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Remarks on stability of time-varying linear systems

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Abstract

The relationships between attractivity and asymptotic stability are fleshed out for homogeneous linear ordinary differential equations with time-varying coefficients.

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AMS Subject Classifications (2020). 34A30, 34D05, 34D23

1. Introduction

The stability of linear ordinary differential equations has, of course, been exhaustively studied, and is included in many standard texts on ordinary differential equations ([Bourlès and Marinescu 2011, §12.4], [Coddington and Levinson 1984, §3.8], [Coppel 1965, §III.2], [Hale 1980, §III.2], [Hahn 1967], [Liao, Wang, and Yu 2007, Chapter 3]) and control theory ([Antsaklis and Michel 1997, §6.5], [Brockett 1970, §29], [Delchamps 1988, §23], [Hinrichsen and Pritchard 2005, §3.3.1], [Kwakernaak and Sivan 1972, §1.4.1], [Sastry 1999, §5.7]). As compared to LTI systems where the notions of stability and uniform stability coincide, for time-varying systems the distinction between these notions become relevant. In this note we flesh out this distinction in a way which seems to have eluded treatment in the standard texts, even occasionally leading to misstated results as a consequence.

Notation and conventions. In the note, we will consider the state-space of a linear ordinary differential equation to be a finite-dimensional normed $\mathbb{R}$-vector space $(\mathsf{V}, \|\cdot\|)$. None of the conclusions of the note depend on the choice of (necessarily equivalent) norm. Thus the differential equation has the form

$$\dot{\xi}(t) = A(t)(\xi(t)), \tag{1}$$

where $A\colon \mathbb{T} \to \mathrm{L}(\mathsf{V}; \mathsf{V})$ is a locally integrable function on an interval $\mathbb{T} \subseteq \mathbb{R}$ taking values in the space $\mathrm{L}(\mathsf{V}; \mathsf{V})$ of linear mappings of V. The solution of the initial value problem

$$\dot{\xi}(t) = A(t)(\xi(t)), \quad \xi(t_0) = x_0,$$

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has the form $\xi(t) = \Phi_A(t,t_0)(x)$, where $\Phi_A\colon \mathbb{T}\times\mathbb{T}\to \mathrm{L}(\mathsf{V};\mathsf{V})$ is the state transition map, which is defined to be the solution of the initial value problem

$$\dot{\Phi}(t) = A(t)\circ\Phi(t), \quad \Phi(t_0) = \mathrm{id}_{\mathsf{V}},$$

id_{V} denoting the identity map. As a consequence, $t\mapsto \Phi_A(t,t_0)$ is absolutely continuous for each $t_0\in\mathbb{T}$.

2. Stability definitions

In this section we first provide definitions for stability of linear ordinary differential equations. The definitions we give are specially contrived for use for linear equations, and are not, at a first glance, equivalent to the usual notions of local stability. Therefore, since we are not aware of all of these results having been proved in the literature, we prove that our linear definitions are equivalent to the usual ones for linear equations.

2.1. Stability definitions for linear ordinary differential equations. We give the various notions of stability that are of interest to us. Our definitions are for the equation (1), keeping in mind that this means, by definition, stability of the zero solution.

1 Definition: (Stability for linear ordinary differential equations) Let $(\mathsf{V},\|\cdot\|)$ be a finite-dimensional normed $\mathbb{R}$-vector space, let $\mathbb{T}\subseteq\mathbb{R}$ be an interval with $\sup\mathbb{T}=\infty$, and let $A\colon\mathbb{T}\to\mathrm{L}(\mathsf{V};\mathsf{V})$ be locally integrable. The differential equation (1) is:

(i) ***stable*** if, for each $t_0\in\mathbb{T}$, there exists $C>0$ such that $\|\Phi_A(t,t_0)(x)\|\le C\|x\|$ for every $x\in\mathsf{V}$ and for every $t\ge t_0$;

(ii) ***attractive*** if, for each $t_0\in\mathbb{T}$ and each $\epsilon>0$, there exists $T>0$ such that $\|\Phi_A(t,t_0)(x)\|\le\epsilon\|x\|$ for every $x\in\mathsf{V}$ and for every $t\ge t_0+T$;

