Invariance of subbundles and nonholonomic trajectories

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Abstract

In the comparison of nonholonomic mechanics and constrained variational mechanics, invariant affine subbundles arise in the determination of the initial conditions where the two methods yield the same trajectories. Motivated by this, differential conditions are considered for invariant affine subbundles as they arise in the comparison of nonholonomic and constrained variational mechanics. First of all, the formal integrability of the resulting linear partial differential equation is determined using Spencer cohomology. Second, iterative formulae are provided that permit the determination of the largest invariant affine subbundle invariant under an affine vector field. Finally, the problem of a disc rolling on an inclined plane with no-slip is considered as an example to illustrate the theory.

Keywords. Nonholonomic mechanics, invariant subbundles, rolling disk

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1. Introduction

We warmly dedicate this paper to the memory of Professor Miguel Muñoz-Lecanda. Miguel was very kind to the first author early in his career; his friendship was greatly valued. The subject of this paper, nonholonomic mechanics, was one in which Professor Muñoz-Lecanda made a number of contributions, e.g., [de León, Lainz, and Muñoz-Lecanda 2021, Gràcia, Marín-Solano, and Muñoz-Lecanda 2003, Muñoz-Lecanda 2018]. Unlike mechanical systems with holonomic constraints, constrained variational trajectories for a system with nonholonomic constraints are not necessarily the ones obtained from nonholonomic mechanics. While the former has the appeal of a variational framework, the latter capture the physics correctly [Lewis 2017]. The case when constrained variational trajectories are nonholonomic trajectories has been considered extensively in the literature, for example in [Fernandez and Bloch 2008], [Jóźwikowski and Respondek 2019], [Lewis 2020], and [Lewis and

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Murray 1995]. As is well-known, the two approaches are equivalent—in the sense that they *always* yield the same physical trajectories—if and only if the constraint is holonomic. The interesting question, and the one we are concerned with in this paper, is when *some* nonholonomic trajectories arise from *some* constrained variational trajectory. For a review of the history of the subject, see [de León 2012].

In the recent exposition of Lewis [2020] using an affine connection formulation for the equations of the extermals, the two sets of trajectories were compared. Although restricted to kinetic energy minus potential energy Lagrangians, previous results for the classification of regular constrained variational trajectories as nonholonomic were recovered and, for the first time, a classification of singular constrained variational trajectories as nonholonomic trajectories was presented. The conditions arrived at in [Lewis 2020] are algebraic and differential in nature and are for the existence of a flow-invariant affine subbundle variety contained in a cogeneralized subbundle for the regular case and of a flow-invariant cogeneralized subbundle contained in a cogeneralized subbundle in the singular case (see Sections 2.2 and 2.3 for the definitions). Our objectives are to explore these conditions more deeply, and to understand how to apply them by considering a fairly simple, but yet illustrative, example.

The following is an outline of the paper.

1. We fix our notation and conventions in Section 2. In this section, we also review the relevant definitions and results from [Lewis 2020].
2. In Section 3 we present the equations governing nonholonomic and constrained variational trajectories, following the refinements of [Lewis 2020] of the initial work in [Kupka and Oliva 2001]. In this section we also indicate how the notions of invariant subbundles from Sections 2.2 and 2.3 arise in the comparison of nonholonomic and constrained variational trajectories.
3. In Section 4 we explore two aspects of the results of [Lewis 2020]: (a) the formal integrability of differential conditions for invariance; (b) the determination of computable infinitesimal characterisations of invariance.
4. We provide a complete geometric formulation of the problem of a disc rolling with no-slip over an inclined plane as suggested by Lemos [2022] in Section 5. For this example, we carry out a detailed analysis of this example *vis-à-vis* the question of comparing nonholonomic and constrained variational mechanics. As we shall see, our results allow for a *complete* characterization of the existence of invariant affine subbundle varieties, something which is typically not done doe to implicit assumptions about the absence of singularities.

2. Preliminaries

In this section we review some elementary geometric constructions for the purpose of fixing notation, and we review the relevant definitions and results of Lewis [2020] concerning subbundles and their invariance under flows of certain sorts of vector fields.

2.1. Notation and elementary constructions. To treat both smooth and real-analytic regularities, we let $r \in \{\infty, \omega\}$.

2.1. Given a linear connection ∇ on a vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$, the tangent bundle TE can be decomposed into horizontal and vertical bundles. These are denoted by HE and VE, respectively. Recall that the horizontal lift isomorphism is given by the map $\mathrm{hl} \colon \pi^*\mathsf{TM} \to \mathsf{HE}$ such that, for $e \in \mathsf{E}$,

$$T\pi(\mathrm{hl}(e, v)) = v, \quad v \in \mathsf{T}_{\pi(e)}\mathsf{M}.$$

The vertical lift isomorphism is the map $\mathrm{vl} : \mathsf{E} \oplus \mathsf{E} \to \mathsf{VE}$ such that, for $e \in \mathsf{E}$,

$$\mathrm{vl}(e, f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e + tf), \quad f \in \mathsf{E}_{\pi(e)}.$$

Let $X_0 \in \Gamma^r(\mathsf{TM})$. The horizontal lift of X_0 is the vector field $X_0^{\mathrm{h}} \in \Gamma^r(\mathsf{TE})$ defined by

$$X_0^{\mathrm{h}}(e) = \mathrm{hl}(e, X_0(\pi(e))), \quad e \in \mathsf{E}.$$

The vertical evaluation of $A \in \Gamma^r(\mathrm{End}(\mathsf{E}))$ is the vector field $A^{\mathrm{e}} \in \Gamma^r(\mathsf{TE})$ defined by

$$A^{\mathrm{e}}(e) = \mathrm{vl}(e, A(\pi(e))(e)), \quad e \in \mathsf{E}.$$

The horizontal lift of $f \in \mathrm{C}^r(\mathsf{M})$ is the pullback using π, that is, $f^{\mathrm{h}} = \pi^* f \in \mathrm{C}^r(\mathsf{E})$. The vertical evaluation of $\lambda \in \Gamma^r(\mathsf{E}^*)$ is the function $\lambda^{\mathrm{e}} \in \mathrm{C}^r(\mathsf{E})$ defined by

$$\lambda^{\mathrm{e}}(e) = \langle \lambda(\pi(e)), e \rangle, \quad e \in \mathsf{E}.$$

2.2. A vector field $X \in \Gamma^r(\mathsf{TE})$ is called a linear (resp. affine) vector field over X_0 if it is a C^r-vector (resp. affine) bundle map over X_0 such that the following diagram commutes

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{X} & \mathsf{TE} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ \mathsf{M} & \xrightarrow[X_0]{} & \mathsf{TM} \end{array}$$

If $X \in \Gamma^r(\mathsf{TE})$ is an affine vector field over X_0, then, using the linear connection ∇ in E, it can be decomposed as

$$X = X_0^{\mathrm{h}} + A^{\mathrm{e}} + b^{\mathrm{v}},$$

where $A \in \Gamma^r(\mathrm{End}(\mathsf{E}))$, and $b \in \Gamma^r(\mathsf{E})$. The case where b is the zero section corresponds to linear vector fields. It is easy to see that a smooth curve $\eta \colon I \to \mathsf{E}$ is an integral curve of $X_0^{\mathrm{h}} + A^{\mathrm{e}} + b^{\mathrm{v}}$ if and only if

$$\nabla_{\gamma'}\eta = A \circ \eta + b \circ \gamma, \tag{2.1}$$

where $\gamma = \pi \circ \eta$.

2.3. The flow of a vector field $Y \in \Gamma^r(\mathsf{TE})$ at $e \in \mathsf{E}$ is denoted by $\mathrm{Fl}_t^Y(e)$. The dual vector field of Y on the dual bundle E^*, is the vector field Y^* on E^* defined by

$$Y^*(\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{Fl}_{-t}^Y)^*(\lambda), \quad \lambda \in \mathsf{E}^*.$$

If Y is a linear vector field over X_0 then Y^* is linear vector field over X_0. Moreover, if $Y = X_0^{\mathrm{h}} + A^{\mathrm{e}}$, then $Y^* = X_0^{\mathrm{h},*} - (A^*)^{\mathrm{e}}$, where $X_0^{\mathrm{h},*}$ is the horizontal lift of X_0 corresponding to the dual connection on E^*.

We need a few elementary results regarding Lie differentiation. We record them in the following elementary lemma. See [Lewis 2020] for the proof.

2.4 Lemma. *Let $X_0 \in \Gamma^r(\mathsf{TM})$, let $A \in \Gamma^r(\mathrm{End}(\mathsf{E}))$, and let $b \in \Gamma^r(\mathsf{E})$. Then, for $k \in \mathbb{Z}_{>0}$, for $U \subseteq \mathsf{M}$ open, for $f \in \mathrm{C}^r(U)$, and for any local section λ of E^* over U, the following holds:*

(i) $\mathscr{L}^k_{X_0^{\mathrm{h}}} f^{\mathrm{h}} = (\mathscr{L}^k_{X_0} f)^{\mathrm{h}}$;
(ii) $\mathscr{L}^k_{A^{\mathrm{e}}} f^{\mathrm{h}} = \mathscr{L}^k_{b^{\mathrm{v}}} f^{\mathrm{h}} = 0$;
(iii) $\mathscr{L}^k_{X_0^{\mathrm{h}}} \lambda^{\mathrm{e}} = \big(\nabla^k_{X_0} \lambda\big)^{\mathrm{e}}$;
(iv) $\mathscr{L}^k_{A^{\mathrm{e}}} \lambda^{\mathrm{e}} = \big((A^*)^k \lambda\big)^{\mathrm{e}}$;
(v) $\mathscr{L}_{b^{\mathrm{v}}} \lambda^{\mathrm{e}} = \langle \lambda, b\rangle^{\mathrm{h}}$.

2.5. Let $\mathbb{G}$ be a C^r-Riemannian metric on M. Then $\mathbb{G}$ defines a musical isomorphism, the flat isomorphism $\mathbb{G}^\flat\colon \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$. This is defined by $\mathbb{G}^\flat(v_x)(u_x) = \mathbb{G}(u_x, v_x)$ for $u_x, v_x \in \mathsf{T}_x\mathsf{M}$. The inverse of $\mathbb{G}^\flat$ is denoted by $\mathbb{G}^\sharp$. We denote by $\overset{\mathbb{G}}{\nabla}$ the Levi-Civita connection associated with $\mathbb{G}$.

2.6. Let $\mathsf{D} \subseteq \mathsf{TM}$ be a C^r-distribution, i.e., a C^r-subbundle of TM. If $\mathbb{G}$ is a C^r-Riemannian metric on M, we denote the $\mathbb{G}$-orthogonal complement of D by $\mathsf{D}^\perp$. The $\mathbb{G}$-orthogonal projections are given by the vector bundle maps $P_{\mathsf{D}}\colon \mathsf{TM} \to \mathsf{D}$ and $P_{\mathsf{D}^\perp}\colon \mathsf{TM} \to \mathsf{D}^\perp$ projecting onto D and $\mathsf{D}^\perp$, respectively. The constrained connection is the affine connection $\overset{\mathsf{D}}{\nabla}$ on M that is defined by

$$\overset{\mathsf{D}}{\nabla}_X Y = \overset{\mathbb{G}}{\nabla}_X Y + (\overset{\mathbb{G}}{\nabla}_X P_{\mathsf{D}^\perp})Y, \quad X, Y \in \Gamma^r(\mathsf{TM}).$$

It follows that $\overset{\mathsf{D}}{\nabla}_X Y = P_{\mathsf{D}} \circ \overset{\mathbb{G}}{\nabla}_X Y$. The second fundamental form for D is the tensor field $S_{\mathsf{D}} \in \Gamma^r(\mathsf{T}^*\mathsf{M} \otimes \mathsf{D}^* \otimes \mathsf{D}^\perp)$ defined by

$$S_{\mathsf{D}}(X, Y) = -(\overset{\mathbb{G}}{\nabla}_X P_{\mathsf{D}^\perp})Y, \quad X \in \Gamma^r(\mathsf{TM}),\ Y \in \Gamma^r(\mathsf{D}).$$

The Frobenius curvature F_{D} is a tensor field $F_{\mathsf{D}} \in \Gamma^r(\bigwedge^2 \mathsf{D}^* \otimes \mathsf{D}^\perp)$ defined by

$$F_{\mathsf{D}}(X, Y) = S_{\mathsf{D}}(X, Y) - S_{\mathsf{D}}(Y, X) \quad X, Y \in \Gamma^r(\mathsf{D}).$$

The geodesic curvature G_{D} is a tensor field $G_{\mathsf{D}} \in \Gamma^r(S^2\mathsf{D}^* \otimes \mathsf{D}^\perp)$ defined by

$$G_{\mathsf{D}}(X,Y) = S_{\mathsf{D}}(X,Y) + S_{\mathsf{D}}(Y,X) \quad X,Y \in \Gamma^r(\mathsf{D}).$$

Related to the above definitions of F_{D} and G_{D}, we consider their transposes. We define $F_{\mathsf{D}}^\star, G_{\mathsf{D}}^\star \in \Gamma^r(\mathsf{D} \otimes \mathsf{D}^*)$ such that, for $\beta \in \Gamma^r(\mathsf{D}^\perp)$ and $X,Y \in \Gamma^r(\mathsf{D})$,

$$\mathbb{G}(X, F_{\mathsf{D}}^\star(\beta)(Y)) = \mathbb{G}(\beta, F_{\mathsf{D}}(X,Y)), \quad \mathbb{G}(X, G_{\mathsf{D}}^\star(\beta)(Y)) = \mathbb{G}(\beta, G_{\mathsf{D}}(X,Y))$$

holds. Similarly, we define the maps F_{D}^* and G_{D}^*:

$$\mathbb{G}(X, F_{\mathsf{D}}^*(Y)(\beta)) = \mathbb{G}(\beta, F_{\mathsf{D}}(X,Y)), \quad \mathbb{G}(X, G_{\mathsf{D}}^*(Y)(\beta)) = \mathbb{G}(\beta, G_{\mathsf{D}}(X,Y)).$$

This can be seen as flipping of arguments, that is

$$F_{\mathsf{D}}^\star(\beta)(Y) = F_{\mathsf{D}}^*(Y)(\beta), \quad G_{\mathsf{D}}^\star(\beta)(Y) = G_{\mathsf{D}}^*(Y)(\beta).$$

2.2. (Co)generalized (affine) subbundles and their invariance. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. We wish to consider generalizations of subbundles for which the rank is allowed to vary. The first sort of such a generalization is the following.

