Degeneracy of velocity constraints in rigid body systems

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Abstract

The equations governing the dynamics of rigid body systems with velocity constraints are singular at degenerate configurations in the constraint distribution. In this report, we describe the causes of singularities in the constraint distribution of interconnected rigid body systems with smooth configuration manifolds. A convention of defining primary velocity constraints in terms of orthogonal complements of one-dimensional subspaces is introduced. Using this convention, linear maps are defined and used to describe the space of allowable velocities of a rigid body. Through the definition of these maps, we present a condition for non-degeneracy of velocity constraints in terms of the one dimensional subspaces defining the primary velocity constraints. A method for defining the constraint subspace and distribution in terms of linear maps is presented. Using these maps, the constraint distribution is shown to be singular at configuration where there is an increase in its dimension.

Contents

Chapter 1

Introduction

In this report we will explore the conditions on velocity constraints that lead to singularities in the constraint distribution. In doing so, we hope to form a better understanding of how one may model interconnected rigid body systems with velocity constraints that do not have locally constant rank.

The remainder of the introduction will be outlined as follows: Section 1.1 is an account of the motivation for studying the topic; Section 1.2 presents an illustrative example that outlines issues when modeling systems with velocity constraint that do not have locally constant rank; Section 1.3 reviews the current practices when modeling systems with velocity constraints; Section 1.4 presents the organization of the report.

1.1. Motivation

The dynamics of rigid body systems with velocity constraints are modeled by differential equations. In order to be able to properly predict the behavior of a rigid body system we require the existence and uniqueness of solutions for the differential equations that describe the system. In general, existence and uniqueness cannot be guaranteed when modeling systems. Singularities may arise in the models for several different reasons. One of these reason, which is the focus for this report, can be the introduction of velocity constraints on interconnected rigid body systems.

The most common method of modeling rigid body systems with velocity constraints is via the Lagrange–D'Alembert principle. To account for the effects of velocity constraints we introduce terms to the unconstrained dynamical system by the method of undetermined Lagrangian multipliers. The terms introduced model the effects of constraint forces for the system to be thought of as unknowns in the equations of motion. Since only one Lagrange multiplier is introduced per velocity constraint, the system remains solvable as long as the constraint forces remain linearly independent. Problems arise when two or more velocity constraints degenerate resulting in a change of rank in the constraint distribution. To circumvent this situation, it is often assumed that the velocity constraints for the system of interest have locally constant rank. This assumption is not always valid as we will see Section 1.2.

Currently there is no method of fully modeling these types of systems. When systems with velocity constraints that do not have locally constant rank are modeled, the singular points of the velocity distribution are omitted from consideration. Here lies the motivation for the report. In studying what causes these singularities, we can gain further understanding of this phenomenon and possibly develop a method of properly modeling the dynamics of these types of interconnected rigid body systems.

1.2. Illustrative example

Here we present an example, modeling the equations of motion for a snake cart, to gain better understanding of the problem at hand. In doing so, we can reveal the problems that arise when dealing with velocity constraints that form an irregular constraint distribution.

A snake cart is similar to a regular cart in that it consists of two wheel assemblies connected at opposite ends of a rigid body structure. The way a snake cart differs from a regular cart is that the wheel assemblies of the snake cart are restricted to move in unison, i.e., they are required to maintain the same angle relative to the body frame of the cart at all time. We begin by presenting a model for this system.

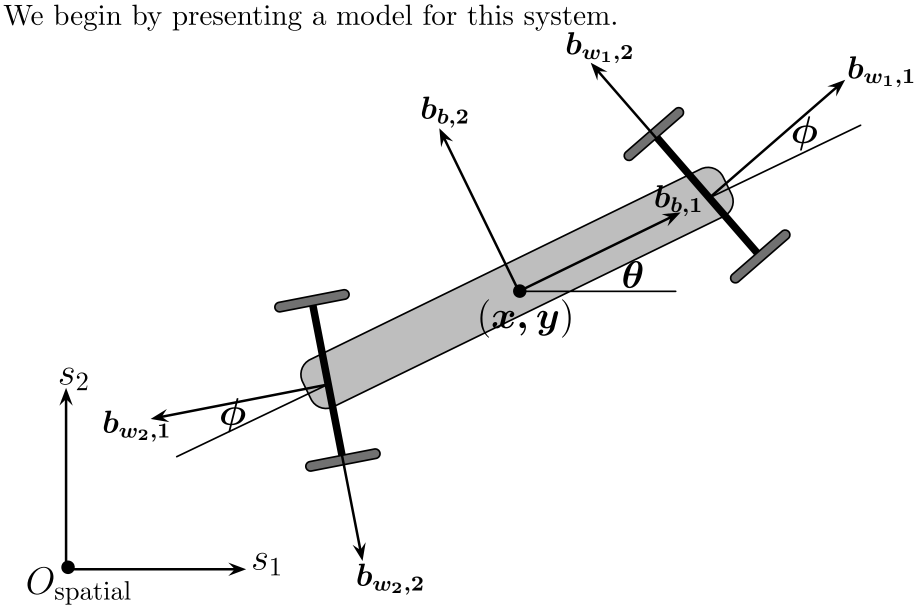

Figure 1.1: The snake cart model

The coordinates $q = (q^1, q^2, q^3, q^4) = (x, y, \theta, \phi)$ will be used to describe the configuration of the system at time t as seen in Figure 1.1. The pair (x, y) describes the location of the centre of mass for the system, which is assumed to lie exactly half way between the two wheel assemblies. The relative angle of the body frame, $(O_{\text{body}}, (b_{b,1}, b_{b,2}, b_{b,3}))$, with the spatial frame, $(O_{\text{spatial}}, (s_1, s_2, s_3))$, is described by the variable θ. Similarly, ϕ describes the angle of the wheels relative to the body frame.

Next we define the following physical parameters:

ℓ distance between the front and back wheels;

m_b mass of the body;

m_w mass of single wheel assembly;

J_b inertia of body about centre of mass;

J_w inertia of single wheel assembly about centre of mass.

Given these, it can be shown that the kinetic energy Lagrangian for the system is

$$L = J_w\dot{\phi}^2(t) + \left(\frac{1}{2}J_b + J_w + \ell^2 m_w\right)\dot{\theta}^2(t) + \left(\frac{1}{2}m_b + m_w\right)\left(\dot{x}^2(t) + \dot{y}^2(t)\right).$$

We require that the body velocity in $b_{\alpha,2}$ direction is zero for body $\alpha \in \{w_1, w_2\}$. From this we get the constraints

$$\begin{aligned}
&\frac{\ell}{2}\cos(\phi(t))\cdot\dot{\theta}(t) + \cos(\theta(t)+\phi(t))\cdot\dot{y}(t) - \sin(\theta(t)+\phi(t))\cdot\dot{x}(t) = 0,\\
&\frac{\ell}{2}\cos(\phi(t))\cdot\dot{\theta}(t) - \cos(\theta(t)-\phi(t))\cdot\dot{y}(t) + \sin(\theta(t)-\phi(t)))\cdot\dot{x}(t) = 0.
\end{aligned} \tag{1.2.1}$$

The subspace generated by (1.2.1) is called the constraint distribution D.

Recall, by the Lagrange–D'Alembert principle,

$$\sum_{i=1}^{4}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right)\delta q^i = 0, \tag{1.2.2}$$

for all $\delta q \in \mathsf{D}$, i.e., for all infinitesimal variations in the constraint distribution.

Using (1.2.2), we can show the equations of motion are given by

$$\begin{bmatrix} (m_b + 2m_w)\ddot{x}(t) \\ (m_b + 2m_w)\ddot{y}(t) \\ \left(J_b + 2J_w + \frac{1}{2}\ell^2 m_w\right)\ddot{\theta}(t) \\ \ddot{\phi}(t) \end{bmatrix} = \begin{bmatrix} -\sin(\theta(t)+\phi(t)) & \sin(\theta(t)-\phi(t)) \\ \cos(\theta(t)+\phi(t)) & -\cos(\theta(t)-\phi(t)) \\ \frac{\ell}{2}\cos(\phi(t)) & \frac{\ell}{2}\cos(\phi(t)) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \tag{1.2.3}$$

where λ_1 and λ_2 are undetermined Lagrangian multipliers. Here is where the problem lies. At certain configurations, specifically when $\phi = \pm\frac{\pi}{2}$, (1.2.3) reduces to

$$\begin{bmatrix} (m_b + 2m_w)\ddot{x}(t) \\ (m_b + 2m_w)\ddot{y}(t) \\ \left(J_b + 2J_w + \frac{1}{2}\ell^2 m_w\right)\ddot{\theta}(t) \\ \ddot{\phi}(t) \end{bmatrix} = \begin{bmatrix} \pm\cos(\theta(t)) & \pm\cos(\theta(t)) \\ \mp\sin(\theta(t)) & \mp\sin(\theta(t)) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{1.2.4}$$

We note that the matrix on the right hand side of (1.2.4) no longer has full rank. Thus, in general, the undetermined Lagrangian multipliers cannot be unambiguously solved. One should note that the singularities at $\phi = \pm\frac{\pi}{2}$ are directly related to the configurations in which the constraint distribution D does not have locally constant rank.

This example raises a question that other rigid body systems without locally constant velocity constraints share. Is there some sort of physical law that dictates a method to solve for the Lagrange multipliers of rigid body systems at singularities in the constraint distribution? Before this question can be answered, we need to understand the cause of these singularities.

1.3. Review of literature

Looking to the literature, it seem no work has been done regarding the modeling of interconnected rigid body systems with irregular constraint distributions. In fact, Lewis' paper [4] and Bullo and Lewis' book [2] both remark on the lack of work done in this area. When modeling rigid body systems with multiple velocity constraints, it is commonly assumed that the constraint distribution is smooth. Often these assumptions are made implicitly, as it is done in [5, pg. 411], [6, pg. 75], and [7, pg. 215], where it is assumed that the Lagrangian multipliers can be solved for, while others explicitly state this, see [1, pg. 220-221], and [3, pg.47], where velocity constraints are assumed to be independent, or [2, pg. 200-201], and [4, pg. 55], which only consider locally constant rank velocity constraints.

