

The bountiful intersection of differential geometry, mechanics, and control theory

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Abstract

The areas of mechanics and control theory have a rich and productive history of interaction with the broad mathematical subject of differential geometry. An overview is given for these sorts of interplay in the areas of Riemannian and affine differential geometry, and the geometry of vector distributions. Emphasis is given to areas where differential geometric methods have played a crucial rôle in solving problems whose solutions are difficult to achieve without access to these methods. Emphasis is also given to a concise and elegant presentation of the approach, rather than a detailed and concrete presentation. The results overviewed, while forming a coherent and elegant body of work, are limited in scope. The paper is closed with a discussion of why the approach is limited, and a brief discussion of issues that must be resolved before the results of the type presented in the paper can be extended.

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1. Introduction

In this introduction, we start by giving a brief overview of the rôle of differential geometry in mechanics and control theory. We do not intend to give a thorough history of this subject, in no small part because the author is not sufficiently knowledgeable to be able to do this without omissions. The intent, rather, is to set the stage for the rest of the paper. In this introduction we also provide a summary of what is in the paper, and give a few motivational examples to which we shall refer subsequently in the paper.

1.1. An incomplete history of geometric mechanics and geometric control theory. The subject of geometric mechanics, in its modern guise, was arguably born with the first edition of Abraham and Marsden's *Foundations of Mechanics* in 1967, which was significantly expanded as a second edition, and which is now an inarguable classic in the field [[Abraham and Marsden 1978](#)]. Since this time, the subject of geometric mechanics has blossomed, giving rise to numerous books [e.g., [Arnol'd 1978](#), [Godbillon 1969](#), [Libermann and Marle 1987](#),

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[Spivak 2010], innumerable papers, and a journal, the *Journal of Geometric Mechanics*, dedicated to the subject. What has resulted from the merging of mechanics and differential geometry has been a deep understanding of the structures that contribute to the mathematical foundations of mechanics. Mathematically, tools from areas such as variational analysis, symplectic and Poisson geometry, and Riemannian and affine differential geometry all have contributed to an elegant picture of classical mechanics. Moreover, mechanics has given life and breadth to some areas of differential geometry, just as one might hope with the interplay of mathematics and its applications.

It is more difficult to locate the origins of geometric control theory since this subject area sprouted from more or less isolated research papers that steadily grew into a well developed research area. An example of a piece of very early work in the area is that of Hermann [1963], but overall we can do nothing better than refer to the paper of Brockett [2014] for an overview of the early history of the subject of geometric control theory. Over time, the tools of differential geometry have become included under the umbrella of “nonlinear control theory,” as represented by classical texts such as [Isidori 1995, Nijmeijer and van der Schaft 1990]. Moreover, the distinct subject of geometric control theory is now one that is well established, and is presented in various texts [e.g., Agrachev and Sachkov 2004, Jurdjevic 1997, Sastry 1999]. Unlike the area of geometric mechanics, whose basic foundations may be argued to be quite fully fleshed out, there are many fundamental problems of geometric control theory that are poorly understood, despite the substantial efforts of many researchers over a period of fifty or so years. In particular, we point to the structural problems of controllability and stabilisability as being in a surprisingly incomplete state. We shall have more to say about this in Section 5.

The merging of geometric mechanics and geometric control theory did not really take place until the 1980’s, and it was only in the 1990’s that there was a consistent effort to merge two areas that obviously needed merging. A very early paper connecting control theory and mechanics is that of Brockett [1977]. An early and complete presentation of an Hamiltonian point of view of mechanics and control theory is given in a series of papers by van der Schaft [1982, 1985, 1986, 1981/82]. Summaries of developments of control theory and mechanics were developed in plenary papers presented by Murray [1997] and Leonard [1998]. Finally, in the early 2000’s, books by Bloch [2003] and Bullo and Lewis [2004] presented a fairly complete picture of certain aspects of geometric control theory and geometric mechanics. In addition to the development of the foundations of the topic, there were innumerable papers that put these tools of geometric mechanics and geometric control theory to use in solving problems. We shall make reference to some such work in the text of the paper, at appropriate moments.

Our intent in this paper is to overview in some depth, but little detail, a subset of the areas where differential geometry, geometric mechanics, and geometric control theory have come together to the benefit of all three areas. We emphasise a geometric and conceptual treatment of the subject. A cost of this is that, to a newcomer, things may not appear to be computable. However, everything we present comes with simple computational formulae. We do not present these due to space limitations, but instead refer the reader to the literature where examples, worked out in detail, illustrate that the theory we present is a practical theory, despite possible appearances to the contrary. We make absolutely no claims to being complete, even within the narrow confines of the selected subject matter. Moreover, we shall blithely omit reference to substantial areas of activity in this intersection

of differential geometry, geometric mechanics, and geometric control theory. This should not be taken as reflecting any sort of bias by the author, but rather an acknowledgement by the author of the limits of space and of his knowledge of a subject so expansive that it is not possible to present a coherent overview of it in all of its facets.

1.2. A summary of the content of the paper. We narrowly focus our presentation on the rôle of two aspects of differential geometry in mechanics and control theory, namely the areas of Riemannian and affine differential geometry, and of vector distributions. The connections between mechanics, and Riemannian and affine differential geometry have been understood since the inception of the notion of an affine connection, as presented in the book of [Levi-Civita \[1927\]](#). This venerable geometric approach to the modelling of mechanical systems is elucidated at length in modern language in the book of [Bullo and Lewis \[2004\]](#). Vector distributions, i.e., subbundles of the tangent bundle of a manifold, feature prominently in control theory, mechanics, and control theory for mechanical systems. Our intent is to outline the myriad and intertwined ways in which the preceding two topics in differential geometry show up in control theory and mechanics, but also to give some means by which control theory and mechanics give back to differential geometry by better illuminating ideas that are known and by revealing new geometric ideas. All of what we present here is known, but the interconnections that exist are explicitly made here in one place for the first time. Moreover, we will on occasion point out features that are sometimes missing in standard presentations of the material. The picture we present of control theory and mechanics is, in some sense, complete, polished, and well understood in and of themselves. However, it is very far from a complete picture of the subject, as the results we overview are what one might call “low-order” (in a sense we shall make precise). Extensions of the results we present to what one might call “higher-order” results are very much not clear. At the end of the paper we discuss why the results of the type that we have presented have not been extended, and indicate a sort of change of viewpoint that is likely going to be required to obtain more comprehensive results in the same vein as the results we have presented.

Here is an outline of the paper.

After presenting a few motivational examples in [Section 1.3](#), we begin the technical content of the paper by discussing vector distributions in [Section 2](#). After presenting the definitions and the basic geometric properties of vector distributions, we provide in [Section 2.2](#) the well-known connections between the geometric properties of vector distributions and the problems of controllability theory. In mechanics, vector distributions arise when modelling velocity constraints such as arise when modelling rolling constraints. In [Section 2.3](#) we present this modelling, and we illustrate connections between control theoretic interpretations of vector distributions and mechanical properties of velocity constraints.

We turn to affine connections in [Section 3](#). We make the well-known connection between Riemannian geometry and unconstrained mechanics in [Section 3.2](#) and the less well-known connection between affine differential geometry and constrained mechanics in [Section 3.3](#). Here we emphasise multiple interpretations of similar ideas that appear in both mechanics and affine differential geometry.

In [Section 4](#) we consider control theory for mechanical systems, in the form of a particular class of systems we call “affine connection control systems.” The results we present explore various interconnected ideas from differential geometry, control theory, and mechanics. We arrive, at the end of this discussion, to a fairly polished set of results. However, the results

are restricted in their applicability, in some sense. In Section 5 we discuss why a change of viewpoint may be necessary to extend the ideas in the paper to more complete and comprehensive results along similar lines.

1.3. A few simple examples. By way of motivating the sorts of problems we are working with, in this section we present four fairly simple examples, and present a few questions that arise from these examples. Some of these questions have answers that will be well-known to readers familiar with standard geometric mechanics, but other questions have nontrivial answers that reveal the interconnections between affine connections and vector distributions.