2.7 Definition. A *generalized subbundle* of E is a subset $\mathsf{F} \subseteq \mathsf{E}$ such that, for every $x \in \mathsf{M}$, there exists an open neighborhood U of x and a collection of sections $\{\xi_i\}_{i \in I} \subseteq \Gamma^r(\pi^{-1}(U))$ such that

$$\pi^{-1}(y) \cap \mathsf{F} = \operatorname{span}_{\mathbb{R}} \{\xi_i(y) \mid i \in I\}, \quad y \in U.$$

The local sections are called the *local generators* of F. Without loss of generality, these can be taken to be a finite collection of global sections of E [Lewis 2020, Corollary 2.18]. For each $x \in \mathsf{M}$, the subspace $\pi^{-1}(x) \cap \mathsf{F}$ is denoted by F_x and its annihilator by $\Lambda(\mathsf{F})_x \subseteq \mathsf{E}_x^*$. We denote the union of all annihilators by $\Lambda(\mathsf{F}) \subseteq \mathsf{E}^*$, that is, $\mathsf{E}_x^* \cap \Lambda(\mathsf{F}) = \Lambda(\mathsf{F})_x$. For any subset $S \subseteq \mathsf{M}$, the set $\mathsf{F} \cap \pi^{-1}(S)$ is denoted by $\mathsf{F}|S$.

Using annihilators, we can define another sort of a generalization of a subbundle.

2.8 Definition. A *cogeneralized subbundle* is a subset $\mathsf{F} \subseteq \mathsf{E}$ such that $\Lambda(\mathsf{F})$ is generalized subbundle of E^*.

For a (co)generalized subbundle $\mathsf{F} \subset \mathsf{E}$, a point $x \in \mathsf{M}$ is a *regular point* of F if there exists a neighborhood U of x such that $\mathsf{F}|U$ has constant rank. A *singular point* of F is a point which is not a regular point.

2.9 Remark. For any generalized or cogeneralized subbundle F, there exists an open dense set $U \subseteq \mathsf{M}$ consisting of regular points of F [Lewis 2020, Lemma 2.20]. That is, $\mathsf{F}|U$ is a subbundle of $\mathsf{E}|U$. In particular, all points are regular for subbundles of E.

As we will be working with affine vector fields, the notion of an invariant subbundle is of less interest than that of an invariant *affine* subbundle. Thus we need to define what we mean by an affine subbundle.

2.10 Definition. A *generalized affine subbundle* is a subset $\mathsf{B} \subset \mathsf{E}$ for which there exists a generalized subbundle $\mathsf{F} \subset \mathsf{E}$ and a section $\xi_0 \in \Gamma^r(\mathsf{TM})$ such that

$$\mathsf{B}_x := \mathsf{B} \cap \mathsf{E}_x = \xi_0(x) + \mathsf{F}_x.$$

One can phrase the definition of a generalized affine subbundle in terms of generators [Lewis 2020, Lemma 2.23].

There is also a cogeneralized version of an affine subbundle.

2.11 Definition. A *cogeneralized affine subbundle* is a subset $\mathsf{B} \subset \mathsf{E}$ for which there exists a cogeneralized subbundle $\mathsf{F} \subset \mathsf{E}$ and a section $\xi_0 \in \Gamma^r(\mathsf{TM})$ such that

$$\mathsf{B}_x := \mathsf{B} \cap \mathsf{E}_x = \xi_0(x) + \mathsf{F}_x.$$

Now we give the definitions for invariance of (co)generalized (affine) subbundles under affine vector fields. In [Lewis 2020], two types of invariance are discussed, "flow-invariance" and "invariance," and much of the work of the paper is devoted to giving the relationships between these. The notion of "flow-invariance" is exactly what one would expect: a set is *flow-invariant* under a vector field if the integral curves through points in the set remain in the set. The notion of "invariance" is more delicate, and involves the ideal sheaf of the set, i.e., the sheaf of functions that vanish on the set. Our interest is in (flow-)invariant sets that are (co)generalized (affine) subbundles. To this end, we make two comments.

1. For subbundles and affine subbundles, the ideal sheaf can be characterized by particular classes of functions, namely linear and affine functions, respectively.
2. The ideal sheaf is only meaningful for cogeneralized (affine) subbundles because these are naturally defined by the vanishing of suitable classes of functions.

We refer to [Lewis 2020, §4] for a detailed discussion of this. What we shall do here is give the results that characterize the invariant objects that are of interest to us here. To do so, we denote by $\mathscr{C}^r_{\mathsf{M}}$ the sheaf of C^r-functions on M and by $\mathscr{G}^r_{\mathsf{F}}$ the sheaf of sections of a generalized subbundle F.

The following is a characterization of flow-invariance in terms of algebraic and differential conditions. See [Lewis 2020, §4] for the details.

2.12 Proposition. *Let $\mathsf{B} = \xi_0 + \mathsf{F}$ be a cogeneralized affine subbundle. If B is flow-invariant under $X^{\mathrm{aff}} = X_0^{\mathrm{h}} + A^{\mathrm{e}} + b^{\mathrm{v}}$, then the following conditions hold:*

(i) $A(x) \in \mathrm{End}(\mathsf{F}_x)$ *for all* $x \in \mathsf{M}$*;*

(ii) $\nabla_{X_0}(\mathscr{G}^r_{\Lambda(\mathsf{F})}) \subseteq \mathscr{G}^r_{\Lambda(\mathsf{F})}$*;*

(iii) $(\nabla_{X_0}\xi_0 - A \circ \xi_0 - b)(x) \in \mathsf{F}_x$, $x \in \mathsf{M}$.

The converse is true in the real analytic case and if F *is a subbundle in the smooth case.*

The same result holds for generalized subbundles, but the differential condition is on F not its annihilator. That is, the second condition $\nabla_{X_0}(\mathcal{G}^r_{\mathsf{F}}) \subseteq \mathcal{G}^r_{\mathsf{F}}$. Note that the conditions of the proposition are differential/algebraic in nature. One of the goals of this paper is to show that the differential part of the result defines a linear partial differential equation that is formally integrable under certain hypotheses.

2.3. Affine subbundle varieties and their invariance. The essential problem of interest in [Lewis 2020] is the following: given a cogeneralized subbundle $\mathsf{F} \subset \mathsf{E}$ and an affine vector field X^{aff} on E, find a cogeneralized affine subbundle B that is (a) invariant under X^{aff} and (b) contained in F. The difficulty with this sort of problem is that the affine subbundle may be empty, or have nonempty fibers over a strict subset of M. To facilitate the analysis in this case, it turns out to be most convenient to recast the problem from one of solutions to linear equations to the linear equations themselves. In this way, one can talk about the equation, even when it has no solutions.

We begin with the algebraic setting.

2.13. Let V be a finite-dimensional $\mathbb{R}$-vector space, let $A \in \mathrm{End}(\mathsf{V})$, and let $b \in \mathsf{V}$. Consider the set

$$\mathrm{Sol}(A, b) = \{v \in \mathsf{V} \mid A(v) + b = 0\}.$$

Using the dual A^*, we define the subspace

$$\mathrm{Sol}^*(A, b) = \{(A^*(\lambda), \langle \lambda, b\rangle) \in \mathsf{V}^* \oplus \mathbb{R} \mid \lambda \in \mathsf{V}^*\}.$$

This subspace has a positive codimension in $\mathsf{V}^* \oplus \mathbb{R}$. Alternatively, if $\Delta \subseteq \mathsf{V}^* \oplus \mathbb{R}$ is a subspace of positive codimension, then $\Delta = \mathrm{Sol}^*(A, b)$ for some $A \in \mathrm{End}(\mathsf{V})$ and $b \in \mathsf{V}$. Furthermore, one can easily see that

$$\mathrm{Sol}(A, b) = \{v \in \mathsf{V} \mid (v, 1) \in \Lambda(\Delta)\}.$$

We remark that the set $\mathrm{Sol}(A, b)$ uniquely defines Δ, but does not uniquely define A and b [Lewis 2020, Lemma 2.29].

Now we adapt this to the geometric setting. By $\mathbb{R}_{\mathsf{M}}$ we denote the trivial line bundle over M.

2.14 Definition. (i) A generalized subbundle $\Delta \subseteq \mathsf{E}^* \oplus \mathbb{R}_{\mathsf{M}}$ is a *defining subbundle* if Δ_x has positive codimension in $\mathsf{E}^*_x \oplus \mathbb{R}$ for all $x \in \mathsf{M}$.

(ii) A subset $\mathsf{A} \subseteq \mathsf{E}$ is called an *affine subbundle variety* if there exists a defining subbundle Δ such that

$$\mathsf{A} = \mathsf{A}(\Delta) := \{e \in \mathsf{E} \mid \lambda(e) + a = 0,\ (\lambda, a) \in \Delta_{\pi(e)}\}.$$

(iii) The subset

$$\mathsf{S}(\mathsf{A}) = \{x \in \mathsf{M} \mid \mathsf{A} \cap \mathsf{E}_x \neq \emptyset\}$$

is called the *base variety* of A.

A defining subbundle Δ is *total* if $\mathsf{S}(\mathsf{A}(\Delta)) = \mathsf{M}$, *partial* if $\emptyset \neq \mathsf{S}(\mathsf{A}(\Delta)) \subsetneq \mathsf{M}$, and *null* if $\mathsf{S}(\mathsf{A}(\Delta)) = \emptyset$.

The idea of the terminology of total (resp. partial, null) is that the linear equation defined by Δ has solutions for all (resp. some, no) points in M.

Suppose that A is an affine subbundle variety. Let A_x denote the affine subspace $\mathsf{A} \cap \mathsf{E}_x$, for each $x \in \mathsf{S}(\mathsf{A})$. It follows that there exists a unique subspace $\Delta_x \subseteq \mathsf{E}_x^* \oplus \mathbb{R}$ such that $\mathsf{A}_x = \{e \in \mathsf{E}_x \mid (e, 1) \in \Lambda(\Delta_x)\}$. Let $\Delta_{1,x}$ denote the image of Δ_x under the projection

$$\mathsf{E}_x^* \oplus \mathbb{R} \to (\mathsf{E}_x^* \oplus \mathbb{R})/(\{0\} \oplus \mathbb{R}) \simeq \mathsf{E}_x^*,$$

and set $\Delta_1 = \cup_{x \in \mathsf{M}} \Delta_{1,x}$.