1.4. Organization of report

This report is organized as follows. Section 2 outlines all the necessary background for the report, the bulk of which is a summary of Lewis [4, Sections 1, 2 and 5]. Section 3 gives a method for properly defining the configuration manifold and its tangent space for systems with interconnection constraints, and discusses the assumptions being made regarding the configuration manifolds of our systems. Section 4 defines primary velocity constraints and discusses the effects they have on the dimension of allowable velocities for a single rigid body. Section 5 defines the constraint subspace and distribution, then presents a proposition and a corollary regarding the location of degenerate configurations in the constraint distribution.

Chapter 2

Background

In the following sections we outline the mathematical frame work we will be using throughout the report. This is a summary of the relevant material from Section 1, 2, and 5 of Lewis [4]. The outline of this section is as follows: Section 2.1 serves as a reference for the main mathematical concepts that we will be using throughout the report; Section 2.2 covers necessary background regarding affine spaces and maps; Section 2.3 presents the manner in which we will describe transformations and motions of rigid bodies; Section 2.4 introduces the interconnected rigid body systems, the configuration manifold, and the tangent space of the configuration manifold.

2.1. Background and notation

This section presents background material that may be useful as a reference, and also introduce some notation that will be used throughout the report.

We begin with a definition of a linear map.

Definition 2.1 (Linear map)**.** *Let* $\mathbb{F}$ *be a field, and let* U *and* V *be* $\mathbb{F}$*-vector spaces. A map* $\mathsf{A} : \mathsf{U} \to V$ *is said to be linear if it has the following properties:*

(i) $\mathsf{A}(u_1 + u_2) = \mathsf{A}(u_1) + \mathsf{A}(u_2)$ *for all* $u_1, u_2 \in \mathsf{U}$*;*

(ii) $\alpha\mathsf{A} = \mathsf{A}(\alpha u)$ *for all* $u \in \mathsf{U}$ *and* $\alpha \in \mathbb{F}$*.*

In this report we are interested in $\mathbb{R}$-vector spaces. Thus, for $\mathbb{R}$-vector spaces U and V, we denote the space of $\mathbb{R}$-linear maps between U and V by $\mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$. Using this notation, we define $\mathrm{End}_{\mathbb{R}}(\mathsf{V}) = \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}; \mathsf{V})$.

We now define dual spaces and the duals of linear maps.

Definition 2.2 (Dual space)**.** *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space. The dual space to* V *is* $\mathsf{V}^* = \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}; \mathbb{R})$*.*

Definition 2.3 (Dual of a $\mathbb{R}$-linear map)**.** *Let* U *and* V *be a finite* $\mathbb{R}$*-vector space and let* $\mathsf{A} \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$*. The linear map* $\mathsf{A}^* : \mathsf{V}^* \to \mathsf{U}^*$ *defined by the property*

$$\langle \mathsf{A}^*(\beta); u \rangle = \langle \beta; \mathsf{A}(u) \rangle, \qquad u \in \mathsf{U},\ \beta \in \mathsf{V}^*,$$

is the ***dual*** *of* A*.*

Here we introduce the concept of alternating k-forms.

Definition 2.4 (Alternating k-form)**.** *Let* V *be a* $\mathbb{R}$*-vector spaces. An* ***alternating k-form*** *is a map*

$$A : \underbrace{\mathsf{V} \times \cdots \times \mathsf{V}}_{k \ \text{times}} \to \mathbb{R}$$

with the following properties:

(i) A *is* $\mathbb{R}$*-linear in each of the* k *components when the others are kept constant;*

(ii) $\mathsf{A}(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\mathsf{A}(v_1, \ldots, v_k)$ *for any permutations* σ *of the set* $(1, \ldots, k)$*, where* $\operatorname{sign}(\sigma)$ *denotes the parity of* σ*.*

We denote the space of alternating k-forms on V *by* $\bigwedge^k(V^*)$*.*

Now we can define an orientation on a vector space.

Definition 2.5 (Orientation)**.** *Let* V *be an* n*-dimensional* $\mathbb{R}$*-vector space. An* ***orientation*** *of* V *is an equivalence class in* $\bigwedge^n(V^*)\backslash\{0\}$*, where* $\theta, \Theta \in \bigwedge^n(V^*)\backslash\{0\}$ *are equivalent if* $\theta = \alpha\Theta$ *for some* $\alpha \in \mathbb{R}_{>0}$*.*

We will designate the choice of an orientation by a single $\theta \in \bigwedge^n(V^*)\backslash\{0\}$, understanding this to mean the equivalence class

$$\left\{ \Theta \in \bigwedge^n(V^*)\backslash\{0\} \;\middle|\; \theta = \alpha\Theta, \text{ for some } \alpha \in \mathbb{R}_{>0} \right\}.$$

We will denote n-dimensional oriented inner product spaces by a triple of the form $(\mathsf{V}, \mathrm{g}, \theta)$, where V is a n-dimensional vector space with inner product g, and $\theta \in \bigwedge^n(V^*)\backslash\{0\}$ defines an orientation on V .

The following is our definition for an orientation-preserving map.

Definition 2.6 (Orientation-preserving map)**.** *Let* $(\mathsf{U}, \mathrm{G}, \Theta)$ *and* $(\mathsf{V}, \mathrm{g}, \theta)$ *be* n*-dimensional oriented inner product spaces. Let* $\mathsf{A} \in \operatorname{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ *be an invertible map.* A *is* ***orientation-preserving*** *if* $\mathsf{A}^*\theta$ *and* Θ *belong to the same equivalence class.*

Next we define a linear isometry.

Definition 2.7 (Linear-isometry)**.** *Let* $(\mathsf{V}, \mathrm{g}, \theta)$*, and* $(\mathsf{U}, \mathrm{G}, \Theta)$ *be finite-dimensional oriented inner product spaces. A map* $\mathsf{A} \in \operatorname{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ *is a* ***linear-isometry*** *if*

$$\mathrm{g}(\mathsf{A}(u_1), \mathsf{A}(u_2)) = \mathrm{G}(u_1, u_2)$$

for any $u_1, u_2 \in \mathsf{U}$*.*

The following are the conditions for a skew symmetric map.

Definition 2.8 (g-skew-symmetric)**.** *Let* $(\mathsf{V}, \mathrm{g}, \theta)$ *be a finite-dimensional oriented inner product space. We define a map* $\mathsf{A} \in \operatorname{End}_{\mathbb{R}}(\mathsf{V})$ *to be* **g*-skew-symmetric*** *if*

$$\mathrm{g}(\mathsf{A}(v_1), v_2) = -\mathrm{g}(v_1, \mathsf{A}(v_2)), \qquad v_1, v_2 \in \mathsf{V}.$$

By $\mathfrak{so}(\mathsf{V}, \mathrm{g}, \theta)$ *we denote the subspace of* g*-skew-symmetric maps on* V*.*

We now present the definition of the hat-map.

Definition 2.9 (The hat-map). *Let* $(\mathsf{V}, \mathrm{g}, \theta)$ *be a 3-dimensional oriented inner product space. Let* $\times$ *denotes the vector cross-product. Define the linear map* $\hat{\ }: \mathsf{V} \to \mathfrak{so}(\mathsf{V}, \mathrm{g}, \theta)$ *by* $\hat{u}(v) = u \times v$ *for any* $u, v \in \mathsf{V}$*. The inverse map we denote* $\check{\ }: \mathfrak{so}(\mathsf{V}, \mathrm{g}, \theta) \to \mathsf{V}$*.*

The map $\hat{\ }$ is easily be shown to be g-skew-symmetric using the identity

$$\mathrm{g}(u \times v, w) = \mathrm{g}(w \times u, v), \qquad u, v, w \in \mathsf{V}, \tag{2.1.1}$$

which will be useful in later sections of the report.

We now introduce some basic differential geometry concepts and notation. By $\pi_{\mathsf{TM}} : \mathsf{TM} \to \mathsf{M}$ we denote the tangent bundle of a manifold M. The fibre of this bundle at $x \in \mathsf{M}$ is denoted by $\mathsf{T}_x\mathsf{M}$. For a subset $A \subseteq \mathsf{M}$, we denote

$$\mathsf{TM}|A = \{v_x \in \mathsf{TM} \mid x \in A\}.$$

If $\Phi : \mathsf{M} \to \mathsf{N}$ is a differentiable mapping of manifolds, we denote its derivative by $\mathsf{T}\Phi : \mathsf{TM} \to \mathsf{TN}$. We also denote $\mathsf{T}_x\Phi = \mathsf{T}\Phi|\mathsf{T}_x\mathsf{M}$.

Now we present the following lemma that will be used to prove theorems in Sections 4.4 and 4.5.

Lemma 2.10 (Orthogonal space identity). *Let* V *be an inner product space, and let* $\mathsf{S}_1, \dots, \mathsf{S}_j$ *be subspaces of* V*. Then*

$$\bigcap_{j=1}^{n} \mathsf{S}_j^{\perp} = \left(\sum_{j=1}^{n} \mathsf{S}_j\right)^{\perp}.$$

Proof. Suppose $s \in \left(\sum_{j=1}^{n} \mathsf{S}_j\right)^{\perp}$. Then, for any $v \in \bigcap_{j=1}^{n} \mathsf{S}_j^{\perp}$, we have $s \perp v$. Furthermore, $s \perp v_j$ for each $v_j \in \mathsf{S}_j$, $j \in \{1, \dots, n\}$. Hence

$$\left(\sum_{j=1}^{n} \mathsf{S}_j\right)^{\perp} \subseteq \bigcap_{j=1}^{n} \mathsf{S}_j^{\perp}.$$

Now suppose $s \in \bigcap_{j=1}^{n} \mathsf{S}_j^{\perp}$. $s \perp v_j$ for all $v_j \in \mathsf{V}_j$, $j \in \{1, \dots, n\}$. Hence,

$$s \perp (\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $v_j \in \mathsf{V}_j$, $j \in \{1, \dots, n\}$. Thus,

$$\bigcap_{j=1}^{n} \mathsf{S}_j^{\perp} \subseteq \left(\sum_{j=1}^{n} \mathsf{S}_j\right)^{\perp}.$$

$\square$

2.2. Affine spaces and maps

In this section we will discuss some background and properties regarding affine spaces and maps.

We begin by recalling the following definitions.

Definition 2.11 (Faithful and transitive actions)**.** *An action* $\phi : \mathsf{A} \times \mathsf{V} \to \mathsf{A}$ *of group* V *on set* A *is*

(i) ***faithful*** *if, for any* $v \in \mathsf{V} \backslash \{\mathrm{id}_{\mathsf{V}}\}$*, there exists some* $x \in \mathsf{A}$ *such that* $\phi(x, v) \neq x$*;*
(ii) ***transitive*** *if, for any* $x, y \in \mathsf{A}$*, there exists some* $v \in \mathsf{V}$ *such that* $\phi(x, v) = y$*.*

With these, we can make the following definition.