In Figure 1 we depict four fairly simple examples of mechanical control systems. All of

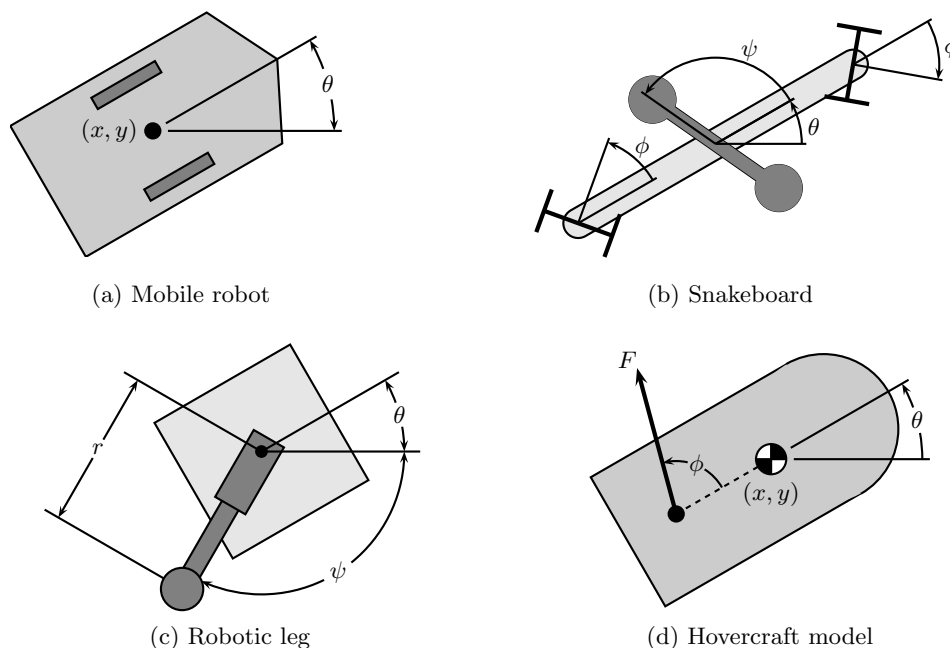


Figure 1: Motivating examples

these have been examined quite thoroughly in the literature, so we shall not have anything new to say about them here. Instead, our objective is to understand the examples deeply, examine their similarities and differences, and show how the interesting features of the examples are completely determined by general ideas about affine differential geometry and vector distributions.

First of all, there is a natural control problem associated with all of the above examples.

1.1 Question: For all four examples: is it possible to steer the system from being at rest in one configuration to being at rest in another configuration? •

We shall give the answer to this question for all four systems. However, we shall significantly flesh out the *way* that the four systems are controllable in the sense of the preceding question. In order to set the table for this, let us consider the four systems in a little detail.

The system depicted in Figure 1a is a simple mobile robot of a sort that has been studied by many people in many contexts. A recent book dedicated to the general subject of “mobile robots” is [Tzafestas 2014]. It is possible, even natural, to treat the mobile robot as a kinematic system. That is to say, one might ignore the dynamics of the robot, and treat the inputs to the robot as being velocities of the wheels. As such, the mobile robot as a control system is modelled as a driftless system.

The next example, in Figure 1b, is the snakeboard. This was a commercially available toy that poses some interesting dynamics and control problems. The idea is that the rider (modelled by the barbell-shaped body) can move their body and can also actuate the wheels at either end of the snakeboard, thereby imparting motion without the rider having to make contact with the ground. The snakeboard was originally studied in [Lewis, Ostrowski, Murray, and Burdick 1994], and subsequently examined by a number of authors [e.g., Bullo and Lewis 2003, Shamma and Olíveira 2012]. While it is, like the mobile robot, a planar wheeled vehicle, it seems somehow different. For example, it is not immediately reasonable that (and is indeed not true that) the system can be effectively modelled using velocities as inputs.

This leads to our first question.

1.2 Question: How can one characterise the difference between the mobile robot and the snakeboard as concerns the validity of their being modelled as systems with velocity inputs? ●

The next example is the robotic leg depicted in Figure 1c. In this system, the “leg” is extensible and it is also possible to control the angle between the leg and the body to which it is attached. This is an example that arises in early work on legged robots [Raibert 1986]. A consideration of this example from a point of view of interest to us is given by Li and Montgomery [1990]. The example is a simple version of the so-called “falling cat,” which is a mathematical model capturing the fact that a cat will “always land on its feet” when dropped [Montgomery 1993].¹ This is an area of research that marks the beginning of substantial efforts in the area of control of mechanical systems.

The final example we consider, depicted in Figure 1d, can be thought of as a simple model of a hovercraft actuated by a single fan. This example was first considered in the work of Lewis and Murray [1997], and a more complete analysis was carried out subsequently by Bullo and Lewis [2005]. One might somehow feel that, between the two examples of the robotic leg and the simple hovercraft, the robotic leg is an easier system to control.

Attached to the last two examples is a vague question that has a precise answer.

1.3 Question: In what way, if any, is the planar body of Figure 1d more difficult to control than the robotic leg of Figure 1c? ●

It turns out that there is a central theme that unifies the questions above, and the theme itself comes from seeking answers to a few questions.

1.4 Questions: 1. It seems like there is sometimes a connection between mechanical control systems and driftless control systems. Is it possible to understand this? ●
 2. For mechanical control systems that are related (in some way) to driftless control systems, is there a way of doing motion planning for mechanical control systems using methods of motion planning for driftless control systems? ●

¹No animals were harmed in the course of this research.

We shall see that one can obtain complete answers to the questions above... and the answers come in terms of properties of affine connections and vector distributions!

1.4. Prerequisites and notation. We do not attempt, in this paper, to make the case for the differential geometric approach to either mechanics or control theory; we have attempted to do this elsewhere [Lewis 2007]. We assume that the reader is already sold on this idea, and is knowledgeable about differential geometry, mechanics, and control theory. This allows us to focus on higher-level objectives. A useful reference that presents in depth all of the background we require is [Bullo and Lewis 2004].

Let us overview the notation and conventions of the paper.

By \mathbb{Z} we denote the set of integers, and $\mathbb{Z}_{>0}$ denotes the set of positive integers and $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers. By \mathbb{R} we denote the set of real numbers and $\mathbb{R}_{>0}$ denotes the set of positive real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers.

If V is a \mathbb{R} -vector space and if $S \subseteq V$, then $\text{span}(S)$ denotes the span of S , i.e., the smallest subspace of V containing S . We denote by $\text{conv}(S)$ the convex hull of S , i.e., the smallest convex subset of V containing S . By V^* we denote the algebraic dual (which equals the continuous dual of V , when $\dim(V) < \infty$). If $B: V \times V \rightarrow \mathbb{R}$ is bilinear, denote by $B^\flat: V \rightarrow V^*$ the induced linear map. If B^\flat is invertible, we denote its inverse by $B^\sharp: V^* \rightarrow V$. If $S \subseteq V$, then we denote by

$$\text{ann}(S) = \{\alpha \in V^* \mid \alpha(v) = 0, v \in S\}$$

the *annihilator* of S . Similarly, if $\Lambda \subseteq V^*$, we denote by

$$\text{coann}(\Lambda) = \{v \in V \mid \alpha(v) = 0, \alpha \in \Lambda\}$$

the *coannihilator* of Λ . Note that the annihilator and coannihilator are always subspaces.

If $A \subseteq X$ is a subset of a topological space, we denote by $\text{int}(A)$ its interior, i.e., the largest open set contained in A .

We shall work with manifolds that are either of class C^∞ (smooth) or of class C^ω (real analytic). We will use the letter “ r ” to represent either ∞ or ω , so that, when convenient, we can talk about the smooth and real analytic cases simultaneously. The set of C^r -functions on a manifold M we denote by $C^r(M)$. The tangent bundle of a manifold M is denoted by $\pi_{\text{TM}}: \text{TM} \rightarrow M$. The tangent space at $x \in M$ is denoted by $T_x M$. The set of C^r -vector fields on a manifold M is denoted by $\Gamma^r(\text{TM})$. By $\mathcal{L}_X f$ we denote the Lie derivative of $f \in C^r(M)$ with respect to $X \in \Gamma^r(\text{TM})$. By $[X, Y]$ we denote the Lie bracket of $X, Y \in \Gamma^r(\text{TM})$, this being defined by

$$\mathcal{L}_{[X, Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$$

for $f \in C^r(M)$. We refer the reader to [Bullo and Lewis 2004, Equation (3.8)] for the coordinate formula for the Lie bracket. The cotangent bundle of M is denoted by $\pi_{\text{T}^*M}: \text{T}^*M \rightarrow M$, with T_x^*M denoting the cotangent space at $x \in M$. A one-form on M , then, is a section of T^*M . Given $f \in C^r(M)$, we denote the differential of f to be the one-form df .

We shall talk about curves, and the domain of a curve is an interval that we shall typically denote by I . If $\gamma: I \rightarrow M$ is a curve on a manifold, we denote by $\gamma'(t)$ the tangent vector to the curve at times t for which the curve is differentiable.

2. Vector distributions in differential geometry, control theory, and mechanics

In this section we review the theory of vector distributions as they arise in differential geometry, control theory, and mechanics. We emphasise properties of distributions that show up in common in all three areas.

2.1. Vector distributions. Let us first give the definitions that are relevant for us.

2.1 Definition: (Vector distribution) Let $r \in \{\infty, \omega\}$ and let M be a C^r -manifold.

- (i) A *vector distribution* on M is a subset $D \subseteq TM$ such that $D_x \triangleq D \cap T_xM$ is a subspace of T_xM .
- (ii) A vector distribution D is of *class C^r* if, for each $x \in M$, there exists a neighbourhood \mathcal{U} of x and C^r -vector fields $(X_a)_{a \in A}$ on \mathcal{U} such that

$$D_y = \text{span}(X_a(y) \mid a \in A)$$

for each $y \in \mathcal{U}$.

