(x)\},$$

is called the s -**fiber** at $x \in B$.

The **isotropy group** of \mathcal{G} at $x \in B$ is defined as

$$\Omega_x^x = \Omega^x \cap \Omega_x.$$

One can show that, for every $y \in \Omega_x^x$, we have $y \in \Omega \otimes \Omega$. This means that the partial multiplication of the groupoid \mathcal{G} is a multiplication on Ω_x^x . Therefore, the isotropy group is actually a group with respect to groupoid partial multiplication.

The **orbit** of \mathcal{G} at $x \in B$ is defined as

$$\text{Orb}_{\mathcal{G}}(x) = t(\Omega_x) = \{t(\xi) \mid \xi \in \Omega_x\}.$$

Definition 2.5.3. Suppose that \mathcal{G} is a groupoid. Then we define an equivalence relation \sim on B by

$$x \sim y \iff y \in \text{Orb}_{\mathcal{G}}(x).$$

We denote the equivalent class of $x \in B$ by $[x]$ and the set of all equivalent classes of \sim is denoted by M/\sim . We define

$$(M, \mathcal{G}) = \dot{\bigcup}_{[x] \in M/\sim} [x].$$

Definition 2.5.4. Let X be a topological space. A **pseudogroup** \mathfrak{h} acting on X is a set of homeomorphisms $F : U \rightarrow V$ where U and V are open subsets of X such that:

1. **composition:** if $F_1 : U_1 \rightarrow V_1$ and $F_2 : U_2 \rightarrow V_2$ are members of \mathfrak{h} , then $F_1 \circ F_2 : F_2^{-1}(U_1 \cap V_2) \rightarrow F_1^{-1}(U_1 \cap V_2)$ is a member of \mathfrak{h} ;
2. **inversion:** for every homeomorphism $F : U \rightarrow V$ in \mathfrak{h} , the homeomorphism $F^{-1} : V \rightarrow U$ is also in \mathfrak{h} ;

3. **restriction:** for every element $F : U \rightarrow V$ and every open subset $W \subseteq X$, the map $F|_{U \cap W} : U \cap W \rightarrow F(U \cap W)$ is a member of \mathfrak{h} ;
4. **covering:** if $F : U \rightarrow V$ is a homeomorphism and $\{U_i\}_{i \in I}$ is an open cover of U such that $F|_{U_i}$ is in \mathfrak{h} , for all $i \in I$, then F is in \mathfrak{h} .

Definition 2.5.5. Suppose that A is a set of homeomorphisms between open subsets of X . Then we denote the pseudogroup generated by A , by $\Gamma(A)$. A homeomorphism $f : U \rightarrow V$ belongs to $\Gamma(A)$ if, for every $x \in X$, there exists a neighbourhood W of x and $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in \{1, -1\}$ and $h_1, h_2, \dots, h_k \in A$ such that $f|_W = h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \dots \circ h_k^{\epsilon_k}$.

Now suppose that we have a family of local homeomorphisms \mathcal{P} , we denote the pseudogroup generated by \mathcal{P} by $\Gamma(\mathcal{P})$.

Definition 2.5.6. Suppose that \mathcal{P} is a collection of local homeomorphisms of X . Then **the groupoid associated to \mathcal{P}** , which is denoted by $G_{\mathcal{P}}$, is defined by $B = M$ and $\Omega = \{(P, x) \mid P \in \Gamma(\mathcal{P}), x \in \text{Dom}P\}$. The five structure maps $(t, s, \text{id}, (\cdot)^{-1}, *)$ for this groupoid are defined as follows:

$$\begin{aligned}
 s((P, x)) &= x, \\
 t((P, x)) &= P(x), \\
 \text{id}(x) &= (\text{id}_M, x), \\
 (P, x)^{-1} &= (P^{-1}, P(x)).
 \end{aligned} \tag{2.5.1}$$

If we have $x = P'(x')$, the partial composition map $*$ is defined as

$$(P, x) * (P', x') = (P \circ P', x'). \tag{2.5.2}$$

Note that, by definition, the orbit of $G_{\mathcal{P}}$ passing through $x \in M$ is

$$G_{\mathcal{P}}(x) = t \circ s^{-1}(x).$$

Therefore, we have

$$s^{-1}(x) = \{(P, x) \mid P \in \Gamma(\mathcal{P}), x \in \text{Dom}(P)\}.$$

This means that

$$G_{\mathcal{P}}(x) = t(s^{-1}(x)) = \{P(x) \mid P \in \Gamma(\mathcal{P}), x \in \text{Dom}(P)\}.$$

Chapter 3

Time-varying vector fields and their flows

3.1. Introduction

In this chapter, we develop an operator approach for studying time-varying vector fields and their flows. The idea of this approach originated (for us) with the so-called chronological calculus of Agrachev and Gamkrelidze [3]. The cornerstone of the chronological calculus developed in [3] and [4] is to consider a complete vector field on M and its flow as linear operators on the \mathbb{R} -algebra $C^\infty(M)$. By this correspondence, one can easily see that the governing nonlinear differential equation for the flow of a time-varying vector field is transformed into a “linear” differential equation on the infinite-dimensional locally convex space $L(C^\infty(M); C^\infty(M))$. In [3] and [4], this approach has been used to study flows of smooth and real analytic vector fields. Although, this representation is the most convenient one for studying smooth vector fields and their flows, in the real analytic case, it does not appear to be natural. This can be seen in the proof of convergence of the sequence of Picard iterations for the flow of a time-varying vector field [3, §2, Proposition 2.1]. Here it has been shown that, for a locally integrable time-varying real analytic vector field on \mathbb{R}^n with bounded holomorphic extension, the sequence of Picard iterations for the “linear” differential equation converges. In [4] it has been stated that the sequence of Picard iterations for a locally integrable real analytic vector field converges in $L(C^\infty(M); C^\infty(M))$. However, the sequence of Picard iterations for the locally integrable time-varying smooth vector fields which are not real analytic, never converges. This different behaviour of the “linear” differential equation in smooth and real analytic case, suggests that one may benefit from a new setting for studying time-varying real analytic vector fields.

In the framework we develop in this section, we consider ν to be in the set $\{\infty, \omega, \text{hol}\}$ and $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then, one can show that C^ν -vector field and its flow are operators on the \mathbb{F} -algebra $C^\nu(M)$. This way we get a unified framework in which operators are consistent with the regularity of their corresponding maps. We start this chapter by defining the C^ν -topology on the vector space $\Gamma^\nu(E)$, where (E, M, π) is a

C^ν -vector bundle. This, in particular, defines the C^ν -topology on the vector spaces $C^\nu(M)$ when the vector bundle (E, M, π) is the trivial bundle $(M \times \mathbb{R}, M, \text{pr}_1)$. For the reasons we will see later, we need this topology to make $\Gamma^\nu(E)$ into a complete topological vector space. For the smooth case, this topology on the space $\Gamma^\infty(E)$ is well-known and has been explored in detail [3], [36], [61], and [51]. For the real analytic case, it is interesting to note that $\Gamma^\omega(E)$ equipped with the subspace topology from $\Gamma^\infty(E)$ is not a complete topological vector space (example 3.2.8). However, by considering real analytic sections as germs of holomorphic sections, one can define a topology on the space $\Gamma^\omega(E)$ which makes it into a complete locally convex vector space [59].

We then proceed in section 3.3 by considering C^ν -vector fields on M as derivations on $C^\nu(M)$. It is interesting to see that there is a one-to-one correspondence between derivations on $C^\nu(M)$ and C^ν -vector fields on M . We finally show that vector fields, as operators on $C^\nu(M)$, are continuous with respect to the C^ν -topology. Similarly, one can show that C^ν -maps between two manifolds are in one-to-one correspondence with unital \mathbb{F} -algebra homomorphisms between their spaces of C^ν -functions.

In order to define a setting where vector fields and their flows are treated the same way, we define $L(C^\nu(M); C^\nu(M))$ as the space of linear operators between $C^\nu(M)$ and $C^\nu(M)$. It is easy to see that both C^ν -maps and C^ν -vector fields are subsets of this space. Then one can consider a time-varying vector field and its flow as curves on the space of linear operators $L(C^\nu(M); C^\nu(M))$. In order to study properties of these curves, we equip the space $L(C^\nu(M); C^\nu(M))$ with the topology of pointwise convergence. Since the space of all C^ν -vector fields is a subset of $L(C^\nu(M); C^\nu(M))$, this induces a topology on the space of C^ν -vector fields (as derivations between $C^\nu(M)$ and $C^\nu(M)$). On the other hand, C^ν -vector fields are exactly C^ν -sections of the tangent bundle (TM, M, π) . Therefore, one can see that, algebraically, the space of C^ν -vector fields on M is the same as $\Gamma^\nu(TM)$. It is interesting to note that the topology of pointwise convergence that we defined above on the space of C^ν -vector fields coincide with the C^ν -topology on $\Gamma^\nu(TM)$ defined in section 3.2.

In section 3.6, using the topology of pointwise convergence on the vector space $L(C^\nu(M); C^\nu(M))$, we define and characterize essential boundedness, integrability and absolute continuity of curves on $L(C^\nu(M); C^\nu(M))$.

It is well-known that every real analytic function on \mathbb{R}^n can be extended to a holomorphic function on some open set in \mathbb{C}^n containing \mathbb{R}^n . In section 3.7, we study the extension of time-varying real analytic vector fields to holomorphic ones. In general, it is not true that every time-varying real analytic vector field can be extended to a time-varying holomorphic vector field (example 3.7.1). However, we will show that locally integrable time-varying real analytic vector fields on M can be extended to a locally integrable time-varying holomorphic vector field on a manifold containing M . We refer to this result as the global extension of time-varying real analytic vector fields. The reason is that it shows the existence of an extension over the whole domain M .

In some applications, one would like to know if there exists a domain such that

“all” members of a family of locally integrable real analytic vector field can be extended to holomorphic vector fields on that domain. Unfortunately, the global extension theorem is indecisive in this situation. However, we show that, for every compact set $K \subseteq M$, one can extend a “bounded” family of locally integrable real analytic vector field to a bounded family of holomorphic vector fields on a domain containing K .

In section 3.8, using the operator representation of time-varying vector fields and their flows, we translate the “nonlinear” differential equation governing the flow of a time-varying vector field on M into a “linear” differential equation on the infinite-dimensional space $L(C^\nu(M); C^\nu(U))$, for some open set $U \subseteq M$. In the holomorphic and real analytic cases, we will show that, for the locally integrable time-varying vector fields, the sequence of the Picard iterations for the “linear” differential equation converges. The limit of this sequence gives us an absolutely continuous curve on $L(C^\nu(M); C^\nu(U))$, which is the flow of the vector field.

Finally, in section 3.9, we study the exponential mapping which takes a locally integrable time-varying vector field and gives us its absolutely continuous flow. Using the local extension result for time-varying real analytic vector fields, we show that real analytic exponential map is a sequential homeomorphism.

3.2. Topology on the space $\Gamma^\nu(E)$

In this section, we define a topology on space of C^ν -sections of the C^ν -vector bundle (E, M, π) . Topologies on spaces of differentiable mapping has been studied in detail in many references (for example [36], [61], and [51]). In the cases of smooth and holomorphic sections, these topologies has been defined in the literature using a family of seminorms [51], [61]. We will show that these locally convex spaces have many nice properties including being complete, separable, and satisfying the Heine–Borel property.

In the real analytic case, using the fact that real analytic sections are germs of holomorphic sections, one can define two representations for a locally convex topology on the space of real analytic sections [59]. Moreover, one can find a family of generating seminorms for this space [81]. We will also show that the space of real analytic sections with this topology has nice properties including being complete, separable, and Heine–Borel.

Finally, when (E, M, π) is the trivial bundle, the space of C^ν -sections of E can be identified with the space of C^ν -functions on M . However, one can see that the space of C^ν -functions on M has an additional structure of an \mathbb{F} -algebra. It is reasonable, therefore, to study algebra multiplication in $C^\nu(M)$ with respect to C^ν -topology. We will show that the algebra multiplication of $C^\nu(M)$ is continuous for the C^ν -topology on $C^\nu(M)$.

3.2.1. Topology on $\Gamma^\infty(E)$. The Whitney topology for the space of smooth mappings on M has been studied in [36] and [61].

Definition 3.2.1. Let (E, M, π) be a C^∞ -vector bundle, (U, ϕ) be a coordinate chart on M , $K \subseteq U$ be a compact set, and $m \in \mathbb{N}$. We define the seminorm $p_{K,m}^\infty$ on $\Gamma^\infty(E)$ as

$$p_{K,m}^\infty(X) = \sup\{\|D^{(r)}X(x)\| \mid x \in K, |r| \leq m\}, \quad \forall X \in \Gamma^\infty(E).$$

The family of seminorms $\{p_{K,m}^\infty\}$ generates a locally convex topology on the space $\Gamma^\infty(E)$. This topology is called C^∞ -**topology**.

This topological vector space has many nice properties. The next theorem states some of the most used properties of this space.

Theorem 3.2.2. *The space $\Gamma^\infty(E)$ with the C^∞ -topology is a Hausdorff, separable, complete, nuclear, and metrizable space. It also satisfies Heine–Borel property.*

Proof. The fact that $\Gamma^\infty(E)$ is a nuclear completely metrizable space is shown in [51, §30.1] and the fact that $\Gamma^\infty(E)$ is separable is proved in [42, §3.2]. Since $\Gamma^\infty(E)$ is metrizable, it is Hausdorff. Finally, the Heine–Borel property of $\Gamma^\infty(E)$ follows from the fact that $\Gamma^\infty(E)$ is nuclear. \square

3.2.2. Topology on $\Gamma^{\text{hol}}(E)$. The natural topology on the space of germs of holomorphic sections on a complex manifold M has been studied in [51, §30.4].

Definition 3.2.3. Let (E, M, π) be a C^{hol} -vector bundle. For every compact set $K \subseteq M$, we define the seminorm p_K^{hol} as

$$p_K^{\text{hol}}(X) = \sup\{\|X(x)\| \mid x \in K\}, \quad \forall X \in \Gamma^{\text{hol}}(E).$$

The family of seminorms $\{p_K^{\text{hol}}\}$ defines a locally convex topology on the vector space $\Gamma^{\text{hol}}(E)$. We call this topology the C^{hol} -**topology**.

Theorem 3.2.4. *The space $\Gamma^{\text{hol}}(E)$ with the C^{hol} -topology is a Hausdorff, separable, complete, nuclear, and metrizable space. It also satisfies Heine–Borel property.*

Proof. The fact that the C^{hol} -topology on $\Gamma^{\text{hol}}(E)$ is complete, nuclear, and metrizable has been proved in [51, §30.5]. In order to prove that $\Gamma^{\text{hol}}(E)$ is separable, note that $\Gamma^{\text{hol}}(E)$ is a subspace of $\Gamma^\infty(E^{\mathbb{R}})$. However, by Theorem 3.2.2, we know that $\Gamma^\infty(E^{\mathbb{R}})$ is separable and by [84, Theorem 16.11], subspaces of separable metric spaces are separable. This implies that $\Gamma^{\text{hol}}(E)$ is separable. Since $\Gamma^{\text{hol}}(E)$ is nuclear, it satisfies the Heine–Borel property. \square

The Topology on the space of germs of holomorphic functions has been studied in [51, §8.3]. We generalize the setting in [51] to include germs of holomorphic sections. Let $A \subseteq M$ be a subset. The family of all neighbourhoods of A in M is denoted by \mathcal{N}_A . One can easily show that \mathcal{N}_A is a directed set with respect to the set inclusion. For every pair $U_A, V_A \in \mathcal{N}_A$ such that $U_A \subseteq V_A$, we can define the restriction map $i_{U_A, V_A} : \Gamma^{\text{hol}}(V_A) \rightarrow \Gamma^{\text{hol}}(U_A)$ as

$$i_{U_A, V_A}(X) = X|_{U_A}.$$

The pair $\{\Gamma^{\text{hol}}(U_A), i_{U_A, V_A}\}_{U_A \subseteq V_A}$ is a directed system. The direct limit of this directed system is denoted by $(\mathcal{G}_A^{\text{hol}}, \{i_{\bar{U}_A}\}_{\bar{U}_A \in \mathcal{N}_A})$. One can define the topology on $\mathcal{G}_A^{\text{hol}}$ as the finest locally convex topology which makes all the maps $i_{U_A} : \Gamma^{\text{hol}}(U_A) \rightarrow \mathcal{G}_A^{\text{hol}}$ continuous.

The special case when $A = K \subseteq M$ is compact has a significant role in studying the space of real analytic sections. One can show that, when $K \subseteq M$ is compact, the inductive limit $\varinjlim_{U_K \in \mathcal{N}_K} \Gamma^{\text{hol}}(U_K) = \mathcal{G}_K^{\text{hol}}$ is countable [23]. It follows that, for every compact set $K \subseteq M$, one can choose a sequence of open sets $\{U_n\}_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, we have

$$\text{cl}(U_{n+1}) \subseteq U_n,$$

and $\bigcap_{i=1}^{\infty} U_i = K$. Then we have $\varinjlim_{n \rightarrow \infty} \Gamma^{\text{hol}}(U_n) = \mathcal{G}_K^{\text{hol}}$.

Definition 3.2.5. Let $U \subseteq M$ be an open set. We define the map $p_U : \Gamma^{\text{hol}}(U) \rightarrow [0, \infty]$ as

$$p_U(X) = \sup\{\|X(x)\| \mid x \in U\}, \quad \forall X \in \Gamma^{\text{hol}}(U).$$

Then $\Gamma_{\text{bdd}}^{\text{hol}}(U)$ is a subspace of $\Gamma^{\text{hol}}(U)$ defined as

$$\Gamma_{\text{bdd}}^{\text{hol}}(U) = \{X \in \Gamma^{\text{hol}}(U) \mid p_U(X) < \infty\}.$$

We equip $\Gamma_{\text{bdd}}^{\text{hol}}(U)$ with the norm p_U and define the inclusion $\rho_U : \Gamma_{\text{bdd}}^{\text{hol}}(U) \rightarrow \Gamma^{\text{hol}}(U)$ as

$$\rho_U(X) = X, \quad \forall f \in \Gamma_{\text{bdd}}^{\text{hol}}(U).$$

Theorem 3.2.6. *The space $(\Gamma_{\text{bdd}}^{\text{hol}}(U), p_U)$ is a Banach space and the map $\rho_U : \Gamma_{\text{bdd}}^{\text{hol}}(U) \rightarrow \Gamma^{\text{hol}}(U)$ is a compact continuous map.*

Proof. Let K be a compact subset of U . Then, for every $X \in \Gamma_{\text{bdd}}^{\text{hol}}(U)$, we have $p_K^{\text{hol}}(\rho_U(X)) = p_K^{\text{hol}}(X) \leq p_U(X)$, which implies that ρ_U is continuous. Now consider the open set $p_U^{-1}([0, 1])$ in $\Gamma_{\text{bdd}}^{\text{hol}}(U)$. The set $p_U^{-1}([0, 1])$ is bounded and ρ_U is continuous. So

$$\rho_U(p_U^{-1}([0, 1])),$$

is bounded in $\Gamma^{\text{hol}}(U)$. Since $\Gamma^{\text{hol}}(U)$ has the Heine–Borel property, the set $\rho_U(p_U^{-1}([0, 1]))$ is relatively compact in $\Gamma^{\text{hol}}(U)$. So ρ_U is compact.

Now we show that $(\Gamma_{\text{bdd}}^{\text{hol}}(U), p_U)$ is a Banach space. Let $\{X_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\Gamma_{\text{bdd}}^{\text{hol}}(U)$. It suffices to show that there exists $X \in \Gamma_{\text{bdd}}^{\text{hol}}(U)$ such that $\lim_{n \rightarrow \infty} X_n = X$ in the topology induced by p_U on $\Gamma_{\text{bdd}}^{\text{hol}}(U)$. Since ρ_U is continuous, the sequence $\{X_n\}_{n \in \mathbb{N}}$ is Cauchy in $\Gamma^{\text{hol}}(U)$. Since $\Gamma^{\text{hol}}(U)$ is complete, there exists $X \in \Gamma^{\text{hol}}(U)$ such that $\lim_{n \rightarrow \infty} X_n = X$ in the C^{hol} -topology. Now we show that $\lim_{n \rightarrow \infty} X_n = X$ in the topology of $(\Gamma_{\text{bdd}}^{\text{hol}}(U), p_U)$ and $X \in \Gamma_{\text{bdd}}^{\text{hol}}(U)$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that, for every $n, m > N$, we have

$$p_U(X_n - X_m) < \frac{\epsilon}{2}.$$

This implies that, for every $z \in U$ and every $n, m > N$, we have

$$\|X_n(z) - X_m(z)\| < \frac{\epsilon}{2}.$$

So, for every $z \in U$ and every $n > N$, we choose $m_z > N$ such that

$$\|X_{m_z}(z) - X(z)\| < \frac{\epsilon}{2}, \quad \forall m \geq m_z.$$

This implies that, for every $z \in U$, we have

$$\|X(z) - X_n(z)\| < \|X_n(z) - X_{m_z}(z)\| + \|X_{m_z}(z) - X(z)\| < \epsilon.$$

So, for every $n > N$, we have

$$p_U(X_n - X) < \epsilon.$$

This completes the proof. \square

Theorem 3.2.7. *Let K be a compact set and $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of neighbourhoods of K such that*

$$\text{cl}(U_{n+1}) \subseteq U_n, \quad \forall n \in \mathbb{N},$$

and $\bigcap_{n \in \mathbb{N}} U_n = K$. Then we have $\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(U_n) = \mathcal{G}_K^{\text{hol}}$. Moreover, the inductive limit is compact.

Proof. For every $n \in \mathbb{N}$, we define $r_n : \Gamma^{\text{hol}}(U_n) \rightarrow \Gamma_{\text{bdd}}^{\text{hol}}(U_{n+1})$ as

$$r_n(X) = X|_{U_{n+1}}, \quad \forall X \in \Gamma^{\text{hol}}(U_n).$$

For every compact set K with $U_{n+1} \subseteq K \subseteq U_n$, we have $p_{U_n}(X) \leq p_K^{\text{hol}}(X)$. This implies that the map r_n is continuous and we have the following diagram:

$$\Gamma_{\text{bdd}}^{\text{hol}}(U_n) \xrightarrow{\rho_{U_n}} \Gamma^{\text{hol}}(U_n) \xrightarrow{r_n} \Gamma_{\text{bdd}}^{\text{hol}}(U_{n+1}) \xrightarrow{\rho_{U_{n+1}}} \Gamma^{\text{hol}}(U_{n+1}).$$

Since all maps in the above diagram are linear and continuous, by the universal property of the inductive limit of locally convex spaces, we have

$$\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(U_n) = \varinjlim_{n \rightarrow \infty} \Gamma^{\text{hol}}(U_n) = \mathcal{G}_K^{\text{hol}}.$$

Moreover, for every $n \in \mathbb{N}$, the map ρ_{U_n} is compact and r_n is continuous. So the composition $r_n \circ \rho_{U_n}$ is also compact [43, §17.1, Proposition 1]. This implies that the direct limit $\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(U_n) = \mathcal{G}_K^{\text{hol}}$ is compact. \square

3.2.3. Topology on $\Gamma^\omega(E)$. Since $\Gamma^\omega(E) \subseteq \Gamma^\infty(E)$, one can define a topology on the space of real analytic sections using the C^∞ -topology. This topology makes $\Gamma^\omega(E)$ into a topological subspace of $\Gamma^\infty(E)$. However, the following example shows that the restriction of C^∞ -topology to the space of real analytic sections is not complete.

Example 3.2.8. Let $f \in C^\infty(\mathbb{S}^1)$ be a smooth function which is not real analytic. Recall that the n th partial sum of the Fourier series of f is given by

$$\begin{aligned} s_n(f)(t) &= \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)e^{iks} ds \\ &= \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)e^{ik(t-s)} ds = \sum_{k=-n}^n \left(\int_{-\pi}^{\pi} f(s)e^{-iks} ds \right) e^{ikt}. \end{aligned}$$

For every $k \in \mathbb{N}$, the function e^{ikt} is real analytic on \mathbb{S}^1 . This implies that, for every $n \in \mathbb{N}$, $s_n(f) \in C^\omega(\mathbb{S}^1)$. Since $f \in C^1(\mathbb{S}^1)$, the sequence $\{s_n(f)\}_{n \in \mathbb{N}}$ converges uniformly on \mathbb{S}^1 to f [40, Chapter 1, §1, Corollary III]. Let us define $g \in C^\infty((-\pi, \pi))$ by

$$g(t) = f(t), \quad \forall t \in (-\pi, \pi).$$

Similarly, for every $n \in \mathbb{N}$, we define $g_n \in C^\infty((-\pi, \pi))$ by

$$g_n(t) = s_n(f)(t), \quad \forall t \in (-\pi, \pi).$$

Now consider the sequence $\{Dg_n\}_{n \in \mathbb{N}}$. Note that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} D(g_n)(t) &= D \left(\sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t-s)e^{iks} ds \right) \\ &= \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} Dg(t-s)e^{iks} ds, \quad \forall t \in (-\pi, \pi). \end{aligned}$$

Since we have $g \in C^\infty((-\pi, \pi))$, $Dg \in C^1((-\pi, \pi))$. So, we can again use [40, Chapter 1, §1, Corollary III] to show that the sequence $\{Dg_n\}_{n \in \mathbb{N}}$ converges to Dg uniformly on $(-\pi, \pi)$. If we continue this procedure, we will see that, for every $i \in \mathbb{N}$, the sequence $\{D^i(g_n)\}_{n \in \mathbb{N}}$ converges to $D^i(g)$ uniformly on $(-\pi, \pi)$. This implies that the sequence $\{g_n\}_{n \in \mathbb{N}}$ converges to g in the C^∞ -topology. Therefore, we have a sequence of real analytic functions $\{g_n\}_{n \in \mathbb{N}}$ which converges in the C^∞ -topology to a smooth but not real analytic function g . This implies that the space $C^\omega((-\pi, \pi))$ with C^∞ -topology is not complete.

We will define a topology on $\Gamma^\omega(E)$ which makes it into a complete topological vector space. This topology on $\Gamma^\omega(E)$ has been first studied comprehensively in [59]. Recently, this topology has been studied throughly in operator theory [23]. A special case of what Martineau did in [59] can be used to define a topology on the space

of real analytic sections on an open set $U \subseteq \mathbb{R}^n$. Martineau defined two topologies on $\Gamma^\omega(E)$, using the fact that every real analytic section on U can be extended to a holomorphic section on some neighbourhood $\bar{U} \subseteq \mathbb{C}^n$ of U . He also showed that these two topologies are the same. Many properties of this topology have been studied in [23]. The generalization of some of these ideas to the case when M is a real analytic manifold is straightforward [23]. This requires extension of the domain of a section $X \in \Gamma^\omega(E)$ to a complex manifold. We first define a specific class of real submanifolds of a complex manifold. One can consider an n -dimensional complex manifold M as a $2n$ -dimensional smooth manifold $M^{\mathbb{R}}$. Although the real submanifolds of $M^{\mathbb{R}}$ are not necessarily complex submanifolds of M , there exists a specific class of submanifolds of $M^{\mathbb{R}}$ which has many nice properties. These submanifolds are called totally real. We start by defining a totally real subspace of a complex vector space.

Definition 3.2.9. Let V be a complex vector space with an almost complex structure J . A real subspace U of V is called **totally real** if we have $J(U) \cap U = \{0\}$.

Definition 3.2.10. Let M be a complex manifold with an almost complex structure J . A submanifold N of M is called a **totally real submanifold** if, for every $p \in N$, we have $J(T_p N) \cap T_p N = \{0\}$.

Definition 3.2.11. Let (F, N, ξ) be a holomorphic vector bundle with an almost complex structure J on its fibers. A generalized vector subbundle (E, M, π) of (F, N, ξ) is called a **totally real subbundle** if, M is a totally real submanifold of N , E is a totally real submanifold of F , and, for every $p \in N$, we have $J(F_p) \cap F_p = \{0\}$.

Let M be a complex manifold. Then every real analytic map on a totally real submanifold N of M can be extended to a holomorphic map on a neighbourhood of N in M [18, Lemma 5.40].

Theorem 3.2.12. *Let M and W be two complex manifold and N be a real analytic totally real submanifold of M , with $\dim_{\mathbb{C}} M = \dim_{\mathbb{R}} N$. Suppose that $f : N \rightarrow W$ is a real analytic map. Then, there exists a sufficiently small neighbourhood \bar{U} of N in M and a holomorphic map $\bar{f} : \bar{U} \rightarrow W$ such that*

$$\bar{f}(x) = f(x), \quad \forall x \in N.$$

Moreover, this holomorphic extension is unique in the following sense: if \bar{f} is a holomorphic extension of f to \bar{U} containing N and \bar{g} is another holomorphic extension of f to \bar{W} containing N , then there exists $\bar{V} \subseteq \bar{W} \cap \bar{U}$ such that we have

$$\bar{f}(x) = \bar{g}(x), \quad \forall x \in \bar{V}.$$

We call the holomorphic function \bar{f} a **holomorphic extension** of f .

What happens if we start with a real analytic manifold M ? It can be shown that, for every real analytic manifold M , there exists a complex manifold $M^{\mathbb{C}}$ which contains M as a totally real submanifold [83].

Theorem 3.2.13. *Let M be a real analytic manifold. There exists a complex manifold $M^{\mathbb{C}}$ such that $\dim_{\mathbb{C}} M^{\mathbb{C}} = \dim_{\mathbb{R}} M$ and M is a totally real submanifold of $M^{\mathbb{C}}$.*

The complex manifold $M^{\mathbb{C}}$ is called a **complexification** of the real analytic manifold M . Grauert showed that $M^{\mathbb{C}}$ can be chosen to be a Stein manifold [27, §3]. Moreover, he showed that, given any neighbourhood \bar{U} of M in $M^{\mathbb{C}}$, there exists a Stein neighbourhood of M inside \bar{U} [27, §3].

Theorem 3.2.14. *Let M be a real analytic manifold. There exists a Stein manifold $M^{\mathbb{C}}$ such that $\dim_{\mathbb{C}} M^{\mathbb{C}} = \dim_{\mathbb{R}} M$, and M is a totally real submanifold of $M^{\mathbb{C}}$. Moreover, for every neighbourhood \bar{U} of M in $M^{\mathbb{C}}$, there exists a Stein neighbourhood \bar{S} of M in $M^{\mathbb{C}}$ such that $\bar{S} \subseteq \bar{U}$.*

One can similarly show that for a real analytic vector bundle (E, M, π) , there exists a holomorphic vector bundle $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$ which contains (E, M, π) as a totally real subbundle.

Theorem 3.2.15. *Let (E, M, π) be a real analytic vector bundle. There exists a holomorphic vector bundle $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$, where $E^{\mathbb{C}}$ and $M^{\mathbb{C}}$ are complexifications of E and M respectively. Moreover, (E, M, π) is a totally real subbundle of $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$.*

Proof. The proof of this theorem is similar to the proof given in [83] for complexification of manifolds. \square

The vector bundle $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$ is called a **complexification** of the real analytic vector bundle (E, M, π) . One can show that every real analytic section on (E, M, π) can be extended to a holomorphic section on $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$.

Theorem 3.2.16. *Let (E, M, π) be a real analytic vector bundle, $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$ be a complexification of (E, M, π) , and $\sigma : M \rightarrow E$ be a real analytic section on (E, M, π) . Then there exists a neighbourhood $\bar{U} \subseteq M^{\mathbb{C}}$ containing M and a holomorphic section $\bar{\sigma} : \bar{U} \rightarrow E^{\mathbb{C}}$ such that*

$$\sigma(x) = \bar{\sigma}(x), \quad \forall x \in M.$$

Proof. Since E is a totally real submanifold of $E^{\mathbb{C}}$, there exists a real analytic map $i : E \rightarrow E^{\mathbb{C}}$. Thus, the composition $i \circ \sigma : M \rightarrow E^{\mathbb{C}}$ is a real analytic map. By Theorem 3.2.12, there exists a neighbourhood $\bar{V} \subseteq M^{\mathbb{C}}$ of M and a holomorphic map $\bar{\sigma} : \bar{V} \rightarrow E^{\mathbb{C}}$ such that

$$\bar{\sigma}(x) = i \circ \sigma(x) = \sigma(x), \quad \forall x \in M.$$

Note that for the map $\pi^{\mathbb{C}} \circ \bar{\sigma} : \bar{V} \rightarrow M^{\mathbb{C}}$, we have

$$\pi^{\mathbb{C}} \circ \bar{\sigma}(x) = \pi^{\mathbb{C}}(\sigma(x)) = \pi \circ \sigma(x) = x, \quad \forall x \in M.$$

Therefore the map $\pi^{\mathbb{C}} \circ \bar{\sigma}$ is the identity on M . Thus, by the uniqueness part of Theorem 3.2.12, there exists a neighbourhood $\bar{U} \subseteq \bar{V}$ of M such that

$$\pi^{\mathbb{C}} \circ \bar{\sigma}(z) = z, \quad \forall z \in \bar{U}.$$

This implies that $\pi^{\mathbb{C}} \circ \bar{\sigma}|_{\bar{U}} = \text{id}_{\bar{U}}$. This means that $\bar{\sigma}|_{\bar{U}}: \bar{U} \rightarrow E^{\mathbb{C}}$ is a holomorphic section of $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$ such that

$$\bar{\sigma}|_{\bar{U}}(x) = \sigma(x), \quad \forall x \in M.$$

This completes the proof. \square

The section $\bar{\sigma}$ is called a **holomorphic extension** of σ .

We are now able to define two topologies on the space of real analytic sections. The definition of these topologies relies on the following characterization of the vector space $\Gamma^{\omega}(E)$ by the space of holomorphic sections.

Definition 3.2.17. Let (E, M, π) be a real analytic vector bundle and $(E^{\mathbb{C}}, M^{\mathbb{C}}, \pi^{\mathbb{C}})$ be a complexification of (E, M, π) . We define $\Gamma^{\text{hol}, \mathbb{R}}(E^{\mathbb{C}}) \subseteq \Gamma^{\text{hol}}(E^{\mathbb{C}})$ as

$$\Gamma^{\text{hol}, \mathbb{R}}(E^{\mathbb{C}}) = \{X \in \Gamma^{\text{hol}}(E^{\mathbb{C}}) \mid X(x) \in E_x, \quad \forall x \in M\}.$$

Theorem 3.2.18. *The space $\Gamma^{\text{hol}, \mathbb{R}}(E^{\mathbb{C}})$ with the C^{hol} -topology is Hausdorff, separable, complete, nuclear, and metrizable. Moreover, it satisfies the Heine–Borel property.*

Proof. In view of Theorem 3.2.4, it suffices to show that $\Gamma^{\text{hol}, \mathbb{R}}(E^{\mathbb{C}})$ is a closed subspace of $\Gamma^{\text{hol}}(E^{\mathbb{C}})$. However, this is clear from the definition and the fact that E_x is closed in $E_x^{\mathbb{C}}$. \square

Let $A \subseteq M$. We denote the set of all neighbourhoods of A in $M^{\mathbb{C}}$ by \mathcal{N}_A . For every $\bar{U}_A, \bar{V}_A \in \mathcal{N}_A$ with $\bar{U}_A \subseteq \bar{V}_A$, we define the map $i_{\bar{U}_A, \bar{V}_A}^{\mathbb{R}}: \Gamma^{\text{hol}, \mathbb{R}}(\bar{V}_A) \rightarrow \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_A)$ as

$$i_{\bar{U}_A, \bar{V}_A}^{\mathbb{R}}(X) = X|_{\bar{U}_A}, \quad \forall X \in \Gamma^{\text{hol}, \mathbb{R}}(\bar{V}_A).$$

The pair $(\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_A), i_{\bar{U}_A, \bar{V}_A}^{\mathbb{R}})_{\bar{U}_A \subseteq \bar{V}_A}$ is a directed system. The direct limit of this system is $(\mathcal{G}_A^{\text{hol}, \mathbb{R}}, i_{\bar{U}_A}^{\mathbb{R}})_{\bar{U}_A \in \mathcal{N}_A}$. One can define a topology on $\mathcal{G}_A^{\text{hol}, \mathbb{R}}$ as the finest locally convex topology which makes all the maps $i_{\bar{U}_A}^{\mathbb{R}}: \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_A) \rightarrow \mathcal{G}_A^{\text{hol}, \mathbb{R}}$ continuous.

Theorem 3.2.19. *Let M be a real analytic manifold. The vector space $\Gamma^{\omega}(M)$ is isomorphic to the vector space $\mathcal{G}_M^{\text{hol}, \mathbb{R}}$.*

Proof. We just need to find a linear bijective map between these two spaces. Let us define the map $i_M: \Gamma^{\omega}(E) \rightarrow \mathcal{G}_M^{\text{hol}, \mathbb{R}}$ as

$$i_M(X) = [\bar{X}]_M, \quad \forall X \in \Gamma^{\omega}(E),$$

where \bar{X} is a holomorphic extension of X to a neighbourhood of M . The linearity of i_M is clear. By existence and uniqueness of the holomorphic extension (Theorem 3.2.12), i_M is well-defined and injective. To prove surjectivity, note that, by definition of $\mathcal{G}_M^{\text{hol}, \mathbb{R}}$, for every $h \in \mathcal{G}_M^{\text{hol}, \mathbb{R}}$, there exists a neighbourhood \bar{U} containing M and a

holomorphic section $H \in \Gamma^{\text{hol},\mathbb{R}}(\bar{U})$ such that $[H]_M = h$. Now, if we define $g : M \rightarrow E$ as

$$g(x) = H(x), \quad \forall x \in M,$$

then, by Theorem 2.1.5, it is clear that $g \in \Gamma^\omega(E)$. Moreover, H is a holomorphic extension of g . So we have $i_M(g) = [H]_M = h$. This completes the proof of the theorem. \square

Theorem 3.2.19 shows that the space of real analytic sections on a real analytic manifold M is isomorphic with the space of germs of holomorphic sections on complex neighbourhoods of M . We first define an inductive topology on the space of germs of holomorphic sections around an arbitrary set $A \subseteq M^{\mathbb{C}}$. Then using Theorem 3.2.19, we induce a topology on $\Gamma^\omega(E)$ [59].

Definition 3.2.20. Let $A \subseteq M$ be a set, $M^{\mathbb{C}}$ be a Stein manifold, and \mathcal{N}_A be the family of neighbourhoods of A in $M^{\mathbb{C}}$. The **inductive topology** on $\mathcal{G}_A^{\text{hol},\mathbb{R}}$ is defined as the finest locally convex topology which makes all the maps $\{i_{\bar{U}}^{\mathbb{R}}\}_{\bar{U} \in \mathcal{N}_A}$ continuous.

Definition 3.2.21. Let $\bar{U} \subseteq M^{\mathbb{C}}$ be an open neighbourhood of M . We define the map $p_{\bar{U}} : \Gamma^{\text{hol},\mathbb{R}}(\bar{U}) \rightarrow [0, \infty]$ as

$$p_{\bar{U}}(X) = \sup\{\|X(x)\| \mid x \in U\}.$$

Then $\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U})$ is the subspace of $\Gamma^{\text{hol},\mathbb{R}}(\bar{U})$ defined as

$$\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}) = \{X \in \Gamma^{\text{hol},\mathbb{R}}(\bar{U}) \mid p_{\bar{U}}(X) < \infty\}.$$

We equip $\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U})$ with the norm $p_{\bar{U}}$ and define the inclusion $\rho_{\bar{U}}^{\mathbb{R}} : \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}) \rightarrow \Gamma^{\text{hol},\mathbb{R}}(\bar{U})$ as

$$\rho_{\bar{U}}^{\mathbb{R}}(X) = X, \quad \forall X \in \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}).$$

Similar to the Theorem 3.2.6, one can show the following theorem.