Definition 2.12 (Affine space)**.** *Let* V *be a* $\mathbb{R}$*-vector space. An* ***affine space*** *modeled on* V *is a set* A *with the faithful and transitive action*

$$\mathsf{A} \times \mathsf{V} \ni (x, v) \mapsto x + v \in \mathsf{A}$$

of the Abelian group V *on* A*.*

One may think of an affine space as a vector space without an origin. Thus the elements of an affine space A are not vectors but the differences of elements in A are. If we fix a point $x \in \mathsf{A}$, then A becomes isomorphic to the vector space V. We will denote this vector space A_x. We can now make the following definition.

Definition 2.13 (Affine map)**.** *Let* A *and* B *be affine spaces modelled on* V*. A map* $\phi : \mathsf{A} \to \mathsf{B}$ *is an* ***affine map*** *if, for each* $x \in \mathsf{A}$*, we have* $\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{A}_x; \mathsf{B}_{\phi(x)})$*.*

2.3. Space, motion, and velocity

In this section we present the manner in which we will describe motion and velocities of rigid bodies. This section is outlined as follows: we begin with a presentation of our models for physical space and a reference model in which a rigid body may reside; we then introduce the concept of a rigid motion along with the necessary background; we conclude the section with our definitions of rigid body velocities.

We begin by defining our model for physical space.

Definition 2.14 (Newtonian space model)**.** *A* ***Newtonian space model*** *is a quadruple* $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$*, where* $(\mathsf{V}, \mathrm{g}, \theta)$ *is a 3-dimensional oriented inner product space and* S *is an affine space modeled on* V*.*

Here our affine space S will act as our model for points in space.

Similarly we can define a reference space in which rigid bodies may reside.

Definition 2.15 (Body reference space)**.** *A* ***body reference space*** *is a quadruple* $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$*, where* $(\mathsf{U}, \mathrm{G}, \Theta)$ *is a 3-dimensional oriented inner product space and* B *is an affine space modeled on* U*.*

It should be noted that a rigid motion is described by relating a Newtonian space model $\mathscr{S}$ to a body reference space $\mathscr{B}$. Hence a body is not required to define a motion.

Our understanding of rigid motion begins with the definition of rigid transformations.

Definition 2.16 (Rigid transformation)**.** *For a Newtonian space model $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ and a body reference space $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$, a* ***rigid transformation*** *of $\mathscr{B}$ in $\mathscr{S}$ is an affine map $\Phi : \mathsf{B} \to \mathsf{S}$ defined by*

$$\Phi(X) = \Phi(X_0) + R_\Phi(X - X_0), \qquad X \in \mathsf{B}, \tag{2.3.1}$$

for any $X_0 \in \mathsf{B}$, where $R_\Phi \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ is a linear orientation-preserving isometry. We will denote the set of rigid transformations of $\mathscr{B}$ in $\mathscr{S}$ by $\mathrm{Rgd}(\mathscr{B}; \mathscr{S})$.

By $\mathrm{Isom}^+(\mathscr{B}, \mathscr{S})$ we will denote the set of linear orientation-preserving isometries from U to V. Hence, from the previous definition, $R_\Phi \in \mathrm{Isom}^+(\mathscr{B}, \mathscr{S})$.

The following lemma, [4, Lemma 2.8], helps us gain better understanding of rigid transformations.

Lemma 2.17 (Rigid transformations with respect to origins)**.** *Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space. Let $x_0 \in \mathsf{S}$ and let $X_0 \in \mathsf{B}$. If $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$, then there exist $r_\Phi \in \mathsf{V}$ and $R_\Phi \in \mathrm{Isom}^+(\mathscr{B}; \mathscr{S})$ such that*

$$\Phi(X) = x_0 + (r_\Phi + R_\Phi(X - X_0)), \qquad X \in \mathsf{B}. \tag{2.3.2}$$

Moreover, R_Φ is uniquely determined by Φ and does not depend on x_0 or X_0.

We interpret Lemma 2.17 as follows. Given a choice of body origin $X_0 \in \mathsf{B}$ and spatial origin $x_0 \in \mathsf{S}$, rigid transformations can be thought of as a rotation followed by a translation.

Before we define motion we need the notion of time.

Definition 2.18 (Time)**.** *A* ***time axis*** *is an affine space $\mathbb{T}$ modelled on $\mathbb{R}$. A* ***time interval*** *is a subset $\mathbb{T}' \subseteq \mathbb{T}$ of the form*

$$\mathbb{T}' := t_0 + t, \qquad t \in I$$

for some $t_0 \in \mathbb{T}$ and for some interval $I \subseteq \mathbb{R}$.

We can now define rigid motion.

Definition 2.19 (Rigid motion)**.** *Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space, and let $\mathbb{T}$ be a time axis. A* ***rigid motion*** *of $\mathscr{B}$ in $\mathscr{S}$ is a curve $\phi : \mathbb{T}' \to \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$ defined on a time interval $\mathbb{T}' \subseteq \mathbb{T}$.*

We present [4, Corollary 2.14] that will be useful to us when representing velocities associated to rigid motions of a rigid body.

Corollary 2.20 (A convenient representation of the velocity of a rigid motion)**.** *Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space, and let $\mathbb{T}$ be a time axis with $\mathbb{T}' \subset \mathbb{T}$ a time interval. Let $X_0 \in \mathsf{B}$ be the body origin and $x_0 \in \mathsf{S}$ be the spatial origin. Then there is an injective vector bundle mapping*

$$\begin{aligned} \mathsf{T}(\mathrm{Rgd}(\mathscr{B}; \mathscr{S})) &\to \mathrm{Rgd}(\mathscr{B}; \mathscr{S}) \times (\mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V}) \oplus \mathsf{V}) \\ \dot{\phi}(t) &\mapsto (\phi(t), (\dot{R}_\phi(t), \dot{r}_\phi(t))) \end{aligned}$$

depending only on X_0 for every differentiable rigid motion $\phi : \mathbb{T}' \to \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$.

This corollary is useful because, upon choice of a body origin, we can extract the necessary information used to calculate the rigid body velocities of a differential rigid motion. We will often make use of this property without mention of the corollary.

Given this background, we can define spatial and body velocities as follows.

Definition 2.21 (Rigid body velocities). *Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space. Let $\mathbb{T}$ be a time axis with $\mathbb{T}' \subset \mathbb{T}$ a time interval. Let $x_0 \in \mathsf{S}$ be our spatial origin and $X_0 \in \mathsf{B}$ be our body origin. Given a rigid motion $\phi : \mathbb{T}' \to \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$ defined by*

$$\phi(t)(X) = x_0 + (r_\phi(t) + R_\phi(t)(X - X_0)), \qquad t \in \mathbb{T}',$$

we can make the following definitions:

*(i) The **spatial angular velocity** for the motion is*

$$t \mapsto \omega_\phi(t) := \overbrace{\dot{R}_\phi(t) R_\phi^T}(t) \in \mathsf{V}.$$

*(ii) The **body angular velocity** for the motion is*

$$t \mapsto \Omega_\phi(t) := \overbrace{R_\phi^T(t) \dot{R}_\phi}(t) \in \mathsf{U}.$$

*(iii) The **spatial translational velocity** for the motion is*

$$t \mapsto v_\phi(t) := \dot{r}_\phi(t) + r_\phi(t) \times \omega_\phi(t) \in \mathsf{V}.$$

*(iv) The **body translational velocity** for the motion is*

$$t \mapsto V_\phi(t) := R_\phi^T(\dot{r}_\phi(t)) \in \mathsf{U}.$$

For a more thorough construction of these velocities, we refer to [4, Section 2].

2.4. Configuration manifold and interconnected rigid body systems

In this section we build up the necessary background to define the configuration manifold and interconnected rigid body systems.

As the name implies, interconnected rigid body systems are a class of mechanical system derived from the interconnections of multiple rigid bodies. As such, we consider a Newtonian space model $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ and a finite collection of body reference spaces $\mathscr{B}_a = (\mathsf{B}_a, \mathsf{U}_a, \mathrm{G}_a, \Theta_a)$, for $a \in \{1, \ldots, m\}$. In each of the body reference spaces $\mathscr{B}_a$ we have a rigid body $\mathcal{B}_a$, where $\mathcal{B}_a$ is a compact set in B_a.

The configuration manifold is a differentiable manifold that represents the allowable configurations of our interconnected rigid body system, or simply put, it represents all possible positions of system. We begin with discussion of the free configuration manifold. This characterizes the artificial situation where multiple bodies can move freely and even occupy the same points in physical space.

Definition 2.22 (Free configuration manifold)**.** *Let $\mathscr{S}$ be a Newtonian space model, let there be body reference spaces $\mathscr{B}_a$ with rigid bodies $\mathcal{B}_a$ for $a \in \{1, \dots, m\}$. The* ***free configuration manifold*** *for the system is $\mathsf{Q}_{free} = \prod_{a=1}^{m} \mathrm{Rgd}(\mathscr{B}_a; \mathscr{S})$.*

Now we discuss admissible physical configurations of the bodies $(\mathcal{B}_1, \dots, \mathcal{B}_m)$ in physical space, and the maps of the rigid bodies from their reference spaces to admissible physical configurations.

Definition 2.23 (Physical configuration space, configuration space)**.** *Let $\mathscr{S}$ be a Newtonian space model, let there be body reference spaces $\mathscr{B}_a$ with rigid bodies $\mathcal{B}_a$ for $a \in \{1, \dots, m\}$.*

(i) A ***physical configuration*** *of the bodies in $\mathscr{S}$ is a subset*

$$\Phi_1(\mathcal{B}_1) \times \cdots \times \Phi_m(\mathcal{B}_m) \subseteq \prod_{a=1}^{m} \mathsf{S},$$

for some $\Phi_a \in \mathrm{Rgd}(\mathscr{B}_a; \mathscr{S})$, $a \in \{1, \dots, m\}$.

