# The Fundamental Theorem of Dynamical Systems: all at once and all in the same place

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**Abstract**

The so-called Fundamental Theorem of Dynamical Systems—which (1) relates attractors and repellers to the chain recurrent set and (2) gives the existence of a complete Lyapunov function—can be seen as a means of separating out "recurrent" and "transient" dynamics. An overview of this theorem is given in its various guises, continuous-time/discrete-time and flows/semiflows. As part of this overview, a unified approach is developed for working simultaneously with both the continuous-time and discrete-time frameworks for topological dynamics. Additionally, a complete Lyapunov function is provided for the first time for continuous-time flows and semiflows.

**Keywords.** Chain recurrence, complete Lyapunov function

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## 1. Introduction

The behaviour of dynamical systems is too complicated to admit any sort of very useful classification. However, one of the basic sorts of classifications one might seek is to separate dynamics that is "steady-state" from dynamics that is "transient." The exact meaning of "steady state" is rather unclear at the outset, e.g., should it be equilibrium dynamics? period dynamics? After some thought, it becomes clear that "recurrence" is a good general concept for capturing "steady-state." However, there is a hierarchy of notions of recurrence, with the most elementary notion being that of a fixed point. As one ascends (or descends, depending on one's point of view) through the hierarchy, one naturally wonders whether the hierarchy has finitely many steps, or whether there are just more and more subtle notions of recurrence that become increasingly incomprehensible. One way to view the Fundamental Theorem of Dynamical systems—the term coined by Norton [1995] for the initial work of Conley [1978]—is that it says that this hierarchy can be terminated with the notion of recurrence known as "chain recurrence." Of course, this resolution of the difference between transient and recurrent dynamics hardly resolves the problem of completely understanding dynamics: chain recurrent dynamics can be extremely complicated. Nonetheless, this Fundamental Theorem of Dynamical Systems provides genuine insights into the general structure of topological dynamical systems.

**1.1. Contribution.** In this paper, we shall provide a comprehensive overview of chain recurrence and the Fundamental Theorem of Dynamical Systems. We feel as if this overview is warranted because (1) the essential results are scattered across many papers and (2) there are a few places in the literature where some fundamental misconceptions lead to misleading or erroneous statements. The reason for the theory being scattered across many papers is simple: the notion of chain recurrence and the statement of the Fundamental Theorem of Dynamical Systems applies to $2 \times 2 = 4$ classes of dynamical systems: continuous-time/discrete-time and flows/semiflows. Sometimes the differences in the way various classes are handled can be passed off to the Latin abbreviation "cf." without incurring too much of a loss; typically the differences between "flow" and "semiflow" fall into this category. Other times, a proof in one case is simply inapplicable to another case; the construction of a complete Lyapunov function in the discrete-time case being one such example since the construction is not immediately helpful in the continuous-time case. What we wish to do, therefore, is to present the theory in a complete and unified way (as much as this is possible). Most of the techniques we use will be familiar to those familiar with the theory. However, we hope that presenting this theory in a unified way will have a *per se* benefit.

**1.2. Historical developments.** Let us give an historical overview of the development of this theory. The terminology we use is given precise definitions in Section 2. A reader completely new to the ideas we present here may benefit from first reading this section. As we have indicated, the theory is initially given by Conley [1978], working with continuous-time flows on compact metric spaces. From this initial work, there arises a few natural extensions: (1) the inclusion of noncompact spaces; (2) the adaptation to discrete-time; (3) the adaptation to semiflows. Note that extensions (2) and (3) are essential for the theory to be applied to the important setting of the dynamics of a continuous mapping. These adaptations were carried out in a series of papers during the 1980's and 90's. When carrying out extensions of Conley's initial work, there are a few different aspects of the work that can be extended, and normally these extensions are not carried out together, but piecemeal. These aspects are the following.

1. *Chain recurrence:* The Fundamental Theorem of Dynamical Systems is connected with the particular recurrence notion of chain recurrence. There are different definitions of chain recurrence, depending on the class of dynamical system with which one is working. As well, chain recurrence, and particularly the connected notion of chain transitivity, has its own properties that can be important to understand, independently of the Fundamental Theorem of Dynamical Systems.
2. *The Conley decomposition:* Part of the Fundamental Theorem of Dynamical Systems is a decomposition of the state space into (a) a subset on which the dynamics is chain recurrent and (b) the complement on which the dynamics is gradient-like.
3. *Complete Lyapunov functions:* Another part of the Fundamental Theorem of Dynamical Systems concerns the existence of a complete Lyapunov function that decreases along trajectories, and is constant on the chain recurrent set.

We shall consider how all three of these aspects have developed, either together or separately.

As we indicated above, chain recurrence is part of a theory of recurrence for topological dynamical systems. Other notions of recurrence include limit sets, Poincaré recurrent sets, and nonwandering sets. A notion of "weak nonwandering point" for ordinary differential equations is introduced by Sharkovskii and Dobrynsky [1973], and this notion can apparently be shown to be equivalent to chain recurrence as introduced by [Conley 1978] (we have seen this "on the internet" but know of no precise reference). The (fairly straightforward) adaptation to discrete-time semiflows seems to have first been given by Block and Franke [1985], where connections to other forms of recurrence is proved in some special cases. The discrete-time setting is also considered in [Easton 1989] and reflections are made on numerical aspects of dynamics. An expository presentation is given by [Franks 1988] that considers chain recurrence in relation to other recurrence notions.

As introduced by Conley, chain recurrence is defined for compact metric spaces. In its original form, for noncompact spaces the notion of chain recurrence is a metric notion, not a topological notion. We shall see an instance of this in Example 3.8. An approach to making the concept less connected to the metric was introduced in [Hurley 1991]. In this work, the setting is discrete-time semiflows on locally compact metric spaces. As well as giving a definition of chain recurrence in this setting, Hurley also shows that the Conley decomposition holds. The local compactness is relaxed to general metric spaces in [Hurley 1992], still in the discrete-time semiflow framework. Also in this work, the existence of a complete Lyapunov function is proved for locally compact, second countable state spaces.

The extension to general metric spaces is developed further by Hurley [1995] to include the continuous-time framework as well as the discrete-time. Additionally, Hurley gives two useful alternative characterisations of continuous-time chain recurrence; for instance, chain recurrence for a continuous-time semiflow is related to chain recurrence for its time-one map. The work [Hurley 1995] contained a few errors that were corrected by [Choi, Chu, and Park 2002], while introducing a few other errors themselves. We hope we have corrected these errors in our presentation. The existence of a complete Lyapunov function is proved by [Hurley 1998] for discrete-time semiflows on separable metric spaces. This is extended in [Patrão 2011] to the continuous-time case, although the proof has errors that we correct in our presentation. For flows defined by smooth ordinary differential equations, Hafstein and Suhr [2021] show the existence of smooth complete Lyapunov functions.

Extensions of chain recurrence and the Fundamental Theorem of Dynamical Systems beyond the metric space setting are also possible, although we do not consider these in our overview. A thorough development of these notions for uniform spaces is given in the manuscript [Akin and Wiseman 2017]. Chain recurrence and related notions are considered for discrete-time semiflows on arbitrary topological spaces in [Block and Coppel 1992]. This treatment is extended to continuous-time semiflows in [Oprocha 2005]. An "open cover" framework for chain recurrence is presented by Patrão and San Martin [2006].

Recent developments in the concepts of chain recurrence and the Fundamental Theorem of Dynamical Systems include the applications to linear dynamical systems, including those in infinite-dimensions [Antunes, Mantovani, and Varão 2022, Bernardes Jr and Peris 2024], control theory [Colonius, Santana, and Viscovini 2024], and hybrid dynamical systems [Kvalheim, Gustafson, and Koditschek 2021].

## 2. Dynamical systems and related concepts

In this section, we give our definitions for the classes of dynamical systems we use throughout the paper, as well as a few of the concepts from topological dynamics to which we refer. As the reader will see, we unite the continuous- and discrete-time in our presentation as much as this is possible. This has a variety of conceptual benefits. We do not, however, go as far as Akin [1993] who uses general relations in place of continuous- and discrete-time.

**2.1. Flows and semiflows.** We use the symbol $\mathbb{T}$ to stand for either $\mathbb{R}$ or $\mathbb{Z}$. If we do not explicitly specify, we intend that $\mathbb{T}$ can be either of the two possibilities. For $t_0 \in \mathbb{T}$ and for an interval $I \subseteq \mathbb{R}$, We denote

$$\begin{aligned}
\mathbb{T}_{>t_0} &= \{t \in \mathbb{T} \mid\ t > t_0\}, \\
\mathbb{T}_{\geq t_0} &= \{t \in \mathbb{T} \mid\ t \geq t_0\}, \\
\mathbb{T}_{<t_0} &= \{t \in \mathbb{T} \mid\ t < t_0\}, \\
\mathbb{T}_{\leq t_0} &= \{t \in \mathbb{T} \mid\ t \leq t_0\}, \\
\mathbb{T}_I &= \{t \in \mathbb{T} \mid\ t \in I\}.
\end{aligned}$$

Let us give the definition of the dynamical systems we work with.

**2.1 Definition: (Flow, semiflow)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space.

(i) A ***topological semiflow*** on $\mathcal{X}$ is a continuous mapping $\Phi\colon \mathbb{T}_{\geq 0} \times \mathcal{X} \to \mathcal{X}$ satisfying

  (a) $\Phi(0, x) = x$, $x \in \mathcal{X}$, and

  (b) $\Phi(t_1, \Phi(t_2, x)) = \Phi(t_1 + t_2, x)$, $t_1, t_2 \in \mathbb{T}_{\geq 0}$, $x \in \mathcal{X}$.

(ii) A ***topological flow*** on $\mathcal{X}$ is a continuous mapping $\Phi\colon \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ satisfying

  (a) $\Phi(0, x) = x$, $x \in \mathcal{X}$, and

  (b) $\Phi(t_1, \Phi(t_2, x)) = \Phi(t_1 + t_2, x)$, $t_1, t_2 \in \mathbb{T}$, $x \in \mathcal{X}$.

If $\mathbb{T} = \mathbb{R}$, the topological semiflow or flow is ***continuous-time***, and otherwise it is ***discrete-time***. •

**2.2 Remark: (Discrete-time flows and semiflows)** Discrete-time flows and semiflows correspond to the dynamics of continuous mappings and homeomorphisms, respectively.

1. If $\Phi$ is a discrete-time topological semiflow, then we can define $\phi(x) = \Phi(1, x)$ and easily very that
$$\Phi(t, x) = \phi^t(x) \triangleq \underbrace{\phi \circ \cdots \circ \phi}_{t \text{ times}}(x), \qquad (t, x) \in \mathbb{Z}_{\geq 0} \times \mathcal{X},$$
where it is understood that $\phi^0 = \mathrm{id}_{\mathcal{X}}$.
2. Similarly, if $\Phi$ is a discrete-time topological flow and if $\phi(x) = \Phi(1, x)$, then $\phi$ is an homeomorphism with $\phi^{-1}(x) = \Phi(-1, x)$. Here we have $\Phi(t, x)$ as above for $t \in \mathbb{Z}_{\geq 0}$, and
$$\Phi(-t, x) = \phi^{-t}(x) \triangleq \underbrace{\phi^{-1} \circ \cdots \circ \phi^{-1}}_{t \text{ times}}(x), \qquad (t, x) \in \mathbb{Z}_{\geq 0} \times \mathcal{X},$$

Note that our use of "flow" or "semiflow" in the discrete-time case is nonstandard. However, there is a unifying benefit to using the same terminology for both the continuous- and discrete-time settings. •

There are somewhat surprising relationships between chain recurrence for a continuous-time flow or semiflow, and chain recurrence for the discrete-time flow or semiflow defined by its time-one map. For this reason, we make the following definition.

**2.3 Definition: (Time-$T$ discretisation of continuous-time flow or semiflow)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological continuous-time topological flow (resp. semiflow) on $\mathcal{S}$. The ***$T$-discretisation*** of $\Phi$ is the discrete-time topological flow (resp. semiflow) $\Phi^{\mathrm{d},T}$ on $\mathcal{X}$ defined by requiring that
$$\Phi^{\mathrm{d},T}(1, x) = \Phi(T, x), \qquad x \in \mathcal{X},$$
cf. Remark 2.2. •

It is evident that, if $\Phi$ is a topological flow, then its restriction to $\mathbb{T}_{\geq 0} \times \mathcal{X}$ is topological semiflow. If $\Phi$ is a topological flow (resp. semiflow) and if $t \in \mathbb{T}$ (resp. $t \in \mathbb{T}_{\geq 0}$), then we have the homeomorphism (resp. continuous mapping)
$$\begin{aligned} \Phi_t\colon \mathcal{X} &\to \mathcal{X} \\ x &\mapsto \Phi(t, x). \end{aligned}$$

In like manner, if $x \in \mathcal{X}$, then

$$\begin{aligned} \Phi^{+,x} \colon \mathbb{T}_{\geq 0} &\to \mathcal{X} \\ t &\mapsto \Phi(t,x) \end{aligned}$$

is the ***forward trajectory*** of $x$ and, if $\Phi$ is a flow, then

$$\begin{aligned} \Phi^{x} \colon \mathbb{T} &\to \mathcal{X} \\ t &\mapsto \Phi(t,x) \end{aligned}$$

is the ***trajectory*** of $x$.

Let us also define the notion of orbits.

**2.4 Definition: (Orbit, forward orbit, backward orbit)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Let $A \subseteq \mathcal{X}$.

(i) The ***forward orbit*** of $A$ is

$$\mathrm{Orb}^{+}(A) = \{\Phi(t,x) \mid t \in \mathbb{T}_{\geq 0},\ x \in A\}.$$

(ii) The ***backward orbit*** of $A$ is

$$\mathrm{Orb}^{-}(A) = \{x \in \mathcal{X} \mid \Phi(t,x) \in A \text{ for some } t \in \mathbb{T}_{\geq 0}\}.$$

If $\Phi$ is a topological flow, then

(iii) the ***orbit*** of $A$ is

$$\mathrm{Orb}(A) = \{\Phi(t,x) \mid t \in \mathbb{T},\ x \in A\}.$$ •

If $A = \{x\}$ for some $x \in \mathcal{X}$, we abbreviate

$$\mathrm{Orb}^{+}(x) = \mathrm{Orb}^{+}(\{x\}),\ \ \mathrm{Orb}^{-}(x) = \mathrm{Orb}^{-}(\{x\}),\ \ \mathrm{Orb}(x) = \mathrm{Orb}(\{x\}).$$

Note that, for a topological flow $\Phi$, the backward orbit of $A$ is

$$\mathrm{Orb}^{-}(A) = \{\Phi(t,x) \mid t \in \mathbb{T}_{\leq 0},\ x \in A\}.$$

The difference in how one should view the backward orbit for a flow and a semiflow will come up in our treatment of the Conley decomposition.

We shall make use of the following invariance notions.

**2.5 Definition: (Invariant, forward-invariant)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow. Let $A \subseteq \mathcal{X}$.

(i) The set $A$ is ***forward-invariant*** for $\Phi$ if $\mathrm{Orb}^{+}(A) \subseteq A$.

(ii) If $\Phi$ is a flow, the set $A$ is ***invariant*** for $\Phi$ if $\mathrm{Orb}(A) \subseteq A$. •

The following elementary lemma will account for some differences in how certain proofs work for flows versus semiflows.

**2.6 Lemma: (Invariance of the complement of an invariant set)** *Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow on $\mathcal{X}$. If $A \subseteq X$ is invariant for $\Phi$, then $\mathcal{X} \setminus A$ is invariant for $\Phi$.*

Proof: Let $x \in \mathcal{X} \setminus A$ and let $t \in \mathbb{T}$. If $\Phi(t, x) \in A$, then $x = \Phi(-t, x) \in A$ by invariance of $A$. Therefore, we must have $\Phi(t, x) \in \mathcal{X} \setminus A$. ■

Note that the lemma is false for forward-invariance and semiflows; think of a semiflow for which every trajectory ends up at a fixed point in finite time.

## 2.2. Attracting sets, repelling sets, and basins.

The Conley decomposition relates the chain recurrent set (which we define in Section 3) to attracting sets and their basins (for semiflows) or attracting sets and repelling sets (for flows), In this section we introduce the necessary terminology.

**2.7 Definition: (Trapping region)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. A nonempty subset $\mathcal{T} \subseteq \mathcal{X}$ is a ***trapping region*** for $\Phi$ if there exists $T \in \mathbb{T}_{>0}$ such that

$$\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathrm{int}(\mathcal{T}). \quad \bullet$$

**2.8 Remark: (Trapping regions in particular cases)** Different definitions and terminology can be found for what we call a trapping region. For the continuous-time case, sometimes "preattractor" is used. In the discrete-time case, the terminology "attractor block" is sometimes used. Again, we prefer unified terminology. Let us also examine more serious differences in the definitions that can be found in the literature.

1. Note that, in the discrete-time case, our definition of a trapping region is *not* the usual definition. The usual definition in the discrete-time case is that $\mathrm{cl}(\Phi_1(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$. We shall call a subset $\mathcal{T}$ a ***strong trapping region*** for the discrete-time flow or semiflow $\Phi$. Note that a strong trapping region is a trapping region. Note, also, that a strong trapping region is forward-invariant, which leads to some simplifications for strong trapping regions compared to trapping regions. It is easy to build examples for which the two definitions of trapping region are not the same.
2. Somewhat in keeping with the notion of a strong trapping region in the discrete-time case, one can consider a notion of trapping region in the continuous-time case where the requirement is that there exists $T \in \mathbb{R}_{>0}$ such that $\mathrm{cl}(\Phi_T(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$. In the case when $\mathcal{X}$ is compact, this condition agrees with our notion of a trapping region. This is proved for flows by Conley [1978, page 33, C]; we prove this here for semiflows as well. Let $T \in \mathbb{R}_{>0}$ be such that $\mathrm{cl}(\Phi_T(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$. Because metric spaces are normal [Willard 1970, Example 15.3(c)], let $\mathcal{N}$ be an open set for which

$$\Phi_T(\mathrm{cl}(\mathcal{T})) \subseteq \mathrm{cl}(\Phi_T(\mathcal{T})) \subseteq \mathcal{N} \subseteq \mathrm{cl}(\mathcal{N}) \subseteq \mathrm{int}(\mathcal{T}).$$

   Note that $\Phi_T(\mathcal{T})$ is compact, being a closed subset of a compact space. By this compactness and by the continuity of the dynamics, there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\Phi((T - \delta, T + \delta) \times \mathrm{cl}(\mathcal{T})) \subseteq \mathcal{N}.$$

Let $T' = \frac{T}{2\delta}$. If $t \geq T'$, then $t$ is a finite sum $t = t_1 + \cdots + t_k$ where $t_j \in (T-\delta, T+\delta)$, $j \in \{1, \dots, k\}$. Note that

$$\Phi_{t_k}(\mathrm{cl}(\mathcal{T})) \subseteq \mathcal{N} \subseteq \mathrm{int}(\mathcal{T}) \subseteq \mathrm{cl}(\mathcal{T}).$$

Thus

$$\Phi_{t_{k-1}} \circ \Phi_{t_k}(\mathrm{cl}(\mathcal{T})) \subseteq \Phi_{t_{k-1}}(\mathrm{cl}(\mathcal{T})) \subseteq \mathcal{N} \subseteq \mathrm{int}(\mathcal{T}) \subseteq \mathrm{cl}(\mathcal{T}).$$

Inductively,

$$\Phi_t(\mathrm{cl}(\mathcal{T})) = \Phi_{t_1} \circ \Phi_{t_2} \circ \Phi_{t_k}(\mathrm{cl}(\mathcal{T})) \subseteq \Phi_{t_1}(\mathrm{cl}(\mathcal{T})) \subseteq \mathcal{N}.$$

Therefore,

$$\begin{aligned} \mathrm{cl}(\Phi([T', \infty) \times \mathcal{T})) &\subseteq \mathrm{cl}\left(\bigcup_{t \in [T', \infty)} \Phi_t(\mathcal{T})\right) \\ &\subseteq \mathrm{cl}\left(\bigcup_{t \in [T', \infty)} \Phi_t(\mathrm{cl}(\mathcal{T}))\right) \\ &\subseteq \mathrm{cl}(\mathcal{N}) \subseteq \mathrm{int}(\mathcal{T}), \end{aligned}$$

and so $\mathcal{T}$ is a trapping region.

Hurley [1995, Example 1] gives a simple continuous-time flow which shows that compactness is required for this assertion. •

Trapping regions give rise to attracting sets and (in the case of flows) repelling sets. Note that we refrain from calling these "attractors" and "repellers" since one normally wants to reserve this terminology for attracting sets and repelling sets with some minimality or transitivity property.

**2.9 Definition: (Attracting set, repelling set)** Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow.

(i) An ***attracting set*** for $\Phi$ is a subset $A$ such that there exists a trapping region $\mathcal{T}$ for which

$$A = \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T})).$$

Suppose now that $\Phi$ is a flow.

(ii) A ***repelling set*** for $\Phi$ is a subset $R$ such that there exists a trapping region $\mathcal{T}$ for which

$$R = \bigcap_{t \in \mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t} \times (\mathcal{X} \setminus \mathcal{T}))).$$

(iii) An ***attracting-repelling pair*** for $\Phi$ is a pair $(A, R)$ of subsets such that there exists a trapping region $\mathcal{T}$ for which

$$A = \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T})), \quad R = \bigcap_{t \in \mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t} \times (\mathcal{X} \setminus \mathcal{T}))).$$ •

If we wish to indicate that an attracting set or a repelling set comes from a particular trapping region $\mathcal{T}$, we may write $A_{\mathcal{T}}$ or $R_{\mathcal{T}}$, respectively. It is entirely possible that attracting and repelling sets may be empty.

**2.10 Examples: (Attracting and repelling sets)**

1. Let $\mathcal{X} = [0,1]$ and let $\Phi$ be the continuous-time flow on $\mathcal{X}$ obtained by restricting to $\mathcal{X}$ the flow associated with the ordinary differential equation

$$\dot{x}(t) = x(t)(1 - x(t))$$

for on $\mathbb{R}$. Then the trapping region $\mathcal{T} = (\frac{1}{2}, 1]$ has the attracting set $A_{\mathcal{T}} = \{1\}$ and the repelling set $R_{\mathcal{T}} = \{0\}$.

2. On $\mathcal{X} = (-\infty, 0]$, consider the continuous-time flow $\Phi(t,x) = x\mathrm{e}^{-t}$. Then the trapping region $\mathcal{T} = (-1, 0]$ has the attracting set $A_{\mathcal{T}} = \{0\}$ and the repelling set $R_{\mathcal{T}} = \varnothing$.
3. On $\mathcal{X} = [0, \infty)$, consider the continuous-time flow $\Phi(t,x) = x\mathrm{e}^{t}$. Then the trapping region $\mathcal{T} = (1, \infty)$ has the attracting set $A_{\mathcal{T}} = \varnothing$ and the repelling set $R_{\mathcal{T}} = \{0\}$.
4. On $\mathcal{X} = \mathbb{R}$, consider the continuous-time flow $\Phi(t,x) = x + t$. Then the trapping region $\mathcal{T} = (0, \infty)$ has the attracting set $A_{\mathcal{T}} = \varnothing$ and the repelling set $R_{\mathcal{T}} = \varnothing$. •

Let us show that trapping regions can be taken to be either open or closed.

**2.11 Proposition: (Trapping regions can be taken to be open or closed)** *Let* $(\mathcal{X}, \mathscr{O})$ *be a topological space and let* $\Phi$ *be an autonomous topological flow or semiflow. If* $\mathcal{T}$ *is a trapping region, then*

$$A_{\mathcal{T}} = A_{\mathrm{cl}(\mathcal{T})} = A_{\mathrm{int}(\mathcal{T})}.$$

*If* $\Phi$ *is a flow, then*

$$R_{\mathcal{T}} = R_{\mathrm{cl}(\mathcal{T})} = R_{\mathrm{int}(\mathcal{T})}.$$

Proof: First we show that $\mathrm{int}(\mathcal{T})$ and $\mathrm{cl}(\mathcal{T})$ are trapping regions. Let $T \in \mathbb{T}_{>0}$ be such that

$$\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathrm{int}(\mathcal{T}).$$

This, in particular, implies that

$$\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathrm{int}(\mathcal{T}))) \subseteq \mathrm{int}(\mathcal{T})$$

and

$$\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathrm{cl}(\mathcal{T}))) \subseteq \mathrm{cl}(\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))) = \mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathrm{int}(\mathcal{T}) \subseteq \mathrm{int}(\mathrm{cl}(\mathcal{T})),$$

which shows that $\mathrm{int}(\mathcal{T})$ and $\mathrm{cl}(\mathcal{T})$ are indeed trapping regions.

Now note that

$$\Phi_T(\mathrm{cl}(\mathcal{T})) \subseteq \mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathrm{int}(\mathcal{T}),$$

and so

$$\Phi_{T+t}(\mathrm{cl}(\mathcal{T})) = \Phi_t(\Phi_T(\mathrm{cl}(\mathcal{T}))) \subseteq \Phi_t(\mathrm{int}(\mathcal{T})), \qquad t \in \mathbb{T}_{\geq 0}.$$

Now calculate

$$\begin{aligned} A_{\mathcal{T}} = \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T})) &\subseteq \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathrm{cl}(\mathcal{T}))) \\ \subseteq \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq T+t} \times \mathrm{cl}(\mathcal{T}))) &\subseteq \bigcap_{t \in \mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathrm{int}(\mathcal{T}))) \subseteq A_{\mathcal{T}}, \end{aligned} \tag{2.1}$$

and so, in particular,

$$A_{\mathcal{T}} = \bigcap_{t\in\mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathrm{cl}(\mathcal{T}))) = \bigcap_{t\in\mathbb{T}_{\geq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathrm{int}(\mathcal{T}))),$$

which is the first part of the result.