2.15. Before defining the flow-invariance of defining subbundles and affine subbundle varieties, we replace the affine vector field X^{aff} on E with a linear vector field $\widehat{X^{\mathrm{aff}}}$ on $\mathsf{E} \oplus \mathbb{R}_{\mathsf{M}}$. Let $X^{\mathrm{aff}} = X_0^{\mathrm{h}} + A^{\mathrm{e}} + b^{\mathrm{v}}$ be an affine vector field over X_0 and set $X^{\mathrm{lin}} = X_0^{\mathrm{h}} + A^{\mathrm{e}}$. Let $\widehat{\nabla}$ be the connection on $\mathsf{E} \oplus \mathbb{R}_{\mathsf{M}}$ obtained by taking the direct sum of ∇ in E and the flat connection in $\mathbb{R}_{\mathsf{M}}$. That is

$$\widehat{\nabla}_X(\xi, f) = (\nabla_X \xi, \mathscr{L}_X f), \quad X \in \Gamma^r(\mathsf{TM}),\ (\xi, f) \in \Gamma^r(\mathsf{E} \oplus \mathbb{R}_{\mathsf{M}}).$$

We define a linear vector field on $\mathsf{E} \oplus \mathbb{R}_{\mathsf{E}}$ by

$$\widehat{X^{\mathrm{aff}}} = X_0^{\mathrm{h}} + \widehat{(A, b)}^{\mathrm{e}},$$

where the horizontal lift of X_0 is done using $\widehat{\nabla}$, and $\widehat{(A, b)} \in \Gamma^r(\mathrm{End}(\mathsf{E} \oplus \mathbb{R}_{\mathsf{M}}))$ is defined by

$$\widehat{(A, b)}(\xi, f) = (A\xi + fb, 0), \quad (\xi, f) \in \Gamma^r(\mathsf{E} \oplus \mathbb{R}_{\mathsf{M}}).$$

We note that the dual of this linear vector field is

$$\widehat{X^{\mathrm{aff}}}^* = X_0^{\mathrm{h}} - \widehat{(A, b)}^{*,\mathrm{e}}.$$

The importance of the vector field $\widehat{X^{\mathrm{aff}}}$ is explained by the following result [Lewis 2020, Lemma 4.18].

2.16 Proposition. *Let Δ be a defining subbundle with A the associated affine subbundle variety. Then the following statements are equivalent:*

(i) $\mathsf{A} \subset \mathsf{E}$ *is flow-invariant under* X^{aff}*;*

(ii) $\{(e, 1) \in \mathsf{E} \times \mathbb{R}_{\mathsf{M}} \mid e \in \mathsf{A}\}$ *is flow-invariant under* $\widehat{X^{\mathrm{aff}}}$*;*

(iii) Δ *is flow-invariant under* $\widehat{X^{\mathrm{aff}}}^*$*.*

The previous result allows us to replace the quite subtle notion of invariance of an affine subbundle variety under an affine vector field with the notion of invariance of an affine subbundle variety under a linear vector field on an extended bundle, and then to replace this with the notion of invariance of a subbundle on the dual bundle of the extended bundle under a linear vector field. By working with Δ rather than $\mathsf{A}(\Delta)$, one does not have to concern oneself with whether $\mathsf{A}(\Delta)$ is nonempty when discussing invariance.

2.17. The constructions from the proceeding paragraph allow for a convenient test for invariance of affine subbundle varieties. As we mentioned at the beginning of this section, we are interested in invariant affine subbundle varieties that are contained in a cogeneralized subbundle F. To intertwine invariance and containment in F, we introduce the following notation for a cogeneralized subbundle F and a defining subbundle Δ:

$$\widehat{\Lambda(\Delta)} = \Lambda(\Delta) \cap (\mathsf{E} \times \{1\}), \quad \widehat{\mathsf{F}} = \{(e,1) \in \mathsf{E} \oplus \mathbb{R}_{\mathsf{M}} \mid e \in \mathsf{F}\}.$$

It is easy to see that the affine subbundle variety associated with Δ is

$$\mathsf{A} = \{e \in \mathsf{E} \mid (e,1) \in \widehat{\Lambda(\Delta)}\}$$

and that $\mathsf{A} \subset \mathsf{F}$ if and only if $\widehat{\Lambda(\Delta)} \subset \widehat{\mathsf{F}}$.

We need the following result [Lewis 2020, Theorem 4.26].

2.18 Proposition. *Let F be a cogeneralized subbundle and suppose that Δ is a defining subbundle that is flow-invariant under X^{aff}. If $\widehat{\Lambda(\Delta)} \subseteq \widehat{\mathsf{F}}$, then the following statements hold:*

(i) $A(\Lambda(\Delta_{1,x})) \subseteq \mathsf{F}_x$ for $x \in \mathsf{M}$;

(ii) $\nabla_{X_0}(\mathscr{G}^r_{\Lambda(\mathsf{F})}) \subseteq \mathscr{G}^r_{\Delta_1}$.

If $\widehat{\Lambda(\Delta)} \cap \widehat{\mathsf{F}} \neq \emptyset$, then the converse is true in the real analytic case and if F is a subbundle in the smooth case.

2.19 Remark. In particular, if Δ and F satisfy the hypothesis of Proposition 2.18 (including the hypothesis that $\widehat{\Lambda(\Delta)} \cap \widehat{\mathsf{F}} \neq \emptyset$), then $\mathsf{A}(\Delta)$ is flow-invariant and contained in F.

Following the previous characterization, we have the following definition for defining subbundles for which the associated affine subbundle variety is flow-invariant and contained in F.

2.20 Definition. A defining subbundle Δ is $(X^{\mathrm{aff}}, \mathsf{F})$-*linearly admissible* if it is flow-invariant under X^{aff} and the following conditions hold:

(i) $A(\Lambda(\Delta_{1,x})) \subseteq \mathsf{F}_x$ for $x \in \mathsf{M}$;

(ii) $\nabla_{X_0}(\mathscr{G}^r_{\Lambda(\mathsf{F})}) \subseteq \mathscr{G}^r_{\Delta_1}$.

If, additionally, $\widehat{\Lambda(\Delta)} \neq \emptyset$, then we say that Δ is $(X^{\mathrm{aff}}, \mathsf{F})$-*admissible*.

One of the objectives of the paper is to find a constructive means to find flow-invariant affine subbundle varieties.

3. Nonholonomic and constrained variational mechanics

In this section we give the governing differential equations for the two types of mechanics we are comparing, nonholonomic mechanics and constrained variational mechanics. We only consider mechanical systems with "kinetic minus potential energy" Lagrangians. These are certainly of significant interest, and also have enough structure to be able to draw deeper conclusions that one will be able to draw with general Lagrangians. The equations we present here are an evolution presented in [Lewis 2020], based on the work in [Kupka and Oliva 2001].

We begin with the definition of the mechanical systems we consider.

3.1 Definition. A C^r-*constrained (simple) mechanical system* is a quadruple $(\mathsf{M}, \mathbb{G}, V, \mathsf{D})$ of a C^r-manifold M, representing the configuration manifold, a C^r-Riemannian metric $\mathbb{G}$ representing the kinetic energy, a C^r-potential function V representing the potential energy, and a C^r-distribution representing the constraints.

Now we consider the two different equations of motion. Note that, in the case of constrained variational mechanics we have *two* cases for the equations of motion, corresponding to normal (called "regular" in the definition) and abnormal (called "singular" in the definition) extremals. The connection between these definitions and variational formulations is established by [Lewis 2020, Theorems 5.18 and 5.22].

3.2 Definition. Let $I \subset \mathbb{R}$ be an interval and let $\gamma\colon I \to \mathsf{M}$ be an absolutely continuous curve such that $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ a.e. on I.

(i) The curve γ is a *nonholonomic trajectory* if it is C^1 with an absolutely continuous derivative and

$$\overset{\mathsf{D}}{\nabla}_{\gamma'} \gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = 0. \tag{NH}$$

(ii) The curve γ is a D-*regular constrained variational trajectory* if it is C^1 with an absolutely continuous derivative and if there exists an absolutely continuous curve $\lambda\colon I \to \mathsf{D}^\perp$ over γ such that

$$\begin{aligned}
&\overset{\mathsf{D}}{\nabla}_{\gamma'} \gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = F^*_{\mathsf{D}}(\gamma')(\lambda),\\
&\overset{\mathsf{D}^\perp}{\nabla}_{\gamma'} \lambda = \frac{1}{2} G_{\mathsf{D}}(\gamma', \gamma') + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma + \frac{1}{2} G^\star_{\mathsf{D}^\perp}(\gamma')(\lambda) + \frac{1}{2} F^\star_{\mathsf{D}^\perp}(\gamma')(\lambda).
\end{aligned} \tag{RCV}$$

(iii) The curve γ is a D-*singular constrained variational trajectory* if there exists a nowhere zero, absolutely continuous curve $\lambda\colon I \to \mathsf{D}^\perp$ over γ such that

$$\begin{aligned}
&F^*_{\mathsf{D}}(\gamma')(\lambda) = 0,\\
&\overset{\mathsf{D}^\perp}{\nabla}_{\gamma'} \lambda = \frac{1}{2} G^\star_{\mathsf{D}^\perp}(\gamma')(\lambda) + \frac{1}{2} F^\star_{\mathsf{D}^\perp}(\gamma'), (\lambda).
\end{aligned} \tag{SCV}$$

The curve λ over γ is the *adjoint field*.

We are primarily interested in this paper with comparing nonholonomic trajectories with D-regular constrained variational trajectories. Note that the first of equations (RCV) agrees with the equation (NH) on the left-hand side, while the former has a term involving the Frobenius curvature on the right-hand side. Essentially, we wish to consider under what initial conditions for the second of equations (RCV) will the term on the right in the first equation vanish. Indeed, in such cases, the corresponding D-regular constrained variational trajectory will also be a nonholonomic trajectory.

To develop the framework for this, we pull back the second of the equations (RCV) to D to get a vector field over the vector field defining the nonholonomic trajectories. We let $\pi_{\mathsf{D}}\colon \mathsf{D}\to\mathsf{M}$ be the restriction of the tangent bundle projection. We let $\pi_{\mathsf{D}}^*\mathsf{D}$ and $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$ denote the pull-back bundles. We use $\pi_{\mathsf{D}}\colon \mathsf{D}\to\mathsf{M}$ to pull back the tensor fields F_{D}^*, $F_{\mathsf{D}^\perp}^\star$, and $G_{\mathsf{D}^\perp}^\star$ to get

$$\begin{aligned}
\hat{F}_{\mathsf{D}}^* :&\pi_{\mathsf{D}}^*\mathsf{D}^\perp \to \pi_{\mathsf{D}}^*\mathsf{D}\\
&(v_x,\alpha_x)\mapsto (v_x, F_{\mathsf{D}}^*(v_x)(\alpha_x)),\\
\hat{F}_{\mathsf{D}^\perp}^\star :&\pi_{\mathsf{D}}^*\mathsf{D}^\perp \to \pi_{\mathsf{D}}^*\mathsf{D}^\perp\\
&(v_x,\alpha_x)\mapsto (v_x, F_{\mathsf{D}^\perp}^\star(v_x)(\alpha_x)),\\
\hat{G}_{\mathsf{D}^\perp}^\star :&\pi_{\mathsf{D}}^*\mathsf{D}^\perp \to \pi_{\mathsf{D}}^*\mathsf{D}^\perp\\
&(v_x,\alpha_x)\mapsto (v_x, G_{\mathsf{D}^\perp}^\star(v_x)(\alpha_x)).
\end{aligned}$$

We also pull back the connection $\overset{\mathsf{D}^\perp}{\nabla}$ to get the connection $\overset{\mathsf{D}^\perp}{\nabla}{}^*$ in $\pi_{\mathsf{D}}^*\pi_{\mathsf{D}^\perp}\colon \pi_{\mathsf{D}}^*\mathsf{D}^\perp\to\mathsf{D}$ defined by

$$\overset{\mathsf{D}^\perp}{\nabla}{}^*_{\omega}\, \pi_{\mathsf{D}}^*\alpha = (v, \overset{\mathsf{D}^\perp}{\nabla}_{T_\nu\pi_{\mathsf{D}}(\omega)}\,\alpha),\quad \nu\in\mathsf{D},\omega\in\mathsf{T}_\nu\mathsf{D},\ \alpha\in\mathsf{D}.$$

Denote by $X_{\mathsf{D}}^{\mathrm{nh}}\in\Gamma^r(\mathsf{TD})$ the vector field whose integral curves are nonholonomic trajectories. Define $b_{\mathsf{D}}\in\Gamma^r(\pi_{\mathsf{D}}^*\mathsf{D}^\perp)$ and $A_{\mathsf{D}}\in\Gamma^r(\mathrm{End}(\pi_{\mathsf{D}}^*\mathsf{D}^\perp))$ by

$$b_{\mathsf{D}}(v_x) = \left(v_x, \frac{1}{2}G_{\mathsf{D}}(v_x,v_x) + P_{\mathsf{D}^\perp}\circ \mathrm{grad}V\right),\quad v_x\in\mathsf{D}, \tag{3.1}$$

and

$$A_{\mathsf{D}}(v_x,\alpha_x) = \left(v_x, \frac{1}{2}\hat{G}^\star_{D^\perp}(v_x,\alpha_x) + \frac{1}{2}\hat{F}^\star_{D^\perp}(v_x,\alpha_x)\right),\quad v_x\in\mathsf{D}, (v_x,\alpha_x)\in\pi_{\mathsf{D}}^*\mathsf{D}^\perp. \tag{3.2}$$

Then set

$$\begin{aligned}
X_{\mathsf{D}}^{\mathrm{sing}} &= (X_{\mathsf{D}}^{\mathrm{nh}})^{\mathrm{h}} + A_{\mathsf{D}}^{\mathrm{e}},\\
X_{\mathsf{D}}^{\mathrm{reg}} &= (X_{\mathsf{D}}^{\mathrm{nh}})^{\mathrm{h}} + A_{\mathsf{D}}^{\mathrm{e}} + b_{\mathsf{D}}^{\mathrm{v}},
\end{aligned}\tag{3.3}$$

where $(X_{\mathsf{D}}^{\mathrm{nh}})^{\mathrm{h}}\in\Gamma^r(\mathsf{T}(\pi_{\mathsf{D}}^*\mathsf{D}^\perp))$ is the horizontal lift of $X_{\mathsf{D}}^{\mathrm{nh}}$ by $\overset{\mathsf{D}^\perp}{\nabla}{}^*$.