Theorem 3.2.22. *The space $(\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}), p_{\bar{U}})$ is a Banach space and the map $\rho_{\bar{U}}^{\mathbb{R}} : \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}) \rightarrow \Gamma^{\text{hol},\mathbb{R}}(\bar{U})$ is a compact continuous map.*

Similar to Theorem 3.2.7, the following result holds.

Theorem 3.2.23. *Let K be a compact set in M and $\{\bar{U}_n\}_{n \in \mathbb{N}}$ be a sequence of neighbourhoods of K in $M^{\mathbb{C}}$ such that*

$$\text{cl}(\bar{U}_{n+1}) = \bar{U}_n, \quad \forall n \in \mathbb{N},$$

and $\bigcap_{n \in \mathbb{N}} \bar{U}_n = K$. Then we have $\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{U}_n) = \mathcal{G}_K^{\text{hol},\mathbb{R}}$. Moreover, the inductive limit is compact and as a result the final topology on $\mathcal{G}_K^{\text{hol},\mathbb{R}}$ and the locally convex topology on $\mathcal{G}_K^{\text{hol},\mathbb{R}}$ coincide.

Using Theorem 3.2.19, one can define the inductive topology on $\Gamma^\omega(E)$. Let M be a real analytic manifold, $M^{\mathbb{C}}$ be a complexification of M , and \mathcal{N}_M be the family of all neighbourhoods of M in $M^{\mathbb{C}}$. Let $\bar{U} \in \mathcal{N}_M$ and let $i_M : \Gamma^\omega(E) \rightarrow \Gamma_M^{\text{hol},\mathbb{R}}$ be defined as

$$i_M(X) = [\bar{X}]_M, \quad \forall X \in \Gamma^\omega(E).$$

Theorem 3.2.19 shows that i_M is a bijective linear map. So, for every $\bar{U} \subseteq \mathcal{N}_M$, one can define $i_{\bar{U},M} : \Gamma^{\text{hol},\mathbb{R}}(\bar{U}) \rightarrow \Gamma^\omega(E)$ as

$$i_{\bar{U},M} = i_M^{-1} \circ i_{\bar{U}}^{\mathbb{R}}.$$

Definition 3.2.24. The **inductive topology** on $\Gamma^\omega(E)$ is defined as the finest locally convex topology which makes all the maps $\{i_{\bar{U},M}\}_{\bar{U} \subseteq \mathcal{N}_M}$ continuous.

Although the definition of inductive topology on $C^\omega(M)$ is natural, characterization of properties of $\Gamma^\omega(E)$ using this topology is not easy. The main reason is that, for non-compact M , the inductive limit $\varinjlim_{\bar{U} \in \mathcal{N}_M} \Gamma^{\text{hol},\mathbb{R}}(\bar{U}) = \Gamma^\omega(E)$ is not necessarily countable [23, Fact 14]. However, one can define another topology on the space of real analytic sections which is representable by countable inductive and projective limits [59]. Let M be a real analytic manifold and let A be a subset of M . Then we define the projection $j_A : \Gamma^\omega(E) \rightarrow \mathcal{G}_A^{\text{hol},\mathbb{R}}$ as

$$j_A(X) = [\bar{X}]_A,$$

where \bar{X} is a holomorphic extension of $X \in \Gamma^\omega(E)$ to a complex neighbourhood $\bar{U} \subseteq M^{\mathbb{C}}$ of A .

Theorem 3.2.25. *Let $A \subseteq M$. Then the map j_A is well-defined and linear.*

Proof. Let \bar{X} and \bar{Y} be two holomorphic extensions of X on neighbourhoods \bar{U} and \bar{V} , respectively. Then, by existence and uniqueness of holomorphic extension (Theorem 3.2.12), there exists an open set $\bar{W} \subseteq \bar{U} \cap \bar{V}$ such that $M \subseteq \bar{W}$ and

$$\bar{X}(x) = \bar{Y}(x), \quad \forall x \in \bar{W}.$$

So we have $[\bar{X}]_A = [\bar{Y}]_A$. This implies that the map p_A is well-defined. Let $X, Y \in \Gamma^\omega(E)$ and $\alpha \in \mathbb{R}$. Then there exist neighbourhoods \bar{U} and \bar{V} , and holomorphic sections $\bar{X} \in \Gamma^{\text{hol},\mathbb{R}}(\bar{U})$ and $\bar{Y} \in \Gamma^{\text{hol},\mathbb{R}}(\bar{V})$ which are holomorphic extensions of X and Y respectively. Let $\bar{W} = \bar{U} \cap \bar{V}$. Then we define $\bar{h} : \Gamma^{\text{hol},\mathbb{R}}(\bar{W})$ as

$$\bar{h}(x) = \bar{X}(x) + \alpha \bar{Y}(x), \quad \forall x \in \bar{W}.$$

It is clear that we have

$$j_A(X + \alpha Y) = [\bar{h}]_A = [\bar{X} + \alpha \bar{Y}]_A = [\bar{X}]_A + \alpha [\bar{Y}]_A = j_A(X) + \alpha j_A(Y).$$

□

Definition 3.2.26. Let M be a real analytic manifold and let $\{K_n\}_{n \in \mathbb{N}}$ be a compact exhaustion for M . The **projective topology** on the space of real analytic sections $\Gamma^\omega(E)$ is defined as the coarsest topology which makes all the projections $j_{K_n} : \Gamma^\omega(E) \rightarrow \mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}}$ continuous.

It is easy to see that the projective topology does not depend on the specific choice of compact exhaustion $\{K_n\}_{n \in \mathbb{N}}$. One can also show that the inductive topology and projective topology on $\Gamma^\omega(E)$ are the same [59, Theorem 1.2(a)].

Theorem 3.2.27. *The inductive and projective topology on $\Gamma^\omega(E)$ coincide.*

We denote this topology on the space $\Gamma^\omega(E)$ by the C^ω -**topology**.

Theorem 3.2.28. *The space $\Gamma^\omega(E)$ with the C^ω -topology is Hausdorff, separable, complete, and nuclear. It also satisfies Heine–Borel property.*

Proof. Let $\{K_n\}_{n \in \mathbb{N}}$ be a compact exhaustion for M . Note that, by theorem 3.2.22, the inductive limit

$$\varinjlim \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_{K_n}) = \mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}}$$

is compact. Note that, for every $n \in \mathbb{N}$, the vector space $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_{K_n})$ is a complete locally convex space. This implies that, for every $n \in \mathbb{N}$, the vector space $\mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}}$ is complete [7, Theorem 4]. Since $\Gamma^\omega(E)$ is the projective limit of the family $\{\mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}}\}_{n \in \mathbb{N}}$, it is a complete locally convex vector space [69, Chapter II, §5.3]. By [51, §30.4], for every $\overline{U}_M \subseteq M^{\mathbb{C}}$ a neighbourhood of M , the vector space $\Gamma^{\text{hol}}(\overline{U}_M)$ is nuclear. Therefore $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_M)$ is nuclear because $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_M)$ is a closed subspace of $\Gamma^{\text{hol}}(\overline{U}_M)$ [69, Chapter II, Theorem 7.4]. According to [69, Chapter II, Theorem 7.4], the inductive limit of a family of nuclear spaces is nuclear. So $\Gamma^\omega(E)$ is nuclear. The fact that $\Gamma^\omega(E)$ is nuclear implies that it satisfies Heine–Borel property. Finally, the fact that $\Gamma^\omega(E)$ is separable has been shown in [23, Theorem 16]. \square

Every locally convex topology can be represented by a family of generating seminorms. So, it would be useful to find families of generating seminorms for the C^ω -topology on $\Gamma^\omega(E)$. In this section, we construct two families of seminorms that generates the C^ω -topology on $\Gamma^\omega(E)$.

Let $C \subseteq M$ be a compact set and \mathcal{A} be a C^ω -atlas on the manifold M . For every coordinate chart $(U, \phi) \in \mathcal{A}$, every local trivialization $\eta : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$, every compact set $K \subseteq U$, and every $\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$, we define the seminorm $p_{K, \mathbf{a}, \phi, \eta}$ on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$ as

$$p_{K, \mathbf{a}, \phi}([X]_C) = \sup \left\{ \frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} X(x)\|_{(U, \phi, \eta)} \mid x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}.$$

Theorem 3.2.29. *Let M be a real analytic manifold, $C \subseteq M$ be a compact set and \mathcal{A} be a C^ω -atlas on M . The family of seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ generates the inductive limit topology on the space $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$.*

Proof. We first show that the topology induced by these seminorms on $\mathcal{G}_C^{\text{hol},\mathbb{R}}$ is independent of the C^ω -atlas \mathcal{A} . Let \mathcal{A}_1 and \mathcal{A}_2 be two equivalent C^ω -atlas on M . Let $(U, \phi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$ be such that $U \cap V \neq \emptyset$. Let $\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$ and $K \subseteq U \cap V$. By Sublemma 3 from the proof of Lemma 5.2 in [42], there exist $C, \sigma > 0$ such that

$$\begin{aligned} & \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \phi^{-1})(x)\| \mid |r| \leq m \right\} \\ & \leq C(\sigma)^m \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \psi^{-1})(\psi \circ \phi^{-1}(x))\| \mid |r| \leq m \right\} \end{aligned}$$

for every $[X]_C \in \mathcal{G}_C^{\text{hol},\mathbb{R}}$ and every $x \in \phi(K)$. Now if we take the supremum over the compact set $\phi(K)$, for every $X \in \mathcal{G}_C^{\text{hol},\mathbb{R}}$, we get

$$\begin{aligned} & \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \phi^{-1})(x)\| \mid |r| \leq m, x \in \phi(K) \right\} \\ & \leq C(\sigma)^m \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \psi)(x)\| \mid |r| \leq m, x \in \psi(K) \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \phi^{-1})(x)\| \mid |r| \leq m, x \in \phi(K) \right\} \\ & \leq C(\sigma)^m \sup \left\{ \frac{1}{(r)!} \|D^{(r)}(\eta \circ X \circ \psi)(x)\| \mid |r| \leq m, x \in \psi(K) \right\}. \end{aligned}$$

Multiplying both side of the equality by $a_0 a_1 \dots a_m$, we get

$$\begin{aligned} & \sup \left\{ \frac{a_0 a_1 \dots a_m}{(r)!} \|D^{(r)}(\eta \circ X \circ \phi^{-1})(x)\| \mid |r| \leq m, x \in \phi(K) \right\} \\ & \leq C \sup \left\{ \frac{(\sigma a_0)(\sigma a_1) \dots (\sigma a_m)}{(r)!} \|D^{(r)}(\eta \circ X \circ \psi^{-1})(x)\| \mid |r| \leq m, x \in \psi(K) \right\}. \end{aligned}$$

Taking supremum over all multi-indices, we have

$$p_{K, \mathbf{a}, \phi, \eta}(X) \leq p_{\psi(K), \sigma \mathbf{a}, \psi, \eta}(X), \quad \forall X \in \mathcal{G}_C^{\text{hol},\mathbb{R}}.$$

Now assume that $K \subseteq U$. Then, for every $x \in K$, there exists a chart $(V_x, \phi_x) \in \mathcal{A}_2$ such that $x \in V_x$. Consider a compact set $K_x \subseteq V_x$ such that $x \in K_x$ and $\text{int}(K_x) \neq \emptyset$. Since K is compact, there exists a finite collection of points $x_1, x_2, \dots, x_n \in K$ such that

$$\bigcup_{i=1}^n \text{int}(K_{x_i}) = K.$$

This implies that

$$K \subseteq \bigcup_{i=1}^n K_{x_i}.$$

Note that, for every $i \in \{1, 2, \dots, n\}$, we have $K_x \in U \cap V_x$. Therefore, for every $i \in \{1, 2, \dots, n\}$, there exist $\sigma_i, C_i > 0$ such that

$$p_{K_{x_i}, \mathbf{a}, \phi, \eta}(X) \leq p_{\psi_{x_i}(K_{x_i}), \sigma \mathbf{a}, \psi_{x_i}, \eta}(X), \quad \forall X \in \mathcal{G}_C^{\text{hol}, \mathbb{R}}.$$

Thus, we have

$$p_{K, \mathbf{a}, \phi, \eta}(X) \leq \sum_{i=1}^n p_{K_{x_i}, \mathbf{a}, \phi, \eta}(X) \leq \sum_{i=1}^n p_{\psi_{x_i}(K_{x_i}), \sigma \mathbf{a}, \psi_{x_i}, \eta}(f), \quad \forall X \in \mathcal{G}_C^{\text{hol}, \mathbb{R}}.$$

This shows that the topology generated by the family of seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ for $(U, \phi) \in \mathcal{A}_1$ is coarser than the topology generated by the family of seminorms $\{p_{K, \mathbf{a}, \psi, \eta}\}$ for $(V, \psi) \in \mathcal{A}_2$. Similarly, we can show that topology generated by the family of seminorms $\{p_{K, \mathbf{a}, \psi, \eta}\}$ for $(V, \psi) \in \mathcal{A}_2$ is coarser than the topology generated by the family of seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ for $(U, \phi) \in \mathcal{A}_1$. This shows that the two topologies are the same. Now one can fix a C^ω -atlas \mathcal{A} on M and consider the seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ for $(U, \phi) \in \mathcal{A}$. The fact that the seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ generates the topology on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$ has been proved in [81]. \square

Remark 3.2.30. Since the topology generated by the family of seminorms $\{p_{K, \mathbf{a}, \phi, \eta}\}$ does not depend on the C^ω -atlas on M , one can fix a C^ω -atlas on M . Unless it is explicitly mentioned, we assume that a coordinate chart (U, ϕ) for M and a local trivialization (E, M, π) is attached to K . Therefore we can usually denote the seminorm $p_{K, \mathbf{a}, \phi, \eta}$ by $p_{K, \mathbf{a}}$ without any confusion, assuming that the choice of the coordinate chart and local trivialization is clear from the context.

One can show that the set $\{p_{K, \mathbf{a}}\}_{\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{> 0})}$ is uncountable. Although this family of seminorms generates the topology on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$, it doesn't mean that one needs every seminorms in the family $\{p_{K, \mathbf{a}}\}$ to generate the topology on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$. In fact, the topology on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$ can be generated by a much smaller subfamily. In the next theorem, we choose a specific subfamily of $\{p_{K, \mathbf{a}}\}$, which turns out to be useful for our future computations.

Let $d > 0$ be a positive real number. We define $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{> 0}, d)$ as the subset of $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{> 0})$ given by

$$\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{> 0}, d) = \{\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{> 0}) \mid a_m \leq d, \quad \forall m \in \mathbb{Z}_{\geq 0}\}.$$

Theorem 3.2.31. *Let $C \subseteq M$ be a compact set. Then the family of seminorms $\{p_{K, \mathbf{a}}\}$, $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{> 0}, d)$, generates the inductive limit topology on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$.*

Proof. Let $\mathbf{b} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Then we define $\mathbf{b}' = (b'_0, b'_1, b'_2, \dots) \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}, d)$ as

$$b'_i = \min\{d, b_i\}$$

Since $\{b_i\}_{i \in \mathbb{N}}$ converges to zero, there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} b'_i &= b_i, & \forall i > m, \\ b'_i &= d, & \forall i \leq m. \end{aligned}$$

This implies that, for every multi-index (r) with $|r| \leq m$, we have

$$\frac{b'_0 b'_1 \dots b'_{|r|}}{(r)!} = \frac{d^{|r|}}{(r)!} \geq \left(\frac{d^m}{b_0 b_1 \dots b_m} \right) \frac{b_0 b_1 \dots b_{|r|}}{(r)!},$$

and, for every multi-index (r) with $|r| > m$, we have

$$\frac{b'_0 b'_1 \dots b'_{|r|}}{(r)!} = \frac{d^m b'_{m+1} b'_{m+2} \dots b'_{|r|}}{(r)!} = \frac{d^m b_{m+1} b_{m+2} \dots b_{|r|}}{(r)!} = \left(\frac{d^m}{b_0 b_1 \dots b_m} \right) \frac{b_0 b_1 \dots b_{|r|}}{(r)!}.$$

So, for every compact set $K \subseteq U$, we have

$$p_{K, \mathbf{b}}([X]_C) \leq \left(\frac{b_0 b_1 \dots b_m}{d^m} \right) p_{K, \mathbf{b}'}([X]_C) \quad \forall [X]_C \in \mathcal{G}_C^{\text{hol}, \mathbb{R}}.$$

□

In many applications it is more convenient to work with another family of seminorms on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$.

Definition 3.2.32. Let \mathcal{A} be an atlas on M , $(U, \phi) \in \mathcal{A}$ be a coordinate chart on M , $\eta : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ is a local trivialization for (E, M, π) , $K \subseteq U$ be a compact set, and $\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$. Then we define the seminorm $\tilde{p}_{K, \mathbf{a}}$ on $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$ as

$$\tilde{p}_{K, \mathbf{a}}([X]_C) = \sup \left\{ \frac{a_0 a_1 \dots a_{|r|}}{|r|!} \|D^{(r)}(X)(x)\|_{(U, \phi, \eta)} \mid x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}.$$

Theorem 3.2.33. *The family of seminorms $\{\tilde{p}_{K, \mathbf{a}}\}$ generates the inductive limit topology on the space $\mathcal{G}_C^{\text{hol}, \mathbb{R}}$.*

Proof. Note that for every multi-index (r) , we have $(r)! \leq |r|!$. This implies that

$$\frac{a_0 a_1 \dots a_{|r|}}{|r|!} \|D^{(r)} X(x)\|_{(U, \phi, \eta)} \leq \frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} X(x)\|_{(U, \phi, \eta)}$$

So we have

$$\tilde{p}_{K, \mathbf{a}}(X) \leq p_{K, \mathbf{a}}(X), \quad \forall \mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}).$$

So the topology generated by the family of seminorms $\{\tilde{p}_{K,\mathbf{a}}\}$ is finer than the inductive limit topology. Moreover, we have

$$\begin{aligned} \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)} &= \frac{|r|!}{(r)!} \frac{a_0 a_1 \cdots a_{|r|}}{|r|!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)} \\ &\leq N^{|r|} \frac{a_0 a_1 \cdots a_{|r|}}{|r|!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)} = \frac{(Na_0)(Na_1) \cdots (Na_{|r|})}{|r|!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)}. \end{aligned}$$

Thus we get

$$p_{K,\mathbf{a}}(X) \leq \tilde{p}_{K,N\mathbf{a}}(X), \quad \forall \mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}).$$

This shows that the topology generated by the family of seminorms $\{\tilde{p}_{K,\mathbf{a}}\}$ is coarser than the inductive limit topology. This completes the proof. \square

Using families of seminorms $\{p_{K,\mathbf{a}}\}$ and $\{\tilde{p}_{K,\mathbf{a}}\}$ on $\mathcal{G}_C^{\text{hol},\mathbb{R}}$, one can easily define two families of seminorms on $\Gamma^\omega(E)$. Let (U, ϕ) be a coordinate chart on M , $\eta : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ be a local trivialization for (E, M, π) , $K \subseteq U$ be a compact set, $d > 0$ be a positive real number, and $\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then we define the seminorm $p_{K,\mathbf{a}}^\omega$ as

$$p_{K,\mathbf{a}}^\omega(X) = \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)} \mid x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}.$$

Similarly, we define

$$\tilde{p}_{K,\mathbf{a}}^\omega(X) = \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{|r|!} \|D^{(r)} X(x)\|_{(U,\phi,\eta)} \mid x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}.$$

One can show that each of the families $\{p_{K,\mathbf{a}}^\omega\}$ and $\{\tilde{p}_{K,\mathbf{a}}^\omega\}$ is a generating family of seminorms for the C^ω -topology on the space $\Gamma^\omega(E)$.

Theorem 3.2.34. *Let $d > 0$ be a positive real number, K be a compact set in a coordinate neighbourhood on M , and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then the family of seminorms $\{p_{K,\mathbf{a}}^\omega\}$ generates the C^ω -topology on the space $\Gamma^\omega(E)$.*

Proof. Let (U, ϕ) be a coordinate chart on M . Then, for all compact sets C and K such that $K \subseteq C \subseteq U$ and every $\mathbf{a} \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we have

$$p_{K,\mathbf{a}}^\omega = p_{K,\mathbf{a}} \circ j_C.$$

So $p_{K,\mathbf{a}}^\omega$ is a continuous seminorm on C^ω -topology on $\Gamma^\omega(E)$. This means that the topology generated by the family of seminorms $\{p_{K,\mathbf{a}}^\omega\}$ on $\Gamma^\omega(E)$ is coarser than the C^ω -topology. Let $\{K_n\}_{n \in \mathbb{N}}$ be a compact exhaustion for the manifold M . We have $\varprojlim \mathcal{G}_{K_n}^{\text{hol},\mathbb{R}} = \Gamma^\omega(E)$. So, for every open set $W \subset \Gamma^\omega(E)$, there exists $m \in \mathbb{N}$ and $W_i \in \mathcal{G}_{K_i}^{\text{hol},\mathbb{R}}$ for all $i \in \{1, 2, \dots, m\}$, such that

$$\bigcap_{i=1}^m j_{K_i}^{-1}(W_i) \subseteq W.$$

Note that, for every $n \in \mathbb{N}$, the family of seminorms $\{p_{K,\mathbf{a}}\}$ generates the topology on $\mathcal{G}_{K_n}^{\text{hol},\mathbb{R}}$. Therefore, there exist compact sets C_1, C_2, \dots, C_m such that $C_i \subseteq K_i$ for every $i \in \{1, 2, \dots, m\}$ and sequences $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ such that $\mathbf{a}_i \in \mathbf{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ for every $i \in \{1, 2, \dots, m\}$, and we have

$$p_{C_i, \mathbf{a}_i}^{-1}([0, 1]) \subseteq W_i, \quad \forall i \in \{1, 2, \dots, m\}.$$

This implies that

$$\bigcap_{i=1}^m j_{K_i}^{-1}(p_{C_i, \mathbf{a}_i}^{-1}([0, 1])) = \bigcap_{i=1}^m (p_{C_i, \mathbf{a}_i}^\omega)^{-1}([0, 1]) \subseteq W.$$

This means that the topology generated by family of seminorms $\{p_{K,\mathbf{a}}^\omega\}$ is finer than the C^ω -topology on $\Gamma^\omega(E)$. \square

Theorem 3.2.35. *Let $d > 0$ be a positive real number, K be a compact set in a coordinate neighbourhood on M , and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then the family of seminorms $\{\tilde{p}_{K,\mathbf{a}}^\omega\}$ generates the C^ω -topology on the space $\Gamma^\omega(E)$.*

Proof. The proof is similar to the proof of Theorem 3.2.34. \square

Let M be a C^ν -manifold. For the C^ν -vector bundle $(M \times \mathbb{R}, M, \text{pr}_1)$, we know that C^ν -sections of $(M \times \mathbb{R}, M, \text{pr}_1)$ are exactly C^ν -functions of M . Therefore, using the above analysis, we can study the topology on the space $C^\nu(M)$. In particular, we can define a family of seminorms on $C^\nu(M)$. However, the space of C^ν -functions has an extra algebra structure that the space of C^ν -sections doesn't have. One can define a multiplication $\cdot : C^\nu(M) \times C^\nu(M) \rightarrow C^\nu(M)$ on the vector space $C^\nu(M)$ by

$$(f \cdot g)(x) = f(x)g(x), \quad \forall x \in M.$$

This multiplication make the vector space $C^\nu(M)$ into an \mathbb{F} -algebra. It is interesting that this algebra structure on $C^\nu(M)$ is consistent with the C^ν -topology on it.

Theorem 3.2.36. *The algebra multiplication $\cdot : C^\nu(M) \times C^\nu(M) \rightarrow C^\nu(M)$ is continuous.*

Proof. The proof consists of three different cases:

1. ($\nu = \infty$):

Let (U, ϕ) be a coordinate chart on M and $K \subseteq U$ be a compact set. Note that, for all multi-indices (r) with $|r| \leq m$, we have

$$D^{(r)}(fg)(x) = \sum_{(s) \leq (r)} \binom{(r)}{(s)} D^{(r-s)}f(x)D^{(s)}g(x), \quad \forall x \in U.$$

So we have

$$\|D^{(r)}(fg)(x)\| \leq \sum_{(s) \leq (r)} \binom{(r)}{(s)} \|D^{(r-s)}f(x)\| \|D^{(s)}g(x)\|, \quad \forall x \in U.$$

Note that, for all multi-indices $(s), (r)$ with $(s) \leq (r)$, we have

$$\|D^{(r-s)}f(x)\| \leq \sup \{ \|D^{(r)}f(x)\| \mid |r| \leq m \}, \quad \forall x \in U,$$

and

$$\|D^{(s)}g(x)\| \leq \sup \{ \|D^{(r)}g(x)\| \mid |r| \leq m \}, \quad \forall x \in U.$$

This implies that

$$\begin{aligned} & \|D^{(r)}(fg)(x)\| \\ & \leq 2^{|r|} \sup \{ \|D^{(r)}f(x)\| \mid |r| \leq m \} \sup \{ \|D^{(r)}g(x)\| \mid |r| \leq m \}, \quad \forall x \in U. \end{aligned}$$

Taking the sup of the left hand side of the above inequality over $x \in K$ and (r) with $|r| \leq m$, we get

$$\begin{aligned} p_{K,m}^\infty(fg) &= \sup \{ \|D^{(r)}(fg)(x)\| \mid x \in K, |r| \leq m \} \\ &\leq 2^m \sup \{ \|D^{(r)}f(x)\| \mid |r| \leq m \} \sup \{ \|D^{(r)}g(x)\| \mid |r| \leq m \} \\ &\leq 2^m p_{K,m}^\infty(f) p_{K,m}^\infty(g). \end{aligned}$$

2. ($\nu = \text{hol}$):

Note that, for every compact set $K \subseteq M$, we have

$$\begin{aligned} p_K^{\text{hol}}(fg) &= \sup \{ \|f(x)g(x)\| \mid x \in K \} \\ &\leq \sup \{ \|f(x)\| \mid x \in K \} \sup \{ \|g(x)\| \mid x \in K \} = p_K^{\text{hol}}(f) p_K^{\text{hol}}(g). \end{aligned}$$

3. ($\nu = \omega$):

Let (U, ϕ) be a coordinate chart on M , $K \subseteq U$ be a compact set, and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Note that, for every multi-index (r) , we have

$$D^{(r)}(fg)(x) = \sum_{s \leq r} D^{(r-s)}f(x) D^{(s)}g(x), \quad \forall x \in U.$$

Since the sequence $\{a_i\}_{i \in \mathbb{N}}$ is decreasing, we have

$$\begin{aligned} & \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(fg)(x)\| \\ & \leq \sum_{s \leq r} \frac{a_0 a_1 \cdots a_{|r-s|}}{(r-s)!} \|D^{(r-s)}f(x)\| \frac{a_0 a_1 \cdots a_{|s|}}{(s)!} \|D^{(s)}g(x)\|, \quad \forall x \in U. \end{aligned}$$

Note that, for every multi-index (s) such that $(s) \leq (r)$, we get

$$\begin{aligned} & \frac{a_0 a_1 \cdots a_{|r-s|}}{(r-s)!} \|D^{(r-s)}(f)(x)\| \\ & \leq \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(f)(x)\| \mid |r| \in \mathbb{Z}_{\geq 0} \right\}, \quad \forall x \in U, \end{aligned}$$

and

$$\begin{aligned} & \frac{a_0 a_1 \cdots a_{|s|}}{(s)!} \|D^{(s)}(g)(x)\| \\ & \leq \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(g)(x)\| \mid |r| \in \mathbb{Z}_{\geq 0} \right\}, \quad \forall x \in U. \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(fg)(x)\| \leq 2^{|r|} \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(f)(x)\| \mid |r| \in \mathbb{Z}_{\geq 0} \right\} \\ & \quad \times \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{(r)!} \|D^{(r)}(g)(x)\| \mid |r| \in \mathbb{Z}_{\geq 0} \right\}, \quad \forall x \in U. \end{aligned}$$

Taking the sup over $x \in K$ and $|r| \in \mathbb{Z}_{\geq 0}$, we get

$$p_{K,\mathbf{a}}^\omega(fg) \leq p_{K,\sqrt{2}\mathbf{a}}^\omega(f) p_{K,\sqrt{2}\mathbf{a}}^\omega(g).$$

□

3.3. C^ν -vector fields as derivations on $C^\nu(M)$

In this section, we study C^ν -vector fields on M as derivations on the \mathbb{F} -algebra $C^\nu(M)$. For a smooth manifold M , it is a well-known fact that every derivation on the \mathbb{R} -algebra $C^\infty(M)$ is the derivation associated to a C^∞ -vector field on M . However, it is not generally true that a derivation on the \mathbb{C} -algebra $C^{\text{hol}}(M)$ is the derivation associated to a C^{hol} -vector field on M . When M is a Stein manifold, this one-to-one correspondence has been proved in [26]. Using this result, it can be shown that, for a real analytic manifold M , there is a one-to-one correspondence between derivations on the \mathbb{R} -algebra $C^\omega(M)$ and the C^{hol} -vector fields on M [26]. Moreover, we will show that, with the C^ν -topology on $C^\nu(M)$, vector fields are continuous operators.

Let M be a C^ν -manifold and let $X : M \rightarrow TM$ be a C^ν -vector field on M . Then we define the corresponding linear map $\widehat{X} : C^\nu(M) \rightarrow C^\nu(M)$ as

$$\widehat{X}(f) = df(X), \quad \forall f \in C^\nu(M).$$

Using the Leibniz rule, this linear map can be shown to be a derivation on the \mathbb{F} -algebra $C^\nu(M)$.

More interestingly, one can show there is a one-to-one correspondence between C^ν -vector fields on M and derivations on the \mathbb{F} -algebra $C^\nu(M)$.

Theorem 3.3.1. *Let M be a C^ν -manifold, where $\nu \in \{\infty, \omega, \text{hol}\}$. Additionally assume that M is a Stein manifold when $\nu = \text{hol}$. If X is a C^ν -vector field, then \widehat{X} is a derivation on the \mathbb{F} -algebra $C^\nu(M)$. Moreover, for every derivation $D : C^\nu(M) \rightarrow C^\nu(M)$, there exists a C^ν -vector field X such that $\widehat{X} = D$.*

Proof. The proof of this theorem is different for the smooth case ($\nu = \infty$), the real analytic case ($\nu = \omega$), and holomorphic case ($\nu = \text{hol}$). For the smooth case the proof is given in [4, Proposition 2.4]. For the holomorphic case, this is proved in [26, Theorem 3.2]. For the real analytic case, the idea of the proof is similar to the holomorphic case, and the sketch of proof is given in [26, Theorem 4.1] \square

Theorem 3.3.2. *Let X be a C^ν -vector field. Then $\widehat{X} : C^\nu(M) \rightarrow C^\nu(M)$ is a continuous linear map.*

Proof. The fact that \widehat{X} is linear is easy. To show that \widehat{X} is continuous, we consider three different cases:

1. ($\nu = \infty$):

Assume that the compact set K is inside a coordinate chart $(U, \eta = (x^1, x^2, \dots, x^N))$. In this coordinate chart, we can write

$$X(f) = \sum_{i=1}^N X(x^i) \frac{\partial f}{\partial x^i}.$$

For every $i \in \{1, 2, \dots, N\}$, we denote $X(x^i)$ by X^i . Let $m \in \mathbb{Z}_{>0}$, then we have

$$p_{K,m}^\infty(X(f)) = \sup \{ \|D^{(r)}(X(f))(x)\| \mid |r| \leq m, x \in K \}.$$

Note that we have

$$D^{(r)}(X(f))(x) = D^{(r)} \left(\sum_{i=1}^N X(x^i) \frac{\partial f}{\partial x^i} \right) = \sum_{l \leq r} \sum_{i=1}^N (D^{(l)} X^i(x)) (D^{(r-l+\widehat{i})} f(x)).$$

This implies that

$$p_{K,m}^\infty(X(f)) \leq 2^m N \max_i \{ p_{K,m}^\infty(X^i) \} p_{K,m+1}^\infty(f).$$

This completes the proof of continuity of \widehat{X} .

2. ($\nu = \text{hol}$):

Let $(U, \eta = (x^1, x^2, \dots, x^N))$ be a coordinate chart such that U is relatively compact and $K \subseteq U$ be a compact set. For every $x \in K$, there exists $r_x > 0$ such that $\text{cl}(D_{(r_x)}(x)) \subseteq U$. Since K is compact, there exists $x_1, x_2, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n \text{cl}(D_{(r_{x_i})}(x_i))$. Note that for every $i \in \{1, 2, \dots, n\}$, the

set $\text{cl}(D_{(r_{x_i})}(x_i))$ is compact. Therefore, the set $K' = \bigcup_{i=1}^n \text{cl}(D_{(r_{x_i})}(x_i))$ is also compact.

In this coordinate chart, we can write

$$X(f) = \sum_{i=1}^N X(x^i) \frac{\partial f}{\partial x^i}.$$

For every $i \in \{1, 2, \dots, N\}$, we denote $X(x^i)$ by X^i . Then, we have

$$p_K^{\text{hol}}(X(f)) = \sup \{ \|X(f)(x)\| \mid x \in K \}.$$

Note that we have

$$\|X(f)(x)\| \leq \sum_{i=1}^N \|X^i(x)\| \left\| \frac{\partial f}{\partial x^i} \right\|$$

If we set $r = \min\{r_1, r_2, \dots, r_n\}$, using the Cauchy inequality, we have

$$\left\| \frac{\partial f}{\partial x^i}(x) \right\| \leq \frac{1}{r} \sup \{ \|f(x)\| \mid x \in K' \}, \quad \forall x \in K.$$

This implies that

$$p_K^{\text{hol}}(X(f)) \leq \frac{N}{r} \max_i \{ p_K^{\text{hol}}(X^i) \} p_{K'}^{\infty}(f).$$

This completes the proof for the holomorphic case.