For the second, note that $\mathrm{cl}(\Phi_T(A)) = \Phi_T(\mathrm{cl}(A))$ and $\mathrm{int}(\Phi_T(A)) = \Phi_T(\mathrm{int}(A))$ for any $A \subseteq \mathcal{X}$ since $\Phi_T$ is an homeomorphism when $\Phi$ is a flow. Also note that $\Phi_T(\mathcal{X} \setminus A) = \mathcal{X} \setminus \Phi_T(A)$ for any $A \subseteq \mathcal{X}$ since $\Phi_T$ is a bijection. Therefore,

$$\begin{aligned}\mathrm{cl}(\Phi_T(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T}) \implies \mathcal{X} \setminus \mathrm{int}(\mathcal{T}) &\subseteq \mathcal{X} \setminus \mathrm{cl}(\Phi_T(\mathcal{T})) = \mathrm{int}(\mathcal{X} \setminus \Phi_T(\mathcal{T})) \\ &= \mathrm{int}(\Phi_T(\mathcal{X} \setminus \mathcal{T})) = \Phi_T(\mathrm{int}(\mathcal{X} \setminus \mathcal{T})).\end{aligned}$$

Therefore,

$$\Phi_{-T}(\mathrm{cl}(\mathcal{X} \setminus \mathcal{T})) = \Phi_{-T}(\mathcal{X} \setminus \mathrm{int}(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{X} \setminus \mathcal{T}).$$

Thus, by a modification of the arguments in the previous part of the proof, we have

$$\begin{aligned}R_{\mathcal{T}} &= \bigcap_{t\in\mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t} \times (\mathcal{X} \setminus \mathcal{T}))) \subseteq \bigcap_{t\in\mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t} \times (\mathcal{X} \setminus \mathrm{int}(\mathcal{T})))) \\ &\subseteq \bigcap_{t\in\mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t-T} \times (\mathrm{cl}(\mathcal{X} \setminus \mathcal{T})))) \subseteq \bigcap_{t\in\mathbb{T}_{\leq 0}} \mathrm{cl}(\Phi(\mathbb{T}_{\leq t} \times (\mathrm{int}(\mathcal{X} \setminus \mathcal{T})))) \subseteq R_{\mathcal{T}},\end{aligned}$$

giving the result for repelling sets. ■

The definitions of attracting sets and repelling sets suggest a "duality" between these notions, depending on whether time goes forwards or backwards. The following result makes this suggestion precise. We denote by $\Phi^\sigma$ the time-reversed flow of a flow $\Phi$ defined by $\Phi^\sigma(t, x) = \Phi(-t, x)$.

**2.12 Proposition: (Attracting sets as repelling sets, and vice versa)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow on $\mathcal{X}$. If $\mathcal{T}$ is a trapping region with attracting-repelling pair $(A, R)$, then $\mathcal{X} \setminus \mathcal{T}$ is a trapping region for $\Phi^\sigma$ and $(R, A)$ is the attracting-repelling pair associated to $\mathcal{X} \setminus \mathcal{T}$.*

Proof: Let $T \in \mathbb{T}_{>0}$ be such that $\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$. Denote

$$\mathcal{N} = \mathcal{X} \setminus \mathrm{cl}\left(\bigcup_{t\in\mathbb{T}_{[0,T]}} \Phi_{-t}(\mathcal{T})\right).$$

The following facts about $\mathcal{N}$ are useful.

**1 Lemma:** *The following statements hold:*

*(i)* $\mathrm{cl}(\mathcal{N}) \subseteq \mathrm{int}(\mathcal{X} \setminus \mathcal{T})$*;*

*(ii)* $\Phi_{-t}(\mathcal{X} \setminus \mathrm{int}(\mathcal{T})) \subseteq \mathcal{N}$, $t \in \mathbb{T}_{\geq 2T}$*.*

Proof: (i) Since metric spaces are normal [Willard 1970, Example 15.3(c)], let $\mathcal{N}'$ be an open set such that

$$\Phi_T(\mathrm{cl}(\mathcal{T})) \subseteq \mathcal{N}' \subseteq \mathrm{cl}(\mathcal{N}') \subseteq \mathcal{T}.$$

Thus

$$\mathrm{cl}(\mathcal{T}) \subseteq \Phi_{-T}(\mathcal{N}') \subseteq \mathrm{cl}(\Phi_{-T}(\mathcal{N}')) \subseteq \Phi_{-T}(\mathcal{T}),$$

and so

$$\mathcal{X} \setminus \mathrm{cl}(\mathcal{T}) \supseteq \mathcal{X} \setminus \Phi_{-T}(\mathcal{N}') \supseteq \mathcal{X} \setminus \Phi_{-T}(\mathcal{T}) \supseteq \mathcal{X} \setminus \mathrm{cl}\left(\bigcup_{t \in \mathbb{T}_{[0,T]}} \Phi_{-t}(\mathcal{T})\right) = \mathcal{N}.$$

Since the set $\mathcal{X} \setminus \Phi_{-T}(\mathcal{N}')$ is closed, we obtain

$$\mathrm{cl}(\mathcal{N}) \subseteq \mathcal{X} \setminus \mathrm{cl}(\mathcal{T}) = \mathrm{int}(\mathcal{X} \setminus \mathcal{T}),$$

as desired.

(ii) Let $t \in \mathbb{T}_{\geq 2T}$. Then

$$\begin{aligned}\Phi_t(\mathcal{X} \setminus \mathcal{N}) = \; & \mathrm{cl}\left(\Phi_t\left(\bigcup_{t' \in \mathbb{T}_{[0,T]}} \Phi_{-t'}(\mathcal{T})\right)\right) = \mathrm{cl}\left(\bigcup_{t' \in \mathbb{T}_{[0,T]}} \Phi_{t-t'}(\mathcal{T})\right) \\ \subseteq \; & \mathrm{cl}\left(\bigcup_{s \in \mathbb{T}_{\geq T}} \Phi_s(\mathcal{T})\right) \subseteq \mathrm{int}(\mathcal{T});\end{aligned}$$

here we have used the fact that $\Phi_t$ commutes with closure since it is an homeomorphism and commutes with union since it is a bijection. Therefore,

$$\mathcal{X} \setminus \mathcal{N} \subseteq \Phi_{-t}(\mathrm{int}(\mathcal{T})) = \mathcal{X} \setminus (\mathcal{X} \setminus \Phi_{-t}(\mathrm{int}(\mathcal{T}))),$$

whereupon $\Phi_{-t}(\mathcal{X} \setminus \mathrm{int}(\mathcal{T})) \subseteq \mathcal{N}$, as claimed. ▼

The proof of the proposition is now straightforward. First of all

$$\begin{aligned}\mathrm{cl}(\Phi^\sigma(\mathbb{T}_{\geq 2T} \times (\mathcal{X} \setminus \mathcal{T}))) \subseteq \; & \mathrm{cl}\left(\bigcup_{t \in \mathbb{T}_{\leq -2T}} \Phi_t(\mathcal{X} \setminus \mathcal{T})\right) \\ \subseteq \; & \mathrm{cl}\left(\bigcup_{t \in \mathbb{T}_{\leq -2T}} \Phi_t(\mathrm{cl}(\mathcal{X} \setminus \mathcal{T}))\right) \\ = \; & \mathrm{cl}(\mathcal{N}) \subseteq \mathrm{int}(\mathcal{X} \setminus \mathcal{T}),\end{aligned}$$

which shows that $\mathcal{X} \setminus \mathcal{T}$ is a trapping region for $\Phi^\sigma$. Also,

$$A_{\mathcal{T}} = \bigcap_{t \in \mathbb{R}_{\geq 0}} \Phi_t(\mathcal{T}) = \bigcap_{t \in \mathbb{R}_{\leq 0}} \Phi_{-t}(\mathcal{X} \setminus (\mathcal{X} \setminus \mathcal{T})) = R_{\mathcal{X} \setminus \mathcal{T}}$$

and

$$R_{\mathcal{T}} = \bigcap_{t \in \mathbb{R}_{\leq 0}} \Phi_t(\mathcal{X} \setminus \mathcal{T}) = \bigcap_{t \in \mathbb{R}_{\geq 0}} \Phi_{-t}(\mathcal{X} \setminus \mathcal{T}) = A_{\mathcal{X} \setminus \mathcal{T}},$$

where $A_{\mathcal{X} \setminus \mathcal{T}}$ and $R_{\mathcal{X} \setminus \mathcal{T}}$ denote the attracting and repelling sets, respectively, for $\Phi^\sigma$ associated to the trapping region $\mathcal{X} \setminus \mathcal{T}$. ■

The following properties of attracting and repelling sets will be used.

**2.13 Proposition: (Properties of attracting and repelling sets)** *Let $(\mathcal{X}, \mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow. Let $A \subseteq \mathcal{X}$ be an attracting set and, when $\Phi$ is a flow, let $R$ be a repelling set. Then the following statements hold:*

*(i) $A$ is closed;*

*(ii) if $\Phi$ is a flow, then $A$ and $R$ are closed;*

*(iii) $A$ and $R$ (resp. $A$) are invariant (resp. forward-invariant).*

Proof: (i) and (ii) follow since $A$ is (or $A$ and $R$ are) an intersection of closed sets.

(iii) We let $\mathcal{T}$ be a trapping region with attracting set $A$. Let $x \in A$ and let $s \in \mathbb{T}_{\geq 0}$. Then, for $t \in \mathbb{T}_{\geq 0}$, we have $x \in \operatorname{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T}))$. Since

$$\Phi_s(\operatorname{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T}))) \subseteq \operatorname{cl}(\Phi_s(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T}))) = \operatorname{cl}(\Phi(\mathbb{T}_{\geq s+t} \times \mathcal{T})),$$

we have $\Phi_s(x) \in \operatorname{cl}(\Phi(\mathbb{T}_{\geq s+t} \times \mathcal{T}))$. As this holds for every $t \in \mathbb{T}_{\geq 0}$, we have

$$\Phi_s(x) \in \bigcap_{t \in \mathbb{T}_{\geq 0}} \operatorname{cl}(\Phi(\mathbb{T}_{\geq s+t} \times \mathcal{T})) = A.$$

Thus $A$ is always forward-invariant.

Now suppose that $\Phi$ is a flow and let $s \in \mathbb{T}_{<0}$. In this case, for $t \in \mathbb{T}_{\geq 0}$, we have $x \in \operatorname{cl}(\Phi(\mathbb{T}_{\geq t-s} \times \mathcal{T}))$. Since

$$\Phi_s(\operatorname{cl}(\Phi(\mathbb{T}_{\geq t-s} \times \mathcal{T}))) \subseteq \operatorname{cl}(\Phi_s(\Phi(\mathbb{T}_{\geq t-s} \times \mathcal{T})) = \operatorname{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T})),$$

we have $\Phi_s(x) \in \operatorname{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T}))$. As this holds for every $t \in \mathbb{T}_{\geq 0}$, we have

$$\Phi_s(x) \in \bigcap_{t \in \mathbb{T}_{\geq 0}} \operatorname{cl}(\Phi(\mathbb{T}_{\geq t} \times \mathcal{T})) = A.$$

One similarly shows that $R$ is invariant in case $\Phi$ is a flow. ■

Of course, the notion of a repelling set is not applicable to semiflows, by our definition. However, sometimes it is useful to think of $\mathcal{X} \setminus \operatorname{Orb}^-(\mathcal{T})$ as being the "repelling set" for a trapping region $\mathcal{T}$. The following simple lemma indicates that this is a reasonable way to think of repelling sets for semiflows.

**2.14 Lemma: (A characterisation of repelling sets)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow on $\mathcal{X}$. If $\mathcal{T}$ is an open trapping region with attracting-repelling pair $(A, R)$, then $R = \mathcal{X} \setminus \operatorname{Orb}^-(\mathcal{T})$.*

Proof: If $x \in R$ and $t \in \mathbb{T}_{\geq 0}$, then $\Phi(t, x) \in R$ by Proposition 2.13(iii). Since $R \subseteq \mathcal{X} \setminus \mathcal{T}$, $\Phi(t, x) \notin \mathcal{T}$ and so $x \notin \operatorname{Orb}^-(\mathcal{T})$. Next let $x \notin R$ and let $t \in \mathbb{T}_{\geq 0}$. Since $\Phi$ is a flow (and so $\Phi_t$ is an homeomorphism for all $t \in \mathbb{T}$) and since $\mathcal{X} \setminus \mathcal{T}$ is closed, we have

$$R = \bigcap_{s \in \mathbb{T}_{\leq 0}} \Phi(\mathbb{T}_{\leq s} \times (\mathcal{X} \setminus \mathcal{T})).$$

Thus $x \notin \Phi_{-t}(\mathcal{X} \setminus \mathcal{T})$. However, $\mathcal{X}$ is the disjoint union of $\Phi_{-t}(\mathcal{X} \setminus \mathcal{T})$ and $\Phi_{-t}(\mathcal{T})$ (again since $\Phi$ is a flow). Therefore, $x \in \Phi_{-t}(\mathcal{T}) \subseteq \operatorname{Orb}^-(\mathcal{T})$. ■

## 3. Chains, chain recurrence, and chain equivalence

In this section we define chains and associated notions such as chain recurrence and chain equivalence. We begin by enumerating useful properties of so-called error functions, and then, after introducing the definitions for chains and related notions, give the proof of Hurley [1995] of equivalent characterisations of chain equivalence. Hurley actually characterises chain *recurrence*, but the proofs can be adapted to chain equivalence. Moreover, we shall see that these characterisations of chain equivalence are essential to our proof of the existence of complete Lyapunov functions for continuous-time flows and semiflows.

**3.1. Error functions.** The important observation of [Hurley 1992] was that one can replace the constant $\epsilon$'s in the usual definition of chain recurrence (see the definitions below) with positive continuous functions. The use of nonconstant functions, in combination with the metric, is reminiscent of the construction of the so-called fine topology for the space of continuous functions on a metric space [McCoy, Kundu, and Jindal 2018]. In this section we collect a few useful technical results for positive continuous functions.

First we give a descriptive name to the set positive continuous functions. By $\mathrm{C}^0(\mathcal{A};\mathcal{B})$, we denote the space of continuous functions from the topological space $\mathcal{A}$ to the topological space $\mathcal{B}$.

**3.1 Definition: (Error function)** For a topological space $(\mathcal{X},\mathscr{O})$, an ***error function*** is an element of $\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$. •

Our first result is a general result concerning approximations of semicontinuous functions by continuous function.

**3.2 Lemma: (Bounding upper and lower semicontinuous functions by continuous functions)** *Let $(\mathcal{X},\mathscr{O})$ be a paracompact topological space and let $\underline{f},\overline{f}\colon\mathcal{X}\to\mathbb{R}$ be upper (resp. lower) semicontinuous functions satisfying $\underline{f}(x)<\overline{f}(x)$, $x\in\mathcal{X}$. Then there exists a continuous function $f\colon\mathcal{X}\to\mathbb{R}$ such that*

$$\underline{f}(x)<f(x)<\overline{f}(x),\qquad x\in\mathcal{X}.$$

Proof: For $q\in\mathbb{Q}$, denote

$$\mathcal{O}_q=\{x\in\mathcal{X}\mid\ \underline{f}<q\}\cup\{x\in\mathcal{X}\mid\ \overline{f}(x)>q\},$$

this set being open. Since, for each $x\in\mathcal{X}$, there exists $q\in\mathbb{Q}$ such that $\underline{f}(x)<q<\overline{f}(x)$, it follows that $(\mathcal{O}_q)_{q\in\mathbb{Q}}$ is an open cover of $\mathcal{X}$. Let $(\phi_q)_{q\in\mathbb{Q}}$ be a partition of unity subordinate to $(\mathcal{O}_q)_{q\in\mathbb{Q}}$ and define $f=\sum_{q\in\mathbb{Q}}q\phi_q$. By local finiteness, $f$ is continuous. Also, for $x\in\mathcal{X}$, let $q_1,\dots,q_k\in\mathbb{Q}$ be such that $x\in\operatorname{supp}(\phi_q)$ if and only if $q\in\{q_1,\dots,q_k\}$. Therefore, $x\in\cap_{j=1}^k\mathcal{O}_{q_j}$ and so $\underline{f}(x)<q_j<\overline{f}(x)$, $j\in\{1,\dots,k\}$. Therefore,

$$\underline{f}(x)=\underline{f}(x)\sum_{j=1}^k\phi_{q_j}(x)<\sum_{j=1}^k q_j\phi_{q_j}(x)=f(x)<\overline{f}(x)\sum_{j=1}^k\phi_{q_j}(x)=\overline{f}(x),$$

as desired. ■

Next we give a sort of approximation lemma for functions in $\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$. For a metric space $(\mathcal{X},\mathrm{d})$, $\mathsf{B}_{\mathrm{d}}(r,x)$ denotes the ball of redius $r\in\mathbb{R}_{>0}$ centred at $x\in\mathcal{X}$.

**3.3 Lemma: (Approximation in $\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$)** *Let $(\mathcal{X},\mathrm{d})$ be a metric space. Then, for $\varepsilon\in\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$, there exists $\delta\in\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ such that*

$$\mathrm{d}(x,y)<\delta(x) \implies \tfrac{1}{2}\varepsilon(x)<\varepsilon(y)<\tfrac{3}{2}\varepsilon(x).$$

Proof: First we give a $\delta$ that gives rise to the lower bound. Let $\beta\colon\mathcal{X}\to\mathbb{R}_{>0}$ be defined by

$$\beta(x)=\sup\{\eta\in\mathbb{R}_{>0} \mid \text{ there exists } \alpha\in(\tfrac{1}{2},1) \text{ such that } \mathrm{d}(x,y)<\eta \implies \varepsilon(y)>\alpha\varepsilon(x)\}.$$

Fix $\alpha\in(\frac{1}{2},1)$ so that $\alpha\varepsilon(x)<\varepsilon(x)$ for $x\in\mathcal{X}$. By continuity of $\varepsilon$ at $x\in\mathcal{X}$, there exists $\eta\in\mathbb{R}_{>0}$ be such that

$$\mathrm{d}(x,y)<\eta \implies \varepsilon(y)<\alpha\varepsilon(x).$$

Therefore, $\beta(x)\geq\eta>0$.

We claim that $\beta$ is lower semicontinuous. Let $x\in\mathcal{X}$ and let $a<\beta(x)$. By definition of $\beta$, let $\eta\in\mathbb{R}_{>0}$ be such that $a<\eta$ and such that there exists $\alpha\in(\frac{1}{2},1)$ for which

$$\mathrm{d}(x,y)<\eta \implies \varepsilon(y)>\alpha\varepsilon(x).$$

Let $\gamma\in(\frac{1}{2},\alpha)$. Since $\gamma<\alpha$, $\gamma^{-1}(\alpha-\gamma)\varepsilon(x)>0$. Let $\xi\in(0,\gamma^{-1}(\gamma-\alpha)\varepsilon(x))$. Let $\mathcal{V}$ be a neighbourhood of $x$ such that

$$y\in\mathcal{V} \implies \varepsilon(y)<\varepsilon+\xi.$$

Then we compute, for $y\in\mathcal{V}$,

$$\gamma\varepsilon(y)<\gamma\varepsilon(x)+\gamma\xi<\gamma\varepsilon(x)+(\gamma-\alpha)\varepsilon(x)=\alpha\varepsilon(x). \tag{3.1}$$

Let $\zeta\in(a,\eta)$ and let $r\in(0,\eta-\zeta)$ be sufficiently small that $\mathcal{U}\triangleq\mathsf{B}_{\mathrm{d}}(r,x)\subseteq\mathcal{V}$. We claim that, if $y\in\mathcal{U}$, then $\mathsf{B}_{\mathrm{d}}(\zeta,y)\subseteq\mathsf{B}_{\mathrm{d}}(\eta,x)$. Indeed, let $y\in\mathcal{U}$ and let $z\in\mathsf{B}_{\mathrm{d}}(\zeta,y)$. Then

$$\mathrm{d}(x,z)\leq\mathrm{d}(x,y)+\mathrm{d}(y,z)\leq r+\zeta<\eta-\zeta+\zeta=\eta,$$

as claimed. Thus, if $y\in\mathcal{U}$ and if $z\in\mathsf{B}_{\mathrm{d}}(y,\zeta)$, then $\mathrm{d}(x,z)<\eta$ implying that $\varepsilon(z)>\alpha\varepsilon(x)$. Let $y\in\mathcal{U}\subseteq\mathcal{V}$. Then, as in (3.2), $\gamma\varepsilon(y)<\alpha\varepsilon(x)$, which immediately gives $\varepsilon(z)>\gamma\varepsilon(y)$. Since $\gamma\in(0,\frac{1}{2})$, the definition of $\beta$ implies that we must have $\beta(y)\geq\zeta>a$. This gives the claimed lower semicontinuity of $\beta$.

By Lemma 3.2, let $\delta\in\mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ be such that $\delta(x)\in(0,\beta(x))$ for $x\in\mathcal{X}$. The definition of $\beta$ then implies that there exists $\eta\in\mathbb{R}_{>0}$ such that $\delta(x)<\eta$ and such that

$$\mathrm{d}(x,y)<\eta \implies \varepsilon(y)>\alpha(x)$$

for some $\alpha\in(\frac{1}{2},1)$. Thus we have

$$\mathrm{d}(x,y)<\delta(x)<\eta \implies \varepsilon(y)>\alpha\varepsilon(x)>\tfrac{1}{2}\varepsilon(x),$$

as desired.

Now we give $\delta$ that gives rise to the upper bound. The argument is similar to that in the first part of the proof, but we give it for completeness. Let $\beta\colon\mathcal{X}\to\mathbb{R}_{>0}$ be defined by

$$\beta(x)=\sup\{\eta\in\mathbb{R}_{>0} \mid \text{ there exists } \alpha\in(1,\tfrac{3}{2}) \text{ such that } \mathrm{d}(x,y)<\eta \implies \varepsilon(y)<\alpha\varepsilon(x)\}.$$

Let $\alpha \in (1, \frac{3}{2})$. Since $\varepsilon$ is continuous at $x \in \mathcal{X}$, there exists $\eta \in \mathbb{R}_{>0}$ be such that

$$\mathrm{d}(x, y) < \eta \implies \varepsilon(y) < \alpha\varepsilon(x).$$

By definition of $\beta$, we have $\beta(x) \geq \eta > 0$.

We claim that $\beta$ is lower semicontinuous. Let $x \in \mathcal{X}$ and let $a < \beta(x)$. By definition of $\beta$, let $\eta \in \mathbb{R}_{>0}$ be such that $a < \eta$ and such that there exists $\alpha \in (1, \frac{3}{2})$ for which

$$\mathrm{d}(x, y) < \eta \implies \varepsilon(y) < \alpha\varepsilon(x).$$

Let $\gamma \in (\alpha, \frac{3}{2})$. Since $\gamma > \alpha$, $\gamma^{-1}(\gamma - \alpha)\varepsilon(x) > 0$. Let $\xi \in (0, \gamma^{-1}(\gamma - \alpha)\varepsilon(x))$. Let $\mathcal{V}$ be a neighbourhood of $x$ such that

$$y \in \mathcal{V} \implies \varepsilon(y) > \varepsilon - \xi.$$

Then we compute, for $y \in \mathcal{V}$,

$$\gamma\varepsilon(y) > \gamma\varepsilon(x) - \gamma\xi > \gamma\varepsilon(x) - (\gamma - \alpha)\varepsilon(x) = \alpha\varepsilon(x). \tag{3.2}$$

Let $\zeta \in (a, \eta)$ and let $r \in (0, \eta - \zeta)$ be sufficiently small that $\mathcal{U} \triangleq \mathsf{B}_{\mathrm{d}}(r, x) \subseteq \mathcal{V}$. We claim that, if $y \in \mathcal{U}$, then $\mathsf{B}_{\mathrm{d}}(\zeta, y) \subseteq \mathsf{B}_{\mathrm{d}}(\eta, x)$. Indeed, let $y \in \mathcal{U}$ and let $z \in \mathsf{B}_{\mathrm{d}}(\zeta, y)$. Then

$$\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z) \leq r + \zeta < \eta - \zeta + \zeta = \eta,$$

as claimed. Thus, if $y \in \mathcal{U}$ and if $z \in \mathsf{B}_{\mathrm{d}}(y, \zeta)$, then $\mathrm{d}(x, z) < \eta$ implying that $\varepsilon(z) < \alpha\varepsilon(x)$. Let $y \in \mathcal{U} \subseteq \mathcal{V}$. Then, as in (3.2), $\alpha\varepsilon(x) < \gamma\varepsilon(y)$, which immediately gives $\varepsilon(z) < \gamma\varepsilon(y)$. Since $\gamma \in (1, \frac{3}{2})$, the definition of $\beta$ implies that we must have $\beta(y) \geq \zeta > a$. This gives the claimed lower semicontinuity of $\beta$.

By Lemma 3.2, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that $\delta(x) \in (0, \beta(x))$ for $x \in \mathcal{X}$. The definition of $\beta$ then implies that there exists $\eta \in \mathbb{R}_{>0}$ such that $\delta(x) < \eta$ and such that

$$\mathrm{d}(x, y) < \eta \implies \varepsilon(y) < \alpha(x)$$

for some $\alpha \in (1, \frac{3}{2})$. Thus we have

$$\mathrm{d}(x, y) < \delta(x) < \eta \implies \varepsilon(y) < \alpha\varepsilon(x) < \tfrac{3}{2}\varepsilon(x),$$

as desired.

By choosing the min of the $\delta$'s giving rise to the lower and upper bound, we can ensure a $\delta$ that simultaneously gives rise to both bounds. ■

Next we show how a positive function behaves under continuous maps. The proof has an entirely similar flavour to the preceding lemma.

**3.4 Lemma: (Positive continuous functions and continuous mappings of metric spaces)** *Let* $(\mathcal{X}, \mathrm{d})$ *and* $(\mathcal{X}', \mathrm{d}')$ *be metric spaces and let* $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X}')$. *Then, for* $\varepsilon \in \mathrm{C}^0(\mathcal{X}'; \mathbb{R}_{>0})$, *there exists* $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ *such that*

$$\mathrm{d}(x, y) < \delta(x) \implies \mathrm{d}'(\phi(x), \phi(y)) < \varepsilon(\phi(x)).$$

Proof: For $x \in \mathcal{X}$, define

$$\beta(x) = \sup\{\eta \in \mathbb{R}_{>0} \mid \text{ there exists } \alpha \in (0, \varepsilon(\phi(x))) \text{ such that } \phi(\mathsf{B}_{\mathrm{d}}(\eta, x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x))\},$$

this making sense by continuity of $\phi$. Let $x \in \mathcal{X}$ and let $\alpha \in (0, \varepsilon(\phi(x)))$. Then there exists $\eta \in \mathbb{R}_{>0}$ such that

$$\phi(\mathsf{B}_{\mathrm{d}}(\eta, x) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x)),$$

and so we can conclude that $\beta(x) \geq \eta > 0$.