Suppose that we have a curve γ satisfying (NH) and let $\Upsilon = \gamma'$. Let $\hat{\lambda}\colon I \to \pi_{\mathsf{D}}^*\mathsf{D}^\perp$ be a curve over Υ and write $\hat{\lambda} = (\Upsilon, \lambda)$. The second of the equations in (RCV) and (SCV) for the adjoint field λ can be written as

$$\overset{\mathsf{D}^\perp}{\nabla}{}^*_{\Upsilon'}\,\hat{\lambda} = A_{\mathsf{D}} \circ \hat{\lambda} + b_{\mathsf{D}} \circ \Upsilon \quad \text{and} \quad \overset{\mathsf{D}^\perp}{\nabla}{}^*_{\Upsilon'}\,\hat{\lambda} = A_{\mathsf{D}} \circ \hat{\lambda}, \tag{3.4}$$

respectively. Together with (NH), these equations are equivalent to $\hat{\lambda}$ being an integral curve for $X_{\mathsf{D}}^{\mathrm{sing}}$ or $X_{\mathsf{D}}^{\mathrm{reg}}$, respectively, that projects to the nonholonomic trajectory γ. By examining equations (3.4), we see that they resemble an affine and a linear differential equation, respectively. We can also make the following observations.

1. If $\hat{\lambda}(t) \in \ker(\hat{F}_{\mathsf{D}}^*)_{\Upsilon(t)}$ and $\hat{\lambda}$ is an integral curve of $X_{\mathsf{D}}^{\mathrm{reg}}$, then Υ is an integral curve of $X_{\mathsf{D}}^{\mathrm{nh}}$, and γ and λ satisfy (RCV).
2. If $\lambda(t) \neq 0$ for all $t \in I$, $\hat{\lambda}(t) \in \ker(\hat{F}_{\mathsf{D}}^*)_{\Upsilon(t)}$, and $\hat{\lambda}$ is an integral curve of $X_{\mathsf{D}}^{\mathrm{sing}}$ then Υ is an integral curve of $X_{\mathsf{D}}^{\mathrm{nh}}$, and γ and λ satisfy (SCV).

The following two results align with these observations [Lewis 2020, Theorems 7.8 and 7.9]. Indeed, we are looking for flow-invariant cogeneralized subbundles and affine subbundle varieties which are contained in $\ker(\hat{F}_{\mathsf{D}}^*)$.

3.3 Theorem. *Suppose that either (a) $r = \omega$ and $X_{\mathsf{D}}^{\mathrm{nh}}$ is complete or (b) $r = \infty$ and $\ker(\hat{F}_{\mathsf{D}}^*)$ is a subbundle. Then the following statements are equivalent:*

(i) some (resp. all) nonholonomic trajectories are D-regular constrained variational trajectories;

(ii) there exists a partial (resp. total) $(X_{\mathsf{D}}^{\mathrm{reg}}, \ker(\hat{F}_{\mathsf{D}}^*))$-admissible C^r-defining subbundle $\Delta \subseteq (\pi_{\mathsf{D}}^*\mathsf{D}^\perp)^* \oplus \mathbb{R}_{\mathsf{D}}$.

3.4 Theorem. *Suppose that either (a) $r = \omega$ and $X_{\mathsf{D}}^{\mathrm{nh}}$ is complete or (b) $r = \infty$ and $\ker(\hat{F}_{\mathsf{D}}^*)$ is a subbundle. Then the following are equivalent:*

(i) some nonholonomic trajectories are D-singular constrained variational trajectories;

(ii) there exists a flow-invariant cogeneralized subbundle under $X_{\mathsf{D}}^{\mathrm{sing}}$ that is contained in $\ker(\hat{F}_{\mathsf{D}}^*)$.

Together with the (flow-)invariance results stated in Sections 2.2 and 2.3, these results are of fundamental importance in understanding questions of when constrained variational trajectories are also nonholonomic trajectories.

4. The differential conditions for invariance of (co)generalized subbundles

As we remarked after the statement of Proposition 2.12, and as can also be seen in Proposition 2.18, the conditions for flow-invariance of an affine subbundle variety under an affine vector field involve differential and algebraic conditions. In this section we prove two results that are useful in applications of the general theory. The first result is concerned with the possibility that, upon differentiation of the differential condition, further algebraic conditions may reveal themselves. We give conditions under which this is not the case by using the theory of linear overdetermined partial differential equations developed in [Goldschmidt 1967] and presented in [Pommaret 1978]. The second result provides a concrete methodology for constructing an invariant affine subbundle variety.

4.1. Formal integrability. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle and let $\mathsf{F} \subset \mathsf{E}$ be a subbundle of E. We let ∇ be a linear connection on E and consider a linear vector field $X^{\mathrm{lin}} = X_0^{\mathrm{h}} + A^{\mathrm{e}}$ over $X_0 \in \Gamma^r(\mathsf{TM})$. We are concerned with the analogue for generalized subbundles of condition (ii) in Proposition 2.12. That is to say, we are concerned with the condition $\nabla_{X_0}(\mathscr{G}^r_{\mathsf{F}}) \subseteq \mathscr{G}^r_{\mathsf{F}}$. We let $p_{\mathsf{E}/\mathsf{F}}\colon \mathsf{E} \to \mathsf{E}/\mathsf{F}$ be the canonical projection, noting that E/F is a vector bundle as we are assuming that F is a subbundle. (We can just as well use the projection onto a complement of F in E.) In this case, the condition $\nabla_{X_0}(\mathscr{G}^r_{\mathsf{F}}) \subseteq \mathscr{G}^r_{\mathsf{F}}$ can be written as $P_{\mathsf{F}^\perp} \circ \nabla_{X_0}\xi = 0$ for $\xi \in \Gamma^r(\mathsf{F})$.

The next result gives two situations in which we have formal integrability of this linear partial differential equation in the real analytic case.

4.1 Proposition. *Let $\pi : \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle with a real analytic linear connection ∇ and fiber-wise inner product, let $X_0 \in \Gamma^\omega(\mathsf{TM})$, let $\mathsf{F} \subseteq \mathsf{E}$ be a C^ω-subbundle, and let $\mathsf{F}^\perp$ denote its (orthogonal) complement subbundle. Denote the canonical projection by $P_{\mathsf{F}^\perp}\colon \mathsf{E} \to \mathsf{F}^\perp$, and define a vector bundle map $\Phi\colon \mathsf{J}_1\mathsf{E} \to \mathsf{F}^\perp$ by $\Phi(j_1\xi(x)) = P_{\mathsf{F}^\perp}(\nabla_{X_0}(\xi(x)))$. Suppose that $\ker(\Phi)$ is a subbundle. Then $P_{\mathsf{F}^\perp} \circ \nabla_{X_0}$ is formally integrable if either of the following conditions holds:*

(i) X_0 is nowhere vanishing;
(ii) $P_{\mathsf{F}^\perp} \circ \nabla \circ P_{\mathsf{F}^\perp} \circ \nabla\xi \in S^2\mathsf{T}^*\mathsf{M} \otimes \mathsf{F}^\perp$ for every $\xi \in \Gamma^\omega(\mathsf{E})$ satisfying $\Phi \circ j_1\xi = 0$.

Proof. We shall first make some constructions that apply to both cases before considering each case separately.

Denote $\mathsf{R}_1 = \ker(\Phi) \subset \mathsf{J}_1\mathsf{E}$. Let $k \in \mathbb{Z}_{\geq 0}$. Let $\rho_k(\Phi)\colon \mathsf{J}^{k+1}\mathsf{E} \to \mathsf{J}^k\mathsf{F}^\perp$ be the kth prolongation of Φ and denote $\mathsf{R}_{k+1} = \ker(\rho_k(\Phi))$. Let $\sigma(\Phi)\colon \mathsf{T}^*\mathsf{M} \otimes \mathsf{E} \to \mathsf{F}^\perp$ be the symbol of Φ and denote by $\sigma_k(\Phi)\colon S^{k+1}\mathsf{T}^*\mathsf{M} \otimes \mathsf{E} \to S^k\mathsf{T}^*\mathsf{M} \otimes \mathsf{F}^\perp$ be the kth prolongation of the symbol (or, equivalently, the symbol of the kth prolongation). Let us give an explicit formula for $\sigma_k(\Phi)$. Denote by

$$\iota^k_{X_0}\colon S^{k+1}\mathsf{T}^*\mathsf{M} \to S^k\mathsf{T}^*\mathsf{M}$$

the contraction by X_0. The symbol of Φ is then $\sigma(\Phi) = \iota^0_{X_0} \otimes P_{\mathsf{F}^\perp}$ and the kth prolongation of the symbol is $\sigma_k(\Phi) = \iota^k_{X_0} \otimes P_{\mathsf{F}^\perp}$. The symbol of R_1 is then $\mathsf{G}_1 = \ker(\sigma(\Phi)) = \mathsf{T}^*\mathsf{M} \otimes \mathsf{F}$ and the symbol of R_{k+1} is $\mathsf{G}_{k+1} = \ker(\sigma_k(\Phi)) = S^{k+1}\mathsf{T}^*\mathsf{M} \otimes \mathsf{F}$.

We claim that G_1 is involutive. Indeed, the sequences

$$0 \longrightarrow S^m\mathsf{T}^*\mathsf{M}\otimes\mathsf{F} \xrightarrow{\delta} \mathsf{T}^*\mathsf{M}\otimes S^{m-1}\mathsf{T}^*\mathsf{M}\otimes\mathsf{F} \xrightarrow{\delta} \bigwedge^2\mathsf{T}^*\mathsf{M}\otimes S^{m-2}\mathsf{T}^*\mathsf{M}\otimes\mathsf{F} \xrightarrow{\delta} \cdots \\ \xrightarrow{\delta} \bigwedge^{m-1}\mathsf{T}^*\mathsf{M}\otimes\mathsf{T}^*\mathsf{M}\otimes\mathsf{F} \xrightarrow{\delta} \bigwedge^{m}\mathsf{T}^*\mathsf{M}\otimes\mathsf{F},$$

for $m\in\mathbb{Z}_{>0}$ are simply the δ-sequences for the vector bundle F, and these sequences are exact by the δ-Poincaré Lemma [Pommaret 1978, Proposition 3.1.5].

Since G_1 is involutive and G_2 is a subbundle, the only remaining ingredient to verifying the conditions of [Goldschmidt 1967, Theorem 4.1] is to show that R_2 projects surjectively onto R_1. We do this separately for the two conditions in the statement of the proposition. In each case, we work with the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S^2\mathsf{T}^*\mathsf{M}\otimes\mathsf{E} & \xrightarrow{\sigma_1(\Phi)} & \mathsf{T}^*\mathsf{M}\otimes\mathsf{F}^\perp & \xrightarrow{\tau} & \mathsf{K} \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \\
0 & \longrightarrow & \mathsf{R}_2 & \longrightarrow & \mathsf{J}_2\mathsf{E} & \xrightarrow{\rho_1(\Phi)} & \mathsf{J}_1\mathsf{F}^\perp & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{R}_1 & \longrightarrow & \mathsf{J}_1\mathsf{E} & \xrightarrow{\Phi} & \mathsf{F}^\perp & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Note that $P_{\mathsf{F}^\perp}\circ\nabla$ is a linear connection in $\mathsf{F}^\perp$, cf. the constrained connection from 2.6. Therefore, we can use this connection to give a splitting

$$\mathsf{J}_1\mathsf{F}^\perp \simeq \mathsf{F}^\perp\oplus(\mathsf{T}^*\mathsf{M}\otimes\mathsf{F}^\perp),$$

this being a simple case of [Lewis 2023b, Lemma 2.15]. Let $j_1\xi(x)\in\mathsf{R}_1$. Define a map $\kappa\colon\mathsf{R}_1\to\mathsf{K}$ by $\kappa(j_1\xi(x)) = \tau(A)$, where $A\in\mathsf{T}^*\mathsf{M}\otimes\mathsf{F}^\perp$ is such that $\epsilon(A) = \rho_1(\Phi)(j_2\xi(x))$. This is possible by exactness of the third column. To show that R_2 projects onto R_1, we need to show that η can be chosen so that $j_1\xi = j_1\eta$ and $j_2\eta(x)\in\mathsf{R}_2$. By [Pommaret 1978, Theorem 2.4.1], this is equivalent to $\kappa(j_1\xi(x)) = 0$. Therefore, to complete the proof we show that we can choose ξ so that $\kappa(j_1\xi(x)) = 0$ in each of the two cases from the statement of the proposition.