- (iii) We denote by

$$\Gamma^r(D) = \{X \in \Gamma^r(TM) \mid X(x) \in D_x, x \in M\}$$

the set of D -valued vector fields.

- (iv) If $\mathcal{F} \subseteq \Gamma^r(TM)$ is a family of vector fields, then we define the *vector distribution* generated by \mathcal{F} to be the C^r -distribution $D(\mathcal{F})$ defined by

$$D(\mathcal{F})_x = \text{span}(X(x) \mid X \in \mathcal{F}).$$

- (v) The *rank* of a vector distribution D is the function

$$\begin{aligned} \text{rank}_D: M &\rightarrow \mathbb{Z}_{\geq 0} \\ x &\mapsto \dim(D_x). \end{aligned}$$

- (vi) A vector distribution D has *locally constant rank* if, for each $x \in M$, there exists a neighbourhood \mathcal{U} of x such that $\text{rank}_D|_{\mathcal{U}}$ is a constant function. •

Just as we can define a vector distribution as a subset of TM , we can define a *covector distribution* as a subset $\Lambda \subseteq T^*M$ with properties analogous to those of vector distributions. As with vector distributions, we denote by $\Gamma^r(\Lambda)$ the set of one-forms taking values in Λ . Moreover, given a vector distribution D , there is the covector distribution $\text{ann}(D)$, and, given a covector distribution Λ , there is a vector distribution $\text{coann}(\Lambda)$.

The rank of a C^0 -vector distribution is lower semicontinuous. This means that, if $\text{rank}_D(x) = k$, then there exists a neighbourhood \mathcal{U} of x such that $\text{rank}_D(y) \geq k$ for all $y \in \mathcal{U}$. Similarly, the rank of a C^0 -covector distribution is lower semicontinuous. However, if D is a C^0 -distribution, then its annihilator has upper semicontinuous rank (and hence $\text{ann}(D)$ is not C^0 if it does not have locally constant rank) and, if Λ is a C^0 -covector distribution, then its coannihilator has upper semicontinuous rank (and hence $\text{coann}(\Lambda)$ is not C^0 if it does not have locally constant rank). These observations will come up when we talk about velocity constraints below.

Note that, for $\mathcal{F} \subseteq \Gamma^r(\text{TM})$, one obviously has

$$\mathcal{F} \subseteq \Gamma^r(\text{D}(\mathcal{F})),$$

but the converse inclusion does not generally hold. However, it is true that

$$\text{D}(\mathcal{F}) = \text{D}(\Gamma^r(\text{D}(\mathcal{F}))). \quad (2.1)$$

We refer to [Bullo and Lewis 2004, Remark 3.118–3(b)] for an example that illustrates the strictness of the inclusion above.

Note that, if $X \in \Gamma^r(\text{TM})$, then there is defined the vector distribution $\text{D}(\{X\})$. In this (loose) sense, a vector distribution is a multidimensional generalisation of a vector field. As vector fields possess integral curves, vector distributions possibly possess integral manifolds.

2.2 Definition: (Integral manifold for a vector distribution, integrable vector distribution) Let $r \in \{\infty, \omega\}$ and let M be a C^r -manifold.

- (i) An *integral curve* of a vector distribution D on M is a locally absolutely continuous curve $\gamma: I \rightarrow M$ satisfying $\gamma'(t) \in D_{\gamma(t)}$ for almost every $t \in I$.
- (ii) An *integral manifold* of a C^r -vector distribution is an immersed C^r -submanifold S of M such that $T_x S = D_x$ for every $x \in S$.
- (iii) A C^r -vector distribution is *integrable* if, for every $x \in M$, there exists an integral manifold S of D such that $x \in S$. •

From the point of view of geometric structure, integrability of a vector distribution is perhaps a good thing, because it implies that the vector distribution has lots of structure. However, from the point of view of control theory and (less so, perhaps) mechanics, integrability of a vector distribution is not a good thing. Let us explore why this might be, and as well understand integrability better.

We begin with the following notions.

2.3 Definition: (Lie algebras associated with sets of vector fields) Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, and let $\mathcal{F} \subseteq \Gamma^r(\text{TM})$ be a set of C^r -vector fields.

- (i) By $\mathcal{L}(\mathcal{F})$ we denote the Lie algebra generated by \mathcal{F} , i.e., the smallest Lie subalgebra of $\Gamma^r(\text{TM})$ (meaning that $[X, Y] \in \mathcal{L}(\mathcal{F})$ if $X, Y \in \mathcal{L}(\mathcal{F})$) containing \mathcal{F} .
- (ii) By $L(\mathcal{F})$ we denote the distribution defined by

$$L(\mathcal{F})_x = \{X(x) \mid X \in \mathcal{L}(\mathcal{F})\}.$$

- (iii) A C^r -vector distribution D is *involutive* if $L(\Gamma^r(D)) = D$. •

Note that, if $\mathcal{F} \subseteq \Gamma^r(\text{TM})$, then

$$\mathcal{L}(\mathcal{F}) \subseteq \mathcal{L}(\Gamma^r(\text{D}(\mathcal{F}))).$$

Moreover, in contrast with (2.1), it is *not* generally the case that

$$L(\mathcal{F}) = L(\Gamma^r(\text{D}(\mathcal{F}))). \quad (2.2)$$

This *is* true in cases when either (1) $r = \infty$ and $\text{D}(\mathcal{F})$ and $L(\mathcal{F})$ are of locally constant rank or (2) $r = \omega$. Examples for which this equality does not hold require more thought to conjure, and even more thought to prove [Lewis 2012b, Example 8.7].

One now has the well-known connection between integrability and involutivity of vector distributions.

2.4 Theorem: (Frobenius’s Theorem) *Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, and let D be a C^r -vector distribution on M . Then the following statements hold:*

- (i) $r = \infty$: *if D has locally constant rank, then it is integrable if and only if it is involutive;*
- (ii) $r = \omega$: *D is integrable if and only if it is involutive.*

The smooth version of the preceding result is due to Frobenius [1877], while the real analytic version is due to Hermann [1960] and Nagano [1966]. While the real analytic result is more difficult to prove, it is the more useful since problems arising in applications tend to be real analytic if they are smooth. We comment that modern proofs of Frobenius’s Theorem often rely on the Orbit Theorem of Sussmann [1973]. However, it is typical for there to be a small omission in the proof of the Orbit Theorem in the real analytic case, this having to do with what is meant by “finitely generated.” The proofs can be made correct, but only by a nontrivial application of the theory of coherent sheaves [Lewis 2012b, Corollary 6.9].

While integrable vector distributions are very structured, a generic vector distribution will have the following property.

2.5 Definition: (Maximally noninvolutive vector distribution) *Let $r \in \{\infty, \omega\}$ and let M be a C^r -manifold. A C^r -vector distribution D on M is **maximally noninvolutive** if $L(\Gamma^r(D)) = TM$. •*

There is an interesting connection between involutive vector distributions and certain Laplacian-type second-order partial differential equations, as explained by Hörmander [1967]. For this reason, there is a subset of the world for whom a maximally noninvolutive distribution is said to “satisfy Hörmander’s condition.” For our purposes, however, the following theorem provides the essential character of maximally noninvolutive distributions.

2.6 Theorem: (Rashevsky–Chow Theorem) *Let $r \in \{\infty, \omega\}$, let M be a connected C^r -manifold, and let D be a C^r -vector distribution on M . Consider the following statements:*

- (i) *D is maximally involutive;*
- (ii) *for $x_1, x_2 \in M$, there exists an integral curve $\gamma: [0, 1] \rightarrow M$ for D such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.*

Then the following implications hold:

- (iii) $r = \infty$: (i) \implies (ii);
- (iv) $r = \omega$: (i) \iff (ii).

The smooth version of the theorem was first proved by Rashevsky [1938] and Chow [1940/1941]. Modern proofs of the Rashevsky–Chow Theorem are typically made using the Orbit Theorem of Sussmann [1973], and hold in the real analytic case as well.

2.2. Vector distributions in control theory. In this section we recall how the theory of vector distributions relates to control theory. The material here is, more or less, standard [Nijmeijer and van der Schaft 1990].

Although we are not substantially interested in general control-affine systems, it is worth framing some of our discussion in terms of such systems.

2.7 Definition: (Control-affine system, driftless system) Let $r \in \{\infty, \omega\}$. A C^r -*control-affine system* is a triple $\Sigma = (M, \mathcal{F} = \{f_0, f_1, \dots, f_m\}, U)$ where

- (i) M is a C^r -manifold (the *state manifold*),
- (ii) $f_0, f_1, \dots, f_m \in \Gamma^r(TM)$, (f_0 is the *drift vector field* and f_1, \dots, f_m are the *control vector fields*), and
- (iii) $U \subseteq \mathbb{R}^m$ (the *control set*).