Now we shall show that $\beta$ is lower semicontinuous. Let $x \in \mathcal{X}$ and let $a < \beta(x)$. Let $\eta \in \mathbb{R}_{>0}$ satisfy $a < \eta$ and

$$\phi(\mathsf{B}_{\mathrm{d}}(\eta, x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x))$$

for some $\alpha \in (0, \varepsilon(\phi(x)))$. Let $\gamma \in (\alpha, \varepsilon(\phi(x)))$. By continuity of $\phi$ and $\varepsilon \circ \phi$, let $\mathcal{U}$ be a neighbourhood of $x$ such that

1. $\mathrm{d}(\phi(x), \phi(y)) < \gamma - \alpha$ and
2. $\varepsilon(\phi(y)) > \gamma$

for $y \in \mathcal{U}$. Let $y \in \mathcal{U}$. Note that

$$\mathrm{d}(z, \phi(x)) < \alpha \implies \mathrm{d}(z, \phi(y)) \leq \mathrm{d}(\phi(x), \phi(y)) + \mathrm{d}(\phi(x), z) < \gamma - \alpha + \alpha = \gamma,$$

whence $\mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\gamma, \phi(y))$. Next choose $\xi \in (a, \eta)$ and choose $b \in (0, \eta - \xi)$ so that $\mathsf{B}_{\mathrm{d}}(b, x) \subseteq \mathcal{U}$. For $y \in \mathcal{U}$ we have

$$\mathrm{d}(z, y) < \xi \implies \mathrm{d}(z, x) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z) < b + \xi < \eta - \xi + \xi = \eta,$$

whereupon $\mathsf{B}_{\mathrm{d}}(\xi, y) \subseteq \mathsf{B}_{\mathrm{d}}(\eta, x)$. We also have

$$\phi(\mathsf{B}_{\mathrm{d}}(\xi, y)) \subseteq \phi(\mathsf{B}_{\mathrm{d}}(\eta, x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\gamma, \phi(y)).$$

From this we deduce that $\beta(y) \geq \xi > a$, which gives lower semicontinuity of $\beta$.

By Lemma 3.2, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that $\delta(x) \in (0, \beta(x))$ for $x \in \mathcal{X}$. Let $x \in \mathcal{X}$. Since $\delta(x) < \beta(x)$, there exists $\eta \in \mathbb{R}_{>0}$ such that $\delta(x) < \eta$ and

$$\phi(\mathsf{B}_{\mathrm{d}}(\eta, x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x))$$

for some $\alpha \in (0, \varepsilon(\phi(x)))$. Then we have

$$\mathrm{d}(x, y) < \delta(x) \implies y \in \mathsf{B}_{\mathrm{d}}(\eta, x),$$

which in turn gives

$$\phi(y) \in \phi(\mathsf{B}_{\mathrm{d}}(\eta, x)) \subseteq \mathsf{B}_{\mathrm{d}'}(\alpha, \phi(x)) \implies \mathrm{d}'(\phi(y), \phi(x)) < \alpha < \varepsilon(\phi(x)),$$

as desired. ■

Our final technical lemma that is of interest to us is the following.

**3.5 Lemma: (Approximation in $C^0(\mathcal{X};\mathbb{R})$ along trajectories)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Let $\varepsilon \in C^0(\mathcal{X};\mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$. Then there exists $\delta \in C^0(\mathcal{X};\mathbb{R}_{>0})$ such that*

$$\mathrm{d}(x, y) < \delta(x) \implies \mathrm{d}(\Phi(t, x), \Phi(t, y)) < \varepsilon(\Phi(t, x)), \qquad t \in \mathbb{T}_{[0,T]}.$$

Proof: We record a lemma (sometimes referred to as the integral continuity condition) for use later in the proof.

**1 Sublemma:** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Let $x_1 \in \mathcal{X}$ and $T \in \mathbb{T}_{>0}$. Then, for $\epsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that*

$$\mathrm{d}(x_1, x_2) < \delta \implies \mathrm{d}(\Phi(t, x_1), \Phi(t, x_2)) < \epsilon, \qquad t \in \mathbb{T}_{[0,T]}.$$

Proof: For concreteness, we suppose that $\Phi$ is a semiflow. For flows, the proof is the same, with the only change being the domain of $\Phi$.

Let $\epsilon \in \mathbb{R}_{>0}$, let $x_1 \in \mathcal{X}$, and let $T \in \mathbb{T}_{>0}$. Note that

$$(t, x_2) \mapsto \mathrm{d}(\Phi(t, x_1), \Phi(t, x_2))$$

is continuous since it is a composition of the continuous maps

$$\begin{aligned}
&\mathbb{T}_{\geq 0} \times \mathcal{X} \ni (t, x_2) \mapsto (t, (t, x_2)) \in \mathbb{T}_{\geq 0} \times (\mathbb{T}_{\geq 0} \times \mathcal{X}),\\
&(s, (t, x_2)) \mapsto (\Phi(s, x_1), \Phi(t, x_2)),\\
&\mathcal{X} \times \mathcal{X} \ni (y_1, y_2) \mapsto \mathrm{d}(y_1, y_2) \in \mathbb{R}_{\geq 0}.
\end{aligned}$$

For each $t \in \mathbb{T}_{[0,T]}$, there is a neighbourhood $\mathcal{U}_t$ of $x_1$ and an open (relatively in $\mathbb{T}$) interval $I_t$ around $t$ such that

$$\mathrm{d}(\Phi(s, x), \Phi(s, x_1)) < \epsilon, \qquad (s, x) \in I_t \times \mathcal{U}_t,$$

by continuity. Note that $(I_t)_{t\in\mathbb{T}_{[0,T]}}$ is an open cover of $\mathbb{T}_{[0,T]}$ and so there exist $t_1, \dots, t_k \in \mathbb{T}_{[0,T]}$ such that $\mathbb{T}_{[0,T]} \subseteq \cup_{t_j} I_{t_j}$. Let $\mathcal{U} = \cap_{j=1}^k \mathcal{U}_{t_j}$. For $t \in \mathbb{T}_{[0,T]}$ and $x_2 \in \mathcal{U}$, we have $(t, x_2) \in I_{t_j} \times \mathcal{U}_{t_j}$ for some $j \in \{1, \dots, k\}$, whence

$$\mathrm{d}(\Phi(t, x_1), \Phi(t, x_2)) < \epsilon.$$

The result follows by taking $\delta$ sufficiently small that $\mathsf{B}_{\mathrm{d}}(\delta, x_1) \subseteq \mathcal{U}$. ▼

For $x, y \in \mathcal{X}$, denote

$$M_x = \inf\{\varepsilon(\Phi(t, x)) \mid t \in \mathbb{T}_{[0,T]}\}$$

and

$$\rho(x, y) = \sup\{\mathrm{d}(\Phi(t, x), \Phi(t, y)) \mid t \in \mathbb{T}_{[0,T]}\},$$

and define $\beta\colon \mathcal{X} \to \mathbb{R}_{>0}$ by

$$\beta(x) = \sup\{\eta \in \mathbb{R}_{>0} \mid \text{there exists } \alpha \in (0, 1) \text{ such that } \mathrm{d}(x, y) < \eta \implies \rho(x, y) < \alpha M_x\}.$$

Let $\alpha \in (0, 1)$ so $\alpha M_x > 0$. Let $\eta \in \mathbb{R}_{>0}$ be such that

$$\mathrm{d}(x, y) < \eta \implies \mathrm{d}(\Phi(t, x), \Phi(t, y)) < \alpha M_x, \qquad t \in \mathbb{T}_{[0,T]};$$

this is possible by Sublemma 1. Note that, if $\rho(x,y) < \alpha M_x$, then $\beta(x) \geq \eta > 0$ by definition of $\beta$.

We claim that $\beta$ is lower semicontinuous. Let $x \in \mathcal{X}$ and let $a < \beta(x)$. By definition of $\beta$, let $\eta \in \mathbb{R}_{>0}$ satisfy $a < \eta$ and satisfy

$$\mathrm{d}(x,y) < \eta \implies \rho(x,y) < \alpha M_x$$

for some $\alpha \in (0,1)$.

We claim that, for $\zeta \in \mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{V}$ of $x$ such that

$$\varepsilon(\Phi(t,x)) - \zeta < \varepsilon(\Phi(t,y)) < \varepsilon(\Phi(t,x)) + \zeta, \qquad (t,y) \in \mathbb{T}_{[0,T]} \times \mathcal{V}.$$

To see this, note that continuity of $\varepsilon \circ \Phi$ implies that, for $t \in \mathbb{T}_{[0,T]}$, there is a neighbourhood $\mathcal{V}_t$ of $x$ and an open (relatively in $\mathbb{T}$) set $I_t$ about $t$ such that

$$|\varepsilon(\Phi(t,x)) - \varepsilon(\Phi(s,y))| < \frac{\zeta}{2}, \qquad (s,y) \in I_t \times \mathcal{V}_t.$$

By compactness of $\mathbb{T}_{[0,T]}$, let $t_1, \dots, t_k \in \mathbb{T}_{[0,T]}$ be such that $\mathbb{T}_{[0,T]} \subseteq \cup_{j=1}^k I_{t_j}$. Denote $\mathcal{V} = \cap_{j=1}^k \mathcal{V}_{t_j}$. Let $(t,y) \in \mathbb{T}_{[0,T]} \times \mathcal{V}$. Then $t \in I_{t_j}$ for some $j \in \{1,\dots,k\}$. We also have $x, y \in \mathcal{V}_{t_j}$. Then

$$\begin{aligned} &|\varepsilon(\Phi(t_j,x)) - \varepsilon(\Phi(t,x))|, |\varepsilon(\Phi(t_j,x)) - \varepsilon(\Phi(t,y))| < \frac{\zeta}{2} \\ \implies &|\varepsilon(\Phi(t,y)) - \varepsilon(\Phi(t,x))| < \zeta, \end{aligned}$$

establishing our claim.

Let $\gamma, \kappa \in \mathbb{R}_{>0}$ satisfy

$$0 < \alpha < \gamma < \kappa < 1, \quad \frac{\kappa - \gamma}{\kappa} M_x > 0.$$

As per the preceding paragraph, there exists a neighbourhood $\mathcal{V}$ of $x$ such that, for $(t,y) \in \mathbb{T}_{[0,T]} \times \mathcal{V}$,

$$\begin{aligned} \varepsilon(\Phi(t,y)) &> \varepsilon(\Phi(t,x)) - \frac{\kappa - \gamma}{\kappa} M_x \\ &\geq M_x - \frac{\kappa - \gamma}{\kappa} M_x = \frac{\gamma}{\kappa} M_x. \end{aligned}$$

This gives the inequalities

$$M_y > \frac{\gamma}{\kappa} M_x \implies \gamma M_x < \theta M_y, \qquad (\gamma - \alpha) M_x > 0 \tag{3.3}$$

for $y \in \mathcal{V}$.

Since $\gamma - \alpha \in (0,1)$, let $\xi \in \mathbb{R}_{>0}$ be such that

$$\mathrm{d}(x,y) < \xi \implies \rho(x,y) < (\gamma - \alpha) M_x;$$

as above, this is possible by Sublemma 1.

Let $\zeta \in \mathbb{R}_{>0}$ be such that, if $0 < r < \min\{\eta - \zeta, \xi\}$, then $\mathcal{U} \triangleq \mathsf{B}_{\mathrm{d}}(r, x) \subseteq \mathcal{V}$. We claim that $\mathsf{B}_{\mathrm{d}}(\zeta, y) \subseteq \mathsf{B}_{\mathrm{d}}(\eta, x)$ if $y \in \mathcal{U}$. Indeed, if $\mathrm{d}(y, z) < \zeta$, then

$$\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z) < \eta - \zeta + \zeta = \eta,$$

as claimed. Thus, if $y \in \mathcal{U}$ and $z \in \mathsf{B}_{\mathrm{d}}(\zeta, y)$, then

$$\mathrm{d}(x, z) < \eta \implies \rho(x, z) < \alpha M_x.$$

If $y \in \mathcal{U}$ then

$$\mathrm{d}(x, y) < b < \xi \implies \rho(x, y) < (\gamma - \alpha) M_x.$$

By (3.3), $\gamma M_x < \kappa M_y$ if $y \in \mathcal{U}$. Thus, if $y \in \mathcal{U}$ and $z \in \mathsf{B}_{\mathrm{d}}(\zeta, y)$, then

$$\rho(y, z) \leq \rho(y, x) + \rho(x, z) < (\gamma - \alpha) M_x + \alpha M_x = \gamma M_x < \kappa M_y.$$

The definition of $\beta$ then gives $\beta(y) \geq \zeta > a$, giving the desired lower semicontinuity of $\beta$.

By Lemma 3.2, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that $0 < \delta(x) < \beta(x)$, $x \in \mathcal{X}$. The definition of $\beta$ implies that, for a given $x \in \mathcal{X}$, there exists $\eta \in \mathbb{R}_{>0}$ such that $\eta > \delta(x)$ such that

$$\mathrm{d}(x, y) < \eta \implies \rho(x, y) < \alpha M_x$$

for some $\alpha \in (0, 1)$. The definitions of $\rho$ and $M_x$ give

$$\mathrm{d}(\Phi(t, x), \Phi(t, y)) \leq \rho(x, y) < \alpha M_x < \alpha \varepsilon(\Phi(t, x)), \qquad t \in \mathbb{T}_{[0,T]},$$

which is the desired conclusion. ∎

**3.2. Chain recurrence.** We next introduce chains and chain recurrence. The notion of chain we use is the following.

**3.6 Definition: ($(\varepsilon, T)$-chain, $\varepsilon$-$T$-chain)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. For $x, y \in \mathcal{X}$, $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, and $T \in \mathbb{T}_{>0}$, an ***$(\varepsilon, T)$-chain*** for $\Phi$ from $x$ to $y$ is two finite sequences

$$x_0, x_1, \dots, x_k, \quad t_0, t_1, \dots, t_{k-1}$$

with

(i) $x_0, x_1, \dots, x_k \in \mathcal{X}$,

(ii) $t_0, t_1, \dots, t_{k-1} \in \mathbb{T}_{\geq T}$,

(iii) $x_0 = x$ and $x_k = y$, and

(iv) $\mathrm{d}(\Phi(t_j, x_j), x_{j+1}) < \varepsilon(\Phi(t_j, x_j))$, $j \in \{0, 1, \dots, k-1\}$.

For such an $(\varepsilon, T)$-chain, its ***length*** is $k$. An ***$\varepsilon$-$T$-chain*** is an $(\varepsilon, T)$ chain

$$x_0, x_1, \dots, x_k, \quad t_0, t_1, \dots, t_{k-1}$$

for which $t_j = T$, $j \in \{0, 1, \dots, k-1\}$. •

In Figure 1 we depict a chain. We can then define an associated notion of recurrence.

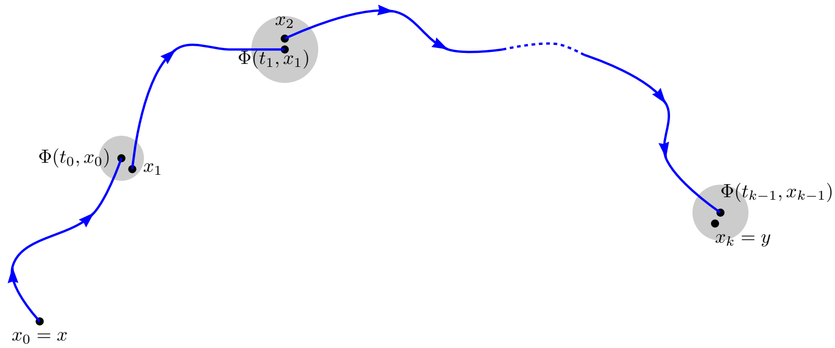


**Figure 1.** An $(\varepsilon, T)$-chain (the depiction is of the continuous-time case)

**3.7 Definition: (Chain recurrence)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$.

(i) A point $x \in \mathcal{X}$ is ***chain recurrent*** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$, there exists an $(\varepsilon, T)$-chain from $x$ to itself.

(ii) Let $T \in \mathbb{T}_{>0}$. A point $x \in \mathcal{X}$ is ***T-chain recurrent*** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, there exists an $(\varepsilon, T)$-chain from $x$ to itself.

(iii) Let $T \in \mathbb{T}_{>0}$. A point $x \in \mathcal{X}$ is ***exactly T-chain recurrent*** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, there exists an $\varepsilon$-$T$-chain from $x$ to itself.

(iv) We denote by $\mathrm{ChRec}(\Phi)$ the set of chain recurrent points for $\Phi$.

(v) We denote by $\mathrm{ChRec}_{\geq T}(\Phi)$ the set of $T$-chain recurrent points for $\Phi$.

(vi) We denote by $\mathrm{ChRec}_{=T}(\Phi)$ the set of exactly $T$-chain recurrent points for $\Phi$. •

In the original definition of chain recurrence by Conley [1978], constant error functions are used. For compact spaces, it is easy to see that the definitions for constant and non-constant error functions are equivalent. For noncompact spaces, however, they are not the same, as the following example shows.

**3.8 Example: (Chain recurrence with constant error functions is metric-dependent)** Let $\mathcal{X} = \mathbb{R} \times \mathbb{R}_{>0}$ be the upper half-plane. We let d be the standard Euclidean metric for $\mathcal{X}$ inherited from $\mathbb{R}^2$ and we let $\mathrm{d}_{\mathrm{h}}$ be the so-called hyperbolic metric, i.e., that metric whose geodesics are arcs of half-circles in $\mathcal{X}$ with centres on the line $\mathbb{R} \times \{0\}$. Note that the topology defined by these two metrics is the same. On $\mathcal{X}$, we consider the continuous-time flow defined by $\Phi(t, (x, y)) = (x+t, y)$. Let $\mathrm{ChRec}'(\Phi)$ be the chain recurrent set where chains are defined using constant error functions and using the metric d. Let $\mathrm{ChRec}'_{\mathrm{h}}(\Phi)$ be the chain recurrent set where chains are defined using constant error functions and using the metric $\mathrm{d}_{\mathrm{h}}$. It is not difficult to show that $\mathrm{ChRec}'(\Phi) = \varnothing$ and that $\mathrm{ChRec}'_{\mathrm{h}}(\Phi) = \mathcal{X}$. The idea in proving the second of these formulae is to observe that horizontal distances in the hyperbolic metric become much larger than their Euclidean counterparts as one gets close (in the Euclidean sense) to $\mathbb{R} \times \{0\}$. The details are given in [Alongi and Nelson 2007, Example 2.7.13]. •

The example illustrates the dangers of using constant error functions in defining chains. Specifically, chain recurrence becomes a metric-dependent concept in this case. There is not necessarily anything wrong with this, of course; perhaps one is content to have notions that depend on metric, e.g., on normed vector spaces. However, the Conley decomposition relates chain recurrence to attracting and repelling sets, and these latter are purely topological concepts. Thus one cannot expect that the Conley decomposition is valid when chains are defined using constant error functions. Nonetheless, the use of constant error functions to define chains is common in the literature, even for noncompact states spaces.

Let us enumerate some properties of the chain recurrent set. We begin by relating chain recurrence to one of the classical notions of recurrence, that of a nonwandering point.

**3.9 Definition: (Nonwandering point)** Let $(\mathcal{X},\mathscr{O})$ be a topological space and let $\Phi$ be a topological flow or semiflow. A point $x_0 \in \mathcal{X}$ is ***nonwandering*** for $\Phi$ if there exists $T \in \mathbb{T}_{>0}$ such that, for each neighbourhood $\mathcal{U}$ of $x_0$, $\Phi_t(\mathcal{U}) \cap \mathcal{U} \neq \varnothing$ for some $t \in \mathbb{T}_{\geq T}$. We denote by $\mathrm{NWnd}(\Phi)$ the set forward nonwandering points for $\Phi$. •

The following characterisation of the set of nonwandering points will be useful.

**3.10 Lemma: (Characterisation of nonwandering set for first countable topological spaces)** *Let $(\mathcal{X},\mathscr{O})$ be a first countable Hausdorff topological space and let $\Phi$ be a topological flow or semiflow. Let $x \in \mathrm{NWnd}(\Phi)$. Then, for a neighbourhood $\mathcal{U}$ of $x$ and for $T \in \mathbb{T}_{>0}$, there exists $t \in \mathbb{T}_{\geq T}$ such that $\mathcal{U} \cap \Phi_t(\mathcal{U}) \neq \varnothing$.*

Proof: We prove the contrapositive. Thus, we let $x \in \mathcal{X}$, we assume that there exists a neighbourhood $\mathcal{U}$ of $x$ and $T \in \mathbb{T}_{>0}$ such that $\mathcal{U} \cap \Phi_t(\mathcal{U}) = \varnothing$ for $t \in \mathbb{T}_{\geq T}$, and we prove that $x \notin \mathrm{NWnd}(\Phi)$.

Note that the assumptions ensure that $x$ is not a periodic point. We claim that this implies that, for any $S \in \mathbb{T}_{>0}$, there exists a neighbourhood $\mathcal{V}$ of $x$ such that $\mathcal{V} \cap \Phi_t(\mathcal{V}) = \varnothing$ for all $t \in [S,T]$. We prove this by the contrapositive. Thus we let $x \in \mathcal{X}$ and $S \in \mathbb{T}_{>0}$, we assume that, for any neighbourhood $\mathcal{V}$ of $x$, there exists $t \in [S,T]$ such that $\mathcal{V} \cap \Phi_t(\mathcal{V}) \neq \varnothing$, and we prove that $x$ is a periodic point. By first countability of $(\mathcal{X},\mathscr{O})$, let $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ be a neighbourhood base for $x$. Our hypotheses ensure that, for each $j \in \mathbb{Z}_{>0}$, there exists $x_j \in \mathcal{X}$ and $t_j \in [S,T]$ such that

$$x_j \in \mathcal{U}_j,\ \Phi(t_j,x_j) \in \mathcal{V}_j \cap \Phi_{t_j}(\mathcal{V}_j), \qquad j \in \mathbb{Z}_{>0}.$$

In particular, $\lim_{j\to\infty} x_j = x$. Since the sequence of times $(t_j)_{j\in\mathbb{Z}_{>0}}$ resides in the compact interval $[S,T]$, there exists a subsequence $(t_{j_k})_{k\in\mathbb{Z}_{>0}}$ that converges to some $\tau \in [S,T]$. Then we have

$$\Phi(\tau,x) = \lim_{j_k\to\infty} \Phi(t_{j_k},x_{j_k}) = x.$$

This shows that $x$ is a periodic point.

Now, combining the first two paragraphs of the proof, for any $S \in \mathbb{T}_{>0}$, there exist neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of $x$ such that

$$(\mathcal{U} \cap \mathcal{V}) \cap \Phi_t(\mathcal{U} \cap \mathcal{V}) = \varnothing$$

for all $t \in \mathbb{T}_{\geq S}$. That is to say, $x \notin \mathrm{NWnd}(\Phi)$. ■

We now show that nonwandering points are chain recurrent.

**3.11 Proposition: (Nonwandering points as chain recurrent points)** *Let* $(\mathcal{X}, \mathrm{d})$ *be a metric space and let* $\Phi$ *be a topological flow or semiflow. Then* $\mathrm{NWnd}(\Phi) \subseteq \mathrm{ChRec}(\Phi)$*.*

Proof: Let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and let $T \in \mathbb{T}_{>0}$. Let $x \in \mathrm{NWnd}(\Phi)$. Since $\Phi_T$ and $\varepsilon$ are continuous, let $\delta \in (0, \frac{1}{2}\varepsilon(x))$ be such that

$$\mathrm{d}(x, y) < \delta \implies \mathrm{d}(\Phi(T, x), \Phi(T, y)) < \varepsilon(\Phi(T, x)).$$

and

$$\mathrm{d}(x, y) < \delta \implies |\varepsilon(y) - \varepsilon(x)| < \tfrac{1}{2}\varepsilon(x).$$

By Lemma 3.10, noting that metric spaces are first countable and Hausdorff, let $t_1 \in \mathbb{T}_{>2T}$ be such that $\mathsf{B}_{\mathrm{d}}(\delta, x) \cap \Phi_{t_1}(\mathsf{B}_{\mathrm{d}}(\delta, x)) \neq \varnothing$, and let $y \in \mathsf{B}_{\mathrm{d}}(\delta, x)$ be such that $\Phi(t_1, y) \in \mathsf{B}_{\mathrm{d}}(\delta, x)$. Then

$$\mathrm{d}(x, y) < \delta \implies \mathrm{d}(\Phi(T, x), \Phi(T, y)) < \varepsilon(\Phi(T, x)).$$

Also,

$$\mathrm{d}(\Phi(t_1 - T, \Phi(T, y)), x) = \mathrm{d}(\Phi(t_1, y), x) < \delta < \tfrac{1}{2}\varepsilon(x).$$

Thus,

$$\varepsilon(\Phi(t_1, y)) - \varepsilon(x) > -\tfrac{1}{2}\varepsilon(x) \implies \varepsilon(\Phi(t_1, y)) > \tfrac{1}{2}\varepsilon(x),$$

and so

$$\mathrm{d}(\Phi(t_1 - T, \Phi(T, y)), x) < \varepsilon(\Phi(t_1, y)).$$

We conclude, then, that

$$x, \Phi(T, y), x, \quad T, t_1 - T$$

is an $(\varepsilon, T)$-chain, and so $x \in \mathrm{ChRec}(\Phi)$. ∎

The invariance and closedness of the chain recurrent set will be needed in our results below.

**3.12 Proposition: (Properties of sets of chain recurrent points)** *Let* $(\mathcal{X}, \mathrm{d})$ *be a metric space and let* $\Phi$ *be a topological flow (resp. semiflow) on* $\mathcal{X}$*. Then the following statements hold:*

*(i)* $\mathrm{ChRec}(\Phi)$ *is invariant (resp. forward-invariant) for* $\Phi$*;*

*(ii)* $\mathrm{ChRec}(\Phi)$ *is closed;*

Proof: (i) Let $x_0 \in \mathrm{ChRec}(\Phi)$ and let $t \in \mathbb{T}$. Let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and let $T \in \mathbb{T}_{>0}$.