3. ($\nu = \omega$):

Let (U, ϕ) be a coordinate chart on M and $K \subseteq U$ be a compact set. We first prove that, for every multi-index (r) , we have

$$\begin{aligned} \|D^{(r)}(fg)(x)\| &\leq \sum_{j=0}^{|r|} \binom{|r|}{j} \sup \{ \|(D^{(l)}f(x))\| \mid |l| = j \} \\ &\quad \times \sup \{ \|(D^{(l)}g(x))\| \mid |l| = |r| - j \}, \quad \forall x \in U. \end{aligned}$$

We prove this by induction on $|r|$. If $|r| = 1$, then it is clear that we have

$$\begin{aligned} \left\| \frac{\partial}{\partial x^i}(fg)(x) \right\| &= \left\| \frac{\partial f}{\partial x^i}(x)g(x) + \frac{\partial g}{\partial x^i}(x)f(x) \right\| \\ &\leq \left\| \frac{\partial f}{\partial x^i}(x)g(x) \right\| + \left\| \frac{\partial g}{\partial x^i}(x)f(x) \right\|, \quad \forall x \in U. \end{aligned}$$

Now suppose that, for every (r) such that $|r| \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned} \|D^{(r)}(fg)(x)\| &\leq \sum_{j=0}^{|r|} \binom{|r|}{j} \sup \{ \|(D^{(l)} f(x))\| \mid |l| = j \} \\ &\quad \times \sup \{ \|(D^{(l)} g(x))\| \mid |l| = |r| - j \}, \quad \forall x \in U. \end{aligned}$$

Let (l) be a multi-index with $|l| = k + 1$. Then there exists $i \in \{1, 2, \dots, N\}$ and (r) with $|r| = k$ such that $(l) = (r) + (\hat{i})$. So we have

$$\begin{aligned} \|D^{(l)}(fg)(x)\| &= \left\| D^{(r)} \left(\frac{\partial}{\partial x^i} (fg) \right) (x) \right\| \\ &\leq \left\| D^{(r)} \left(\frac{\partial f}{\partial x^i} g \right) (x) \right\| + \left\| D^{(r)} \left(\frac{\partial g}{\partial x^i} f \right) (x) \right\| \\ &\leq \sum_{j=0}^{|r|} \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} \frac{\partial f}{\partial x^i})(x) \right\| \mid |l| = j \right\} \sup \{ \|(D^{(l)} g(x))\| \mid |l| = |r| - j \} + \\ &\quad \binom{|r|}{j} \sup \{ \|(D^{(l)} f(x))\| \mid |l| = j \} \sup \left\{ \left\| (D^{(l)} \frac{\partial g}{\partial x^i})(x) \right\| \mid |l| = |r| - j \right\} \\ &= \sum_{j=0}^{|r|} \left(\binom{|r|}{j-1} + \binom{|r|}{j} \right) \sup \{ \|(D^{(l)} f(x))\| \mid |l| = j \} \\ &\quad \times \sup \{ \|(D^{(l)} g(x))\| \mid |l| = |r| - j + 1 \} \\ &= \sum_{j=0}^{|r|} \binom{|r|+1}{j} \sup \{ \|(D^{(l)} f(x))\| \mid |l| = j \} \\ &\quad \times \sup \{ \|(D^{(l)} g(x))\| \mid |l| = |r| - j + 1 \}, \quad \forall x \in U. \end{aligned}$$

This completes the induction. Thus, by noting that in a coordinate chart we have

$$X(f) = \sum_{i=1}^N X(x^i) \frac{\partial f}{\partial x^i},$$

we get

$$\begin{aligned} \|D^{(r)}(X(f))(x)\| &\leq \sum_{j=0}^{|r|} \sum_{i=1}^N \binom{|r|}{j} \sup \{ \|(D^{(l)} X^i(x))\| \mid |l| = |r| - j \} \\ &\quad \times \sup \left\{ \left\| D^{(l)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |l| = j \right\}, \quad \forall x \in U. \quad (3.3.1) \end{aligned}$$

Now let $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$. Then we have

$$\begin{aligned} \frac{a_0 a_1 \cdots a_{|r|}}{|r|!} \|D^{(r)}(X(f))(x)\| &\leq \\ \frac{1}{a_0} \sum_{j=0}^{|r|} \sum_{i=1}^N \sup \left\{ \frac{a_0 a_1 \cdots a_j}{j!} \left\| D^{(l)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |l| = |r| - j \right\} &\times \\ \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|-j}}{(|r|-j)!} \|(D^{(l)} X^i(x))\| \mid |l| = |r| - j \right\}, &\quad \forall x \in U. \end{aligned}$$

We define the sequence $\mathbf{p} = (p_0, p_1, \dots)$ as

$$p_m = \begin{cases} a_0 a_1, & m = 0, \\ \binom{m+1}{m} a_{m-1}, & m \geq 1. \end{cases}$$

Then it is clear that $\lim_{m \rightarrow \infty} p_m = 0$. This implies that $\mathbf{p} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$. Moreover, we have

$$\frac{p_0 p_1 \cdots p_{m+1}}{(m+1)!} = \frac{a_0 a_1 \cdots a_m}{m!}, \quad \forall m \in \mathbb{Z}_{\geq 0}.$$

Thus we have

$$\begin{aligned} \frac{a_0 a_1 \cdots a_{|r|}}{|r|!} \|D^{(r)}(X(f))(x)\| &\leq \\ \frac{1}{a_0} \sum_{j=0}^{|r|} \sum_{i=1}^N \sup \left\{ \frac{p_0 p_1 \cdots p_{j+1}}{(j+1)!} \left\| D^{(l)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |l| = |r| - j \right\} &\times \\ \sup \left\{ \frac{a_0 a_1 \cdots a_{|r|-j}}{(|r|-j)!} \|(D^{(l)} X^i(x))\| \mid |l| = |r| - j \right\}, &\quad \forall x \in U. \end{aligned}$$

Taking the supremum of both side of the above inequality over $x \in K$, we have

$$p_{K,\mathbf{a}}^\omega(X(f)) \leq \frac{N}{a_0} \sup_i \{p_{K,\mathbf{a}}^\omega(X^i)\} p_{K,\mathbf{p}}^\omega(f)$$

This completes the proof of continuity of \widehat{X} in real analytic case. □

Now, we prove a specific approximation for the seminorms on $C^\omega(M)$. In the next sections, we will see that this approximation plays an essential role in studying flows of time-varying real analytic vector fields. Let $d > 0$ be a positive real number and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. For every $n \in \mathbb{N}$, we define the sequence $\mathbf{a}_n = (a_{n,0}, a_{n,1}, \dots, a_{n,m}, \dots)$ as

$$a_{n,m} = \begin{cases} \left(\frac{m+1}{m}\right)^n a_m, & m > n, \\ \left(\frac{m+1}{m}\right)^m a_m, & m \leq n. \end{cases}$$

Associated to every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we define the sequence $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$ as

$$b_{n,m} = \begin{cases} a_{n,m}, & m = 0, m = 1, \\ \left(\frac{(m+1)(m+2)}{(m-1)(m)} \right) a_{n,m}, & m > 1. \end{cases}$$

Lemma 3.3.3. *Let $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then, for every $n \in \mathbb{Z}_{\geq 0}$, we have $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$ and, for every $m, n \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} a_{n,m} &\leq ea_m, \\ \frac{(m+1)}{(n+1)} &\leq \frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})}, \end{aligned}$$

where e is the Euler constant. Moreover, for every $n \in \mathbb{Z}_{\geq 0}$ we have $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$ and, for every $m > 1$, we have

$$\begin{aligned} b_{n,m} &\leq 6ea_m, \\ \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,m})}{(m-2)!} &= \frac{(b_{n,0})(b_{n,1}) \cdots (b_{n,m})}{m!}. \end{aligned}$$

Proof. Let $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then by definition of \mathbf{a}_n , for $n < m$, we have

$$a_{n,m} = \left(\frac{m+1}{m} \right)^n a_m \leq \left(\frac{m+1}{m} \right)^m a_m \leq ea_m$$

For $n \geq m$, we have

$$a_{n,m} = \left(\frac{m+1}{m} \right)^m a_m \leq ea_m.$$

This implies that $\lim_{m \rightarrow \infty} a_{n,m} = 0$. Moreover, for every $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$a_{n,m} \leq ea_m \leq ed.$$

So we have $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$. Let $m, n \in \mathbb{Z}_{\geq 0}$ be such that $n+1 > m+1$. Then we have

$$\frac{a_{n+1,m+1}}{a_{n,m+1}} = 1.$$

So we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq 1.$$

Since we have $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$, we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq 1 \geq \frac{m+1}{n+1}.$$

Now suppose that $m, n \in \mathbb{Z}_{\geq 0}$ are such that $n + 1 \leq m + 1$. Then we have

$$\frac{a_{n+1,m+1}}{a_{n,m+1}} = \left(\frac{m+1}{m} \right).$$

Therefore, we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} = \left(\frac{n+2}{n+1} \right) \left(\frac{n+3}{n+2} \right) \cdots \left(\frac{m+2}{m+1} \right) = \frac{m+2}{n+1} > \frac{m+1}{n+1}.$$

Since we have $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$, we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m})} \geq \frac{m+1}{n+1}.$$

So, for all $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq \frac{m+1}{n+1}.$$

Finally, since $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$ and we have $\frac{(m+2)(m+1)}{m(m-1)} \leq 6$, for all $m > 1$, we get

$$b_{n,m} = \frac{(m+2)(m+1)}{m(m-1)} a_{n,m} \leq 6a_{n,m}.$$

So we have $\lim_{m \rightarrow \infty} b_{n,m} = 6 \lim_{m \rightarrow \infty} a_{n,m} = 0$. Moreover, we have

$$b_{n,m} \leq 6a_{n,m} \leq 6ea_m \leq 6ed.$$

Thus we get $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$. This completes the proof of the lemma. \square

Theorem 3.3.4. *Let M be a real analytic manifold of dimension N , $X \in \Gamma^\omega(E)$, and $f \in C^\omega(M)$. Let U be a coordinate neighbourhood in M and $K \subseteq U$ is compact. For every $d > 0$, every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}, d)$, and every $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\tilde{p}_{K,\mathbf{a}_n}^\omega(X(f)) \leq 4N(n+1) \max_i \{ \tilde{p}_{K,\mathbf{b}_n}^\omega(X^i) \} \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f). \quad (3.3.2)$$

Proof. Now let $d > 0$ and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then by multiplying both sides of equation (3.3.1) by $\frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!}$ we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \|D^{(r)}(X(f))(x)\| \leq \\ & \sum_{i=1}^N \sum_{l=0}^{|r|} \left(\frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{l!} \sup \left\{ \left\| D^{(s)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |s| = l \right\} \right) \\ & \times \left(\frac{(a_{n,l+2})(a_{n,l+3}) \cdots (a_{n,|r|})}{(|r|-l)!} \sup \left\{ \|D^{(s)} X^i(x)\| \mid |s| = |r| - l \right\} \right), \quad \forall x \in U. \end{aligned}$$

Since the sequence \mathbf{a}_n is decreasing, we have

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \|D^{(r)}(X(f))(x)\| \\ & \leq \sum_{i=1}^N \sum_{l=0}^{|r|} \left(\frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{l!} \sup \left\{ \left\| D^{(s)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |s| = l \right\} \right) \\ & \times \left(\frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|-l-2})}{(|r|-l)!} \sup \left\{ \|D^{(s)} X^i(x)\| \mid |s| = |r|-l \right\} \right), \quad \forall x \in U. \end{aligned}$$

Using the above lemma, we have

$$\begin{aligned} \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{(l)!} & \leq (n+1) \frac{(a_{n+1,0})(a_{n+1,1} \cdots (a_{n+1,l+1}))}{(l+1)!}, \\ \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|-l-2})}{(|r|-l-2)!} & = \frac{(b_{n+1,0})(b_{n+1,1}) \cdots (b_{n+1,|r|-l})}{(|r|-l)!} \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \|D^{(r)}(X(f))(x)\| \\ & \leq \sum_{i=1}^N \sum_{l=0}^{|r|} \frac{(n+1)}{(|r|-l)(|r|-l-1)} \left(\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,l+1})}{(l+1)!} \sup \left\{ \|D^{(s)} f(x)\| \mid |s| = l+1 \right\} \right) \\ & \quad \left(\frac{(b_{n,0})(b_{n,1}) \cdots (b_{n,|r|-l})}{(|r|-l)!} \sup \left\{ \|D^{(s)} X^i(x)\| \mid |s| = |r|-l \right\} \right), \quad \forall x \in U. \end{aligned}$$

Thus, by taking supremum over $l \in \mathbb{Z}_{\geq 0}$ and $x \in K$ of the two term in the right hand side of the above inequality, we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \|D^{(r)}(X(f))(x)\| \\ & \leq N(n+1) \tilde{p}_{K, \mathbf{a}_{n+1}}^\omega(f) \tilde{p}_{K, \mathbf{b}_n}^\omega(X^i) \sum_{l=0}^{|r|} \frac{1}{(|r|-l)(|r|-l-1)} \\ & \leq 4N(n+1) \tilde{p}_{K, \mathbf{a}_{n+1}}^\omega(f) \tilde{p}_{K, \mathbf{b}_n}^\omega(X^i), \quad \forall x \in U. \end{aligned}$$

By taking the supremum of the left hand side of the above inequality over $|r| \in \mathbb{N}$ and $x \in K$, for every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}, d)$, we get

$$\tilde{p}_{K, \mathbf{a}_n}^\omega(X(f)) \leq 4N(n+1) \max_i \{ \tilde{p}_{K, \mathbf{b}_n}^\omega(X^i) \} \tilde{p}_{K, \mathbf{a}_{n+1}}^\omega(f).$$

□

3.4. C^ν -maps as unital algebra homomorphisms on $C^\nu(M)$

In this section, we study C^ν -maps between manifolds. One can easily associate to every C^ν -map between two manifolds M and N , a unital \mathbb{F} -algebra homomorphism between the vector spaces $C^\nu(N)$ and $C^\nu(M)$. For $\nu = \infty$, this correspondence can be shown to be one-to-one [4, Proposition 2.1]. For the $\nu = \text{hol}$, by adding the extra assumption that M and N are Stein manifolds, one can show that C^{hol} -maps from M to N are in one-to-one correspondence with unital \mathbb{C} -algebra homomorphisms between $C^{\text{hol}}(N)$ and $C^{\text{hol}}(M)$ [30, Theorem J.1]. For the real analytic case, this correspondence has shown to be one-to-one for the case when the manifolds are open subsets of \mathbb{R}^n [24, Theorem 2.1]. In Theorem 3.4.2, we give a unifying proof to the fact that, for $\nu \in \{\infty, \omega\}$ and for $\nu = \text{hol}$ with the extra assumption of M and N being Stein manifolds, there is a one-to-one correspondence between C^ν -maps between two manifolds M and N and unital \mathbb{F} -algebra homomorphisms between $C^\nu(N)$ and $C^\nu(M)$. This result, in particular, generalize Theorem 2.1 in [24] to arbitrary real analytic manifolds. Finally, we will show that, with the C^ν -topology on the vector spaces $C^\nu(N)$ and $C^\nu(M)$, C^ν -maps are continuous operators.

Let $\phi : M \rightarrow N$ be a C^ν -map. Then we can define the map $\widehat{\phi} : C^\nu(N) \rightarrow C^\nu(M)$ as

$$\widehat{\phi}(f) = f \circ \phi.$$

It is easy to see that $\widehat{\phi}$ is an \mathbb{F} -algebra homomorphism. For every $x \in M$, one can define the unital \mathbb{F} -algebra homomorphism $\text{ev}_x : C^\nu(M) \rightarrow \mathbb{F}$ as

$$\text{ev}_x(f) = f(x).$$

The map ev_x is called the **evaluation map** at $x \in M$. It is natural to ask whether the evaluation maps are continuous with respect to C^ν -topology.

Theorem 3.4.1. *For every $x \in M$, the map $\text{ev}_x : C^\nu(M) \rightarrow \mathbb{F}$ is continuous with respect to C^ν -topology.*

Proof. If p_K^ν is one of the seminorms $p_{K,m}^\infty$, p_K^{hol} , or $p_{K,\mathbf{a}}^\omega$, we have

$$p_K^\nu(f) \leq C |\text{ev}_x(f)|,$$

where $C = 1$ for $\nu \in \{\infty, \text{hol}\}$ and $C = a_0$ for $\nu = \omega$. □

The evaluation map plays an essential role in characterizing unital \mathbb{F} -algebra homomorphisms. The following result is of significant importance.

Theorem 3.4.2. *Let M be a C^ν -manifold where $\nu \in \{\infty, \text{hol}, \omega\}$. If $\nu = \text{hol}$, additionally assume that M is a Stein manifold. Let $\phi : C^\nu(M) \rightarrow \mathbb{F}$ be a nonzero and unital \mathbb{F} -algebra homomorphism. Then there exists $x \in M$ such that $\phi = \text{ev}_x$.*

Proof. For the smooth case, the proof is given in [4, Proposition 2.1]. For the holomorphic case, the proof is the special case of [30, Theorem J.1]. For the real analytic case, when M and N are open subsets of euclidean spaces, a proof for this theorem is given in [24, Theorem 2.1]. However, it seems that this proof cannot be generalized to include the general real analytic manifolds. Using the techniques and ideas in [60, Proposition 12.5], we present a unified proof of this theorem for all $\nu \in \{\infty, \omega, \text{hol}\}$. Let $\phi : C^\omega(M) \rightarrow \mathbb{R}$ be a unital \mathbb{R} -algebra homomorphism. It is easy to see that $\text{Ker}(\phi)$ is a maximal ideal in $C^\omega(M)$. For every $f \in C^\omega(M)$, we define

$$Z(f) = \{x \in M \mid f(x) = 0\}.$$

Lemma. *Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{Ker}(\phi)$. Then we have*

$$\bigcap_{i=1}^n Z(f_i) \neq \emptyset.$$

Proof. Suppose that we have

$$\bigcap_{i=1}^n Z(f_i) = \emptyset.$$

Then we can define a function $g \in C^\omega(M)$ as

$$g(x) = \frac{1}{(\sum_{i=1}^n (f_i(x))^2)}, \quad \forall x \in M.$$

Then it is clear that we have

$$\left(\sum_{i=1}^n (f_i)^2 \right) (g) = \mathbf{1},$$

where $\mathbf{1} : C^\nu(M) \rightarrow \mathbb{F}$ is a unital \mathbb{F} -algebra homomorphism defined as

$$\mathbf{1}(f) = 1.$$

Since $\text{Ker}(\phi)$ is an ideal in $C^\omega(M)$, we have $\mathbf{1} \in \text{Ker}(\phi)$. This implies that $\phi = 0$, which is a contradiction of ϕ being unital. \square

Since M is a C^ν -manifold, there exists a C^ν -embedding of M into some \mathbb{R}^N (for smooth case, we use Whitney's embedding theorem [82, §8, Theorem 1] with $N = 2n + 1$, for the holomorphic case, one can use Remmert's embedding theorem [29], [9] with $N = 2n + 1$, and for the real analytic case, we use Grauert's embedding theorem with $N = 4n + 2$). Let x_1, x_2, \dots, x_N be the standard coordinate functions on \mathbb{R}^N and $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$ be their restrictions to M . Now, for every $i \in \{1, 2, \dots, N\}$, consider the functions $\hat{x}_i - \phi(\hat{x}_i)\mathbf{1} \in C^\nu(M)$. It is easy to see that

$$\phi(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1}) = \phi(\hat{x}_i) - \phi(\hat{x}_i)\phi(\mathbf{1}) = 0, \quad \forall i \in \{1, 2, \dots, N\}.$$

This implies that, for every $i \in \{1, 2, \dots, N\}$, we have $\widehat{x}_i - \phi(\widehat{x}_i)\mathbf{1} \in \text{Ker}(\phi)$. So, by the above Lemma, we get

$$\bigcap_{i=1}^N Z(\widehat{x}_i - \phi(\widehat{x}_i)\mathbf{1}) \neq \emptyset.$$

Since x_1, x_2, \dots, x_N are coordinate functions, it is easy to see that $\bigcap_{i=1}^N Z(\widehat{x}_i - \phi(\widehat{x}_i)\mathbf{1})$ is just a one-point set. So we set $\bigcap_{i=1}^N Z(\widehat{x}_i - \phi(\widehat{x}_i)\mathbf{1}) = \{x\}$.

Now we proceed to prove the theorem. Note that, for every $f \in \text{Ker}(\phi)$, we have

$$Z(f) \cap \{x\} = Z(f) \bigcap \left(\bigcap_{i=1}^N Z(\widehat{x}_i - \phi(\widehat{x}_i)\mathbf{1}) \right).$$

So, by the above Lemma, we have

$$Z(f) \cap \{x\} \neq \emptyset, \quad \forall f \in \text{Ker}(\phi).$$

This implies that

$$\{x\} \subseteq Z(f), \quad \forall f \in \text{Ker}(\phi).$$

This means that

$$\{x\} \subseteq \bigcap_{f \in \text{Ker}(\phi)} Z(f).$$

This implies that $\text{Ker}(\phi) \subseteq \text{Ker}(\text{ev}_x)$. Since $\text{Ker}(\text{ev}_x)$ and $\text{Ker}(\phi)$ are both maximal ideals, we have

$$\text{Ker}(\text{ev}_x) = \text{Ker}(\phi).$$

Now let $f \in C^\nu(M)$, so we have $f - f(x)\mathbf{1} \in \text{Ker}(\phi)$. This implies that

$$0 = \phi(f - f(x)\mathbf{1}) = \phi(f) - f(x).$$

So, for every $f \in C^\nu(M)$,

$$\phi(f) = f(x).$$

Therefore, we have $\phi = \text{ev}_x$. □

Theorem 3.4.3. *Let M be a C^ν -manifold where $\nu \in \{\infty, \text{hol}, \omega\}$. If $\nu = \text{hol}$, additionally assume that M is a Stein manifold. We define the map $\text{ev} : M \rightarrow (C^\nu(M))'$ by*

$$\text{ev}(x) = \text{ev}_x.$$

The image of ev is $\text{Hom}_{\mathbb{R}}(C^\nu(M); \mathbb{F})$. Moreover, this map is a homeomorphism onto its image.

Proof. To show that ev_x is continuous, it suffice to show that, for every $f \in C^\nu(M)$, the set

$$\{x \in M \mid \|f(x)\| < 1\}$$

is open in M . But this is clear since we have $C^\nu(M) \subset C^0(M)$. To show that ev is a homeomorphism onto its image, we prove that the topology induced by $C^\nu(M)'$ on M

using the map ev is finer than the original topology of M . Suppose that $\dim(M) = n$. For $\nu \in \{\infty, \text{hol}, \omega\}$, there is a C^ν -embedding $i : M \rightarrow \mathbb{R}^s$ for some $s \in \mathbb{N}$ (for smooth case, it is exactly Whitney's embedding theorem [82, §8, Theorem 1] with $s = 2n + 1$, for the holomorphic case, it is Remmert's embedding theorem [29], [9] with $s = 2n + 1$, and for the real analytic case, it is Grauert's embedding theorem with $s = 4n + 2$). This implies that, without loss of generality, one can assume M to be an embedded submanifold of \mathbb{R}^s . Let (x^1, x^2, \dots, x^s) be the standard coordinate chart on M and $U \subseteq M$ be a neighborhood of x_0 . There exists $c > 0$ such that

$$\{x \in M \mid \sum_{i=1}^s \|x^i - (x_0)^i\|^2 < c\} \subseteq U.$$

However, we have

$$\begin{aligned} M \cap \left(\bigcap_{i=1}^s p_{x^i}^{-1} \left(\left[0, \sqrt{\frac{c}{s}} \right] \right) \right) &= \left\{ x \in M \mid \|x^i - (x_0)^i\| < \sqrt{\frac{c}{s}}, \forall i \in \{1, 2, \dots, s\} \right\} \\ &\subseteq \{x \in M \mid \sum_{i=1}^s \|x^i - (x_0)^i\|^2 < c\}. \end{aligned}$$

Therefore, we have

$$M \cap \left(\bigcap_{i=1}^s p_{x^i}^{-1} \left(\left[0, \sqrt{\frac{c}{s}} \right] \right) \right) \subseteq U.$$

□

Theorem 3.4.4. *Let M and N be C^ν -manifolds and $\nu \in \{\infty, \text{hol}, \omega\}$. If $\nu = \text{hol}$, additionally assume that M and N are Stein manifolds. Then, for every \mathbb{F} -algebra map $A : C^\nu(M) \rightarrow C^\nu(N)$, there exists a C^ν -map $\phi : N \rightarrow M$ such that*

$$\widehat{\phi} = A.$$

Proof. For every $x \in N$, consider the unital \mathbb{F} -algebra homomorphism $\text{ev}_x \circ A : C^\nu(M) \rightarrow \mathbb{F}$. By Theorem 3.4.2, there exists $y_x \in M$ such that $\text{ev}_x \circ A = \text{ev}_{y_x}$. We define $\phi : N \rightarrow M$ as

$$\phi(x) = y_x, \quad \forall x \in N.$$

Let $(U, \eta = (z^1, z^2, \dots, z^m))$ be a coordinate neighbourhood on M around y_x . Then, by using the embedding theorems (for $\nu = \infty$, we use Whitney's embedding theorem [82, §8, Theorem 1], for $\nu = \text{hol}$, we use Remmert's embedding theorem [29], [9], and for $\nu = \omega$, we use Grauert's embedding theorem), there exist functions $\tilde{z}^1, \tilde{z}^2, \dots, \tilde{z}^m$ such that, for every $i \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} \tilde{z}^i &\in C^\nu(N), \\ \tilde{z}^i|_U &= z^i. \end{aligned}$$

Thus, for every $x \in U$, we have

$$y_x^i = \text{ev}_x \circ A(\tilde{z}^i) = A(\tilde{z}^i)(x), \quad \forall i \in \{1, 2, \dots, m\}.$$

However, for every $i \in \{1, 2, \dots, m\}$, we have $A(\tilde{z}^i) \in C^\nu(N)$. This implies that, for every $i \in \{1, 2, \dots, m\}$, the function y_x^i is of class C^ν with respect to x on the neighborhood U . Therefore, the map ϕ is of class C^ν . One can easily check that $\widehat{\phi} = A$. □

Theorem 3.4.5. *Let $\phi \in C^\nu(N, M)$ be a C^ν -map. Then $\widehat{\phi}: C^\nu(M) \rightarrow C^\nu(N)$ is a continuous linear map .*

Proof. It is easy to see that $\widehat{\phi}$ is a linear map. To show that it is continuous, note that we need to separate the cases $\nu = \infty$, $\nu = \text{hol}$, and $\nu = \omega$.

1. ($\nu = \infty$):

Consider a coordinate chart (U, η) on M and a coordinate chart (V, ξ) on N . Then, for every compact set $K \subseteq V$ and every $m \in \mathbb{Z}_{\geq 0}$, we have

$$p_{K,m}^\infty(f \circ \phi) = \sup\{\|D^{(r)}(f \circ \phi)(x)\| \mid x \in K, |r| \leq m\}.$$

Note that we have

$$\begin{aligned} & \|D^{(r)}(f \circ \phi)(x)\| \\ & \leq 2^{|r|} \sup\{\|D^{(j)}\phi(x)\|^{|j|} \mid x \in K, |j| \leq |r|\} \sup\{\|D^{(j)}f(x)\| \mid x \in K, |j| < |r|\}. \end{aligned}$$

This implies that

$$p_{K,m}^\infty(\phi) \leq 2^m \sup\{\|D^{(j)}\phi(x)\|^{|j|} \mid x \in K, |j| \leq m\} p_{K,m}^\infty(f).$$

This completes the proof for the smooth case.

2. ($\nu = \text{hol}$):

For every compact set $K \subseteq M$, we have

$$p_{K,f}^{\text{hol}}(\phi) = \sup\{\|(f \circ \phi)(x)\| \mid x \in K\} = \sup\{\|f(z)\| \mid z \in \phi(K)\} = p_{\phi(K)}^{\text{hol}}(f).$$

This completes the proof for the holomorphic case.

3. ($\nu = \omega$):

Let (U, η) be a coordinate chart on M , (V, ξ) be a coordinate chart on N such that $\eta(U) \subseteq V$, and $K \subseteq V$ be a compact set. Since $\eta \circ \phi \circ \xi^{-1}: \xi(V \cap \phi^{-1}(U)) \rightarrow \eta(\phi(V) \cap U)$ is real analytic, by Sublemma 3 of [42], there exist $C > 0, \sigma > 0$ such that, for every $f \in C^\omega(M)$ and every multi-index (r) , we have

$$\sup\left\{\frac{1}{(r)!}\|D^{(r)}(f \circ \phi)(x)\| \mid x \in K\right\} \leq C\sigma^m \sup\left\{\frac{1}{(r)!}\|D^{(r)}f(x)\| \mid x \in \phi(K)\right\}.$$

Let $\mathbf{a} = (a_0, a_1, \dots) \in \mathbf{c}_0(\mathbb{Z}_{>0}; \mathbb{R}_{\geq 0})$. We multiply both side of the above inequality by $a_0 a_1 \dots a_{|r|}$ and we get

$$\begin{aligned} \sup \left\{ \frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)}(f \circ \phi)(x)\| \mid x \in K \right\} \\ \leq C \sigma^m \sup \left\{ \frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} f(x)\| \mid x \in \phi(K) \right\}, \end{aligned}$$

Now by taking the sup over $|r| \in \mathbb{Z}_{>0}$, we have

$$p_{K, \mathbf{a}, f}^\omega(\phi) \leq C p_{K, \sigma \mathbf{a}}^\omega(f), \quad \forall f \in C^\omega(M).$$

This completes the proof for the real analytic case. □

Definition 3.4.6. The set of all unital \mathbb{R} -algebra homomorphisms from $C^\nu(N)$ to $C^\nu(M)$ is denoted by $\text{Hom}_{\mathbb{R}}(C^\nu(N); C^\nu(M))$.

By Theorem 3.4.4, there is a one-to-one correspondence between $C^\nu(M; N)$ and $\text{Hom}_{\mathbb{R}}(C^\nu(N); C^\nu(M))$.

3.5. Topology on $L(C^\nu(M); C^\nu(N))$

In this section, using the C^ν -topology on the vector space $C^\nu(M)$, we equip the space of linear maps from $C^\nu(M)$ to $C^\nu(N)$ with the pointwise convergence topology. We will show that $L(C^\nu(M); C^\nu(N))$ with this topology is a locally convex topological vector space. In particular, we find a family of defining seminorms for this space. We then proceed by studying properties of the topological vector spaces $L(C^\nu(M); C^\nu(N))$ and $\text{Der}(C^\nu(M))$.

Definition 3.5.1. For $f \in C^\nu(M)$, we define the map $\mathcal{L}_f : L(C^\nu(M); C^\nu(N)) \rightarrow C^\nu(N)$ as

$$\mathcal{L}_f(X) = X(f).$$

The C^ν -**topology** on $L(C^\nu(M); C^\nu(N))$ is the projective topology with respect to the family $\{C^\nu(N), \mathcal{L}_f\}_{f \in C^\nu(M)}$.

One can show that this topology coincides with the topology of pointwise-convergence on $L(C^\nu(M); C^\nu(N))$.

Theorem 3.5.2. *The C^ν -topology and the topology of pointwise convergence on $L(C^\nu(M); C^\nu(N))$ are the same. Moreover, $L(C^\nu(M); C^\nu(N))$ is a closed subspace of $C^\nu(N)^{C^\nu(M)}$.*

Proof. The fact that C^ν -topology and the topology of pointwise convergence on $L(C^\nu(M); C^\nu(N))$ are the same is clear from the definition. We show that $L(C^\nu(M); C^\nu(N))$ is a closed subspace of $C^\nu(N)^{C^\nu(M)}$, if we equip the latter space with its natural topology of pointwise convergence. Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a converging net in $L(C^\nu(M); C^\nu(N))$ with the limit $X \in C^\nu(N)^{C^\nu(M)}$. We show that X is linear and continuous. Let $f, g \in C^\nu(M)$ and $c \in \mathbb{F}$. Then we have

$$X_\alpha(f + cg) = X_\alpha(f) + cX_\alpha(g), \quad \forall \alpha \in \Lambda.$$

By taking limit on α , we get

$$X(f + cg) = X(f) + cX(g).$$

This implies that X is linear. □

Theorem 3.5.3. *The locally convex space $L(C^\nu(M); C^\nu(N))$ with the C^ν -topology is Hausdorff, separable, complete, and nuclear.*

Proof. Since $C^\nu(N)$ is Hausdorff, it is clear that $C^\nu(N)^{C^\nu(M)}$ is Hausdorff. This implies that $L(C^\nu(M); C^\nu(N)) \subseteq C^\nu(N)^{C^\nu(M)}$ is Hausdorff. Let \mathfrak{c} be the cardinality of the continuum. Note that $C^\nu(M) \subseteq C^0(M)$ and M is separable. This implies that the cardinality of $C^0(M)$ is \mathfrak{c} [38, Chapter 5, Theorem 2.6(a)]. Therefore, the cardinality of $C^\nu(M)$ is at most \mathfrak{c} . The product of \mathfrak{c} separable spaces is separable [84, Theorem 16.4(c)]. This implies that $C^\nu(N)^{C^\nu(M)}$ is separable. Since $L(C^\nu(M); C^\nu(N))$ is a closed subspace of $C^\nu(N)^{C^\nu(M)}$, it is separable [84, Theorem 16.4]. Note that $C^\nu(N)$ is complete. This implies that $C^\nu(N)^{C^\nu(M)}$ is complete [69, Chapter II, §5.3]. Since $L(C^\nu(M); C^\nu(N))$ is a closed subspace of $C^\nu(N)^{C^\nu(M)}$, it is complete. The product of any arbitrary family of nuclear locally convex vector spaces is nuclear [69, Chapter III, §7.4]. This implies that $C^\nu(N)^{C^\nu(M)}$ is nuclear. Since every subspace of nuclear space is nuclear [69, Chapter III, §7.4], $L(C^\nu(M); C^\nu(N))$ is nuclear. □

Since locally convex spaces can be characterized using a family of seminorms, one would like to find a family of generating seminorms for the spaces $L(C^\nu(M); C^\nu(N))$.

Definition 3.5.4. Let $\{p_i\}_{i \in I}$ be a generating family of seminorms on $C^\nu(N)$. Then, for every $f \in C^\omega(M)$, we define the family of seminorms $\{p_{i,f}\}_{i \in I}$ as

$$p_{i,f}(X) = p_i(X(f)).$$

Theorem 3.5.5. *Let $\{p_i\}_{i \in I}$ be a generating family of seminorms on $C^\nu(N)$. Then the family of seminorms $\{p_{i,f}\}$, where $i \in I$ and $f \in C^\nu(M)$, generates the C^ν -topology on $L(C^\nu(M); C^\nu(N))$.*

Proof. Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a net in $L(C^\nu(M); C^\nu(N))$ which converges to X in the C^ν -topology. Since C^ν -topology and topology of pointwise convergence coincide on $L(C^\nu(M); C^\nu(N))$, for every $f \in C^\nu(M)$, we have

$$\lim_{\alpha} X_\alpha(f) = X(f).$$

This implies that, for every $i \in I$, we have

$$\lim_{\alpha} p_i (X_{\alpha}(f) - X(f)) = 0.$$

This means that, for every $f \in C^{\nu}(N)$ and every $i \in I$, we have

$$\lim_{\alpha} p_{i,f} (X_{\alpha} - X) = 0.$$

This means that $\lim_{\alpha} X_{\alpha} = X$ in the topology generated by $\{p_{i,f}\}$.

Now let $\{X_{\alpha}\}_{\alpha \in \Lambda}$ be a net in $L(C^{\nu}(M); C^{\nu}(N))$ which converges to X in the topology generated by $\{p_{i,f}\}_{i \in I, f \in C^{\nu}(M)}$. This implies that, for every $f \in C^{\nu}(N)$ and every $i \in I$, we have

$$\lim_{\alpha} p_i (X_{\alpha}(f) - X(f)) = 0.$$

Since the family of seminorms $\{p_i\}_{i \in I}$ generates the topology on $L(C^{\nu}(M); C^{\nu}(N))$, we have $\lim_{\alpha} X_{\alpha} = X$ in the topology of pointwise convergence on $L(C^{\nu}(M); C^{\nu}(N))$. \square

Remark 3.5.6. Using Theorem 3.5.5, one can define a family of generating seminorms for $L(C^{\infty}(M); C^{\infty}(N))$, $L(C^{\omega}(M); C^{\omega}(N))$, and $L(C^{\text{hol}}(M); C^{\text{hol}}(N))$.

1. Let (U, ϕ) be a coordinate chart on N , $K \subseteq U$ be a compact set, and $m \in \mathbb{Z}_{\geq 0}$. Then we define

$$p_{K,m,f}^{\infty}(X) = p_{K,m}^{\infty}(X(f)), \quad \forall f \in C^{\infty}(M).$$

By Theorem 3.5.5, the family of seminorms $\{p_{K,m,f}^{\infty}\}$ generates the C^{∞} -topology on $L(C^{\infty}(M); C^{\infty}(N))$.

2. Let $K \subseteq N$ be a compact set. Then we define

$$p_{K,f}^{\text{hol}}(X) = p_K^{\text{hol}}(X(f)), \quad \forall f \in C^{\text{hol}}(M).$$

By Theorem 3.5.5, the family of seminorms $\{p_{K,f}^{\text{hol}}\}$ generates the C^{hol} -topology on $L(C^{\text{hol}}(M); C^{\text{hol}}(N))$.

3. Let (U, ϕ) be a coordinate chart on N , $K \subseteq U$ be a compact set, $d > 0$ be a positive real number, and $\mathbf{a} \in \mathbf{c}_0^{\downarrow}(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then we define

$$p_{K,\mathbf{a},f}^{\omega}(X) = p_{K,\mathbf{a}}^{\omega}(X(f)), \quad \forall f \in C^{\omega}(M).$$

By Theorem 3.5.5, the family of seminorms $\{p_{K,\mathbf{a},f}^{\omega}\}$ generates the C^{ω} -topology on $L(C^{\omega}(M); C^{\omega}(N))$.

Theorem 3.5.7. *Let M, N, P be C^{ν} -manifolds and $V \in L(C^{\nu}(M); C^{\nu}(N))$. We define the operator $\mathcal{L}_V : L(C^{\nu}(P); C^{\nu}(M)) \rightarrow L(C^{\nu}(P); C^{\nu}(N))$ as*

$$\mathcal{L}_V(W) = V \circ W.$$

For every $V \in L(C^{\nu}(M); C^{\nu}(N))$, the operator \mathcal{L}_V is continuous.

Proof. Let $\{W_\alpha\}_{\alpha \in \Lambda}$ be a converging net in $L(C^\nu(P); C^\nu(M))$ such that $\lim_\alpha W_\alpha = W$. By definition of the C^ν -topology on $L(C^\nu(P); C^\nu(M))$, for every $f \in C^\nu(P)$, we have

$$\lim_\alpha W_\alpha(f) = W(f).$$

Since $V \in L(C^\nu(M); C^\nu(N))$ is continuous, for every net $\{g_\beta\}$ in $C^\nu(M)$ such that $\lim g_\beta = g$, we have

$$\lim_\alpha V(g_\alpha) = V(g).$$

By choosing $g_\alpha = W_\alpha(f)$, we have

$$\lim_\alpha V(W_\alpha(f)) = V(W(f)).$$

This implies that $\lim_\alpha \mathcal{L}_V(W_\alpha) = \lim_\alpha V \circ W_\alpha = V \circ W = \mathcal{L}_V(W)$. This means that \mathcal{L}_V is a continuous map. \square

Since $\text{Der}(C^\nu(M))$ is a subspace of $L(C^\nu(M); C^\nu(M))$, one can easily define the C^ν -topology on $\text{Der}(C^\nu(M))$ as the subspace topology.

Theorem 3.5.8. *The space $\text{Der}(C^\nu(M))$ is Hausdorff, separable, complete, and nuclear.*

Proof. We first show that $\text{Der}(C^\nu(M))$ is a closed subspace of $L(C^\nu(M); C^\nu(M))$. Let X_α be a net in $\text{Der}(C^\nu(M))$ such that $\lim_\alpha X_\alpha = X$. Then we have

$$\begin{aligned} X(fg) &= \lim_\alpha X_\alpha(fg) = \lim_\alpha fX_\alpha(g) + \lim_\alpha X_\alpha(f)g \\ &= fX(g) + X(f)g, \quad \forall f, g \in C^\nu(M). \end{aligned}$$

This implies that $X \in \text{Der}(C^\nu(M))$. So $\text{Der}(C^\nu(M))$ is a closed subset of $L(C^\nu(M); C^\nu(M))$. As a result, $\text{Der}(C^\nu(M))$ is Hausdorff, separable [84, Theorem 16.4], complete, and nuclear [69, Chapter III, §7.4]. \square

Let M be a C^ν -manifold, where $\nu \in \{\infty, \text{hol}, \omega\}$. If $\nu = \text{hol}$, additionally assume that M is a Stein manifold. In Theorem 3.3.1, we showed that one can identify $\text{Der}(C^\nu(M))$ with $\Gamma^\nu(TM)$ algebraically. Now that we defined the C^ν -topology on $\text{Der}(C^\nu(M))$ using the C^ν -topology on $C^\nu(M)$, one would like to see the relationship between the C^ν -topology on $\text{Der}(C^\nu(M))$ and the C^ν -topology on $\Gamma^\nu(TM)$.

Theorem 3.5.9. *The C^ν -topology on $\Gamma^\nu(TM)$ coincide with C^ν -topology on $\text{Der}(C^\nu(M))$.*

Proof. This has been shown in [42, Theorem 3.5, Theorem 4.5, Theorem 5.8]. \square

3.6. Curves on the space $L(C^\nu(M); C^\nu(N))$

In the previous section, we defined the C^ν -topology on $L(C^\nu(M); C^\nu(N))$. We showed that $L(C^\nu(M); C^\nu(N))$ endowed with C^ν -topology is a locally convex topological vector space. In this section, using the fact that $L(C^\nu(M); C^\nu(N))$ is complete and separable, we are able to characterize Bochner integrable curves on $L(C^\nu(M); C^\nu(N))$ using a family of generating seminorm on $C^\nu(N)$. Then, we proceed by defining topologies on the space of locally Bochner integrable and continuous using appropriate families of seminorms. These topologies will be important in studying the relationship between time-varying vector fields and their flows. Finally, using the C^ν -topology, we prove that a locally absolutely continuous curve on $L(C^\nu(M); C^\nu(N))$ is almost everywhere differentiable.

3.6.1. Bochner integrable curves. In Section 2.3.5, we defined Bochner integrable and locally Bochner integrable curves on a locally convex space. In this section, using the fact that the locally convex space $L(C^\nu(M); C^\nu(N))$ is complete and separable, we characterize Bochner integrability of a curve on $L(C^\nu(M); C^\nu(N))$ in terms of a generating family of seminorms on $C^\nu(N)$.

Theorem 3.6.1. *Let $\{p_i\}_{i \in I}$ be a family of generating seminorms on $C^\nu(N)$. A curve $\xi : \mathbb{T} \rightarrow L(C^\nu(M); C^\nu(N))$ is locally Bochner integrable if and only if, for every $i \in I$ and every $f \in C^\nu(M)$, there exists $g \in L^1_{\text{loc}}(\mathbb{T})$ such that*

$$p_i(\xi(t)(f)) \leq g(t), \quad \forall t \in \mathbb{T}.$$

Proof. Since the space $L(C^\nu(M); C^\nu(N))$ is complete and separable, the proof follows from Theorem 2.3.40. □

A time-varying vector field can be considered as a curve on the space $L(C^\nu(M); C^\nu(M))$. Let $V : \mathbb{T} \times M \rightarrow TM$ be a time-varying C^ν -vector field, then we define $\widehat{V} : \mathbb{T} \rightarrow L(C^\nu(M); C^\nu(M))$ as

$$\widehat{V}(t)(f) = \widehat{V}(t)(f), \quad \forall f \in C^\nu(M).$$

3.6.2. Space $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$. Let $\{p'_i\}_{i \in I}$ be a family of generating seminorms for $C^\nu(N)$ and $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For every compact subinterval $\mathbb{I} \subseteq \mathbb{T}$, we define the seminorm $p'_{i,f,\mathbb{I}}$ on $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ as

$$p'_{i,f,\mathbb{I}}(X) = \int_{\mathbb{I}} p'_i(X(\tau)(f)) d\tau, \quad \forall X \in L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N))).$$

The family of seminorms $\{p'_{i,f,\mathbb{I}}\}$ generates a locally convex topology on the space $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$.