A C^r -*driftless system* is a C^r -control-affine system for which the drift vector field is zero. We shall denote a driftless system by $\Sigma = (M, \mathcal{F} = \{f_1, \dots, f_m\}, U)$. •

A *controlled trajectory* for a control-affine system Σ is a pair (ξ, μ) where $\mu: I \rightarrow U$ is locally integrable and where $\xi: I \rightarrow M$ satisfies

$$\xi'(t) = f_0(\xi(t)) + \sum_{a=1}^m \mu^a(t) f_a(\xi(t))$$

for almost every $t \in I$. For $x_0 \in M$, $T \in \mathbb{R}_{>0}$, and a neighbourhood \mathcal{U} of x_0 , the *reachable set* from x_0 in time T and in \mathcal{U} is

$$\mathcal{R}_\Sigma(x_0, T, \mathcal{U}) = \{\xi(T) \mid (\xi, \mu) \text{ is a controlled trajectory defined on } [0, T] \text{ with } \xi(0) = x_0 \text{ and } \text{image}(\xi) \subseteq \mathcal{U}\}$$

and the *reachable set* from x_0 in time at most T and in \mathcal{U} is

$$\mathcal{R}_\Sigma(x_0, \leq T, \mathcal{U}) = \cup_{t \in [0, T]} \mathcal{R}_\Sigma(x_0, t, \mathcal{U}).$$

With these notions of reachable sets, let us define a couple of forms of controllability in which we will be interested.

2.8 Definition: (Accessibility and controllability of control-affine systems) Let $\Sigma = (M, \mathcal{F}, U)$ be a C^r -control-affine system and let $x_0 \in M$.

- (i) The system Σ is *small-time locally accessible (STLA)* from x_0 if there exists $T_0 \in \mathbb{R}_{>0}$ and a neighbourhood \mathcal{U}_0 of x_0 such that, for each $T \in (0, T_0]$ and each neighbourhood $\mathcal{U} \subseteq \mathcal{U}_0$ of x_0 , $\text{int}(\mathcal{R}_\Sigma(x_0, \leq T, \mathcal{U})) \neq \emptyset$.
- (ii) The system Σ is *small-time locally controllable (STLC)* from x_0 if there exists $T_0 \in \mathbb{R}_{>0}$ and a neighbourhood \mathcal{U}_0 of x_0 such that, for each $T \in (0, T_0]$ and each neighbourhood $\mathcal{U} \subseteq \mathcal{U}_0$ of x_0 , $x_0 \in \text{int}(\mathcal{R}_\Sigma(x_0, \leq T, \mathcal{U}))$. •

We can now state the well-known characterisations of accessibility (for control-affine systems) and controllability (for driftless systems).

2.9 Theorem: (Accessibility for control-affine systems) Let $r \in \{\infty, \omega\}$ and let $\Sigma = (M, \mathcal{F}, U)$ be a C^r -control-affine system, supposing that $0 \in \text{int}(\text{conv}(U))$. For $x_0 \in M$, the following statements hold:

- (i) $r = \infty$: Σ is STLA from x_0 if $L(\mathcal{F})_{x_0} = T_{x_0}M$;
- (ii) $r = \omega$: Σ is STLA from x_0 if and only if $L(\mathcal{F})_{x_0} = T_{x_0}M$.

2.10 Theorem: (Controllability for driftless systems) *Let $r \in \{\infty, \omega\}$ and let $\Sigma = (M, \mathcal{F}, U)$ be a C^r -driftless system, supposing that $0 \in \text{int}(\text{conv}(U))$. For $x_0 \in M$, the following statements hold:*

- (i) $r = \infty$: Σ is STLC from x_0 if $L(\mathcal{F})_{x_0} = T_{x_0}M$;
- (ii) $r = \omega$: Σ is STLC from x_0 if and only if $L(\mathcal{F})_{x_0} = T_{x_0}M$.

Let us say a few words about the preceding results. The accessibility results date to the work of Sussmann and Jurdjevic [1972] and Hermann and Krener [1977]. There are necessary and sufficient conditions for STLA (for control-affine systems) and for STLC (for driftless systems) in the smooth case, and these come from the Orbit Theorem of Sussmann [1973] in the smooth case. The controllability results follow, more or less, from the Rashevsky–Chow Theorem. The matter of controllability for control-affine systems is presently very much up in the air. While a great deal of work has been done in the area, we are still very far from having a satisfactory description of the mechanisms for controllability of control-affine systems. We shall overview in Section 4.2 some results for so-called “low-order” controllability for a class of mechanical control systems. For controllability of control-affine systems, we shall only refer to [Sussmann 1987] as representative of work that is deep and useful, but ultimately unsatisfying in ways that we will discuss in some detail in Section 5.

While the results of this section are not as powerful as one would like—since they do not give one controllability results, such as one would ultimately need to do useful control theory—they do have the benefit of being tractable, requiring only the computation of the Lie bracket. We refer to [Bullo and Lewis 2004, Example 7.11] for a few worked out examples.

2.3. Vector distributions in mechanics. Vector distributions arise in mechanics when modelling velocity constraints arising from rolling contact, without slipping. The modelling of such phenomenon is considered in detail in [Neimark and Fufaev 1972]. Here we shall restrict ourselves to geometric representations of the modelling, after the physics has been done properly. We refer the reader to [Lewis 2017] for a detailed description of physical modelling for mechanical systems, from a geometric mechanics point of view. For a mechanical system, we shall denote by Q its configuration manifold, that we consider to be of class C^∞ or class C^ω .

The definition we use is the following.

2.11 Definition: (Velocity constraint) Let $r \in \{\infty, \omega\}$ and let Q be a C^r -manifold. A C^r -*velocity constraint* on Q is a vector distribution $D = \text{coann}(\Lambda_D)$ on Q , where Λ_D is a C^r -covector distribution on Q . We shall say that D is *regular* if it has locally constant rank. •

A reader is justified in wondering why we define a velocity constraint, not by defining a vector distribution, but by defining it to be the coannihilator of a covector distribution. The reason for this is that, in cases where velocity constraints do not have locally constant rank, the changes of rank arise because constraint forces align. This can be seen in the snakeboard from Figure 1b, where such a singularity arises when $\phi = \frac{\pi}{2}$ since, in this case, the constraint forces maintaining the no slip constraint become collinear. This phenomenon can be shown to be the natural way in which singularities in velocity constraints arise [Briggs 2016]. Thus

velocity constraints are upper semicontinuous, and so cannot be modelled correctly by C^0 -vector distributions, unless the velocity constraint is regular. This has mathematical as well as physical implications. Mathematically, a velocity constraint D that is not regular will have the property that

$$D(\Gamma^r(D)) \subseteq D,$$

but this inclusion will be strict at points where the rank function is not locally constant. This, for example, makes the Lie algebra constructions regarding involutivity nontrivial to interpret. Physically, the author does not know how to determine the correct equations of motion for mechanical systems with velocity constraints that are not regular. For this reason, we shall sometimes restrict ourselves to velocity constraints that are regular, and so are of class C^r as vector distributions.

Let us make the connection from topics concerning distributions to physical concepts. This really amounts to using mechanical language to describe mathematical concepts.

2.12 Definition: (Holonomic, nonholonomic, and totally nonholonomic velocity constraints) Let $r \in \{\infty, \omega\}$ and let Q be a C^r -manifold with D a C^r -velocity constraint defined by the C^r -covector distribution Λ_D .

- (i) A locally absolutely continuous curve $\gamma: I \rightarrow Q$ is **D-admissible** if, for every $\lambda \in \Gamma^r(\Lambda)$, $\langle \lambda(\gamma(t)); \gamma'(t) \rangle = 0$ for almost every $t \in I$.
- (ii) The velocity constraint D is **holonomic** if, for each $q \in Q$, there exists a neighbourhood \mathcal{U} of q and $f^1, \dots, f^k \in C^r(Q)$ such that

$$\Lambda_{D,p} = \text{span}(df^1(p), \dots, df^k(p))$$

for every $p \in \mathcal{U}$.

- (iii) The velocity constraint D is **nonholonomic** if it is not holonomic.
- (iv) The velocity constraint D is **totally nonholonomic** if, for each $q \in Q$, there exists a neighbourhood \mathcal{U} of q such that, for each $p \in \mathcal{U}$, there exists a D -admissible curve $\gamma: [0, 1] \rightarrow Q$ such that $\gamma(0) = q$ and $\gamma(1) = p$. •

The idea, physically, is that for an holonomic velocity constraint, D -admissible curves $\gamma: I \rightarrow Q$ actually satisfy the equations

$$f^1 \circ \gamma(t) = \dots = f^k \circ \gamma(t) = 0, \quad t \in I,$$

i.e., the configurations are constrained, not the velocities. For a totally nonholonomic velocity constraint, the idea is that the velocity constraints impose no restrictions on the configurations.