Assume first that $t \in \mathbb{T}_{>0}$. Let $T' = T + t$. Let $\mathcal{U}$ be a neighbourhood of $\Phi(t, x_0)$, chosen so that

$$y \in \mathcal{U} \implies \begin{cases} \mathrm{d}(y, \Phi(t, x_0)) < \frac{1}{2}\varepsilon(\Phi(t, x_0)), \\ \varepsilon(y) > \frac{1}{2}\varepsilon(\Phi(t, x_0)). \end{cases}$$

Let $\delta \in \mathbb{R}_{>0}$ be small enough that

$$\mathrm{d}(x_0, x) < \delta \implies \Phi(t, x) \in \mathcal{U}.$$

Take $\eta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ to be $\eta(x) = \min\{\varepsilon(x), \delta\}$, $x \in \mathcal{X}$. Let

$$x_0, x_1, \dots, x_n = x_0, \quad t_0, t_1, \dots, t_{n-1}$$

be an $(\eta, T')$-chain and denote

$$y_0 = \Phi(t, x_0), y_1 = x_1, \dots, y_{n-1} = x_{n-1}, y_n = \Phi(t, x_0),$$
$$s_0 = t_0 - t, s_1 = t_1, \dots, s_{n-2} = t_{n-2}, s_{n-1} = t_{n-1} + t.$$

Note that

$$\mathrm{d}(\Phi(s_0, y_0), y_1) = \mathrm{d}(\Phi(t_0 - t, \Phi(t, x_0)), x_1) = \mathrm{d}(\Phi(t_0, x_0), x_1).$$

Thus we have

$$\mathrm{d}(\Phi(s_0, y_0), y_1) < \eta(\Phi(t_0, x_0)) \leq \varepsilon(\Phi(s_0, y_0)).$$

Clearly we have

$$\mathrm{d}(\Phi(s_j, y_j), y_{j+1}) < \varepsilon(\Phi(s_j, y_j)), \qquad j \in \{1, \dots, n-2\}.$$

Finally, we have

$$\mathrm{d}(\Phi(s_{n-1}, y_{n-1}), y_n) = \mathrm{d}(\Phi(t_{n-1} + t, y_{n-1}), y_n) = \mathrm{d}(\Phi(t, \Phi(t_{n-1}, x_{n-1})), \Phi(t, x_0)).$$

From this and the various definitions we calculate

$$\begin{aligned}
& \mathrm{d}(\Phi(t_{n-1}, x_{n-1}), x_0) < \eta(\Phi(t_{n-1}, x_{n-1})) < \delta \\
\implies & \Phi(t, \Phi(t_{n-1}, x_{n-1})) \in \mathcal{U} \\
\implies & \begin{cases} \mathrm{d}(\Phi(t, \Phi(t_{n-1}, x_{n-1})), \Phi(t, x_0)) < \frac{1}{2}\varepsilon(\Phi(t, x_0)), \\ \varepsilon(\Phi(t, \Phi(t_{n-1}, x_{n-1}))) > \frac{1}{2}\varepsilon(\Phi(t, x_0)) \end{cases} \\
\implies & \mathrm{d}(\Phi(t, \Phi(t_{n-1}, x_{n-1})), \Phi(t, x_0)) < \varepsilon(\Phi(t, \Phi(t_{n-1}, x_{n-1}))) \\
\implies & \mathrm{d}(\Phi(s_{n-1}, y_{n-1}), y_n) < \varepsilon(\Phi(t, \Phi(t_{n-1}, x_{n-1}))).
\end{aligned}$$

Therefore,

$$y_0, y_1, \dots, y_n = y_0, \quad s_0, s_1, \dots, s_{n-1}$$

is an $(\varepsilon, T)$-chain from $\Phi(t, x)$ to $\Phi(t, x)$.

If $\Phi$ is a flow and if $t \in \mathbb{T}_{<0}$, then let $T' = T - t$. One can then proceed exactly as in the preceding paragraph to deduce that $\Phi(t, x_0) \in \mathrm{ChRec}(\Phi)$.

(ii) Let $x_0 \in \mathrm{cl}(\mathrm{ChRec}(\Phi))$, and let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$. Let $\delta \in \mathbb{R}_{>0}$ be such that

$$\mathrm{d}(x_0, x) < \delta \implies \begin{cases} \mathrm{d}(\Phi(T, x_0), \Phi(T, x)) < \frac{1}{2}\varepsilon(\Phi(T, x_0)), \\ \varepsilon(x) > \frac{1}{2}\varepsilon(x_0). \end{cases}$$

Since $x_0 \in \mathrm{cl}(\mathrm{ChRec}(\Phi))$, let $x \in \mathrm{ChRec}(\Phi)$ be such that

$$\mathrm{d}(x_0, x) < \tfrac{1}{2} \min\{\delta, \varepsilon(\Phi(T, x_0)), \tfrac{1}{2}\varepsilon(x_0)\}.$$

Let $\eta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be defined by $\eta(x) = \frac{1}{2}\min\{\delta, \varepsilon(x)\}$.

First note that

$$x_0, x_1 = \Phi(T, x), \quad T$$

is an $(\varepsilon, T)$-chain from $x_0$ to $\Phi(T, x)$.

Now, since $x \in \mathrm{ChRec}(\Phi)$, let

$$y_0 = x, y_1, \dots, y_n = x, \quad s_0, s_1, \dots, s_{n-1}$$

be an $(\eta, 2T)$-chain from $x$ to itself. Since

$$\Phi(s_0 - T, \Phi(T, x)) = \Phi(s_0, x),$$

it follows immediately that

$$\Phi(T, x), y_1, \dots, y_{n-1}, \quad s_0 - T, s_1, \dots, s_{n-2}$$

is an $(\varepsilon, T)$-chain from $\Phi(T, x)$ to $y_{n-1}$.

Finally, we claim that

$$y_{n-1}, x_0, \quad s_{n-1}$$

is an $(\varepsilon, T)$-chain from $y_{n-1}$ to $x_0$. First note that,

$$\begin{aligned} \mathrm{d}(\Phi(s_{n-1}, y_{n-1}), x_0) &\le \mathrm{d}(\Phi(s_{n-1}, y_{n-1}), x) + \mathrm{d}(x, x_0) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Therefore,

$$\varepsilon(\Phi(s_{n-1}, y_{n-1})) > \tfrac{1}{2}\varepsilon(x_0),$$

and so

$$\begin{aligned} \mathrm{d}(\Phi(s_{n-1}, y_{n-1}), x_0) &\le \mathrm{d}(\Phi(s_{n-1}, y_{n-1}), x) + \mathrm{d}(x, x_0) \\ &< \tfrac{1}{2}\varepsilon(\Phi(s_{n-1}, y_{n-1})) + \tfrac{1}{4}\varepsilon(x_0) \\ &< \varepsilon(\Phi(s_{n-1}, y_{n-1})), \end{aligned}$$

giving our claim.

Putting the above three constructions together, we see that

$$x_0, \Phi(T, x), y_1, \dots, y_{n-1}, x_0, \quad T, s_0 - T, s_1, \dots, s_{n-1}$$

is an $(\varepsilon, T)$-chain from $x_0$ to itself, whence $x_0 \in \mathrm{ChRec}(\Phi)$. ■

**3.3. Chain equivalence.** On the set of chain recurrent points of a topological flow or semiflow, there is an important equivalence relation.

**3.13 Definition: (Chain equivalence)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$.

(i) Points $x, y \in \mathcal{X}$ are ***chain equivalent*** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and for each $T \in \mathbb{T}_{>0}$, there exist $(\varepsilon, T)$-chains

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

and

$$y_0 = y, y_1, \dots, y_m = x, \quad s_0, s_1, \dots, s_{m-1}.$$

(ii) Let $T \in \mathbb{T}_{>0}$. Points $x, y \in \mathcal{X}$ are ***T*-chain equivalent** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, there exist $(\varepsilon, T)$-chains

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

and

$$y_0 = y, y_1, \dots, y_m = x, \quad s_0, s_1, \dots, s_{m-1}.$$

(iii) Let $T \in \mathbb{T}_{>0}$. Points $x, y \in \mathcal{X}$ are ***exactly T-chain equivalent*** for $\Phi$ if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, there exist $\varepsilon$-$T$-chains

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

and

$$y_0 = y, y_1, \dots, y_m = x, \quad s_0, s_1, \dots, s_{m-1}.$$

(iv) A ***chain component*** for $\Phi$ is a subset $C \subseteq \mathcal{X}$ such that, if $x, y \in C$, then $x$ and $y$ are chain equivalent.

(v) Let $T \in \mathbb{T}_{>0}$. A ***T-chain component*** for $\Phi$ is a subset $C \subseteq \mathcal{X}$ such that, if $x, y \in C$, then $x$ and $y$ are $T$-chain equivalent.

(vi) Let $T \in \mathbb{T}_{>0}$. An ***exact T-chain component*** for $\Phi$ is a subset $C \subseteq \mathcal{X}$ such that, if $x, y \in C$, then $x$ and $y$ are exactly $T$-chain equivalent. •

Note that the relation of chain equivalence is not generally an equivalence relation on $\mathcal{X}$; it is symmetric and transitive, but not generally reflexive. However, chain equivalence is an equivalence relation on $\mathrm{ChRec}(\Phi)$. Therefore, $\mathrm{ChRec}(\Phi)$ is a disjoint union of chain components. This fact features significantly in the Fundamental Theorem of Dynamical Systems.

Let us prove a few essential properties of chain components.

**3.14 Proposition: (Properties of chain components)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow (resp. semiflow) on $\mathcal{X}$. Then the following statements hold:*

*(i) if $C$ is a chain component for $\Phi$, then it is closed;*

*(ii) if $C$ is a chain component for $\Phi$, then it is invariant (resp. forward-invariant).*

Proof: (i) Let $C \subseteq \mathrm{ChRec}(\Phi)$ be a chain component. To prove closedness of $C$, we shall show that, if $y \in \mathrm{cl}(C)$, then $y \in C$. To do this, we will let $x \in C$, $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, and $T \in \mathbb{T}_{>0}$ and construct two $(\varepsilon, T)$-chains, one from $x$ to $y$ and one from $y$ to $x$. Since $x \in \mathrm{ChRec}(\Phi)$, there is also an $(\varepsilon, T)$-chain from $x$ to itself. By concatenating chains, one concludes that (1) $y \in \mathrm{ChRec}(\Phi)$ and (2) $y$ is chain equivalent to any point in $C$. This suffices to show that $y \in C$.

By Lemma 3.4, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(x_1, x_2) < \delta(x_1) \implies \mathrm{d}(\Phi(T, x_1), \Phi(T, x_2)) < \varepsilon(\Phi(T, x_1)).$$

Let $x' \in C$ be such that $\mathrm{d}(x', y) < \delta(y)$; this is possible since $y \in \mathrm{cl}(C)$. Since $x, x' \in C$, let

$$x_0 = x', x_1, \dots, x_k = x, \quad t_0, t_1, \dots, t_{k-1}$$

be an $(\varepsilon, 2T)$-chain from $x'$ to $x$. Note that

$$\mathrm{d}(x', y) < \delta(y) \implies \mathrm{d}(\Phi(T, x'), \Phi(T, y)) < \varepsilon(\Phi(T, y)),$$

and from this we easily deduce that

$$y_0 = y, y_1 = \Phi(T, x'), y_2 = x_1, \dots, y_{k+1} = x_k = x, \quad s_0 = T, s_1 = t_0 - T, \dots, s_k = t_{k-1}$$

is an $(\varepsilon, T)$-chain from $y$ to $x$.

Now we find an $(\varepsilon, T)$-chain from $x$ to $y$. By Lemma 3.3, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(x_1, x_2) < \delta(x_1) \implies \frac{1}{2}\varepsilon(x_1) < \varepsilon(x_2) < \frac{3}{2}\varepsilon(x_1).$$

Let $x' \in C$ satisfy $\mathrm{d}(y, x') < \max\{\delta(y), \frac{1}{6}\varepsilon(y)\}$, this being possible since $y \in \mathrm{cl}(C)$. Let

$$x_0 = x, x_1, \dots, x_k = x', \quad t_0, t_1, \dots, t_{k-1}$$

be a $(\frac{1}{2}\varepsilon, T)$-chain from $x$ to $x'$. We have

$$\begin{aligned}\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) &\leq \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), x') + \mathrm{d}(y, x') \\ &< \frac{1}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})) + \frac{1}{6}\varepsilon(y)\end{aligned}$$

We have

$$\mathrm{d}(x', y) < \delta(y) \implies \frac{1}{6}\varepsilon(y) < \frac{1}{3}\varepsilon(x')$$

and

$$\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), x') < \delta(\Phi(t_{k-1}, x_{k-1})) \implies \frac{1}{3}\varepsilon(x') < \frac{1}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})).$$

Putting this all together, we have

$$\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) < \varepsilon(\Phi(t_{k-1}, x_{k-1})).$$

From this, we conclude that

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

is an $(\varepsilon, T)$-chain from $x$ to $y$.

(ii) Let $C$ be a chain component, let $x \in C$, and let $t \in \mathbb{T}_{>0}$ (if $\Phi$ is a semiflow) and $t \in \mathbb{T}$ (if $\Phi$ is a flow). As in the preceding part of the proof, to show that $\Phi(t, x) \in C$, it suffices to show that, for any $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$, there exist two $(\varepsilon, T)$-chains, one from $x$ to $\Phi(t, x)$ and one from $\Phi(t, x)$ to $x$.

Since $C \subseteq \mathrm{ChRec}(\Phi)$, let

$$x_0 = x, x_1, \dots, x_k = x, \quad t_0, t_1, \dots, t_{k-1}$$

be an $(\varepsilon, T + |t|)$-chain from $x$ to itself. Then it is evident that

$$\Phi(t, x), x_1, \dots, x_k = x, \quad t_0 - t, t_1, \dots, t_{k-1}$$

is an $(\varepsilon, T)$-chain from $\Phi(t, x)$ to $x$.

To construct an $(\varepsilon, T)$-chain from $x$ to $\Phi(t, x)$, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(x_1, x_2) < \delta(x_1) \implies \mathrm{d}(\Phi(t, x_1), \Phi(t, x_2)) < \varepsilon(x_1).$$

Let

$$x_0 = x, x_1, \dots, x_k = x, \quad t_0, t_1, \dots, t_{k-1}$$

be a $(\min\{\varepsilon, \delta\}, T + |t|)$-chain from $x$ to itself; this is possible since $x \in C \subseteq \mathrm{ChRec}(\Phi)$. Since

$$\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), x) < \delta(\Phi(t_{k-1}, x_{k-1})),$$

we have

$$\begin{aligned}\mathrm{d}(\Phi(\Phi(t_{k-1} + t), x_{k-1}), \Phi(t, x)) &= \mathrm{d}(\Phi(t, \Phi(t_{k-1}, x_{k-1})), \Phi(t, x)) \\ &< \varepsilon(\Phi(t_{k-1}, x_{k-1})).\end{aligned}$$

Thus we conclude that

$$x_0 = 0, x_1, \dots, x_{k-1}, \Phi(t, x), \quad t_0, t_1, \dots, t_{k-2}, t_{k-1} + t$$

is an $(\varepsilon, T)$-chain from $x$ to $\Phi(t, x)$. ■

**3.4. Alternative characterisations of chain equivalence.** Our objective in this section is to give two alternative characterisations of chain equivalence, as per [Hurley 1995]. There are two principal simplifications we consider. First we shall show that, for chain equivalence, it is possible to fix $T$, provided that $T$ possesses a multiplicative inverse in $\mathbb{T}$. Next we shall show that the precise choices of the times $t_0, t_1, \dots, t_{k-1}$ can also be fixed to be the same time $T$, provided again that $T$ possesses a multiplicative inverse in $\mathbb{T}$. In proving that these simplifications can be made without loss of generality, we make extensive use of the technical results about error functions from Section 3.1.

The result we prove is the following.

**3.15 Theorem: (Equivalent characterisations of chain equivalence)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Then the following statements are equivalent for $x, y \in \mathcal{X}$ and for $T_0 \in \mathbb{T}_{>0}$ possessing a multiplicative inverse in $\mathbb{T}$:*

*(i) $x$ and $y$ are chain equivalent for $\Phi$;*

*(ii) $x$ and $y$ are $T_0$-chain equivalent for $\Phi$;*

*(iii) $x$ and $y$ are exactly $T_0$-chain equivalent for $\Phi$.*

Proof: (i) $\implies$ (ii) This is clear from the definitions.

(ii) $\implies$ (iii) We break the proof into two parts, first for the discrete-time case then for the continuous-time case.

*The discrete-time case*

The discrete-time case is almost immediate. The requirement that $T_0$ possess a multiplicative inverse in $\mathbb{T}$ means that $T_0 = 1$ in the discrete-time case. In this case, however, every $(\varepsilon, T)$-chain gives rise to a $\varepsilon$-1-chain, simply by adding zero jumps at times for which there is not already a jump.

*The continuous-time case*

In this case, the requirement that $T_0 \in \mathbb{T}_{>0}$ possess a multiplicative inverse in $\mathbb{T}$ places no restriction on $T_0$. Let $x$ and $y$ satisfy the hypotheses of (ii). We claim that this implies that, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ and for each $\tau \in \mathbb{R}_{>0}$, there exists an $(\varepsilon, T_0)$ chain

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

and $N \in \mathbb{Z}_{>0}$ such that $t_j \in [T_0, 2T_0)$, $j \in \{0, 1, \dots, k-1\}$, and $|\sum_{j=1}^{k} t_j - NT_0| < \tau$.

To prove this, let $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ and $\tau \in \mathbb{R}_{>0}$. Note that the hypotheses ensure the existence of $(\frac{1}{4}\varepsilon, T_0)$-chains

$$u_0 = x, u_1, \dots, u_l = y, \quad r_0, r_1, \dots, r_{l-1}$$

and

$$v_0 = y, v_1, \dots, v_m = x, \quad s_0, s_1, \dots, s_{m-1}.$$

First of all, we can assume that $r_a \in [T_0, 2T_0)$, $a \in \{0, 1, \dots, l-1\}$, and $s_b \in [T_0, 2T_0)$, $b \in \{0, 1, \dots, m-1\}$. Indeed, if this is not so, then we can add jump times with zero jumps to ensure that these conditions are met. Let us abbreviate

$$R = r_0 + r_1 + \dots + r_{l-1}, \quad S = s_0 + s_1 + \dots + s_{m-1}.$$

By Lemma 3.3, let $\delta \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \delta(z_1) \implies \frac{1}{2}\varepsilon(z_1) < \varepsilon(z_2).$$

Let $\sigma \in \mathbb{R}_{>0}$ be such that

$$|r_0' - r_0| < \sigma \implies \mathrm{d}(\Phi(r_0, u_0), \Phi(r_0', u_0)) < \min\{\tfrac{1}{4}\varepsilon(\Phi(r_0, u_0), \delta(\Phi(r_0, u_0))\}.$$

Choose $r_0' \in [r_0, r_0 + \sigma)$ such that $R + S - r_0 + r_0'$ is irrational and such that $r_0' \in [T_0, 2T_0)$. Then, since

$$\mathrm{d}(\Phi(r_0, u_0), \Phi(r_0', u_0)) < \delta(\Phi(r_0, u_0)),$$

we have

$$\begin{aligned}
\mathrm{d}(\Phi(r_0', u_0), u_1) &\le \mathrm{d}(\Phi(r_0', u_0), \Phi(r_0, u_0)) + \mathrm{d}(\Phi(r_0, u_0), u_1) \\
&< \frac{1}{4}\varepsilon(\Phi(r_0, u_0)) + \frac{1}{4}\varepsilon(\Phi(r_0, u_0)) \\
&< \varepsilon(\Phi(r_0', u_0)),
\end{aligned}$$

and we deduce that

$$u_0 = x, u_1, \dots, u_l = y, \quad r_0', r_1, \dots, r_{l-1}$$

is an $(\varepsilon, T_0)$-chain from $x$ to $y$. Abbreviate

$$R' = r_0' + r_1 + \dots + r_{l-1}.$$

By irrationality of $R' + S$, there exist $M, N \in \mathbb{Z}_{>0}$ such that

$$|R' + M(R' + S) - NT_0| < \tau.$$

Now build an $(\varepsilon, T_0)$ chain as follows:

$$u_0 = x, u_1, \dots, u_l = y = \underbrace{v_0, v_1, \dots, v_m, u_0, u_1, \dots, u_l, \dots, v_0, v_1, \dots, v_m, u_0, u_1, \dots, u_l}_{v_0,v_1,\dots,v_m,u_0,u_1,\dots,u_l \text{ repeated } M \text{ times}},$$
$$r'_0, r_1, \dots, r_{l-1}, \underbrace{s_0, s_1, \dots, s_{m-1}, r'_0, r_1, \dots, r_{l-1}, \dots, s_0, s_1, \dots, s_{m-1}, r'_0, r_1, \dots, r_{l-1}}_{s_0,s_1,\dots,s_{m-1},r'_0,r_1,\dots,r_{l-1} \text{ repeated } M \text{ times}}.$$

This is the desired $(\varepsilon, T_0)$-chain whose total duration is within $\tau$ of $NT_0$.

Now, with this construction, we prove the desired implication. We let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$. By Lemma 3.3, let $\delta_1 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \delta_1(z_1) \implies \frac{1}{2}\varepsilon(z_1) < \varepsilon(z_2) < \frac{3}{2}\varepsilon(z_1). \tag{3.4}$$

By Lemma 3.5, let $\delta_2 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\begin{aligned}\mathrm{d}(z_1, z_2) < \delta_2(z_1) \implies & \mathrm{d}(\Phi(t, z_1), \Phi(t, z_2)) \\ & < \min\left\{\frac{1}{4}\varepsilon(\Phi(t, z_1)), \frac{1}{2}\delta_1(\Phi(t, z_1))\right\}, \qquad t \in [0, 3T_0].\end{aligned} \tag{3.5}$$

Let

$$u_0 = x, u_1, \dots, u_l = y, \quad r_0, r_1, \dots, r_{l-1} \tag{3.6}$$

and

$$v_0 = y, v_1, \dots, v_m = x, \quad s_0, s_1, \dots, s_{m-1} \tag{3.7}$$

be $(\min\{\frac{1}{32}\varepsilon, \frac{1}{4}\delta_2\}, T_0)$-chains. As we saw above, we can suppose that $r_a \in [T_0, 2T_0)$, $a \in \{0, 1, \dots, l-1\}$, and $s_b \in [T_0, 2T_0)$, $b \in \{0, 1, \dots, m-1\}$. Our construction above gives rise, for every $\tau' \in \mathbb{R}_{>0}$, to $N \in \mathbb{Z}_{>0}$ and a $(\min\{\frac{1}{8}\varepsilon, \delta_2\}, T_0)$-chain

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1} \tag{3.8}$$

satisfying $|T - N| < \tau'$, where $T = \sum_{j=0}^{k-1} t_j$. Furthermore, we can see from our constructions that $t_{k-1} = r_{l-1}$ and $x_{k-1} = u_{l-1}$, independently of $\tau'$. We now specify a suitable value of $\tau'$.

To do so, denote

$$\begin{aligned} K_a &= \{\Phi(t, u_a) \mid t \in [0, r_a]\}, \qquad a \in \{0, 1, \dots, l-1\}, \\ L_b &= \{\Phi(t, v_b) \mid t \in [0, s_b]\}, \qquad a \in \{0, 1, \dots, m-1\}, \\ K &= \left(\bigcup_{a=0}^{l-1} K_a\right) \cup \left(\bigcup_{b=0}^{m-1} L_b\right). \end{aligned}$$

Note that $K_a$, $a \in \{0, 1, \dots, l-1\}$, and $L_b$, $b \in \{0, 1, \dots, m-1\}$, are compact, and thus so too is $K$. Importantly, note that $K$ depends only on the chains (3.6) and (3.7), i.e., not on the choice of any $\tau'$. Let

$$\alpha = \inf\left\{\min\left\{\frac{1}{8}\varepsilon(z), \frac{1}{2}\delta_1(z), \delta_2(z)\right\} \;\middle|\; z \in K\right\},$$

noting that $\alpha > 0$. We claim that we can choose $\tau \in (0, T_0)$ such that

$$|s| < \tau \implies \mathrm{d}(\Phi(s, x), x) < \alpha \qquad x \in K. \tag{3.9}$$

Indeed, the function

$$[-1, 1] \times K \ni (t, z) \mapsto \Phi(t, z)$$

is uniformly continuous (its domain being compact), and so there exists $\eta \in \mathbb{R}_{>0}$ such that

$$(|\tau_1 - \tau_2| < \eta,\ \mathrm{d}(z_1, z_2) < \eta) \implies \mathrm{d}(\Phi(\tau_1, z_2), \Phi(\tau_2, z_2)) < \alpha.$$

Applying this formula to $\tau_1 = s$, $\tau_2 = 0$, and $z_1 = z_2 = z$ gives the claim. Thus we choose $\tau \in (0, T_0)$ so that (3.9) holds and choose the chain (3.8) so that $|T - NT_0| < \tau$, where $T = \sum_{j=0}^{k-1} t_j$. Note that we can, without loss of generality, assume that $T > NT_0$, and so $T - NT_0 \in [0, \tau)$.

We now build points $y_0, y_1, \dots, y_N \in \mathcal{X}$ such that

1. $y_0 = x$,
2. $y_N = y$, and
3. $\mathrm{d}(\Phi(T_0, y_j), y_{j+1}) < \varepsilon(\Phi(T_0, y_j))$, $j \in \{0, 1, \dots, N-1\}$.

We take $y_0 = x_0 = x$. If $t_0 = t_1 = \dots = t_j = T_0$ for some $j \in \{0, 1, \dots, j-2\}$, then we can take $y_a = x_a$, $a \in \{0, 1, \dots, j\}$. Our constructions then begin with the next jump. We can, therefore, simplify life notationally, and not lose generality, by simply supposing that $t_0 \in (T_0, 2T_0)$. (We recommend that a reader draw pictures of trajectories with times labelled to make sense of the constructions that follow.)