(iii) ***exponentially attractive*** if, for each $t_0\in\mathbb{T}$ and each $\epsilon>0$, there exists $M,\sigma>0$ such that $\|\Phi_A(t,t_0)(x)\|\le M\mathrm{e}^{-\sigma(t-t_0)}\|x\|$ for every $x\in\mathsf{V}$ and for every $t\ge t_0$;

(iv) ***asymptotically stable*** if it is stable and attractive;

(v) ***exponentially stable*** if it is stable and exponentially attractive;

(vi) ***uniformly stable*** if there exists $C>0$ such that $\|\Phi_A(t,t_0)(x)\|\le C\|x\|$ for every $(t_0,x)\in\mathbb{T}\times\mathsf{V}$ and for every $t\ge t_0$;

(vii) ***uniformly attractive*** if, for each $\epsilon>0$, there exists $T>0$ such that $\|\Phi_A(t,t_0)(x)\|\le\epsilon\|x\|$ for every $(t_0,x)\in\mathbb{T}\times\mathsf{V}$ and for every $t\ge t_0+T$;

(viii) ***uniformly exponentially attractive*** if there exists $M,\sigma>0$ such that $\|\Phi_A(t,t_0)(x)\|\le M\mathrm{e}^{-\sigma(t-t_0)}\|x\|$ for every $(t_0,x)\in\mathbb{T}\times\mathsf{V}$ and for every $t\ge t_0$;

(ix) ***uniformly asymptotically stable*** if it is uniformly stable and uniformly attractive;

(x) ***uniformly exponentially stable*** if it is uniformly stable and uniformly exponentially attractive. •

We have made the definitions of "asymptotically stable," "exponentially stable," "uniformly asymptotically stable," "uniformly exponentially stable" in such a way that they are visually symmetric. Moreover, for nonlinear equations, the form of the definitions we give agree with the standard ones (as we shall prove in the next section). However, as we shall see, some of the definitions are logically redundant for linear equations. Indeed, this is the point of the note.

2.2. Equivalence with standard general definitions. Since we are not aware of complete proofs in the existing literature of the equivalence between our stability definitions for linear equations and the usual ones for general equations, in this section we prove this equivalence.

First we give the general definitions, just so we are clear about which of the myriad possible definitions we are using. The definitions are made for an ordinary differential equation

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{F}(t, \boldsymbol{\xi}(t))$$

for right-hand side $\boldsymbol{F}$ defined on $\mathbb{T} \times \mathcal{U}$ for an interval $\mathbb{T} \subseteq \mathbb{R}$ and $\mathcal{U} \subseteq \mathbb{R}^n$ open, and that is locally integrable in t and locally Lipschitz in state with a locally integrally bounded Lipschitz constant. The solution to the initial value problem with initial condition $\boldsymbol{\xi}(t_0) = \boldsymbol{x}_0$ we denote by $\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$.

2 Definition: (General stability definitions) Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval and let $\mathcal{U} \subseteq \mathbb{R}^n$ be open. Let $\boldsymbol{F} \colon \mathbb{T} \times \mathcal{U} \to \mathbb{R}^n$ define a system of ordinary differential equations (as above) and suppose that $\sup \mathbb{T} = \infty$. Let $\boldsymbol{x}_0 \in \mathcal{U}$ be an equilibrium point for $\boldsymbol{F}$. The equilibrium point $\boldsymbol{x}_0$ is:

(i) ***locally stable*** if, for any $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists $\delta > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0\| < \epsilon$ for every $\boldsymbol{x} \in \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0$;

(ii) ***locally attractive*** if, for every $t_0 \in \mathbb{T}$, there exists $\delta > 0$ such that, for $\epsilon > 0$, there exists $T > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0\| < \epsilon$ for every $\boldsymbol{x} \in \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0 + T$;

(iii) ***locally exponentially attractive*** if, for every $t_0 \in \mathbb{T}$, there exists $M, \delta, \sigma > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0\| \leq M\mathrm{e}^{-\sigma(t-t_0)}$ for every $\boldsymbol{x} \in \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0$.