Let us connect these mechanical ideas with the geometric attributes of velocity distributions. We emphasise that the following result holds only for regular velocity constraints. The analogues of such results in the nonregular case will be more complicated, relying on the notion of integrable covector distributions [Freeman 1984].

2.13 Theorem: (Characterisations of velocity constraints) Let $r \in \{\infty, \omega\}$, let Q be a C^r -manifold, and let D be a regular C^r -velocity constraint. Then the following statements hold.

- (i) *Characterisation of holonomic velocity constraints:*

- (a) $r = \infty$: D is holonomic if it is involutive;
- (b) $r = \omega$: D is holonomic if and only if it is involutive.
- (ii) Characterisation of totally nonholonomic velocity constraints:
 - (a) $r = \infty$: D is totally nonholonomic if it is maximally noninvolutive;
 - (b) $r = \omega$: D is totally nonholonomic if and only if it is maximally noninvolutive.

There is, of course, a substantial similarity between the accessibility results of Section 2.2 and the characterisations of velocity constraints above. And, like the accessibility results, the results in this section are typically easy to apply. For example, one can show that the mobile robot of 1a and the snakeboard of 1b are totally nonholonomic; see [Laumond 1993] and [Bullo and Lewis 2003], respectively.

3. Affine connections and mechanics

Our next order of business is to introduce the notion of an affine connection. We first give an hasty presentation of the mathematics, then indicate the rôle played by affine connections in mechanics. We expect that many readers will be familiar with the constructions and results regarding vector distributions from the preceding section. However, possibly fewer readers will be familiar with the constructions regarding affine connections from this section. However, affine connections are *crucial* to our objectives here. We refer reader to [Lewis 2007] for a more detailed justification of the rôle of affine differential geometry in mechanics. For readers wanting to see how the constructions in this section can actually be carried out in concrete examples (for example, those from Figure 1), we can do no better than point to [Bullo and Lewis 2004, Chapter 4] where this is worked out in great detail.

3.1. Affine connections. We first give the definition.

3.1 Definition: (Affine connection) Let $r \in \{\infty, \omega\}$ and let M be a C^r -manifold. A C^r -*affine connection* on M is a mapping

$$\Gamma^r(TM) \times \Gamma^r(TM) \ni (X, Y) \mapsto \nabla_X Y \in \Gamma^r(TM)$$

satisfying the following conditions:

- (i) the mapping is \mathbb{R} -bilinear;
- (ii) $\nabla_{fX} Y = f \nabla_X Y$ for $f \in C^r(M)$ and $X, Y \in \Gamma^r(TM)$;
- (iii) $\nabla_X(fY) = f \nabla_X Y + (\mathcal{L}_X f)Y$ for $f \in C^r(M)$ and $X, Y \in \Gamma^r(TM)$.

We call $\nabla_X Y$ the *covariant derivative* of Y with respect to X . •

Property (ii) ensures that $\nabla_X Y(x)$ depends only on the value of X at x . Thus, if $v_x \in T_x M$ and if $Y \in \Gamma^r(TM)$, then the expression $\nabla_{v_x} Y \in T_x M$ makes sense. Therefore, if $\gamma: I \rightarrow M$ is differentiable at $t \in I$ and if $Y \in \Gamma^r(TM)$, then we can make sense of $\nabla_{\gamma'(t)} Y \in T_{\gamma(t)} M$. What's more, $\nabla_{\gamma'(t)} Y$ depends only on the values of Y at points along γ nearby t . In other words, suppose that $\gamma: I \rightarrow M$ and that $Y: I \rightarrow TM$ satisfies $Y(t) \in T_{\gamma(t)} M$. If both γ and Y are differentiable at $t \in I$, then one can make sense of $\nabla_{\gamma'(t)} Y(t) \in T_{\gamma(t)} M$. This can, in particular, be applied to the case where γ' is locally absolutely continuous and where $Y = \gamma'$ to give $\nabla_{\gamma'(t)} \gamma'(t) \in T_{\gamma(t)} M$ at times t where γ is twice differentiable. In such a case, the curve γ is a *geodesic* when $\nabla_{\gamma'(t)} \gamma'(t) = 0$.

An important affine connection in both mathematics and applications is given in the following result. We recall that a **C^r -Riemannian metric** \mathbf{G} on a C^r -manifold M is an assignment to each $x \in M$ an inner product $\mathbf{G}(x)$ with the assignment being of class C^r in the sense that the function $x \mapsto \mathbf{G}(x)(X(x), Y(x))$ is of class C^r for every $X, Y \in \Gamma^r(\text{TM})$.

3.2 Theorem: (The Fundamental Theorem of Riemannian Geometry) *Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, and let \mathbf{G} be a C^r -Riemannian metric on M . Then there exists a unique C^r -affine connection $\nabla^{\mathbf{G}}$ on M satisfying the conditions*

- (i) $\mathcal{L}_Z(\mathbf{G}(X, Y)) = \mathbf{G}(\nabla_Z^{\mathbf{G}}X, Y) + \mathbf{G}(X, \nabla_Z^{\mathbf{G}}Y)$ for $X, Y, Z \in \Gamma^r(\text{TM})$ and
- (ii) $\nabla_X^{\mathbf{G}}Y - \nabla_Y^{\mathbf{G}}X = [X, Y]$ for $X, Y \in \Gamma^r(\text{TM})$.

The affine connection $\nabla^{\mathbf{G}}$ is called the **Levi-Civita affine connection**. The way to read the two defining conditions of the Levi-Civita affine connection are the following: condition (i) means that the Levi-Civita affine connection “leaves the metric invariant;” condition (ii) means that the Levi-Civita affine connection is “torsion-free.” When one is studying a Riemannian manifold with its Levi-Civita affine connection, one is engaged in the act of doing “Riemannian geometry.”

The topics of Riemannian geometry and affine differential geometry, as initiated in the few paragraphs above, have a significant life of their own, and we refer the interested reader to the classic [Kobayashi and Nomizu 1963] for the basics. We shall be able to restrict ourselves to the few concepts we introduce above.

Let us consider a nonstandard construction in affine differential geometry. Given an affine connection ∇ on M , the **symmetric product** is the mapping

$$\Gamma^r(\text{TM}) \times \Gamma^r(\text{TM}) \ni (X, Y) \mapsto \langle X : Y \rangle \triangleq \nabla_X Y + \nabla_Y X \in \Gamma^r(\text{TM}).$$

This rather innocuous looking object was first studied in [Crouch 1981] during the course of studying gradient control systems. Its geometric meaning was presented by Lewis [1998], with additional context being given by Barbero-Liñán and Lewis [2012]. As we shall see, the symmetric product features prominently in the study of controllability for certain classes of mechanical control systems. The geometric interpretation is interesting, so let us present it here in a way that mirrors our presentation of involutive distributions.

The first thing we do is provide the following definition, which is to be thought of as being analogous to the notion of an integrable vector distribution.

3.3 Definition: (Geodesically invariant vector distribution) Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, let D be a C^r -distribution on M , and let ∇ be a C^r -affine connection on M . The vector distribution D is **geodesically invariant** with respect to ∇ if, for any geodesic $\gamma: I \rightarrow M$ satisfying $\gamma'(t_0) \in D_{\gamma(t_0)}$ for some $t_0 \in I$, it holds that $\gamma'(t) \in D_{\gamma(t)}$ for every $t \in I$. •

Another way of thinking of a geodesically invariant vector distribution $D \subseteq \text{TM}$ is as being an invariant submanifold for the second-order dynamics for geodesics. For our purposes, the interesting thing is that one can characterise geodesically invariant vector distributions using an differential characterisation like involutivity for integrable vector distributions.

3.4 Definition: (Symmetric subalgebras associated with sets of vector fields) Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, let ∇ be a C^r -affine connection on M , and let $\mathcal{F} \subseteq \Gamma^r(TM)$ be a set of C^r -vector fields.

- (i) By $\mathcal{S}(\mathcal{F})$ we denote the symmetric subalgebra generated by \mathcal{F} , i.e., the smallest symmetric subalgebra of $\Gamma^r(TM)$ (meaning that $\langle X : Y \rangle \in \mathcal{S}(\mathcal{F})$ if $X, Y \in \mathcal{S}(\mathcal{F})$) containing \mathcal{F} .
- (ii) By $S(\mathcal{F})$ we denote the vector distribution defined by

$$S(\mathcal{F})_x = \{X(x) \mid X \in \mathcal{S}(\mathcal{F})\}.$$

- (iii) A vector distribution D is *symmetric* with respect to ∇ if $S(\Gamma^r(D)) = D$. •

We now have the following theorem.