Theorem 3.6.2. *There is a canonical isomorphism between $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ and $L^1(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N))$. In particular, the vector space $L^1(\mathbb{T}; \text{Der}(C^\nu(M)))$ is complete.*

Proof. Since $L(C^\nu(M); C^\nu(N))$ is a locally convex space, this theorem follows from Theorem 2.3.60. \square

3.6.3. Space $L^1(\mathbb{T}; \text{Der}(C^\nu(M)))$. Let $\{p_i^\nu\}_{i \in I}$ be a family of generating seminorms for $C^\nu(N)$ and $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For every compact subinterval $\mathbb{I} \subseteq \mathbb{T}$, we define the seminorm $p_{i,f,\mathbb{I}}^\nu$ as

$$p_{i,f,\mathbb{I}}^\nu(X) = \int_{\mathbb{I}} p_i^\nu(X(\tau)(f)) d\tau, \quad \forall X \in \text{Der}(C^\nu(M))$$

The family of seminorms $\{p_{i,f,\mathbb{I}}^\nu\}$ generate a locally convex topology on the space $L^1(\mathbb{T}; \text{Der}(C^\nu(M)))$.

Theorem 3.6.3. *There is a canonical isomorphism between $L^1(\mathbb{T}; \text{Der}(C^\nu(M)))$ and $L^1(\mathbb{T}) \widehat{\otimes}_\pi \text{Der}(C^\nu(M))$. In particular, the vector space $L^1(\mathbb{T}; \text{Der}(C^\nu(M)))$ is complete.*

Proof. Since $\text{Der}(C^\nu(M))$ is a locally convex space, this theorem follows from Theorem 2.3.60. \square

3.6.4. Space $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$. Let $\{p_i^\nu\}_{i \in I}$ be a family of generating seminorms for $C^\nu(N)$ and $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Then, for every compact subinterval $\mathbb{I} \subseteq \mathbb{T}$, we define the seminorm $r_{i,f,\mathbb{I}}^\nu$ as

$$r_{i,f,\mathbb{I}}^\nu(X) = \sup\{p_i^\nu(X(t)(f)) \mid t \in \mathbb{I}\}, \quad \forall X \in L(C^\nu(M); C^\nu(N)).$$

The family of seminorms $\{r_{i,f,\mathbb{I}}^\nu(X)\}$ defines a locally convex topology on $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$.

Theorem 3.6.4. *There is a canonical isomorphism between $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ and $C^0(\mathbb{T}) \widehat{\otimes}_\epsilon \text{Der}(C^\nu(M))$. In particular, the vector space $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ is complete.*

Proof. Since $L(C^\nu(M); C^\nu(N))$ is a locally convex space, this theorem follows from Theorem 2.3.61. \square

3.6.5. Absolutely continuous curves. In Section 2.3.5, we defined absolutely continuous and locally absolutely continuous curves on a locally convex space. In this section, we show that locally absolutely continuous curves on $L(C^\nu(M); C^\nu(N))$ are almost everywhere differentiable.

Theorem 3.6.5. *Let $\xi : \mathbb{T} \rightarrow L(C^\nu(M); C^\nu(N))$ be a locally absolutely continuous curve on $L(C^\nu(M); C^\nu(N))$. Then ξ is differentiable for almost every $t \in \mathbb{T}$.*

Proof. Without loss of generality, we assume that \mathbb{T} is compact. Then there exists $\eta \in L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ such that

$$\xi(t) = \xi(t_0) + \int_{t_0}^t \eta(\tau) d\tau, \quad \forall t \in \mathbb{T}.$$

Therefore, it suffice to show that, for almost every $t_0 \in \mathbb{T}$, we have

$$\limsup_{t \rightarrow t_0} \frac{1}{t - t_0} \int_{t_0}^t (\eta(\tau) - \eta(t_0)) d\tau = 0, \quad t > t_0.$$

Since $C^0(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, by Theorem 2.3.57, the set $C^0(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N))$ is dense in $L^1(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N))$. Since the locally convex space $L(C^\nu(M); C^\nu(N))$ is complete, by Theorem 2.3.60 and Theorem 2.3.61, we have

$$\begin{aligned} C^0(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N)) &= C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N))), \\ L^1(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N)) &= L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N))). \end{aligned}$$

This implies that $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ is dense in $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$. Let $\{p_i\}_{i \in I}$ be a generating family of seminorms for $L(C^\nu(M); C^\nu(N))$. For $\epsilon > 0$ and $i \in I$, there exists $g \in C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ such that

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau < \epsilon.$$

So, we can write

$$\begin{aligned} \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau &\leq \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \\ &\quad + \frac{1}{t - t_0} \int_{t_0}^t p_i(g(\tau) - g(t_0)) d\tau + p_i(g(t_0) - \eta(t_0)). \end{aligned} \quad (3.6.1)$$

Since g is continuous, we get

$$\limsup_{t \rightarrow t_0} \frac{1}{t - t_0} \int_{t_0}^t p_i(g(\tau) - g(t_0)) d\tau = 0.$$

If we take limsup of both side of (3.6.1), we have

$$\begin{aligned} \limsup_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) \\ \leq \limsup_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) + p_i(g(t_0) - \eta(t_0)). \end{aligned}$$

Now suppose that there exists a set A such that $\mathbf{m}(A) \neq 0$ and we have

$$\limsup_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) \neq 0, \quad \forall t_0 \in A.$$

This implies that, there exists $\alpha > 0$ such that the set B defined as

$$B = \left\{ t_0 \in \mathbb{T} \mid \limsup_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) > \alpha \right\}.$$

has positive Lebesgue measure. However, we have

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau = \int_C p_i(g(\tau) - \eta(\tau)) d\tau + \int_D p_i(g(\tau) - \eta(\tau)) d\tau.$$

Where $C, D \subseteq \mathbb{T}$ are defined as

$$\begin{aligned} C &= \{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\}, \\ D &= \{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) \leq \frac{\alpha}{2}\}. \end{aligned}$$

This implies that

$$\int_C p_i(g(\tau) - \eta(\tau)) d\tau \geq \mathbf{m}\{C\} \frac{\alpha}{2}.$$

Therefore we have

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau \geq \int_C p_i(g(\tau) - \eta(\tau)) d\tau \geq \mathbf{m}\{C\} \frac{\alpha}{2}.$$

This means that

$$\mathbf{m}\left\{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\right\} \leq \frac{2}{\alpha} \int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau < \frac{2\epsilon}{\alpha}.$$

Also, by [25, Chapter 1, Theorem 4.3(a)], we have

$$\begin{aligned} \mathbf{m}\left\{t_0 \in \mathbb{T} \mid \limsup_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) > \frac{\alpha}{2}\right\} \\ \leq \frac{4}{\alpha} \int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau < \frac{4\epsilon}{\alpha}. \end{aligned}$$

So this implies that

$$\begin{aligned} \mathbf{m}(B) &\leq \mathbf{m}\left\{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\right\} \\ &\quad + \mathbf{m}\left\{t_0 \in \mathbb{T} \mid \lim_{t \rightarrow t_0} \left(\frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) > \frac{\alpha}{2}\right\} \leq \frac{6\epsilon}{\alpha}. \end{aligned}$$

Since ϵ can be chosen arbitrary small, this is a contradiction. \square

3.6.6. Space $\text{AC}(\mathbb{T}; \text{L}(C^\nu(M); C^\nu(N)))$. Let $\{p_i^\nu\}_{i \in I}$ be a family of generating seminorms for $\text{L}(C^\nu(M); C^\nu(N))$ and $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For every compact subinterval $\mathbb{I} \subseteq \mathbb{T}$, we define the seminorm $q_{K,f,\mathbb{I}}^\nu$ as

$$q_{i,f,\mathbb{I}}^\nu(X) = \int_{\mathbb{I}} p_i^\nu \left(\frac{dX}{d\tau}(\tau)(f) \right) d\tau.$$

The family of seminorms $\{p_{i,f,\mathbb{I}}^\nu, q_{i,f,\mathbb{I}}^\nu\}$ generate a locally convex topology on the space $\text{AC}(\mathbb{T}; \text{L}(C^\nu(M); C^\nu(N)))$.

3.7. Extension of real analytic vector fields

It is well-known that every real analytic function can be extended to a holomorphic function on a complex manifold. Similar to the case of real analytic functions, one can show that a real analytic vector field can also be extended to a holomorphic vector field on an appropriate complex manifold. We then proceed to study real analytic time-varying vector fields. Considering a time-varying real analytic vector field on M with some regularity in terms of time, one would expect that it can be extended to a holomorphic vector field on a complex manifold containing M . Unfortunately this is not generally true. As the following example shows, a measurable time-varying real analytic vector field may not even have a local holomorphic extension to a complex manifold.

Example 3.7.1. Let $X : \mathbb{R} \times \mathbb{R} \rightarrow T\mathbb{R}$ be a time-varying vector field defined as

$$X(t, x) = \begin{cases} \frac{t^2}{t^2+x^2} \frac{\partial}{\partial x} & x \neq 0 \text{ or } t \neq 0, \\ 0 & x, t = 0. \end{cases}$$

Then X is a time-varying vector field on \mathbb{R} which is locally integrable with respect to t and real analytic with respect to x . However, there does not exist a connected neighbourhood \bar{U} of $x = 0$ in \mathbb{C} on which X can be extended to a holomorphic function. To see this, let $\bar{U} \subseteq \mathbb{C}$ be a connected neighbourhood of $x = 0$ and let $\mathbb{T} \subseteq \mathbb{R}$ be a neighbourhood of $t = 0$. Let $\bar{X} : \mathbb{T} \times \bar{U} \rightarrow T\mathbb{C}$ be a time-varying vector field which is measurable in time and holomorphic in state such that

$$\bar{X}(t, x) = X(t, x) \quad \forall x \in \mathbb{R} \cap \bar{U}, \forall t \in \mathbb{T}.$$

Since $0 \in \mathbb{T}$, there exists $t \in \mathbb{T}$ such that $\text{cl}(D_t(0)) \subseteq \bar{U}$. Let us fix this t and define the real analytic vector field $X_t : \mathbb{R} \rightarrow T\mathbb{R}$ as

$$X_t(x) = \frac{t^2}{t^2+x^2} \frac{\partial}{\partial x}, \quad \forall x \in \mathbb{R},$$

and the holomorphic vector field $\bar{X}_t : \bar{U} \rightarrow T\mathbb{C}$ as

$$\bar{X}_t(z) = \bar{X}(t, z) \quad \forall z \in \bar{U},$$

Then it is clear that \overline{X}_t is a holomorphic extension of X_t . However, one can define another holomorphic vector field $Y : D_t(0) \rightarrow T\mathbb{C}$ by

$$Y(z) = \frac{t^2}{t^2 + z^2} \frac{\partial}{\partial z}, \quad \forall z \in D_t(0),$$

It is easy to observe that Y is also a holomorphic extension of X_t . Thus, by the identity theorem, we should have $Y(z) = \overline{X}_t(z)$, for all $z \in D_t(0)$. Moreover, we should have $\overline{U} \subseteq D_t(0)$. However, this is a contradiction with the fact that $\text{cl}(D_t(0)) \subseteq \overline{U}$.

However, one can show that every locally Bochner integrable time-varying real-analytic vector field can be extended to a locally Bochner integrable time-varying holomorphic vector field. This extension result will show its significance in the next sections in proving convergence of the sequence of Picard iterations and continuity of the exponential map.

3.7.1. Global extension of real analytic vector fields. As mentioned in the introduction, not every time-varying real analytic vector field can be extended to a holomorphic one on a complex neighbourhood of its domain. However, by imposing some appropriate joint condition on time and state, one can show that such an extension exists. In this section, we show that every “locally Bochner integrable” time-varying real analytic vector field on a real analytic manifold M , can be extended to a locally Bochner integrable, time-varying holomorphic vector field on a complex neighbourhood of M . Moreover, we show that if X is a continuous time-varying real analytic vector field, then its extension \overline{X} is a continuous time-varying holomorphic vector field.

We state the following lemma which turns out to be useful in studying extension of real analytic vector fields. The proof of the first lemma is given in [39, Corollary 1].

Lemma 3.7.2. *Let Λ be a directed set and $(E_\alpha, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$ be a directed system of locally convex spaces with locally convex inductive limit $(E, \{i_\alpha\}_{\alpha \in \Lambda})$. Let F be a subspace of E such that, for every $\alpha \in \Lambda$, we have*

$$E_\alpha = \text{cl}_{E_\alpha}(i_\alpha^{-1}(F)).$$

Then F is a dense subset of E .

Having a directed set Λ and a locally convex directed system $(E_\alpha, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$, for every $\beta \succeq \alpha$, one can define $\tilde{i}_{\alpha\beta} : L^1(\mathbb{T}; E_\alpha) \rightarrow L^1(\mathbb{T}; E_\beta)$ as

$$\tilde{i}_{\alpha\beta}(f)(t) = i_{\alpha\beta}(f(t)), \quad \forall t \in \mathbb{T}.$$

We can also define the map $\tilde{i}_\alpha : L^1(\mathbb{T}; E_\alpha) \rightarrow L^1(\mathbb{T}; E)$ as

$$\tilde{i}_\alpha(f)(t) = i_\alpha(f(t)).$$

Then it is clear that $(L^1(\mathbb{T}; E_\alpha), \{\tilde{i}_{\alpha\beta}\}_{\beta \succeq \alpha})$ is a directed system of locally convex spaces.

Lemma 3.7.3. *Let $\mathbb{T} \subseteq \mathbb{R}$ be a compact interval, Λ be a directed set, and $(E_\alpha, \{i_{\alpha\beta}\})_{\beta, \alpha \in \Lambda}$ be a directed system of locally convex spaces with locally convex inductive limit $(E, \{i_\alpha\}_{\alpha \in \Lambda})$. Then $(L^1(\mathbb{T}; E_\alpha), \{\tilde{i}_{\alpha\beta}\})_{\beta, \alpha \in \Lambda}$ is a directed system of locally convex spaces with locally convex inductive limit $(L^1(\mathbb{T}; E), \{\tilde{i}_\alpha\}_{\alpha \in \Lambda})$.*

Proof. Since $L^1(\mathbb{T})$ is a normable space, by [43, Corollary 4, §15.5], we have $\varinjlim_\alpha L^1(\mathbb{T}) \otimes_\pi E_\alpha = L^1(\mathbb{T}) \otimes_\pi E$. Let $F = L^1(\mathbb{T}) \otimes_\pi E$. Then, for every $\alpha \in \Lambda$, we have

$$L^1(\mathbb{T}) \otimes_\pi E_\alpha \subseteq \tilde{i}_\alpha^{-1}(F).$$

This implies that

$$L^1(\mathbb{T}; E_\alpha) = \text{cl}(\tilde{i}_\alpha^{-1}(F)).$$

Then by using Lemma 3.7.2, we have that F is a dense subset of $\varinjlim_\alpha L^1(\mathbb{T}; E_\alpha)$. This means that $\varinjlim_\alpha L^1(\mathbb{T}; E_\alpha) = L^1(\mathbb{T}; E)$. \square

Using Lemmata 3.7.2 and 3.7.3, one can deduce the following result which we refer to as the global extension of real analytic vector fields.

Theorem 3.7.4. *Let M be a real analytic manifold and let \mathcal{N}_M be the family of all neighbourhoods of M . Then we have*

$$\varinjlim_{\bar{U}_M \in \mathcal{N}_M} L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_M)) = L^1(\mathbb{T}; \Gamma^\omega(TM)).$$

Corollary 3.7.5. *Let $X \in L^1(\mathbb{T}; \Gamma^\omega(TM))$. There exists a Stein neighbourhood \bar{U}_M of M and a locally Bochner integrable time-varying holomorphic vector field $\bar{X} \in L^1(\mathbb{T}; \Gamma^{\text{hol}}(\bar{U}_M))$ such that $\bar{X}(t, x) = X(t, x)$, for every $t \in \mathbb{T}$ and every $x \in M$.*

Similarly, one can study the extension of continuous time-varying real analytic vector fields. While a continuous time-varying real analytic vector fields is locally Bochner integrable, it has a holomorphic extension to a suitable domain. However, this raises the question of whether the holomorphic extension of a “continuous” time-varying real analytic vector field is a “continuous” time-varying holomorphic vector field or not. Using the following lemma, we show that the answer to the above question is positive.

Lemma 3.7.6. *Let K be a compact topological space, Λ be a directed set, and $(E_\alpha, \{i_{\alpha\beta}\})_{\beta \succeq \alpha}$ be a directed family of nuclear locally convex spaces with locally convex inductive limit $(E, \{i_\alpha\}_{\alpha \in \Lambda})$. Then $(C^0(K; E_\alpha), \{\hat{i}_{\alpha\beta}\})_{\beta \succeq \alpha}$ is a directed system of locally convex spaces with inductive limit $(C^0(K; E), \{\hat{i}_\alpha\}_{\alpha \in \Lambda})$.*

Proof. Since $C^0(K)$ is a normable space, by [43, Corollary 4, §15.5], we have $\varinjlim_\alpha C^0(K) \otimes_\pi E_\alpha = C^0(K) \otimes_\pi E$. For every $\alpha \in \Lambda$, the space E_α is nuclear. Therefore, by Theorem 2.3.67, we have

$$C^0(K) \otimes_\pi E_\alpha = C^0(K) \otimes_\epsilon E_\alpha, \quad \forall \alpha \in \Lambda.$$

Moreover, the space E is nuclear. So, by Theorem 2.3.67, we have

$$C^0(K) \otimes_{\pi} E = C^0(K) \otimes_{\epsilon} E.$$

This implies that

$$\varinjlim_{\alpha} C^0(K) \otimes_{\epsilon} E_{\alpha} = C^0(K) \otimes_{\epsilon} E.$$

We set $F = C^0(K) \otimes_{\epsilon} E$. Then, for every $\alpha \in \Lambda$, we have

$$C^0(K) \otimes_{\epsilon} E_{\alpha} \subseteq \hat{i}_{\alpha}^{-1}(F).$$

This implies that

$$C^0(K; E_{\alpha}) \subseteq \text{cl} \left(\hat{i}_{\alpha}^{-1} F \right).$$

Then, by using Lemma 3.7.2, we have that F is a dense subset of $\varinjlim_{\alpha} C^0(K; E_{\alpha})$. This means that we have $\varinjlim_{\alpha} C^0(K; E_{\alpha}) = C^0(K; E)$. \square

Theorem 3.7.7. *Let K be a compact topological space, M be a real analytic vector field and \mathcal{N}_M be the family of all neighbourhoods of M , which is a directed set under inclusion. Then we have*

$$\varinjlim_{\bar{U}_M \in \mathcal{N}_M} C^0(K; \Gamma^{\text{hol}}(\bar{U}_M)) = C^0(K; \Gamma^{\omega}(TM)).$$

Proof. Let Λ be a directed set and $(E_{\alpha}, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$ be a directed system of locally convex spaces. Then, for every $\beta \succeq \alpha$, one can define $\hat{i}_{\alpha\beta} : C^0(K; E_{\alpha}) \rightarrow C^0(K; E_{\beta})$ as

$$\hat{i}_{\alpha\beta}(f)(u) = i_{\alpha\beta}(f(u)), \quad \forall u \in K.$$

For every $\alpha \in \Lambda$, we can also define the map $\hat{i}_{\alpha} : C^0(K; E_{\alpha}) \rightarrow C^0(K; E)$ as

$$\hat{i}_{\alpha}(f)(u) = i_{\alpha}(f(u)), \quad \forall u \in K.$$

Then it is clear that $(C^0(K; E_{\alpha}), \{\hat{i}_{\alpha\beta}\}_{\beta \succeq \alpha})$ is a directed system of locally convex spaces. The result follows from the above lemma. \square

3.7.2. Local extension of real analytic vector fields. In the previous section, we proved that every locally Bochner integrable real analytic vector field on M has a holomorphic extension on a neighbourhood of M . However, this result is true for extending one vector field. It is natural to ask that, if we have a family of locally integrable real analytic vector fields on M , can we extend every member of the family to holomorphic vector fields on one neighbourhood of M ? In order to answer this question, we need a finer result for the extension of real analytic vector fields. We will see that the projective limit representation of the space of real analytic vector fields helps us to get this extension result.

Theorem 3.7.8. *Let $K \subseteq M$ be a compact set and $\{\bar{U}_n\}_{n \in \mathbb{N}}$ be a sequence of Stein neighbourhoods of M such that*

$$\text{cl}(\bar{U}_{n+1}) \subseteq \bar{U}_n, \quad \forall n \in \mathbb{N}.$$

and $\bigcap_{n \in \mathbb{N}} \bar{U}_n = K$. Then we have $\varinjlim_{n \rightarrow \infty} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol}, \mathbb{R}})$. Moreover the direct limit is weakly compact and boundedly retractive.

Proof. We know that, by Theorem 3.2.6, for every $n \in \mathbb{N}$, the map $\rho_{\bar{U}_n}^{\mathbb{R}} : \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n) \rightarrow \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$ is a compact continuous map. Note that every $n \in \mathbb{N}$, the map $\text{id} \otimes \rho_{\bar{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}) \otimes_{\pi} \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n) \rightarrow L^1(\mathbb{T}) \otimes_{\pi} \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$ is defined by

$$\text{id} \otimes \rho_{\bar{U}_n}^{\mathbb{R}}(\xi(t) \otimes \eta) = \xi(t) \otimes \rho_{\bar{U}_n}^{\mathbb{R}}(\eta).$$

Since $L^1(\mathbb{T}) \otimes_{\pi} \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n)$ is a dense subset of $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n))$, one can extend the map $\text{id} \otimes \rho_{\bar{U}_n}^{\mathbb{R}}$ into the map $\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n)) \rightarrow L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))$. We show that $\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}$ is weakly compact.

In order to show that $\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}$ is weakly compact, it suffices to show that for a bounded set $B \subset L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n))$, the set $\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}(B)$ is relatively weakly compact in $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))$. Since $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))$ is a complete locally convex space, by Theorem 2.3.25, the set

$$\text{cl} \left(\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}(B) \right)$$

is weakly compact if it is weakly sequentially compact. Therefore, it suffices to show that $\text{cl} \left(\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}(B) \right)$ is weakly sequentially compact. Let $\{f_n\}_{n=1}^{\infty}$ in $\text{cl} \left(\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}(B) \right)$. Since $\text{cl} \left(\text{id} \widehat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}(B) \right)$ is bounded, for every seminorm p on $\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$, there exists $M > 0$ such that

$$p \left(\int_{\mathbb{T}} f_n(\tau) d\tau \right) \leq \int_{\mathbb{T}} p(f_n(\tau)) d\tau \leq M.$$

This implies that the sequence $\left\{ \int_{\mathbb{T}} f_n(\tau) d\tau \right\}_{n=1}^{\infty}$ is bounded in $\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$. Since $\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$ is a nuclear locally convex space, the sequence $\left\{ \int_{\mathbb{T}} f_n(\tau) d\tau \right\}_{n=1}^{\infty}$ is relatively compact in $\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$. Therefore, there is a subsequence $\{f_{n_r}\}_{r=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that

$$\left\{ \int_{\mathbb{T}} f_{n_r}(\tau) d\tau \right\}_{r=1}^{\infty}$$

is Cauchy in $\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n)$.

By Theorems 2.3.68 and 2.3.41, the strong dual of $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))$ is exactly $L^{\infty}(\mathbb{T}) \widehat{\otimes}_{\pi} (\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))'_{\beta}$. We first show that, for every $\xi \otimes \eta \in L^{\infty}(\mathbb{T}) \otimes (\Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_n))'$, the sequence

$$\{\xi \otimes \eta(f_{n_r})\}_{r=1}^{\infty}$$

is Cauchy in \mathbb{R} . Note that we have

$$\begin{aligned} \xi \otimes \eta(f_{n_r} - f_{n_s}) &= \int_{\mathbb{T}} \xi(t) \eta(f_{n_s}(t) - f_{n_r}(t)) dt \\ &\leq M \int_{\mathbb{T}} \eta(f_{n_s}(t) - f_{n_r}(t)) dt = M \eta \left(\int_{\mathbb{T}} (f_{n_s}(t) - f_{n_r}(t)) dt \right). \end{aligned}$$

Since the sequence $\left\{ \int_{\mathbb{T}} f_{n_r}(\tau) d\tau \right\}_{r=1}^{\infty}$ is Cauchy in $\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n)$, this implies that the sequence $\left\{ \xi \otimes \eta(f_{n_r}) \right\}_{r=1}^{\infty}$ is Cauchy in \mathbb{R} . Now we show that, for every $\lambda \in L^{\infty}(\mathbb{T}) \widehat{\otimes} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'$, the sequence

$$\left\{ \lambda(f_{n_r}) \right\}_{r=1}^{\infty}$$

is Cauchy in \mathbb{R} . Note that $L^{\infty}(\mathbb{T}) \otimes_{\pi} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'_{\beta}$ is a dense subset of $L^{\infty}(\mathbb{T}) \widehat{\otimes}_{\pi} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'_{\beta}$. So there exist nets $\{\xi_{\alpha}\}_{\alpha \in \Lambda}$ in $L^{\infty}(\mathbb{T})$ and $\{\eta_{\alpha}\}_{\alpha \in \Lambda}$ in $(\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'$ such that

$$\lim_{\alpha} \xi_{\alpha} \otimes \eta_{\alpha} = \lambda.$$

Thus, for every $\epsilon > 0$, there exists $\theta > 0$ such that

$$\|\xi_{\theta} \otimes \eta_{\theta}(v) - \lambda(v)\| \leq \frac{\epsilon}{3}, \quad \forall v \in \text{cl} \left(\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B) \right).$$

Since the sequence $\left\{ \xi_{\theta} \otimes \eta_{\theta}(f_{n_r}) \right\}_{r=1}^{\infty}$ is Cauchy in \mathbb{F} , for every $\epsilon > 0$, there exists $\tilde{N} > 0$ such that

$$\|\xi_N \otimes \eta_{\theta}(f_{n_s} - f_{n_r})\| < \frac{\epsilon}{3}, \quad \forall r, s > \tilde{N}.$$

Thus, for every $\epsilon > 0$, there exists $\tilde{N} > 0$ such that

$$\begin{aligned} \|\lambda(f_{n_s} - f_{n_r})\| &\leq \|\lambda(f_{n_s} - f_{n_r}) - \xi_{\theta} \otimes \eta_N(f_{n_s} - f_{n_r})\| + \|\xi_{\theta} \otimes \eta_N(f_{n_s} - f_{n_r})\| \\ &\leq \|\lambda(f_{n_s}) - \xi_{\theta} \otimes \eta_N(f_{n_s})\| + \|\lambda(f_{n_r}) - \xi_N \otimes \eta_{\theta}(f_{n_r})\| + \|\xi_N \otimes \eta_{\theta}(f_{n_s} - f_{n_r})\| < \epsilon. \end{aligned}$$

Therefore, the sequence $\{f_{n_r}\}_{r=1}^{\infty}$ is weakly Cauchy in $L^1(\mathbb{T}; \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))$. This completes the proof of weak compactness of the map $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\overline{U}_n)) \rightarrow L^1(\mathbb{T}; \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))$. Recall that in the proof of Theorem 3.2.7, for every $n \in \mathbb{N}$, we defined the continuous linear map $r_n^{\mathbb{R}} : \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n) \rightarrow \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\overline{U}_{n+1})$ by

$$r_n^{\mathbb{R}}(X) = X|_{\overline{U}_{n+1}}.$$

Then we have the following diagram:

$$\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_n) \xrightarrow{\rho_{\overline{U}_n}^{\mathbb{R}}} \Gamma^{\text{hol},\mathbb{R}}(U_n) \xrightarrow{r_n^{\mathbb{R}}} \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_{n+1}).$$

Therefore, we have the following diagram:

$$L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_n)) \xrightarrow{\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}} L^1(\mathbb{T}; \Gamma^{\text{hol}}(U_n)) \xrightarrow{\text{id} \widehat{\otimes} r_n^{\mathbb{R}}} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(U_{n+1})).$$

Since, $\text{id} \hat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}}$ is weakly compact, by [43, §17.2, Proposition 1], the composition $\text{id} \hat{\otimes} \rho_{\bar{U}_n}^{\mathbb{R}} \circ \text{id} \hat{\otimes} r_{\bar{U}_n}^{\mathbb{R}}$ is weakly compact. Therefore, the connecting maps in the inductive limit $\lim_{\substack{\longrightarrow \\ n \rightarrow \infty}} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol}, \mathbb{R}})$ are weakly compact.

Using Theorem 2.3.53, if we can show that the direct limit satisfies condition (M), then it would be boundedly retractive. Since the inductive limit $\lim_{\substack{\longrightarrow \\ n \rightarrow \infty}} \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n) = \mathcal{G}_K^{\text{hol}, \mathbb{R}}$ is compact, by Theorem 2.3.51, it satisfies condition (M). This means that there exists a sequence $\{V_n\}_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, V_n is an absolutely convex neighbourhood of 0 in $\Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n)$ and there exists $M_n > 0$ such that, for every $m > M_n$, the topologies induced from $\Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_m)$ on V_n are all the same. Now consider the sequence $\{L^1(\mathbb{T}; V_n)\}_{n \in \mathbb{N}}$. It is clear that, for every $n \in \mathbb{N}$, $L^1(\mathbb{T}; V_n)$ is an absolutely convex neighbourhood of 0 in $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n))$. For every seminorm p on $\Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n)$ and every $m > M_n$, there exists a seminorm q_m on $\Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_m)$ such that

$$p(v) \leq q_m(v), \quad \forall v \in V_n.$$

This implies that, for every $X \in L^1(\mathbb{T}; V_n)$, we have

$$\int_{\mathbb{T}} p(X(\tau)) d\tau \leq \int_{\mathbb{T}} q_m(X(\tau)) d\tau.$$

So, for every $m > M_n$, the topology induced on $L^1(\mathbb{T}; V_n)$ from $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_m))$ is the same as its original topology. Therefore, the inductive limit $\lim_{\substack{\longrightarrow \\ n \rightarrow \infty}} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol}, \mathbb{R}})$ satisfies condition (M) and it is boundedly retractive. \square

Using the local extension theorem developed here, we can state the following result, which can be considered as generalization of Corollary 3.7.5.

Corollary 3.7.9. *Let $B \subseteq L^1(\mathbb{T}; \Gamma^\omega(TM))$ be a bounded set. Then, for every compact set $K \subseteq M$, there exists a Stein neighbourhood \bar{U}_K of K and a bounded set $\bar{B} \in L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\bar{U}_n))$ such that, for every $X \in B$, there exists a $\bar{X} \in \bar{B}$ such that*

$$\bar{X}(t, x) = X(t, x) \quad \forall t \in \mathbb{T}, \forall x \in K.$$

Let M be a real analytic manifold and let $U \subseteq M$ be a relatively compact subset of M . Then, by the local extension theorem, for every $f \in C^\omega(M)$, there exists a neighbourhood $\bar{V} \subseteq M^{\mathbb{C}}$ of U such that f can be extended to a bounded holomorphic function $\bar{f} \in C_{\text{bdd}}^{\text{hol}}(\bar{V})$. It is useful to study the relationship between the seminorms of f and the seminorms of its holomorphic extension \bar{f} .

Theorem 3.7.10. *Let M be a real analytic manifold and U be a relatively compact subset of M . Then, for every neighbourhood $\bar{V} \subseteq M^{\mathbb{C}}$ of $\text{cl}(U)$, there exists $d > 0$ such that, for every $f \in C^\omega(M)$ with a holomorphic extension $\bar{f} \in C_{\text{bdd}}^{\text{hol}}(\bar{V})$, we have*

$$p_{K, \mathbf{a}}^\omega(f) \leq p_{\bar{V}}(\bar{f}), \quad \forall \mathbf{a} \in \mathbf{c}_0^\dagger(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d), \forall \text{ compact } K \subseteq U.$$

Proof. Since \bar{f} is a holomorphic extension of f , we have

$$\bar{f}(x) = f(x), \quad \forall x \in \text{cl}(U).$$

Since $\text{cl}(U)$ is compact, one can choose $d > 0$ such that, for every $x \in \text{cl}(U)$, we have $D_{(d)}(x) \subseteq \bar{V}$, where $(d) = (d, d, \dots, d)$. We set $D = \bigcup_{x \in U} D_{(d)}(x)$. Then we have $D \subseteq \bar{V}$. Using Cauchy's estimate, for every multi-index (r) and for every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we have

$$\frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} f(x)\| \leq \frac{a_0}{d} \frac{a_1}{d} \dots \frac{a_{|r|}}{d} \sup \{ \|\bar{f}(x)\| \mid x \in D \} \leq p_{\bar{V}}(\bar{f}), \quad \forall x \in U.$$

This implies that, for every compact set $K \subseteq U$ and every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we have

$$p_{K, \mathbf{a}}^\omega(f) \leq p_{\bar{V}}(\bar{f}).$$

□

3.8. Flows of time-varying vector fields

As mentioned in the previous sections, a time-varying C^ν -vector field can be considered as a curve on the locally convex space $L(C^\nu(M); C^\nu(M))$. Following the analysis in [3], the flow of a time-varying C^ν -vector field X can be considered as a curve $\zeta : \mathbb{T} \rightarrow L(C^\nu(M); C^\nu(U))$ which satisfies the following initial value problem on the locally convex space $L(C^\nu(M); C^\nu(U))$:

$$\begin{aligned} \frac{d\zeta}{dt}(t) &= \zeta(t) \circ X(t), & a.e. \ t \in \mathbb{T} \\ \zeta(0) &= \text{id}. \end{aligned} \tag{3.8.1}$$

Therefore, one can reduce the problem of studying the flow of a time-varying vector field to the problem of studying solutions of a linear differential equation on a locally convex vector space. The theory of ordinary differential equations on locally convex spaces is different in nature from the classical theory of ordinary differential equations on Banach spaces. While in the theory of differential equations on Banach spaces, there are many general results about existence, uniqueness and properties of the flows, which hold independently of the underlying Banach space, the theory of ordinary differential equations on locally convex spaces heavily depends on the nature of their underlying space. Many methods in the classical theory of ordinary differential equations in Banach spaces have no counterpart in the theory of ordinary differential equations on locally convex spaces [56].

In [3], the initial value problem (3.8.1) for smooth and real analytic time-varying vector fields has been studied. In the real analytic case, X is assumed to be a locally integrable time-varying C^ω -vector field on \mathbb{R}^n such that it can be extended to a bounded holomorphic vector field on a neighbourhood $\Omega \subseteq \mathbb{C}^n$ of \mathbb{R}^n . Using the

C^{hol} -topology on the space of holomorphic vector fields, it has been shown that the well-known sequence of Picard iterations for the initial value problem (3.8.1) converges and gives us the unique solution of (3.8.1) [3, §2, Proposition 2.1]. In the smooth case, the existence and uniqueness of solutions of (3.8.1) has been shown. However, for smooth but not real analytic vector fields, the sequence of Picard iterations associated to the initial value problem (3.8.1) does not converge [4, §2.4.4].

In this section, using the framework we developed in this chapter, we study the initial value problem (3.8.1) for the holomorphic and the real analytic cases. Our proof for the existence of the solution of (3.8.1) in the holomorphic case is similar to the one given in [3]. In the real analytic case, using the local extension theorem (3.7.5) and estimates for seminorms on the space of real analytic functions, we provide a direct method for proving and studying the convergence of sequence of Picard iterations. This method helps us to generalize the result of [3, §2, Proposition 2.1] to arbitrary locally Bochner integrable real analytic vector fields.

Theorem 3.8.1. *Let $X : \mathbb{T} \rightarrow \text{Der}(C^\nu(M))$ be a locally Bochner integrable time-varying vector field. Then, for every $t_0 \in \mathbb{T}$ and every $x_0 \in M$, there exists an interval $\mathbb{T}' \subseteq \mathbb{T}$ containing t_0 and an open set $U \subseteq M$ containing x_0 such that there exists a unique locally absolutely continuous curve $\zeta : \mathbb{T}' \rightarrow L^1(C^\nu(M); C^\nu(U))$ which satisfies the following initial value problem:*

$$\begin{aligned} \frac{d\zeta}{dt}(t) &= \zeta(t) \circ X(t), & \text{a.e. } t \in \mathbb{T}', \\ \zeta(t_0) &= \text{id}, \end{aligned} \tag{3.8.2}$$

and, for every $t \in \mathbb{T}'$, we have

$$\zeta(t)(fg) = \zeta(t)(f)\zeta(t)(g), \quad \forall f, g \in C^\nu(M).$$

Proof. We study two different cases.

1. $\nu = \infty$

In the smooth case, this theorem is just a restatement of the classical existence, local uniqueness, and C^∞ -dependence on initial condition proved in [42, Theorem 6.6].

2. $\nu \in \{\text{hol}, \omega\}$

Let $N = \dim(M)$ and $(V, (x^1, x^2, \dots, x^N))$ be a coordinate chart around x_0 . Without loss of generality, we can assume that \mathbb{T} is a compact interval. Let U be a relatively compact set such that $\text{cl}(U) \subseteq V$, $K \subseteq U$ be a compact set. For every $k \in \mathbb{N}$, we define $\phi_k : \mathbb{R} \rightarrow L(C^\nu(M); C^\nu(U))$ inductively as

$$\begin{aligned} \phi_0(t)(f) &= f|_U, & \forall t \in [t_0, T], \\ \phi_k(t)(f) &= f|_U + \int_{t_0}^t \phi_{k-1}(\tau) \circ X(\tau)(f) d\tau, & \forall t \in [t_0, T]. \end{aligned}$$

If p_K^ν is one of the seminorms p_K^{hol} or $\tilde{p}_{K,\mathbf{a}}^\omega$, we have the following lemma.

Lemma. *If p_K^ν is one of the seminorms p_K^{hol} or $\tilde{p}_{K,\mathbf{a}}^\omega$. Then, there exist a real number $T > t_0$ and locally integrable function $m \in L_{\text{loc}}^1(\mathbb{T})$ such that, for every $f \in C^\nu(M)$, there exist constants $M_f, \tilde{M}_f \in \mathbb{R}^{\geq 0}$*

$$\begin{aligned} p_{K,f}^\nu(\phi_n(t) - \phi_{n-1}(t)) &\leq (M(t))^n M_f, & \forall t \in [t_0, T], \forall n \in \mathbb{N}. \\ p_{K,f}^\nu((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) &\leq m(t)(M(t))^n \tilde{M}_f, & \forall t \in [t_0, T], \forall n \in \mathbb{N}. \end{aligned}$$

where $M : [t_0, T] \rightarrow \mathbb{R}$ is defined as

$$M(t) = \int_{t_0}^t m(\tau) d\tau, \quad \forall t \in [t_0, T].$$

Proof. (a) $\nu = \text{hol}$

We set:

$$d = \max\{\|x - y\| \mid x \in \text{cl}(U), y \in K\}.$$

Since X is locally Bochner integrally, by Theorem 3.6.1, there exists $m \in L_{\text{loc}}^1(\mathbb{T})$ such that

$$\frac{d}{2N} \max_i \{p_{\text{cl}(U)}^{\text{hol}}(X^i(t))\} \leq m(t).$$

Then we have

$$p_{K,f}^{\text{hol}}(X(t)) = p_K^{\text{hol}}\left(X^i(t) \frac{\partial f}{\partial x^i}\right) \leq N \max_i \{p_K^{\text{hol}}(X^i(t))\} \max_i \left\{p_K^{\text{hol}}\left(\frac{\partial f}{\partial x^i}\right)\right\}.$$

Using Cauchy's estimate, we have

$$\sup \left\{ \left\| \frac{\partial f}{\partial x^i}(x) \right\| \mid x \in K \right\} \leq \frac{1}{d} \sup\{\|f(x)\| \mid x \in \text{cl}(U)\}.$$

Therefore, we get

$$p_{K,f}^{\text{hol}}(X(t)) \leq \frac{N}{d} \max_i \{p_K^{\text{hol}}(X^i(t))\} p_{\text{cl}(U)}^{\text{hol}}(f) \leq \frac{1}{2} m(t) p_{\text{cl}(U)}^{\text{hol}}(f).$$

We set $M_f = p_{\text{cl}(U)}^{\text{hol}}(f)$. Moreover since $M(t) = \int_{t_0}^t m(\tau) d\tau$, there exists $T > t_0$ such that $|M(T)| < 1$.