(iv) ***locally asymptotically stable*** if it is locally stable and locally attractive;

(v) ***locally exponentially stable*** if it is locally stable and locally exponentially attractive;

(vi) ***uniformly locally stable*** if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0(t)\| < \epsilon$ for every $(t_0, \boldsymbol{x}) \in \mathbb{T} \times \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0$;

(vii) ***uniformly locally attractive*** if there exists $\delta > 0$ such that, for $\epsilon > 0$, there exists $T > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0\| < \epsilon$ for every $(t_0, \boldsymbol{x}) \in \mathbb{T} \times \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0 + T$;

(viii) ***uniformly locally exponentially attractive*** if there exists $M, \sigma, \delta > 0$ such that $t \mapsto \Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0)$ is defined on $[t_0, \infty)$ and satisfies $\|\Phi^{\boldsymbol{F}}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_0\| \leq M\mathrm{e}^{-\sigma(t-t_0)}$ for every $(t_0, \boldsymbol{x}) \in \mathbb{T} \times \mathcal{U}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and for every $t \geq t_0$.

(ix) ***uniformly locally asymptotically stable*** if it is uniformly locally stable and uniformly locally attractive;

(x) ***uniformly locally exponentially stable*** if it is uniformly locally stable and uniformly locally exponentially attractive;

(xi) ***unstable*** if it is not stable. •

To simplify reading, let us introduce abbreviations.

Definition 1	Definition 2
S: stable	LS: locally stable
A: attractive	LA: locally attractive
EA: exponentially attractive	LEA: locally exponentially attractive
AS: asymptotically stable	LAS: locally asymptotically stable
ES: exponentially stable	LES: locally exponentially stable
US: uniformly stable	ULS: uniformly locally stable
UA: uniformly attractive	ULA: uniformly locally attractive
UEA: uniformly exponentially attractive	ULEA: uniformly locally exponentially attractive
UAS: uniformly asymptotically stable	ULAS: uniformly locally asymptotically stable
UES: uniformly exponentially stable	ULES: uniformly locally exponentially stable

Now the following result proves the equivalence of the two possibly different collections of stability definitions.

3 Proposition: (Equivalence of stability definitions for linear equations) *Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval for which $\sup \mathbb{T} = \infty$, let V be a finite-dimensional $\mathbb{R}$-vector space, and let $F\colon \mathbb{T} \times \mathsf{V} \to \mathsf{V}$ be a linear ordinary differential equation: $F(t,x) = A(t)(x)$. We have the following equivalences:*

(i) S *of F* $\iff$ 0 *is* LS*;*

(ii) A *of F* $\iff$ 0 *is* LA*;*

(iii) EA *of F* $\iff$ 0 *is* LEA*;*

(iv) AS *of F* $\iff$ 0 *is* LAS*;*

(v) ES *of F* $\iff$ 0 *is* LES*;*

(vi) US *of F* $\iff$ 0 *is* ULS*;*

(vii) UA *of F* $\iff$ 0 *is* ULA*;*

(viii) UEA *of F* $\iff$ 0 *is* ULEA*;*

(ix) UAS *of F* $\iff$ 0 *is* ULAS*;*

(x) UES *of F* $\iff$ 0 *is* ULES*;*

Proof: (S $\Longrightarrow$ LS) Let $t_0 \in \mathbb{T}$ and let $C > 0$ be such that $\|\Phi_A(t,t_0)(x)\| \leq C\|x\|$ for $x \in \mathsf{V}$ and $t \geq t_0$. Let $\epsilon > 0$ and take $\delta = \frac{\epsilon}{2C}$. Now let $x \in \mathsf{V}$ satisfy $\|x\| < \delta$. We then have

$$\|\Phi_A(t,t_0)(x)\| \leq C\|x\| = \frac{\epsilon}{2\delta}\|x\| < \epsilon,$$

for $t \geq t_0$, giving local stability of 0.