(ii) A ***physical configuration space*** *is a subset $\mathsf{P} \subseteq \prod_{a=1}^{m} \mathbf{2}^{\mathsf{S}}$ of physical configurations, where $\mathbf{2}^{\mathsf{S}}$ is the power set of S. A point in P is called an* ***admissible physical configuration****.*

(iii) Given a physical configuration space P, the ***configuration space*** *is the subset $\mathsf{Q} \subseteq \mathsf{Q}_{free}$ defined by*

$$\mathsf{Q} := \{(\Phi_1, \dots, \Phi_m) |\ \Phi_1(\mathcal{B}_1) \times \cdots \times \Phi_m(\mathcal{B}_m) \in \mathsf{P}\}.$$

A point in Q is called an ***admissible configuration****.*

We will use the following definition for interconnected rigid body systems.

Definition 2.24 (Interconnected rigid body system)**.** *An* ***interconnected rigid body system*** *consists of the following data:*

(i) a Newtonian space model $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$;

(ii) body references spaces $\mathscr{B}_a = (\mathsf{B}_a, \mathsf{U}_a, \mathrm{G}_a, \Theta_a)$ with rigid bodies $\mathcal{B}_a$, $a \in \{1, \dots, m\}$;

(iii) a physical configuration space P where the associated configuration space Q is a submanifold of Q_{free}, called the ***configuration manifold****.*

Now let us consider motions of interconnected rigid body systems.

Definition 2.25 (Motion)**.** *Consider an interconnected rigid body system with Newtonian space model $\mathscr{S}$, body reference spaces $\mathscr{B}_1, \dots, \mathscr{B}_m$ with rigid bodies $(\mathcal{B}_1, \dots, \mathcal{B}_m)$, and with configuration manifold Q. A* ***motion*** *for the system is a curve $\phi : \mathbb{T}' \to \mathsf{Q}$ whose domain is a time interval in a time axis $\mathbb{T}$.*

The tangent space $\mathsf{TQ}_{\mathrm{free}}$ represents the subspace of free motions of the system, that is, the motions that are not required to remain in the configuration manifold Q. The tangent space TQ represents the subspace of allowable configurations whose motion remain in the configuration manifold Q.

By choosing an arbitrary reference point in each body (commonly the centre of mass) as an origin for its body reference space and repeated application of the of Corollary 2.20, we define a vector bundle monomorphism

$$\begin{aligned} \mathsf{TQ}_{\text{free}} &\to \Big(\prod_{a=1}^{m} \mathrm{Rgd}(\mathscr{B}_a;\mathscr{S})\Big) \times \Big(\bigoplus_{a=1}^{m} (\mathrm{Hom}_{\mathbb{R}}(\mathsf{U}_a;\mathsf{V}) \oplus \mathsf{V})\Big) \\ (\dot{\phi}_1(t),\dots,\dot{\phi}_m(t)) &\mapsto ((\phi_1(t),\dots,\phi_m(t)),((A_1,v_1),\dots,(A_m,v_m))), \end{aligned} \tag{2.4.1}$$

where restriction of the map to $\mathsf{T}(\mathrm{Rgd}(\mathscr{B}_a;\mathscr{S}))$ would be the map of Corollary 2.20 for each $a \in \{1,\dots,m\}$. Since $\mathsf{Q} \subseteq \mathsf{Q}_{\text{free}}$, we have $\mathsf{TQ} \subseteq \mathsf{TQ}_{\text{free}}$. We can restrict the vector bundle monomorphism (2.4.1) to a vector bundle monomorphism with domain TQ and the same codomain. This map will be useful in defining rigid body velocities for interconnected rigid body systems, we will often make use of this property without mention.

Chapter 3

Interconnections and the configuration manifold

Interconnection constraints are prescribed by restricting the points of two rigid bodies to share the same point in physical space for all allowable configurations. We can use this information, in part, to describe the space of allowable physical configuration for the system. The description of the physical configuration space can then be used to define the configuration manifold and to describe the allowable motions in the configuration manifold. In Section 3.1 we define the configuration manifold for a single constraint between two bodies, and in Section 3.2 we follow a similar construction for multiple bodies with multiple constraints. Not all configuration manifolds are defined by the interconnection of points in rigid bodies. In Section 3.3 we remark on situations where this is the case and state the assumptions we make regarding the configuration manifold.

3.1. Configuration manifold given a single interconnection

In this section we will develop the configuration manifold for an interconnected rigid body system with two rigid bodies that share one interconnection constraint. We also present a method for defining the space of allowable motion for this system. We note that, throughout this section, it is assumed that the interconnection of the two points is the only constraint acting on the system.

Let $\mathscr{S}$ be a Newtonian space model, and let there be body reference spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ with rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. The free configuration manifold for the system is defined to be

$$\mathsf{Q}_{\text{free}} = \mathrm{Rgd}(\mathscr{B}_1, \mathscr{S}) \times \mathrm{Rgd}(\mathscr{B}_2, \mathscr{S}).$$

Given interconnected points $X_1 \in \mathcal{B}_1$ and $X_2 \in \mathcal{B}_2$, the set of admissible physical configuration of bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined by

$$\mathsf{P} = \{(\Phi_1(\mathcal{B}_1), \Phi_2(\mathcal{B}_2)) \subseteq \mathsf{S} \times \mathsf{S} \mid \Phi_1(X_1) = \Phi_2(X_2)\},$$

for some $\Phi_j \in \mathrm{Rgd}(\mathscr{B}_j; \mathscr{S})$, $j \in \{1, 2\}$. Given our definition of the physical configuration space P, we can define the configuration manifold to be

$$\mathsf{Q} = \{(\Phi_1, \Phi_2) \in \mathsf{Q}_{\text{free}} \mid (\Phi_1(\mathcal{B}_1), \Phi_2(\mathcal{B}_2)) \in \mathsf{P}\}.$$

To understand the effects of this interconnection on the space of allowable motions for $\mathcal{B}_1$ and $\mathcal{B}_2$, we consider continuously differential motions $\phi_j : \mathbb{T}' \to \mathrm{Rgd}(\mathscr{B}_j, \mathscr{S})$, $j \in \{1,2\}$, such that $(\phi_1(0), \phi_2(0)) = (\Phi_1, \Phi_2)$ and $(\Phi_1, \Phi_2) \in \mathsf{Q}$. Here the spatial motion of the interconnected points X_1 and X_2 are given by

$$t \mapsto \phi_j(t)(X_j) = x_0 + (r_{\phi_j}(t) + R_{\phi_j}(t)(X_j - X_{j,0})), \qquad \text{for } j \in \{1,2\},$$

where $X_{1,0} \in \mathcal{B}_1$ and $X_{2,0} \in \mathcal{B}_2$ are body origins for bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, and $x_0 \in \mathsf{S}$ is the spatial origin. Differentiating these expressions at $t = 0$ gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi_j(t)(X_j)\right)\big|_{t=0} = \dot{r}_{\phi_j}(0) + \dot{R}_{\phi_j}(0)(X_j - X_{j,0}), \qquad \text{for } j \in \{1,2\},$$

where $\dot{r}_{\phi_j}(0) \in \mathsf{V}$ and $\dot{R}_{\phi_j}(0) \in \mathrm{Hom}(\mathsf{U}_j; \mathsf{V})$ for $j \in \{1,2\}$. For the motions of the interconnected rigid body system to remain in Q we require

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi_1(X_1)(t)\right)\big|_{t=0} = \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi_2(X_2)(t)\right)\big|_{t=0},$$

implying that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi_1(X_1)(t)\right)\big|_{t=0} - \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi_2(X_2)(t)\right)\big|_{t=0} = 0. \tag{3.1.1}$$

Next we define the map

$$\begin{aligned} \iota_{\boldsymbol{\Phi}, X_1, X_2} &: \mathsf{T}_{\boldsymbol{\Phi}}\mathsf{Q}_{\mathrm{free}} \to \mathsf{V} \\ ((A_1, v_1), (A_2, v_2)) &\mapsto v_1 - v_2 + A_1(X_1 - X_{1,0}) - A_2(X_2 - X_{2,0}), \end{aligned} \tag{3.1.2}$$

for $\boldsymbol{\Phi} = (\Phi_1, \Phi_2) \in \mathsf{Q}$. It should be noted that the kernel of (3.1.2) defines the subspace of motions in (3.1.1). Hence (3.1.2) is defined such that

$$\ker(\iota_{\boldsymbol{\Phi}, X_1, X_2}) = \mathsf{T}_{\boldsymbol{\Phi}}\mathsf{Q},$$

where $\mathsf{T}_{\boldsymbol{\Phi}}\mathsf{Q}$ represents the allowable motions for configurations $\boldsymbol{\Phi} \in \mathsf{Q}$.

Under the assumption that the interconnection of points X_1 and X_2 is the only constraint on the system, we can then define the tangent space of the configuration manifold to be the kernel of the map

$$\begin{aligned} \iota &: \mathsf{T}\mathsf{Q}_{\mathrm{free}} \to \mathsf{Q}_{\mathrm{free}} \times \mathsf{V} \\ (\boldsymbol{\Phi}, (\mathbf{A}, \mathbf{v})) &\mapsto ((\Phi_1, \Phi_2), \iota_{\boldsymbol{\Phi}, X_1, X_2}((A_1, v_1), (A_2, v_2)), \end{aligned}$$

where $(\boldsymbol{\Phi}, (\mathbf{A}, \mathbf{v})) = ((\Phi_1, \Phi_2), ((A_1, v_1), (A_2, v_2)))$, i.e.,

$$\mathsf{T}\mathsf{Q} = \ker(\iota_{X_1, X_2}).$$

3.2. Configuration manifold given multiple interconnections

Using a similar process as in Section 3.1, we give a description of the configuration manifold for multiple bodies with multiple constraints, and define the tangent space of the configuration manifold. Again, we note that interconnections are assumed to be the only constraints acting on the system throughout the section.