3.5 Theorem: (Characterisation of geodesically invariant vector distributions) Let $r \in \{\infty, \omega\}$, let M be a C^r -manifold, let D be a C^r -distribution on M , and let ∇ be a C^r -affine connection on M . Then we have the following statements:

- (i) $r = \infty$: if D has locally constant rank, then it is geodesically invariant if and only if it is symmetric;
- (ii) $r = \omega$: D is geodesically invariant if and only if it is symmetric.

We hope that it barely needs pointing out the similarities between this theorem and Frobenius's Theorem.

3.2. Unconstrained mechanics and the Levi-Civita connection. Let us provide the well-known connections between Riemannian geometry and mechanics. We suppose that we have a mechanical system with a C^r -configuration manifold Q , $r \in \{\infty, \omega\}$, that arises from an interconnection of rigid bodies. The inertial properties of the system can be encoded by a Riemannian metric G on Q that is defined by the fact that the kinetic energy, as a function on the tangent bundle, is given by

$$KE(v_q) = \frac{1}{2}G(v_q, v_q).$$

We call G the *kinetic energy metric*. Let us initially suppose that the system is subject to no external forces (but is subject to internal forces required to maintain the interconnections). In this case, there are various ways one might determine the equations of motion, all of which are required to agree with the basic momentum balance equations of Newton and Euler. Let us call these balance equations the *Newton–Euler equations*. These equations may be shown to be equivalent to the Euler–Lagrange equations for the kinetic energy Lagrangian. In turn, the Euler–Lagrange equations for the kinetic energy Lagrangian are equivalent to the geodesic equations for the Levi-Civita affine connection. In short, we have the following theorem.

3.6 Theorem: (Physical motions and geodesics for the Levi-Civita affine connection) Let $r \in \{\infty, \omega\}$, and let Q be a C^r -manifold that is the configuration manifold for a mechanical system with kinetic energy metric G and that is subject to no external forces. For a curve $\gamma: I \rightarrow Q$, the following statements are equivalent:

- (i) γ satisfies the Newton–Euler equations;

- (ii) γ satisfies the Euler–Lagrange equations for the Lagrangian function KE;
- (iii) γ is a geodesic for $\nabla^{\mathbf{G}}$.

One can include forces and torques in this framework, and it turns out that forces and torques that depend on time, configuration, and velocity give rise to a mapping $F: I \times \mathbb{T}\mathbb{Q} \rightarrow \mathbb{T}^*\mathbb{Q}$. Moreover, the effects of these forces appear in the equations of motion according to the formula

$$\nabla_{\gamma'(t)}^{\mathbf{G}} \gamma'(t) = \mathbf{G}^{\sharp} \circ F(t, \gamma'(t)),$$

that we call the **forced geodesic equations**. The correspondence between these equations and the momentum balance equations is carried out in any book on analytical or classical mechanics [e.g., Goldstein 1951]. The gap between these classical approaches and the geometric approach is filled in [Lewis 2017].

3.3. Constrained mechanics and the constrained connection. Now we include velocity constraints in our geometric formulation for mechanics. Thus we suppose that we have a configuration manifold \mathbb{Q} , a kinetic energy metric \mathbf{G} , external forces F , and a regular velocity constraint \mathbb{D} . To prescribe the equations of motion in this setting, one must do two things: (1) restrict to curves that are integral curves for \mathbb{D} ; (2) add to the equations of motion a constraint force, to be determined, that is \mathbf{G} -orthogonal to \mathbb{D} . It turns out that, when $F = 0$, the resulting equations are the geodesic equations for a particular affine connection. To define that affine connection, we need some notation.

We denote by \mathbb{D}^{\perp} the \mathbf{G} -orthogonal complement to \mathbb{D} . We denote by $P_{\mathbb{D}}, P_{\mathbb{D}}^{\perp}: \mathbb{T}\mathbb{Q} \rightarrow \mathbb{T}\mathbb{Q}$ the \mathbf{G} -orthogonal projections onto \mathbb{D} and \mathbb{D}^{\perp} , respectively.

We now have the following characterisation of the constrained equations of motion in the absence of external forces [Lewis 1998, Lewis 2000].

3.7 Theorem: (Physical motions and geodesics for the constrained connection)

Let $r \in \{\infty, \omega\}$, let \mathbb{Q} be a C^r -manifold that is the configuration manifold for a mechanical system with kinetic energy metric \mathbf{G} , and let \mathbb{D} be a regular C^r -velocity constraint. Suppose that the system is not subject to external forces. For a curve $\gamma: I \rightarrow \mathbb{Q}$, the following statements are equivalent:

- (i) γ satisfies the Newton–Euler equations;
- (ii) there exists $\lambda: I \rightarrow \mathbb{D}^{\perp}$ such that

$$\begin{aligned} \nabla_{\gamma'(t)}^{\mathbf{G}} \gamma'(t) &= \lambda(t), \\ P_{\mathbb{D}}(\gamma'(t)) &= 0; \end{aligned}$$

- (iii) $\gamma'(t_0) \in \mathbb{D}_{\gamma(t_0)}$ for some $t_0 \in I$ and γ is a geodesic for the **constrained connection**, which is the affine connection $\nabla^{\mathbb{D}}$ defined by

$$\nabla_X^{\mathbb{D}} Y = \nabla_X^{\mathbf{G}} Y + (\nabla_X^{\mathbf{G}} P_{\mathbb{D}}^{\perp})(Y).$$

As with the unconstrained equations, one can add forces and torques to obtain the mapping $F: I \times \mathbb{T}\mathbb{Q} \rightarrow \mathbb{T}^*\mathbb{Q}$, and these appear in the constrained equations in the following form:

$$\nabla_{\gamma'(t)}^{\mathbb{D}} \gamma'(t) = P_{\mathbb{D}} \circ \mathbf{G}^{\sharp} \circ F(t, \gamma'(t)).$$

These are the *forced, constrained geodesic equations*, and model *all* of the sorts of systems considered in “standard” mechanics. Readers enamoured of compact elegant equations will be pleased with the form of these equations. It should be noted, however, that if one writes these equations explicitly for a complicated example, the expressions will be extremely bulky in many cases, e.g., the snakeboard system of Figure 1b. Sometimes one can use special structure of the system to simplify the equations [Bullo and Žefran 2002]. Moreover, casting aside such practical matters as actually writing the equations for specific systems, the geometric structure of the systems permits one to study mechanical control systems in ways that would otherwise be inaccessible. It is to this that we now turn our attention.

4. Affine connection control systems

We present in this section the class of mechanical control systems that we study in detail. For these systems, we emphasise results that are (1) geometric and (2) difficult to arrive at by means that are not geometric.

4.1. Definitions and basic properties. We work with a mechanical system with configuration manifold Q , kinetic energy metric G , and regular velocity constraint D . Apart from constraint forces, the only forces we consider are control forces, and these we consider to be given by

$$F(t, v_q) = \sum_{a=1}^m u^a(t) F^a(q)$$

for one-forms F^1, \dots, F^m on Q , and where $t \mapsto \mathbf{u}(t) \in U \subseteq \mathbb{R}^m$ are the controls. This means that the directions that are actuated form a covector distribution spanned by F^1, \dots, F^m . The governing equations for the system are thus

$$\nabla_{\gamma'(t)}^D \gamma'(t) = \sum_{a=1}^m u^a(t) P_D \circ G^\sharp \circ F^a(\gamma(t)).$$

These equations govern the behaviour of all four examples in Figure 1, provided that one considers control forces as the only external forces. We abstract the preceding system according to the following definition.

4.1 Definition: (Affine connection control system) Let $r \in \{\infty, \omega\}$. A C^r -*affine connection control system* is a 4-tuple $\Sigma = (Q, \nabla, \mathcal{Y} = \{Y_1, \dots, Y_m\}, U)$ where

- (i) Q is a C^r -manifold (the *configuration manifold*),
- (ii) ∇ is a C^r -affine connection;
- (iii) $Y_1, \dots, Y_m \in \Gamma(TQ)$ (the *control vector fields*), and
- (iv) $U \subseteq \mathbb{R}^m$ (the *control set*). •

A *controlled trajectory* for an affine connection control system Σ is a pair (γ, μ) where $\mu: I \rightarrow U$ is locally integrable and where $\gamma: I \rightarrow Q$ satisfies

$$\nabla_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^m \mu^a(t) Y_a(\gamma(t)) \tag{4.1}$$

for almost every $t \in I$. For $q_0 \in \mathbb{Q}$, $T \in \mathbb{R}_{>0}$, and a neighbourhood \mathcal{U} of q_0 , the **reachable configurations** from q_0 in time T and in \mathcal{U} is

$$\mathcal{R}_\Sigma(q_0, T, \mathcal{U}) = \{\gamma(T) \mid (\gamma, \boldsymbol{\mu}) \text{ is a controlled trajectory defined on } [0, T] \text{ with } \gamma'(0) = 0 \text{ and } \text{image}(\gamma) \subseteq \mathcal{U}\}$$

and the **reachable configurations** from q_0 in time at most T and in \mathcal{U} is

$$\mathcal{R}_\Sigma(q_0, \leq T, \mathcal{U}) = \cup_{t \in [0, T]} \mathcal{R}_\Sigma(q_0, t, \mathcal{U}).$$

With these notions of reachable sets, let us define a couple of forms of controllability in which we will be interested.