(LS $\Longrightarrow$ S) Let $t_0 \in \mathbb{T}$ and let $\delta > 0$ have the property that $\|\Phi_A(t,t_0)(x)\| < 1$ for every $x \in \mathsf{V}$ such that $\|x\| < \delta$ and for every $t \geq t_0$. Define $C = \frac{2}{\delta}$. Let $x \in \mathsf{V}$. First suppose that $x \neq 0$ and define $\hat{x} = \delta\frac{x}{2\|x\|}$ so that $\|\hat{x}\| = \frac{\delta}{2} < \delta$. Thus $\|\Phi_A(t,t_0)(\hat{x})\| < 1$ for $t \geq t_0$. However,

$$\Phi_A(t,t_0)(x) = \Phi_A(t,t_0)\left(\frac{2\|x\|}{\delta}\hat{x}\right) = \frac{2\|x\|}{\delta}\Phi_A(t,t_0)(\hat{x}) = C\|x\|\Phi_A(t,t_0)(\hat{x}).$$

Therefore,

$$\|\Phi_A(t,t_0)(x)\| = C\|x\|\,\|\Phi(t,t_0)(\hat{x})\| \leq C\|x\|$$

for $t \geq t_0$. Since we also have $\|\Phi_A(t,t_0)(0)\| = 0 \leq C\|0\|$, we conclude that F is stable.

(A $\Longrightarrow$ LA) Let $t_0 \in \mathbb{T}$ and let $\delta > 0$. Let $\epsilon > 0$, and take $T > 0$ such that $\|\Phi_A(t,t_0)(x)\| \leq \frac{\epsilon}{\delta}\|x\|$ for every $x \in \mathsf{V}$ and for every $t \geq t_0 + T$. Now suppose that $\|x\| < \delta$ and note that $\|\Phi_A(t,t_0)(x)\| \leq \frac{\epsilon}{\delta}\|x\| < \epsilon$ for every $t \geq t_0 + T$. This shows that 0 is locally attractive.

($\mathsf{LA} \implies \mathsf{A}$) Let $t_0 \in \mathbb{T}$ and let $\delta > 0$ have the property that, given $\epsilon > 0$, there exists $T > 0$ such that $\|\Phi_A(t, t_0)(x)\| < \epsilon$ for every $x \in \mathsf{V}$ such that $\|x\| < \delta$ and for every $t \geq t_0 + T$.

Let $\epsilon > 0$ and let $T > 0$ be such that $\|\Phi_A(t, t_0)(x)\| < \frac{\epsilon\delta}{2}$ for every $x \in \mathsf{V}$ such that $\|x\| < \delta$ and for every $t \geq t_0 + T$. Let $x \in \mathsf{V} \setminus \{0\}$ and denote $\hat{x} = \delta \frac{x}{2\|x\|}$. Since $\|\hat{x}\| = \frac{\delta}{2} < \delta$, $\|\Phi_A(t, t_0)(\hat{x})\| < \frac{\epsilon\delta}{2}$ for $t \geq t_0 + T$. We also have

$$\Phi_A(t, t_0)(x) = \Phi_A(t, t_0)\left(\frac{2\|x\|}{\delta}\hat{x}\right) = \frac{2\|x\|}{\delta}\Phi_A(t, t_0)(\hat{x}).$$

Thus

$$\|\Phi_A(t, t_0)(x)(t)\| = \frac{2}{\delta}\|x\|\,\|\Phi_A(t, t_0)(\hat{x})\| < \epsilon\|x\|,$$

for $t \geq t_0 + T$, and so F is attractive.

($\mathsf{EA} \implies \mathsf{LEA}$) Let $t_0 \in \mathbb{T}$ and let $\tilde{M}, \tilde{\sigma} > 0$ be such that $\|\Phi_A(t, t_0)(x)\| \leq \tilde{M}\|x\|\mathrm{e}^{-\tilde{\sigma}(t-t_0)}$ for every $x \in \mathsf{V}$ and for every $t \geq t_0$. Now let $\delta > 0$ and take $M = \tilde{M}\delta$ and $\sigma = \tilde{\sigma}$. Then, for $\|x\| < \delta$, we have

$$\|\Phi_A(t, t_0)(x)\| \leq \tilde{M}\|x\|\mathrm{e}^{\tilde{\sigma}(t-t_0)} \leq \frac{\tilde{M}}{\delta}\delta\|x\|\mathrm{e}^{\tilde{\sigma}(t-t_0)} \leq M\mathrm{e}^{-\sigma(t-t_0)},$$

showing that 0 is locally exponentially attractive.