(i) Here we claim that the vector bundle map $\sigma_1(\Phi)$ is surjective. Indeed, let $\alpha\in\mathsf{T}^*\mathsf{M}\otimes\mathsf{F}$. We can consider a local trivialization given in terms of local sections $\{e_1,\dots,e_k\}$, where k is the rank of E, such that $\{e_1,\dots,e_m\}$ spans F, where m is the rank of F, and $\{e_{m+1},\dots,e_k\}$ spans $\mathsf{F}^\perp$. Moreover, we consider a chart for M such that the chart domain coincides with the neighbourhood associated with the trivialization. Since X_0 is nowhere vanishing by assumption, we can choose the chart for M so that $X_0 = \partial_1$. We can write $\alpha = \alpha^a_j \, dx^j \otimes e_a$. We want to find $A \in S^2\mathsf{T}^*\mathsf{M}\otimes\mathsf{E}$ such that $P_{\mathsf{F}^\perp}(A(X_0)) = \alpha$. We construct A locally. Trivially, we set $A^a_{ij} = 0$, for $a \in \{1, \dots, m\}$, and

$$A^a_{1j}X^1_0 = A^a_{j1} = \alpha^a_j,$$

for $a \in \{m + 1, \dots, k\}$ and $j \in \{1, \dots, \dim(\mathsf{M})\}$. These components give rise to a tensor A which is mapped to α by $\sigma_1(\Phi)$.

Now we note that, if $\sigma_1(\Phi)$ is surjective, then $\mathsf{K} = 0$, and so τ is the zero map. Therefore, $\kappa(j_1\xi(x)) = 0$ for any ξ for which $j_1\xi(x) \in \ker(\Phi)$.

(ii) In this case, we explain how to find $A \in \mathsf{T}^*\mathsf{M}\otimes\mathsf{F}^\perp$ so that $\epsilon(A) = \rho_1(\Phi)(j_2\xi(x))$ and $\tau(A) = 0$. We have

$$\rho_1(\Phi)(j_2\xi(x)) = j_1(P_{\mathsf{F}^\perp} \circ \nabla_{X_0}\xi(x))$$

by definition of prolongation. By noting that the fiber of $\mathsf{J}_1\mathsf{F}^\perp \simeq \mathsf{F}^\perp \oplus (\mathsf{T}^*\mathsf{M} \otimes \mathsf{F}^\perp)$ over $P_{\mathsf{F}^\perp}(\nabla_{X_0}\xi(x))$ is just $\mathsf{T}^*_x\mathsf{M} \otimes \mathsf{F}^\perp_x$, we get

$$\kappa(j_1\xi(x)) = \tau(P_{\mathsf{F}^\perp} \circ \nabla P_{\mathsf{F}^\perp}(\nabla_{X_0}\xi(x))),$$

since $P_{\mathsf{F}^\perp} \circ \nabla$ is a connection in $\mathsf{F}^\perp$. Thus $\kappa(j_1\xi(x)) = 0$ if and only if

$$P_{\mathsf{F}^\perp} \circ \nabla P_{\mathsf{F}^\perp}(\nabla_{X_0}\xi(x)) \in \ker(\tau) = \operatorname{image}(\sigma_1(\Phi)).$$

By definition of $\sigma_1(\Phi)$, this condition will be met when $P_{\mathsf{F}^\perp} \circ \nabla \circ P_{\mathsf{F}^\perp} \circ \nabla\xi$ is symmetric for ξ satisfying $\Phi \circ j_1\xi(x) = 0$. ■

4.2. Iterative infinitesimal constructions. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, let $\mathsf{F} \subset \mathsf{E}$ be a cogeneralized subbundle, and let X^{aff} be an affine vector field on E over a vector field X_0 on M. We suppose that we have a linear connection ∇ in E, and write $X^{\mathrm{aff}} = X^{\mathrm{h}}_0 + A^{\mathrm{e}} + b^{\mathrm{v}}$. Motivated by Theorem 3.3, we wish to determine "the largest X^{aff}-invariant affine subbundle variety contained in F." The existence of such an object is established in [Lewis 2020, Theorem 4.23]. However, the matter of constructing this affine subbundle variety is by no means clear. Here we provide an infinitesimal construction based on the following observations.

1. The cogeneralized subbundle F is defined by its annihilating generalized subbundle $\Lambda(\mathsf{F})$, in the sense that

$$\mathsf{F} = \{w \in \mathsf{E} \mid \lambda(w) = 0,\ \lambda \in \Gamma^r(\Lambda(\mathsf{F}))\} = \bigcap_{\lambda\in\Gamma^r(\Lambda(\mathsf{F}))} (\lambda^{\mathrm{e}})^{-1}(0).$$

   Therefore, the functions λ^{e} are in the ideal sheaf of any variety contained in F.
2. If $\lambda \in \Gamma^r(\Lambda(\mathsf{F}))$, then $\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}}$, $k \in \mathbb{Z}_{>0}$, will vanish on any flow-invariant variety contained in F.

3. The function $\mathscr{L}_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}}$, and more generally the functions $\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}}$, $k \in \mathbb{Z}_{>0}$, are affine functions on E, and so their zero level sets will be affine subbundle contained in F. The intersection of these zero level sets,
$$\bigcap_{k\in\mathbb{Z}_{\geq 0}} \bigcap_{\lambda\in\Gamma^r(\Lambda(\mathsf{F}))} (\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}})^{-1}(0), \tag{4.1}$$
will then be something like an affine subbundle variety invariant under X^{aff}. Because the intersection of affine subspaces is an affine subspace that is possibly empty, the projection of this affine subbundle to M might be a strict subset of M, or even empty.
4. Under suitable regularity hypotheses, the intersection (4.1) will define the sought after largest X^{aff}-invariant affine subbundle contained in F.

The difficulty with this procedure, and the one we address in this section, is to come up with a suitable characterization of the iterated Lie derivates of λ^{e} with respect to X^{aff}.

We note that
$$\mathscr{L}_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}} = ((\nabla_{X_0} + A^*)\,\lambda)^{\mathrm{e}} + \langle \lambda\circ\pi; b\rangle^{\mathrm{h}},$$
using Lemma 2.4. The following result records a recursive formulation of the iterated Lie derivatives..

4.2 Proposition. *Set $L_0 = \mathrm{id}$ and $c_0 = 0$. For $k \in \mathbb{Z}_{>0}$, recursively define L_k and c_k by*
$$L_k = (\nabla_{X_0} + A^*)^k \quad \text{and} \quad c_k(\lambda) = \sum_{j=1}^{k} \mathscr{L}^{j-1}_{X_0}\langle L_{k-j}\lambda, b\rangle\,,$$
for $U \subseteq \mathsf{M}$ open and $\lambda \in \mathscr{G}^r_{\mathsf{E}^*}(U)$. Then the following statements hold:

(i) $c_{k+l}(\lambda) = c_l(L_k(\lambda)) + \mathscr{L}^l_{X_0}c_k(\lambda)$, for $l,k\in\mathbb{Z}_{\geq 0}$;

(ii) $\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^e = (L_k(\lambda))^{\mathrm{e}} + (c_k(\lambda))^{\mathrm{h}}$, for $k \in \mathbb{Z}_{\geq 0}$.

Proof. (i) Compute
$$\begin{aligned} c_{k+l}(\lambda) &= \sum_{j=1}^{k+l} \mathscr{L}^{j-1}_{X_0}\langle L_{k+l-j}\lambda, b\rangle \\ &= \sum_{j=1}^{l} \mathscr{L}^{j-1}_{X_0}\langle L_{l-j}(L_k(\lambda)), b\rangle + \sum_{j'=1}^{k} \mathscr{L}^{l+j'-1}_{X_0}\langle L_{k-j'}(\lambda), b\rangle \\ &= c_l(L_k(\lambda)) + \mathscr{L}^l_{X_0}c_k(\lambda). \end{aligned}$$

(ii) We proceed by induction; for $k = 1$, we compute
$$\begin{aligned} \mathscr{L}_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}} &= (\nabla_{X_0}\lambda)^{\mathrm{e}} + (A^*\lambda)^{\mathrm{e}} + (\langle\lambda, b\rangle)^{\mathrm{h}} \\ &= (L_1(\lambda))^{\mathrm{e}} + (\langle\lambda, b\rangle)^{\mathrm{h}}. \end{aligned}$$

Suppose this is true for $k-1$ and compute

$$\begin{aligned}\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}} &= \mathscr{L}_{X^{\mathrm{aff}}}\left((L_{k-1}(\lambda))^{\mathrm{e}} + (c_{k-1}(\lambda))^{\mathrm{h}}\right)\\ &= (L_k(\lambda))^{\mathrm{e}} + \langle L_{k-1}(\lambda), b\rangle^{\mathrm{h}} + (\mathscr{L}_{X_0}c_{k-1}(\lambda))^{\mathrm{h}}\\ &= (L_k(\lambda))^{\mathrm{e}} + \langle L_{k-1}(\lambda), b\rangle^{\mathrm{h}} + (c_k(\lambda) - c_1(L_{k-1}\lambda))^{\mathrm{h}}\\ &= (L_k(\lambda))^{\mathrm{e}} + (c_k(\lambda))^{\mathrm{h}},\end{aligned}$$

where we use linearity of pullback, (i), and the definition of $c_1(\lambda)$. ■

Of course, the preceding result does not give a very concrete representation of the iterated Lie derivatives $\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}}$, but it is all one can expect. In terms of using these iterated Lie derivatives to find a flow-invariant affine subbundle variety, we have the following result.

4.3 Proposition. *Let $\mathsf{F} \subset \mathsf{E}$ be a C^r-subbundle and let $\lambda^1, \dots, \lambda^m \in \Gamma^r(\Lambda(\mathsf{E}))$ be generators for $\Gamma^r(\Lambda(\mathsf{F}))$ as a C^r-module. Suppose that there exists $N \in \mathbb{Z}_{>0}$ such that*

$$\bigcap_{k\in\mathbb{Z}_{\geq 0}} \bigcap_{\lambda\in\Gamma^r(\Lambda(\mathsf{F}))} (\mathscr{L}^k_{X^{\mathrm{aff}}}\lambda^{\mathrm{e}})^{-1}(0) = \mathsf{A}(X^{\mathrm{aff}}, \mathsf{F}) := \bigcap_{k=0}^{N}\bigcap_{j=1}^{m}(\mathscr{L}^k_{X^{\mathrm{aff}}}(\lambda^j)^{\mathrm{e}})^{-1}(0),$$

and that either $r = \omega$ or that $r = \infty$ and $\mathsf{A}(X^{\mathrm{aff}}, \mathsf{F})$ is a submanifold. Then $\mathsf{A}(X^{\mathrm{aff}}, \mathsf{F})$ is a flow-invariant affine subbundle variety.

Proof. By [Lewis 2020, Proposition 4.3], to show flow-invariance it suffices to show invariance. To show invariance, it suffices to show that $\mathscr{L}_{X^{\mathrm{aff}}}f$ vanishes on $\mathsf{A}(X^{\mathrm{aff}}, \mathsf{F})$ for

$$f \in \{\mathscr{L}^k_{X^{\mathrm{aff}}}(\lambda^j)^{\mathrm{e}} \mid k \in \{0, 1, \dots, N\},\ j \in \{1, \dots, m\}\}.$$

This, however, holds by hypothesis.

To show that $\mathsf{A}(X^{\mathrm{aff}}, \mathsf{F})$ is an affine subbundle variety, note that the functions $\mathscr{L}^k_{X^{\mathrm{aff}}}(\lambda^j)^{\mathrm{e}}$, $k \in \{0, 1, \dots, N\}$, $j \in \{1, \dots, m\}$, are affine functions, and so can be identified as sections of $\mathsf{E}^* \oplus \mathbb{R}$. Indeed, these are generators for a defining subbundle of $\mathsf{E}^* \oplus \mathbb{R}$ whose associated affine subbundle variety is exactly $\mathsf{A}(X^{\mathrm{aff}}, \mathsf{F})$. ■

The global generators hypothesized in the statement of the result exist by an appropriate version of the Serre–Swan Theorem [e.g., Lewis 2023a, Theorem 20].

5. Example: Rolling disc on an inclined plane

In this section, we consider a rolling disc over an inclined plane without slip, as depicted in Figure 5. A classical treatment of this example is given in [Lemos 2022]. As we shall see, by properly dealing with the singularity in the generalized subbundle

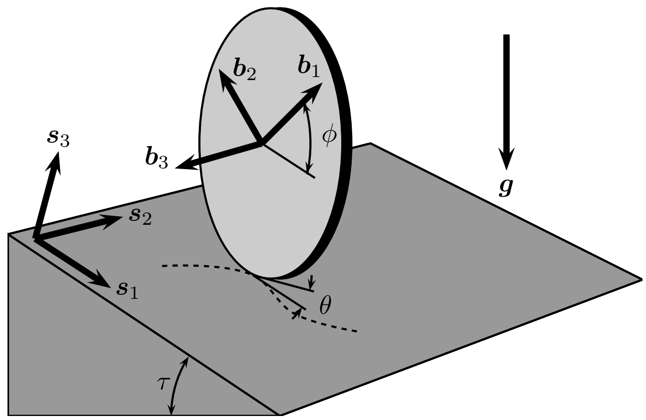

Figure 1: Disc rolling on an inclined plane

spanned by sections of D and their Lie brackets, we are able to refine the analysis in [Lemos 2022].