We take $y_1 = \Phi(T_0, y_0)$, whereupon

$$\mathrm{d}(\Phi(T_0, y_0), y_1) = 0 < \varepsilon(\Phi(T_0, y_0)).$$

To define $y_2$, we first note that

$$t_0 \in (T_0, 2T_0) \implies t_0 - T_0 \in (0, T_0).$$

Thus we can follow the trajectory through $\Phi(T_0, y_0)$ for time $t_0 - T_0$ and then jump to $x_1$ and follow the trajectory through $x_1$ for time $T_0 - (t_0 - T_0) = 2T_0 - t_0$. Since $2T_0 - t_0 \in (0, T_0)$, we have $t_1 > 2T_0 - t_0$, and so we do not need to jump from the trajectory through $x_1$ and so we can define $y_2 = \Phi(2T_0 - t_0, x_1)$.

Since

$$\mathrm{d}(\Phi(t_0, x_0), x_1) < \delta_2(\Phi(t_0, x_0))$$

and $2T_0 - t_0 \le 2T_0$, by (3.5) we have

$$\mathrm{d}(\Phi(2T_0 - t_0, \Phi(t_0, x_0)), \Phi(2T_0 - t_0, x_1)) < \frac{1}{4}\varepsilon(\Phi(2T_0 - t_0, \Phi(t_0, x_0))).$$

Therefore,

$$\begin{aligned}
\mathrm{d}(\Phi(T_0, y_1), y_2) &= \mathrm{d}(\Phi(2T_0, y_0), \Phi(2T_0 - t_0, x_1)\\
&= \mathrm{d}(\Phi(2T_0 - t_0, \Phi(t_0, x_0)), \Phi(2T_0 - t_0, x_1))\\
&< \frac{1}{4}\varepsilon(\Phi(2T_0 - t_0, \Phi(t_0, x_0)))\\
&= \frac{1}{4}\varepsilon(\Phi(2T_0, x_0)) < \varepsilon(\Phi(T_0, y_0)).
\end{aligned}$$

Note that $y_2$ lies on the trajectory through $x_1$ of time duration $t_1$. Specifically, $y_2 = \Phi(s, x_1)$ with $s = 2T_0 - t_0$. To define $y_3$, we need to follow the chain for time $T_0$, and we need to bookkeep if and when we need to jump to $x_2$. There are three cases to consider.

1. $t_1 - s > T_0$: In this case we do not need to jump, and can immediately define $y_3 = \Phi(T_0, y_2)$. In this case we have

$$\mathrm{d}(\Phi(T_0, y_2), y_3) = 0 < \varepsilon(\Phi(T_0, y_2)).$$

2. $t_1 - s = T_0$: In this case, we jump at the end of the time interval of duration $T_0$. That is, we take $y_3 = x_2$. Note that

$$\Phi(T_0, y_2) = \Phi(t_1, x_1).$$

Thus we have

$$\begin{aligned}\mathrm{d}(\Phi(T_0, y_2), y_3) &= \mathrm{d}(\Phi(t_1, x_1), x_2) < \frac{1}{4}\varepsilon(\Phi(t_1, x_1)) \\ &= \frac{1}{4}\varepsilon(\Phi(T_0, y_2)) < \varepsilon(\Phi(T_0, y_2)).\end{aligned}$$

3. $t_1 - s < T_0$: In this case, we follow the trajectory through $y_2$ for time $t_1 - s$, jump to $x_2$, and then take $y_3 = \Phi(T_0 - (t_1 - s), x_2)$. Note that

$$\Phi(T_0, y_2) = \Phi(T_0, \Phi(s, x_1)) = \Phi(T_0 + s, x_1).$$

Since

$$\mathrm{d}(\Phi(t_1, x_1), x_2) < \delta_2(\Phi(t_1, x_1))$$

and since $T_0 - (t_1 - s) \le 2T_0$, by (3.5) we have

$$\mathrm{d}(\Phi(T_0 - (t_1 - s), \Phi(t_1, x_1)), \Phi(T_0 - (t_1 - s), x_2)) < \frac{1}{4}\varepsilon(\Phi(T_0 - (t_1 - s), \Phi(t_1, x_1))).$$

Therefore,

$$\begin{aligned}\mathrm{d}(\Phi(T_0, y_2), y_3) &= \mathrm{d}(\Phi(T_0 + s, x_1), \Phi(T_0 - (t_1 - s), x_2)) \\ &= \mathrm{d}(\Phi(T_0 - (t_1 - s), \Phi(t_1, x_1)), \Phi(T_0 - (t_1 - s), x_2)) \\ &< \frac{1}{4}\varepsilon(\Phi(T_0 - (t_1 - s), \Phi(t_1, x_1))) \\ &= \frac{1}{4}\varepsilon(\Phi(T_0, y_2)) < \varepsilon(\Phi(T_0, y_2)).\end{aligned}$$

Now one can proceed as in the construction of $y_3$ to define $y_4, \ldots, y_{N-1}$. Note that, having done this, we have followed the chain (3.8) for time duration $(N-1)T_0$. Thus the time remaining along the chain is $T - (N-1)T_0 = T_0 + (T - NT_0) \in [T_0, T_0 + \tau)$. That is to say, the time remaining in the chain is within $\tau$ of $T_0$.

We take $y_N = y$ and consider two cases.

1. $y_{N-1} = \Phi(s, x_{k-1})$ for some $s \in [0, t_{k-1})$: In this case, the time left to travel along the chain (3.8) is $t_{k-1} - s = T_0 + (T - NT_0)$. Thus

$$\Phi(T_0, y_{N-1}) = \Phi(T_0 + s, x_{k-1}) = \Phi(t_{k-1} - (T - NT_0), x_{k-1}).$$

Since $|NT_0 - T| < \tau$, by (3.9) and (3.4) we have

$$\begin{aligned}
&\Phi(T_0, y_{N-1}) = \Phi(NT_0 - T, \Phi(t_{k-1}, x_{k-1})) \\
\Longrightarrow\ & \mathrm{d}(\Phi(T_0, y_{N-1}), \Phi(t_{k-1}, x_{k-1})) < \alpha \leq \delta_1(\Phi(t_{k-1}, x_{k-1})) \\
\Longrightarrow\ & \frac{1}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})) < \varepsilon(\Phi(T_0, y_{N-1})) < \frac{3}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})).
\end{aligned}$$

Similarly, since $|T - NT_0| < \tau$ and $t_{k-1} \leq 2$, by (3.9) and (3.5) we have

$$\begin{aligned}
&\mathrm{d}(x_{k-1}, \Phi(NT_0 - T, x_{k-1})) < \alpha \leq \delta_2(x_{k-1}) \\
\Longrightarrow\ & \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), \Phi(t_{k-1}, \Phi(NT_0 - T, x_{k-1}))) < \frac{1}{4}\varepsilon(\Phi(t_{k-1}, x_{k-1})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}(\Phi(T_0, y_{N-1}), y_N) &= \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), x_{k-1}), y) \\
&\leq \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), x_{k-1}), \Phi(t_{k-1}, x_{k-1}) \\
&\quad + \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) \\
&= \mathrm{d}(\Phi(t_{k-1}, \Phi(NT_0 - T, x_{k-1})), \Phi(t_{k-1}, x_{k-1})) \\
&\quad + \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) \\
&< \frac{1}{4}\varepsilon(\Phi(t_{k-1}, x_{k-1})) + \frac{1}{4}\varepsilon(\Phi(t_{k-1}, x_{k-1})) \\
&= \frac{1}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})) < \varepsilon(\Phi(T_0, y_{N-1})).
\end{aligned}$$

2. $y_{N-1} = \Phi(s, x_{k-2})$ for some $s \in [0, t_{k-2})$: In this case, to get to $y_N$, we must jump to the trajectory through $x_{k-1}$ after time $t_{k-2} - s$, and then follow this trajectory for time $t_{k-1} - (T - NT_0)$. Note that

$$t_{k-1} - (T - NT_0) + t_{k-2} - s = T_0 \implies T_0 + s = t_{k-1} + t_{k-2} - (T - NT_0).$$

Also note that

$$(t_{k-1} \in [T_0, 2T_0),\ T - NT_0 \in [0, \tau)) \implies t_{k-1} - (T - NT_0) \leq 2T_0.$$

Since

$$\mathrm{d}(\Phi(t_{k-2}, x_{k-2}), x_{k-1}) < \delta_2(\Phi(t_{k-2}, x_{k-2})),$$

and $t_{k-1} - (T - NT_0) \leq 2T_0$, by (3.5) we have

$$\begin{aligned}
\frac{1}{4}\varepsilon(\Phi(T_0{+}s, x_{k-2})) &= \frac{1}{4}\varepsilon(\Phi(t_{k-1} + t_{k-2} - (T - NT_0), x_{k-2})) \\
&> \mathrm{d}(\Phi(t_{k-1} + t_{k-2} - (T - NT_0), x_{k-2}), \Phi(t_{k-1} - (T - NT_0), x_{k-1})) \\
&= \mathrm{d}(\Phi(T_0 + s, x_{k-2}), \Phi(t_{k-1} - (T - NT_0), x_{k-1})). \qquad (3.10)
\end{aligned}$$

Since $|NT_0 - T| < \tau$, by (3.9) we have

$$\begin{aligned}&\Phi(t_{k-1} - (T - NT_0), x_{k-1}) = \Phi(NT_0 - T, \Phi(t_{k-1}, x_{k-1}))\\ \Longrightarrow\ &\mathrm{d}(\Phi(t_{k-1} - (NT_0 - T), x_{k-1})), \Phi(t_{k-1}, x_{k-1})) < \alpha \leq \frac{1}{8}\varepsilon(\Phi(t_{k-1}, x_{k-1})).\end{aligned} \tag{3.11}$$

Using the definition of $\alpha$, the same argument gives

$$\mathrm{d}(\Phi(t_{k-1} - (T - NT_0), x_{k-1}), \Phi(t_{k-1}, x_{k-1}) < \alpha \leq \frac{1}{2}\delta_1(\Phi(t_{k-1}, x_{k-1})).$$

Since

$$\mathrm{d}(\Phi(t_{k-2}, x_{k-2}), x_{k-1}) < \delta_2(\Phi(t_{k-2}, x_{k-2}))$$

and $t_{k-1} - (T - NT_0) \leq 2T_0$, by (3.5) we have

$$\begin{aligned}&\mathrm{d}(\Phi(t_{k-1} - (T - NT_0), \Phi(t_{k-2}, x_{k-2})), \Phi(t_{k-1} - (T - NT_0), x_{k-1}))\\ &\qquad < \frac{1}{2}\delta_1(\Phi(t_{k-1} - (T - NT_0), \Phi(t_{k-2}, x_{k-2}))) = \frac{1}{2}\delta_1(T_0 + s, x_{k-2}).\end{aligned}$$

Putting the above together, we have

$$\begin{aligned}&\mathrm{d}(\Phi(t_{k-1}{+}t_{k-2} - (T - NT_0), x_{k-2}), \Phi(t_{k-1}, x_{k-1}))\\ &\quad = \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), \Phi(t_{k-2}, x_{k-2})), \Phi(t_{k-1}, x_{k-1})\\ &\quad \leq \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), x_{k-1}), \Phi(t_{k-1}, x_{k-1})\\ &\qquad + \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), \Phi(t_{k-2}, x_{k-2})), \Phi(t_{k-1} - (T - NT_0), x_{k-1}))\\ &\quad < \frac{1}{2}\delta_1(\Phi(t_{k-1}, x_{k-1})) + \frac{1}{2}\delta_1(T_0 + s, x_{k-2})\\ &\quad \leq \max\{\delta_1(\Phi(t_{k-1}, x_{k-1})), \delta_1(\Phi(T_0 + s, x_{k-2}))\},\end{aligned}$$

using the standard relation $\|\cdot\|_1 \leq n\|\cdot\|_\infty$ between the 1- and $\infty$-norms for $\mathbb{R}^n$. Keeping in mind that $T_0 + s = t_{k-1} + t_{k-2} - (T - NT_0)$, we now consider two cases.

(a) $\delta_1(\Phi(T_0 + s, x_{k-2})) \leq \delta_1(\Phi(t_{k-1}, x_{k-1}))$: In this case,

$$\max\{\delta_1(\Phi(t_{k-1}, x_{k-1})), \delta_1(\Phi(T_0 + s, x_{k-2}))\} = \delta_1(\Phi(t_{k-1}, x_{k-1})).$$

By (3.4) we have

$$\begin{aligned}&\mathrm{d}(\Phi(T_0 + s, x_{k-2}), \Phi(t_{k-1}, x_{k-1})) < \delta_1(\Phi(t_{k-1}, x_{k-1}))\\ \Longrightarrow\ &\frac{1}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1})) < \varepsilon(\Phi(T_0 + s, x_{k-2})) < \frac{3}{2}\varepsilon(\Phi(t_{k-1}, x_{k-1}))\end{aligned}$$

(b) $\delta_1(T_0 + s, y_{N-1}) > \delta_1(\Phi(t_{k-1}, x_{k-1}))$: In this case,

$$\max\{\delta_1(\Phi(t_{k-1}, x_{k-1})), \delta_1(\Phi(T_0 + s, y_{N-1}))\} = \delta_1(\Phi(T_0 + s, x_{k-2}))$$

and, again by (3.4), we have

$$\begin{aligned}&\mathrm{d}(\Phi(T_0 + s, x_{k-2}), \Phi(t_{k-1}, x_{k-1})) < \delta_1(\Phi(T_0 + s, x_{k-s}))\\ \Longrightarrow\ &\frac{1}{2}\varepsilon(\Phi(T_0 + s, x_{k-2})) < \varepsilon(\Phi(t_{k-1}, x_{k-1})) < \frac{3}{2}\varepsilon(\Phi(T_0 + s, x_{k-2})).\end{aligned}$$

Therefore, considering both cases together,

$$\varepsilon(\Phi(t_{k-1}, x_{k-1})) < 2\varepsilon(\Phi(T_0 + s, x_{k-2})). \tag{3.12}$$

Finally, we assemble (3.10), (3.11), and (3.12):

$$\begin{aligned}
\mathrm{d}(\Phi(T_0, y_{N-1}), y_N) &= \mathrm{d}(\Phi(T_0 + s, x_{k-2}), y) \\
&\le \mathrm{d}(\Phi(T_0 + s, x_{k-2}), \Phi(t_{k-1} - (T - NT_0), x_{k-1})) \\
&\quad + \mathrm{d}(\Phi(t_{k-1} - (T - NT_0), x_{k-1}), \Phi(t_{k-1}, x_{k-1}))) \\
&\quad + \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) \\
&< \frac{1}{4}\varepsilon(\Phi(T_0 + s, x_{k-2})) + \frac{1}{8}\varepsilon(\Phi(t_{k-1}, x_{k-1})) \\
&\quad + \frac{1}{8}\varepsilon(\Phi(t_{k-1}, x_{k-1})) \\
&< \frac{3}{4}\varepsilon(\Phi(T_0 + s, x_{k-2})) < \varepsilon(\Phi(T_0, y_{N-1})).
\end{aligned}$$

Thus the points $y_0, y_1, \dots, y_N$ have the desired properties.

(iii) $\Longrightarrow$ (i) Under the stated hypotheses, let $\varepsilon \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ and let $T \in \mathbb{T}_{>0}$. Without loss of generality, suppose that $T = MT_0$ for some $M \in \mathbb{Z}_{>0}$. By Lemma 3.3, let $\varepsilon_{2M-1} \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ be such that $\varepsilon_{2M-1} < \frac{1}{2}\varepsilon$ and such that

$$\mathrm{d}(z_1, z_2) < \varepsilon_{2M-1}(z_1) \implies \varepsilon(z_2) < \tfrac{3}{2}\varepsilon(z_1). \tag{3.13}$$

Then, by Lemma 3.4, let $\eta_{2M-2} \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \eta_{2M-2}(z_1) \implies \mathrm{d}(\Phi(T_0, z_1), \Phi(T_0, z_2)) < \varepsilon_{2M-1}(\Phi(T_0, z_1)). \tag{3.14}$$

Then, using Lemmata 3.3 and 3.4, recursively define $\varepsilon_{2M-1}, \dots, \varepsilon_2 \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ and $\eta_{2M-2}, \dots, \eta_1 \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ such that $\varepsilon_j < \frac{1}{2}\eta_j$, $j \in \{2, \dots, 2M-2\}$, $x \in \mathfrak{X}$, and such that

$$\mathrm{d}(z_1, z_2) < \varepsilon_j(z_1) \implies \eta_j(z_2) < \tfrac{3}{2}\eta_j(z_1), \qquad j \in \{2, \dots, 2M-1\}, \tag{3.15}$$

and

$$\mathrm{d}(z_1, z_2) < \eta_j(z_1) \implies \mathrm{d}(\Phi(T_0, z_1), \Phi(T_0, z_2)) < \varepsilon_{j+1}(\Phi(T_0, z_1)), \qquad j \in \{1, \dots, 2M-2\}. \tag{3.16}$$

Define $\delta \in \mathrm{C}^0(\mathfrak{X}; \mathbb{R}_{>0})$ by

$$\delta = \min\{\eta_1, \tfrac{1}{3}\eta_2, \dots, \tfrac{1}{3}\eta_{N-1}, \tfrac{1}{3}\varepsilon\}.$$

Now, by hypothesis, let

$$x_0 = x, x_1, \dots, x_k = y, \quad T_0, T_0, \dots, T_0$$

and

$$y_0 = y, y_1, \dots, y_m = x, \quad T_0, T_0, \dots, T_0$$

be $\varepsilon$-$T_0$-chains, i.e.,

$$\mathrm{d}(\Phi(T_0, x_j), x_{j+1}) < \delta(\Phi(T_0, x_j)), \qquad j \in \{1, \dots, k\}, \tag{3.17}$$

and

$$\mathrm{d}(\Phi(T_0, y_l), y_{l+1}) < \delta(\Phi(T_0, y_l)), \qquad l \in \{1, \dots, m\}. \tag{3.18}$$

From the two finite sequences $x_0, x_1, \dots, x_k$ and $y_0, y_1, \dots, y_m$, build another finite sequence

$$x_0 = x, x_1, \dots, x_{k-1}, \\ \underbrace{y_0, y_1, \dots, y_{m-1}, x_0, x_1, \dots, x_{k-1}, \dots, y_0, y_1, \dots, y_{m-1}, x_0, x_1, \dots, x_{k-1}}_{y_0, y_1, \dots, y_{m-1}, x_0, x_1, \dots, x_{k-1} \text{ repeated } M \text{ times}}, y,$$

this of length $k + M(k+m) + 1$. Let us use this sequence to assemble an $\varepsilon$-$T_0$-chain

$$z_0, z_1, \dots, z_{k+M(k+m)}, \quad T_0, T_0, \dots, T_0,$$

i.e.,

$$\mathrm{d}(\Phi(T_0, z_j), z_{j+1}) < \delta(\Phi(T_0, z_j)), \qquad j \in \{0, 1, \dots, k + M(k+m) - 1\}.$$

By the Euclidean Algorithm, let $N', R' \in \mathbb{Z}_{>0}$ be such that

$$k + M(k+m) = N'M + R', \quad R' \in \{0, 1, \dots, M-1\}.$$

and take $N = N' - 1$ and $R = R' + M$ so that

$$k + m(k+m) = N'M + R' = (N+1)M + R - M = NM + R, \\ R \in \{M, M+1, \dots, 2M-1\}.$$

Define

$$z'_j = \begin{cases} z_{jM}, & j \in \{0, 1, \dots, N-1\}, \\ y, & j = N \end{cases}$$

and

$$t_j = \begin{cases} T, & j \in \{0, 1, \dots, N-2\}, \\ R, & j = N-1. \end{cases}$$

We claim that

$$z'_0, z'_1, \dots, z'_N, \quad t_0, t_1, \dots, t_{N-1}$$

is an $(\varepsilon, T)$-chain from $x$ to $y$.

To establish this, we first claim that, for $j \in \{0, 1, \dots, N\}$ and for $l \in \{1, \dots, R\}$, we have

$$\mathrm{d}(\Phi(lT_0, z_{jM}), z_{jM+l}) < \eta_l(\Phi(lT_0, z_{jM})). \tag{3.19}$$

For $l = 1$, we have

$$\mathrm{d}(\Phi(T_0, z_{jM}), z_{jM+1}) < \delta(\Phi(T_0, z_{jM})) < \eta_1(\Phi(T_0, z_{jM})),$$

verifying the claim in this case. To argue inductively, suppose that

$$\mathrm{d}(\Phi(T_0, z_{jM}), z_{jM+l}) < \eta_l(\Phi(lT_0, z_{jM}))$$

for some $l \in \{2, \dots, R-1\}$. From this inequality and by (3.16), we have

$$\mathrm{d}(\Phi((l+1)T_0, z_{jM}), \Phi(T_0, z_{jM+l})) < \varepsilon_{l+1}(\Phi((l+1)T_0, z_{jM})) < \tfrac{1}{2}\eta_{l+1}(\Phi((l+1)T_0, z_{jM})).$$

Similarly, by (3.15) we have

$$\eta_{l+1}(\Phi(T_0, z_{jM+l})) < \tfrac{3}{2}\eta_{l+1}(\Phi((l+1)T_0, z_{jM})).$$

Thus, using (3.17), (3.18), and the definition of $\delta$,

$$\begin{aligned} \mathrm{d}(\Phi(T_0, z_{jM+l}), z_{jM+l+1}) &< \delta(\Phi(T_0, z_{jM+l})) < \tfrac{1}{3}\eta_{l+1}(\Phi(T_0, z_{jM+l})) \\ &< \tfrac{1}{2}\eta_{l+1}(\Phi((l+1)T_0, z_{jM})). \end{aligned}$$

Putting this together,

$$\begin{aligned} \mathrm{d}(\Phi((l+1)T_0, z_{jM}), z_{jM+l+1}) &\le \mathrm{d}(\Phi((l+1)T_0, z_{jM}), \Phi(T_0, z_{jM+l})) \\ &\quad + \mathrm{d}(\Phi(T_0, z_{jM}), z_{jM+l+1}) \\ &< \eta_{l+1}(\Phi((l+1)T_0, z_{jM})). \end{aligned}$$

This establishes (3.19) by induction.

In the particular case of $l = N$, (3.19) gives

$$\mathrm{d}(\Phi(NT_0, z'_j), z'_{j+l}) < \eta_N(\Phi(NT_0, z'_j)) < \varepsilon(\Phi(NT_0, z'_j)).$$

If $l = R$ and $j = N - 1$, (3.19) gives

$$\mathrm{d}(\Phi(RT_0, z'_{N-1}), z'_N) < \eta_{N-1}(\Phi(RT_0, z'_{N-1})) < \varepsilon(\Phi(RT_0, z'_{N-1})).$$

Of course, the same argument gives, for $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$, an $(\varepsilon, T)$-chain from $y$ to $x$, which gives this part of the result. ■

The following corollary is immediate since a point is chain recurrent if and only if it is chain equivalent to itself.

**3.16 Corollary: (Equivalent characterisations of chain recurrent set)** *Let* $(\mathcal{X}, \mathrm{d})$ *be a metric space and let* $\Phi$ *be a topological flow or semiflow on* $\mathcal{X}$*. Then, for* $T_0 \in \mathbb{T}_{>0}$ *possessing a multiplicative inverse in* $\mathbb{T}$*,*

$$\mathrm{ChRec}(\Phi) = \mathrm{ChRec}_{\ge T_0}(\Phi) = \mathrm{ChRec}_{=T_0}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},T_0}).$$

Note that the theorem and the corollary indicate why, when working with discrete-time flows and semiflows, one can ignore the switching times and simply take them to be 1. This leads to the following simplified notion of a chain in these cases. Indeed, this is the usual definition of a chain for discrete-time flows and semiflows.

**3.17 Definition: ($\varepsilon$-chain)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a discrete-time topological flow or semiflow on $\mathcal{X}$ with $\phi = \Phi_1$. For $x, y \in \mathcal{X}$ and $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, an ***$\varepsilon$-chain*** for $\Phi$ from $x$ to $y$ is a finite sequence

$$x_0, x_1, \dots, x_k,$$

with

(i) $x_0, x_1, \dots, x_k \in \mathcal{X}$,

(ii) $x_0 = x$ and $x_k = y$, and

(iii) $\mathrm{d}(\phi(x_j), x_{j+1}) < \varepsilon(\phi(x_j))$, $j \in \{0, 1, \dots, k-1\}$. •

Staying with discrete-time flows and semiflows for a moment, let us introduce some notation for these that will facilitate a comparison of the chain recurrent set of a mapping with the chain recurrent set of its iterate mappings. Let $\Phi$ be a discrete-time flow or semiflow on a topological space $(\mathcal{X}, \mathscr{O})$ and let $k \in \mathbb{Z}_{>0}$. Then define the discrete-time flow or semiflow $\Phi^k$ on $\mathcal{X}$ by

$$\begin{aligned} \Phi^k \colon \mathbb{T} \times \mathcal{X} &\to \mathcal{X} \\ (j, x) &\mapsto \Phi(jk, x). \end{aligned}$$

With this notation, we have the following result, which is essentially corollary to Theorem 3.15.

**3.18 Corollary: (The chain recurrent set of a mapping agrees with the chain recurrent set of its iterates)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Then $\mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^k)$ for every $k \in \mathbb{Z}_{>0}$.*

Proof: If $x \in \mathrm{ChRec}(\Phi^k)$, then an elementary argument like that given in the discrete-time case of the implication (ii) $\Longrightarrow$ (iii) from the proof of Theorem 3.15 shows that $x \in \mathrm{ChRec}(\Phi)$.

Now suppose that $x \in \mathrm{ChRec}(\Phi)$. Let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and, using an inductive proof like that from the proof of the implication (iii) $\Longrightarrow$ (i) from Theorem 3.15, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that, if

$$y_0, y_1, \dots, y_k$$

is a $\delta$-chain for $\Phi$, then

$$\mathrm{d}(y_k, \Phi^k(y_0)) < \varepsilon(\Phi^k(y_0)).$$

By Theorem 3.15, suppose that

$$x_0, x_1, \dots, x_m$$

is a $\delta$-chain from $x$ to $x$ for $\Phi$. Then

$$\underbrace{x_0, x_1, \dots, x_{m-1}, x_0, x_1, \dots, x_m, \dots, x_0, x_1, \dots, x_m}_{k \text{ times}}$$

is a $\delta$-chain from $x$ to $x$ for $\Phi$, and the choice of $\delta$ ensures that

$$x_0, x_k, \dots, x_{mk}$$

is a $\varepsilon$-chain from $x$ to $x$ for $\Phi^k$. Thus, by Theorem 3.15, $x \in \mathrm{ChRec}(\Phi^k)$. ■

## 4. The Conley decomposition

Now we turn to the first part of the Fundamental Theorem of Dynamical Systems, the so-called Conley decomposition. This gives a decomposition of the state space for a flow or semiflow into chain recurrent dynamics on the chain recurrent set and gradient-like dynamics off the chain recurrent set, as (roughly) one flows from repelling sets to attracting sets. We first spend some time understanding the relationship between chains and trapping regions, as this is essential for any sort of understanding of how the Fundamental Theorem of Dynamical Systems works. After this, we prove the decomposition theorem.