($\mathsf{LEA} \implies \mathsf{EA}$) Let $t_0 \in \mathbb{T}$ and let $\tilde{M}, \delta, \tilde{\sigma} > 0$ be such that $\|\Phi_A(t, t_0)(x)\| \leq \tilde{M}\mathrm{e}^{-\tilde{\sigma}(t-t_0)}$ for every $x \in \mathsf{V}$ such that $\|x\| < \delta$ and for every $t \geq t_0$.

Take $M = \frac{2\tilde{M}}{\delta}$ and $\sigma = \tilde{\sigma}$. Now let $x \in \mathsf{V}$ and denote $\hat{x} = \delta\frac{x}{2\|x\|}$. Since $\|\hat{x}\| = \frac{\delta}{2} < \delta$, $\|\Phi_A(t, t_0)(\hat{x})\| \leq \tilde{M}\mathrm{e}^{-\tilde{\sigma}(t-t_0)}$ for $t \geq t_0$. Then, as in the proof that $\mathsf{AS} \implies \mathsf{GAS}$,

$$\Phi_A(t, t_0)(x) = \frac{2\|x\|}{\delta}\Phi_A(t, t_0)(\hat{x}),$$

and so

$$\|\Phi_A(t, t_0)(x)\| = \frac{2}{\delta}\|x\|\,\|\Phi_A(t, t_0)(\hat{x})\| \leq \frac{2\tilde{M}}{\delta}\|x\|\mathrm{e}^{-\tilde{\sigma}(t-t_0)} = M\|x\|\mathrm{e}^{-\sigma(t-t_0)},$$

for $t \geq t_0$, showing that F is exponentially attractive.

The fact that ($\mathsf{AS} \iff \mathsf{LAS}$) and ($\mathsf{ES} \iff \mathsf{LES}$) follows immediately from the preceding parts of the proof.

The remainder of the proof concerns the results we have already proved, but with the property "uniform" being applied to all hypotheses and conclusions. The proofs are entirely similar to those above. We shall, therefore, only work this out in one of the three cases, the other two following in an entirely similar manner.

($\mathsf{US} \implies \mathsf{ULS}$) Let $C > 0$ be such $\|\Phi_A(t, t_0)(x)\| \leq C\|x\|$ for every $x \in \mathsf{V}$ and for every $t \geq t_0$. Let $\epsilon > 0$ and take $\delta = \frac{\epsilon}{2C}$. Now let $x \in \mathsf{V}$ satisfy $\|x\| < \delta$ and note that

$$\|\Phi_A(t, t_0)(x)\| \leq C\|x\| = \frac{\epsilon}{2\delta}\|x\| < \epsilon,$$

for $t \geq t_0$, giving uniform local stability of 0.

(ULS $\Longrightarrow$ US) Let $\delta > 0$ have the property that $\|\Phi_A(t, t_0)(x)\| < 1$ for every $x \in \mathsf{V}$ such that $\|x\| < \delta$ and for every $t \geq t_0$. Define $C = \frac{2}{\delta}$. Now let $x \in \mathsf{V}$ and define $\hat{x} = \delta\frac{x}{2\|x\|}$ so that $\|\hat{x}\| = \frac{\delta}{2} < \delta$. Thus $\|\Phi(t, t_0)(\hat{x})\| < 1$ for $t \geq t_0$. Then,

$$\Phi_A(t, t_0)(x) = \Phi_A(t, t_0)\left(\frac{2\|x\|}{\delta}\hat{x}\right) = \frac{2\|x\|}{\delta}\Phi_A(t, t_0)(\hat{x}) = C\|x\|\Phi_A(t, t_0)(\hat{x}).$$

Therefore,

$$\|\Phi_A(t, t_0)(x)\| = C\|x\|\,\|\Phi_A(t, t_0)(\hat{x})\| \leq C\|x\|$$

for $t \geq t_0$. Since we also have $\|\Phi_A(t, t_0)(0)\| = 0 \leq C\|0\|$, we conclude that F is uniformly stable. ■

3. Two trivial results and a nontrivial example

In this section are contained the principal observations of this note, which are that the rôle of "stability" in the definitions of "asymptotic stability" differ in the nonuniform and uniform cases.