We shall first present a few of the calculations that are needed, and then apply the main results of the paper to draw conclusions concerning which constrained variational trajectories are also nonholonomic trajectories. We use MATHEMATICA® for symbolic computations. The modelling procedure we give is as outlined in Chapter 3 of [Bullo and Lewis 2004].

5.1. The spatial frame $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is chosen so that $\mathbf{s}_3$ is perpendicular to the plane on which the disc rolls. The spin angle is denoted by θ, the rolling angle is denoted by ϕ, and the inclination angle for the plane is denoted by τ. The body frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is located at the geometric center of the disc, with $\mathbf{b}_3$ chosen to be the axis about which the disc rolls.

5.2. From the problem setup, the configuration manifold is

$$\mathsf{Q} = \left\{ \begin{bmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) & \sin(\theta) \\ \cos(\phi)\sin(\theta) & -\sin(\phi)\sin(\theta) & -\cos(\theta) \\ \sin(\phi) & \cos(\phi) & 0 \end{bmatrix}, (x, y, R) \right\} \subseteq \mathrm{SO}(3) \times \mathbb{R}^3.$$

It is easy to see that $\mathsf{Q} \simeq \mathbb{T}^2 \times \mathbb{R}^2$. We carry out our calculations in a single chart with coordinates $(\theta, \phi, x, y) \in (-\pi, \pi)^2 \times \mathbb{R}^2$. We abuse the notation and identify the chart domain with Q. We will denote a typical point in Q as q. It will be convenient at times to use the coordinate functions as functions, and write $x(q)$ or $\phi(q)$, for example.

5.3. We assume that the inertia tensor of the disc is represented by a diagonal matrix $\mathbb{I}$ given by $\mathbb{I}_{11} = \mathbb{I}_{22} = J_s$ and $\mathbb{I}_{33} = J_r$, where J_s denotes the spinning principal inertia associated with $\mathbf{b}_1$ and $\mathbf{b}_2$, and J_r is the principal inertia related to rolling. We assume that $\mathbf{b}_3$ is an axis of symmetry of the body. We denote the mass of the disc by m.

One then readily ascertains that

$$\mathbb{G} = m dx \otimes dx + m dy \otimes dy + J_s d\theta \otimes d\theta + J_r d\phi \otimes d\phi.$$

Clearly all Christoffel symbols for the associated Levi-Civita connection vanish.

5.4. The potential energy is given by $V(\theta, \phi, x, y) = mg(R - x\sin(\tau))$, where g is the acceleration due to gravity. Hence we compute

$$\text{grad } V = \mathbb{G}^{\#} \circ dV = -\frac{g\sin(\tau)}{m}\partial_x.$$

5.5. By assumption, the disc rolls with no-slip on the inclined plan. Hence, for any non-holonomic trajectory $t \mapsto (\theta(t), \phi(t), x(t), y(t))$, we must have

$$\begin{aligned} \dot{x}(t) &= R\dot{\phi}(t)\cos(\theta(t)), \\ \dot{y}(t) &= R\dot{\phi}(t)\sin(\theta(t)). \end{aligned}$$

We define a vector bundle map $F\colon \mathsf{TQ} \to \mathsf{Q} \times \mathbb{R}^2$ by

$$F(\theta, \phi, x, y, v_\theta, v_\phi, v_x, v_y) = (\theta, \phi, x, y, v_x - Rv_\phi\cos(\theta), v_y - Rv_\phi\sin(\theta)).$$

We set $\mathsf{D} = \ker(F)$. It is clear that D has constant rank and D is the constraint distribution.

5.6. Let us determine a suitable basis for TQ adapted to the distribution D and its $\mathbb{G}$-orthogonal complement $\mathsf{D}^\perp$. By construction, the vector fields

$$\{\partial_\theta, \partial_\phi + R\cos(\theta)\partial_x + R\sin(\theta)\partial_y\}$$

span D. In fact, one can see that the given vector fields are $\mathbb{G}$-orthogonal. One can thus normalize to obtain $\mathbb{G}$-orthonormal vector fields $X_1, X_2 \in \Gamma^r(\mathsf{D})$ spanning D. That is,

$$X_1(\theta, \phi, x, y) = \frac{1}{\sqrt{J_s}}\partial_\theta$$

and

$$X_2(\theta, \phi, x, y) = \frac{1}{\sqrt{J_r + mR^2}}\left(\partial_\phi + R\cos(\theta)\partial_x + R\sin(\theta)\partial_y\right).$$

A convenient orthonormal basis for $\mathsf{D}^\perp$ is verified to be given by $\{X_3, X_4\}$ with

$$X_3(\theta, \phi, x, y) = \frac{\sin(\theta)}{\sqrt{m}}\partial_x - \frac{\cos(\theta)}{\sqrt{m}}\partial_y$$

and

$$X_4(\theta, \phi, x, y) = \frac{\sqrt{J_r}\cos(\theta)}{\sqrt{m}\sqrt{J_r + mR^2}}\partial_x + \frac{\sqrt{J_r}\sin(\theta)}{\sqrt{m}\sqrt{J_r + mR^2}}\partial_x - \frac{\sqrt{m}}{\sqrt{J_r}\sqrt{J_r + mR^2}}\partial_\phi.$$

5.7. The bases $\{X_1, X_2\}$ and $\{X_3, X_4\}$ for D and $\mathsf{D}^\perp$, respectively, allow us to introduce coordinates for D and $\mathsf{D}^\perp$. We write a typical point in D as $v_s X_1 + v_r X_2$, where v_s is the "spin velocity" and v_r is the "roll velocity." Thus we use coordinates $(\theta, \phi, x, y, v_r, v_s)$ for D.

We will denote a typical point in D by v and, as we indicated above for coordinates for Q, we will think of coordinates for D as functions on D. Thus we will sometimes write $y(v)$, or $\theta(v)$, or $v_r(v)$, for example.

5.8. Note that the basis vector fields for D and $\mathsf{D}^\perp$ give basis vector fields for the pull-back bundles $\pi_{\mathsf{D}}^*\mathsf{D}$ and $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$, and we denote these bases by

$$\{\pi_{\mathsf{D}}^* X_1, \pi_{\mathsf{D}}^* X_2\}, \quad \{\pi_{\mathsf{D}}^* X_3, \pi_{\mathsf{D}}^* X_4\},$$

respectively. There are also dual bases that we denote by

$$\{\pi_{\mathsf{D}}^* X_1^*, \pi_{\mathsf{D}}^* X_2^*\}, \quad \{\pi_{\mathsf{D}}^* X_3^*, \pi_{\mathsf{D}}^* X_4^*\}.$$

Using the basis for $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$, we write a typical point in $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$ as $p_1 \pi_{\mathsf{D}}^* X_3 + p_2 \pi_{\mathsf{D}}^* X_4$. This gives coordinates $(\theta, \phi, x, y, v_s, v_r, p_1, p_2)$ for $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$. A typical point in $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$ we denote by p, and we think of the coordinates as functions on $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$. Thus we may write $x(p)$, or $\phi(p)$, or $v_s(p)$, or $p_1(p)$.

5.9. We calculate

$$\begin{aligned}
\overset{\mathbb{G}}{\nabla}_{X_1} X_2 &= -\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_3,\\
\overset{\mathbb{G}}{\nabla}_{X_1} X_3 &= \frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_2 + \frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_4,\\
\overset{\mathbb{G}}{\nabla}_{X_1} X_4 &= \frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_3,
\end{aligned}$$

and all other covariant derivatives of the basis vector fields are zero. From these computations, we make a few immediate conclusions.

1. The Christoffel symbols of the linear connection $\overset{\mathsf{D}}{\nabla} = P_{\mathsf{D}} \circ \overset{\mathbb{G}}{\nabla}$ in D with respect to the frames $\{X_1, X_2, X_3, X_4\}$ for TQ and $\{X_1, X_2\}$ for D are zero.
2. The Christoffel symbols of the linear connection $\overset{\mathsf{D}^\perp}{\nabla} = P_{\mathsf{D}^\perp} \circ \overset{\mathbb{G}}{\nabla}$ in $\mathsf{D}^\perp$ with respect to the frames $\{X_1, X_2, X_3, X_4\}$ for TQ and $\{X_3, X_4\}$ $\mathsf{D}^\perp$ are determined by

$$\overset{\mathsf{D}^\perp}{\nabla}_{X_1} X_3 = \frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_4, \quad \overset{\mathsf{D}^\perp}{\nabla}_{X_1} X_4 = \frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_3.$$

3. The Frobenius curvature F_{D} is determined by

$$F_{\mathsf{D}}(X_1, X_2) = -F_D(X_2, X_1) = -\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_3.$$

4. The geodesic curvature G_{D} is determined by

$$G_{\mathsf{D}}(X_1, X_2) = G_D(X_2, X_1) = -\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}} X_3.$$

5. The Frobenius and geodesic curvatures $F_{\mathsf{D}^\perp}$ and $G_{\mathsf{D}^\perp}$ both vanish; thus $\mathsf{D}^\perp$ is integrable and geodesically invariant for $\overset{\mathbb{G}}{\nabla}$.

5.10. Let us next calculate $\hat{F}^*_{\mathsf{D}}$, which is a vector bundle map from $\pi^*_{\mathsf{D}}\mathsf{D}^\perp$ to $\pi^*_{\mathsf{D}}\mathsf{D}$. We represent this bundle map as a 2×2 matrix representing this map in the bases $\{\pi^*_{\mathsf{D}}X_3, \pi^*_{\mathsf{D}}X_4\}$ and $\{\pi^*_{\mathsf{D}}X_1, \pi^*_{\mathsf{D}}X_2\}$. Using the computation of F_{D} from above, we ascertain that

$$[\hat{F}^*_{\mathsf{D}}] = \begin{bmatrix} -\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r+mR^2}} v_r & 0 \\ \frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r+mR^2}} v_s & 0 \end{bmatrix}.$$

We observe that

$$\ker(\hat{F}^*_{\mathsf{D}})_v = \begin{cases} \operatorname{span}\{\pi^*_{\mathsf{D}}X_4\}, & v_s^2 + v_r^2 \neq 0, \\ (\pi^*_{\mathsf{D}}\mathsf{D}^\perp)_v, & v_s = v_r = 0, \end{cases}$$

where $v \in \mathsf{D}$.

5.11. Let us next calculate A_{D} and b_{D}. Since $G_{\mathsf{D}^\perp}$ and $F_{\mathsf{D}^\perp}$ are zero, A_{D} is also zero. We note that b_{D} is a section of the pullback bundle $\pi^*_{\mathsf{D}}\mathsf{D}^\perp$. We represent b_{D} in terms of its components in the basis provided by $\pi^*_{\mathsf{D}}X_3$ and $\pi^*_{\mathsf{D}}X_4$. A calculation gives

$$b_{\mathsf{D}} = -\left(\sqrt{m}g\sin(\tau)\sin(\theta) + \frac{2R}{\sqrt{J_s}\sqrt{J_r + mR^2}} v_s v_r\right)\pi^*_{\mathsf{D}}X_3 - \frac{g\sqrt{m}\sqrt{J_r}}{\sqrt{J_r + mR^2}}\sin(\tau)\cos(\theta)\pi^*_{\mathsf{D}}X_4.$$

5.12. Let us determine the vector field $X^{\mathrm{nh}}_{\mathsf{D}}$ using the "Poincaré representation" explained in [Bullo and Lewis 2004, §4.6.4]. We can use the computations above to ascertain that

$$\begin{aligned} &X^{\mathrm{nh}}_{\mathsf{D}}(\theta, \phi, x, y, v_s, v_r) \\ &= \frac{v_s}{\sqrt{J_s}}\partial_\theta + \frac{v_r}{\sqrt{J_r + mR^2}}\partial_\phi + \frac{R\cos(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_x + \frac{R\sin(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_y + \frac{mgR\sin(\tau)\cos(\theta)}{\sqrt{J_r + mR^2}}\partial_{v_r}. \end{aligned}$$

Thus the associated differential equations are

$$\begin{aligned}
\dot{x}(t) &= \frac{R\cos(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{y}(t) &= \frac{R\sin(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{\theta}(t) &= \frac{v_s(t)}{\sqrt{J_s}}, \\
\dot{\phi}(t) &= \frac{v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{v}_s(t) &= 0, \\
\dot{v}_r(t) &= \frac{mgR\sin(\tau)\cos(\theta(t))}{\sqrt{J_r + mR^2}}.
\end{aligned}$$