Now, for every $n \in \mathbb{Z}_{>0}$, for every $t \in [t_0, T]$, we have

$$\begin{aligned} (\phi_n(t) - \phi_{n-1}(t))(f) &= \\ &\int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{n-1}} X(t_1) \circ X(t_2) \circ \dots \circ X(t_n)(f) dt_n dt_{n-1} \dots dt_1. \end{aligned}$$

This implies that, for every $t \in [t_0, T]$,

$$\begin{aligned} & p_{K,f}^{\text{hol}}(\phi_n(t) - \phi_{n-1}(t)) \\ & \leq \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} p_{K,f}^{\text{hol}}(X(t_1) \circ X(t_2) \circ \cdots X(t_n)) dt_n dt_{n-1} \cdots dt_1. \end{aligned}$$

Using induction on n , one can show that

$$p_{K,f}^{\text{hol}}(X(t_1) \circ X(t_2) \circ \cdots X(t_n)) \leq \frac{(2N)^n n!}{d^n} \left(\prod_{j=1}^n m(t_j) \right) p_{\text{cl}(U)}^{\text{hol}}(f).$$

This implies that, for every $t \in [t_0, T]$, we have

$$\begin{aligned} & \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} p_{K,f}^{\text{hol}}(X(t_1) \circ X(t_2) \circ \cdots X(t_n)) dt_n dt_{n-1} \cdots dt_1 \\ & \leq \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} \left(\frac{(2N)^n n!}{d^n} \prod_{i=1}^n m(t_i) \right) p_{\text{cl}(U)}^{\text{hol}}(f) dt_n dt_{n-1} \cdots dt_1 \\ & = (M(t))^n p_{\text{cl}(U)}^{\text{hol}}(f). \end{aligned}$$

Thus we have

$$p_{K,f}^{\text{hol}}(\phi_n(t) - \phi_{n-1}(t)) \leq (M(t))^n M_f.$$

Moreover, we have

$$\begin{aligned} & (\phi_n(t) - \phi_{n-1}(t)) \circ X(t) \\ & \leq \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} X(t_1) \circ X(t_2) \circ \cdots X(t_n) \circ X(t) dt_n dt_{n-1} \cdots dt_1. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & p_{K,f}^{\text{hol}}((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \\ & \leq \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} p_{K,f}^{\text{hol}}(X(t_1) \circ X(t_2) \circ \cdots X(t_n) \circ X(t)) dt_n dt_{n-1} \cdots dt_1 \\ & \leq \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} \left(\frac{(2N)^n n!}{d^n} \prod_{i=1}^n m(t_i) \right) p_{\text{cl}(U)}^{\text{hol}}(X(t)f) dt_n dt_{n-1} \cdots dt_1 \\ & \leq (M(t))^n \max_i \{ p_{\text{cl}(U)}^{\text{hol}}(X^i(t)) \} \max_i \left\{ p_{\text{cl}(U)}^{\text{hol}} \left(\frac{\partial f}{\partial x^i}(x) \right) \right\} \end{aligned}$$

If we set

$$\tilde{M}_f = \frac{2N}{d} \max_i \left\{ p_{\text{cl}(U)}^{\text{hol}} \left(\frac{\partial f}{\partial x^i}(x) \right) \right\}$$

Then, for every $n \in \mathbb{N}$, we have

$$p_{K,f}^{\text{hol}}((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in [t_0, T]$$

This completes the proof of the lemma for the holomorphic case.

(b) $\nu = \omega$

Since X is locally Bochner integrable, by Corollary 3.7.9, there exist a neighbourhood \bar{V} of U , a locally Bochner integrable vector field $\bar{X} \in L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{V}))$, and a function $\bar{f} \in C_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{V})$ such that \bar{X}_t and \bar{f} are the holomorphic extension of X and f over \bar{V} , respectively. Then, by Theorem 3.7.10, there exists $d > 0$ such that, for every compact set $K \subseteq U$ and every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$, we have

$$\begin{aligned} \tilde{p}_{K, \mathbf{a}}^\omega(f) &\leq p_{\bar{V}}(\bar{f}), \\ \max_i \{\tilde{p}_{K, \mathbf{a}}^\omega(X^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}(t))\}, \quad \forall t \in \mathbb{T}, \end{aligned}$$

Since X is locally Bochner integrable, there exists $m \in L^1(\mathbb{T})$ such that

$$4N \max_i \{p_{\bar{V}}(\bar{X}^i(t))\} \leq m(t), \quad \forall t \in \mathbb{T},$$

Then we define $M : \mathbb{T} \rightarrow \mathbb{R}$ as

$$M(t) = \int_{t_0}^t m(\tau) d\tau.$$

Since M is continuous, there exists $T \in \mathbb{T}$ such that

$$M(t) < 1, \quad \forall t \in [t_0, T].$$

Let $K \subseteq U$ be a compact set and let $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. We show by induction that, for every $n \in \mathbb{N}$, we have

$$\tilde{p}_{K, \mathbf{a}, f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \leq (M(t))^{n+1} \tilde{p}_{K, \mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in [t_0, T],$$

where, for every $n \in \mathbb{N}$, the sequence $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$ is defined as in Lemma 3.3.3:

$$a_{n,m} = \begin{cases} \left(\frac{m+1}{m}\right)^n a_m & n < m, \\ \left(\frac{m+1}{m}\right)^m a_m & n \geq m. \end{cases}$$

First note that for $n = 1$ we have

$$\phi_1(t) - \phi_0(t) = \int_{t_0}^t X(\tau) d\tau, \quad \forall t \in [t_0, T].$$

This implies that

$$\tilde{p}_{K, \mathbf{a}, f}^\omega(\phi_1(t) - \phi_0(t)) \leq \int_{t_0}^t \tilde{p}_{K, \mathbf{a}}^\omega(X(\tau)f) d\tau, \quad \forall t \in [t_0, T].$$

By inequality (3.3.2), we have

$$\tilde{p}_{K, \mathbf{a}}^\omega(X(t)f) \leq 4N \max_i \{\tilde{p}_{K, \mathbf{b}_1}^\omega(X^i(t))\} \tilde{p}_{K, \mathbf{a}_1}^\omega(f), \quad \forall t \in [t_0, T].$$

Therefore we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_1(t) - \phi_0(t)) &\leq \int_{t_0}^t 4N \max_i \{\tilde{p}_{K,\mathbf{b}_1}^\omega(X^i(\tau))\} \tilde{p}_{K,\mathbf{a}_1}^\omega(f) d\tau \\ &\leq M(t) \tilde{p}_{K,\mathbf{a}_1}^\omega(f). \end{aligned}$$

Now suppose that, for every $k \in \{1, 2, \dots, n-1\}$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{k+1}(t) - \phi_k(t)) \leq (M(t))^{k+1} \tilde{p}_{K,\mathbf{a}_{k+1}}^\omega(f), \quad \forall t \in [t_0, T].$$

By definition of ϕ_n , we have

$$\phi_{n+1}(t) - \phi_n(t) = \int_{t_0}^t (\phi_n(\tau) \circ X(\tau) - \phi_{n-1}(\tau) \circ X(\tau)) d\tau, \quad \forall t \in [t_0, T].$$

Taking $\tilde{p}_{K,\mathbf{a},f}^\omega$ of both side of the above equality, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \\ \leq \int_{t_0}^t \tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n(\tau) - \phi_{n-1}(\tau)) \circ X(\tau)) d\tau, \quad \forall t \in [t_0, T]. \end{aligned}$$

However, we know that by the induction hypothesis

$$\tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \leq (M(t))^n \tilde{p}_{K,\mathbf{a}_n}^\omega(X(t)f), \quad \forall t \in [t_0, T].$$

Moreover, by the inequality (3.3.2), we have

$$\tilde{p}_{K,\mathbf{a}_n}^\omega(X(t)f) \leq 4N(n+1) \max_i \{\tilde{p}_{K,\mathbf{b}_n}^\omega(X^i(t))\} \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in [t_0, T].$$

By Lemma 3.3.3, for every $n \in \mathbb{N}$, we have $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$. This implies that, for every $n \in \mathbb{N}$, we have

$$\max_i \{\tilde{p}_{K,\mathbf{b}_n}^\omega(X^i(t))\} \leq \max_i \{p_V(\bar{X}^i(t))\} < \frac{1}{4N} m(t), \quad \forall t \in [t_0, T].$$

Therefore, for every $n \in \mathbb{N}$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \leq (n+1)m(t)M^n(t) \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f).$$

Thus we get

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \\ \leq \int_{t_0}^t (n+1)(M(\tau))^n m(\tau) \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f) d\tau \\ = (M(t))^{n+1} \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in [t_0, T]. \end{aligned}$$

This completes the induction. Note that by Lemma 3.3.3, for every $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$a_{n,m} \leq ea_m \leq 6ed$$

This implies that, for every $n \in \mathbb{N}$, we have

$$\tilde{p}_{K,\mathbf{a}_n}^\omega(f) \leq p_{\bar{V}}(\bar{f}).$$

If we set $M_f = p_{\bar{V}}(\bar{f})$ then, for every $n \in \mathbb{N}$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \leq (M(t))^{n+1} M_f, \quad \forall t \in [t_0, T].$$

Moreover, for every $n \in \mathbb{N}$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \leq (M(t))^n \tilde{p}_{K,\mathbf{a}_n}^\omega(X(t)f), \quad \forall t \in [t_0, T].$$

However, by inequality (3.3.2), we have

$$\tilde{p}_{K,\mathbf{a}_n}^\omega(X(t)f) \leq 4N \max_i \{ \tilde{p}_{K,\mathbf{b}_n}^\omega \} \tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in [t_0, T].$$

Noting that we have

$$\max_i \{ \tilde{p}_{K,\mathbf{b}_n}^\omega(X^i(t)) \} \leq \max_i \{ p_{\bar{V}}(\bar{X}^i(t)) \} < \frac{1}{4N} m(t), \quad \forall t \in [t_0, T],$$

and

$$\tilde{p}_{K,\mathbf{a}_{n+1}}^\omega(f) \leq p_{\bar{V}}(\bar{f}), \quad \forall t \in [t_0, T].$$

Therefore, if we set $\tilde{M}_f = p_{\bar{V}}(\bar{f})$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n(t) - \phi_{n-1}(t)) \circ X(t)) \leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in [t_0, T].$$

This completes the proof of the lemma for the real analytic case. \square

We now show that, for every $n \in \mathbb{N}$, $\phi_n \in \text{AC}([t_0, T], \text{L}(C^\nu(M); C^\nu(U)))$. For every $n \in \mathbb{N}$, consider the following inequality:

$$\begin{aligned} p_{K,f}^\nu(\phi_{n-1}(t) \circ X(t)) &\leq p_{K,f}^\nu(X(t)) + \sum_{i=1}^{n-1} p_{K,f}^\nu((\phi_i(t) - \phi_{i-1}(t)) \circ X(t)) \\ &\leq p_{K,f}^\nu(X(t)) + \sum_{i=1}^{n-1} m(t)(M(t))^{i+1} \tilde{M}_f \\ &\leq m(t) \left(\sum_{i=0}^{n-1} (M(t))^i \right) \tilde{M}_f, \quad \forall t \in [t_0, T]. \end{aligned}$$

The function $g_n : [t_0, T] \rightarrow \mathbb{R}$ defined as

$$g_n(t) = m(t) \left(\sum_{i=0}^{n-1} M^i(t) \right), \quad \forall t \in [t_0, T],$$

is locally integrable. Thus, by Theorem 3.6.1, $\phi_{n-1} \circ X$ is locally Bochner integrable. So, by Definition 2.3.39, ϕ_n is absolutely continuous. Therefore, for every $n \in \mathbb{N}$, we have

$$p_{K,f}^\nu(\phi_n(t) - \phi_{n-1}(t)) \leq |M(T)|^n M_f, \quad \forall t \in [t_0, T].$$

Since $|M(T)| < 1$, one can deduce that the sequence $\{\phi_n\}_{n \in \mathbb{N}}$ converges uniformly on $[t_0, T]$ in $L(C^\nu(M); C^\nu(U))$. Since uniform convergence implies L^1 -convergence and the space $L^1([t_0, T]; L(C^\nu(M); C^\nu(U)))$ is complete, there exists $\phi \in L^1([t_0, T]; L(C^\nu(M); C^\nu(U)))$ such that

$$\lim_{n \rightarrow \infty} \phi_n = \phi,$$

where the limit is in L^1 -topology on $L^1([t_0, T]; L(C^\nu(M); C^\nu(U)))$. Now we need to show that ϕ satisfies the initial value problem (3.8.2). We prove that

$$\lim_{n \rightarrow \infty} \int_{t_0}^t \phi_n(\tau) \circ X(\tau) d\tau = \int_{t_0}^t \phi(\tau) \circ X(\tau) d\tau, \quad \forall t \in [t_0, T].$$

Note that, for every $n \in \mathbb{N}$, we have

$$p_{K,f}^\nu(\phi(t) - \phi_n(t)) \leq \sum_{k=n+1}^{\infty} (M(t))^k M_f.$$

This implies that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \int_{t_0}^t p_{K,f}^\nu((\phi(\tau) - \phi_n(\tau)) \circ X(\tau)) d\tau &\leq \int_{t_0}^t \sum_{k=n+1}^{\infty} m(\tau) (M(\tau))^k \tilde{M}_f \\ &\leq N(T - t_0) \sum_{i=n+1}^{\infty} |M(T)|^i \tilde{M}_f, \quad \forall t \in [t_0, T]. \end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \int_{t_0}^t \phi_n(\tau) \circ X(\tau) d\tau = \int_{t_0}^t \phi(\tau) \circ X(\tau) d\tau, \quad \forall t \in [t_0, T].$$

This implies that we have

$$\phi(t) = \lim_{n \rightarrow \infty} \phi_n(t) = \lim_{n \rightarrow \infty} \int_{t_0}^t \phi_{n-1}(\tau) \circ X(\tau) d\tau = \int_{t_0}^t \phi(\tau) \circ X(\tau) d\tau.$$

Thus ϕ satisfies the initial value problem (3.8.2). On the other hand

$$\phi \in L^1([t_0, T]; L(C^\nu(M); C^\nu(U))),$$

and

$$X \in L^1([t_0, T]; L(C^\nu(M); C^\nu(M))),$$

Therefore

$$\phi \circ X \in L^1([t_0, T]; L(C^\nu(M); C^\nu(U)))$$

By definition 2.3.39, ϕ is in the space $AC([t_0, T]; L(C^\nu(M); C^\nu(U)))$. This completes the proof.

One can also show that the sequence $\{\phi_n\}_{n \in \mathbb{N}}$ converges to ϕ in $AC([t_0, T]; L(C^\nu(M); C^\nu(U)))$. In order to show this, it suffices to show that, for every compact set $K \subseteq U$ and every $f \in C^\nu(M)$, we have

$$\lim_{n \rightarrow \infty} \int_{t_0}^t p_{K,f}^\nu \left(\frac{d\phi_{n+1}}{dt} - \frac{d\phi_n}{dt} \right) = 0, \quad \forall t \in [t_0, T].$$

Note that, for every $n \in \mathbb{N}$, we have

$$\frac{d\phi_{n+1}}{dt} = \phi_n(t) \circ X(t), \quad \text{a.e., } t \in [t_0, T].$$

Therefore, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{t_0}^t p_{K,f}^\nu (\phi_n(t) \circ X(t) - \phi_{n-1}(t) \circ X(t)) = 0, \quad \forall t \in [t_0, T].$$

But we know that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} p_{K,f}^\nu (\phi_n(t) \circ X(t) - \phi_{n-1}(t) \circ X(t)) &\leq \\ &m(t)(M(t))^n \tilde{M}_f \leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in [t_0, T]. \end{aligned}$$

So we have

$$\begin{aligned} \int_{t_0}^t p_{K,f}^\nu (\phi_n(t) \circ X(t) - \phi_{n-1}(t) \circ X(t)) &\leq \frac{d}{(n+1)N} (M(T))^{n+1} \tilde{M}_f \\ &\leq \frac{d}{(n+1)N} (M(T))^{n+1} \tilde{M}_f. \end{aligned}$$

This completes the proof of convergence of $\{\phi_n\}_{n \in \mathbb{N}}$ in $AC([t_0, T]; L(C^\nu(M); C^\nu(U)))$.

To show the uniqueness of the solutions, we assume that there exists two solutions $\phi, \psi \in \text{AC}(\mathbb{T}; L(C^\nu(M); C^\nu(U)))$ for the initial value problem (3.8.2). If we choose $T > 0$ as in the lemma above, then we have

$$\phi(t) - \psi(t) = \int_{t_0}^t (\phi(\tau) - \psi(\tau)) \circ X(\tau) d\tau, \quad \forall t \in [t_0, T].$$

Therefore, for every $i \in \{1, 2, \dots, N\}$ and every convex open set $W \subseteq U$ such that $\text{cl}(W) \subseteq U$, we have

$$p_{\text{cl}(W), 0, x^i}^\infty(\phi(t) - \psi(t)) \leq \int_{t_0}^t p_{\text{cl}(W), 0, x^i}^\infty((\phi(\tau) - \psi(\tau)) \circ X(\tau)) d\tau, \quad \forall t \in [t_0, T].$$

Note that

$$p_{\text{cl}(W), 0, x^i}^\infty((\phi(t) - \psi(t)) \circ X(t)) = \sup \{ \|X^i(t, \phi(t, x)) - X^i(t, \psi(t, x))\| \mid x \in \text{cl}(W) \}.$$

Now, by using the mean value inequality, for every $x \in \text{cl}(W)$, there exists $r_x \in \text{cl}(W)$ such that

$$\|X^i(t, \phi(t, x)) - X^i(t, \psi(t, x))\| \leq \left\| \frac{\partial X^i}{\partial x^j}(t, r_x) \right\| \|\phi^j(t, x) - \psi^j(t, x)\|$$

Taking supremum over $x \in \text{cl}(W)$, we have

$$\begin{aligned} & \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty((\phi(t) - \psi(t)) \circ X(t)) \} \\ & \leq \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty(\phi(t) - \psi(t)) \} \max_i \{ p_{\text{cl}(W), 1, x^i}^\infty(X(t)) \}, \quad \forall t \in [t_0, T]. \end{aligned}$$

Since X is locally Bochner integrable, there exists $m \in L_{\text{loc}}^1(\mathbb{T})$ such that

$$\max_i \{ p_{\text{cl}(W), 1, x^i}^\infty(X(t)) \} \leq m(t), \quad \forall t \in [t_0, T].$$

Therefore, we have

$$\begin{aligned} & \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty((\phi(t) - \psi(t)) \circ X(t)) \} \\ & \leq m(t) \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty(\phi(t) - \psi(t)) \}, \quad \forall t \in [t_0, T]. \end{aligned}$$

This implies that

$$\begin{aligned} & \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty(\phi(t) - \psi(t)) \} \\ & \leq \int_{t_0}^t m(\tau) \max_i \{ p_{\text{cl}(W), 0, x^i}^\infty(\phi(\tau) - \psi(\tau)) \} d\tau, \quad \forall t \in [t_0, T]. \end{aligned}$$

Using the Grönwall inequality, we will get

$$\max_i \{p_{\text{cl}(W),0,x^i}^\infty(\phi(t) - \psi(t))\} \leq \max_i \{p_{\text{cl}(W),0,x^i}^\infty(\phi(t_0) - \psi(t_0))\} e^{\int_{t_0}^t m(\tau) d\tau}.$$

Since we have $\phi(t_0) = \psi(t_0) = \text{id}$, we get

$$\max_i \{p_{\text{cl}(W),0,x^i}^\infty(\phi(t) - \psi(t))\} = 0, \quad \forall t \in [t_0, T], \quad \forall \text{ convex open set } W \subseteq U.$$

So we have

$$\phi(t)(x) = \psi(t)(x), \quad \forall t \in [t_0, T], \quad \forall x \in \text{cl}(W).$$

This implies the local uniqueness of the solution of (3.8.2), on the interval $[t_0, T]$. \square

3.9. The exponential map

In this section, we study the relationship between locally integrable time-varying real analytic vector fields and their flows. In order to define such a map connecting time-varying vector fields and their flows, one should note that there may not exist a fixed interval $\mathbb{T} \subseteq \mathbb{R}$ containing t_0 and a fixed open neighbourhood $U \subseteq M$ of x_0 , such that the flow of “every” locally Bochner integrable time-varying vector field $X \in L^1(\mathbb{R}, \Gamma^\nu(TM))$ is defined on time interval \mathbb{T} and on neighbourhood U . The following example shows this for a family of vector fields.

Example 3.9.1. Consider the family of vector fields $\{X_n\}_{n \in \mathbb{N}}$, where $X_n : \mathbb{R} \times \mathbb{R} \rightarrow T\mathbb{R} \simeq \mathbb{R}^2$ is defined as

$$X_n(t, x) = (x, nx^2), \quad \forall t \in \mathbb{T}, \quad \forall x \in \mathbb{R}.$$

Let $\mathbb{T} = [-1, 1]$. Then, for every $n \in \mathbb{N}$, the flow of X_n is defined as

$$\phi^{X_n}(t, x) = \frac{x}{1 - nxt}.$$

This implies that ϕ^{X_n} is only defined for $x \in [-\frac{1}{n}, \frac{1}{n}]$. Therefore, there does not exist an open neighbourhood U of 0 such that, for every $n \in \mathbb{N}$, ϕ^{X_n} is defined on U .

The above example suggest that it is natural to define the connection between vector fields and their flows on their germs around t_0 and x_0 . Let $\mathbb{T} \subseteq \mathbb{R}$ be a compact interval containing $t_0 \in \mathbb{R}$ and $U \subseteq M$ be an open set containing $x_0 \in M$. We define

$$L_{(t_0, x_0)}^{1, \nu} = \varinjlim L^1(\mathbb{T}; \text{Der}(C^\nu(U))),$$

and

$$\text{AC}_{(t_0, x_0)}^\nu = \varinjlim \text{AC}(\mathbb{T}; L(C^\nu(M); C^\nu(U))).$$

These direct limits are in the category of topological spaces. We define the exponential map $\exp : L_{(t_0, x_0)}^{1, \nu} \rightarrow \text{AC}_{(t_0, x_0)}^\nu$ as

$$\exp([X]_{(t_0, x_0)}) = [\phi^X]_{(t_0, x_0)}, \quad \forall [X]_{(t_0, x_0)} \in L_{(t_0, x_0)}^{1, \nu}.$$

Theorem 3.9.2. *The exponential map is sequentially continuous.*

Proof. To show that $\exp : L_{(t_0, x_0)}^{1, \nu} \rightarrow AC_{(t_0, x_0)}^\nu$ is a sequentially continuous map, it suffices to prove that, for every sequence $\{X_n\}_{n \in \mathbb{N}}$ in $L^1(\mathbb{T}; \Gamma^\nu(TM))$ which converges to $X \in L^1(\mathbb{T}; \Gamma^\nu(TM))$, the sequence $\{[\phi^{X_n}]_{(t_0, x_0)}\}$ converges to $[\phi^X]_{(t_0, x_0)}$ in $AC_{(t_0, x_0)}^\nu$. Since the sequence $\{X_n\}_{n \in \mathbb{N}}$ is converging, it is bounded in $L^1(\mathbb{T}; \Gamma^\nu(TM))$. So, by Theorem 3.8.1, there exists $T > t_0$ and a relatively compact coordinate neighbourhood U of x_0 such that $[t_0, T] \subseteq \mathbb{T}$ and, for every $n \in \mathbb{N}$, we have $\phi^{X_n} \in AC([t_0, T]; L(C^\nu(M); C^\nu(U)))$. Therefore, it suffices to show that, for the sequence $\{X_n\}_{n \in \mathbb{N}}$ in $L^1(\mathbb{T}; \Gamma^\nu(TM))$ converging to $X \in L^1(\mathbb{T}; \Gamma^\nu(TM))$, the sequence $\{\phi^{X_n}\}$ converges to ϕ^X in $AC([t_0, T]; L(C^\nu(M); C^\nu(U)))$. We separate the proof for the cases $\nu = \infty$, $\nu = \text{hol}$, and $\nu = \omega$.

1. $\nu = \infty$:

The proof for the smooth case follows from Theorem 2.1 in [70] and Theorem 6.6 in [42].

2. $\nu = \text{hol}$:

Let $(V, \eta = (z^1, z^2, \dots, z^N))$ be a coordinate chart on M . There exists $r > 0$ such that

$$W = \{z \mid \|z\| \leq r\} \subseteq V.$$

We define $\Lambda = L^1(\mathbb{T}; \Gamma^{\text{hol}}(TM))$ and $F : \mathbb{T} \times V \times \Lambda \rightarrow \mathbb{C}^N$ as

$$F(t, \mathbf{x}, X) = X(t, \mathbf{x})$$

Since $X \in L^1(\mathbb{T}; \Gamma^{\text{hol}}(TM))$, one can easily check that F satisfies conditions of Theorem 1.1 in [70]. Therefore, there exist $T > 0$ and $U \subseteq K$ such that

$$\phi^F \in C^0([t_0, T] \times U \times \Lambda; \mathbb{C}^N)$$

where $\phi^F : [t_0, T] \times U \times \Lambda \rightarrow \mathbb{C}^N$ is the solution of the following differential equation

$$\frac{d}{dt} \phi^F(t, \mathbf{x}, X) = X(t, \phi^F(t, \mathbf{x}, X)), \quad \forall x \in U, \forall X \in \Lambda.$$

Using the identity

$$C^0([t_0, T] \times U \times \Lambda; \mathbb{C}^N) = C^0(\Lambda; C^0([t_0, T] \times U; \mathbb{C}^N)),$$

we can easily see that, if $\lim_{n \rightarrow \infty} X_n = X$ in $L^1(\mathbb{T}; \Gamma^{\text{hol}}(TM))$, then $\lim_{n \rightarrow \infty} \phi^{X_n} = \phi^X$ in $C^0([t_0, T] \times U; \mathbb{C}^N)$. Therefore, for every compact set $K \subseteq U$, every nonzero $f \in C^{\text{hol}}(M)$, and every $\epsilon > 0$, there exists $M > 0$ such that

$$\begin{aligned} & \sup \left\{ \left\| (\phi^X)^j(t, z) - (\phi^{X_n})^j(t, z) \right\| \mid z \in K, t \in [t_0, T] \right\} \\ & \leq \frac{\epsilon}{\sup \left\{ \left\| \frac{\partial f}{\partial x^j}(x) \right\| \mid x \in W \right\} (T - t_0)}, \quad \forall n > M. \end{aligned}$$

This implies that, for every $t \in [t_0, T]$, we have

$$\begin{aligned} p_{K,f}^{\text{hol}}(\phi^{X_n}(t) - \phi^X(t)) &= \sup\{\|f(\phi^{X_n}(t, z)) - f(\phi^X(t, z))\| \mid z \in K, t \in [t_0, T]\} \\ &\leq \sup\left\{\left\|\frac{\partial f}{\partial x^j}(x)\right\| \mid x \in W\right\} \times \\ &\sup\left\{\left\|(\phi^X)^j(t, z) - (\phi^{X_n})^j(t, z)\right\| \mid z \in K, t \in [t_0, T]\right\} < \frac{\epsilon}{T - t_0}, \quad \forall n > M. \end{aligned}$$

Therefore, we get

$$\int_{t_0}^t p_{K,f}^{\text{hol}}(\phi^{X_n}(\tau) - \phi^X(\tau)) d\tau < \epsilon, \quad \forall n > M.$$

This completes the proof of continuity of exponential map in the holomorphic case.

3. $\nu = \omega$:

Let $f \in C^\omega(M)$ be a real analytic function and suppose that we have

$$\lim_{m \rightarrow \infty} X_m = X$$

in $L^1(\mathbb{T}; \Gamma^\omega(U))$. By Theorems 3.7.4 and 3.7.8, there exists a neighbourhood $\bar{V} \subseteq M^{\mathbb{C}}$ of U such that the bounded sequence of locally integrable real analytic vector fields $\{X_m\}_{m \in \mathbb{N}}$, the real analytic vector field X , and the real analytic function f can be extended to a converging sequence of locally integrable holomorphic vector fields $\{\bar{X}_m\}_{m \in \mathbb{N}}$, a locally integrable holomorphic vector field X , and a holomorphic function \bar{f} respectively. Moreover, by Theorem 3.7.8, the inductive limit

$$\varinjlim L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n)) = L^1(\mathbb{T}; \Gamma^\omega(TM))$$

is boundedly retractive. Therefore, we have

$$\lim_{m \rightarrow \infty} \bar{X}_m = \bar{X}$$

in $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{V}))$. Now, according to Theorem 3.7.10, there exists $d > 0$, such that for every compact set $K \subseteq U$, every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, and every $t \in \mathbb{T}$, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a}}^\omega(f) &\leq p_{\bar{V}}(\bar{f}), \\ \max_i \{\tilde{p}_{K,\mathbf{a}}^\omega(X^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}^i(t))\}, \\ \max_i \{\tilde{p}_{K,\mathbf{a}}^\omega(X^i(t) - X_m^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}^i(t) - \bar{X}_m^i(t))\}. \end{aligned}$$

Since \bar{X} is locally integrable, by Theorem 3.6.1, there exists $g \in L^1(\mathbb{T})$ such that

$$\max_i \{p_{\bar{V}}(\bar{X}(t))\} < g(t), \quad \forall t \in \mathbb{T}.$$

This implies that, for every compact set $K \subseteq U$ and every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we have

$$\max_i \{p_{K,\mathbf{a}}^\omega(X^i(t))\} \leq \max_i \{p_{\bar{V}}(\bar{X}^i(t))\} < g(t), \quad \forall t \in \mathbb{T}.$$

This means that, for every $\epsilon > 0$, there exists $C \in \mathbb{N}$ such that

$$\int_{t_0}^t \max_i \{p_{\bar{V}}(\bar{X}_m^i(\tau) - \bar{X}^i(\tau))\} d\tau < \epsilon, \quad \forall m > C, t \in \mathbb{T}.$$

Therefore, if $m > C$, we have

$$\max_i \{p_{\bar{V}}(\bar{X}_m^i(t))\} \leq \max_i \{p_{\bar{V}}(\bar{X}^i(t))\} + \epsilon \leq g(t) + \epsilon, \quad \forall t \in \mathbb{T}, \forall m > C.$$

We define $m \in L^1(\mathbb{T})$ as

$$m(t) = g(t) + \epsilon, \quad \forall t \in \mathbb{T}.$$

We also define $\tilde{m} \in C(\mathbb{T})$ as

$$\tilde{m}(t) = \int_{t_0}^t (4N)m(\tau)d\tau, \quad \forall t \in \mathbb{T}.$$

We choose $T > t_0$ such that $|\tilde{m}(T)| < \frac{1}{2}$.

Lemma. *Let $K \subseteq U$ be a compact set and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$. Then, for every $n \in \mathbb{N}$, we have*

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) &\leq \left(\sum_{r=0}^{n-1} (r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f) \right) \times \\ &\int_{t_0}^t \max_i \{p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau))\} d\tau, \quad \forall t \in [t_0, T], \forall m > C, \end{aligned}$$

where \mathbf{a}_k is as defined in Lemma 3.3.3.

Proof. We prove this lemma using induction on $n \in \mathbb{N}$. We first check the case $n = 1$. For $n = 1$, using Theorem 3.3.4, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_1^X(t) - \phi_1^{X_m}(t)) &= \tilde{p}_{K,\mathbf{a},f}^\omega \left(\int_{t_0}^t X(\tau) - X_m(\tau) d\tau \right) \\ &\leq \int_{t_0}^t \tilde{p}_{K,\mathbf{a},f}^\omega(X(\tau) - X_m(\tau)) d\tau \\ &\leq \tilde{p}_{K,\mathbf{a}_1}^\omega(f) \int_{t_0}^t \max_i \{p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau))\} d\tau, \quad \forall t \in [t_0, T], \forall m > C, \end{aligned}$$

Now assume that, for $j \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_j^X(t) - \phi_j^{X_m}(t)) &\leq \sum_{r=0}^{j-1} ((r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f)) \times \\ &\int_{t_0}^t \max_i \left\{ p_V(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C. \end{aligned}$$

We want to show that

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) &\leq \sum_{r=0}^n ((r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f)) \times \\ &\int_{t_0}^t \max_i \left\{ p_V(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C. \end{aligned}$$

Note that one can write

$$\begin{aligned} \phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t) &= \int_{t_0}^t (\phi_n^X(\tau) \circ X(\tau) - \phi_n^{X_m}(\tau) \circ X_m(\tau)) d\tau \\ &= \int_{t_0}^t (\phi_n^X(\tau) - \phi_n^{X_m}(\tau)) \circ X(\tau) d\tau \\ &\quad + \int_{t_0}^t \phi_n^{X_m}(\tau) \circ (X(\tau) - X_m(\tau)) d\tau \quad \forall t \in [t_0, T], \quad \forall m > C. \end{aligned}$$

Therefore, for every compact set $K \subseteq U$ and every $\mathbf{a} \in \mathbf{c}_0^\dagger(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) &\leq \int_{t_0}^t \tilde{p}_{K,\mathbf{a},f}^\omega((\phi_n^X(\tau) - \phi_n^{X_m}(\tau)) \circ X(\tau)) d\tau \\ &\quad + \int_{t_0}^t \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^{X_m}(\tau) \circ (X(\tau) - X_m(\tau))) d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C. \end{aligned}$$

Note that, for every $X, Y \in L^1([t_0, T]; \Gamma^\omega(TM))$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) \circ Y(t)) = \tilde{p}_{K,\mathbf{a},f}^\omega(Y(t)) + \sum_{r=1}^n \tilde{p}_{K,\mathbf{a},f}^\omega((\phi_r^X(t) - \phi_{r-1}^X(t)) \circ Y(t))$$

Since, for every $r \in \mathbb{N}$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_r^X(t) - \phi_{r-1}^X(t)) \leq (\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_r}^\omega(f), \quad \forall t \in [t_0, T]$$

for every $X, Y \in L^1([t_0, T]; \Gamma^\omega(TM))$, we have

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) \circ Y(t)) \leq \sum_{r=0}^n (\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_r}^\omega(Y(t)), \quad \forall t \in [t_0, T].$$

This implies that, for every $t \in [t_0, T]$ and every $m > C$, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^{X_m}(t) \circ (X(t) - X_m(t))) &\leq \sum_{r=0}^n (\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_r,f}^\omega(X(t) - X_m(t)) \\ &\leq \sum_{r=0}^n ((r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f)) \max_i \left\{ p_{\bar{V}}(\bar{X}^i(t) - \bar{X}_m^i(t)) \right\}. \end{aligned}$$

Therefore, for every $t \in [t_0, T]$ and every $m > C$, we get

$$\begin{aligned} &\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) \\ &\leq \int_{t_0}^t \sum_{r=0}^{n-1} ((r+1)(r+2)(\tilde{m}(t))^r m(t) \tilde{p}_{K,\mathbf{a}_{r+2}}^\omega(f)) \int_{t_0}^t \max_i \left\{ p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) \right\} d\tau \\ &\quad + \int_{t_0}^t \sum_{r=0}^n ((r+1)(\tilde{m}(\tau))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f)) \max_i \left\{ p_{\bar{V}}(\bar{X}^i(t) - \bar{X}_m^i(t)) \right\} d\tau. \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} \tilde{p}_{K,\mathbf{a},f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) &\leq \sum_{r=0}^n (r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f) \times \\ &\quad \int_{t_0}^t p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \forall m > C. \end{aligned}$$

This completes the proof of the lemma \square

Thus, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} &\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) \\ &\leq \sum_{r=0}^{n-1} (r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f) \left(\int_{t_0}^t p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) d\tau \right), \quad \forall t \in [t_0, T], \forall m > C. \end{aligned}$$

Since, for every $t \in [t_0, T]$, we have

$$|\tilde{m}(t)| < \frac{1}{2},$$

the series

$$\sum_{r=0}^{\infty} (r+1)(\tilde{m}(t))^r \tilde{p}_{K,\mathbf{a}_{r+1}}^\omega(f)$$

converges to a function $h(t)$, for every $t \in [t_0, T]$. By Lebesgue's monotone convergence theorem, h is integrable. This implies that, for every $n \in \mathbb{N}$ and every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$,

$$\tilde{p}_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) \leq h(t) \int_{t_0}^t p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \forall m > C.$$

Therefore, by taking the limit as n goes to infinity of the left hand side of the inequality, we have

$$\tilde{p}_{K,\mathbf{a},f}^{\omega}(\phi^X(t) - \phi^{X_m}(t)) \leq h(t) \int_{t_0}^t p_{\bar{V}}(\bar{X}^i(\tau) - \bar{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \forall m > C.$$

This completes the proof of sequential continuity of \exp .

□

Chapter 4

Tautological control systems

4.1. Introduction

In geometric control theory, a control system can be described by a family of parametrized vector field $\{F^u\}_{u \in \mathcal{U}}$, where \mathcal{U} is the set of all controls. The evolution of this system is studied using trajectories of the system, i.e., solutions of the differential equations

$$\frac{dx(t)}{dt} = F^{u(t)}(x(t)), \quad (4.1.1)$$

for admissible controls $t \mapsto u(t)$. Many fundamental properties of a control system such as controllability, stabilizability, and reachability are defined using trajectories of the system. However, finding the solutions for the differential equations (4.1.1) is usually very hard, if not impossible. Therefore, one would like to get some information about these properties using the family of parametrized vector fields $\{F^u\}_{u \in \mathcal{U}}$. This has been an area of research for more than four decades in control theory and many deep and fundamental results about control systems have been proved. For instance, the accessibility of analytic control systems has been completely characterized in two independent works [79] and [50]. In [33], [34], [75], [77], [78] the problem of controllability of a system has been studied using nilpotent approximations and geometric methods. In [14], [76], [20], and [22] many deep and fundamental results in stabilizability of systems have been developed. While in the most of the these papers the analysis of control systems is done in the geometric framework mentioned above, this framework has some deficiencies.

First, one can easily notice that two different families of parametrized vector fields can generate the same family of trajectories. Therefore, in order to get characterization of fundamental properties of control systems, the conditions on $\{F^u\}_{u \in \mathcal{U}}$ should be parameter-invariant. Unfortunately, most of the criteria in the literature for studying fundamental properties of control systems depend on a specific parametrization of the system. The following example shows this fact about sufficient controllability conditions developed in [78].