First a simple result.

4 Proposition: ("Stability" is redundant in the definition of "asymptotic stability") *Let* $(\mathsf{V}, \|\cdot\|)$ *be a finite-dimensional normed* $\mathbb{R}$*-vector space, let* $\mathbb{T} \subseteq \mathbb{R}$ *be an interval with* $\sup \mathbb{T} = \infty$*, and let* $A\colon \mathbb{T} \to \mathrm{L}(\mathsf{V}; \mathsf{V})$ *be locally integrable. Then the differential equation (1) is stable if it is attractive.*

Proof: Let $t_0 \in \mathbb{T}$. Since the differential equation is attractive, there exists $T > 0$ such that $\|\Phi_A(t, t_0)(x)\| \leq \|x\|$ for all $x \in \mathsf{V}$ and for every $t \geq t_0 + T$. This means that, for $t \in [t_0, t_0 + T]$,

$$\|\Phi_A(t, t_0)(x)\| \leq \sup\{|||\Phi_A(t, t_0)||| \mid t \in [t_0, t_0 + T]\}\|x\|,$$

where $|||\cdot|||$ is the induced norm on $\mathrm{L}(\mathsf{V}; \mathsf{V})$. Note that the supremum in the preceding equation is finite, since $t \mapsto \Phi_A(t, t_0)$ is continuous. Thus, letting

$$C = \max\{1, \sup\{|||\Phi_A(t, t_0)||| \mid t \in [t_0, t_0 + T]\}\},$$

we have $\|\Phi_A(t, t_0)(x)\| \leq C\|x\|$ for all $x \in \mathsf{V}$ and for all $t \geq t_0$. Thus the differential equation (1) is stable. ■

Next we give an example that is, in some sense, the main contribution of the note.

5 Example: ("Uniform stability" is not redundant in the definition of "uniform asymptotic stability") We shall construct a scalar linear ordinary differential equation that is uniformly attractive but not uniformly stable. To do this we construct a locally integrable function $a\colon [0, \infty) \to \mathbb{R}$ and work with the differential equation

$$\dot{\xi}(t) = a(t)\xi(t). \tag{2}$$

Let us define a.

1. Define sequences (α_k), (β_k), and (Δ_k) as follows:

(a) $\Delta_k = 2^{-k-1}$;
(b) $\beta_k = k2^{k+1}$;
(c) define $\alpha_0 = 1$ and then define α_k, $k \geq 1$, by

$$\beta_{k-1}\Delta_{k-1} - \alpha_k(1-\Delta_k) + \beta_k\Delta_k + \beta_{k+1}\Delta_{k+1} = -1.$$

2. If $t \in \mathbb{T}$, let k be a nonnegative integer such that $t \in [k, k+1)$, and then define

$$a(t) = \begin{cases} -\alpha_k, & t \in [k, (k+1)-\Delta_k), \\ \beta_k, & t \in [(k+1)-\Delta_k, k+1). \end{cases}$$

To show that (2) has the desired properties, we first show that it is not uniformly stable. For $k \geq 1$ define $t_k = (k+1) - \Delta_k$ and $t_{0,k} = k+1$. We consider the initial condition $1 \in \mathsf{V}$ and note that

$$|\Phi_a(t_k, t_{0,k})(1)| = \left|\mathrm{e}^{\int_{t_{0,k}}^{t_k} a(\tau)\,\mathrm{d}\tau}\right| = \mathrm{e}^k.$$

This prohibits uniform global stability.