5.13. We can similarly ascertain the equations governing the regular constrained variational trajectories. Here we again use the Poincaré representation for the vector fields. We compute the vector field on $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$ representing the dynamics (RCV) as

$$\begin{aligned}
X_{\mathsf{D}}^{\mathrm{rcv}} = {} & \frac{R\cos(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_x + \frac{R\sin(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_y + \frac{v_s}{\sqrt{J_s}}\partial_\theta + \frac{v_r}{\sqrt{J_r + mR^2}}\partial_\phi \\
& - \frac{\sqrt{m}v_r p_1}{\sqrt{J_s}\sqrt{J_r + mR^2}}\partial_{v_s} + \frac{mgR\sqrt{J_s}\cos(\theta)\sin(\tau) + \sqrt{m}v_s p_1}{\sqrt{J_s}\sqrt{J_r + mR^2}}\partial_{v_r} \\
& + \left(\frac{\sqrt{J_r}v_s p_2}{\sqrt{J_s}\sqrt{J_r + mR^2}} - \frac{2\sqrt{m}Rv_s v_r}{\sqrt{J_s}\sqrt{J_r + mR^2}} - \sqrt{m}g\sin(\tau)\sin(\theta)\right)\partial_{p_1} \\
& - \left(\frac{\sqrt{J_r}v_s p_1}{\sqrt{J_s}\sqrt{J_r + mR^2}} + \frac{\sqrt{m}\sqrt{J_r}g\sin(\tau)\cos(\theta)}{\sqrt{J_r + mR^2}}\right)\partial_{p_2}.
\end{aligned}$$

The differential equations are then computed to be

$$\begin{aligned}
\dot{x}(t) &= \frac{R\cos(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}},\\
\dot{y}(t) &= \frac{R\sin(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}},\\
\dot{\theta}(t) &= \frac{v_s(t)}{\sqrt{J_s}},\\
\dot{\phi}(t) &= \frac{v_r(t)}{\sqrt{J_r + mR^2}},\\
\dot{v}_s(t) &= -\frac{\sqrt{m}Rv_r(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}},\\
\dot{v}_r(t) &= \frac{mgR\sqrt{J_s}\cos(\theta(t))\sin(\tau) + \sqrt{m}Rv_s(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}},\\
\dot{p}_1(t) &= \frac{\sqrt{J_r}v_s(t)p_2(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}} - \frac{2\sqrt{m}Rv_s(t)v_r(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}} - \sqrt{m}g\sin(\tau)\sin(\theta(t)),\\
\dot{p}_2(t) &= -\frac{\sqrt{J_r}v_s(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}} - \frac{\sqrt{m}\sqrt{J_r}g\sin(\tau)\cos(\theta(t))}{\sqrt{J_r + mR^2}}.
\end{aligned}$$

5.14. One can also ascertain that the singular constrained trajectories as prescribed by (SCV) are determined by curves $t \mapsto \gamma(t) \in \mathsf{Q}$ and nonzero sections $t \mapsto \lambda(t)$ of $\mathsf{D}^\perp$ over γ satisfying the algebraic equation

$$\begin{aligned}
\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}}v_r(t)p_1(t) = 0,\\
\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}}v_s(t)p_2(t) = 0
\end{aligned}$$

(of course, one can eliminate the coefficient $\frac{\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r+mR^2}}$, but we elect to keep it because it reminds us where these conditions come from), and the differential equation

$$\begin{aligned}
\dot{p}_1(t) &= \frac{\sqrt{J_r}v_s(t)p_2(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}},\\
\dot{p}_2(t) &= -\frac{\sqrt{J_r}v_s(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}}.
\end{aligned}$$

5.15. We now use Proposition 4.3 to find the largest $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety of $\pi_{\mathsf{D}}^*\mathsf{D}^\perp$ contained in $\ker(\hat{F}_{\mathsf{D}}^*)$, as required by Theorem 3.3 to determine the regular constrained variational trajectories that are also nonholonomic trajectories.

We have

$$X_{\mathsf{D}}^{\text{reg}} = \frac{R\cos(\theta)v_r}{\sqrt{J_r+mR^2}}\partial_x + \frac{R\sin(\theta)v_r}{\sqrt{J_r+mR^2}}\partial_y + \frac{v_s}{\sqrt{J_s}}\partial_\theta + \frac{v_r}{\sqrt{J_r+mR^2}}\partial_\phi + \frac{mgR\sqrt{J_s}\sin(\tau)\cos(\theta)}{\sqrt{J_s}\sqrt{J_r+mR^2}}\partial_{v_r}$$
$$+ \left(\frac{\sqrt{J_r}v_sp_2}{\sqrt{J_s}\sqrt{J_r+mR^2}} - \frac{2\sqrt{m}Rv_sv_r}{\sqrt{J_s}\sqrt{J_r+mR^2}} - \sqrt{m}g\sin(\tau)\sin(\theta)\right)\partial_{p_1}$$
$$- \left(\frac{\sqrt{J_r}v_sp_1}{\sqrt{J_s}\sqrt{J_r+mR^2}} + \frac{\sqrt{m}\sqrt{J_r}g\sin(\tau)\cos(\theta)}{\sqrt{J_r+mR^2}}\right)\partial_{p_2},$$

and the associated differential equations are

$$\begin{aligned}
\dot{x}(t) &= \frac{R\cos(\theta(t))v_r(t)}{\sqrt{J_r+mR^2}},\\
\dot{y}(t) &= \frac{R\sin(\theta(t))v_r(t)}{\sqrt{J_r+mR^2}},\\
\dot{\theta}(t) &= \frac{v_s(t)}{\sqrt{J_s}},\\
\dot{\phi}(t) &= \frac{v_r(t)}{\sqrt{J_r+mR^2}},\\
\dot{v}_s(t) &= 0,\\
\dot{v}_r(t) &= \frac{mgR\sqrt{J_s}\sin(\tau)\cos(\theta(t))}{\sqrt{J_s}\sqrt{J_r+mR^2}},\\
\dot{p}_1(t) &= \frac{\sqrt{J_r}v_s(t)p_2(t)}{\sqrt{J_s}\sqrt{J_r+mR^2}} - \frac{2\sqrt{m}Rv_s(t)v_r(t)}{\sqrt{J_s}\sqrt{J_r+mR^2}} - \sqrt{m}g\sin(\tau)\sin(\theta(t)),\\
\dot{p}_2(t) &= -\frac{\sqrt{J_r}v_s(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r+mR^2}} - \frac{\sqrt{m}\sqrt{J_r}g\sin(\tau)\cos(\theta(t))}{\sqrt{J_r+mR^2}}.
\end{aligned} \tag{5.1}$$

We note that

$$\ker(\hat{F}_{\mathsf{D}}^*) = (\lambda^{\mathrm{e}})^{-1}(0),$$

where $\lambda \in (\pi_{\mathsf{D}}^*\mathsf{D}^\perp)^*$ is given by

$$\lambda = (v_s^2+v_r^2)\pi_{\mathsf{D}}^*X_3^*.$$

One computes

$$\mathscr{L}_{X_{\mathsf{D}}^{\text{reg}}}\lambda^{\mathrm{e}} = \frac{2mgR\sin(\tau)\cos(\theta)}{\sqrt{J_r+mR^2}}v_rp_1$$
$$+ (v_s^2+v_r^2)\left(\frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r+mR^2}}v_sp_2 - \sqrt{m}g\sin(\tau)\sin(\theta) - \frac{2\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r+mR^2}}v_sv_r\right).$$

One can also compute $\mathscr{L}^k_{X_{\mathsf{D}}^{\text{reg}}}\lambda^{\mathrm{e}}$ for $k \geq 2$, but the symbolic expressions are too unwieldy to record.

Our strategy for finding invariant affine subbundle varieties is the following. For $k \in \mathbb{Z}_{\geq 0}$, denote

$$\mathsf{A}_k(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) = \bigcap_{j=0}^{k} (\mathscr{L}_{X_{\mathsf{D}}^{\mathrm{reg}}}^{k} \lambda^{\mathrm{e}})^{-1}(0).$$

We shall sequentially consider conditions on $p \in \pi_{\mathsf{D}}^{*}\mathsf{D}^{\perp}$ that ensure that $p \in \mathsf{A}_k(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}})$, $k \in \mathbb{Z}_{\geq 0}$. We do this explicitly and in detail for $k \in \{0, 1\}$, and then use our conclusions, along with a recording of conclusions deduced from MATHEMATICA® and the differential equations (5.1), to finalize the conditions on p.

R1. We first take $\sin(\tau) \neq 0$. If $p \in \mathsf{A}_0(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}})$, then either (a) $p_1(p) = 0$ or (b) $v_s(p) = v_r(p) = 0$. We consider these possibilities in turn.

(a) If $p_1(p) = 0$, then $p \in \mathsf{A}_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}})$ if and only if either (i) $v_s(p) = v_r(p) = 0$ or (ii)

$$\frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}} v_s(p) p_2(p) - \sqrt{m} g \sin(\tau)\sin(\theta(p)) - \frac{2\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}} v_s(p) v_r(p) = 0.$$

Let us denote by $\zeta(p)$ the expression on the left in the preceding equation. We consider the preceding possibilities in turn.

(i) If

$$p \in \mathsf{A}'_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) := \{p' \in \pi_{\mathsf{D}}^{*}\mathsf{D}^{\perp} \mid p_1(p') = 0,\ v_s(p') = 0,\ v_r(p') = 0\},$$

then we calculate

$$\begin{aligned} \mathscr{L}_{X_{\mathsf{D}}^{\mathrm{reg}}}^{2} \lambda^{\mathrm{e}}(p) &= 0, \\ \mathscr{L}_{X_{\mathsf{D}}^{\mathrm{reg}}}^{3} \lambda^{\mathrm{e}}(p) &= -\frac{6m^{5/2}g^3R^2\sin(\tau)^3}{J_r + mR^2}\cos(\theta(p))^2\sin(\theta(p)). \end{aligned}$$

This last expression vanishes exactly when either $\sin(\theta(p)) = 0$ or $\cos(\theta(p)) = 0$. If we take

$$\begin{aligned} \mathsf{A} = &\{p \in \pi_{\mathsf{D}}^{*}\mathsf{D}^{\perp} \mid p_1(p) = 0,\ v_s(p) = 0,\ v_r(p) = 0,\ \sin(\theta(p)) = 0\} \\ &\cup \{p \in \pi_{\mathsf{D}}^{*}\mathsf{D}^{\perp} \mid p_1(p) = 0,\ v_s(p) = 0,\ v_r(p) = 0,\ \cos(\theta(p)) = 0\}, \end{aligned}$$

then the differential equations (5.1) are easily examined to see that A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. We observe that p_2 is constant along integral curves of $X_{\mathsf{D}}^{\mathrm{reg}}$ with initial conditions in A. (We remark that the submanifold defined by the condition $v_s(p) = 0$ is $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant. That is, integral curves of $X_{\mathsf{D}}^{\mathrm{reg}}$ with an initial condition with $v_s(0) = 0$ will satisfy $v_s(t) = 0$ for all t. In this case, one also sees that θ is constant. This is a starting point for much of the analysis of the differential equations (5.1) that we reference below.)

(ii) For the condition that $\zeta(p) = 0$ we consider two cases, namely (I) $v_s(p) = 0$ and (II) $v_s(p) \neq 0$. Let us consider these cases in turn.

(I) If

$$p \in \mathsf{A}'_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) := \{p' \in \pi_{\mathsf{D}}^*\mathsf{D}^{\perp} \mid p_1(p') = 0,\ v_s(p') = 0\},$$

then we calculate

$$\mathscr{L}_{X_{\mathsf{D}}^{\mathrm{reg}}}\lambda^{\mathrm{e}}(p) = -\sqrt{m}g\sin(\tau)\sin(\theta(p))v_r(p)^2.$$

Excluding the condition $v_r(p) = 0$ which we have already considered above, the vanishing of the expression on the right requires that $\sin(\theta(p)) = 0$. If we take

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^{\perp} \mid p_1(p) = 0,\ v_s(p) = 0,\ \sin(\theta(p)) = 0\},$$

then the differential equations (5.1) are easily examined to see that A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. The evolution of p_2 along integral curves is as determined by the p_2-component of the differential equations (5.1), noting that $p_1 = 0$ in this case.