**4.1. Chains and trapping regions.** We begin with some notation. Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $S \subseteq \mathcal{X}$. The function

$$\begin{aligned} \mathrm{dist}_S \colon \mathcal{X} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \inf\{\mathrm{d}(x,y) \mid y \in S\} \end{aligned}$$

is the ***distance function to $S$***. We shall also denote $\mathrm{dist}(x,S) = \mathrm{dist}_S(x)$. Evidently, if $S$ is closed and $x \notin S$, $\mathrm{dist}_S(x) \in \mathbb{R}_{>0}$. We use this notation in the proof of the following lemma that shows how trapping regions give natural error functions with useful properties.

**4.1 Lemma: (Error functions and trapping regions)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. For an open set $\mathcal{T} \subseteq \mathcal{X}$ and for $T \in \mathbb{T}_{>0}$, the following statements are equivalent:*

*(i) $\mathcal{T}$ is a trapping region with $\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathcal{T}$;*

*(ii) there exists $\varepsilon \in \mathrm{C}^0(\mathcal{X}; (0,1])$ such that*

$$\mathsf{B}_{\mathrm{d}}(\varepsilon(\Phi(t,x)), \Phi(t,x)) \subseteq \mathcal{T}, \qquad t \in \mathbb{T}_{\geq T},\ x \in \mathcal{T}.$$

Proof: (i) $\Longrightarrow$ (ii) Let

$$\varepsilon'(x) = \frac{1}{2}\left(\mathrm{dist}_{\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))}(x) + \mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(x)\right),$$

noting that $\varepsilon'$ is continuous [Aliprantis and Border 2006, Theorem 3.16]. Also, $\varepsilon'$ takes values in $\mathbb{R}_{>0}$ since, if $x \in \mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))$, then $x \notin \mathcal{X} \setminus \mathcal{T}$. For $x \in \mathcal{T}$ and $t \in \mathbb{T}_{\geq T}$, we have

$$\Phi(t,x) \in \Phi(\mathbb{T}_{\geq T} \times \mathcal{T}) \implies \mathrm{dist}_{\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))}(\Phi(t,x)) = 0.$$

Thus, if $y \in \mathsf{B}_{\mathrm{d}}(\varepsilon'(\Phi(t,x)), \Phi(t,x))$, we have

$$\mathrm{d}(\Phi(t,x), y) < \varepsilon'(\Phi(t,x)) = \frac{1}{2}\mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(\Phi(t,x)).$$

Also,

$$\begin{aligned} \mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(\Phi(t,x)) =&\ \inf\{\mathrm{d}(\Phi(t,x), z) \mid z \in \mathcal{X} \setminus \mathcal{T}\} \\ \leq&\ \inf\{\mathrm{d}(\Phi(t,x), y) + \mathrm{d}(z,y) \mid x \in \mathcal{X} \setminus \mathcal{T}\} \\ =&\ \mathrm{d}(\Phi(t,x), y) + \mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(y). \end{aligned}$$

Putting this together,

$$2\mathrm{d}(\Phi(t,x), y) < \mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(\Phi(t,x)) \leq \mathrm{d}(\Phi(t,x), y) + \mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(y).$$

Then

$$\mathrm{dist}_{\mathcal{X} \setminus \mathcal{T}}(y) > \mathrm{d}(y, \Phi(t,x)) \geq 0 \implies y \in \mathcal{T},$$

showing that $\mathsf{B}_{\mathrm{d}}(\varepsilon'(\Phi(t,x)), \Phi(t,x)) \subseteq \mathcal{T}$. Taking

$$\varepsilon(x) = \min\{\varepsilon'(x), 1\}$$

gives this part of the lemma.

(ii) $\implies$ (i) Let $\varepsilon$ be as stated. Let $y \in \mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))$ and let $((t_j, x_j))_{j \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{T}_{\geq T} \times \mathcal{T}$ for which $y = \lim_{j \to \infty} \Phi(t_j, x_j)$. By continuity of $\varepsilon$, let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\varepsilon(\Phi(t_j, x_j)) \geq \frac{1}{2}\varepsilon(y)$, $j \geq N$. Also suppose that $N$ is large enough that $\mathrm{d}(\Phi(t_j, x_j), y) < \frac{1}{4}\varepsilon(y)$ for $j \geq N$. Let $z \in \mathsf{B}_{\mathrm{d}}(\frac{1}{4}\varepsilon(y), y)$. Then

$$\begin{aligned}\mathrm{d}(\Phi(t_N, x_N), z) &\leq \mathrm{d}(\Phi(t_N, x_N), y) + \mathrm{d}(y, z) \\ &< \frac{1}{4}\varepsilon(y) + \frac{1}{4}\varepsilon(y) \leq \varepsilon(\Phi(t_N, x_N)).\end{aligned}$$

Thus

$$\mathsf{B}_{\mathrm{d}}(\tfrac{1}{4}\varepsilon(y), y) \subseteq \mathsf{B}_{\mathrm{d}}(\varepsilon(\Phi(t_N, x_N)), \Phi(t_N, x_N)) \subseteq \mathcal{T}$$

and so $y \in \mathcal{T}$. ■

The following lemma provides an essential conceptual step in understanding the Fundamental Theorem of Dynamical Systems. It shows how chains and the chain recurrent set give rise to natural trapping regions.

**4.2 Lemma: (Trapping regions from chains)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Let $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$, and let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$ be such that there is no $(\varepsilon, T)$-chain from $x$ to itself. Let $m \in \mathbb{Z}_{>0}$. Then*

$$\mathcal{T} = \{y \in \mathcal{X} \mid \text{ there exists an } (\varepsilon, T)\text{-chain of length } k \geq m \text{ from } x \text{ to } y\}$$

*is an open trapping region satisfying*

*(i) $\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}) \subseteq \mathcal{T}$ and*

*(ii) $x \notin \mathcal{T}$.*

Proof: Clearly, $x \notin \mathcal{T}$ by definition of $\varepsilon$ and $T$.

To prove openness of $\mathcal{T}$, let $y \in \mathcal{T}$, let $k \geq m$ and let

$$x_0 = x, x_1, \dots, x_k = y, \quad t_0, t_1, \dots, t_{k-1}$$

be an $(\varepsilon, T)$-chain from $x$ to $y \in \mathcal{T}$. We claim that, if

$$r = \varepsilon(\Phi(t_{k-1}, x_{k-1})) - \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y),$$

then $\mathsf{B}_{\mathrm{d}}(r, y) \subseteq \mathcal{T}$. Indeed, let $z \in \mathsf{B}_{\mathrm{d}}(r, y)$. Then

$$\begin{aligned}\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), z) &\leq \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) + \mathrm{d}(y, z) \\ &< \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) + \varepsilon(\Phi(t_{k-1}, x_{k-1})) - \mathrm{d}(\Phi(t_{k-1}, x_{k-1}), y) \\ &= \varepsilon(\Phi(t_{k-1}, x_{k-1})).\end{aligned}$$

Thus we have

$$\mathsf{B}_{\mathrm{d}}(r, y) \subseteq \mathsf{B}_{\mathrm{d}}(\varepsilon(\Phi(t_{k-1}, x_{k-1})), \Phi(t_{k-1}, x_{k-1})).$$

Since the ball on the right consists of points $z$ for which there is an $(\varepsilon, T)$-chain from $x$ to $z$, we obtain $\mathsf{B}_{\mathrm{d}}(r, y) \subseteq \mathcal{T}$, giving the desired openness.

Now we show that $\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathcal{T}$. By Lemma 3.3, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \delta(z_1) \implies \frac{1}{2}\varepsilon(z_1) < \varepsilon(z_2).$$

Without loss of generality, we can assume that $\delta \leq \frac{1}{2}\varepsilon$. Let $y \in \mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T}))$. Then

$$\mathsf{B}_{\mathrm{d}}(\delta(y), y) \cap \Phi(\mathbb{T}_{\geq T} \times \mathcal{T}) \neq \varnothing,$$

and so there exists $t \in \mathbb{T}_{\geq T}$ and $z \in \mathcal{T}$ such that $\Phi(t, z) \in \mathsf{B}_{\mathrm{d}}(\delta(y), y)$. By definition of $\mathcal{T}$, let $k \geq m$ and let

$$x_0 = x, x_1, \dots, x_k = z, \quad t_0, t_1, \dots, t_{k-1}$$

be an $(\epsilon, T)$-chain from $x$ to $z$. We have

$$\mathrm{d}(\Phi(t, z), y) < \delta(y) \leq \frac{1}{2}\varepsilon(y) < \varepsilon(\Phi(t, z)),$$

from which we may conclude that

$$x_0 = x, x_1, \dots, x_k = z, y, \quad t_0, t_1, \dots, t_{k-1}, t$$

is an $(\varepsilon, T)$-chain from $x$ to $y$, and so $y \in \mathcal{T}$. This gives $\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathcal{T}$, as desired. ■

**4.2. The decomposition theorem.** With the understanding of the connection between chains and trapping regions from the preceding section, we can prove the Conley decomposition. The following lemma captures an essential part of the theorem. Typically the proof of this lemma is given separately for the continuous-time case and the discrete-time case (where the notion of trapping region is taken to be our notion of strong trapping region as in Remark 2.8–1 and in Definition 5.6 below). Our proof works for both cases, and has the additional benefit of being simpler than the already simple proofs in each of the separate cases.

**4.3 Lemma: (Chain recurrent points whose forward trajectories lie in a trapping region)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. If $\mathcal{T}$ is a trapping region with $A$ its corresponding attracting set, then*

$$\mathrm{ChRec}(\Phi) \cap \mathrm{Orb}^-(\mathcal{T}) = A.$$

Proof: Let $T \in \mathbb{T}_{>0}$ be such that $\mathrm{cl}(\Phi(\mathbb{T}_{\geq T} \times \mathcal{T})) \subseteq \mathcal{T}$ and, by Lemma 4.1, let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathsf{B}_{\mathrm{d}}(\varepsilon(\Phi(t, x)), \Phi(t, x)) \subseteq \mathcal{T}, \qquad t \in \mathbb{T}_{\geq T},\ x \in \mathcal{T}.$$

Let $t \in \mathbb{T}_{\geq T}$ and let $m \in \mathbb{Z}_{>0}$. Let $x \in \mathrm{ChRec}(\Phi) \cap \mathcal{T}$. Let

$$x_0 = x, x_1, \dots, x_k = x, \quad t_0, t_1, \dots, t_{k-1}$$

be a $(\min\{\frac{1}{m}, \varepsilon\}, t)$-chain from $x$ to itself. We have

$$\mathrm{d}(\Phi(t_0, x_0), x_1) < \min\left\{\frac{1}{m}, \varepsilon(\Phi(t, x_0))\right\} \leq \varepsilon(\Phi(t_0, x_0)),$$

which implies that $x_1 \in \mathcal{T}$ by definition of $\varepsilon$. Now suppose that $x_j \in \mathcal{T}$ for $j \in \{0, 1, \dots, k-2\}$, and note that, as above,

$$\mathrm{d}(\Phi(t_j, x_j), x_{j+1}) < \varepsilon(\Phi(t_j, x_j)).$$

Thus $x_{j+1} \in \mathcal{T}$ by definition of $\varepsilon$. Thus $x_1, \dots, x_{k-1} \in \mathcal{T}$. Now we have

$$\mathrm{d}(\Phi(t_{k-1}, x_{k-1}), x) < \min\left\{\frac{1}{m}, \varepsilon(\Phi(t_{k-1}, x_{k-1}))\right\} \leq \frac{1}{m}.$$

Since $t_{k-1} \geq T + t$ and $x_{k-1} \in \mathcal{T}$, we have

$$\Phi(t_{k-1}, x_{k-1}) \in \Phi(\mathbb{T}_{\geq T+t} \times \mathcal{T}),$$

which implies that

$$\mathrm{dist}_{\Phi(\mathbb{T}_{\geq T+t} \times \mathcal{T})}(x) \leq \mathrm{d}(x, \Phi(t_{k-1}, x_{k-1})) < \frac{1}{m}.$$

As this construction can be made for any $m \in \mathbb{Z}_{>0}$, we conclude that $x \in \mathrm{cl}(\Phi(\mathbb{T}_{\geq T+t} \times \mathcal{T}))$. Therefore, as this holds for every $t \in \mathbb{T}_{\geq T}$, we have

$$x \in \bigcap_{t \in \mathbb{T}_{\geq 0}} \Phi(\mathbb{T}_{\geq T+t} \times \mathcal{T}) = A.$$

Let $x \in \mathrm{ChRec}(\Phi) \cap \mathrm{Orb}^{-}(\mathcal{T})$. Let $(\varepsilon, T)$ be as in part (ii) of Lemma 4.1. Suppose that $\Phi(t, x) \in \mathcal{T}$ for $t \in \mathbb{T}_{>0}$. Since $x \in \mathrm{ChRec}(\Phi)$, let

$$x_0 = x, x_1, \dots, x_k = x, \quad t_0, t_1, \dots, t_{k-1}$$

be an $(\varepsilon, T + t)$-chain from $x$ to $x$. Since $\Phi_t(x) \in \mathcal{T}$, $\Phi_{s+t}(x) \in \mathrm{cl}(\mathcal{T}) \subseteq \mathcal{T}$ for every $s \in \mathbb{T}_{\geq T}$; in particular, $\Phi(t_0, x_0) \in \mathcal{T}$. By definition of $\varepsilon$, $x_1 \in \mathcal{T}$. Thus

$$x_1, x_2, \dots, x_k = x, \quad t_1, t_2, \dots, t_{k-1}$$

is an $(\varepsilon, T)$ chain from $x_1 \in \mathcal{T}$ to $x$. As we argued in the first part of the proof, this implies that $x_k = x \in \mathcal{T}$. Thus $x \in \mathrm{ChRec}(\Phi) \cap \mathcal{T} = A$, again from the first part of the proof. ■

Now we can state the decomposition theorem. In the statement of the result, we denote by $\mathscr{T}(\Phi)$ the set of trapping regions for a topological flow or semiflow.

**4.4 Theorem: (The Conley decomposition)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Then*

$$\mathcal{X} \setminus \mathrm{ChRec}(\Phi) = \bigcup_{\mathcal{T} \in \mathscr{T}(\Phi)} \mathrm{Orb}^{-}(\mathcal{T}) \setminus A_{\mathcal{T}}.$$

Proof: Let $x \notin \mathrm{ChRec}(\Phi)$. Let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $T \in \mathbb{T}_{>0}$ be such that there is no $(\varepsilon, T)$-chain from $x$ to itself. Let $\mathcal{T}$ be the associated trapping region with $m = 1$ as in Lemma 4.2 and let $A$ be the attracting set associated with $\mathcal{T}$. Clearly $x \notin A \subseteq \mathcal{T}$. Since

$$x_0 = x, x_1 = \Phi_T(x), \quad T$$

is an $(\varepsilon, T)$ chain, $\Phi_T(x) \in \mathcal{T}$. Therefore, $x \in \mathrm{Orb}^-(\mathcal{T})$. Thus

$$\mathcal{X} \setminus \mathrm{ChRec}(\Phi) \subseteq \mathrm{Orb}^-(\mathcal{T}) \setminus A \subseteq \bigcup_{\mathcal{T}' \in \mathscr{T}(\Phi)} \mathrm{Orb}^-(\mathcal{T}') \setminus A_{\mathcal{T}'}.$$

Now, let $x \in \mathrm{Orb}^-(\mathcal{T}) \setminus A$ for some $\mathcal{T} \in \mathscr{T}(\Phi)$ with $A$ its attracting set. By Lemma 4.3, $x \notin \mathrm{ChRec}(\Phi)$ and so

$$\bigcup_{\mathcal{T}' \in \mathscr{T}(\Phi)} \mathrm{Orb}^-(\mathcal{T}') \setminus A_{\mathcal{T}'} \subseteq \mathcal{X} \setminus \mathrm{ChRec}(\Phi), \qquad \blacksquare$$

There is an insightful rephrasing of this decomposition for flows.

**4.5 Corollary: (A refinement of the Conley decomposition for flows)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. Then*

$$\mathrm{ChRec}(\Phi) = \bigcap_{\mathcal{T} \in \mathscr{T}(\Phi)} \{A \cup R \mid (A, R) \text{ is the attracting-repelling pair for } \mathcal{T} \in \mathscr{T}(\Phi)\}.$$

Proof: From the theorem,

$$\mathrm{ChRec}(\Phi) = \bigcap_{\mathcal{T} \in \mathscr{T}(\Phi)} \mathcal{X} \setminus (\mathrm{Orb}^-(\mathcal{T}) \setminus A_{\mathcal{T}}) = \bigcap_{\mathcal{T} \in \mathscr{T}(\Phi)} (\mathcal{X} \setminus \mathrm{Orb}^-(\mathcal{T})) \cup A_{\mathcal{T}}.$$

The corollary now follows from Lemma 2.14. ■

## 5. Complete Lyapunov functions

In this section we give the second part of the Fundamental Theorem of Dynamical Systems, namely the existence of a complete Lyapunov function. Our construction comes with a few steps. First we work with discrete-time flows and semiflows, with the final results being valid for flows and semiflows on separable metric spaces. In our constructions, we make use of strong trapping regions (as in Remark 2.8–1 and Definition 5.6 below), following [Hurley 1998]. Despite using strong trapping regions in place of the trapping regions from the decomposition theorem, the conclusions refer only to chain notions which are themselves not concerned with whether the trapping regions are strong or not. After these constructions are complete, we show that they imply the existence of complete Lyapunov functions in the continuous-time case for separable metric spaces.

### 5.1. Definitions.

We begin by giving definitions and elementary results around the notion of a complete Lyapunov function.

**5.1 Definition: (Complete Lyapunov function)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow or semiflow on $\mathcal{X}$. A ***complete Lyapunov function*** for $\Phi$ is a continuous function $L \colon \mathcal{X} \to [0, 1]$ such that:

(i) $L \circ \Phi(t_2, x) \le L \circ \Phi(t_1, x)$ for $x \in \mathcal{X}$ and $t_1, t_2 \in \mathbb{T}_{\ge 0}$ satisfying $t_1 < t_2$;

(ii) $L \circ \Phi(t, x) < L(x)$ for $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$ and $t \in \mathbb{T}_{>0}$;

(iii) $L \circ \Phi(t, x) = L(x)$ for $x \in \mathrm{ChRec}(\Phi)$ and $t \in \mathbb{T}_{\ge 0}$;

(iv) points $x, y \in \mathrm{ChRec}(\Phi)$ are chain equivalent if and only if $L(x) = L(y)$. •

Thus we see that a complete Lyapunov function (1) distinguishes the forward chain recurrent set, (2) additionally distinguishes the chain components, and (3) tells us something about the flow or semiflow between the chain components.

Complete Lyapunov functions for flows have particular properties that are sometimes also casually (and falsely) ascribed to complete Lyapunov functions for semiflows. The following lemma gives two particular properties of complete Lyapunov functions for flows.

**5.2 Lemma: (Complete Lyapunov functions for flows)** *Let $(\mathfrak{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a topological flow on $\mathfrak{X}$ with $L$ a complete Lyapunov function for $\Phi$. Then the following statements hold:*

*(i) if $x \in \mathfrak{X} \setminus \mathrm{ChRec}(\Phi)$, then the function $t \mapsto L \circ \Phi(t, x)$ is strictly decreasing on $\mathbb{T}$;*

*(ii) if $x \in \mathrm{ChRec}(\Phi)$, then $L \circ \Phi(t, x) = L(x)$ for $t \in \mathbb{T}$.*

Proof: (i) Note that $\mathrm{ChRec}(\Phi)$ is invariant by Proposition 3.12(i). By Lemma 2.6, $\mathfrak{X} \setminus \mathrm{ChRec}(\Phi)$ is also invariant. Therefore, if $x \in \mathfrak{X} \setminus \mathrm{ChRec}(\Phi)$ and if $t_1, t_2 \in \mathbb{T}$ satisfy $t_1 < t_2$, we have

$$L \circ \Phi(t_2, x) = L \circ \Phi(t_2 - t_1, \Phi(t_1, x)) < L \circ \Phi(t_1, x),$$

since $\Phi(t_1, x) \in \mathfrak{X} \setminus \mathrm{ChRec}(\Phi)$.

(ii) We need only prove this for $t \in \mathbb{T}_{<0}$, so let $x \in \mathrm{ChRec}(\Phi)$ and $t \in \mathbb{T}_{<0}$. Then $\Phi(t, x) \in \mathrm{ChRec}(\Phi)$ by Proposition 3.12(i). By definition,

$$L \circ \Phi(t, x) = L \circ \Phi(-t, \Phi(t, x)) = L(x). \qquad \blacksquare$$

We shall show that every topological flow or semiflow on a separable metric space possesses a complete Lyapunov function. To do so will require some work. First we will establish the result in the discrete-time case, and then use this result to establish the result in the continuous-time case. That the continuous- and discrete-time cases should be related should come as no surprise, given that the chain recurrent set and its chain components are essentially determined by the mapping $\Phi_1$, given Theorem 3.15.

**5.2. Some constructions particular to the discrete-time case.** In this section and the two sections following it, we work with a discrete-time flow or semiflow $\Phi$ on a metric space $(\mathfrak{X}, \mathrm{d})$, and we denote $\phi = \Phi_1 \in \mathrm{C}^0(\mathfrak{X}; \mathfrak{X})$. Where convenient, we will drop the reference to $\Phi$ and simply refer to $\phi$. For instance, we will write $\mathrm{ChRec}(\phi) \triangleq \mathrm{ChRec}(\Phi)$.

First, it is useful to give a refined equivalent characterisation of complete Lyapunov functions in this case.

**5.3 Lemma: (Complete Lyapunov functions for discrete-time flows and semiflows)** *Let $(\mathfrak{X}, \mathrm{d})$ be a metric space and let $\phi = \Phi_1$. A mapping $L \in \mathrm{C}^0(\mathfrak{X}; [0,1])$ is a complete Lyapunov function if and only if*

*(i) $L \circ \phi(x) \le L(x)$ for $x \in \mathfrak{X}$,*

*(ii) $L \circ \phi(x) < L(x)$ for $x \in \mathfrak{X} \setminus \mathrm{ChRec}(\Phi)$,*

*(iii) $L \circ \phi(x) = L(x)$ for $x \in \mathrm{ChRec}(\Phi)$, and*

*(iv) points $x, y \in \mathrm{ChRec}(\Phi)$ are chain equivalent if and only if $L(x) = L(y)$.*

Proof: Since the fourth property in the definition of a complete Lyapunov function and in the statement of the lemma are the same, we need only work with the first three properties in each case.

First suppose that $L$ is a complete Lyapunov function. Then, since $0 < 1$, $L\circ\phi(x) \le L(x)$ for every $x \in \mathcal{X}$. Similarly, $L \circ \phi(x) < L(x)$ for $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$. If $x \in \mathrm{ChRec}(\Phi)$, then $\phi(x) \in \mathrm{ChRec}(\Phi)$ by Proposition 3.12(i). Therefore, $L \circ \phi(x) = L(x)$. This gives the first three properties in the statement of the lemma.

Next suppose that $L$ has the first three properties from the statement of the lemma. Since $L \circ \Phi(t, x) = L \circ \phi^t(x)$, an elementary induction gives each of the first three properties in the definition of a complete Lyapunov function. ■

We shall work with $\varepsilon$-1-chains for $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, which we simply call $\varepsilon$-chains according to Definition 3.17. It turns out to be convenient to modify slightly our notion of chain. As pointed out by Hurley [1998], this minor modification is required by the condition that complete Lyapunov functions *decrease*. If we were happy with them being allowed to *increase*, we could get by with our existing notion of chains.

**5.4 Definition: ($\sigma$chain, $\varepsilon$-$\sigma$chain)** Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X})$. A ***$\sigma$chain*** for $\phi$ is a finite sequence

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)$$

of ordered pairs in $\mathcal{X}$ such that $x_{j+1} = \phi(y_j)$, $j \in \{0, 1, \dots, k-1\}$. The nonnegative integer $k$ is the ***length*** of the $\sigma$chain and the $\sigma$chain is said to be ***from $x_0$ to $y_k$***. If $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, then a $\sigma$chain

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)$$

is an ***$\varepsilon$-$\sigma$chain*** for $\phi$ if $\mathrm{d}(x_j, y_j) < \varepsilon(x_j)$, $j \in \{0, 1, \dots, k\}$. •

Let us clarify the relationship between $\varepsilon$-chains and $\varepsilon$-$\sigma$chains.

**5.5 Lemma: ($\varepsilon$-chains and $\varepsilon$-$\sigma$chains)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X})$. Then the following statements hold:*

*(i) if $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, then there exists $\varepsilon' \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that, if*

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)$$

*is an $\varepsilon'$-$\sigma$chain for $\phi$, then*

$$x_0, x_1, \dots, x_k$$

*is an $\varepsilon$-chain for $\phi$;*

*(ii) if $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and if*

$$x_0, x_1, \dots, x_k$$

*is a $\varepsilon$-chain for $\phi$, then*

$$(x_0, x_0), (\phi(x_0), x_1), \dots, (\phi(x_{k-1}), x_k)$$

*is a $\varepsilon$-$\sigma$chain for $\phi$.*

Proof: (i) By Lemma 3.4, let $\varepsilon' \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ be such that

$$\mathrm{d}(x,y) < \varepsilon'(x) \implies \mathrm{d}(\phi(x),\phi(y)) < \varepsilon(x).$$

This $\delta$ can be verified to have the asserted property.

(ii) This is clear by definition. ■

We shall also work with a modified notion of trapping region. Specifically, we will work with the *usual* notion of trapping region from the literature in the discrete-time case, as discussed in Remark 2.8–1.