4.2 Definition: (Accessibility and controllability of affine connection control systems) Let $\Sigma = (\mathbb{Q}, \nabla, \mathcal{Y}, U)$ be a C^r -affine connection control system and let $q_0 \in \mathbb{Q}$.

- (i) The system Σ is **small-time locally configuration accessible (STLCA)** from q_0 if there exists $T_0 \in \mathbb{R}_{>0}$ and a neighbourhood \mathcal{U}_0 of q_0 such that, for each $T \in (0, T_0]$ and each neighbourhood $\mathcal{U} \subseteq \mathcal{U}_0$ of q_0 , $\text{int}(\mathcal{R}_\Sigma(q_0, \leq T, \mathcal{U})) \neq \emptyset$.
- (ii) The system Σ is **small-time locally configuration controllable (STLCC)** from q_0 if there exists $T_0 \in \mathbb{R}_{>0}$ and a neighbourhood \mathcal{U}_0 of q_0 such that, for each $T \in (0, T_0]$ and each neighbourhood $\mathcal{U} \subseteq \mathcal{U}_0$ of q_0 , $q_0 \in \text{int}(\mathcal{R}_\Sigma(q_0, \leq T, \mathcal{U}))$. •

We shall now state the accessibility results of Lewis and Murray [1997]. The results we state follow from those for control-affine systems (stated above as Theorem 2.9), by noting that the second-order equations (4.1) define a first-order system with state manifold $\mathbb{T}\mathbb{Q}$.

4.3 Theorem: (Accessibility for affine connection control systems) Let $r \in \{\infty, \omega\}$ and let $\Sigma = (\mathbb{Q}, \nabla, \mathcal{Y}, U)$ be a C^r -control-affine system, supposing that $0 \in \text{int}(\text{conv}(U))$. For $q_0 \in \mathbb{Q}$, the following statements hold:

- (i) $r = \infty$: Σ is STLCA from q_0 if $\mathbb{L}(\mathcal{S}(\mathcal{Y}))_{q_0} = \mathbb{T}_{q_0}\mathbb{Q}$;
- (ii) $r = \omega$: Σ is STLCA from q_0 if and only if $\mathbb{L}(\mathcal{S}(\mathcal{Y}))_{q_0} = \mathbb{T}_{q_0}\mathbb{Q}$.

Just as for control-affine systems with drift, there are no “pat” general results for controllability of affine connection control systems. Thus we do not state any controllability results that one might regard as “general.” However, what we will do is outline some particular controllability results for affine connection control systems that illuminate some interesting geometry, as well as answer the questions posed in Section 1.3.

4.2. Low-order controllability results. Let us state the assumptions we make in this section.

4.4 Assumptions: (Low-order controllability assumptions) Let $r \in \{\infty, \omega\}$, let $\Sigma = (\mathbb{Q}, \nabla, \mathcal{Y} = \{Y_1, \dots, Y_m\}, U)$ be an affine connection control system, and let $q_0 \in \mathbb{Q}$. We shall assume the following:

- (i) $0 \in \text{int}(\text{conv}(U))$;
- (ii) $\mathbb{L}(\mathcal{S}(\mathcal{Y}))_{q_0} = \mathbb{T}_{q_0}\mathbb{Q}$;
- (iii) $\mathbb{S}(\mathcal{Y})_{q_0} = \text{span}(\langle Y_a : Y_b \rangle(q_0) \mid a, b \in \{1, \dots, m\})$. •

Assumption (i) is a standard assumption ensuring that the control set provides enough actuation. We make assumption (ii) since we will be talking about controllability, and so one must necessarily have accessibility. The meat of the assumptions is (iii), which ensures that the information about symmetric products for the system at q_0 is contained in symmetric products of pairs of vector fields.

Our controllability results rely on an algebraic construction that neatly encodes the geometry of the affine connection ∇ and the vector distribution $Y = D(\mathcal{Y})$. Indeed, we define

$$B_{\mathcal{Y},q_0} : Y_{q_0} \times Y_{q_0} \rightarrow T_{q_0}Q/Y_{q_0} \\ (u_{q_0}, v_{q_0}) \mapsto \pi_{Y,q_0}(\langle U : V \rangle(q_0)),$$

where U and V are C^∞ -vector fields extending u_{q_0} and v_{q_0} , respectively, and where $\pi_{Y,q_0} : T_{q_0}Q \rightarrow Y_{q_0}$ is the canonical projection onto the quotient. Thus $B_{\mathcal{Y},q_0}$ is a vector space-valued bilinear map. Let us provide some useful attributes of such objects.

4.5 Definition: (Attributes of vector space-valued bilinear maps) Let U and V be finite-dimensional \mathbb{R} -vector spaces and let $B : V \times V \rightarrow U$ be a bilinear map. For $\lambda \in U^*$, denote by $\lambda \cdot B : V \times V \rightarrow \mathbb{R}$ the bilinear map

$$\lambda \cdot B(v_1, v_2) = \lambda(B(v_1, v_2)), \quad v_1, v_2 \in V.$$

Then B is:

- (i) **definite** if there exists $\lambda \in U^*$ such that $\lambda \cdot B$ is positive-definite;
- (ii) **essentially indefinite** if, for each $\lambda \in U^*$, $\lambda \cdot B$ is either zero, or neither positive nor negative semidefinite. •

We then have the following results for controllability of affine connection control systems [Bullo and Lewis 2004].

4.6 Theorem: (Low-order controllability results for affine connection control systems) Let $r \in \{\infty, \omega\}$ and let Σ be an affine connection control system satisfying Assumption 4.4 for $q_0 \in Q$. Then the following statements hold:

- (i) $r = \infty$: if $B_{\mathcal{Y},q_0}$ is essentially indefinite, then Σ is STLCC from q_0 ;
- (ii) $r = \omega$: if q_0 is a regular point for Y , and if $B_{\mathcal{Y},q_0}$ is definite, then Σ is not STLCC from q_0 .

This result can be used to determine the controllability of the four example from Figure 1. Let us record these results.

1. [e.g., Laumond 1993] If one can actuate both wheels in the mobile robot from Figure 1a, then the system is STLCC from every configuration.
2. [Bullo and Lewis 2003] For the snakeboard of Figure 1b, if one can control the angle $\psi - \theta$ between the barbell-shaped rider and the body coupling the wheels, and if one can actuate the wheel angles ϕ , then the system is STLCC from every configuration, except those where $\phi(0) = \pm \frac{\pi}{2}$. Note that, for these latter initial configurations, it is not just that we cannot assert anything about the controllability of the system; we do not even know the equations of motion from these initial configurations!
3. [Lewis and Murray 1997] For the robotic leg from Figure 1c, if one can actuate the angle $\psi - \theta$ between the leg and the body and if one can actuate the length r of the leg, then the system is STLCC from every configuration.

4. [Lewis and Murray 1997] If one can control the magnitude of the force F as well as the angle ϕ through which this force is applied, then the planar body of Figure 1d is STLCC from every configuration. If the angle ϕ is fixed, however, the system is not STLCC from any configuration. Moreover, it is not even STLCA if ϕ is fixed at 0.

4.3. Relationships between affine connection control systems and driftless control systems. In this section we consider the somewhat peculiar problem of comparing affine connection control systems with driftless control systems. On the mathematical surface, it seems unlikely that this should be a fruitful avenue of exploration. However, when faced with specific examples, the question seems to arise in natural ways, as we saw in Section 1.3. This, then, incites one to examine the question closely, and upon doing so one makes interesting connections with the low-order controllability results from the preceding section.

First let us consider the problem we are studying. We let

$$\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y} = \{Y_1, \dots, Y_m\}, \mathbb{R}^m), \quad \Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{F} = \{f_1, \dots, f_k\}, \mathbb{R}^k)$$

be an affine connection control system and a driftless control system, respectively. Note that we do not constrain the controls to reside in a subset of \mathbb{R}^m or \mathbb{R}^k since, if one does this, comparison of affine connection control systems and driftless systems is not really possible. Let us write the governing equations side-by-side:

$$\nabla_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^m u^a(t) Y_a(\gamma(t)), \quad \gamma'(t) = \sum_{j=1}^k v^j(t) f_j(\gamma(t)).$$

One can see that, for an affine connection control system, one is controlling the acceleration of the system, whereas, for a driftless control system, one is controlling the velocity. One of the consequences of this is that the trajectories for the two systems will have different regularity. Specifically, trajectories for driftless systems will be locally absolutely continuous, whereas trajectories for affine connection control systems will have velocities that are locally absolutely continuous. Note that this is not just mere mathematical fussiness; for a driftless system, one can have discontinuous changes in velocity, whereas for affine connection control systems this is not possible, reflecting that such discontinuities would require infinite force. Therefore, in comparing the two sorts of systems, one must keep this in mind.