Consider an interconnected rigid body system with Newtonian space model $\mathscr{S}$, and body reference spaces $(\mathscr{B}_1, \dots, \mathscr{B}_m)$ with rigid bodies $(\mathcal{B}_1, \dots, \mathcal{B}_m)$. The free configuration manifold for the system is $\mathsf{Q}_{\mathrm{free}} = \mathrm{Rgd}(\mathscr{B}_1, \mathscr{S}) \times \cdots \times \mathrm{Rgd}(\mathscr{B}_m, \mathscr{S})$. For this system we assign the following data:

(i) for each pair (a, b) such that $a, b \in \{1, \ldots, m\}$, where $b > a$, we assign an integer $q_{a,b} \in \mathbb{Z}_{\geqslant 0}$, which denotes the number of interconnection constraints between bodies $\mathcal{B}_a$ and $\mathcal{B}_b$.
(ii) for each pair (a, b), and each $j \in \{1, \ldots, q_{a,b}\}$, we assign interconnected points $X_{a,b,j} \in \mathcal{B}_a$ and $X_{b,a,j} \in \mathcal{B}_b$

Given this information, we define the set of admissible physical configuration to be

$$\mathsf{P} = \bigcap_{a=1}^{m} \bigcap_{b=2,b>a}^{m} \bigcap_{j=1}^{q_{a,b}} \left\{ \prod_{i=1}^{m} \Phi_i(\mathcal{B}_i) \subseteq \prod^{m} \mathsf{S} \mid \Phi_a(X_{a,b,j}) = \Phi_b(X_{b,a,j}) \right\},$$

for $(\Phi_1, \ldots, \Phi_m) \in \mathsf{Q}_{\text{free}}$. Here the physical configuration space is defined by the intersection of all the subspaces generated by pairs of interconnected points. We now define the configuration manifold to be

$$\mathsf{Q} = \{(\Phi_1, \ldots, \Phi_m) \in \mathsf{Q}_{\text{free}} \mid (\Phi_1(\mathcal{B}_1), \ldots, \Phi_m(\mathcal{B}_m)) \in \mathsf{P}\}.$$

To properly define the tangent space of the configuration manifold, we begin by defining the vector bundle mapping

$$\iota : \mathsf{TQ} \to \mathsf{Q} \times \left(\bigoplus_{a=1}^{m} \bigoplus_{b=2,b>a}^{m} \bigoplus_{j=1}^{q_{a,b}} \mathsf{V} \right),$$

defined by

$$\iota(\boldsymbol{\Phi}, (\mathbf{A}, \mathbf{v})) = (\Phi_1, \ldots, \Phi_m) \times \left(\bigoplus_{a=1}^{m} \bigoplus_{b=2,b>a}^{m} \bigoplus_{j=1}^{q_{a,b}} \iota_{(\Phi_a,\Phi_b),X_{a,b,j},X_{b,a,j}}((A_a, v_a), (A_b, v_b)) \right),$$

where $(\boldsymbol{\Phi}, (\mathbf{A}, \mathbf{v})) = (\Phi_1, \ldots, \Phi_m), ((A_1, v_1), \ldots, (A_m, v_m))$, and where $\iota_{(\Phi_a,\Phi_b),X_{a,b,j},X_{b,a,j}}$ is defined as in (3.1.2). Using this we define the tangent space of the configuration manifold to be

$$\mathsf{TQ} = \ker(\iota).$$

3.3. Remarks on the configuration manifold

In Sections 3.1 and 3.2 we presented a method in which we could describe the configuration manifold and the tangent space to the configuration manifold. In both of these sections it was assumed that the interconnections were the only constraints on the system. It should be noted that the configuration manifold of an interconnected rigid body system cannot always be defined by interconnection constraints alone. There are other phenomena that are also modeled in the configuration manifold.

A simple example of this would be a system restricted to planar motion, such as the system presented in Section 1.2. In the example we chose our coordinates accordingly so we would be restricted to planar motion. This restriction to planar motion alters the configuration manifold in a way that is not described by interconnection constraints, although it can be modeled quite easily.

There are many more examples in which defining the configuration manifold may be a lot more challenging. The process involved in defining such a configuration manifold is outside the scope of this report. For our purposes, as stated in Definition 2.24, we will work under the assumption that our configuration manifold Q is a smooth submanifold of $\mathsf{Q}_{\mathrm{free}}$. Doing so gives us the useful property that $\mathsf{T}_{\boldsymbol{\Phi}}\mathsf{Q}$ has locally constant rank.

It is possible that one may be interested in interconnected rigid body system whose configuration manifold Q is not a submanifold of $\mathsf{Q}_{\mathrm{free}}$. An example of a system with this property would be any system that may encounter obstacles in its environment. There will be a discontinuities about any configuration that results in a collision.

Chapter 4

Velocity constraints on single rigid bodies

In this chapter we discuss the definitions of primary velocity constraints as well as their effects on the dimension of allowable angular and translational velocities for a single rigid body. The following is a detailed outline of the chapter: Section 4.1 presents the definition of primary velocity constraints and the classifications of primary angular and translational velocity constraints; in Section 4.2 two mappings are introduced which are used to describe the space of allowable velocities of a reference point in a rigid body that is being constrained; Section 4.3 describes the dimension of allowable motions for a rigid body with a single primary velocty constraint; in Section 4.4 we present a theorem that outlines the conditions for degeneracy of rigid bodies with multiple primary angular constraints; in Section 4.5 we present a theorem that outlines the conditions for degeneracy of rigid bodies with multiple primary translational constraints; Section 4.6 explores the degeneracy of rigid bodies with multiple primary angular and translational constraints.

4.1. Primary velocity constraints

In this section we present our definition of a primary velocity constraint and its different classifications.

Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space. When no ambiguity can arise, we will use the convention that $x \in \mathsf{S}$ denotes the image of $X \in \mathcal{B}$ under configuration $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$, i.e., $\Phi(X) = x \in \mathsf{S}$.

We now define a primary velocity constraint.

Definition 4.1 (Primary velocity constraint)**.** *Let $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$ and $X_0 \in \mathcal{B}$. A **primary velocity constraint** for $\mathcal{B}$ at (Φ, X_0) is a subspace $\mathsf{C}_{\Phi,X_0} \subseteq \mathsf{V} \oplus \mathsf{V}$ where, for $(\omega, v) \in \mathsf{C}_{\Phi,X_0}$, v represents a possible spatial translational velocity of the point $\Phi(X_0) = x_0$ and ω is a possible spatial angular velocity of the body $\Phi(\mathcal{B})$ about the point $\Phi(X_0) = x_0$.*

We can further classify primary velocity constraints into two different types, primary angular velocity constraints, and primary translational velocity constraints.

Definition 4.2 (Primary angular velocity constraint)**.** *Let $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$ and $X_0 \in \mathcal{B}$. A **primary angular velocity constraint** for $\mathcal{B}$ at (Φ, X_0) is a subspace $\mathsf{A}_{\Phi,X_0} \oplus \mathsf{V} \subseteq \mathsf{V} \oplus \mathsf{V}$*

where, for $\omega \in \mathsf{A}_{\Phi,X_0}$, ω is a possible spatial angular velocity of the body $\Phi(\mathcal{B})$ about the point $\Phi(X_0) = x_0$, and spatial translational velocities are not restricted for the point $\Phi(X_0) = x_0$.

Definition 4.3 (Primary Translational velocity constraint)**.** *Let $\Phi \in \mathrm{Rgd}(\mathscr{B};\mathscr{S})$ and $X_0 \in \mathcal{B}$. A **primary translation velocity constraint** for $\mathcal{B}$ at (Φ, X_0) is a subspace $\mathsf{V} \oplus \mathsf{T}_{\Phi,X_0} \subseteq \mathsf{V} \oplus \mathsf{V}$ where, for $v \in \mathsf{T}_{\Phi,X_0}$, v represents a possible spatial translational velocity of the point $\Phi(X_0) = x_0$, and spatial angular velocities are not restricted about the point $\Phi(X_0) = x_0$.*

We will use the following convention when describing primary angular and translational velocity constraints. A primary angular velocity constraints is prescribed by the restriction of rotations in a particular axis $\mu_0 \in \mathsf{V}$ about point $\Phi(X_0) = x_0$, i.e.,

$$\mathsf{A}_{\Phi,X_0} \oplus \mathsf{V} = \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(\omega, \mu_0) = 0\}.$$

Similarly, a primary translational velocity constraint is prescribed by the restriction of translations of the point $\Phi(X_0) = x_0$ in a particular vector $u_0 \in \mathsf{V}$, i.e.,

$$\mathsf{V} \oplus \mathsf{T}_{\Phi,X_0} = \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(v, u_0) = 0\}.$$

Given this convention, any type of velocity constraint can be described by one of these constraints or a combination of multiple of these constraints. For example, a constraint on the angular velocities at point $\Phi(X_0)$, such that the set of allowable angular velocities is described by rotations about a single vector $v \in \mathsf{V}$, can be described by the intersection of two primary angular velocity constraints

$$\mathsf{A}_{\Phi,X_{0,j}} = \{\omega \in \mathsf{V} |\ \mathrm{g}(\omega, \mu_j) = 0\}, \qquad j \in \{1, 2\},$$

where $v \in (\mathrm{span}(\mu_1, \mu_2))^{\perp_{\mathrm{g}}}$, and $\mu_1 \notin \mathrm{span}(\mu_2)$. Similarly, translational velocity constraints restricting translations to a one-dimensional subspace can also be described by two primary translational velocity constraints.

4.2. Effects of primary constraints on allowable velocities

In this section we are interested in describing the set of allowable angular and translational velocities of a point $x = \Phi(X) \in \mathsf{S}$ given primary velocity constraints.

Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space with rigid body $\mathcal{B}$. Let us suppose that we have a differentiable rigid motion $\phi : \mathbb{T}' \to \mathrm{Rgd}(\mathscr{B};\mathscr{S})$ such that $\phi(0) = \Phi$. The spatial motion of a point $X \in \mathcal{B}$ is given by

$$t \mapsto \phi(t)(X) = \phi(t)(X_j) + R_\phi(t)(X - X_j), \qquad j \in \{1, 2\}.$$

Differentiating at $t = 0$ gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t)(X)\right)\big|_{t=0} = \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t)(X_j)\right)\big|_{t=0} + \omega_\phi(0) \times \left(R_\phi(0)(X - X_j)\right), \qquad j \in \{1, 2\}.$$

From the above equations we can see

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t)(X_1)\right)\big|_{t=0} &= \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t)(X_2)\right)\big|_{t=0} + \omega_\phi(0) \times \left(R_\phi(0)(X_1 - X_2)\right) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t)(X_2)\right)\big|_{t=0} + \omega_\phi(0) \times (x_1 - x_2),\end{aligned}$$

where $x_j = \Phi(X_j)$, $j \in \{1, 2\}$. Using this expression, we can express the spatial translational velocity of point $x_1 = \Phi(X_1)$ in terms of the spatial velocities of point $x_2 = \Phi(X_2)$. We will now use this information to define two maps that will help describe the set of allowable velocities at any given point.