(II) In this case, the condition $\zeta(p) = 0$ uniquely specifies $p_2(p)$, and we denote this value of p_2 by $p_2^*(v)$, noting that this value depends only on $v = \pi_{\mathsf{D}}^*\pi_{\mathsf{D}^{\perp}}(p)$. If

$$p \in \mathsf{A}'_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) := \{p' \in \pi_{\mathsf{D}}^*\mathsf{D}^{\perp} \mid p_1(p') = 0,\ p_2(p') = p_2^*(v')\},$$

then we calculate

$$\mathscr{L}^2_{X_{\mathsf{D}}^{\mathrm{reg}}}\lambda^{\mathrm{e}}(p) = \alpha_1(v)\cos(\theta(p)),$$

where α_1 is a complicated nowhere zero θ-independent function of the points $v \in \mathsf{D}$ satisfying $v_s(v) \neq 0$. Now we let

$$\begin{aligned} p &\in \mathsf{A}'_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) \\ &:= \{p' \in \pi_{\mathsf{D}}^*\mathsf{D}^{\perp} \mid p_1(p') = 0,\ p_2(p') = p_2^*(v'),\ \cos(\theta(p')) = 0\}, \end{aligned}$$

then we calculate

$$\mathscr{L}^3_{X_{\mathsf{D}}^{\mathrm{reg}}}\lambda^{\mathrm{e}}(p) = \alpha_2(v)\sin(\theta(p)),$$

where α_2 is a complicated nowhere zero θ-independent function of the points $v \in \mathsf{D}$ satisfying $v_s(v) \neq 0$. Since sin and cos have no common zeros, we see that the subset

$$\{p \in \pi_{\mathsf{D}}^*\mathsf{D}^{\perp} \mid p_1(p) = 0,\ v_s(p) \neq 0\}$$

contains no $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle varieties.

(b) If
$$p \in \mathsf{A}_1'(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}}) := \{p' \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0\},$$
then
$$\mathscr{L}^2_{X_{\mathsf{D}}^{\mathrm{reg}}}\lambda^{\mathrm{e}}(p) = \frac{2m^2g^2R^2\sin(\tau)^2}{J_r + mR^2}\cos(\theta(p))^2 p_1(p).$$
Setting aside the condition that $p_1(p) = 0$ that has already been considered, the vanishing of the expression on the right requires that $\cos(\theta(p)) = 0$. Now, if we let
$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0,\ \cos(\theta(p)) = 0\},$$
then a consideration of the differential equations (5.1) shows that A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. The evolution of p_1 and p_2 along integral curves is as determined by the p_1- and p_2-components of the differential equations (5.1).

R2. Now we consider the case of $\sin(\tau) = 0$. As above, for $p \in \mathsf{A}_0(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}})$, we have the two cases (a) $p_1(p) = 0$ or (b) $v_s(p) = v_r(p) = 0$, which we consider in turn.

(a) If $p_1(p) = 0$, then $p \in \mathsf{A}_1(X_{\mathsf{D}}^{\mathrm{reg}}, \lambda^{\mathrm{e}})$ if and only if either (i) $v_s(p) = v_r(p) = 0$ or (ii)
$$\frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}}v_s(p)p_2(p) - \frac{2\sqrt{m}R}{\sqrt{J_s}\sqrt{J_r + mR^2}}v_s(p)v_r(p) = 0.$$
Let us again abbreviate the expression on the left by $\zeta(p)$. We consider the two preceding cases.

(i) If we take
$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid p_1(p) = 0,\ v_s(p) = 0,\ v_r(p) = 0\},$$
then we can immediately deduce from the differential equations (5.1) that A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. We observe that p_2 is constant along integral curves of $X_{\mathsf{D}}^{\mathrm{reg}}$ with initial conditions in A.

(ii) For the condition that $\zeta(p) = 0$ we consider two cases, namely (I) $v_s(p) = 0$ and (II) $v_s(p) \neq 0$. Let us consider these cases in turn.

(I) In this case, we can immediately verify from the differential equations (5.1) that
$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid p_1(p) = 0,\ v_s(p) = 0\}$$
is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. We observe that p_2 is constant along integral curves of $X_{\mathsf{D}}^{\mathrm{reg}}$ with initial conditions in A.

(II) In this case, the condition $\zeta(p)$ uniquely specifies $p_2(p)$, say $p_2(p) = p_2^*(v)$. One can see that, if we take

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^* \mathsf{D}^\perp \mid p_1(p) = 0,\ p_2(p) = p_2^*(v)\},$$

then both $\dot{p}_1$ and $\dot{p}_2$ are zero on A. Thus A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety.

(b) In this case, we take

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^* \mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0\}$$

and note directly from the differential equations (5.1) that A is an $X_{\mathsf{D}}^{\mathrm{reg}}$-invariant affine subbundle variety. We observe that both p_1 and p_2 are constant along integral curves of $X_{\mathsf{D}}^{\mathrm{reg}}$ with initial conditions in A.

5.16. We can also find the largest $X_{\mathsf{D}}^{\mathrm{sing}}$-invariant subbundle of $\pi_{\mathsf{D}}^* \mathsf{D}^\perp$ contained in $\ker(\hat{F}_{\mathsf{D}}^*)$, as required by Theorem 3.4 to determine the singular constrained variational trajectories that are also nonholonomic trajectories. We have

$$X_{\mathsf{D}}^{\mathrm{sing}} = \frac{R\cos(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_x + \frac{R\sin(\theta)v_r}{\sqrt{J_r + mR^2}}\partial_y + \frac{v_s}{\sqrt{J_s}}\partial_\theta + \frac{v_r}{\sqrt{J_r + mR^2}}\partial_\phi + \frac{mgR\sqrt{J_s}\sin(\tau)\cos(\theta)}{\sqrt{J_s}\sqrt{J_r + mR^2}}\partial_{v_r} + \frac{\sqrt{J_r}v_s p_2}{\sqrt{J_s}\sqrt{J_r + mR^2}}\partial_{p_1} - \frac{\sqrt{J_r}v_s p_1}{\sqrt{J_s}\sqrt{J_r + mR^2}}\partial_{p_2},$$

and the associated differential equations are

$$\begin{aligned}
\dot{x}(t) &= \frac{R\cos(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{y}(t) &= \frac{R\sin(\theta(t))v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{\theta}(t) &= \frac{v_s(t)}{\sqrt{J_s}}, \\
\dot{\phi}(t) &= \frac{v_r(t)}{\sqrt{J_r + mR^2}}, \\
\dot{v}_s(t) &= 0, \\
\dot{v}_r(t) &= \frac{mgR\sqrt{J_s}\sin(\tau)\cos(\theta(t))}{\sqrt{J_s}\sqrt{J_r + mR^2}}, \\
\dot{p}_1(t) &= \frac{\sqrt{J_r}v_s(t)p_2(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}}, \\
\dot{p}_2(t) &= -\frac{\sqrt{J_r}v_s(t)p_1(t)}{\sqrt{J_s}\sqrt{J_r + mR^2}}.
\end{aligned} \tag{5.2}$$

One computes

$$\mathscr{L}_{X_{\mathsf{D}}^{\mathrm{sing}}}\lambda^{\mathrm{e}} = \frac{2mgR\sin(\tau)\cos(\theta)}{\sqrt{J_r + mR^2}}v_r p_1 + (v_s^2 + v_r^2)\frac{\sqrt{J_r}}{\sqrt{J_s}\sqrt{J_r + mR^2}}v_s p_2.$$

We follow the same strategy as in the preceding paragraph to determine the cogeneralized subbundles that are invariant under $X_{\mathsf{D}}^{\text{sing}}$.

S1. We first take $\sin(\tau) \neq 0$. If $p \in \mathsf{A}_0(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}})$, then either (a) $p_1(p) = 0$ or (b) $v_s(p) = v_r(p) = 0$. We consider these possibilities in turn.

(a) If $p_1(p) = 0$, then $p \in \mathsf{A}_1(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}})$ if and only if

$$v_s(p)p_2(p)(v_s(p)^2 + v_r(p)^2) = 0,$$

and this holds if and only if either (i) $v_s(p) = 0$ or (ii) $p_2(p) = 0$. We consider these possibilities in turn.

(i) In this case we see that

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid p_1(p) = 0,\ v_s(p) = 0\}$$

is an $X_{\mathsf{D}}^{\text{sing}}$-invariant subbundle and the value of p_2 is constant along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

(ii) In this case, if we take

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid p_1(p) = 0,\ p_2(p) = 0\},$$

then A is an $X_{\mathsf{D}}^{\text{sing}}$-invariant affine subbundle variety, and the values of p_1 and p_2 are zero along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

(b) Let us denote

$$\mathsf{A}_1'(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}}) = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0\}.$$

For $p \in \mathsf{A}_1'(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}})$ we compute

$$\mathscr{L}^2_{X_{\mathsf{D}}^{\text{sing}}}\lambda^{\text{e}}(p) = \frac{2m^2g^2R^2\sin(\tau)^2}{J_r + mR^2}\cos(\theta(p))^2 p_1(p).$$

Setting aside the condition that $p_1(p) = 0$ which has already been considered, we see that for the expression on the right to vanish we must have $\cos(\theta(p)) = 0$. One can then see that, if we define

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^*\mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0,\ \cos(\theta(p)) = 0\},$$

then A is an $X_{\mathsf{D}}^{\text{sing}}$-invariant affine subbundle variety. We can additionally see that the values of p_1 and p_2 are constant along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

S2. Now we consider the case of $\sin(\tau) = 0$. As above, for $p \in \mathsf{A}_0(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}})$, we have the two cases (a) $p_1(p) = 0$ or (b) $v_s(p) = v_r(p) = 0$, which we consider in turn.

(a) If $p_1(p) = 0$, then $p \in \mathsf{A}_0(X_{\mathsf{D}}^{\text{sing}}, \lambda^{\text{e}})$ if and only if either (i) $v_s(p) = 0$ or (ii) $p_2(p) = 0$. We consider the two preceding cases.

(i) Here we define

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^* \mathsf{D}^\perp \mid p_1(p) = 0,\ v_s(p) = 0\},$$

and note that A is an $X_{\mathsf{D}}^{\text{sing}}$-invariant affine subbundle variety. The value of p_2 is constant along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

(ii) If we define

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^* \mathsf{D}^\perp \mid p_1(p) = 0,\ p_2(p) = 0\},$$

then A is an $X_{\mathsf{D}}^{\text{sing}}$-invariant affine subbundle variety, and the values of p_1 and p_2 are zero along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

(b) Finally, if we take

$$\mathsf{A} = \{p \in \pi_{\mathsf{D}}^* \mathsf{D}^\perp \mid v_s(p) = 0,\ v_r(p) = 0\},$$

then A is an $X_{\mathsf{D}}^{\text{sing}}$-invariant affine subbundle variety, and the values of p_1 and p_2 are constant along integral curves of $X_{\mathsf{D}}^{\text{sing}}$ with initial conditions in A.

5.17. Let us now assemble the detailed analysis of the preceding two paragraphs into final results. We first consider the case where nonholonomic trajectories are also regular constrained variational trajectories.

5.18 Proposition. *(i) When* $\sin(\tau) \neq 0$, *the following initial conditions give all nonholonomic trajectories that are regular constrained variational trajectories for suitable choices of* $p_1(0)$ *and* $p_2(0)$*:*

(a) $\sin(\theta(0)) = 0$, $v_s(0) = 0$, $v_r(0) = 0$;

(b) $\cos(\theta(0)) = 0$, $v_s(0) = 0$, $v_r(0) = 0$;

(c) $\sin(\theta(0)) = 0$, $v_s(0) = 0$.

(ii) When $\sin(\tau) = 0$, *all initial conditions give nonholonomic trajectories that are regular constrained variational trajectories for suitable choices of* $p_1(0)$ *and* $p_2(0)$.

Next we consider the case where nonholonomic trajectories are also singular constrained variational trajectories.

5.19 Proposition. *(i) When* $\sin(\tau) \neq 0$, *the following initial conditions give all nonholonomic trajectories that are singular constrained variational trajectories for suitable choices of* $p_1(0)$ *and* $p_2(0)$*:*

(a) $v_s(0) = 0$;

(b) $\cos(\theta(0)) = 0$, $v_s(0) = 0$, $v_r(0) = 0$.

(ii) When $\sin(\tau) = 0$, *the following initial conditions give all nonholonomic trajectories that are singular constrained variational trajectories for suitable choices of* $p_1(0)$ *and* $p_2(0)$*:*

(a) $v_s(0) = 0$.

These results improve the analysis in the literature for this example in various ways. For example, our analysis includes the singular case for the first time, even in the oft-studied case when $\sin(\tau) = 0$ [Cortés, de León, Martín de Diego, and Martínez 2002, Example 6.6]. We also are able to carry out a more detailed analysis in the regular case when $\sin(\tau) \neq 0$ than is carried out in [Lemos 2022]. For example, in [Lemos 2022] it is indicated that (here we paraphrase to convert to our terminology), "except for the trivial case $v_s(t) = 0$, for which the constraints are actually holonomic, regular constrained variational trajectories are never nonholonomic trajectories." As we have seen in our analysis, this "trivial case" is actually not quite trivial since it arises at singularities of the cogeneralized subbundle $\ker(\hat{F}^*_{\mathsf{D}})$. These singularities preclude most nonholonomic trajectories with $v_s(t) = 0$ from being regular constrained variational trajectories. It does, however, permit such nonholonomic trajectories to be *singular* constrained variational trajectories.

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