**5.6 Definition: (Strong trapping region)** Let $(\mathcal{X},\mathrm{d})$ be a metric space and let $\phi \in \mathrm{C}^0(\mathcal{X};\mathcal{X})$. A ***strong trapping region*** for $\phi$ is a subset $\mathcal{T}$ for which $\mathrm{cl}(\phi(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$. •

Clearly a strong trapping region is a trapping region. Thus a strong trapping region has an associated attracting set and, in the case of flows, an associated repelling set.

**5.3. The discrete-time case: weak Lyapunov functions for attracting sets.** In this part of our development, we do much of the technical heavy lifting. We work with a fixed strong trapping region and build a weak Lyapunov function (i.e., a function that is nonincreasing along trajectories) with particular properties relative to the associated attracting set.

Our construction of a weak Lyapunov function for an attracting set is done in stages. First, given $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$ and a $\sigma$chain

$$(x_0,y_0),(x_1,y_1),\dots,(x_k,y_k)$$

for $\phi$ which we denote by $\gamma$, we define

$$D_\varepsilon(\gamma) = \sum_{j=0}^{k} \frac{\mathrm{d}(x_j,y_j)}{\varepsilon(x_j)}.$$

Now, if $C \subseteq \mathcal{X}$ is a nonempty closed set, if $x \in \mathcal{X}$, and if $\phi \in \mathrm{C}^0(\mathcal{X};\mathcal{X})$, denote by $\mathscr{C}_\phi(C;x)$ the set of all $\sigma$chains for $\phi$ from a point in $C$ to $x$. Then, for $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$, define

$$E_{\phi,\varepsilon}(C;x) = \inf\{D_\varepsilon(\gamma) \mid \gamma \in \mathscr{C}_\phi(C;x)\}.$$

It is evident that, since $(x,x) \in \mathscr{C}_\phi(C;x)$ if $x \in C$, that $E_{\phi,\varepsilon}(C;x) = 0$ in this case.

The following properties of $E_{\phi,\varepsilon}(C;x)$ are important for us.

**5.7 Lemma: (Properties of $E_{\phi,\varepsilon}(C;x)$)** *Let $(\mathcal{X},\mathrm{d})$ be a metric space and let $\phi \in \mathrm{C}^0(\mathcal{X};\mathcal{X})$. Let $C \subseteq \mathcal{X}$ be a nonempty closed set, let $x \in \mathcal{X}$, and let $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$. Then the following statements hold:*

*(i) $E_{\phi,\varepsilon}(C;\phi(x)) \le E_{\phi,\varepsilon}(C;x)$;*

*(ii) the mapping $x \mapsto E_{\phi,\varepsilon}(C;x)$ is continuous.*

Proof: (i) Let

$$(x_0,y_0),(x_1,y_1),\dots,(x_k,x)$$

be a $\sigma$chain from $x_0 \in C$ to $x$, denoted by $\gamma$. Define a $\sigma$chain $\gamma'$ from $x_0 \in C$ to $\phi(x)$ by

$$(x_0,y_0),(x_1,y_1),\dots,(x_k,x),(\phi(x),\phi(x)).$$

Since $D_\varepsilon(\gamma) = D_\varepsilon(\gamma')$, we must have $E_{\phi,\varepsilon}(C;\phi(x)) \le E_{\phi,\varepsilon}(C;x)$.

(ii) Suppose first that $x \in C$. In this case, $E_{\phi,\varepsilon}(C;x) = 0$ (as we observed just prior to the statement of the lemma). For $y \in C$, we have the $\sigma$chain $\gamma$ given by

$$(y, x)$$

from a point in $C$ to $x$. Note that

$$E_{\phi,\varepsilon}(C;y) = D_\varepsilon(\gamma) = \frac{\mathrm{d}(x,y)}{\varepsilon(x)},$$

and so $\lim_{y\to x} E_{\phi,\varepsilon}(C;y) = 0 = E_{\phi,\varepsilon}(C;x)$, giving continuity at $x$.

Now suppose that $x \notin C$. Consider a $\sigma$chain $\gamma$

$$(x_0,y_0),(x_1,y_1),\dots,(x_k,x)$$

from a point in $C$ to $x$. Let $y \in \mathcal{X}$ and consider the $\sigma$chain $\gamma'$

$$(x_0,y_0),(x_1,y_1),\dots,(x_k,y)$$

from the same point in $C$ to $y$. We then have

$$|D_\varepsilon(\gamma) - D_\varepsilon(\gamma')| = \left|\frac{\mathrm{d}(x_k,x)}{\varepsilon(x_k)} - \frac{\mathrm{d}(x_k,y)}{\varepsilon(x_k)}\right| \le \frac{\mathrm{d}(x,y)}{\varepsilon(x_k)} \tag{5.1}$$

using a standard metric identity [Searcóid 2007, Theorem 1.1.2]. We claim now that this shows that $x \mapsto E_{\phi,\varepsilon}(C;x)$ is upper semicontinuous. To see this, let $\delta \in \mathbb{R}_{>0}$ and let $\gamma$ be a $\sigma$chain from a point in $C$ to $x$ such that

$$D_\varepsilon(\gamma) < E_{\phi,\varepsilon}(C;x) + \frac{\delta}{2}.$$

For $y \in \mathcal{X}$ sufficiently close to $x$ and with $\gamma'$ as defined above, we can ensure that

$$D_\varepsilon(\gamma') < D_\varepsilon(\gamma') + \frac{\delta}{2}.$$

With $\gamma'$ so chosen, we have

$$E_{\phi,\varepsilon}(C;y) \le D_\varepsilon(\gamma') < D_\varepsilon(\gamma) + \frac{\delta}{2} < E_{\phi,\varepsilon}(C;x) + \delta,$$

from which we can conclude the asserted upper semicontinuity.

Let us suppose that we do not have lower semicontinuity at $x$. Then there exists a sequence $(y_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathcal{X}$ converging to $x$ and $\beta \in \mathbb{R}_{>0}$ such that

$$E_{\phi,\varepsilon}(C;y_j) < E_{\phi,\varepsilon}(C;x) - \beta, \qquad j \in \mathbb{Z}_{>0}.$$

If this last condition holds, then there exists a sequence $\gamma'_j \in \mathscr{C}_\phi(C;y_j)$, $j \in \mathbb{Z}_{>0}$, such that

$$D_\varepsilon(\gamma'_j) < E_{\phi,\varepsilon}(C;x) - \beta, \qquad j \in \mathbb{Z}_{>0}.$$

Denote by $\gamma_j$ the $\sigma$chain obtained as above, replacing $y_j$ with $x$. As in (5.1), we have

$$|D_\varepsilon(\gamma_j) - D_\varepsilon(\gamma_j')| \le \frac{\mathrm{d}(x, y_j)}{\varepsilon(z_j)},$$

where the last pair in $\gamma_j'$ is $(z_j, y_j)$. If it holds that $\liminf_{j\to\infty} \frac{\mathrm{d}(x,y_j)}{\varepsilon(z_j)} = 0$, then we arrive at the contradiction that $D_\varepsilon(\gamma_j) < E_{\phi,\varepsilon}(C; x)$ for some sufficiently large $j$, in contradiction to the definition of $E_{\phi,\varepsilon}(C; x)$. This contradiction would then show that our assumption that we do not have lower semicontinuity at $x$ must be false.

Thus it remains to show that $\liminf_{j\to\infty} \frac{\mathrm{d}(x,y_j)}{\varepsilon(z_j)} = 0$ under the assumption that

$$E_{\phi,\varepsilon}(C; y_j) < E_{\phi,\varepsilon}(C; x) - \beta, \qquad j \in \mathbb{Z}_{>0}.$$

To this end, note that, as we have seen, the assumption implies that

$$E_{\phi,\varepsilon}(C; x) - \beta > D_\varepsilon(\gamma_j') \ge \frac{\mathrm{d}(z_j, y_j)}{\varepsilon(z_j)}.$$

Therefore,

$$\varepsilon(z_j) \ge M \mathrm{d}(z_j, y_j), \qquad j \in \mathbb{Z}_{>0},$$

for a constant $M \in \mathbb{R}_{>0}$ independent of $j$. Therefore,

$$\frac{\mathrm{d}(x, y_j)}{\varepsilon(z_j)} \le M^{-1} \frac{\mathrm{d}(x, y_j)}{\mathrm{d}(z_j, y_j)}, \qquad j \in \mathbb{Z}_{>0}.$$

We now consider two cases: (1) $\lim_{j\to 0} \mathrm{d}(z_j, y_j) = 0$ and (2) (1) does not hold. In the first case, $\lim_{j\to\infty} z_j = x$. Therefore, for $j$ sufficiently large, $\varepsilon(z_j) \ge \frac{1}{2}\varepsilon(x)$, in which case we directly have

$$\lim_{j\to\infty} \frac{\mathrm{d}(x, y_j)}{\varepsilon(z_j)} = 0,$$

giving the desired conclusion in this case. In the second case, we have

$$\liminf_{j\to\infty} \frac{\mathrm{d}(x, y_j)}{\varepsilon(z_j)} \le \liminf_{j\to\infty} M^{-1} \frac{\mathrm{d}(x, y_j)}{\mathrm{d}(z_j, y_j)} = 0,$$

again giving the desired result. ■

The way to view the function $x \mapsto E_\varepsilon(C; x)$ is that it measures the "$\varepsilon$-effort" expended along any $\varepsilon$-$\sigma$chain going from a point in $C$ to the point $x$, where "$\varepsilon$-effort" is characterised by the ratio of the jumps of a $\sigma$chain compared to the maximum jump permitted by $\varepsilon$. (Thus "1" represents neutral effort.) The closed set $C$ in the definition has no relationship to the dynamics.

In the next stage in our construction, we fix a strong trapping region $\mathcal{T}$ and consider our previous "minimum effort" function applied to the sets $\mathrm{cl}(\phi^k(\mathcal{T}))$, $k \in \mathbb{Z}_{>0}$. We tailor $\varepsilon$ according to Lemma 4.1, and furthermore to be bounded above by 1. Throughout the next part of our construction, we understand this $\varepsilon$ to have been chosen and fixed. With such a $\varepsilon$ at hand and for $k \in \mathbb{Z}_{>0}$, we define

$$E_{\varepsilon,k}(x) = E_{\phi^k,\varepsilon}(\mathrm{cl}(\phi^k(\mathcal{T})); x), \qquad x \in \mathcal{X}.$$

Let us list the pertinent properties of $E_{\varepsilon,k}$.

**5.8 Lemma: (Properties of $E_{\varepsilon,k}$)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\phi \in \mathrm{C}^0(\mathcal{X};\mathcal{X})$. For a strong trapping region $\mathcal{T}$ and with $\varepsilon$ as above, and for $k \in \mathbb{Z}_{>0}$, the following statements hold:*

*(i) $E_{\varepsilon,k}$ is nonnegative and continuous;*

*(ii) $E_{\varepsilon,k} \circ \phi^k(x) \leq E_{\varepsilon,k}(x)$, $x \in \mathcal{X}$;*

*(iii) if*

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)$$

*is an $\varepsilon$-$\sigma$chain for $\phi^k$ and if $x_0 \in \mathrm{cl}(\phi^k(\mathcal{T}))$, then $y_k \in \mathcal{T}$;*

*(iv) $E^{-1}_{\varepsilon,k}(0) = \mathrm{cl}(\phi^k(\mathcal{T}))$.*

*Additionally, no longer fixing $k$,*

*(v) if $A$ is the attracting set associated with $\mathcal{T}$ and if $x \in A$, then $E_{\varepsilon,k}(x) = 0$ for every $k \in \mathbb{Z}_{>0}$;*

*(vi) if $x \notin \mathcal{T}$, then $E_{\varepsilon,k}(x) \geq 1$ for every $k \in \mathbb{Z}_{>0}$.*

Proof: Parts (i) and (ii) follow directly from the definitions and Lemma 5.7.

(iii) Since $x_0 \in \mathrm{cl}(\phi^k(\mathcal{T})) \subseteq \mathcal{T}$, then $y_0 \in \mathcal{T}$ by the fact that $\varepsilon$ satisfies the condition of Lemma 4.1. Then $x_1 = \phi^k(y_0) \in \mathrm{cl}(\phi^k(\mathcal{T}))$ and so $y_1 \in \mathcal{T}$ by the same argument. Inductively, $y_k \in \mathcal{T}$.

(iv) As we have observed above when working with $E_{\phi,\varepsilon}(C; x)$, we have

$$\mathrm{cl}(\phi^k(\mathcal{T})) \subseteq E^{-1}_{\varepsilon,k}(0).$$

It suffices, then, to show that $E_{\varepsilon,k}(x) > 0$ for $x \notin \mathrm{cl}(\phi^k(\mathcal{T}))$. If there are no $\varepsilon$-$\sigma$chains from a point in $\mathrm{cl}(\phi^k(\mathcal{T}))$ to $x$, then $E_{\varepsilon,k}(x) \geq 1$, as we shall see in our proof of part (vi) below. So suppose that

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, x)$$

is a $\varepsilon$-$\sigma$chain from a point $x_0 \in \mathrm{cl}(\phi^k(\mathcal{T}))$ to $x$. By part (iii), $x_k \in \mathrm{cl}(\phi^k(\mathcal{T}))$, whereupon

$$D_\varepsilon(\gamma) \geq \frac{\mathrm{d}(x_k, x)}{\varepsilon(x_k)} \geq \mathrm{d}(x_k, x) \geq \mathrm{dist}_{\mathrm{cl}(\phi^k(\mathcal{T}))}(x) > 0,$$

noting that we have assumed that $\varepsilon$ takes values in $(0, 1]$.

(v) If $x \in A$, then $x \in \mathrm{cl}(\phi^k(\mathcal{T}))$ for $k \in \mathbb{Z}_{>0}$. As we argued in the previous part of the proof, this implies that $E_{\varepsilon,k}(x) = 0$ for all $k \in \mathbb{Z}_{>0}$.

(vi) If $x \notin \mathcal{T}$ and if $\gamma \in \mathscr{C}_\phi(\mathrm{cl}(\phi^k(\mathcal{T})); x)$, then $\gamma$ is not an $\varepsilon$-$\sigma$chain by part (iii). Therefore, if $\gamma$ is given by

$$(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k),$$

then, for some $j \in \{0, 1, \dots, k\}$, $\mathrm{d}(x_j, y_j) \geq \varepsilon(x_j)$, from which we have

$$D_\varepsilon(\gamma) \geq \frac{\mathrm{d}(x_j, y_j)}{\varepsilon(x_j)} \geq 1.$$

Therefore, $E_{\phi^k,\varepsilon}(\mathrm{cl}(\phi^k(\mathcal{T})); x) \geq 1$. ■

The next step in our construction is to average $E_{\varepsilon,k}$ over the first $k$ iterates of $\phi$:

$$\overline{E}_{\varepsilon,k}(x) = \frac{1}{k}\sum_{j=0}^{k-1} E_{\varepsilon,k}\circ\phi^j(x).$$

Let us record the salient properties of this function.

**5.9 Lemma: (Properties of $\overline{E}_{\varepsilon,k}$)** *Let $(\mathcal{X},\mathrm{d})$ be a metric space and let $\phi\in\mathrm{C}^0(\mathcal{X};\mathcal{X})$ with strong trapping region $\mathcal{T}$ and associated error function $\varepsilon$ as in Lemma 4.1. Then the following statements hold:*

*(i) $\overline{E}_{\varepsilon,k}$ is continuous and $\mathbb{R}_{\geq 0}$-valued;*

*(ii) $\overline{E}_{\varepsilon,k}\circ\phi(x)\leq\overline{E}_{\varepsilon,k}(x)$.*

Proof: (i) This is clear.

(ii) We have

$$\overline{E}_{\varepsilon,k}\circ\phi(x) - \overline{E}_{\varepsilon,k}(x) = \frac{1}{k}(E_{\varepsilon,k}\circ\phi(x) - E_{\varepsilon,k}(x)) \leq 0. \qquad \blacksquare$$

Now we can give the final part of the constructions we shall make for a fixed strong trapping region $\mathcal{T}$. We define

$$E_{\mathcal{T}}(x) = \sum_{k=1}^{\infty}\frac{\min\{\overline{E}_{\varepsilon,k}(x),1\}}{2^k}, \qquad x\in\mathcal{X},$$

and then

$$L_{\mathcal{T}}(x) = \sum_{j=0}^{\infty}\frac{E_{\mathcal{T}}\circ\phi^j(x)}{2^j}, \qquad x\in\mathcal{X}.$$

Let us record the properties of the second of these functions. In the statement and proof of the lemma (and of many constructions in the remainder of this section), it is insightful to keep in mind the characterisation of repelling sets from Lemma 2.14.

**5.10 Lemma: (Properties of $L_{\mathcal{T}}$)** *Let $(\mathcal{X},\mathrm{d})$ be a metric space and let $\phi\in\mathrm{C}^0(\mathcal{X};\mathcal{X})$. Let $\mathcal{T}$ be an open strong trapping region for $\phi$ with $A$ the associated attracting set. Then the following statements hold:*

*(i) $L_{\mathcal{T}}$ is continuous with values in $[0,1]$;*

*(ii) $L_{\mathcal{T}}^{-1}(0)=A$ and $L_{\mathcal{T}}^{-1}(1)=\mathcal{X}\setminus\mathrm{Orb}^-(\mathcal{T})$;*

*(iii) $L_{\mathcal{T}}\circ\phi(x)<L_{\mathcal{T}}(x)$ for $x\in\mathrm{Orb}^-(\mathcal{T})\setminus A$.*

Proof: (i) The series defining the functions $E_{\mathcal{T}}$ is uniformly convergent by the Weierstrass $M$-test, by virtue of which the limit function is continuous. The function $E_{\mathcal{T}}$ take values in $[0,1]$ since $\sum_{j=1}^{\infty}\frac{1}{2^j}=1$. For the same reason, the function $L_{\mathcal{T}}$ take values in $[0,1]$.

(ii) If $x\in A$, then $\phi^j(x)\in A$ for every $j\in\mathbb{Z}_{\geq 0}$ by Proposition 2.13(ii). Therefore, $E_{\varepsilon,k}\circ\phi^j(x)=0$ for every $k\in\mathbb{Z}_{>0}$ and $j\in\mathbb{Z}_{\geq 0}$ by Lemma 5.8(v). Therefore, $\overline{E}_{\varepsilon,k}\circ\phi^j(x)=0$ for every $k\in\mathbb{Z}_{>0}$ and $j\in\mathbb{Z}_{\geq 0}$ and so $E_{\mathcal{T}}\circ\phi^j(x)=0$ for every $j\in\mathbb{Z}_{\geq 0}$. Therefore, $L_{\mathcal{T}}(x)=0$. Conversely, if $L_{\mathcal{T}}(x)=0$, then $E_{\mathcal{T}}\circ\phi^j(x)=0$ for $j\in\mathbb{Z}_{\geq 0}$. This, in turn, implies that $\overline{E}_{\varepsilon,k}\circ\phi^j(x)=0$ for every $k\in\mathbb{Z}_{>0}$ and $j\in\mathbb{Z}_{\geq 0}$. Considering the case of $j=0$, this implies that $E_{\varepsilon,k}(\phi^l(x)) = 0$ for every $k \in \mathbb{Z}_{>0}$ and $l \in \{0, 1, \dots, k-1\}$. In particular, $E_{\varepsilon,k}(x) = 0$ for every $k \in \mathbb{Z}_{>0}$. Thus $x \in \cap_{k\in\mathbb{Z}_{>0}} \mathrm{cl}(\phi^k(\mathcal{T}))$ by Lemma 5.8(iv). Thus $x \in A$.

Now suppose that $x \in \mathcal{X} \setminus \mathrm{Orb}^-(\mathcal{T})$ and note that $\phi^j(x) \notin \mathcal{T}$ for every $j \in \mathbb{Z}_{\geq 0}$. Therefore, $E_{\varepsilon,k} \circ \phi^j(x) \geq 1$ for every $k \in \mathbb{Z}_{>0}$ and every $j \in \mathbb{Z}_{\geq 0}$ by Lemma 5.8(vi). Therefore, $\overline{E}_{\varepsilon,k}(x) \geq 1$, and so $E_{\mathcal{T}} \circ \phi^j(x) = 1$ for every $k \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$. Thus $L_{\mathcal{T}}(x) = 1$.

Conversely, suppose that $L_{\mathcal{T}}(x) = 1$. It follows that $E_{\mathcal{T}} \circ \phi^j(x) = 1$ for $j \in \mathbb{Z}_{\geq 0}$ and so $\overline{E}_{\varepsilon,k} \circ \phi^j(x) \geq 1$ for every $k \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$. We claim that this prohibits $x \in \mathrm{Orb}^-(\mathcal{T})$ for some $y \in \mathcal{T}$. Indeed, if $x \in \mathrm{Orb}^-(y)$ for some $y \in \mathcal{T}$, then $\phi^j(x) \in \mathcal{T}$ for some $j \in \mathbb{Z}_{\geq 0}$ and so $\phi^{j+k}(x) \in \phi^{j+k}(\mathcal{T}) \subseteq \mathrm{cl}(\phi^{j+k}(\mathcal{T}))$ for $k \in \mathbb{Z}_{>0}$. By Lemma 5.8(iv), this implies that $E_{\varepsilon,k} \circ \phi^j(x) = 0$. This, in turn, implies that $\overline{E}_{\varepsilon,k}(x) < 1$. This shows that, indeed, if $L_{\mathcal{T}}(x) = 1$, then $x \notin \mathrm{Orb}(y)$ for every $y \in \mathcal{T}$. Thus $x \in \mathcal{X} \setminus \mathrm{Orb}^-(\mathcal{T})$.

(iii) By Lemma 5.9(ii),

$$\overline{E}_{\varepsilon,k} \circ \phi^{j+1}(x) \leq \overline{E}_{\varepsilon,k} \circ \phi^j(x), \qquad x \in \mathcal{X},\ k \in \mathbb{Z}_{>0},\ j \in \mathbb{Z}_{\geq 0}.$$

It follows that

$$E_{\mathcal{T}}(\phi^{j+1}(x)) \leq E_{\mathcal{T}}(\phi^j(x)), \qquad x \in \mathcal{X},\ j \in \mathbb{Z}_{\geq 0}.$$

We will have $L_{\mathcal{T}} \circ \phi(x) < L_{\mathcal{T}}(x)$ when there exists $j \in \mathbb{Z}_{\geq 0}$ such that

$$E_{\mathcal{T}}(\phi^{j+1}(x)) < E_{\mathcal{T}}(\phi^j(x)).$$

This will happen when there exists $k \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$ such that

1. $\overline{E}_{\varepsilon,k} \circ \phi^j(x) < 1$ and
2. $\overline{E}_{\varepsilon,k} \circ \phi^{j+1}(x) < \overline{E}_{\varepsilon,k} \circ \phi^j(x)$.

Therefore, to establish this part of the lemma, it suffices to show that, for $x \in \mathrm{Orb}^-(\mathcal{T})$, there exists $k \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$ such that conditions 1 and 2 are satisfied.

First suppose that $x \in \phi(\mathcal{T}) \setminus A$. Since $x \notin A$, there exists $k \in \mathbb{Z}_{>0}$ such that $x \notin \mathrm{cl}(\phi^{k+1}(\mathcal{T}))$ but $x \in \mathrm{cl}(\phi^j(\mathcal{T}))$ for $j \in \{1, \dots, k\}$. Thus $\overline{E}_{\varepsilon,j}(x) = 0$ for $j \in \{1, \dots, k\}$ and $\overline{E}_{\varepsilon,k+1}(x) > 0$ by Lemma 5.8(iv). Since $\phi(x) \in \mathrm{cl}(\phi^j(\mathcal{T}))$ for $j \in \{1, \dots, k+1\}$, we can again use Lemma 5.8(iv) to see that

$$\overline{E}_{\varepsilon,k+1} \circ \phi(x) = 0 < \overline{E}_{\varepsilon,k+1}(x),$$

giving this part of the lemma in the case that $x \ni \phi(\mathcal{T}) \setminus A$. Now suppose that $x \in \mathrm{Orb}^-(\mathcal{T}) \setminus \phi(\mathcal{T})$. Then $\phi^j(x) \in \phi(\mathcal{T})$ for some $j \in \mathbb{Z}_{>0}$, and then the above argument gives

$$\overline{E}_{\varepsilon,k+1} \circ \phi^{j+1}(x) = 0 < \overline{E}_{\varepsilon,k+1} \circ \phi^j(x)$$

for some $k \in \mathbb{Z}_{>0}$. This gives the result. ■

**5.4. The discrete-time case: the complete Lyapunov function.** In the preceding part of our construction, we fixed a strong trapping region $\mathcal{T}$ and constructed a weak Lyapunov function $L_{\mathcal{T}}$ with some useful properties. We now use this construction to build a complete Lyapunov function for a continuous mapping. To do so, we "sum over attracting sets," and so this requires being able to sum in a useful way. Thus we first establish some countability for strong trapping regions. To do this, we shall make a connection between strong trapping regions and points that can be connected by chains, rather as we did in the proof of Theorem 4.4.

This being said, we begin with some constructions with chains.

**5.11 Lemma: (Robust chains)** *Let $(\mathcal{X}, \mathrm{d})$ and let $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X})$. Let $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ and $x, y \in \mathcal{X}$ be such that there is no $3\varepsilon$-chain of length at least 2 for $\phi$ from $x$ to $y$. Then there are neighbourhoods $\mathcal{U}$ of $x$ and $\mathcal{V}$ of $y$, and $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that there is no $\delta$ chain for $\phi$ from a point in $\mathcal{U}$ to a point in $\mathcal{V}$.*

Proof: We first claim that there exists $\delta_1 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that, if

$$x_0, x_1, \dots, x_k$$

is a $\varepsilon$-chain for $\phi$, and if $\mathrm{d}(z, x_0) < \delta_1(x_0)$, then

$$(z, x_1, \dots, x_n)$$

is a $3\varepsilon$-chain for $\phi$. By Lemma 3.3, let $\delta \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that

$$\mathrm{d}(z_1, z_2) < \delta(z_1) \implies \tfrac{1}{2}\varepsilon(z_1) < \varepsilon(z_2) < \tfrac{3}{2}\varepsilon(z_1).$$

Without loss of generality, suppose that $\delta \le \frac{1}{2}\varepsilon$. By Lemma 3.4, let $\delta_1 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \delta_1(z_1) \implies \mathrm{d}(\phi(z_1), \phi(z_2)) < \delta(\phi(z_1)).$$

Note that

$$\mathrm{d}(z, x_0) < \delta_1(x_0) \implies \mathrm{d}(\phi(x_0), \phi(z)) < \delta(\phi(x_0)) \implies \frac{1}{2}\varepsilon(\phi(x_0)) < \varepsilon(\phi(z)).$$

Then, if $\mathrm{d}(z, x_0) < \delta_1(x_0)$, we have

$$\begin{aligned} \mathrm{d}(\phi(z), x_1) &\le \mathrm{d}(\phi(z), \phi(x_0)) + \mathrm{d}(\phi(x_0), x_1) \\ &< \delta(\phi(x_0)) + \varepsilon(\phi(x_0)) \le \frac{3}{2}\varepsilon(\phi(x_0)) \le 3\varepsilon(\phi(z)), \end{aligned}$$

as desired.