We begin with a description of the allowable velocities of a point given a single primary angular velocity constraint. We define the linear map

$$\begin{aligned} \rho_{\Phi,X_0,x} : V \times V &\to \mathbb{R} \\ (\omega, v) &\mapsto \mathrm{g}(\omega, \mu_0), \end{aligned} \tag{4.2.1}$$

where μ_0 represents the constrained axis of rotation for point $x = \Phi(X)$. Thus $\omega \in \ker(\rho_{\Phi,X_0,x})$ are the allowable spatial angular velocities of any point $x \in S$ given primary angular velocity constraint $\mathsf{C}_{\Phi,X_0} = \mathsf{A}_{\Phi,X_0} \oplus \mathsf{V}$. It should be noted that $\ker(\rho_{\Phi,X_0,x}) = \mathsf{A}_{\Phi,X_0} \oplus \mathsf{V}$ for any choice of x.

Next we describe the set of allowable velocities of a point given a single primary translational velocity constraint. We define the linear map

$$\begin{aligned} \tau_{\Phi,X_0,x} : V \times V &\to \mathbb{R} \\ (\omega, v) &\mapsto \mathrm{g}(v + \omega \times \Delta_0, u_0), \end{aligned} \tag{4.2.2}$$

where $\Delta_0 = x - x_0$, and $x_0 = \Phi(X_0)$ is a point whose velocities have been constrained in the direction of u_0. Thus $(\omega, v) \in \ker(\tau_{\Phi,X_0,x})$ are the allowable spatial angular and translational velocities of point x given primary translational velocity constraint $\mathsf{C}_{\Phi,X_0} = \mathsf{V} \oplus \mathsf{T}_{\Phi,X_0}$.

Using these maps, we describe the subspace of allowable velocities of a point $x = \Phi(X) \in \mathsf{S}$ given multiple primary velocity constraints. Suppose we are given primary velocity constraints $\mathsf{C}_{\Phi,X_1}, \ldots, \mathsf{C}_{\Phi,X_{n+m}}$ on body $\mathcal{B}$, where $n, m \in \mathbb{Z}_{\geqslant 0}$ and $n + m \in \mathbb{Z}_{>0}$. Without loss of generality we assume $\mathsf{C}_{\Phi,X_1}, \ldots, \mathsf{C}_{\Phi,X_n}$ are primary angular velocity constraints, and $\mathsf{C}_{\Phi,X_{n+1}}, \ldots, \mathsf{C}_{\Phi,X_{n+m}}$ are primary translational velocity constraints. Given maps (4.2.1) and (4.2.2) we define the allowable velocities for a point $x \in \mathsf{S}$ to be

$$\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_j,x}) \right) \bigcap \left(\bigcap_{j=n+1}^{n+m} \ker(\tau_{\Phi,X_j,x}) \right). \tag{4.2.3}$$

Hence, we define the subspace of allowable spatial velocities of a point in the body to be the intersection of the all the subspaces of allowable velocities generated by each primary constraint at that point.

4.3. Effects of single primary constraints on dimension

In this section we present the dimensions of allowable motion for a rigid body given a single primary velocity constraint.

The dimension of allowable motions of a rigid body can be given by the dimension of the set of allowable spatial velocities for any point in the body. We are interested in the effects of primary velocity constraints on the dimension of allowable velocities of a rigid body. For an unconstrained rigid body the allowable angular velocities, $\omega \in V$, and translational velocities, $v \in \mathsf{V}$, for a given point $\Phi(X) = x$ is the set

$$\{(\omega, v) \in \mathsf{V} \oplus \mathsf{V}\}.$$

Thus the dimension of allowable velocities for the unconstrained rigid body is 6.

Given a single primary angular velocity constraint $\mathsf{A}_{\Phi,X_0} \oplus \mathsf{V}$, the set of allowable velocities for the point x in the rigid body is given by

$$\begin{aligned}\ker(\rho_{\Phi,X_0,x}) &= \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(\omega, \mu_0) = 0\} \\ &= \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(\omega, \mu_0) + \mathrm{g}(v, 0_{\mathsf{V}}) = 0\},\end{aligned} \tag{4.3.1}$$

where 0_{V} is the zero vector in V. Thus $\dim(\ker(\rho_{\Phi,X_0,x})) = 5$, and the dimension of allowable velocities for the rigid body with a single primary angular velocity constraint is 5.

Similarly, given a single primary translational velocity constraint $\mathsf{V} \oplus \mathsf{T}_{\Phi,X_0}$ the set of allowable velocities for the point x in the rigid body is given by

$$\ker(\tau_{\Phi,X_0,x}) = \{(\omega, v) \in V \oplus V |\ \mathrm{g}(v + \omega \times \Delta_0, u_0) = 0\}.$$

$\dim(\ker(\tau_{\Phi,X_0,x})) = 5$, hence the dimension of allowable velocities for the rigid body with a single primary translational velocity constraint is 5.

4.4. Degeneracy of multiple primary angular velocity constraints

In this section we are interested in providing the conditions for degeneracy of the subspace generated by multiple primary angular velocity constraints.

Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space with rigid body $\mathcal{B}$. Let $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$. Let $\mathsf{A}_{\Phi,X_j} \oplus \mathsf{V} \subseteq \mathsf{V} \oplus \mathsf{V}$, $j \in \{1, \dots, n\}$, be primary angular velocity constraint for $\mathcal{B}$ at (Φ, X_j), $j \in \{1, \dots, n\}$. By the results of Section 4.2, the dimension of allowable velocities given n primary angular velocity constraints is equal to

$$\dim\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_j,x})\right).$$

By Section 4.3 we know each primary angular velocity constraint spans a 5-dimensional subspace. We also note, by the definition,

$$\ker(\rho_{\Phi,X_j,x}) = \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(\omega, \mu_j) = 0\},$$

primary angular velocity constraints are only capable of restricting the angular velocities of a given point since the translational velocities always lie in the kernel of the map. Hence the subspace generated by n primary angular velocity constraints is non-degenerate when

$$\dim\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_0,x})\right) = \max\{6 - n, 3\}, \tag{4.4.1}$$

and degenerate if the dimension is greater. We will show the conditions in which a rigid body with n angular velocity constraints is non-degenerate. We present the following theorem.

Theorem 4.4 (Degeneracy of primary angular velocity constraints)**.** *Given n primary angular velocity constraints, $A_{\Phi,X_1} \oplus \mathsf{V}, \ldots, A_{\Phi,X_n} \oplus \mathsf{V}$, the subspace generated by these primary translational velocity constraints is non-degenerate if and only if*

$$\dim\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right) = \min\{n, 3\}.$$

Proof. We begin by noting

$$\begin{aligned}\ker(\rho_{\Phi,X_j,x}) &= \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} |\ \mathrm{g}(\omega, \mu_0) + \mathrm{g}(v, 0_{\mathsf{V}}) = 0\} \\ &= \operatorname{span}(\mu_j, 0_{\mathsf{V}})^{\perp_{\mathrm{g}}},\end{aligned}$$

for $j \in \{1, \ldots, n\}$. By (4.4.1), the dimension of allowable velocities given $n < 3$ primary translational velocity constraints is non-degenerate only if

$$\begin{aligned}\dim\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_0,x})\right) &= \dim\left(\bigcap_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}})^{\perp}\right) \\ &= \dim\left(\left(\bigcap_{j=1}^{n} \operatorname{span}(\mu_j)^{\perp}\right) \oplus \mathsf{V}\right) \\ &= \dim\left(\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right)^{\perp} \oplus \mathsf{V}\right) \qquad (4.4.2) \\ &= \dim\left(\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right)^{\perp}\right) + 3 \\ &= 6 - n,\end{aligned}$$

where the line (4.4.2) follows from Lemma 2.10. From this it follows that

$$\dim\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right) = n.$$

Similarly, the dimension of allowable velocities given $n \geqslant 3$ primary angular velocity constraints is non-degenerate only if

$$\begin{aligned}\dim\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_0,x})\right) &= \dim\left(\bigcap_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}})^{\perp}\right) \\ &= \dim\left(\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right)^{\perp}\right) + 3 \\ &= 3\end{aligned}$$

Hence we have the result

$$\dim\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j)\right) = 3.$$

□

This theorem has a simple interpretation. Non-degeneracy of primary angular velocity constraints is a linear independence condition on the restricted axes of rotation (i.e., the μ_j, $j \in \{1, \dots n\}$). For there to be non-degeneracy between primary angular velocity constraint we require the vectors μ_j, $j \in \{1, \dots, n\}$, be linearly independent when $n \leqslant 3$ or we require three of them remain linearly independent when $n > 3$.

4.5. Degeneracy of multiple primary translational velocity constraints

In this section we will follow the same process as Section 4.4 to show the conditions for degeneracy of the subspace generated by multiple translational angular velocity constraints.

Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space with rigid body $\mathcal{B}$. Let $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$. Let $\mathsf{V} \oplus \mathsf{T}_{\Phi, X_j} \subseteq \mathsf{V} \oplus \mathsf{V}$, $j \in \{1, \dots, n\}$, be primary translational velocity constraint for $\mathcal{B}$ at (Φ, X_j), $j \in \{1, \dots, n\}$. By the results of Section 4.2, the dimension of allowable velocities given n primary translational velocity constraints is equal to

$$\dim \left(\bigcap_{j=1}^{n} \ker(\tau_{\Phi, X_j, x}) \right).$$

By Section 4.3 we know each primary translational velocity constraint spans a 5-dimensional subspace. Unlike angular velocity constraints, primary translational velocity constraints can restrict both angular and translational velocities of a given point. Hence the subspace generated by n primary translational velocity constraints is non-degenerate when

$$\dim \left(\bigcap_{j=1}^{n} \ker(\tau_{\Phi, X_0, x}) \right) = \max\{6 - n, 0\}, \tag{4.5.1}$$

and degenerate if the dimension is greater. We will show the conditions in which a rigid body with n translational velocity constraints is non-degenerate. We present the following theorem.