Example 4.1.1. [15] Consider the following two control systems on \mathbb{R}^3 with the two

inputs. The first one is the system given by

$$\dot{x} = f_0(x) + u_1 f_1(x) + u_2 f_2(x), \quad \forall x \in \mathbb{R}^3 \quad \forall (u_1, u_2) \in [-1, 1]^2, \quad (4.1.2)$$

where

$$\begin{aligned} f_0(x_1, x_2, x_3) &= (x_1^2 - x_2^2) \frac{\partial}{\partial x_3}, \\ f_1(x_1, x_2, x_3) &= \frac{\partial}{\partial x_1}, \\ f_2(x_1, x_2, x_3) &= \frac{\partial}{\partial x_2}. \end{aligned}$$

The second one is the system given by the following equations

$$\dot{x} = g_0(x) + u_1 g_1(x) + u_2 g_2(x), \quad \forall x \in \mathbb{R}^3 \quad \forall (u_1, u_2) \in [-1, 1]^2, \quad (4.1.3)$$

where

$$\begin{aligned} g_0(x_1, x_2, x_3) &= (x_1^2 - x_2^2) \frac{\partial}{\partial x_3}, \\ g_1(x_1, x_2, x_3) &= \frac{1}{2} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \\ g_2(x_1, x_2, x_3) &= \frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right). \end{aligned}$$

It is easy to see that, by the following change of parametrization,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

one would get the control system (4.1.2) from the control system (4.1.3). This implies that two systems (4.1.2) and (4.1.3) have the same trajectories and they should have the same small-time local controllability around $(0, 0, 0)$. In order to study the small-time local controllability of system (4.1.2) from $(0, 0, 0)$, one can apply Theorem 7.3 in [78] and show that the system (4.1.2) is small-time locally controllable from $(0, 0, 0)$. However, when we apply the same theorem for the system (4.1.3), we get that Theorem 7.3 in [78] is indecisive for studying small-time local controllability of the system (4.1.3).

This example shows that the sufficient conditions in the literature for controllability of systems are not parameter-invariant. It also motivates the attempt to get parameter-invariant conditions for studying structural properties of control systems. Although checking that a condition is parameter-invariant is not impossible, it sometimes needs lots of efforts and huge amount of computations.

Secondly, the regularity of maps and functions plays an important role in studying control systems. Many fundamental results in control theory are only true for a

specific class of regularity. For example, in [79] the local accessibility of nonlinear systems with “real analytic” vector fields has been characterized using Lie brackets of vector fields of the system. However, it can be shown that such a characterization does not hold for “smooth” systems [79]. In the geometric control literature, the treatment of regularity is not coherent. For example, while it is very common to assume that $x \mapsto F^u(x)$ are smooth vector fields, one may need some extra joint conditions on regularity of F^u with respect to x and u to ensure nice properties of trajectories of the system. In the literature many different joint regularity conditions on $(x, u) \mapsto F^u(x)$ have been introduced. In [53] and [13], it has been assumed that \mathcal{U} is an open subset of \mathbb{R}^m and the map $u \mapsto F^u(x)$ is C^1 for every state x . However, in [21], it is assumed that this map is smooth. This is maybe because of the absence of a consistent framework for studying regularity of the control systems.

In this chapter, we introduce a model which has the following features.

1. It forgets the labeling of vector fields of the system by controls. This allows one to start with a control system which is parameter-independent at the very level of the definition.
2. Using the suitable topologies on spaces of vector fields developed in chapter 3, this model gives us a unified setting in which control systems of different regularity classes can be treated in the same manner. In particular, in this framework, one can study and analyze the class of “real analytic” systems in a consistent way.

In section 4.2, we give some motivations for defining what we call a C^ν -tautological control system. We then proceed to study the relationship between this new notion of control systems with the classical one. This has been done in sections 4.3, 4.4, and 4.5. Finally, we define the trajectories and reachable of C^ν -tautological control systems in 4.6.

4.2. C^ν -tautological control systems

In this section, following [55], we introduce a mathematical model for studying structure of control systems. At the heart of this model is the notion of a sheaf. Using the sheaf of C^ν -vector fields, this model makes the definition of control systems invariant of control-parameters. To motivate this definition of control system, we revisit the Example 4.1.1 in the introduction.

Example 4.2.1. Consider the two control systems (4.1.2) and (4.1.3) in the Example 4.1.1. It has been shown in the Example 4.1.1 that while these two systems have the same trajectories, Theorem 7.3 in [78] does not imply the same result about their small-time local controllability. Note that the control system (4.1.3) is obtained from the control system (4.1.2) by a linear transformation of parameters (u_1, u_2) . So it seems that the criteria in Theorem 7.3 of [78] depends on how we label our family of

vector fields. We forget the labeling of vector fields in a control system by consider them as a subset of space of C^ν -vector fields. So for the control system 4.1.2, we get the subset $\mathcal{F}_1 \subseteq \Gamma^\omega(T\mathbb{R}^3)$ given by

$$\mathcal{F}_1 = \{f_0 + u_1 f_1 + u_2 f_2 \mid (u_1, u_2) \in [-1, 1]^2\}.$$

Similarly, for the control system 4.1.3, we get the subset $\mathcal{F}_2 \subseteq \Gamma^\omega(T\mathbb{R}^3)$ given by

$$\mathcal{F}_2 = \{g_0 + u_1 g_1 + u_2 g_2 \mid (u_1, u_2) \in [-1, 1]^2\}.$$

It is easy to see that $\mathcal{F}_1 = \mathcal{F}_2$.

The above example shows that by considering vector fields of a control system as a subset of space of vector fields, one can forget the control-parametrization of systems. In order to emphasize the local-global behaviour of the control system, one can consider \mathcal{F}_1 as a subpresheaf of sets of the sheaf Γ^ω .

Definition 4.2.2. A C^ν -tautological control system is a pair (M, \mathcal{F}) where

1. M is a C^ν -manifold called **state manifold**, and
2. \mathcal{F} is a subpresheaf of sets of the sheaf of C^ν -vector fields on M .

A C^ν -tautological control system (M, \mathcal{F}) is **globally generated** if \mathcal{F} is a globally generated presheaf.

We will next study the correspondence between tautological control systems and classical control systems. In order to make this correspondence more clear, we need to define what we call a classical control system. We first define a specific family of C^ν -vector fields. This class of parametrized vector fields turns out to be useful in connecting the notion of tautological control system to the classical control system.

4.3. Parametrized vector fields of class C^ν

As mentioned in the introduction, in the geometric setting, a control system can be considered as a family of parametrized vector fields. However, the dependence of these vector fields on the parameters plays a crucial role in properties of systems. In many applications, it is completely natural to assume that the vector fields depends continuously on control u . This can be made precise in the following weak and strong versions.

Definition 4.3.1. Let \mathcal{U} be a topological space. A map $X : \mathcal{U} \times M \rightarrow TM$ is called a **separately parametrized vector field of class C^ν** if,

1. for every $u \in \mathcal{U}$, the map $X^u : M \rightarrow TM$ defined as

$$X^u(x) = X(u, x),$$

has the property that, for every $x \in M$, $X^u(x) \in T_x M$ and is of class C^ν , and

2. for every $x \in M$, the map $X_x : \mathcal{U} \rightarrow TM$ defined as

$$X_x(u) = X(u, x), \quad \forall u \in \mathcal{U}, \forall x \in M,$$

is continuous.

The class of separately parametrized vector field of class C^ν on M , with parameters in \mathcal{U} is denoted by $\text{SP}\Gamma^\nu(\mathcal{U}, M)$.

Definition 4.3.2. Let \mathcal{U} be a topological space. A map $X : \mathcal{U} \times M \rightarrow TM$ is called a **jointly parametrized vector field of class C^ν** if

1. X is separately parametrized of class C^ν , and
2. the map $\widehat{X} : \mathcal{U} \rightarrow \Gamma^\nu(TM)$ defined as

$$\widehat{X}(u)(x) = X^u(x), \quad \forall x \in M.$$

is continuous.

The class of jointly parametrized vector field of class C^ν on M , with parameters in \mathcal{U} is denoted by $\text{JP}\Gamma^\nu(\mathcal{U}, M)$. By definition, every jointly parametrized vector field of class C^ν is a separately parametrized vector field of class C^ν . So we have

$$\text{JP}\Gamma^\nu(\mathcal{U}, M) \subseteq \text{SP}\Gamma^\nu(\mathcal{U}, M).$$

4.4. C^ν -control systems

Now we are in the position to define a C^ν -control system.

Definition 4.4.1. A C^ν -control system is a triple (M, F, \mathcal{U}) such that

1. M is a manifold called **state manifold**,
2. \mathcal{U} is a topological space called **control set**, and
3. $F : \mathcal{U} \times M \rightarrow TM$ is a jointly parametrized vector field of class C^ν .

Given an admissible control $t \mapsto u(t)$, one would like to study the evolution of the system by applying that control. The notion of “open-loop system” roughly captures what happens to the vector fields of the system when you plug in the admissible control $t \mapsto u(t)$.

Definition 4.4.2. Let $\Sigma = (M, F, \mathcal{U})$ be a C^ν -control system, $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$. Then an **open-loop system associated to u** is a time-varying vector field $F^{(u(t))} : \mathbb{T} \times M \rightarrow TM$ defined as

$$F^{(u(t))}(t, x) = F(u(t), x), \quad \forall t \in \mathbb{T}, \forall x \in M.$$

One can define trajectories of a C^ν -control systems using open-loop system.

Definition 4.4.3. Let $\Sigma = (M, F, \mathcal{U})$ be a C^ν -control system, $t_0 \in \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$ be an interval containing t_0 , $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$, and $x_0 \in M$. Then a **trajectory** of Σ associated to u and starting from x_0 at time t_0 is a locally absolutely continuous curve $\gamma : \mathbb{T}' \rightarrow M$, for some interval $\mathbb{T}' \subseteq \mathbb{T}$ containing t_0 , which satisfies

$$\begin{aligned} \frac{d\gamma(t)}{dt} &= F^{(u(t))}(t, \gamma(t)), \quad \text{a.e } t \in \mathbb{T}', \\ \gamma(t_0) &= x_0, \end{aligned}$$

for the open-loop system $F^{(u(t))}$ associated to $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$.

This definition of open-loops of C^ν -control systems ensures many nice properties for the trajectories of these systems.

Theorem 4.4.4. Let $\Sigma = (M, F, \mathcal{U})$ be a C^ν -control system. Then, for every $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$, the map $\widehat{F}^u : \mathbb{T} \rightarrow \Gamma^\nu(M)$ defined as

$$\widehat{F}^u(t)(x) = F(u(t), x), \quad \forall t \in \mathbb{T}, \forall x \in M,$$

is locally essentially bounded and locally Bochner integrable.

Proof. Since Σ is a C^ν -control system, the map $F : \mathcal{U} \times M \rightarrow TM$ is jointly parametrized of class C^ν . This implies that the map $\widehat{F} : \mathcal{U} \rightarrow \Gamma^\nu(M)$ is continuous. Since $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$, for every compact interval $\mathbb{I} \subseteq \mathbb{T}$, there exists a compact set $K \subseteq \mathcal{U}$ such that

$$\mathfrak{m}\{t \in \mathbb{I} \mid u(t) \notin K\} = 0.$$

Since \widehat{F} is continuous and K is compact, the set $B = \widehat{F}(K)$ is bounded in $C^\nu(M)$. Thus we have

$$\mathfrak{m}\{t \in \mathbb{I} \mid \widehat{F}^u(t) \notin B\} \leq \mathfrak{m}\{t \in \mathbb{I} \mid u(t) \notin K\} = 0.$$

This implies that \widehat{F}^u is locally essentially bounded. Now, we show that \widehat{F}^u is locally Bochner integrable. Let $\{p_i\}_{i \in I}$ be a family of generating seminorms for $\Gamma^\nu(TM)$. Then, for every $i \in I$, there exists $m(t) \in L_{\text{loc}}^\infty(\mathbb{T}; \mathbb{R})$ such that

$$p_i(\widehat{F}^u(t)) \leq m(t), \quad \forall t \in \mathbb{T}.$$

Now since $L_{\text{loc}}^\infty(\mathbb{T}; \mathbb{R}) \subset L_{\text{loc}}^1(\mathbb{T}; \mathbb{R})$, \widehat{F}^u is locally Bochner integrable. \square

Theorem 4.4.5. Let $\Sigma = (M, F, \mathcal{U})$ be a C^ν -control system, $x_0 \in M$, $t_0 \in \mathbb{R}$, and $\mathbb{T} \subseteq \mathbb{R}$ be an interval containing t_0 . Then, for every $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; \mathcal{U})$, the trajectory of Σ associated to u starting from x_0 at time t_0 exists and is locally unique. Moreover, the resulting flow is of class C^ν with respect to the initial condition.

Proof. The proof follows from Theorem 3.8.1. \square

4.4.1. Extension of real analytic control systems. It is well-known that every real analytic function on a real analytic manifold M is a restriction of a holomorphic function on a complexification of M . In chapter 3, we also showed that every locally Bochner integrable time-varying vector field on a real analytic manifold M is restriction of a locally Bochner time-varying holomorphic vector field (Corollary 3.7.5). So it is interesting to see whether a real analytic parametrized vector fields on a real analytic manifold M is a restriction of holomorphic parametrized vector field on a complexification of M . It is easy to see that, in general, this is not the case for jointly parametrized vector fields.

Example 4.4.6. Let $\mathcal{U} = C^\omega(M)$ and consider $F : \mathcal{U} \times M \rightarrow TM$ which is defined as

$$F(X, x) = X(x).$$

Since $\widehat{F} : \mathcal{U} \rightarrow C^\omega(M)$ is the identity map, it is continuous. So F is a jointly parametrized vector field of class C^ω . Now assume that \overline{M} is a complexification of M and $\overline{F} : C^\omega(M) \times \overline{M} \rightarrow T\overline{M}$ is a jointly parametrized vector field of class C^{hol} such that

$$\overline{F}(X, x) = F(X, x), \quad \forall x \in M, \forall X \in C^\omega(M).$$

So $\widehat{\overline{F}} : C^\omega(M) \rightarrow C^{\text{hol}}(\overline{M})$ is defined as

$$\widehat{\overline{F}}(X) = \overline{X},$$

where \overline{X} is the holomorphic extension of X over \overline{M} . But, this is a contradiction and the jointly parametrized real analytic vector field F does not have any holomorphic extension.

One can show that, when \mathcal{U} is locally compact and Hausdorff, a jointly parametrized vector field of class C^ω is a restriction of a jointly parametrized vector field of class C^{hol} .

Theorem 4.4.7. *Let $X : \mathcal{U} \times M \rightarrow TM$ be a separately parametrized vector field of class C^ω .*

- (i) *if \mathcal{U} is locally compact and Hausdorff and X is jointly parametrized vector field of class C^ω , then, for every $u_0 \in \mathcal{U}$, there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{U}$ of u_0 , a complexification \overline{M} of the real analytic manifold M , and a jointly parametrized vector field of class C^{hol} , $\overline{X} : \mathcal{U} \times \overline{M} \rightarrow T\overline{M}$, such that*

$$\overline{X}(u, x) = X(u, x), \quad \forall u \in \mathcal{O}, \forall x \in M,$$

- (ii) *if, for a complexification \overline{M} of the real analytic manifold M , there exists a jointly parametrized vector field of class C^{hol} , $\overline{X} : \mathcal{U} \times \overline{M} \rightarrow T\overline{M}$, such that*

$$\overline{X}(u, x) = X(u, x), \quad \forall u \in \mathcal{U}, \forall x \in M,$$

then X is jointly parametrized vector field of class C^ω .

Proof. (i) Suppose that $u_0 \in \mathcal{U}$. Since \mathcal{U} is locally compact, there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{U}$ of u_0 such that $\text{cl}(\mathcal{O})$ is compact in \mathcal{U} . Since \mathcal{O} is compact and Hausdorff, $C^0(\text{cl}(\mathcal{O}))$ is a Banach space. Therefore, by Theorem 3.7.7, we have

$$\varinjlim C^0(\text{cl}(\mathcal{O}); \Gamma^{\text{hol}}(\bar{U}_M)) = C^0(\text{cl}(\mathcal{O}); \Gamma^\omega(M)).$$

If $X : \mathcal{U} \times M \rightarrow TM$ is a jointly parametrized vector field of class C^ω , then $\widehat{X} : \mathcal{U} \rightarrow \Gamma^\omega(TM)$ is continuous. So we have $\widehat{X}|_{\text{cl}(\mathcal{O})} \in C^0(\text{cl}(\mathcal{O}); \Gamma^\omega(M))$. So

by the above direct limit, there exists a neighbourhood \bar{U}_M of M and $\widehat{X} \in C^0(\text{cl}(\mathcal{O}); \Gamma^{\text{hol}}(\bar{U}_M))$ such that

$$\widehat{X}(u)(x) = \widehat{X}(u)(x), \quad \forall u \in \text{cl}(\mathcal{O}), \forall x \in M.$$

So if we define $\bar{X} : \text{cl}(\mathcal{O} \times \bar{U}_M) \rightarrow T\bar{U}_M$ as

$$\bar{X}(u, x) = \widehat{X}(u)(x), \quad \forall u \in \text{cl}(\mathcal{O}), \forall x \in \bar{U}_M$$

It is easy to see that \bar{X} is separately parametrized vector field of class C^{hol} . Moreover, we have

$$\widehat{\bar{X}} = \widehat{X}.$$

This implies that \bar{X} is a jointly parametrized vector field of class C^{hol} and we have

$$\bar{X}(u, x) = X(u, x), \quad \forall u \in \text{cl}(\mathcal{O}), \forall x \in M.$$

This completes the proof.

- (ii) It suffices to show that $\widehat{X} : \mathcal{U} \rightarrow \Gamma^\omega(TM)$ is continuous. Since \bar{X} is a jointly parametrized vector field of class C^{hol} , $\widehat{\bar{X}} : \mathcal{U} \rightarrow \Gamma^{\text{hol}}(T\bar{M})$ is continuous. Let $f \in C^\omega(M)$, $\mathbf{a} = (a_0, a_1, a_2, \dots) \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$, (U, ϕ) be a coordinate chart on M and $K \subseteq U$ be a compact set. Suppose that there exists a neighbourhood \bar{U} of M in \bar{M} such that f can be extended to a holomorphic function $\bar{f} \in C^{\text{hol}}(\bar{U})$. Let $d > 0$ be such that, for every $x \in K$, we have $D_{(d)}(x) \subseteq \bar{U}$. Since K is compact and $\{D_{(d)}(x)\}_{x \in K}$ is an open cover for K , there exists $x_1, x_2, \dots, x_m \in K$ such that $K \subseteq \bigcup_{i=1}^m D_d(x_i)$. We set $\bar{V} = \bigcup_{i=1}^m D_d(x_i)$. Note that \bar{V} is compact and $\bar{V} \subseteq \bar{U}$. Since $\lim_{i \rightarrow \infty} a_i = 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$, we have

$$a_n < d.$$

On the other hand, by Cauchy's estimate, we have

$$\frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} X f(x)\| \leq \frac{a_0 a_1 \dots a_{|r|}}{d^{|r|}} \sup\{\|\bar{X}\bar{f}(z)\| \mid z \in \bar{V}\}$$

If we set $\frac{a_0 a_1 \dots a_n}{d^n} = C$, then, for every multi-index (r) and every $x \in K$, we have

$$\frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} X f(x)\| \leq C \sup\{\|\bar{X}\bar{f}(z)\| \mid z \in \bar{V}\}$$

This implies that

$$p_{K,\mathbf{a},f}^\omega(X) \leq Cp_{\bar{V},\bar{f}}^{\text{hol}}(\bar{X}).$$

By continuity of \widehat{X} , for every $u_0 \in \mathcal{U}$ and every $\epsilon > 0$, there exists an open neighbourhood $\mathcal{O} \subseteq \mathcal{U}$ such that

$$p_{\bar{V},\bar{f}}^{\text{hol}}(\widehat{X}(u)) < \frac{\epsilon}{C}, \quad \forall u \in \mathcal{O}.$$

This implies that

$$p_{K,\mathbf{a},f}^\omega(\widehat{X}(u)) < \epsilon, \quad \forall u \in \mathcal{O}.$$

This completes the proof. □

4.5. From control systems to tautological control systems and vice versa

In this section, we build a correspondence between C^ν -control systems and C^ν -tautological control systems. Given a C^ν -control system $\Sigma = (M, F, \mathcal{U})$, one can define the assignment \mathcal{F}_Σ as

$$\mathcal{F}_\Sigma(U) = \{F^u|_U \mid u \in \mathcal{U}\}.$$

It is easy to check that (M, \mathcal{F}_Σ) is a globally generated C^ν -tautological control system. The system (M, \mathcal{F}_Σ) is called the C^ν -tautological control system associated to the C^ν -control system Σ . However, as is shown in Example 4.2.1, this correspondence is not one-to-one. This raises this question that, given a C^ν -tautological control system (M, \mathcal{F}) , is it coming from a C^ν -control system? In the next theorem, we will show that the answer is “yes” for globally generated C^ν -tautological control systems.

Theorem 4.5.1. *Let (M, \mathcal{F}) be a globally generated C^ν -tautological control system. Then we define $F_{\mathcal{F}} : \mathcal{F}(M) \times M \rightarrow TM$ as*

$$F_{\mathcal{F}}(X, x) = X(x), \quad \forall x \in M, \forall X \in \mathcal{F}(M).$$

We consider $\mathcal{F}(M)$ as a topological subspace of $\Gamma^\nu(TM)$ with the C^ν -topology. Then the triple $\Sigma_{\mathcal{F}} = (M, F_{\mathcal{F}}, \mathcal{F}(M))$ is a C^ν -control system. Moreover, we have $\mathcal{F}_{\Sigma_{\mathcal{F}}} = \mathcal{F}$.

Proof. It suffice to show that $F_{\mathcal{F}}$ is a jointly parametrized vector field of class C^ν . Note that, for every $X \in \mathcal{F}(U)$, we have

$$F_{\mathcal{F}}(X, x) = X(x) \in T_x M, \quad \forall x \in M.$$

Moreover, the map $\widehat{F}_{\mathcal{F}} : \mathcal{F}(M) \rightarrow \Gamma^\nu(TM)$ is defined as

$$\widehat{F}_{\mathcal{F}}(X)(x) = X(x), \quad \forall x \in M.$$

This implies that $\widehat{F}_{\mathcal{F}}$ is the inclusion map and it is clearly continuous in C^ν -topology on $\mathcal{F}(M)$ and $\Gamma^\nu(TM)$. To show the last part of the theorem, note that we have

$$F_{\mathcal{F}}^X = X, \quad \forall X \in \mathcal{F}(M).$$

This implies that, for every $U \subseteq M$, we have

$$\mathcal{F}_{\Sigma, \mathcal{F}}(U) = \{F_{\mathcal{F}}^X(U) \mid X \in \mathcal{F}(M)\} = \{X|_U \mid X \in \mathcal{F}(M)\} = \mathcal{F}(U).$$

This completes the proof. \square

4.6. Trajectories of C^ν -tautological systems

In previous sections, we studied C^ν -tautological control systems as presheaves of vector fields. In order to study the evolution of C^ν -tautological control systems, one needs to define trajectories for these systems. In the context of C^ν -control systems, one defines trajectories of the system by plugging in an admissible control $u \in L_{\text{loc}}^{\text{cpt}}(\mathbb{T}; U)$ and defining the trajectories of the resulting time-varying vector fields as the trajectories of the control system. In the control literature, the time-varying vector field which is obtained by plugging in the admissible control is called an ‘‘open-loop system’’. Thus, in this language, we define trajectories of a C^ν -control system Σ as the trajectories of open-loop families of Σ . This idea can be generalized to define the trajectories for C^ν -tautological systems. Therefore, one can naively define an open-loop system of a C^ν -tautological control system as a locally Bochner integrable time-varying C^ν -vector field $X : \mathbb{T} \times U \rightarrow TU$ such that

$$\widehat{X}(t) \in \mathcal{F}(U), \quad \forall t \in \mathbb{T}.$$

However, this definition does not cover all the desirable trajectories of the systems. In particular, with this definition of open-loop system, it is possible that concatenation of two trajectories is not a trajectory of the system. This can also affect the fundamental properties of the tautological control systems.

Example 4.6.1. Consider the C^ν -tautological control system $(\mathbb{R}^2, \mathcal{F})$ defined as

$$\mathcal{F}(U) = \begin{cases} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} & (0, 0) \notin U, \\ \left\{ \frac{\partial}{\partial x} \right\} & (0, 0) \in U \end{cases}$$

Consider the two points $(0, 0)$ and $(1, 1)$ in \mathbb{R}^2 . It is clear that by the above definition of open-loop systems, there does not exist a trajectory of (M, \mathcal{F}) starting from $(0, 0)$ and reaching $(1, 1)$. However, one can see that the curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ defined as

$$\gamma(t) = \begin{cases} (0, t) & 0 \leq t \leq \frac{1}{2}, \\ (t - \frac{1}{2}, \frac{1}{2}) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

is concatenation of trajectories of (M, \mathcal{F}) and connect $(0, 0)$ and $(1, 1)$. However, γ itself is not a trajectory of the system.

Definition 4.6.2. Associated to every C^ν -tautological control system (M, \mathcal{F}) , we define a sheaf $\text{LI}\mathcal{F}^\nu$ on open subsets of $\mathbb{R} \times M$. This sheaf is defined for every interval $\mathbb{T} \subseteq \mathbb{R}$ and every open set $U \subseteq M$ as

$$\text{LI}\mathcal{F}^\nu(\mathbb{T} \times U) = L^1(\mathbb{T}; \mathcal{F}(U)).$$

One can show that this define a sheaf on open subsets of $\mathbb{T} \times M$ [58, Chapter II, Theorem 2].

Recall that a local section of $\text{LI}\mathcal{F}^\nu$ is a continuous map $\mathcal{X} : W \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$, for some open set $W \subseteq \mathbb{R} \times M$ such that $\mathcal{X}(t, x) \in \text{LI}\mathcal{F}^\nu_{(t,x)}$, for all $(t, x) \in W$.

Definition 4.6.3. Let $W \subseteq \mathbb{T} \times \mathbb{R}$ be an open set. We define the projection maps $\text{pr}_1 : \mathbb{R} \times M \rightarrow \mathbb{R}$ and $\text{pr}_2 : \mathbb{R} \times M \rightarrow M$ as

$$\begin{aligned} \text{pr}_1(t, x) &= t, \\ \text{pr}_2(t, x) &= x. \end{aligned}$$

For every $t \in \text{pr}_1(W)$, we define $W^t = \{x \in M \mid (t, x) \in W\}$ and similarly for every $x \in \text{pr}_2(W)$, we define $W_x = \{t \in \mathbb{R} \mid (t, x) \in W\}$.

Definition 4.6.4. Let $W \subseteq \mathbb{T} \times \mathbb{R}$ be an open set.

1. Suppose that $(T_1, T_2) = \mathbb{T} \subseteq \mathbb{R}$ is an open interval, $U \subseteq M$ is an open set, and $\mathcal{X} : \mathbb{T} \times U \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$ is a local section of $\text{LI}\mathcal{F}^\nu$ defined on $\mathbb{T} \times U$. Then \mathcal{X} is a **piecewise constant local section of $\text{LI}\mathcal{F}^\nu$** , if there exist $n \in \mathbb{N}$, real numbers $T_1 = t_0 < t_1 < \dots < t_{n-1} < t_n = T_2$, and vector fields $X_i \in \Gamma^\nu(TU)$ for $i \in \{0, 1, 2, \dots, n-1\}$ such that

$$\mathcal{X}(t, x) = [X_i]_x, \quad \forall t \in (t_i, t_{i+1}), \quad \forall x \in U, \quad \forall i \in \{0, 1, 2, \dots, n-1\}.$$

2. Let $\mathcal{X} : W \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)(M)$ is a local section of $\text{LI}\mathcal{F}^\nu$. Then \mathcal{X} is a **piecewise constant local section of $\text{LI}\mathcal{F}^\nu$** , if for every $(t, x) \in W$, there exists an open interval $\mathbb{T}_t \subseteq \mathbb{R}$ and an open set $U_x \subseteq M$ such that $(t, x) \in \mathbb{T}_t \times U_x$ and $\mathcal{X}|_{\mathbb{T}_t \times U_x}$ is piecewise constant.

We prove the following fact about piecewise constant local sections of $\text{Sh}(\text{LI}\mathcal{F}^\nu)$.

Theorem 4.6.5. *Suppose that $\mathcal{X} : W \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$ is a piecewise constant local section of $\text{LI}\mathcal{F}^\nu$. Then, for every open set $V \subseteq W$, $\mathcal{X}|_V$ is also a piecewise constant local section of $\text{LI}\mathcal{F}^\nu$.*

Proof. It is clear that it suffices to prove the theorem for $W = \mathbb{T} \times U$ and $V = \mathbb{T}' \times U'$ where $\mathbb{T} = (T_1, T_2)$ and $\mathbb{T}' = (T'_1, T'_2)$ are open intervals such that $\mathbb{T}' \subset \mathbb{T}$, and U and U' are open subsets of M such that $U' \subseteq U$.

Suppose that $\mathcal{X} : \mathbb{T} \times U \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$ is a piecewise constant local section of $\text{LI}\mathcal{F}^\nu$. Then by definition, there exist $n \in \mathbb{N}$, real numbers $T_1 = t_0 < t_1 < \dots < t_{n-1} < t_n = T_2$, and vector fields $X_i \in \Gamma^\nu(TU)$ for $i \in \{0, 1, 2, \dots, n-1\}$ such that

$$\mathcal{X}(t, x) = [X_i]_x, \quad \forall t \in (t_i, t_{i+1}), \quad \forall x \in U, \quad \forall i \in \{0, 1, 2, \dots, n-1\}.$$

Since $(T'_1, T'_2) \subset (T_1, T_2)$, there exists $i, j \in \mathbb{N}$ such that $T'_1 < t_i < t_{i+1} < \dots < t_{j-1} < T'_2$. Now, we set $j - i = k \in \mathbb{N}$, $T'_1 = s_0$, $t_l = s_{l-i}$, for $l \in \{1, 2, \dots, k-1\}$, and $T'_2 = s_k$. Moreover, we define $Y_l = X_{l-i} |_{U'}$ for $l \in \{0, 1, 2, \dots, k\}$. Then it is clear that, for $\mathcal{X} |_{\mathbb{T}' \times U'}$, there exists $k+1 \in \mathbb{N}$, real numbers $T'_1 = s_0 < s_1 < \dots < s_k = T'_2$ and $Y_i \in \Gamma^\nu(TU'_i)$ for all $i \in \{0, 1, \dots, k\}$ such that

$$\mathcal{X} |_{\mathbb{T}' \times U'}(t, x) = [Y_i]_x, \quad \forall t \in (s_i, s_{i+1}), \quad \forall x \in U', \quad \forall i \in \{0, 1, 2, \dots, k\}.$$

This means that $\mathcal{X} |_{\mathbb{T}' \times U'}$ is piecewise constant. \square

Definition 4.6.6. An **étalé open-loop system** for (M, \mathcal{F}) is a local section of the sheaf $\text{Sh}(\text{LI}\mathcal{F}^\nu)$. An **étalé open loop subfamily** for (M, \mathcal{F}) is an assignment \mathcal{O}_σ to every open set $W \subset \mathbb{R} \times M$ such that

$$\mathcal{O}_\sigma(W) \subseteq \text{Sh}(\text{LI}\mathcal{F}^\nu)(W).$$

with the property that, if $W_1 \subseteq W_2$, then we have

$$\{r_{W_2, W_1}(X) \mid X \in \mathcal{O}_\sigma(W_2)\} \subseteq \mathcal{O}_\sigma(W_1).$$

Definition 4.6.7. 1. The **full étalé open-loop subfamily** of vector fields, denoted by $\mathcal{O}_{\text{full}}$, is the étalé open-loop subfamily for (M, \mathcal{F}) defined as

$$\mathcal{O}_{\text{full}}(W) = \text{Sh}(\text{LI}\mathcal{F}^\nu)(W),$$

for all open sets $W \subseteq \mathbb{R} \times M$.

2. The **piecewise constant étalé open-loop subfamily** of vector fields, denoted by \mathcal{O}_{pwc} , is the étalé open-loop subfamily for (M, \mathcal{F}) defined as the assignment

$$\mathcal{O}_{\text{pwc}}(W) = \{X \in \text{Sh}(\text{LI}\mathcal{F}^\nu)(W) \mid X \text{ is a piecewise constant open-loop system for } (M, \mathcal{F})\},$$

for all $W \subseteq \mathbb{R} \times M$.

Remark 4.6.8. By Theorem 4.6.5, it is clear that piecewise constant étalé open-loop subfamily is an étalé open-loop subfamily for (M, \mathcal{F}) .

Let (M, \mathcal{F}) be a C^ν -tautological control system and suppose that \mathcal{O}_σ is an étalé open-loop subfamily of (M, \mathcal{F}) and $W \subset \mathbb{R} \times M$ be an open set. Then

1. an (W, \mathcal{O}_σ) -**etalé trajectory** of (M, \mathcal{F}) is a locally absolutely continuous curve $\gamma : \mathbb{T} \rightarrow M$ such that there exists an open-loop system $X \in \mathcal{O}_\sigma(W)$ for (M, \mathcal{F}) such that

$$\frac{d\gamma}{dt}(t) = \text{ev}_{(t, \gamma(t))}(X(t, \gamma(t))), \text{ a.e. } t \in \mathbb{T},$$

and

2. an (\mathcal{O}_σ) -**etalé trajectory** of (M, \mathcal{F}) as a locally absolutely continuous curve $\gamma : \mathbb{T} \rightarrow M$ such that there exists open set $W \subseteq \mathbb{R} \times M$ such that γ is a (W, \mathcal{O}_σ) -trajectory of (M, \mathcal{F}) .

The set of all (W, \mathcal{O}_σ) -etalé trajectories of (M, \mathcal{F}) is denoted by $\text{ETraj}(W, \mathcal{O}_\sigma)$ and the set of all (\mathcal{O}_σ) -etalé trajectory of (M, \mathcal{F}) is denoted by $\text{ETraj}(\mathcal{O}_\sigma)$.

Chapter 5

The orbit theorem for tautological control systems

5.1. Introduction

In this chapter, we first define the orbits and reachable sets of a tautological control system. In particular, we show that the étalé trajectory that we define in Chapter 4 is consistent with the orbits of the system. The rest of the chapter focuses on studying the orbits of tautological control systems. We associate to every C^ν -tautological control system (M, \mathcal{F}) , a groupoid $\mathcal{G}_{\mathcal{F}}$ which is generated by the flows of the system. It can be shown that the orbits of this groupoid is the same as the orbits of its corresponding C^ν -tautological control system. We then proceed to study the geometric properties of orbit of C^ν -tautological control systems. In 1974, Sussmann [74] and Stefan [72] independently studied the orbits of a family of C^ν -vector fields on M and showed that the orbits are immersed C^ν -submanifold of M . Moreover, they completely characterized the tangent space to the orbits using the family of vector fields of the system. In this thesis, we generalized these results for C^ν -tautological control systems. In particular, for a C^ν -tautological control system $\Sigma = (M, \mathcal{F})$, we show that orbits of Σ are C^ν -immersed submanifold of M and we characterize the tangent space to orbits of Σ using the presheaf \mathcal{F} .

Moreover, we show that when Σ is a globally generated real analytic tautological control system, the tangent space to orbits of Σ passing through x can be characterized by the Lie brackets of vector fields of \mathcal{F} at point x .

5.2. Reachable sets of C^ν -tautological control systems

In order to study local properties of a tautological control system, one should define the notions of orbits, attainable sets, and reachable sets of the tautological control system. In this section, we generalize the notions of orbits and attainable sets to tautological control systems. Moreover, using the étalé trajectories, we define reachable sets of a tautological control system. We then show that reachable sets

are consistent with orbits and attainable sets. More specifically, we prove that the reachable set by étalé trajectories of piecewise constant vector fields is the same as attainable set of the system.

Definition 5.2.1. Let $T \in \mathbb{R}$. The T -**orbit** of (M, \mathcal{F}) passing through x_0 is the set

$$\text{Orb}_{\mathcal{F}}(T, x_0) = \{\phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_n}^{X_n}(x_0) \mid t_i \in \mathbb{R}, \sum_{i=1}^n t_i = T, \\ X_i \in \mathcal{F}(U_i), U_i \subseteq M, U_i \text{ open in } M, \forall i \in \{1, 2, \dots, n\}, \forall n \in \mathbb{Z}_{\geq 0}\}.$$

One can define the **orbit** of (M, \mathcal{F}) passing through x_0 as

$$\text{Orb}_{\mathcal{F}}(x_0) = \bigcup_{T \in \mathbb{R}} \text{Orb}_{\mathcal{F}}(T, x_0).$$

The T -**attainable set** of (M, \mathcal{F}) passing through x_0 is the set

$$\text{A}_{\mathcal{F}}(T, x_0) = \{\phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_n}^{X_n}(x_0) \mid t_i \in \mathbb{R}_{>0}, \sum_{i=1}^n t_i = T, \\ X_i \in \mathcal{F}(U_i), U_i \subseteq M, U_i \text{ open in } M, \forall i \in \{1, 2, \dots, n\}, \forall n \in \mathbb{Z}_{\geq 0}\}.$$

One can define the **attainable set** of (M, \mathcal{F}) passing through x_0 as

$$\text{A}_{\mathcal{F}}(x_0) = \bigcup_{T \geq 0} \text{A}_{\mathcal{F}}(T, x_0).$$

Definition 5.2.2. Let \mathcal{O}_σ be an étalé open loop subfamily of (M, \mathcal{F}) and suppose that $x_0 \in M$ and $T \in \mathbb{R}_{\geq 0}$. We define the $(T, U, \mathcal{O}_\sigma)$ -**reachable set** of (M, \mathcal{F}) from x_0 as

$$\text{R}_{\mathcal{F}}(T, x_0, U, \mathcal{O}_\sigma) = \{\gamma(T) \mid \gamma : [0, T] \rightarrow M \text{ is a} \\ (\mathcal{O}_\sigma)\text{-étalé trajectory of } (M, \mathcal{F}), \text{Image}(\gamma) \subseteq U, \gamma(0) = x_0\}.$$

We define the $(\leq T, U, \mathcal{O}_\sigma)$ -**reachable set** of (M, \mathcal{F}) from x_0 as

$$\text{R}_{\mathcal{F}}(\leq T, x_0, U, \mathcal{O}_\sigma) = \bigcup_{t \in [0, T]} \text{R}_{\mathcal{F}}(t, x_0, U, \mathcal{O}_\sigma).$$

And we define the (U, \mathcal{O}_σ) -**reachable set** of (M, \mathcal{F}) from x_0 as

$$\text{R}_{\mathcal{F}}(x_0, U, \mathcal{O}_\sigma) = \bigcup_{T \geq 0} \text{R}_{\mathcal{F}}(T, x_0, U, \mathcal{O}_\sigma).$$

Also, we can define (T, \mathcal{O}_σ) -**reachable set** of (M, \mathcal{F}) from x_0 as

$$\text{R}_{\mathcal{F}}(T, x_0, \mathcal{O}_\sigma) = \bigcup_{\substack{U \subseteq M \\ U \text{ is open}}} \text{R}_{\mathcal{F}}(T, x_0, U, \mathcal{O}_\sigma).$$

We define $(\leq T, \mathcal{O}_\sigma)$ -**reachable** set of (M, \mathcal{F}) from x_0 as

$$R_{\mathcal{F}}(\leq T, x_0, \mathcal{O}_\sigma) = \bigcup_{t \in [0, T]} R_{\mathcal{F}}(t, x_0, \mathcal{O}_\sigma).$$

We define the (\mathcal{O}_σ) -**reachable** set of (M, \mathcal{F}) from x_0 as

$$R_{\mathcal{F}}(x_0, \mathcal{O}_\sigma) = \bigcup_{T \geq 0} R_{\mathcal{F}}(T, x_0, \mathcal{O}_\sigma).$$

Definition 5.2.3 (Accessibility definitions). Let (M, \mathcal{F}) be a C^ν -tautological control system and $x_0 \in M$. Suppose that \mathcal{O}_σ is an étalé open-loop subfamily of (M, \mathcal{F}) . Then the étalé open-loop subfamily \mathcal{O}_σ is called

1. **small-time locally accessible** from x_0 if there exist $T > 0$ and open set $U \subset M$ containing x_0 such that, for every $t \in (0, T]$, we have $\text{int}(R_{\mathcal{F}}(\leq t, U, x_0, \mathcal{O}_\sigma)) \neq \emptyset$,
2. **locally accessible** from x_0 if there exists an open set $U \subset M$ containing x_0 such that $\text{int}(R_{\mathcal{F}}(U, x_0, \mathcal{O}_\sigma)) \neq \emptyset$,
3. **small-time accessible** from x_0 if there exists $T > 0$ such that, for every $t \in (0, T]$, we have $\text{int}(R_{\mathcal{F}}(\leq t, U, x_0, \mathcal{O}_\sigma)) \neq \emptyset$,
4. **accessible** from x_0 if $\text{int}(R_{\mathcal{F}}(x_0, \mathcal{O}_\sigma)) \neq \emptyset$.