Next we show that (2) is uniformly attractive. Thus let $\epsilon > 0$ and define $T > 0$ such that $\mathrm{e}^{-T} < \epsilon$. Let $t_0 \in \mathbb{T}$ and let $t \geq t_0 + T$. Let $k_1 \geq 1$ be such that $t_0 \in [k_1, k_1+1)$, let $k_2 \geq 1$ be such that $t \in [k_2, k_2+1)$. Note that

$$t - t_0 \geq T \quad \Longrightarrow \quad k_2 - k_1 - 1 > T.$$

Now we estimate

$$\begin{aligned}
\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau &= \int_{t_0}^{k_1+1} a(\tau)\,\mathrm{d}\tau + \sum_{k=k_1+1}^{k_2-1}\int_{k}^{k+1} a(\tau)\,\mathrm{d}\tau + \int_{k_2}^{t} a(\tau)\,\mathrm{d}\tau \\
&\leq \beta_{k_1}\Delta_{k_1} + \sum_{k=k_1+1}^{k_2-1}(-\alpha_k(1-\Delta_k) + \beta_k\Delta_k) + \beta_{k_2}\Delta_{k_2} \\
&\leq \sum_{k_1+1}^{k_2-1}(\beta_{k-1}\Delta_{k-1} - \alpha_k(1-\Delta_k) + \beta_k\Delta_k + \beta_{k+1}\Delta_{k+1}) \\
&= \sum_{k=k_1+1}^{k_2-1}(-1) = -(k_2 - k_1 - 1) < -T.
\end{aligned}$$

Now let $x \in \mathsf{V}$ and note that

$$|\Phi_a(t, t_0)(x)| = \left|x\mathrm{e}^{\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau}\right| \leq |x|\mathrm{e}^{-T} < \epsilon|x|,$$

for $t \geq t_0 + T$, giving the desired conclusion. •

Finally, we consider another simple result.

6 Proposition: ("Uniform stability" is redundant in the definition of "uniform exponential stability") *Let* $(\mathsf{V}, \|\cdot\|)$ *be a finite-dimensional normed* $\mathbb{R}$*-vector space, let* $\mathbb{T} \subseteq \mathbb{R}$ *be an interval with* $\sup \mathbb{T} = \infty$*, and let* $A\colon \mathbb{T} \to \mathrm{L}(\mathsf{V}; \mathsf{V})$ *be locally integrable. Then the differential equation (1) is uniformly stable if it is uniformly exponentially attractive.*

Proof: Since the differential equation is uniformly exponentially attractive, there exists $M, \sigma > 0$ such that $\|\Phi_A(t, t_0)(x)\| \leq M\mathrm{e}^{-\sigma(t-t_0)}\|x\|$ for every $x \in \mathsf{V}$ and for every $t \geq t_0$. This immediately gives $\|\Phi_A(t, t_0)(x)\| \leq M\|x\|$ for every $x \in \mathsf{V}$ and for every $t \geq t_0$ and so gives uniform stability. ■

4. Discussion

The simple results and the example illustrate that the logically correct, i.e., not redundant, versions of our definitions for linear ordinary differential equations are:

1. "asymptotically stable" if "attractive";
2. "exponentially stable" if "exponentially attractive";
3. "uniformly asymptotically stable" if "uniformly stable" and "uniformly attractive";
4. "uniformly exponentially stable" if "uniformly exponentially attractive".

Thus, when making the definitions, one has a choice between symmetry of the definitions (as we have gone with in Definition 1) and logical correctness. That there is such a choice, and the precise reasons why there is such a choice (mainly Example 5), is something that is sidestepped in the literature on stability of time-varying linear ordinary differential equations, and occasionally gotten wrong. Let us point out how this matter is handled in some places in the literature, to illustrate the propagation of the confusion.