Theorem 4.5 (Degeneracy of primary translational velocity constraints)**.** *Given n primary translational velocity constraints, $\mathsf{V} \oplus T_{\Phi, X_1}, \dots, \mathsf{V} \oplus T_{\Phi, X_n}$, the subspace generated by these primary translational velocity constraints is non-degenerate if and only if*

$$\dim \left(\sum_{j=1}^{n} \mathrm{span}(u_j \times \Delta_j, u_j) \right) = \min\{n, 6\}$$

Proof. We begin by noting that

$$\begin{aligned} \ker(\tau_{\Phi, X_0, x}) &= \{(\omega, v) \in V \oplus V |\ \mathrm{g}(v + \omega \times \Delta_0, u_0) = 0\} \\ &= \{(\omega, v) \in V \oplus V |\ \mathrm{g}(v, u_j) + \mathrm{g}(\omega, u_j \times \Delta_j) = 0\} \\ &= \mathrm{span}(u_j \times \Delta_j, u_j)^{\perp_{\mathrm{g}}} \end{aligned} \tag{4.5.2}$$

where (4.5.2) follows from (2.1.1). By (4.5.1), the dimension of allowable velocities given $n < 6$ primary translational velocity constraints is non-degenerate only if

$$\begin{aligned}\dim\left(\bigcap_{j=1}^{n}\ker(\tau_{\Phi,X_j,x})\right) &= \dim\left(\bigcap_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)^{\perp}\right)\\ &= \dim\left(\left(\sum_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)\right)^{\perp}\right)\\ &= 6-n.\end{aligned}$$

Hence it follows that

$$\dim\left(\sum_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)\right) = n.$$

Similarly, the dimension of allowable velocities given $n \geqslant 6$ primary translational velocity constraints is non-degenerate only if

$$\begin{aligned}\dim\left(\bigcap_{j=1}^{n}\ker(\tau_{\Phi,X_0,x})\right) &= \dim\left(\bigcap_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)^{\perp}\right)\\ &= \dim\left(\left(\sum_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)\right)^{\perp}\right)\\ &= 0.\end{aligned}$$

Once again it follows that

$$\dim\left(\sum_{j=1}^{n}\operatorname{span}(u_j\times\Delta_j,u_j)\right) = 6.$$

□

The interpretation of this theorem is not completely clear. Much like Theorem 4.4, non-degeneracy of primary translational velocity constraints is a linear independence condition on the vectors $(u_j\times\Delta_j,u_j)$, $j\in\{1,\dots n\}$. For there to be non-degeneracy between primary translational velocity constraint we require the vectors $(u_j\times\Delta_j,u_j)$, $j\in\{1,\dots n\}$, be linearly independent when $n\leqslant 6$ or we require six of them remain linearly independent when $n>6$.

In the case that $n=2$, the result has a clear interpretation. For degeneracy between two primary translational velocity constraints $\mathsf{V}\oplus T_{\Phi,X_1}$, and $\mathsf{V}\oplus T_{\Phi,X_2}$ we require $(u_1\times\Delta_1,u_1)$ to be parallel to $(u_2\times\Delta_2,u_2)$. So we must have $(u_1\times\Delta_1,u_1)=\alpha(u_2\times\Delta_2,u_2)$ for some $\alpha\in\mathbb{R}$. Thus $u_1=\alpha u_2$ and

$$u_1\times\Delta_1-\alpha u_2\times\Delta_2=\alpha u_2\times(\Delta_1-\Delta_2)=0.$$

Hence the conditions for degeneracy are as follows:

(i) $u_1 \parallel u_2$;

(ii) $\Delta_1 - \Delta_2 \parallel u_1$.

We note that $\Delta_1 - \Delta_2 = (x - x_1) - (x - x_2) = x_2 - x_1$, where $x_1 = \Phi(X_1)$ and $x_2 = \Phi(X_2)$ and $x \in S$ is some reference point. Thus degeneracy for two primary translational velocity constraints occurs when the two constraints are collinear, that is, both vectors u_1 and u_2 pass through both constrained points points x_1 and x_2.

In the case that $n \geqslant 3$, the result is not as clear. Constraints still degenerate when they are collinear, but it may not be the only cause of degeneracy. Further research into the interpretation of this result is still needed.

4.6. Interaction of translational and angular primary velocity constraints

Now that we understand where degeneracies occur when given a body with n primary angular or translational velocity constraints, we shift our focus to understanding the interaction of angular and translational constraints.

Let $\mathscr{S} = (\mathsf{S}, \mathsf{V}, \mathrm{g}, \theta)$ be a Newtonian space model, and let $\mathscr{B} = (\mathsf{B}, \mathsf{U}, \mathrm{G}, \Theta)$ be a body reference space with rigid body $\mathcal{B}$. Let $\Phi \in \mathrm{Rgd}(\mathscr{B}; \mathscr{S})$. Let $\mathsf{A}_{\Phi,X_j} \oplus \mathsf{V} \subseteq \mathsf{V} \oplus \mathsf{V}$, $j \in \{1, \dots, n\}$, be primary angular velocity constraint for $\mathcal{B}$ at (Φ, X_j), $j \in \{1, \dots, n\}$, and let $\mathsf{V} \oplus \mathsf{T}_{\Phi,X_j} \subseteq \mathsf{V} \oplus \mathsf{V}$, $j \in \{n+1, \dots, n+m\}$, be primary translational velocity constraint for $\mathcal{B}$ at (Φ, X_j), $j \in \{n+1, \dots, n+m\}$. By the results of Section 4.2, the dimension of allowable velocities given $n + m$ primary velocity constraints is equal to

$$\dim\left(\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_j,x})\right) \bigcap \left(\bigcap_{j=n+1}^{n+m} \ker(\tau_{\Phi,X_j,x})\right)\right).$$

By the results from Sections 4.4 and 4.5 this system is non-degenerate when

$$\dim\left(\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_j,x})\right) \bigcap \left(\bigcap_{j=n+1}^{n+m} \ker(\tau_{\Phi,X_j,x})\right)\right) = \max\{6 - (\tilde{n} + \tilde{m}), 0\}, \tag{4.6.1}$$

where $\tilde{n} = \min\{n, 3\}$ and $\tilde{m} = \min\{m, 6\}$, and degenerate if the dimension is greater. We will show the conditions in which a rigid body with these types of constraints will be non-degenerate. We present the following theorem.

Theorem 4.6 (Degeneracy of primary angular and translational velocity constraints)**.** *Given n primary angular velocity constraints, $A_{\Phi,X_1} \oplus \mathsf{V}, \dots, A_{\Phi,X_n} \oplus \mathsf{V}$, and m primary translational velocity constraints, $\mathsf{V} \oplus T_{\Phi,X_{n+1}}, \dots, \mathsf{V} \oplus T_{\Phi,X_{n+m}}$, the subspace generated by these primary velocity constraints is non-degenerate if and only if*

$$\dim\left(\sum_{j=1}^{n} \mathrm{span}(\mu_j, 0_{\mathsf{V}}) + \sum_{j=n+1}^{n+m} \mathrm{span}(u_j \times \Delta_j, u_j)\right) = \min\{\tilde{n} + \tilde{m}, 6\}.$$

Proof. From the proofs of Theorems 4.4 and 4.5 we know

$$\ker(\rho_{\Phi,X_j,x}) = \mathrm{span}(\mu_j, 0_{\mathsf{V}})^{\perp}, \qquad j \in \{1, \dots, n\},$$

and

$$\ker(\tau_{\Phi,X_j,x}) = \operatorname{span}(u_j \times \Delta_j, u_j)^{\perp} \qquad j \in \{n+1, \dots, n+m\}.$$

Hence we have

$$\begin{aligned}
\dim &\left(\left(\bigcap_{j=1}^{n} \ker(\rho_{\Phi,X_j,x}) \right) \bigcap \left(\bigcap_{j=n+1}^{n+m} \ker(\tau_{\Phi,X_j,x}) \right) \right) \\
&= \dim \left(\left(\bigcap_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}})^{\perp} \right) \bigcap \left(\bigcap_{j=n+1}^{n+m} \operatorname{span}(u_j \times \Delta_j, u_j)^{\perp} \right) \right) \\
&= \dim \left(\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}}) + \sum_{j=n+1}^{n+m} \operatorname{span}(u_j \times \Delta_j, u_j) \right)^{\perp} \right), \qquad (4.6.2)
\end{aligned}$$

where (4.6.2) follows from Lemma 2.10. To simplify notation, let

$$A = \sum_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}}) + \sum_{j=n+1}^{n+m} \operatorname{span}(u_j \times \Delta_j, u_j).$$

Since $\dim(A) + \dim(A^{\perp}) = 6$, we must have $\dim(A) = 6 - \dim(A^{\perp})$. Under the assumption that A is not degenerate, the result follows from (4.6.1). □

This theorem tells us that non-degeneracy of multiple primary velocity constraints is a linear independence condition on the vectors $(\mu_j, 0_{\mathsf{V}})$, $j \in \{1, \dots, n\}$, and $(u_j \times \Delta_j, u_j)$, $j \in \{n+1, \dots, n+m\}$. Thus we have a condition that describes where degeneracies of primary constraints occur, however, the physical interpretation of this is not clear and requires more research.

By the previous proof we have one more result. Given n primary angular velocity constraints, $A_{\Phi,X_1} \oplus \mathsf{V}, \dots, A_{\Phi,X_n} \oplus \mathsf{V}$, and m primary translational velocity constraints, $\mathsf{V} \oplus T_{\Phi,X_{n+1}}, \dots, \mathsf{V} \oplus T_{\Phi,X_{n+m}}$, the subspace of allowable velocities for point $x = \Phi(X)$ is

$$\left(\sum_{j=1}^{n} \operatorname{span}(\mu_j, 0_{\mathsf{V}}) + \sum_{j=n+1}^{n+m} \operatorname{span}(u_j \times \Delta_j, u_j) \right)^{\perp_{\mathrm{g}}},$$

which follows from (4.6.2). This is different representation of the result (4.2.3) that has a clearer interpretation of the effects of primary constraints on the set of allowable motions.

Chapter 5

Constraint subspace and distribution

In this chapter we describe the constraint subspace, constraint distribution, and present a theorem regarding the location of singularities in the constraint distribution. The chapter is outlined as follows: in Section 5.1 we define the constraint subspace and two mappings that will be useful in the definition of the constraint distribution; in Section 5.2 we develop the definition of the of the constraint distribution; in Section 5.3 we present a theorem regarding the location of singularities in the constraint distribution.

5.1. Constraint subspace

In this section we present the definition of the constraint subspace and present two maps that can also be used to define the constraint subspace.

We begin this section with the definition of the constraint subspace.