Definition 5.2.4 (Fixed-time accessibility definitions). Let (M, \mathcal{F}) be a C^ν -tautological control system, $x_0 \in M$ and $T \in \mathbb{R}_{>0}$. Suppose that \mathcal{O}_σ is an étalé open-loop subfamily of (M, \mathcal{F}) . Then the étalé open-loop subfamily \mathcal{O}_σ is called

1. **locally T -accessible** from x_0 if there exists an open set $U \subset M$ containing x_0 such that we have $\text{int}(R_{\mathcal{F}}(T, U, x_0, \mathcal{O}_\sigma)) \neq \emptyset$,
2. **T -accessible** from x_0 if $\text{int}(R_{\mathcal{F}}(T, x_0, \mathcal{O}_\sigma)) \neq \emptyset$,
3. **locally strongly accessible** from x_0 if, for every $T > 0$, it is locally T -accessible from x_0 ,
4. **strongly accessible** from x_0 if, for every $T > 0$, it is T -accessible from x_0 .

Definition 5.2.5 (Reachability definitions). Let (M, \mathcal{F}) be a C^ν -tautological control system and $x_0 \in M$. Suppose that \mathcal{O}_σ is an étalé open-loop subfamily of (M, \mathcal{F}) . Then the étalé open-loop subfamily \mathcal{O}_σ is called

1. **small-time locally reachable** from x_0 if there exist $T > 0$ and open set $U \subset M$ such that for every $t \in (0, T]$ we have $x_0 \in \text{int}(R_{\mathcal{F}}(\leq t, U, x_0, \mathcal{O}_\sigma))$,
2. **locally reachable** from x_0 if there exists an open set $U \subset M$ such that $x_0 \in \text{int}(R_{\mathcal{F}}(U, x_0, \mathcal{O}_\sigma))$,

3. **small-time reachable** from x_0 if there exists $T > 0$ such that for every $t \in (0, T]$ we have $x_0 \in \text{int}(\mathbf{R}_{\mathcal{F}}(\leq t, U, x_0, \mathcal{O}_\sigma))$,
4. **reachable** from x_0 if $x_0 \in \text{int}(\mathbf{R}_{\mathcal{F}}(x_0, \mathcal{O}_\sigma))$,
5. **totally reachable** if it is reachable from every point $x \in M$.

Definition 5.2.6 (Fixed-time reachability definitions). Let (M, \mathcal{F}) be a C^ν -tautological control system, $x_0 \in M$ and $T \in \mathbb{T}_{>0}$. Suppose that \mathcal{O}_σ is an étalé open-loop subfamily of (M, \mathcal{F}) . Then the étalé open-loop subfamily \mathcal{O}_σ is called

1. **locally T -reachable** from x_0 if there exists an open set $U \subset M$ containing x_0 such that we have $x_0 \in \text{int}(\mathbf{R}_{\mathcal{F}}(T, U, x_0, \mathcal{O}_\sigma))$,
2. **T -reachable** from x_0 if $x_0 \in \text{int}(\mathbf{R}_{\mathcal{F}}(T, x_0, \mathcal{O}_\sigma))$,
3. **strongly reachable** from x_0 if, for every $T > 0$, it is T -reachable from x_0 .
4. **locally strongly reachable** from x_0 if, for every $T > 0$, it is locally T -reachable from x_0 .

The connection between notion of reachable sets and attainable sets can be expressed in the following theorem.

Theorem 5.2.7. *Suppose that (M, \mathcal{F}) is a C^ν -tautological control system. Then for every $x_0 \in M$, and for every $T \in \mathbb{R}_{\geq 0}$ we have*

$$\mathbf{A}_{\mathcal{F}}(T, x_0) = \mathbf{R}_{\mathcal{F}}(T, x_0, \mathcal{O}_{\text{pwc}}).$$

Proof. Note that, if $y \in \mathbf{A}_{\mathcal{F}}(T, x_0)$, there exists X_1, X_2, \dots, X_k and open sets $U_1, U_2, \dots, U_k \subseteq M$ such that $X_i \in \mathcal{F}(U_i)$, for all $i \in \{1, 2, \dots, k\}$ and $t_1, t_2, \dots, t_k \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^k t_i = T$ and we have

$$\phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(x_0) = y.$$

We set $T_i = 0 + t_1 + t_2 + \dots + t_i$, for all $t \in \{0, 1, 2, \dots, k\}$ and we define $W = \bigcup_{i=1}^k (T_{i-1}, T_i) \times U_i$. We define the piecewise constant local section $X : W \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$ as

$$X(t, x) = [X_i]_x, \quad \forall (t, x) \in (T_{i-1}, T_i) \times U_i, \quad \forall i \in \{1, 2, \dots, k\}.$$

It is clear that X is piecewise constant. So, if $\gamma : [0, T] \rightarrow U$ is the integral curve of X starting from $x_0 \in M$, we have

$$\gamma(T) = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(x_0) = y$$

Therefore, if $y \in \mathbf{A}_{\mathcal{F}}(T, x_0)$, then there exists $X \in \mathcal{O}_{\text{pwc}}$ such that $\gamma(T) = y$, where $\gamma : [0, T] \rightarrow M$ is the integral curve of X starting at $\gamma(0) = x_0$. This means that $y \in \mathbf{R}_{\mathcal{F}}(T, x_0, \mathcal{O}_{\text{pwc}})$. Therefore, we have the inclusion

$$\mathbf{A}_{\mathcal{F}}(T, x_0) \subseteq \mathbf{R}_{\mathcal{F}}(T, x_0, \mathcal{O}_{\text{pwc}}).$$

Now let $y \in R_{\mathcal{F}}(T, x_0, \mathcal{O}_{\text{pwc}})$. Therefore, there exists an \mathcal{O}_{pwc} -étalé trajectory $\xi : [0, T] \rightarrow M$ such that $\xi(0) = x_0$ and $\xi(T) = y$. Since ξ is a \mathcal{O}_{pwc} -étalé trajectory, there exists $X \in \mathcal{O}_{\text{pwc}}$ such that

$$\xi'(t) = \text{ev}_{(t, \xi(t))}(X(t, \xi(t))), \quad a.e. t \in [0, T].$$

If one consider $X : W \rightarrow \text{Et}(\text{LI}\mathcal{F}^\nu)$, then X is piecewise constant. So, for every $t \in [0, T]$, there exists an open interval $\mathbb{T}_t \subseteq \mathbb{R}$ and an open set $U_{\xi(t)} \subseteq M$ such that $(t, \xi(t)) \in \mathbb{T}_t \times U_{\xi(t)}$ and $\mathcal{X} |_{\mathbb{T}_t \times U_{\xi(t)}}$ is piecewise constant. Therefore, for every $t \in [0, T]$, there exists $T_0 < T_1 < \dots < T_n$ and vector fields $X_i \in \Gamma^\nu(TU_{\xi(t)})$, for all $i \in \{0, 1, \dots, n-1\}$, such that

$$\mathcal{X}(t, x) = [X_i]_x, \quad \forall t \in (T_i, T_{i+1}), \forall x \in U_{\xi(t)}, \forall i \in \{0, 1, 2, \dots, n-1\}.$$

Now consider the open cover $\{\mathbb{T}_t \times U_{\xi(t)}\}_{t \in [0, T]}$ of $\text{graph}(\xi)$. Since $\text{graph}(\xi)$ is a compact set, there exists finite subcover $\{\mathbb{T}_{t_i} \times U_{\xi(t_i)}\}_{i=1}^n$ of $\text{graph}(\xi)$. This shows that there exists $k \in \mathbb{N}$ and $0 = s_0 < s_1 < \dots < s_k = T$ and open sets $U_1, U_2, \dots, U_k \subseteq M$ and vector fields $X_i \in \Gamma^\nu(TU_k)$ such that

$$\mathcal{X}(t, x) = [X_i]_x, \quad \forall t \in (s_i, s_{i+1}), \forall x \in U_i, \forall i \in \{0, 1, 2, \dots, k-1\}.$$

So we have

$$\xi'(t) = X_i(\xi(t)), \quad \forall t \in (s_i, s_{i+1}), \forall i \in \{0, 1, \dots, k-1\}$$

By denoting $t_i = s_{i+1} - s_i$ for $i \in \{0, 1, 2, \dots, k-1\}$, one can easily see that

$$\xi(T) = \phi_{t_k}^{X_k} \circ \phi_{t_{k-1}}^{X_{k-1}} \circ \dots \circ \phi_{t_1}^{X_1}(x_0).$$

This shows that $y \in A_{\mathcal{F}}(T, x_0)$. Thus, we have $R_{\mathcal{F}}(T, x_0, \mathcal{O}_{\text{pwc}}) \subseteq A_{\mathcal{F}}(T, x_0)$. This completes the proof. \square

Corollary 5.2.8. *Suppose that (M, \mathcal{F}) is a C^ν -tautological control system. Then, for every $x_0 \in M$, we have*

$$A_{\mathcal{F}}(x_0) = R_{\mathcal{F}}(x_0, \mathcal{O}_{\text{pwc}}).$$

5.3. Algebraic structure of orbits

In this section we associate a groupoid to a C^ν -tautological control system. We will show that the groupoid and the C^ν -tautological control system have the same orbits.

Definition 5.3.1. Let (M, \mathcal{F}) be a C^ν -tautological control system and $U \subseteq M$ be an open set. For every $X \in \mathcal{F}(U)$, we define the **flow** of X as the pair (\mathcal{D}, ϕ^X) , where $\mathcal{D} \subseteq \mathbb{R} \times U$ is open and $\phi^X : \mathcal{D} \rightarrow U$ is the flow of X .

Note that, for every $x \in U$, the set \mathcal{D}^x defined as

$$\mathcal{D}^x = \{t \in \mathbb{R} \mid (t, x) \in \mathcal{D}\},$$

is an open interval containing 0. For every $t \in \mathbb{R}$, we define \mathcal{D}_t as

$$\mathcal{D}_t = \{x \in U \mid (t, x) \in \mathcal{D}\}.$$

Then, for every $t \in \mathbb{R}$, the set \mathcal{D}_t is open in U and the map $\phi_t^X : \mathcal{D}_t \rightarrow \mathcal{D}_{-t}$ defined as

$$\phi_t^X(x) = \phi^X(t, x) \quad \forall x \in \mathcal{D}_t.$$

is a C^ν -diffeomorphism with the inverse ϕ_{-t}^X .

Definition 5.3.2. The family $\mathcal{P}_{\mathcal{F}}$ of local C^ν -diffeomorphisms of (M, \mathcal{F}) is defined as

$$\mathcal{P}_{\mathcal{F}} = \{\phi_t^X \mid X \in \mathcal{F}(U), U \subseteq M, t \in \mathbb{R}\}. \quad (5.3.1)$$

We denote the groupoid associated to $\mathcal{P}_{\mathcal{F}}$ by $\mathcal{G}_{\mathcal{F}}$.

It is interesting to see that this groupoid has the same orbits as the C^ν -tautological control system.

Theorem 5.3.3. *Let (M, \mathcal{F}) be a C^ν -tautological control system and $\mathcal{G}_{\mathcal{F}}$ be the groupoid associated with $\mathcal{P}_{\mathcal{F}}$. Then, for every $x_0 \in M$, we have*

$$\text{Orb}_{\mathcal{F}}(x_0) = \mathcal{G}_{\mathcal{F}}(x_0).$$

Proof. We first show that $\mathcal{G}_{\mathcal{F}}(x_0) \subseteq \text{Orb}_{\mathcal{F}}(x_0)$. Suppose that $y \in \mathcal{G}_{\mathcal{F}}(x_0)$. Then there exists $P \in \Gamma(\mathcal{P})$ such that $x_0 \in \text{Dom}(P)$ and we have $P(x_0) = y$. By definition of $\Gamma(\mathcal{P})$, there exist a neighbourhood U of x_0 , integers $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in \{1, -1\}$, and $h_1, h_2, \dots, h_k \in \mathcal{P}$ such that $P|_U = h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \dots \circ h_k^{\epsilon_k}$. So we have $y = P(x_0) = h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \dots \circ h_k^{\epsilon_k}(x_0)$. Note that, for every $i \in \{1, 2, \dots, k\}$, we have $h_i \in \mathcal{P}$. Therefore, there exist $t_i \in \mathbb{R}$ and open sets $U_i \subseteq M$ and $X_i \in \mathcal{F}(U_i)$ such that $h_i = \phi_{t_i}^{X_i}$. This means that $y = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(x_0)$ and so by definition of orbits, we have $y \in \text{Orb}_{\mathcal{F}}(x_0)$.

On the other hand, if $y \in \text{Orb}_{\mathcal{F}}(x_0)$, then there exists open sets $U_1, U_2, \dots, U_k \subseteq M$ and $t_1, t_2, \dots, t_k \in \mathbb{R}$ and X_1, X_2, \dots, X_n such that $X_i \in \mathcal{F}(U_i)$ for all $i \in \{1, 2, \dots, k\}$ and we have

$$y = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(x_0).$$

By choosing $P = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}$, it is clear that $P \in \Gamma(\mathcal{P})$, and we have $y = P(x_0)$. This means that $y \in \mathcal{G}_{\mathcal{F}}(x_0)$. This implies that $\text{Orb}_{\mathcal{F}}(x_0) \subseteq \mathcal{G}_{\mathcal{F}}(x_0)$. \square

5.4. Geometric structure of orbits

In this section, we study the geometric properties of orbits of a C^ν -tautological control system (M, \mathcal{F}) . The geometric structure of orbits of a family of vector fields has been studied in [74] and [72].

5.4.1. Singular foliations. In his 1973 paper, in order to study orbits of a family of vector fields, Peter Stefan generalized the notion of foliations to “singular” foliations [72]. He showed that orbits of a family of C^∞ -vector fields are leaves of a singular foliation on M . In this section, we recall the notion of singular foliation and its leaves. Following [72], we show that existence of a specific coordinate chart on M called the “privileged coordinate chart”

Definition 5.4.1. Let M be a C^ν -manifold. Then a subset $L \subseteq M$ is called a **leaf** of M if there exists a C^ν -atlas σ on L such that

1. (L, σ) is a connected immersed submanifold of M , and
2. for every locally connected topological space Y and every continuous map $f : Y \rightarrow M$ such that $f(Y) \subseteq L$, the map $f : Y \rightarrow (L, \sigma)$ is continuous.

A leaf $L \subseteq M$ is called a **k-leaf**, if (L, σ) is a k -dimensional manifold.

Definition 5.4.2. Let M be a C^ν -manifold of dimension n and $k \in \mathbb{N}$ be such that $k \leq n$. A **foliation** of dimension k on M is a collection of disjoint k -leaves $\{S_\lambda\}_{\lambda \in \Lambda}$ of M such that

$$\bigcup_{\lambda \in \Lambda} S_\lambda = M$$

and, for every $x \in M$, there exists a chart (U_x, ϕ_x) around x with the following properties:

1. $\phi_x : U_x \rightarrow V_x \times W_x \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$, $\phi_x(x) = (0, 0)$, and
2. $S_\lambda \cap U_x = \phi_x^{-1}(V_x \times l_{x,\lambda})$, for every $\lambda \in \Lambda$,

where $l_{x,\lambda} = \{s \in W \mid \phi_x^{-1}(0, s) \in S_\lambda\}$.

In [72], the notion of foliation with singularity, which is a generalization of foliations, is introduced. Roughly speaking, foliations with singularity are foliations in which leaves can have different dimension.

Definition 5.4.3. Let M be a C^ν -manifold of dimension n . A **foliation with singularities** on M is a collection of disjoint leaves $\{S_\lambda\}_{\lambda \in \Lambda}$ of M such that

$$\bigcup_{\lambda \in \Lambda} S_\lambda = M$$

and, for every $x \in M$, there exist $k \in \mathbb{N}$ and a chart (U_x, ϕ_x) around x with the following properties:

1. $\phi : U_x \rightarrow V_x \times W_x \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$, $\phi_x(x) = (0, 0)$, and
2. $S_\lambda \cap U_x = \phi_x^{-1}(V_x \times l_{x,\lambda})$, for every $\lambda \in \Lambda$,

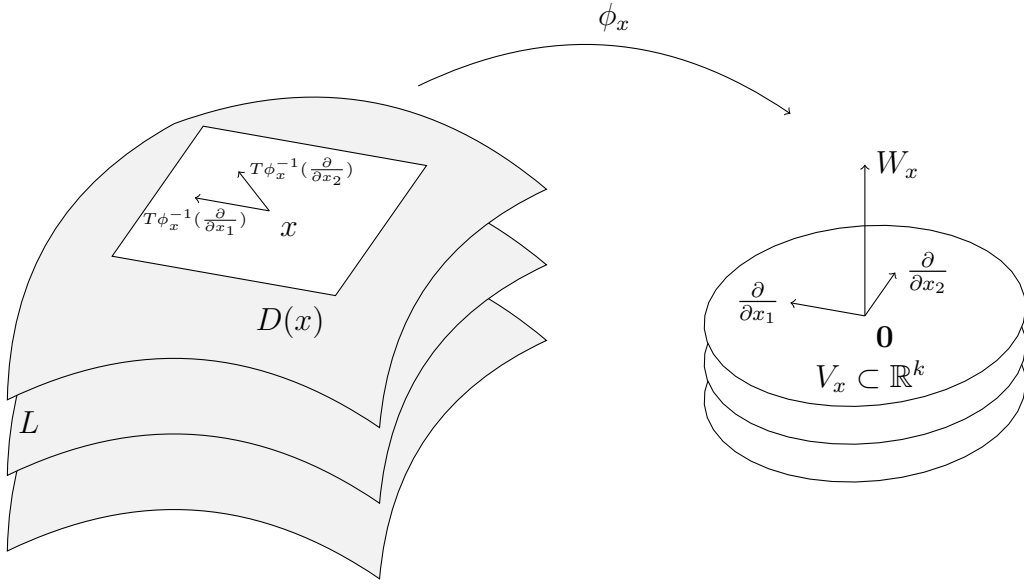


Figure 5.1: Privileged Chart

where $l_{x,\lambda} = \{s \in W \mid \phi_x^{-1}(0, s) \in S_\lambda\}$.

Suppose that $L \subseteq M$ and $D(x)$ is a vector subspace of $T_x M$ for every $x \in L$. In order to prove Stefan’s orbit theorem, we need to define privileged chart on M with respect to L and $D(x)$.

Definition 5.4.4. Suppose that $L \subseteq M$ and, for every $x \in L$, $D(x)$ is a vector subspace of $T_x M$. A chart (U_x, ϕ_x) on M around x such that

1. $\phi_x : U_x \rightarrow V_x \times W_x$, where U_x is an open neighbourhood of x and V_x and W_x are open neighbourhoods of 0 in \mathbb{R}^k and \mathbb{R}^{n-k} respectively,
2. $\phi_x(x) = (0, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$,
3. $L \cap \phi_x^{-1}(V_x \times W_x) = \phi_x^{-1}(V_x \times l_x)$, where $l_x = \{s \in W \mid (0, s) \in \phi_x(L)\}$,
4. $T\phi_x^{-1}(\frac{\partial}{\partial x_i}) \in D(\phi_x^{-1}(t, s))$ for every $(t, s) \in \phi_x(L)$ and for every $i \in \{1, 2, \dots, k\}$,

is called a **privileged chart** on M with respect to L and $D(x)$.

We first prove the following results about privileged charts on M .

Proposition 5.4.5. Let $\phi_x : U_x \rightarrow V_x \times W_x$ be a privileged chart on M with respect to L and $D(x)$. Suppose that N is a connected C^ν -manifold and $f : N \rightarrow M$ is a C^ν -map such that

1. $f(N) \subseteq L \cap \phi_x^{-1}(V_x \times W_x)$, and

2. $T_y f(v) \in D(f(y))$, for every $y \in N$ and every $v \in T_y N$.

Then there exists $s \in W_x$ such that $f(N) \subseteq \phi_x^{-1}(V_x \times \{s\})$.

Proof. We show that the map $\eta : N \rightarrow \mathbb{R}^k$ defined as

$$\eta = \text{pr}_2 \circ \phi_x \circ f,$$

is a constant function. Note that, for every $i \in \{1, 2, \dots, k\}$, we have

$$D_i(\text{pr}_2)(t, s) = 0, \quad \forall (t, s) \in V_x \times W_x.$$

So we can write

$$D_i(\text{pr}_2 \circ \phi_x \circ \phi_x^{-1})(t, s) = T_{\phi_x^{-1}(t, s)}(\text{pr}_2 \circ \phi_x) D_i(\phi_x^{-1})(t, s) = 0, \quad \forall (t, s) \in V_x \times W_x.$$

Since, for every $y \in N$, we have $f(N) \subseteq L \cap \phi_x^{-1}(V_x \times W_x)$, there exists $(t, s) \in V_x \times W_x$ such that $f(y) = \phi_x^{-1}(t, s)$ and $\phi_x^{-1}(t, s) \in L$. On the other hand, by property (4) of privileged charts, we have

$$\text{span}\{D_1\phi_x^{-1}(t, s), D_2\phi_x^{-1}(t, s), \dots, D_k\phi_x^{-1}(t, s)\} \subseteq D(\phi_x^{-1}(t, s)).$$

Note that one can write

$$T_y \eta(v) = T_{\phi_x^{-1}(t, s)}(\text{pr}_2 \circ \phi_x) \circ T_y f(v), \quad \forall y \in N.$$

Since $T_y f(v) \in D(f(y)) = D(\phi_x^{-1}(t, s))$, for every $v \in T_y N$, we can write

$$T_y \eta(v) = 0 \quad \forall y \in N$$

Note that N is connected. Therefore, there exists $s \in W_x$ such that $\eta(N) = \{s\}$. \square

Theorem 5.4.6. *Let M be a C^ν -manifold, L be a subset of M , and, for every $x \in L$, $D(x)$ be a vector subspace of $T_x M$. Suppose that there exists $k \in \mathbb{N}$ such that, for every $x \in L$, $\dim(D(x)) = k$ and, for every $x \in L$, there exists a privileged chart (U_x, ϕ_x) on M with respect to L and $D(x)$. Then there exists a C^ν -atlas σ on L such that*

1. (L, σ) is an immersed submanifold of M with the tangent space $T_x L = D(x)$ for all $x \in L$,
2. for every C^ν -map $f : N \rightarrow M$ such that $f(N) \subseteq L$ and $T_y f(v) \in D(f(y))$ for all $y \in N$, the map $f : N \rightarrow (L, \sigma)$ is of class C^ν , and
3. every connected component of L is a leaf of M .

Proof. We construct an atlas on L . Let $\phi_x : U_x \rightarrow V_x \times W_x$ be a privileged chart on M with respect to L and $D(x)$ and $s \in W_x$ be such that $(0, s) \in \phi_x(L)$. Then we define $\phi_{x,s} : \phi_x^{-1}(V_x \times \{s\}) \rightarrow V_x$ as

$$\phi_{x,s} = \text{pr}_1 \circ \phi_x.$$

By property (3) of privileged charts, we have that $\phi_x^{-1}(V_x \times \{s\}) \subseteq L$.

We show that $\sigma = \{\phi_{x,s}\}_{x \in L, s \in l_x}$ is a C^ν -atlas for L . Note that the domains of elements of σ covers the whole L . The reason is that, for every $x \in L$, we have $x \in \phi_x^{-1}(V_x \times \{0\})$. This implies that $\bigcup_{x \in L} \phi_x^{-1}(V_x \times \{0\}) = L$. Now we need to check that the charts in atlas σ are C^ν -compatible. We first prove the following lemma.

Lemma. *Suppose that $f : N \rightarrow M$ is a C^ν -map such that $f(N) \subseteq L$ and $T_y f(v) \in D(f(y))$, for all $y \in N$ and all $v \in T_y N$. Then $G = f^{-1}(\phi_{x,s}^{-1}(V_x))$ is an open subset of N and the map $\phi_{x,s} \circ f : G \rightarrow \mathbb{R}^k$ is of class C^ν .*

Proof. Let $y \in G$. Since $G = f^{-1}(\phi_{x,s}^{-1}(V_x)) \subseteq f^{-1}(\phi_x^{-1}(V_x \times W_x))$, there exists a connected component H of $f^{-1}(\phi_x^{-1}(V_x \times W_x))$ such that $y \in H$. Then it is clear that H is an open submanifold of N . Note that $f(H) \subseteq \phi_x^{-1}(V_x \times W_x)$ and we have $f(H) \subseteq L$. Therefore, we get

$$f(H) \subseteq L \cap \phi_x^{-1}(V_x \times W_x).$$

By Theorem 5.4.5, one can show that $f(H) \subseteq \phi_x^{-1}(V_x \times \{s\})$. So $H \subseteq f^{-1}(\phi_{x,s}^{-1}(V_x))$. This means that, for every $y \in G$, there exists an open set $H \subseteq f^{-1}(\phi_{x,s}^{-1}(V_x))$ such that $y \in H$. Thus $f^{-1}(\phi_{x,s}^{-1}(V_x))$ is an open subset of N .

To show that $\phi_{x,s} \circ f : G \rightarrow \mathbb{R}^k$ is a C^ν -map, one needs to notice that

$$\phi_{x,s} \circ f = \text{pr}_1(\phi_x \circ f |_G).$$

Since G is open in N , the map $\phi_{x,s} \circ f$ is of class C^ν . □

Now using the above lemma, we show that the charts in atlas \mathcal{A} are C^ν -compatible. Let $\phi_{x,s}, \phi_{y,t} \in \mathcal{A}$ be two coordinate charts. Consider the map $\phi_{x,s}^{-1} : V_x \rightarrow \phi_x^{-1}(V_x \times \{s\}) \subseteq M$. Using the fact that

$$\phi_x^{-1}(V_x \times \{s\}) \subseteq L,$$

we can deduce that

$$\phi_{x,s}^{-1}(V_x) \subseteq L.$$

Also we have

$$T_t \phi_{x,s}^{-1}(v) = T_{(t,s)} \phi_x^{-1} \circ T_t \text{pr}_1^{-1}(v), \quad \forall v \in \mathbb{R}^k.$$

However, it is clear that

$$T_t \text{pr}_1^{-1}(v) = (v, 0), \quad \forall v \in \mathbb{R}^k, 0 \in \mathbb{R}^{n-k}.$$

This implies that

$$T_t \phi_{x,s}^{-1}(v) = T_{(t,s)} \phi_x^{-1}(v, 0), \quad \forall v \in \mathbb{R}^k.$$

Since we know that $T_{(t,s)} \phi_x^{-1}(v, 0) \in D(\phi_x^{-1}(t, s))$, we get

$$T_t \phi_{x,s}^{-1}(v) \in D(\phi_x^{-1}(t, s)) = D(\phi_{x,s}^{-1}(t)), \quad \forall v \in \mathbb{R}^k.$$

So, by the above Lemma, $\phi_{(x,s)} \circ \phi_{(y,t)}^{-1}(V_x)$ is open and the map $\phi_{(y,t)} \circ \phi_{(x,s)}^{-1} : \phi_{(x,s)} \circ \phi_{(y,t)}^{-1}(V_x) \rightarrow \mathbb{R}^k$ is of class C^ν .

The atlas σ defines an immersed submanifold structure on L . By the Lemma above, the assertion (2) clearly holds. To show the assertion (3), suppose that L_0 is a connected component of (L, σ) . Let Y be a locally connected topological space such that $f : Y \rightarrow M$ is continuous and $f(Y) \subseteq L_0$. If we denote the space L_0 with the subspace topology from M by (L_0, τ) , then it is clear that $f : Y \rightarrow (L_0, \tau)$ is continuous. Let (U_x, ϕ_x) be a privileged coordinate chart on M with respect to L_0 and $D(x)$ such that $\phi_x : U_x \rightarrow V_x \times W_x$. Then $L_0 \cap U_x = \phi_x^{-1}(V_x \times l_x)$. Therefore, $\phi_x^{-1}(V_x \times l_x)$ is an open set in (L_0, τ) .

Also, $\{\phi_{x,s}^{-1}(V_x)\}_{s \in l_x}$ is a collection of disjoint open sets of (L_0, σ) . Since M is second-countable and (L_0, σ) is connected, (L_0, σ) is a separable space. This implies that l_x is countable. This means that, for every $s \in l_x$, $V_x \times \{s\}$ is a connected component of $V_x \times l_x$ and $\phi_x^{-1}(V_x \times \{s\})$ is a connected component of $\phi_x^{-1}(V_x \times l_x)$ in (L_0, τ) . Thus $\phi_x^{-1}(V_x \times \{s\})$ is open in (L_0, τ) . Since $f : Y \rightarrow (L_0, \tau)$ is continuous, $f^{-1}(\phi_x^{-1}(V_x \times \{s\})) = f^{-1}(\phi_{x,s}^{-1}(V_x))$ is open in Y . Moreover, $\{\phi_{x,s}^{-1}(V_x)\}_{x \in L_0, s \in l_x}$ is a basis for topology on (L_0, σ) . This implies that the map $f : Y \rightarrow (L_0, \sigma)$ is also continuous. \square

5.4.2. The orbit theorem. In geometric control theory a control system is considered as a family S of parametrized vector fields. Orbits of this family of vector fields is one of the basic objects of interest in control theory. However, any analytic description of orbits requires solving nonlinear differential equations, which is, in the best case, very difficult, if not impossible. Therefore, it would be more reasonable to study orbits using properties of the family of vector fields S .

In 1939 Chow [17] and Rashevskii [65] independently proved a theorem which connects properties of the orbits of S to the Lie brackets of the vector fields in S . This theorem can be considered as one of the first results where the tools and techniques of differential geometry are used in control theory. Let $\text{Lie}(S)$ be the distribution generated by the Lie brackets of the vector fields in S . The Chow–Rashevskii theorem states that, for a connected manifold M , if $\text{Lie}(S)(x)$ is $T_x M$, then the orbit of S passing through x is the whole space M . However, in the case that $\text{Lie}(S)(x)$ is not $T_x M$, this theorem does not give us any information about the structure of orbits of the system.

In 1974 Sussmann [74] and Stefan [72] proved a generalization of Chow–Rashevskii theorem. They showed that, even in the case that $\text{Lie}(S)(x)$ is not $T_x M$, the orbits of S are immersed submanifolds of M . As Sussmann mentions in his 1973 paper [74], a

naive way of generalizing the Chow–Rashevskii theorem is to consider a submanifold of M such that its tangent space at each point x is $\text{Lie}(S)(x)$. Then, one can apply the Chow–Rashevskii theorem to this “integral” submanifold of $\text{Lie}(S)$ and show that the orbit of S passing through x is exactly this submanifold. Unfortunately, this generalization does not generally work. In fact, it is possible that such an integral submanifold for the distribution $\text{Lie}(S)$ does not exist. Sussmann and Stefan defined another distribution P_S using flows of the vector fields in S . They showed that the distribution P_S is always integrable and the integral submanifolds of P_S are exactly the orbits of the family of vector fields S . Stefan also showed that this manifold structure on orbits of S makes M into a singular foliation where the leaves of this foliation are orbits of S [72, Theorem 1].

In this section, following the approach of Stefan [72], we generalize the classical orbit theorem to the tautological framework. Given a tautological control system (M, \mathcal{F}) , one can define another presheaf of modules $\overline{\mathcal{F}}$ using the flows of vector fields in \mathcal{F} . It can be shown that the presheaf $\overline{\mathcal{F}}$ induces a unique manifold structures on the orbits of \mathcal{F} , which makes M into a singular foliation with orbits of \mathcal{F} as the leaves of this foliation. Moreover, for every $x \in M$, the tangent space to the orbit of \mathcal{F} passing through x is the vector space $\overline{\mathcal{F}}(x)$.

Definition 5.4.7. Suppose that (M, \mathcal{F}) is a C^ν -tautological control system. Then we define

$$\mathcal{F}(x) = \{X(x) \mid X \in \mathcal{F}(U), U \text{ is an open subset of } M \text{ containing } x\}.$$

$\text{Lie}(\mathcal{F})$ is the subpresheaf of C^ν -modules of Γ^ν which assigns to every open set $U \subseteq M$, the $C^\nu(U)$ -module $\text{Lie}(\mathcal{F})(U)$ defined as

$$\text{Lie}(\mathcal{F})(U) = \text{span}_{C^\nu(U)} \{[\dots [[X_1, X_2], X_3] \dots, X_n] \mid X_1, X_2, \dots, X_n \in \mathcal{F}(U)\},$$

where, for every family of C^ν -vector fields S , $\text{span}_{C^\nu(U)}(S)$ is the $C^\nu(U)$ -module generated by the vector fields in S . Also $\overline{\mathcal{F}}$ is defined as the subpresheaf of C^ν -modules of Γ^ν which assigns to every open set $U \subseteq M$ the following $C^\nu(U)$ -module:

$$\begin{aligned} \overline{\mathcal{F}}(U) = \text{span}_{C^\nu(U)} \{ \eta^* X \mid \exists V \text{ an open subset of } M \text{ s.t.} \\ X \in \mathcal{F}(V), \eta : U \rightarrow V \in \Gamma(\mathcal{P}_{\mathcal{F}}) \}, \end{aligned} \quad (5.4.1)$$

where, for every family of C^ν -vector fields S , $\text{span}_{C^\nu(U)}(S)$ is the $C^\nu(U)$ -module generated by the vector fields in S .

We will show that $\overline{\mathcal{F}}$ is a subpresheaf of C^ν -modules of Γ^ν . Let $Y \in \overline{\mathcal{F}}(U)$ and $W \subseteq U$ be an open set. Then there exists an open set $V \subseteq M$, a section $X \in \mathcal{F}(U)$, and $\eta : U \rightarrow V \in \Gamma(\mathcal{P}_{\mathcal{F}})$ such that

$$Y = \eta^* X.$$

If we restrict to W , we have

$$Y|_W = (\eta|_W)^* X|_W.$$

By the restriction property of the pseudogroup $\Gamma(\mathcal{P}_{\mathcal{F}})$, we have $\xi = \eta|_W \in \Gamma(\mathcal{P}_{\mathcal{F}})$. Since $X|_W \in \mathcal{F}(W)$, we have $Y|_W \in \overline{\mathcal{F}}(W)$. This shows that \mathcal{F} is a subpresheaf of Γ^ν . The following theorem is an immediate consequence of this fact.

Theorem 5.4.8. *Suppose that (M, \mathcal{F}) is a C^ν -tautological control system. Then $(M, \overline{\mathcal{F}})$ is a C^ν -tautological control system.*

The C^ν -tautological control system $(M, \overline{\mathcal{F}})$ defined as above is the **homogeneous C^ν -tautological control system associated to (M, \mathcal{F})** .

Theorem 5.4.9. *Suppose that (M, \mathcal{F}) is a C^ν -tautological control system. Then $(M, \mathcal{G}_{\mathcal{F}})$ is a foliation with singularities. Moreover, for every $x_0 \in M$ and every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, we have $T_x \text{Orb}_{\mathcal{F}}(x_0) = \overline{\mathcal{F}}(x)$. In particular, for every $x_0 \in M$, $\text{Orb}_{\mathcal{F}}(x_0)$ is a leaf of M and has a unique structure as a connected immersed submanifold of M .*

Proof. Consider the homogeneous C^ν -tautological control system $(M, \overline{\mathcal{F}})$ associated to (M, \mathcal{F}) . By Theorem 5.4.6, it suffices to show that, for every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, there exists $k \in \mathbb{N}$ such that $\dim(\overline{\mathcal{F}}(x)) = k$ and there exists a privileged chart (U_x, ϕ_x) on M with respect to $\text{Orb}_{\mathcal{F}}(x_0)$ and $\overline{\mathcal{F}}(x)$.

Lemma. *There exists $k \in \mathbb{N}$ such that, for every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, we have $\dim(\overline{\mathcal{F}}(x)) = k$.*

Proof. Suppose that $\dim(\overline{\mathcal{F}}(x_0)) = k$. There exists open sets V_1, V_2, \dots, V_k in M and vector fields Y_1, Y_2, \dots, Y_k such that $Y_i \in \overline{\mathcal{F}}(V_i)$, for every $i \in \{1, 2, \dots, k\}$, and we have

$$\text{span}\{Y_1(x_0), Y_2(x_0), \dots, Y_k(x_0)\} = \overline{\mathcal{F}}(x_0).$$

$x \in \text{Orb}_{\mathcal{F}}(x_0)$. Then there exist real numbers $t_1, t_2, \dots, t_n \in \mathbb{R}$, open sets $U_1, U_2, \dots, U_n \subseteq M$, and vector fields X_1, X_2, \dots, X_n such that

$$X_i \in \mathcal{F}(U_i), \quad \forall i \in \{1, 2, \dots, k\},$$

and we have

$$x = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_n}^{X_n}(x_0).$$

Therefore, there exist open sets $W, H \subseteq M$ such that the map $\eta : W \rightarrow H$ defined as

$$\eta = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_n}^{X_n}$$

is a C^ν -diffeomorphism. We set $V = \left(\bigcap_{i=1}^k V_i\right) \cap H$ and $U = W \cap \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_n}^{X_n}(V)$. Then, for every $i \in \{1, 2, \dots, k\}$, we define $Z_i \in \overline{\mathcal{F}}(U)$ as

$$Z_i = \eta^* Y_i|_V.$$

Since η is a C^ν -diffeomorphism and $\eta(x_0) = x$, we have

$$\text{span}\{Z_1(x), Z_2(x), \dots, Z_k(x)\} = \text{span}\{Y_1(x_0), Y_2(x_0), \dots, Y_k(x_0)\} = k.$$

This means that $\dim(\overline{\mathcal{F}}(x_0)) \leq \dim(\overline{\mathcal{F}}(x))$. By symmetry, we have $\dim(\overline{\mathcal{F}}(x)) \leq \dim(\overline{\mathcal{F}}(x_0))$. This implies that, for every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, $\dim(\overline{\mathcal{F}}(x)) = k$. \square

Now we show that, for every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, there exists a privileged chart (U_x, ϕ_x) on M with respect to $\text{Orb}_{\mathcal{F}}(x_0)$ and $\overline{\mathcal{F}}(x)$. Let us fix $x \in M$ and define $Q(x)$ as a $n - k$ dimensional vector subspace of $T_x M$ such that $Q(x) \oplus \overline{\mathcal{F}}(x) = T_x M$. The following lemma has been proved in [47].