The first notion we consider is a strong one, where the trajectories of the affine connection control system and the driftless system agree as well as they possibly can.

4.7 Definition: (Reducibility of an affine connection control system to a driftless system) Let $r \in \{\infty, \omega\}$ and let $\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y}, \mathbb{R}^m)$ and $\Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{F}, \mathbb{R}^k)$ be a C^r -affine connection control system and a C^r -driftless control system, respectively. Then Σ_{dyn} is *reducible to* Σ_{kin} if the following conditions are satisfied:

- (i) for every controlled trajectory (γ, ν) for Σ_{kin} with ν locally absolutely continuous, there exists a locally integrable control μ such that (γ, μ) is a controlled trajectory for Σ_{dyn} ;
- (ii) for every controlled trajectory (γ, μ) for Σ_{dyn} with μ locally integrable and with $\gamma'(t) \in D(\mathcal{F})_{\gamma(t)}$ for some t , there exists a locally absolutely continuous control ν such that (γ, ν) is a controlled trajectory for Σ_{kin} . •

Note that, in the second part of the preceding definition, we do require that the tangent vector to the trajectory at some time reside in the vector distribution spanned by the vector fields \mathcal{F} since, if this is not the case, then it will certainly not be the case that this trajectory can be followed by the driftless system.

A weaker notion than reducibility, but related as we shall see, is afforded by the following definition.

4.8 Definition: (Decoupling vector field) Let $r \in \{\infty, \omega\}$ and let $\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y}, \mathbb{R}^m)$ be a C^r -affine connection control system. A vector field $X \in \Gamma^r(\text{TM})$ is a **decoupling vector field** if, for every integral curve $\gamma: I \rightarrow \mathbb{Q}$ for X and for any reparameterisation $\tau: J \rightarrow I$ of γ , there exists a locally integrable control μ for which $(\gamma \circ \tau, \mu)$ is a controlled trajectory for Σ_{dyn} . •

In order to characterise the preceding concepts, we shall use the vector space-valued quadratic map from the preceding section. To do so, we shall make an additional assumption.

4.9 Assumption: (Reducibility assumption) Let $r \in \{\infty, \omega\}$ and let Σ_{dyn} be an affine connection control system. We assume that the assumption (iii) of Assumption 4.4 holds for every $q_0 \in \mathbb{Q}$. •

This assumption allows us to define a bilinear vector bundle mapping $B_Y: Y \times Y \rightarrow \text{TQ}/Y$ by $B_Y(q) = B_{\mathcal{Y}, q}$.

We can now characterise the various ways of relating affine connection control systems and driftless control systems.

4.10 Theorem: (Characterising when an affine connection control system is related to a driftless system) Let $r \in \{\infty, \omega\}$, let $\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y}, \mathbb{R}^m)$ be a C^r -affine connection control system, and let $\Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{F}, \mathbb{R}^k)$ be a C^r -driftless control system. Suppose that $Y = \text{D}(\mathcal{Y})$ and $F = \text{D}(\mathcal{F})$ have locally constant rank. Then the following statements hold:

- (i) Σ_{dyn} is reducible to Σ_{kin} if and only if $Y = F$ and $B_Y = 0$;
- (ii) $X \in \Gamma^r(\text{TM})$ is a decoupling vector field if and only if $X \in \Gamma^r(Y)$ and $B_Y(X, X) = 0$.

We note that, if an affine connection control system is reducible to a driftless system, then all linear combinations of control vector fields, even with functions as coefficients, are decoupling vector fields. We also note that, if an affine connection control system possesses decoupling vector fields $\{X_1, \dots, X_k\}$ such that the driftless system $(\mathbb{Q}, \{X_1, \dots, X_k\}, \mathbb{R}^k)$ is controllable, then one can often explicitly perform motion planning for the affine connection control system if one can do this for the driftless control system. This is explored in [Bullo and Lewis 2004, Chapter 13].

Additional insight into the connections between Theorems 4.6 and 4.10 can be obtained by the following observation [Bullo and Lewis 2004, Lemma 8.4]: A vector space-valued bilinear map $B: V \times V \rightarrow U$ is essentially indefinite if and only if there exists a basis (e_1, \dots, e_n) for V such that $B(e_j, e_j) = 0$ for $j \in \{1, \dots, n\}$.

We can also use the theorem to answer some of the questions from Section 1.3. Let us record these answers here, referring to the references for details.

1. [Bullo and Lewis 2003] The mobile robot system of Figure 1a is reducible to a driftless system, whereas the snakeboard system of Figure 1b is not, although the snakeboard does possess two decoupling vector fields.

2. [Bullo and Lewis 2005] The robotic leg of Figure 1c is reducible to a driftless system, whereas the hovercraft model of Figure 1d is not, although it does possess two decoupling vector fields.

5. Where to go next

The results from Sections 4.2 and 4.3 offer an elegant, thorough, and self-contained picture of controllability for affine connection control systems satisfying Assumption 4.4. There are no such results for systems not satisfying this assumption. It is worth reflecting, for a moment, on why this is so.

First of all, let us consider some features of the results we have, starting with Theorem 2.9 for accessibility of control-affine systems and Theorem 2.10 for controllability of driftless systems. Both of these theorems have to do with the rank of the vector distribution $L(\mathcal{F})$, and, at least in the real analytic case, this rank is independent of the vector fields \mathcal{F} that generate the distribution $D(\mathcal{F})$.² Thus the conditions of Theorems 2.9 and 2.10 are “geometric” and do not depend on the representation of the system by the vector fields \mathcal{F} . This attribute is inherited by our low-order controllability results of Theorem 4.6 and the reducibility results of Theorem 4.10. This is one reason why all of these results look “good.”

This feature of being “geometric” is one that does not hold up in much of the controllability literature (and, indeed, in much of the nonlinear control literature). This is not to say that there has not been a lot of deep, useful, and interesting work in the problem of nonlinear controllability (or, indeed, in much of the nonlinear control literature); there has been an enormous amount of such work. However, without being independent of representation, it is simply not possible to arrive at results that are comprehensive. This is because one will end up studying, not the system, but a particular representation of the system. This issue arises even at the level of lowly linearisation. To illustrate, consider the two 2-input control-affine systems in \mathbb{R}^3 with the governing equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), & \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), & \dot{x}_3(t) &= u_2(t). \end{aligned}$$

These two systems have the same trajectories. That is, if $(\boldsymbol{\xi}, \boldsymbol{\mu})$ is a controlled trajectory for one of these systems, then there exists a control $\boldsymbol{\mu}'$ such that $(\boldsymbol{\xi}, \boldsymbol{\mu}')$ is a controlled trajectory for the other. Both systems have the equilibrium $(\boldsymbol{x}_0, \boldsymbol{u}_0) = (\mathbf{0}, \mathbf{0})$. However, the system on the left does not have a controllable linearisation at this equilibrium, while that on the right does. It is quite easy to understand what is going on with this example, and to patch it to make oneself believe that everything is alright with life as we know it. However, an alternative idea is to instead listen to what the two systems are saying, and to ask the questions,

1. how does one properly define linearisation?
2. how does one determine when a linearisation is controllable?

²In the smooth case, the fact that the equality (2.2) does not generally hold arises because of the dependence of $L(\mathcal{F})$ on the specific vector fields \mathcal{F} one chooses to generate the distribution $D(\mathcal{F})$.

Moreover, if one reflects upon the problem with the examples, one sees that it is precisely the fact that one has two different representations of the same system that is the root of the problems. This then raises the even more basic question

3. what is a control system?

This is a question that one must correctly answer before one can address fundamental structural problems of control systems, such as controllability and stabilisability. These issues are discussed in [Lewis 2012a] and in [Lewis 2014, Chapter 1]. In the remainder of [Lewis 2014] can be found a development of a formulation of control theory devised expressly to overcome the problems of representation dependence pointed out above. This approach is highly technical, but it is our belief that it provides an answer to question 3 above in such a way that one can initiate a serious effort to address the unresolved structural problems in geometric control theory.

These fundamental structural problems have fallen out of vogue in control theory, and it is the author's opinion that this is regrettable. Indeed, one can see, by the results reviewed in this paper concerning control theory for mechanical systems, that sometimes thinking about these fundamental structural problems can lead one to interesting conclusions about "practical" control systems. Moreover, as also can be seen by our elegant resolution of the physically motivated problems of Section 1.3, applications can drive one to examine questions that may otherwise remain hidden. This interplay of deep theory and application has, we hope, been illustrated by the discussion in the paper.

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