Definition 5.1 (Constraint subspace)**.** *Let $\Phi \in \mathrm{Rgd}(\mathscr{B};\mathscr{S})$ and $X_0 \in \mathcal{B}$. The* ***constraint subspace*** *associated to a primary velocity constraint C_{Φ,X_0} is the subspace $\mathsf{D}_{\Phi,X_0} \subseteq \mathsf{T}_\Phi(\mathrm{Rgd}(\mathscr{B};\mathscr{S}))$ defined by*

$$\mathsf{D}_{\Phi,X_0} = \{(\Phi,(A,v)) | (\widetilde{AR_\Phi^T}, v + A(X_0 - X)) \in \mathsf{C}_{\Phi,X_0}\}.$$

Given a primary angular velocity constraint $\mathsf{A}_{\Phi,X_0} \oplus \mathsf{V}$, defined by

$$\mathsf{A}_{\Phi,X_0} = \{\omega \in \mathsf{V} |\ \mathrm{g}(\omega,\mu_0) = 0\},$$

we can describe the associated constraint subspace D_{Φ,X_0} by the kernel of the map,

$$\begin{aligned} \tilde{\rho}_{\Phi,X_0} : \mathsf{T}_\Phi(\mathrm{Rgd}(\mathscr{B};\mathscr{S}) &\to \mathbb{R} \\ (A,v) &\mapsto \mathrm{g}(\widetilde{AR_\Phi^T},\mu_0). \end{aligned} \tag{5.1.1}$$

Similarly, given a primary translational velocity constraint $\mathsf{V} \oplus \mathsf{T}_{\Phi,X_0}$, defined by

$$\mathsf{T}_{\Phi,X_0} = \{v \in \mathsf{V} |\ \mathrm{g}(v,u_0) = 0\},$$

we can describe the associated constraint subspace D_{Φ,X_0} by the kernel of the vector bundle mapping,

$$\begin{aligned}\tilde{\tau}_{\Phi,X_0} : \mathsf{T}_\Phi(\mathrm{Rgd}(\mathscr{B};\mathscr{S}) &\to \mathbb{R}\\ (A,v) &\mapsto \mathrm{g}(v + A(X_0 - X), u_0).\end{aligned} \tag{5.1.2}$$

It should be noted that the maps (5.1.1) and (5.1.2) are simply the vector bundle version of (4.2.1) and (4.2.2) respectively.

5.2. Constraint distribution

The constraint distribution is the subspace that describes the set of velocities which satisfy all velocity constraints as well as the constraint of remaining in the configuration manifold Q. In this section we construct the definition of the constraint distribution.

Consider an interconnected rigid body system with Newtonian space model $\mathscr{S}$, body reference spaces $(\mathscr{B}_1,\dots,\mathscr{B}_m)$ with rigid bodies $(\mathcal{B}_1,\dots,\mathcal{B}_m)$ and configuration manifold Q. To each $\mathbf{\Phi} = (\Phi_1,\dots,\Phi_m) \in \mathsf{Q}$ we assign the following data:

(i) for each $a \in \{1,\dots,m\}$, we assign two integers $h_{\mathbf{\Phi},a}, k_{\mathbf{\Phi},a} \in \mathbb{Z}_{\geqslant 0}$ which corresponds to the number of angular and translation constraints on body a, respectively;

(ii) for each $a \in \{1,\dots,m\}$ and $j \in \{1,\dots,h_{\mathbf{\Phi},a} + k_{\mathbf{\Phi},a}\}$, a point $X_{\mathbf{\Phi},a,j} \in \mathcal{B}_a$ which is the point being constrained;

(iii) for each $a \in \{1,\dots,m\}$ and $j \in \{1,\dots,h_{\mathbf{\Phi},a}\}$, a primary angular velocity constraint $\mathsf{A}_{\mathbf{\Phi},a,j} \oplus \mathsf{V}$ for $\mathcal{B}_a$ at $(\Phi_a, X_{\mathbf{\Phi},a,j})$.

(iv) for each $a \in \{1,\dots,m\}$ and $j \in \{h_{\mathbf{\Phi},a}+1,\dots,h_{\mathbf{\Phi},a} + k_{\mathbf{\Phi},a}\}$, a primary translational velocity constraint $\mathsf{V} \oplus \mathsf{T}_{\mathbf{\Phi},a,j}$ for $\mathcal{B}_a$ at $(\Phi_a, X_{\mathbf{\Phi},a,j})$.

With this allocation of data we define the map

$$\Lambda : \mathsf{TQ} \to \mathsf{Q} \times \left(\bigoplus_{a=1}^{m} \mathbb{R}^{h_{\phi_a}+k_{\phi_a}}\right) \tag{5.2.1}$$

$$\Lambda\left(\mathbf{\Phi},(\mathbf{A},\mathbf{v})\right) = \left(\left(\Phi_1,\dots,\Phi_m\right), \bigoplus_{a=1}^{m}\left(\left(\bigoplus_{j=1}^{h_{\phi_a}} \tilde{\rho}_{\Phi,X_{\mathbf{\Phi_a},a,j}}\right) \bigoplus \left(\bigoplus_{j=h_{\phi_a}+1}^{h_{\phi_a}+k_{\phi_a}} \tilde{\tau}_{\Phi_a,X_{\mathbf{\Phi},a,j}}\right)\right)\right),$$

for $(\mathbf{\Phi},(\mathbf{A},\mathbf{v})) = ((\Phi_1,\dots,\Phi_m),((A_1,v_1),\dots,(A_m,v_m)))$.

We can now define the constraint distribution.

Definition 5.2 (Constraint distribution)**.** *Consider an interconnected rigid body system as above. The* ***constraint distribution*** *of the system is defined to be*

$$\mathsf{D} = \ker\Lambda,$$

where Λ *is defined by* (5.2.1).

5.3. Singularities in the constraint distribution

In this section we present a proposition and a corollary that describes the location of singularities in the constraint distribution.

We begin by defining singularities in the constraint distribution.

Definition 5.3 (Singularities)**.** *Consider an interconnected rigid body system with Newtonian space model $\mathscr{S}$, body reference spaces $\mathscr{B}_1, \dots, \mathscr{B}_m$ with rigid bodies $\mathcal{B}_1, \dots, \mathcal{B}_m$, and with configuration manifold Q. Let D be the constraint distribution for the system. A configuration $\mathbf{\Phi}_0 \in \mathsf{D}$ is a* ***singularity*** *if there does not exist a neighbourhood $\mathcal{U}$ of $\mathbf{\Phi}_0$ such that*

$$\dim(\mathsf{D}_{\mathbf{\Phi}}) = \dim(\mathsf{D}_{\mathbf{\Phi_0}})$$

for all $\mathbf{\Phi} \in \mathcal{U}$, where

$$\mathsf{D}_{\mathbf{\Phi}} = \ker(\Lambda)|_{\mathsf{T}_{\mathbf{\Phi}}\mathsf{Q}}.$$

We now present a useful property from [8, Proposition 6] that will be useful in proving our final result.

Proposition 5.4 (Upper semicontinuity of the dimension of the kernel)**.** *Let M and N be smooth manifolds. We let $\pi : \mathsf{E} \to \mathsf{M}$ and $\sigma : \mathsf{F} \to \mathsf{N}$ be infinitely differentiable vector bundles with finite-dimensional fibres and we let $f : \mathsf{E} \to \mathsf{F}$ be a infinitely differentiable vector bundle map over $f_0 : \mathsf{M} \to \mathsf{N}$. We have function $\dim(\ker(f)) : \mathsf{M} \to \mathbb{Z}_{\geqslant 0}$ defined by asking that $\dim(\ker(f))(x)$ be the dimension of the kernel of $f|_{\mathsf{E}_x}$. With M, N, E, F, f and f_0 as above the function $\dim(\ker(f))$ is upper semicontinuous.*

With this definition, Proposition 5.4 leads to the following corollary, which gives insight to the locations degenerate configurations.

Corollary 5.5 (Singularities in the constraint distribution)**.** *Let Λ, defined as in* (5.2.1)*, be a smooth vector bundle map. Given an interconnected rigid body system with smooth configuration manifold Q, singularities in the constraint distribution D arise at configurations where* $\operatorname{rank}(\mathsf{D})$ *increases.*

Proof. By definition, singularities of a subspace occur at locations that do not have locally constant rank. By the property presented in Proposition 5.4, given a singular configuration $\mathbf{\Phi} \in \mathsf{Q}$, the dimension of the constraint distribution $\mathsf{D} = \ker(\Lambda)$ is upper semicontinuous in a neighbourhood around this configuration. Hence, we have shown that singularities are a result of an increase of dimension in the constraint distribution. □

Chapter 6

Summary and conclusion

In this report we developed a convention for primary velocity constraints that defines the set of allowable velocities of a rigid body in terms the orthogonal complements of a one dimensional subspace. For primary angular velocity constraints

$$\mathsf{A}_{\Phi,X_j} \oplus \mathsf{V} = \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} | \ \mathrm{g}(\omega, \mu_j) = 0\}, \quad j \in \{1, \dots, n\},$$

and primary translational velocty constraints

$$\mathsf{V} \oplus \mathsf{T}_{\Phi,X_j} = \{(\omega, v) \in \mathsf{V} \oplus \mathsf{V} | \ \mathrm{g}(v, u_j) = 0\}, \quad j \in \{n+1, \dots, n+m\},$$

the subspace of allowable velocities for a point $x = \Phi(X)$ was shown to be given by

$$\left(\sum_{j=1}^{n} \mathrm{span}(\mu_j, 0_{\mathsf{V}}) + \sum_{j=n+1}^{n+m} \mathrm{span}(u_j \times \Delta_j, u_j) \right)^{\perp_{\mathrm{g}}},$$

where $\Delta_j = x - \Phi(X_j)$, and non-degeneracy of this subspace was shown to be a linear independence condition on the vectors $(\mu_j, 0_{\mathsf{V}})$, $j \in \{1, \dots, n\}$, and $(u_j \times \Delta_j, u_j)$, $j \in \{n + 1, \dots, n + m\}$. We then developed the definition of the constraint distribution in terms of the kernel of the vector bundle mapping Λ. Using this, we were able to prove that singular configuration in the constraint distribution arise where there is an increase of dimension in the constraint distribution

Further work must go into the interpretation of the physical meaning of linear independence condition in Theorem 4.6. With a better understanding of this, one may be able to determine how to model interconnected rigid body systems with velocity constraints that do not have locally constant rank. Specifically, one may be able to determine some sort of physical law that dictates a method to solve for the unsolved Lagrange multipliers in the equations of motion for a rigid body systems at singular configurations.

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