1. Antsaklis and Michel [1997]: Although this is a book about linear systems, the definitions for linear equations are specialised from general definitions for nonlinear equations. For nonlinear equations, one does not have to choose between symmetry and logical consistency, since the two choices align in the general case. For linear equations, these definitions are logically redundant, as exhibited by Propositions 4 and 6. In Theorem 5.4, Antsaklis and Michel correctly observe that, for linear equations, uniform asymptotic stability and uniform exponential stability coincide.
2. Bourlès and Marinescu [2011]: Here too, the definitions for linear equations appear as a special case of definitions for nonlinear equations (although this is a text specifically about time-varying linear differential equations), so the definitions are logically redundant as per Propositions 4 and 6. Also, in Proposition 1186, it is proved that, correctly for linear equations, uniform asymptotic stability and uniform exponential stability coincide.
3. Brockett [1970]: Here only the stability of linear differential equations is considered, but only the notions of "uniform stability" and "exponential stability" are defined. Thus the exception of "uniform asymptotic stability" in the definitions does not arise.
4. Coddington and Levinson [1984]: The notions of uniform stability are not discussed.
5. Coppel [1965]: As this is a book on the general notions of stability, the stability definitions for linear equations are adapted from those for nonlinear equations. Thus the redundancies of Propositions 4 and 6 are not discussed. In Theorem III.2.1, uniform asymptotic stability is correctly characterised precisely as uniform exponential stability.

6. Delchamps [1988]: In this text, while only linear equations are considered, the definitions of stability are those for nonlinear equations, and so the redundancies of Propositions 4 and 6 are inherited. While a correct definition is given for uniform asymptotic stability (the one for nonlinear equations), it is incorrectly characterised in Theorem 23.10(d), where the uniform boundedness of the state transition map is not hypothesised, as we have shown in Example 5 that it must be.
7. Hahn [1967]: This is a book on the general theory of stability, with stability of linear systems arising as a special case. The treatment here has the feature of proving (as Theorem 58.7) the usual result that uniform exponential stability is equivalent to uniform asymptotic stability. However, the consideration of the special logical implications between various stability notions for linear systems, such as we consider here, does not receive attention.
8. Hale [1980]: This is another instance where the redundancies of Propositions 4 and 6 are not revealed because linear equations are treated as a special case of nonlinear equations. In Theorem III.2.1, uniform asymptotic stability is correctly characterised as uniform exponential stability.
9. [Hahn 1967]: This is a book on the general theory of stability, with stability of linear systems arising as a special case. The treatment here has the feature of proving (as Theorem 58.7) the usual result that uniform exponential stability is equivalent to uniform asymptotic stability. However, the consideration of the special logical implications between various stability notions for linear systems, such as we consider here, does not receive attention.
10. Hinrichsen and Pritchard [2005]: In this text also, the stability definitions for linear equations are simply those for nonlinear equations, and consequently the redundancies of Propositions 4 and 6 are not discussed. However, in their Proposition 3.3.2(iii), Hinrichsen and Pritchard give an incorrect characterisation of uniform asymptotic stability, omitting the hypothesis that the state transition map must be uniformly bounded, as we show must be the case in Example 5.
11. Kwakernaak and Sivan [1972]: In this text on linear optimal control theory, stability definitions are given for nonlinear equations, and these are inherited for linear equations. Notions of uniform stability are not considered, and only exponential stability of linear equations is discussed in detail.
12. Liao, Wang, and Yu [2007]: The stability definitions for linear equations are inherited from the general definitions for nonlinear equations, as might be expected in a text on the general theory of stability. In Theorem 3.2.5 it is pointed out that uniform asymptotic stability is equivalent to the uniform decay of the state transition map *and* uniform boundedness of this map. This is the only place in the literature where we saw this correspondence stated (although it is logically and nontrivially equivalent to the often stated uniform exponential decay of the state transition map). However, the redundancies of Propositions 4 and 6 are not addressed.
13. Sastry [1999]: In this text on nonlinear systems, the linear definitions of stability are specialised from those for nonlinear equations, and again the fact that these definitions are redundant as per Propositions 4 and 6 is not discussed. In Theorem 5.33 the correct theorem concerning the equivalence of uniform asymptotic stability and uniform exponential stability is stated.

As can be seen, in most texts, while there are no errors, the treatment is made in such a way that the interesting subtleties of uniform asymptotic stability for linear equations are avoided. And, in a few cases, the overlooking of these subtleties has led to erroneously stated results.

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