Lemma. *Let M be a C^ν -manifold, $x \in M$, and D be a vector subspace of $T_x M$. Then there exists an embedded C^ν -submanifold S of M passing through x such that $T_x S = D(x)$.*

Using the above lemma, there exists an embedded submanifold of M called Q such that $x \in Q$ and $T_x Q = Q(x)$. Let (W_x, ψ_x) be a coordinate chart on Q around x . Since $\dim(\overline{\mathcal{F}}(x)) = k$, there exist vector fields X_1, X_2, \dots, X_k and open sets $U_1, U_2, \dots, U_k \subseteq M$ such that

$$X_i \in \overline{\mathcal{F}}(U_i), \quad \forall i \in \{1, 2, \dots, k\},$$

and we have

$$\text{span}\{X_1(x), X_2(x), \dots, X_k(x)\} = \overline{\mathcal{F}}(x).$$

We define a map $\eta_x : V_x \times W_x \rightarrow M$ as

$$\eta_x(t_1, t_2, \dots, t_k, y_1, y_2, \dots, y_{n-k}) = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(\psi_x^{-1}(y_1, y_2, \dots, y_{n-k})).$$

We show that $T_{(0,0)}\eta_x$ is a linear map of rank n . Note that we have

$$T_{(0,0)}\eta_x\left(\frac{\partial}{\partial t_i}\right) = X_i(x), \quad \forall i \in \{1, 2, \dots, k\}.$$

Also we have

$$T_{(0,0)}\eta_x\left(\frac{\partial}{\partial y_i}\right) = T_0\psi_x^{-1}\left(\frac{\partial}{\partial y_i}\right), \quad \forall i \in \{1, 2, \dots, n - k\}.$$

Thus we have

$$\text{rank}(T_{(0,0)}\eta_x) = \dim\{X_1(x), \dots, X_k(x), T_0\psi_x^{-1}\left(\frac{\partial}{\partial y_1}\right), \dots, T_0\psi_x^{-1}\left(\frac{\partial}{\partial y_{n-k}}\right)\} = n.$$

By the inverse function theorem, there exist a neighbourhood $V'_x \subseteq \mathbb{R}^k$ and a neighbourhood $W'_x \subseteq \mathbb{R}^{n-k}$ such that $\eta_x|_{V'_x \times W'_x}$ is a C^ν -diffeomorphism. So, there exists a chart $(V'_x \times W'_x, \phi_x)$ such that

$$\phi_x(y) = \eta_x^{-1}(y), \quad \forall y \in V'_x \times W'_x.$$

Now, we show that $(V'_x \times W'_x, \phi_x)$ is a privileged chart on M with respect to $\text{Orb}_{\mathcal{F}}(x_0)$ and $\overline{\mathcal{F}}(x)$. Note that we have

$$\text{Orb}_{\mathcal{F}}(x_0) \cap \eta_x(V'_x \times W'_x) = \eta_x(V'_x \times \{l_x\}),$$

and by setting $z = \phi_{t_{i+1}}^{X_{i+1}} \circ \dots \circ \phi_{t_k}^{X_k}(\psi_x^{-1}(y_1, \dots, y_{n-k}))$, we have

$$D_i \eta_x(t_1, \dots, t_k, y_1, \dots, y_{n-k}) = T_z(\phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_i}^{X_{i-1}})X_i(z), \quad \forall i \in \{1, 2, \dots, k\}.$$

Thus, it is clear that

$$D_i \eta_x(t_1, \dots, t_k, y_1, \dots, y_{n-k}) \in \overline{\mathcal{F}}(\eta_x(t_1, \dots, t_k, y_1, \dots, y_{n-k})).$$

This implies that $(V'_x \times W'_x, \phi_x)$ is privileged chart on M with respect to $\text{Orb}_{\mathcal{F}}(x_0)$ and $\overline{\mathcal{F}}(x)$. Therefore, by Theorem 5.4.6, $(M, \text{Orb}_{\mathcal{F}})$ is a singular foliation and, for every $x \in M$, we have $T_x \text{Orb}_{\mathcal{F}}(x_0) = \overline{\mathcal{F}}(x)$.

It remains to show that $\text{Orb}_{\mathcal{F}}(x_0)$ is a connected immersed submanifold of M . The fact that $\text{Orb}_{\mathcal{F}}(x_0)$ is an immersed submanifold of M is clear from the above argument. So we only need to show that $\text{Orb}_{\mathcal{F}}(x_0)$ is connected. Let $x \in M$, $U \subseteq M$ be an open set containing x , and $X \in \overline{\mathcal{F}}(U)$. We define a map $\gamma_x^X : \mathcal{D}^x \rightarrow M$ as

$$\gamma_x^X(t) = \phi_t^X(x), \quad \forall t \in \mathcal{D}^x.$$

Since X is time-invariant, $t \mapsto \phi_t^X(x)$ is of class C^ν . This implies that γ_x^X is of class C^ν and in particular, it is continuous. Since $\gamma_x^X(\mathcal{D}^x) \subseteq \text{Orb}_{\mathcal{F}}(x)$, the map $\gamma_x^X : \mathbb{R} \rightarrow \text{Orb}_{\mathcal{F}}(x)$ is continuous. We know that $y \in \text{Orb}_{\mathcal{F}}(x_0)$, then there exists $t_1, t_2, \dots, t_k \in \mathbb{R}_{>0}$, open sets $U_1, U_2, \dots, U_k \subseteq M$ and vector fields X_1, X_2, \dots, X_k such that

$$X_i \in \mathcal{F}(U_i), \quad \forall i \in \{1, 2, \dots, k\},$$

and we have

$$y = \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \dots \circ \phi_{t_k}^{X_k}(x_0).$$

So one can reach y from $x_0 \in M$, by moving along the continuous curves of the form γ_x^X . This shows that $\text{Orb}_{\mathcal{F}}(x_0)$ is connected. \square

5.4.3. The real analytic case. While the classical orbit theorem of Sussmann and Stefan characterizes the tangent space to the orbits of a family of vector fields S using the distribution P_S , computing the distribution P_S requires finding the flows of the system. Therefore, it would be natural to investigate the conditions under which one can characterize the distribution P_S using the Lie brackets of the vector fields in S . Using the Chow–Rashevskii theorem, it is easy to see that if the distribution $\text{Lie}(S)$ is integrable, the distributions P_S and $\text{Lie}(S)$ are identical.

In the differential geometry literature, integrability of distributions has been deeply studied. The most well-known result about integrability of distributions is

the Frobenius theorem. According to the Frobenius theorem, if the rank of the distribution $\text{Lie}(S)$ is locally constant, then it is integrable. In 1963, Hermann realized that the module structure on the family of vector fields $\text{Lie}(S)$ plays a crucial role in the integrability of the distribution $\text{Lie}(S)$. More specifically, he showed that if the $C^\infty(M)$ -module generated by the vector fields in $\text{Lie}(S)$ is locally finitely generated, then the distribution $\text{Lie}(S)$ is integrable [31, 2.1(b)]. He also claimed that if the vector field of the distribution $\text{Lie}(S)$ are real analytic, then this distribution is integrable [31, 2.1(c)]. However, his paper does not contain a complete proof of this claim. In 1966, Nagano proved that for a family of real analytic vector fields S , the distribution $\text{Lie}(S)$ is integrable [62, Theorem 1]. In his 1970 paper, Lobry introduced a weaker condition called “locally of finite type”. He proved that if a distribution is locally of finite type, then it is integrable. In particular, mentioning the fact that the space of real analytic vector fields has Noetherian property, he claimed that for a family of real analytic vector fields S , the distribution $\text{Lie}(S)$ is always locally of finite type and therefore integrable. In [73], Stefan gave a counterexample to this assertion. However, he showed that Lobry’s locally of finite type condition can be modified to give integrability of a distribution [72, Theorem 6].

In the previous section, we proved the orbit theorem for C^ν -tautological control systems. We showed that orbits of a C^ν -tautological control system (M, \mathcal{F}) define the structure of a singular foliations on M and, for every $x_0 \in M$, we have

$$T_x \text{Orb}_{\mathcal{F}}(x_0) = \overline{\mathcal{F}}(x), \quad \forall x \in \text{Orb}_{\mathcal{F}}(x_0).$$

However, computing the sheaf $\overline{\mathcal{F}}$ requires solving for the flows of the vector fields of the system, which is in the best case very difficult, if not impossible. Therefore, similar to the classical orbit theorem, it would be natural to investigate the conditions under which the vector space $\overline{\mathcal{F}}(x)$ is identical with the vector space $\text{Lie}(\mathcal{F})(x)$. It is natural to expect that Hermann’s condition can be generalized the the tautological framework. In other words, if the “module” $\text{Lie}(\mathcal{F})$ is locally finitely generated, then we have $\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x)$. However, Example 5.4.13 shows that this implication is not true for all tautological control systems.

In this section, we show that, for having the equality $\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x)$, the presheaf structure on the family of vector fields $\text{Lie}(\mathcal{F})$ plays an essential role. We prove that if $\text{Lie}(\mathcal{F})$ is a locally finitely generated “presheaf”, then the vector spaces $\overline{\mathcal{F}}(x)$ and $\text{Lie}(\mathcal{F})(x)$ are identical. In particular, we show that for “globally defined” C^ω -tautological control system the presheaf $\text{Lie}(\mathcal{F})$ is locally finitely generated. This shows that for a C^ω -tautological control system, one can characterize the tangent space to the orbits of the system using the Lie brackets of vector fields of the system.

Theorem 5.4.10. *Let (M, \mathcal{F}) be a C^ν -tautological control system such that the presheaf $\text{Lie}(\mathcal{F})$ is locally finitely generated. Then, for every $x \in M$, we have $\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x)$.*

Proof. We first show that, for every $x \in M$, we have $\text{Lie}(\mathcal{F})(x) \subseteq \overline{\mathcal{F}}(x)$. Let us fix $x \in M$. In order to show this inclusion, it suffices to show that, for every open set

$U \subseteq M$ containing x and every $X_1, X_2 \in \text{Lie}(\mathcal{F}(U))$, we have $[X_1, X_2](x) \in \overline{\mathcal{F}}(x)$. We know that

$$[X_1, X_2](x) = \left. \frac{d}{dt} \right|_{t=0} ((\phi_t^{X_1})^* X_2(x)).$$

Since $\overline{\mathcal{F}}(x)$ is finite dimensional, there exist $n \in \mathbb{N}$, a neighbourhood $V \subseteq M$ of x , $\eta_1, \eta_2, \dots, \eta_n \in \Gamma(\mathcal{P}_{\mathcal{F}})$ whose domain contains V , and vector fields $Y_1, Y_2, \dots, Y_n \in \mathcal{F}(V)$ such that the set S defined as

$$S = \{\eta_1^* Y_1(x), \eta_2^* Y_2(x), \dots, \eta_n^* Y_n(x)\}$$

generates the vector space $\overline{\mathcal{F}}(x)$. Let $\phi^{X_1} : \mathcal{D} \rightarrow M$ be the flow of X_1 . Then, there exists $T > 0$ such that $[-T, T] \subseteq (\mathcal{D})^x$. Since S generates the vector space $\overline{\mathcal{F}}(x)$, there exist functions $f_1, f_2, \dots, f_n \in \mathbb{F}^{[-T, T]}$ such that

$$(\phi_t^{X_1})^* X_2(x) = \sum_{j=1}^n f_j(t) \eta_j^* Y_j(x).$$

Since X_1 is a vector field of class C^ν , the map $t \mapsto (\phi_t^{X_1})^* X_2(x)$ is of class C^ν . Therefore the functions f_1, f_2, \dots, f_n are of class C^ν with respect to t . This implies that

$$[X_1, X_2](x) = \left. \frac{d}{dt} \right|_{t=0} ((\phi_t^{X_1})^* X_2(x)) = \left. \frac{d}{dt} \right|_{t=0} \sum_{j=1}^n f_j(t) \eta_j^* Y_j(x) = \sum_{j=1}^n \left. \frac{df_j(t)}{dt} \right|_{t=0} \eta_j^* Y_j(x).$$

Therefore $[X_1, X_2](x) \in \overline{\mathcal{F}}(x)$. This completes the proof of the inclusion $\text{Lie}(\mathcal{F})(x) \subseteq \overline{\mathcal{F}}(x)$.

Now we show that, for every $x \in M$, we have $\overline{\mathcal{F}}(x) \subseteq \text{Lie}(\mathcal{F})(x)$. Let us fix $x \in M$. Suppose that U, V are two open sets in M , where V contains x , $X \in \mathcal{F}(U)$, and $Y \in \mathcal{F}(V)$. We show that, for every $t > 0$ where $(\phi_t^Y)^* X(x)$ is defined, we have

$$(\phi_t^Y)^* X(x) \in \text{Lie}(\mathcal{F})(x).$$

Without loss of generality, we can assume that ϕ_t^Y is defined on $[0, T]$. Since the presheaf $\text{Lie}(\mathcal{F})$ is locally finitely generated, for every $y \in \phi^Y([0, T], x)$, there exist a neighbourhood U_y , sections $X_1^y, X_2^y, \dots, X_m^y \in \text{Lie}(\mathcal{F})(U_y)$ and functions $f_{ij}^y \in C^\nu(U_y)$ for $i, j \in \{1, 2, \dots, m\}$ such that

$$[Y, X_j^y](z) = \sum_{i=1}^m f_{ij}^y(z) X_i^y(z), \quad \forall z \in U_y, \quad \forall j \in \{1, 2, \dots, m\}$$

and, for every $z \in U_y$, the set $\{[X_1^y]_z, [X_2^y]_z, \dots, [X_m^y]_z\}$ generates $\text{Lie}(\mathcal{F})_z$.

Now we consider all open set U_y , where $y \in \phi^Y([0, T], x)$. For every $y \in \phi^Y([0, T], x)$, there exists $t_y \in [0, T]$ such that

$$y = \phi^Y(t_y, x).$$

One can assume that (possibly after shrinking U_y), for every $s \in [0, T]$ such that

$$\inf\{\tau \mid \phi^Y(\tau, x) \in U_y\} < s < \sup\{\tau \mid \phi^Y(\tau, x) \in U_y\},$$

we have

$$\phi^Y(s, x) \in U_y.$$

Since $\phi^Y([0, T], x)$ is compact, there exists $y_1, y_2, \dots, y_n \in \phi^Y([0, T], x)$ such that

$$\phi^Y([0, T], x) \subseteq \bigcup_{i=0}^n U_{y_i}.$$

Without loss of generality, we can assume that, for every $i \in \{1, 2, \dots, n-1\}$, we have

$$\inf\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_i}\} \leq \inf\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_{i+1}}\}.$$

Since $\bigcup_{i=0}^n U_{y_i}$ covers $\phi^Y([0, T], x)$, for every $i \in \{1, 2, \dots, n-1\}$, we have

$$\sup\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_i}\} \geq \inf\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_{i+1}}\}.$$

Therefore, for every $i \in \{1, 2, \dots, n-1\}$, there exists $\tau_i \in [0, T]$ such that

$$\sup\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_{i-1}}\} \geq \tau_i \geq \inf\{t \in [0, T] \mid \phi^Y(t, x) \in U_{y_i}\}.$$

We also set $\tau_0 = 0$ and $\tau_n = T$. The following lemma is essential in the course of the proof.

Lemma. *For every $i \in \{1, 2, \dots, n\}$ and every $j \in \{1, 2, \dots, m\}$, we have*

$$(\phi_{\tau_i}^Y)^* X_j^{y_i}(x) \in \text{span} \left\{ \left(\phi_{\tau_{i-1}}^Y \right)^* X_k^{y_{i-1}}(x) \mid k \in \{1, 2, \dots, m\} \right\}.$$

Proof. Let us fix $i \in \{1, 2, \dots, n\}$. Then, for every $j \in \{1, 2, \dots, m\}$, we have

$$\left. \frac{d}{dt} \right|_{t=s} (\phi_t^Y)^* X_j^{y_i}(x) = (\phi_s^Y)^* [Y, X_j^{y_i}](x), \quad \forall s \in [\tau_i, \tau_{i+1}].$$

Note that we have

$$\phi_s^Y(x) \in U_{y_i}, \quad \forall s \in [\tau_i, \tau_{i+1}].$$

Thus, for every $z \in U_{y_i}$ we have

$$[Y, X_j^{y_i}](\phi_s^Y(x)) = \sum_{k=1}^m f_{kj}^{y_i}(\phi_s^Y(x)) X_k^{y_i}(\phi_s^Y(x)), \quad \forall s \in [\tau_i, \tau_{i+1}].$$

Therefore we have

$$\left. \frac{d}{dt} \right|_{t=s} (\phi_t^Y)^* X_j^{y_i}(x) = \sum_{k=1}^m f_{kj}^{y_i}(\phi_s^Y(x)) (\phi_s^Y)^* X_k^{y_i}(x), \quad \forall s \in [\tau_i, \tau_{i+1}].$$

For every $k, j \in \{1, 2, \dots, m\}$, we have that the map $s \mapsto f_{kj}^{y_i}(\phi_s^Y(x))$ is of class C^ν . For every $j \in \{1, 2, \dots, m\}$, we set

$$a_j(t) = (\phi_t^Y)^* X_j^{y_i}(x), \quad \forall t \in [\tau_i, \tau_{i+1}],$$

and

$$b_{kj}(t) = f_{kj}^{y_i}(\phi_t^Y(x)), \quad \forall t \in [\tau_i, \tau_{i+1}].$$

Therefore, we have the family of linear time-varying differential equations:

$$\frac{da_j(t)}{dt} = \sum_{k=1}^m b_{kj}(t)a_j(t), \quad \forall t \in [\tau_i, \tau_{i+1}].$$

By [19, Theorem 5.1], there exists $\gamma_{kj} \in C^\nu(\mathbb{T})$ such that

$$a_j(t) = \gamma_{kj}(t)a_j(\tau_i), \quad \forall t \in [\tau_i, \tau_{i+1}].$$

Thus, by replacing a_j and b_j^k , we get

$$(\phi_t^Y)^* X_j^{y_i}(x) = \sum_{k=1}^m \gamma_{kj}(t) (\phi_{\tau_{i-1}}^Y)^* X_k^{y_i}(x), \quad \forall t \in [\tau_i, \tau_{i+1}].$$

By setting $t = \tau_i$, we get

$$(\phi_{\tau_i}^Y)^* X_j^{y_i}(x) = \sum_{k=1}^m \gamma_{kj}(\tau_i) (\phi_{\tau_{i-1}}^Y)^* X_k^{y_i}(x).$$

However, by the way we have chosen $\tau_0, \tau_1, \dots, \tau_n$, we have

$$X_k^{y_i}(\phi_{\tau_{i-1}}^Y(x)) \in \text{span} \left\{ X_r^{y_{i-1}}(\phi_{\tau_{i-1}}^Y(x)) \mid r \in \{1, 2, \dots, m\} \right\}.$$

Thus, for every $j \in \{1, 2, \dots, m\}$, there exist real numbers $c_{1j}, c_{2j}, \dots, c_{mj} \in \mathbb{R}$ such that

$$(\phi_{\tau_i}^Y)^* X_j^{y_i}(x) = \sum_{k=1}^m c_{kj} (\phi_{\tau_{i-1}}^Y)^* X_k^{y_{i-1}}(x)$$

This completes the proof of the lemma. \square

Using the above lemma and induction on $i \in \{1, 2, \dots, n\}$, it is easy to see that, for every $j \in \{1, 2, \dots, m\}$, we have

$$(\phi_T^Y)^* X_j^{y_n}(x) \in \text{span} \left\{ (\phi_{\tau_0}^Y)^* X_j^x(x) \mid j \in \{1, 2, \dots, m\} \right\}.$$

However, we know that $\tau_0 = 0$. This implies that

$$(\phi_T^Y)^* X_j^{y_n}(x) \in \text{span} \left\{ X_j^x(x) \mid j \in \{1, 2, \dots, m\} \right\}.$$

Thus, we have

$$(\phi_T^Y)^* X_j^{y_n}(x) \in \text{Lie}(\mathcal{F})(x).$$

Also, since $\text{Lie}(\mathcal{F})$ is locally finitely generated, we know that there exists $g_1, g_2, \dots, g_n \in C^\nu(U_{y_n})$ such that

$$X(\phi_T^Y(x)) = \sum_{i=1}^n g_i(\phi_T^Y(x)) X_j^{y_n}(\phi_T^Y(x))$$

This implies that

$$(\phi_T^Y)^* X(x) \in \text{Lie}(\mathcal{F})(x).$$

Thus, we get $\overline{\mathcal{F}}(x) \subseteq \text{Lie}(\mathcal{F})(x)$. □

Corollary 5.4.11. *Let (M, \mathcal{F}) be a globally generated C^ω -tautological control system. Then, for every $x \in M$, we have $\overline{\mathcal{F}}(x) = \text{Lie}(\mathcal{F})(x)$.*

Proof. In the real analytic case, by Theorem 2.4.13, the presheaf $\text{Lie}(\mathcal{F})$ is locally finitely generated. The result then follows from Theorem 5.4.10. □

One can apply Theorem 5.4.11 to the orbit theorem to get the real analytic version of the orbit theorem.

Corollary 5.4.12. *Let (\mathcal{F}, M) be a globally generated C^ω -tautological control system. Then $(M, \mathcal{G}_{\mathcal{F}})$ is a foliation with singularities. Moreover, for every $x_0 \in M$ and every $x \in \text{Orb}_{\mathcal{F}}(x_0)$, we have $T_x \text{Orb}_{\mathcal{F}}(x_0) = \text{Lie}(\mathcal{F})(x)$. In particular, for every $x_0 \in M$, $\text{Orb}_{\mathcal{F}}(x_0)$ is a leaf of $(M, \mathcal{G}_{\mathcal{F}})$ and it has a unique structure as a connected immersed submanifold of M .*

In the following example, we define a real analytic tautological control system $(\mathbb{R}^2, \mathcal{F})$ with the following properties:

1. the vector fields in \mathcal{F} are not globally defined,
2. for every x , there exists a neighbourhood $U \subseteq \mathbb{R}^2$ of x such that $\text{Lie}(\mathcal{F})(U)$ is a locally finitely generated $C^\omega(U)$ -module, and
3. $\text{Lie}(\mathcal{F})(0,0) \neq \overline{\mathcal{F}}(0,0)$.

Properties (1) and (2) indicate that one cannot remove the condition that \mathcal{F} is globally generated from Theorem 3.2.12. This is because Theorem 2.4.13 fails if \mathcal{F} is not globally generated. Properties (2) and (3) show that the condition that $\text{Lie}(\mathcal{F})$ is a locally finitely generated “module” is not sufficient for the equality of vector spaces $\text{Lie}(\mathcal{F})(x)$ and $\overline{\mathcal{F}}(x)$.

Example 5.4.13. [73] Let $M = \mathbb{R}^2$ and $\mathcal{F} = \{X_1, X_2\}$, where $X_1 : \mathbb{R}^2 \rightarrow T\mathbb{R}^2$ is defined as

$$X_1(x, y) = \frac{\partial}{\partial x},$$

and $X_2 : \mathbb{R}^{>0} \times \mathbb{R} \rightarrow T\mathbb{R}^2$ is defined as

$$X_2(x, y) = \frac{1}{x} \frac{\partial}{\partial y}.$$

It is clear that both X_1 and X_2 are real analytic on their domain of definition. We first compute the presheaf $\text{Lie}(\mathcal{F})$. If we compute the Lie bracket of X_1 and X_2 , we have

$$[X_1, X_2](x, y) = \frac{1}{x^2} \frac{\partial}{\partial y}, \quad \forall (x, y) \in \mathbb{R}^{>0} \times \mathbb{R}.$$

Now we can continue computing the Lie brackets. It is easy to see that the only non-zero Lie brackets are

$$\underbrace{[X_1, [X_1, \dots, [X_1, X_2] \dots]]}_{n}(x, y) = \frac{n}{x^{n+1}} \frac{\partial}{\partial y}, \quad \forall n \in \mathbb{N}.$$

This implies that, for every open sets $U \not\subseteq \mathbb{R}^{>0} \times \mathbb{R}$, we have

$$\text{Lie}(\mathcal{F})(U) = \left\{ f_0 \frac{\partial}{\partial x} \mid f_0 \in C^\omega(U) \right\},$$

and, for every open sets $U \subseteq \mathbb{R}^{>0} \times \mathbb{R}$, we have

$$\text{Lie}(\mathcal{F})(U) = \left\{ f_0 \frac{\partial}{\partial x} + \left(\sum_{i=1}^n \frac{f_i}{x^i} \right) \frac{\partial}{\partial y} \mid n \in \mathbb{N}, f_0, f_1, \dots, f_n \in C^\omega(U) \right\}.$$

We show that, for every $(x_0, y_0) \in \mathbb{R}^2$, there exists a neighbourhood U of (x_0, y_0) such that $\text{Lie}(\mathcal{F})(U)$ is a finitely generated $C^\omega(U)$ -module. Let $(x_0, y_0) \in \mathbb{R}^{>0} \times \mathbb{R}$. Then, we choose an open neighbourhood U of (x_0, y_0) such that $U \subseteq \mathbb{R}^{>0} \times \mathbb{R}$. Thus, we have

$$\begin{aligned} \text{Lie}(\mathcal{F})(U) &= \left\{ f_0 \frac{\partial}{\partial x} + \left(\sum_{i=1}^n \frac{f_i}{x^i} \right) \frac{\partial}{\partial y} \mid n \in \mathbb{N}, f_0, f_1, \dots, f_n \in C^\omega(U) \right\} \\ &= \text{span}_{C^\omega(U)} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}. \end{aligned}$$

Now suppose that $(x_0, y_0) \notin \mathbb{R}^{>0} \times \mathbb{R}$. Then, for every open neighbourhood U of (x_0, y_0) , we have $U \not\subseteq \mathbb{R}^{>0} \times \mathbb{R}$. This implies that

$$\text{Lie}(\mathcal{F})(U) = \left\{ f_0 \frac{\partial}{\partial x} \mid f_0 \in C^\omega(U) \right\} = \text{span}_{C^\omega(U)} \left\{ \frac{\partial}{\partial x} \right\}.$$

Therefore, for every $(x_0, y_0) \in \mathbb{R}^2$, there exists an open neighbourhood U of (x_0, y_0) such that $\text{Lie}(\mathcal{F})(U)$ is a finitely generated $C^\omega(U)$ -module.

One can see that $\text{Orb}_{\mathcal{F}}(0, 0) = \mathbb{R}^2$. Therefore, by the orbit theorem, we have

$$\overline{\mathcal{F}}(0, 0) = T_{(0,0)}\mathbb{R}^2.$$

In particular, we have $\dim(\overline{\mathcal{F}}(0, 0)) = 2$. On the other hand, $\text{Lie}(\mathcal{F})(0, 0) = \{\frac{\partial}{\partial x}\}$ and so $\dim(\text{Lie}(\mathcal{F})(0, 0)) = 1$. This implies that

$$\overline{\mathcal{F}}(0, 0) \neq \text{Lie}(\mathcal{F})(0, 0).$$

Chapter 6

Conclusions and future work

6.1. Conclusions

In this thesis, using topologies on the space of C^ν -functions, we have developed a framework for studying time-varying C^ν -vector fields and their flows. The setting that we constructed in this thesis unifies different classes of regularities in a coherent manner. In particular, it includes the real analytic regularity which is of significance in mathematical control theory. Moreover, we have developed tools and techniques for studying the extension of a time-varying real analytic vector field to a time-varying holomorphic one. Using the suitable topology on the space of real analytic functions, we found a mild sufficient condition to ensure that a time-varying real analytic vector field has a holomorphic extension.

In chapter 4, following [55], we have presented a parameter-invariant model for studying control system called “tautological control system”. Using the notion of presheaf, we developed an appropriate notion of trajectories for tautological control systems. In chapter 5, we generalized the orbit theorem of Sussmann and Stefan for tautological control systems. Using the tautological system approach, we got a natural condition on a tautological control system which ensure that the tangent space to the orbit of the system at a point x is generated by the Lie brackets of vector fields of the system at x . In particular, we showed that globally generated real analytic tautological control systems satisfy this condition.

6.2. Future work

In this section, we mention possible directions for future research.

1. The operator approach developed in chapter 3 can be used to study the local controllability of control systems. While local controllability of a control system is a property of its flows, it is sometimes very hard, if not impossible, to find the flows of a control system. Therefore, in mathematical control theory, one would like to study local controllability of a control system using the vector

fields of the system. Numerous deep and interesting results has been developed in this direction during last four decades. However, many questions still remain unanswered. One of the most interesting open questions is whether local controllability of a control system can be determined using finite number of differentiations of vector fields of the system. One can make this question more rigorous as follows. For a family of vector fields $S = \{f_1, f_2, \dots, f_m\}$, we define a trajectory of S as a piecewise continuous curve such that every continuous piece is a trajectory of one of elements of S . For every $T > 0$, the attainable set of S from $0 \in \mathbb{R}^n$ at time less than equal T is defined as

$$\mathcal{A}_{\leq T}(0) = \{\xi(t) \mid \xi \text{ is a trajectory of } S, \xi(0) = 0, t \leq T\}.$$

The family of vector fields $S = \{f_1, f_2, \dots, f_m\}$ is called small-time locally controllable from 0 if, for every $T > 0$, we have

$$0 \in \text{int}(\mathcal{A}_{\leq T}(0)).$$

In [2], the question of deciding local controllability by a finite number of differentiation has been stated in the following way:

- **Question:** Suppose that we have a family of real analytic vector fields $\{f_1, f_2, \dots, f_m\}$ on \mathbb{R}^n such that it is small-time locally controllable from 0. Does there exists $N \in \mathbb{N}$ such that any other family of real analytic vector fields with the same Taylor polynomial of order N at 0 as the Taylor polynomials of $\{f_1, f_2, \dots, f_m\}$ is small-time locally controllable?

In [5], using suitable variations, this question has been answered affirmative for a specific class of real analytic vector fields. However, for a general family of real analytic vector fields this problem is still open.

We first state the above question in the framework of C^ω -control systems. Consider a C^ω -control system $\Sigma_1 = (\mathbb{R}^n, F, \mathbb{R}^m)$. Suppose that $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow T\mathbb{R}^n$ is defined as

$$F(u, x) = X_0(x) + u_1 X_1(x) + \dots + u_m X_m(x), \quad \forall u \in \mathbb{R}^m, \forall x \in M.$$

Where X_0, X_1, \dots, X_m are real analytic vector fields. Suppose that $u : \mathbb{T} \rightarrow \mathbb{R}$ is a locally essentially bounded and the time-varying vector field $F^u : \mathbb{T} \times \mathbb{R}^n \rightarrow T\mathbb{R}^n$ is defined as

$$F^u(t, x) = F(u(t), x), \quad \forall t \in \mathbb{T}, \forall x \in \mathbb{R}^n.$$

Now assume that $\Sigma_2 = (\mathbb{R}^n, G, \mathbb{R}^m)$ is another C^ω -control system with $G : \mathbb{R}^m \times \mathbb{R}^n \rightarrow T\mathbb{R}^n$ defined as

$$G(u, x) = Y_0(x) + u_1 Y_1(x) + \dots + u_m Y_m(x), \quad \forall u \in \mathbb{R}^m, \forall x \in \mathbb{R}^n.$$

where Y_0, Y_1, \dots, Y_m are real analytic vector fields with the same Taylor polynomial of order N at x_0 as X_0, X_1, \dots, X_m . Then, we say that C^ν -control systems Σ_1 and Σ_2 **agree up to order N at x_0** and we write

$$(\Sigma_1)_{x_0}^N = (\Sigma_2)_{x_0}^N.$$

Now, one can restate the above question as follows:

- **Question:** Let $\Sigma = (\mathbb{R}^n, F, \mathcal{U})$ be a C^ω -control system. Suppose that $x_0 \in \mathbb{R}^n$ and Σ is locally controllable from x_0 . Does there exist $N \in \mathbb{N}$ such that, for every C^ω -control system $\Theta = (\mathbb{R}^n, G, \mathcal{U})$ with $(\Sigma)_{x_0}^N = (\Theta)_{x_0}^N$, Θ is locally controllable from x_0 ?

Since local controllability is a property of flows of the system, it is reasonable to investigate the relation between flows of two real analytic control system which agree up to order N at x_0 . We showed in Theorem 4.4.4 that if $t \mapsto u(t)$ is locally essentially bounded, then F^u is a locally Bochner integrable time-varying real analytic vector field. Therefore, by Theorem 3.8.1, the flow of this vector field can be written as:

$$\phi^{F^u}(t) = \text{id} + \int_{t_0}^t F^u(\tau) d\tau + \int_{t_0}^t \int_{t_0}^{\tau} F^u(\tau) \circ F^u(s) ds d\tau + \dots$$

Since the local controllability of Σ_1 from x_0 only depends on the trajectories of Σ_1 which pass through x_0 at time t_0 , we evaluate ϕ^{F^u} at x_0 .

$$\text{ev}_{x_0} \circ \phi^{F^u}(t) = \text{ev}_{x_0} + \int_{t_0}^t \text{ev}_{x_0}(F^u(\tau)) d\tau + \int_{t_0}^t \int_{t_0}^{\tau} \text{ev}_{x_0}(F^u(\tau) \circ F^u(s)) ds d\tau + \dots$$

Suppose that Σ_2 is a C^ω -control system such that $(\Sigma_1)_{x_0}^N = (\Sigma_2)_{x_0}^N$. Then, for every locally essentially bounded control $t \mapsto u(t)$, for every $i \in \{0, 1, \dots, N\}$, and for every $t_1, t_2, \dots, t_i \in \mathbb{R}$, we have

$$\text{ev}_{x_0}(F^u(t_1) \circ F^u(t_2) \circ \dots \circ F^u(t_i)) = \text{ev}_{x_0}(G^u(t_1) \circ G^u(t_2) \circ \dots \circ G^u(t_i)).$$

Therefore, for every control $t \mapsto u(t)$ which is locally essentially bounded, we have

$$\text{ev}_{x_0} \circ \phi_N^{F^u}(t) = \text{ev}_{x_0} \circ \phi_N^{G^u}(t)$$

This motivates the definition of a C^ν -control system which is N th order approximation of Σ_1 . Unfortunately, such an approximation cannot be a C^ν -control system. The reason is that the map $\phi_N^{F^u}(t)$ is not an algebra homomorphism and as a result, $\phi_N^{F^u}(t)$ is not the flow of any time-varying vector field. However, it is easy to check that $\phi_N^{F^u}(t)$ is an N th order differential operator on $C^\omega(\mathbb{R}^n)$. Let us denote by $L^N(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$ the subspace of $L(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$ consists of N th order differential operators.

Considering flows of a system as unital \mathbb{R} -algebra homomorphism between $C^\omega(\mathbb{R}^n)$ and $C^\omega(\mathbb{R}^n)$, one can get the following restatement of definition of local controllability.

Theorem 6.2.1. *The system Σ_1 is locally controllable from x_0 if and only if ev_{x_0} is in the interior of $\{\text{ev}_{x_0} \circ \phi^{F^u}(T) \mid u \in L_{\text{loc}}^\infty(\mathbb{T}), T \in \mathbb{R}\}$ in $\text{Hom}_{\mathbb{R}}(C^\omega(\mathbb{R}^n); \mathbb{R})$.*

Proof. According to Theorem 3.4.3, the map $\text{ev} : \mathbb{R}^n \rightarrow (C^\omega(\mathbb{R}^n))'$ is a topological homeomorphism onto its image. Using the fact that image of this map is exactly $\text{Hom}_{\mathbb{R}}(C^\omega(\mathbb{R}^n); \mathbb{R})$ (Theorem 3.4.3), the theorem immediately follows. \square

While the above theorem may seem a complicated version of the definition of local controllability of a system from x_0 , it will allow us to generalize the notion of local controllability to the N th order approximation of the system.

In order to modify the definition of the local controllability for the N th order approximation of a system, we define $\widehat{\text{ev}}_{x_0} : L(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n)) \rightarrow (C^\omega(\mathbb{R}^n))'$ as

$$\widehat{\text{ev}}_{x_0}(X) = \text{ev}_{x_0} \circ X, \quad \forall X \in L(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n)).$$

Then it is clear that $\widehat{\text{ev}}_{x_0}(\text{Hom}_{\mathbb{R}}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))) = \text{Hom}_{\mathbb{R}}(C^\omega(\mathbb{R}^n); \mathbb{R})$. We define

$$\widehat{\text{ev}}_{x_0}(L^N(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))) = L_{x_0}^N(C^\omega(M); C^\omega(\mathbb{R}^n)).$$

Note that using the map $\widehat{\text{ev}}_{x_0}$, we get

Theorem 6.2.2. *The system Σ_1 is locally controllable from x_0 if and only if ev_{x_0} is in the interior of $\{\text{ev}_{x_0} \circ \phi^{F^u}(T) \mid u \in L_{\text{loc}}^\infty(\mathbb{T}), T \in \mathbb{R}\}$ in $\widehat{\text{ev}}_{x_0}(\text{Hom}_{\mathbb{R}}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n)))$.*

The above theorem motivates the introduction of the following notion of N th order local controllability.

Definition 6.2.3. A C^ω -control system Σ is called **N th order locally controllable** at x_0 if ev_{x_0} is in the interior of $\{\text{ev}_{x_0} \circ \phi_N^{F^u}(T) \mid u \in L_{\text{loc}}^\infty(\mathbb{T}), T \in \mathbb{R}\}$ in $L_{x_0}^N(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$

It is clear that for a C^ν -control system Σ , N th order local controllability of Σ from x_0 only depends on N th order Taylor polynomial of vector fields of the system around x_0 . Using the above definition, one can ask the following question:

- **Open problem:** Suppose that Σ_1 is locally controllable from x_0 . Does there exist $N \in \mathbb{N}$ such that Σ_1 is N th order locally controllable at x_0 ?

It is obvious that a positive answer to this question will give an affirmative answer to the question mentioned above.

2. We treated the nonlinear differential equation governing the flow of a time-varying C^ν -vector field on M as a linear ODE on the locally convex space $L(C^\nu(M); C^\nu(M))$. Using the holomorphic extension of locally Bochner integrable time-varying real analytic vector fields, we showed that the classical methods for studying linear ODEs on \mathbb{R}^n can be extended to find the solution for the this linear ODE on $L(C^\nu(M); C^\nu(M))$. The evolution of flow a vector field is not the only differential equation that can be translated into an ODE on a locally convex space. In fact, every evolution of partial differential equation and every pseudodifferential equation can be treated as an ODE on some appropriate locally convex space. However, the theory of ODE on locally convex spaces is different in nature from the classical theory of ODE on Banach spaces. While, in the classical theory of ODE on Banach spaces, most of the techniques and tools can be used independently of the geometry of the Banach space, the theory of ODE on locally convex spaces heavily depends on the geometry of the underlying space [56]. The machinery that we developed in chapter 3, including the local and global extension results (Theorems 3.7.8 and 3.7.4) and the family of seminorms for space of real analytic functions (Theorem 3.2.34 and 3.2.35) enables us to study the generalization of the ideas and methods of classical theory of ODE on Banach spaces to ODEs on specific locally convex spaces.
3. The parameter-invariant framework developed in [55] seems to be the right framework for studying fundamental properties of control systems. As mentioned in [54], even the simple linear test for controllability of systems is not parameter-invariant. In [41], a parameter-invariant approach has been used to study linearization of tautological control systems. However, the problem of developing a parameter-invariant theory for small-time local controllability of systems is still open.
4. While in chapter 5 we generalized the orbit theorem for tautological control systems, the proof of the generalized orbit theorem is essentially the same as the classical orbit theorem. In particular, “piecewise constant vector fields” plays a crucial role in the proof of this theorem. With the machinery that we developed in chapter 3, it seems plausible that one can get a new proof of orbit theorem based on general “locally Bochner integrable” vector fields which does not rely on piecewise constant vector fields.

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