Next we claim that there exists $\delta_2 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that, if

$$x_0, x_1, \dots, x_{k-1}, x_k$$

is a $\delta_2$-chain with $k \ge 2$ for $\phi$ and if $\mathrm{d}(x_k, z) < \varepsilon(x_k)$, then

$$x_0, x_1, \dots, x_{k-1}, z$$

is a $3\varepsilon$-chain for $\phi$. Indeed, by Lemma 3.3, let $\delta_2 \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ be such that

$$\mathrm{d}(z_1, z_2) < \delta_2(z_1) \implies \frac{1}{2}\varepsilon(z_1) < \varepsilon(z_2) < \frac{3}{2}\varepsilon(z_1).$$

Without loss of generality, suppose that $\delta_2 < \varepsilon$. Note that

$$\mathrm{d}(\phi(x_{k-1}), x_k) < \varepsilon(\phi(x_{k-1})) \implies \varepsilon(x_k) < \frac{3}{2}\varepsilon(\phi(x_{k-1})).$$

Therefore, if $\mathrm{d}(x_k, z) < \varepsilon(x_k)$, we have

$$\begin{aligned} \mathrm{d}(\phi(x_{k-1}), z) &\le \mathrm{d}(\phi(x_{k-1}), x_k) + \mathrm{d}(x_k, z) \\ &< \delta_2(\phi(x_{k-1})) + \varepsilon(x_k) < \frac{5}{2}\varepsilon(\phi(x_{k-1})) < 3\varepsilon(\phi(x_{k-1})), \end{aligned}$$

as desired.

Now take $\delta = \frac{1}{2}\min\{\delta_1, \delta_2\}$ with $\delta_1$ and $\delta_2$ as in the preceding two paragraphs. Let $\mathcal{U}$ be a neighbourhood of $x$ such that

1. $\mathcal{U} \subseteq \mathsf{B}(\delta(x), x)$ and
2. $2\delta(x') > \delta(x)$ for $x' \in \mathcal{U}$,

and specify a neighbourhood $\mathcal{V}$ of $y$ similarly. Now, if

$$x', x_1, \dots, x_{k-1}, y'$$

is a $\delta$-chain for $\phi$ from $x' \in \mathcal{U}$ to $y' \in \mathcal{V}$, then

$$\mathrm{d}(x, x') < \frac{1}{2}\delta_1(x) < \delta_1(x')$$

and

$$\mathrm{d}(y, y') < \frac{1}{2}\delta_2(y) < \delta_2(y'),$$

our constructions ensure that

$$x, x_1, \dots, x_{k-1}, y$$

is a $3\varepsilon$-chain for $\phi$ from $x$ to $y$. This proves the lemma by contraposition. ■

Using the lemma, we devise a countable subset of strong trapping regions with desired properties. As we shall see—and as we have seen already in our proof of Theorem 4.4—there is a connection between strong trapping regions and chains. Thus the first step in reducing to a countable number of strong trapping regions is to reduce to a countable number of $\varepsilon$'s used to define these strong trapping regions.

**5.12 Lemma: (A useful countable subset of $\mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$)** *Let $(\mathcal{X}, \mathrm{d})$ be a separable metric space and let $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X})$. Then there exists a countable subset $\mathscr{P} \subseteq \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ with the following property: if $x, y \in \mathcal{X}$ and if, for some $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, there is no $\varepsilon$-chain of length at least 2 for $\phi$ from $x$ to $y$, then there exists $\delta \in \mathscr{P}$ such that there is no $\delta$-chain of length at least 2 for $\phi$ from $x$ to $y$.*

Proof: Let us denote

$$\mathscr{NC} = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid \text{ there exists } \varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0}) \text{ such that there is no } \varepsilon\text{-chain of length at least 2 from } x \text{ to } y\}.$$

For $(x, y) \in \mathscr{NC}$, there exist neighbourhoods $\mathcal{U}_x$ of $x$ and $\mathcal{V}_y$ of $y$, and $\delta_{x,y} \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that there is no $\delta_{x,j}$-chain from a point in $\mathcal{U}$ to a point in $\mathcal{V}$. Now note that the collection $\mathcal{U}_x \times \mathcal{V}_y$, $(x, y) \in \mathscr{NC}$, of open sets covers $\mathscr{NC}$. There is a countable collection of points $((x_j, y_j))_{j \in \mathbb{Z}_{>0}}$ from $\mathscr{NC}$ such that the sets $\mathcal{N}_j \triangleq \mathcal{U}_{x_j} \times \mathcal{V}_{y_j}$, $j \in \mathbb{Z}_{>0}$, covers $\mathscr{NC}$ [Willard 1970, Theorem 16.9]. Taking $\mathscr{P} = \{\delta_j \triangleq \delta_{x_j, y_j} \mid j \in \mathbb{Z}_{>0}\}$ gives the result. ■

Let us extract from the proof the notation

$$\mathscr{NC} = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid \text{ there exists } \varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0}) \text{ such that there is no } \varepsilon\text{-chain of length at least 2 from } x \text{ to } y\}.$$

Let $D \subseteq \mathcal{X}$ be a countable dense subset and let $\mathscr{P}$ be as prescribed by the lemma. Now, for $x \in \mathcal{X}$ and $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$, denote

$$\mathcal{T}(x, \varepsilon) = \{y \in \mathcal{X} \mid \text{ there exists an } \varepsilon\text{-chain of length at least 2 from } x \text{ to } y\}.$$

By Lemma 4.2, $\mathcal{T}(x,\varepsilon)$ is a nonempty, open strong trapping region. Denote

$$\mathscr{T} = \{\mathcal{T}(x,\varepsilon) \mid x \in D,\ \varepsilon \in \mathscr{P}\}.$$

As $\mathscr{T}$ is countable, we enumerate its elements as $(\mathcal{T}_j)_{j\in\mathbb{Z}_{>0}}$. We let $A_j$ be the attracting set for $\mathcal{T}_j$ and we let $L_j = L_{\mathcal{T}_j}$ be the Lyapunov function for $\mathcal{T}_j$ as constructed in the preceding section. Then define

$$L(x) = \sum_{j=1}^{\infty} \frac{2L_j(x)}{3^j}. \tag{5.2}$$

Since $L_j$ takes values in $[0,1]$, this series converges uniformly by the Weierstrass $M$-test, and so converges uniformly to a continuous function. Certainly $L\circ\phi(x) \leq L(x)$ for $x \in \mathcal{X}$. The following lemma lists some other pertinent properties of $L$.

**5.13 Lemma: (Properties of $L$)** *Let $(\mathcal{X},\mathrm{d})$ be a separable metric space and let $\phi \in \mathrm{C}^0(\mathcal{X};\mathcal{X})$. With the notation introduced above, the following statements hold for $(x,y) \in \mathscr{N\!C}$:*

*(i) there exist $z \in D$ and $\delta_j \in \mathscr{P}$ such that $y \notin \mathcal{T}(z,\delta_j)$ and $x \in \mathrm{Orb}^-(\mathcal{T}(z,\delta_j))$.*

*With $z \in D$ and $j \in \mathbb{Z}_{>0}$ as in the preceding statement, let $k \in \mathbb{Z}_{>0}$ be such that $\mathcal{T}(z,\delta_j) = \mathcal{T}_k$. Then the following statements hold;*

*(ii) if $y \in \mathrm{ChRec}(\phi)$, then $L_k(x) < L_k(y)$;*

*(iii) if $y = x$, then $L_k\circ\phi(x) < L_k(x)$;*

*(iv) if $x \notin \mathrm{ChRec}(\phi)$, then $L\circ\phi(x) < L(x)$;*

*(v) if $x,y \in \mathrm{ChRec}(\phi)$ but $x$ and $y$ are not chain equivalent, then $L(x) \neq L(y)$;*

*(vi) if $C, C' \subseteq \mathrm{ChRec}(\phi)$ are distinct chain components and if, for each $\varepsilon \in \mathrm{C}^0(\mathcal{X};\mathbb{R}_{>0})$, there is a $\varepsilon$-chain for $\phi$ from a point in $C$ to a point in $C'$, then $L(C) > L(C')$.*

Proof: (i) Adopting the notation from the proof of Lemma 5.12, let $\mathcal{N}_j = \mathcal{U}_{x_j} \times \mathcal{V}_{y_j}$ be such that $(x,y) \in \mathcal{N}_j$. Note that $D \cap \mathcal{U}_{x_j}$ is dense in $\mathcal{U}_{x_j}$. For $z \in D\cap\mathcal{U}_{x_j}$, there is no $\delta_j$-chain from $z$ to $y$, as we can conclude from Lemma 5.12. Thus $y \notin \mathcal{T}(z,\delta_j)$. Now, if $z$ is sufficiently close to $x$, then $\phi(z)$ will be close enough to $\phi(x)$ that

$$z, \phi(x), \phi^2(x)$$

is a $\delta_j$-chain from $z$ to $\phi^2(x)$. Thus $\phi^2(x) \in \mathcal{T}(z,\delta_j)$ and so $x \in \cup_{j\in\mathbb{Z}_{>0}}\phi^{-j}(\mathcal{T}(z,\varepsilon))$.

(ii) If $y \in \mathrm{ChRec}(\phi)$, then $y \in A_k \cup (\mathcal{X}\setminus \mathrm{Orb}^-(\mathcal{T}_k))$ by Theorem 4.4. Since $y \notin \mathcal{T}_k \supseteq A_k$, we must have $y \in \mathcal{X}\setminus\mathrm{Orb}^-(\mathcal{T}_k)$. Since $x \in \mathrm{Orb}^-(\mathcal{T}_k)$, $L_k(x) < 1$, cf. the proof of Lemma 5.10(ii), which gives this part of the result.

(iii) With the stated hypotheses, we have $x \in \mathrm{Orb}(\mathcal{T}_k)\setminus\mathcal{T}_k$, whereupon $L_k\circ\phi(x) < L_k(x)$ by Lemma 5.10(iii).

(iv) Note that $x \in \mathrm{ChRec}(\phi)$ if and only if $(x,x) \in \mathscr{N\!C}$. This being the case, this part of the result follows from the previous one.

(v) Note that $x$ and $y$ are not chain equivalent if and only if $(x,y) \in \mathscr{N\!C}$ or $(y,x) \in \mathscr{N\!C}$. Let us consider the case $(x,y) \in \mathscr{N\!C}$. By part (ii), $L_k(x) < L_y(y)$. By Lemma 5.10(ii), $L_k(x) = 0$ and $L_k(y) = 1$. Thus $L(x)$ and $L(y)$ do not agree since they necessarily have different ternary (that is, base 3) expansions.

(vi) Note that

$$C, C' \subseteq \mathrm{ChRec}(\phi) = \bigcap_{\mathcal{T} \in \mathscr{T}} \{A \cup (\mathcal{X} \setminus \mathrm{Orb}^-(\mathcal{T})) \mid A \text{ is the attracting set for } \mathcal{T}\}$$

by Theorem 4.4. Suppose that $C \subseteq A_k$ for some $k \in \mathbb{Z}_{>0}$. We claim that, with the hypotheses of this part of the lemma, $C' \subseteq A_k$. Indeed, if $C' \not\subseteq A_k$, then $C_k \subseteq \mathcal{X} \setminus \mathrm{Orb}^-(\mathcal{T}_k)$. However, by Lemma 4.1, there exists $\varepsilon \in \mathrm{C}^0(\mathcal{X}; \mathbb{R}_{>0})$ such that every $\varepsilon$-chain starting in $\mathcal{T}_k$ ends in $\mathcal{T}_k$, in contradiction with the current hypotheses. Thus, indeed, $C' \subseteq A_k$. This shows that, if $L_k(C) = 0$, then $L_k(C') = 0$. Since, for each $k \in \mathbb{Z}_{>0}$, $L_k(C), L_k(C') \in \{0, 1\}$ by Lemma 5.10(ii), we have $L_k(C) \geq L_k(C')$ for each $k \in \mathbb{Z}_{>0}$. This part of the result now follows from the previous part. ■

This gives the following theorem which is the second part of the Fundamental Theorem of Dynamical Systems for discrete-time flows and semiflows.

**5.14 Theorem: (Complete Lyapunov functions for mappings)** *Let $(\mathcal{X}, \mathrm{d})$ be a separable metric space and let $\phi \in \mathrm{C}^0(\mathcal{X}; \mathcal{X})$. Then there exists a complete Lyapunov function for $\phi$.*

**5.5. The continuous-time case: the complete Lyapunov function.** Now we use the preceding results concerning mappings to obtain the existence of complete Lyapunov functions for flows. We use the idea of Patrão [2011]. However, the proof of Patrão contains a number of errors, including making use of connectedness of chain components (as far as we know, this has only been proved in the compact case) and using Lemma 2.6 for semiflows (the lemma is not true for semiflows).

The theorem we prove is the following.

**5.15 Theorem: (Complete Lyapunov functions for flows)** *Let $(\mathcal{X}, \mathrm{d})$ be a metric space and let $\Phi$ be a continuous-time topological flow or semiflow on $\mathcal{X}$. If $\ell \in \mathrm{C}^0(\mathcal{X}; [0, 1])$ is a complete Lyapunov function for $\Phi^{\mathrm{d},1}$, then the function $L\colon \mathcal{X} \to \mathbb{R}$ given by*

$$L(x) = \int_0^1 \ell \circ \Phi(s, x)\, \mathrm{d}s$$

*is a complete Lyapunov function for $\Phi$. In particular, if $\mathcal{X}$ is separable, then there exists a complete Lyapunov function for $\Phi$.*

Proof: We note that the complete Lyapunov function $\ell$ that we constructed in the proof of Theorem 5.14 has a property of which we shall make use. Namely, in the definition (5.2) of $\ell$, we see that points in $\mathrm{ChRec}(\Phi^{\mathrm{d},1})$, being points where the functions $L_j$ take value 0, are points whose ternary (i.e., base 3) expansion contains only 1s and 2s. This means that $\ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$ is a subset of the classical middle-thirds Cantor set, which is closed and nowhere dense.

First note that $L$ is continuous by standard results concerning swapping limits and integrals [e.g., Rudin 1976, Theorem 7.16]. Since $\ell(x) \in [0, 1]$ for every $x \in \mathcal{X}$, we also have $L(x) \in [0, 1]$ for every $x \in \mathcal{X}$.

The following calculation will be useful in the remainder of the proof. Let $t \in \mathbb{R}$ (if $\Phi$ is a flow) or $t \in \mathbb{R}_{\geq 0}$ (if $\Phi$ is a semiflow) and suppose that $t \in [j, j+1)$ for some $j \in \mathbb{Z}$. Let $x \in \mathcal{X}$. Then we have

$$\begin{aligned} L \circ \Phi(t,x) &= \int_0^1 \ell \circ \Phi(s, \Phi(t-j+j, x))\,\mathrm{d}s \\ &= \int_0^1 \ell \circ \Phi(s+t-j, \Phi(j,x))\,\mathrm{d}s \\ &= \int_{t-j}^{1+t-j} \ell \circ \Phi(s, \Phi(j,x))\,\mathrm{d}s \\ &= \int_{t-j}^{1} \ell \circ \Phi(s, \Phi(j,x))\,\mathrm{d}s + \int_1^{1+t-j} \ell \circ \Phi(s, \Phi(j,x))\,\mathrm{d}s \\ &= \int_{t-j}^{1} \ell \circ \Phi(s, \Phi(j,x))\,\mathrm{d}s + \int_0^{t-j} \ell \circ \Phi(1+s, \Phi(j,x))\,\mathrm{d}s. \end{aligned} \tag{5.3}$$

Let $x \in \mathcal{X}$ and let $t \in [0,1)$. Then

$$\ell \circ \Phi(1+s, x) \leq \ell \circ \Phi(s,x), \qquad s \in \mathbb{R}_{\geq 0},$$

since $\ell$ is a complete Lyapunov function for $\Phi^{\mathrm{d},1}$. Then, using (5.3) for $j=0$, we have

$$\begin{aligned} L \circ \Phi(t,x) &= \int_t^1 \ell \circ \Phi(s,x)\,\mathrm{d}s + \int_0^t \ell \circ \Phi(1+s,x)\,\mathrm{d}s \\ &\leq \int_t^1 \ell \circ \Phi(s,x)\,\mathrm{d}s + \int_0^t \ell \circ \Phi(s,x)\,\mathrm{d}s \\ &= \int_0^1 \Phi(s,x)\,\mathrm{d}s = L(x). \end{aligned}$$

In particular, $L \circ \Phi(1,x) \leq L(x)$. Now, if $t \in \mathbb{R}_{>0}$ satisfies $t \in [j, j+1)$ for some $j \in \mathbb{Z}_{\geq 0}$, then

$$L \circ \Phi(t,x) = L \circ \Phi(t-j, \Phi(j,x)) \leq L \circ \Phi(j,x),$$

and so, by an elementary induction, $L \circ \Phi(t,x) \leq L(x)$.

Let $t \in \mathbb{R}$ (if $\Phi$ is a flow) or $t \in \mathbb{R}_{\geq 0}$ (if $\Phi$ is a semiflow) satisfy $t \in [j, j+1)$ for some $j \in \mathbb{Z}$, let $x \in \mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},1})$, this last equality by Corollary 3.16. By Proposition 3.12(i) and Corollary 3.16 we have

$$\Phi(s, \Phi(k,x)) \in \mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},1}), \\ k \in \mathbb{Z} \text{ (for flows) or } k \in \mathbb{Z}_{\geq 0} \text{ (for semiflows)}, \ s \in [0,1).$$

Therefore, by Theorem 5.14 we have

$$\ell \circ \Phi(1+s, \Phi(j,x)) = \ell \circ \Phi(s, \Phi(j,x)), \qquad s \in [0,1).$$

Thus we have, by (5.3),

$$\begin{aligned} L \circ \Phi(t,x) &= \int_{t-j}^1 \ell \circ \Phi(s, \Phi(j,x)))\,\mathrm{d}s + \int_0^{t-j} \ell \circ \Phi(1+s, \Phi(j,x))\,\mathrm{d}s \\ &= \int_0^1 \ell \circ \Phi(s, \Phi(j,x))\,\mathrm{d}s = \int_0^1 \ell \circ \Phi(s,x)\,\mathrm{d}s = L(x). \end{aligned}$$

Thus $L$ is constant along forward trajectories through points in $\mathrm{ChRec}(\Phi)$.

Let $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$. Since $\mathcal{X} \setminus \mathrm{ChRec}(\Phi)$ is open by Proposition 3.12(ii), there exists $\tau \in \mathbb{R}_{>0}$ such that

$$\Phi(t,x) \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi) = \mathcal{X} \setminus \mathrm{ChRec}(\Phi^{\mathrm{d},1}), \qquad t \in [t_1, t_1 + \tau].$$

Therefore,

$$\ell \circ \Phi(1+t,x) < \ell(\Phi(t,x)), \qquad t \in [0,\tau]$$

since $\ell$ is a complete Lyapunov function for $\Phi^{\mathrm{d},1}$. Therefore, for $t \in [0,\tau]$, we can use (5.3) to get

$$\begin{aligned} L \circ \Phi(t,x) &= \int_t^1 \ell \circ \Phi(s,x)\,\mathrm{d}s + \int_0^t \ell \circ \Phi(1+s,x)\,\mathrm{d}s \\ &< \int_t^1 \ell \circ \Phi(s,x)\,\mathrm{d}s + \int_0^t \ell \circ \Phi(s,x)\,\mathrm{d}s = L(x). \end{aligned}$$

If $t > \tau$, we have

$$L \circ \Phi(t,x) = L \circ \Phi(t-\tau, \Phi(\tau,x)) \le L \circ \Phi(\tau,x) < L(x)$$

since $\Phi(t-\tau,x) \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$.

Now let $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$. Since $\ell$ is a complete Lyapunov function for $\Phi^{\mathrm{d},1}$, $\ell^{-1}(\alpha)$ is a chain component for $\Phi^{\mathrm{d},1}$, and so also a chain component of $\Phi$ by Theorem 3.15. By Proposition 3.14(ii), $\ell^{-1}(\alpha)$ is invariant under $\Phi$. Thus $\Phi(t,x) \in \ell^{-1}(\alpha)$ for all $t \in \mathbb{R}$ (if $\Phi$ is a flow) or for all $t \in \mathbb{R}_{\ge 0}$ (if $\Phi$ is a semiflow) if $x \in \ell^{-1}(\alpha)$. Therefore, for $x \in \ell^{-1}(\alpha)$, we have

$$L(x) = \int_0^1 \ell \circ \Phi(t,x)\,\mathrm{d}t = \alpha,$$

and so

$$\ell^{-1}(\alpha) \subseteq L^{-1}(\alpha), \qquad \alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1})). \tag{5.4}$$

To show that this inclusion is equality, we claim that it is sufficient to show that $L^{-1}(\alpha) \subseteq \mathrm{ChRec}(\Phi)$. To prove this claim, we proceed by contradiction. Thus suppose that

1. there exists $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$ such that $\ell^{-1}(\alpha) \subset L^{-1}(\alpha)$ and
2. $L^{-1}(\alpha) \subseteq \mathrm{ChRec}(\Phi)$ for each $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$.

Since chain equivalence defines an equivalence relation on $\mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},1})$, and since the equivalence classes are determined by the unique value that $\ell$ takes on each equivalence class, the assumptions 1 and 2 imply that there exists $\alpha' \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$ such that $\alpha \neq \alpha'$ and such that $L^{-1}(\alpha) \cap \ell^{-1}(\alpha') \neq \varnothing$. Thus there exists $x \in \mathrm{ChRec}(\Phi^{\mathrm{d},1})$ such that $L(x) = \alpha$ and $\ell(x) = \alpha'$. However, since $\ell^{-1}(\alpha') \subseteq L^{-1}(\alpha')$, this implies that $L(x) = \alpha' \neq \alpha$. This contradiction shows that, if we can show that $L^{-1}(\alpha) \subseteq \mathrm{ChRec}(\Phi)$ for every $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$, it will follow that $\ell^{-1}(\alpha) = L^{-1}(\alpha)$ for every $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$.

To show that $L^{-1}(\alpha) \subseteq \mathrm{ChRec}(\Phi)$ for every $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1})$, let $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$. We claim that $L(x) \notin \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$. Let

$$\tau_x = \sup\{t \in \mathbb{R} \mid \ \Phi(t,x) \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)\},$$

noting that $\tau_x \in \mathbb{R}_{>0}$ since $\mathcal{X} \setminus \mathrm{ChRec}(\Phi)$ is open by Proposition 3.12(ii). If $\tau_x < \infty$, we claim that $\Phi(\tau_x, x) \in \mathrm{ChRec}(\Phi)$. Indeed, if $\Phi(\tau_x.x) \notin \mathrm{ChRec}(\Phi)$, then there must be $\tau' > \tau_x$ with $\Phi(t, x) \notin \mathrm{ChRec}(\Phi)$ for $t \in [\tau_x, \tau']$ by openness of $\mathcal{X} \setminus \mathrm{ChRec}(\Phi)$, contradicting the definition of $\tau_x$. Let $\alpha = \ell \circ \Phi(\tau_x, x) \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$. Thus, using (5.4),

$$\Phi(\tau_x, x) \in \ell^{-1}(\alpha) \subseteq L^{-1}(\alpha),$$

and so $L \circ \Phi(\tau_x, x) = \alpha$. By the Mean Value Theorem for integrals [Rudin 1976, Theorem 5.10], there exists $s_0 \in (0, 1)$ such that

$$L(x) = \int_0^1 \ell \circ \Phi(s, x)\, \mathrm{d}s = \ell \circ \Phi(s_0, x)(1 - 0) = \ell \circ \Phi(s_0, x). \tag{5.5}$$

If $\tau_x \geq 1$, (5.5) immediately gives $L(x) \notin \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$ since the preceding formula holds with $s_0 < 1 < \tau_x$ and using the definition of $\tau_x$. On the other hand, if $\tau_x \in (0, 1)$, we proceed as follows. Since $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$, we have

$$L(x) > L \circ \Phi(\tau_x, x)$$

using already proved properties of $L$. By (5.5) we thus have, for some $s_0 \in (0, 1)$,

$$L(x) = \ell \circ \Phi(s_0, x) > \alpha = \ell \circ \Phi(\tau_x, x).$$

We claim that $s_0 \in (0, \tau_x)$. To see this, first note that

$$L \circ \Phi(t, x) \leq L \circ \Phi(\tau_x, x) = \alpha, \qquad t \in [\tau_x, 1), \tag{5.6}$$

using the fact, already proved, that $L$ is nonincreasing along trajectories. Suppose now that there exists $t_0 \in [\tau_x, 1)$ such that $\ell \circ \Phi(t_0, x) > \alpha$. By Proposition 3.12(i) and the fact that $\Phi(\tau_x, x) \in \mathrm{ChRec}(\Phi)$, we have

$$\Phi(t_0, x) \in \mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},1}) \implies \alpha' \triangleq \ell \circ \Phi(t_0, x) \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1})).$$

Therefore, by (5.4), we have

$$\ell \circ \Phi(t_0, x) = \alpha' \implies \Phi(x, t_0) \in \ell^{-1}(\alpha') \subseteq L^{-1}(\alpha') \implies L \circ \Phi(t_0, x) = \alpha' > \alpha,$$

in contradiction with (5.6). Therefore, we have shown that, for any $x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi)$ and irregardless of the value of $\tau_x \in \mathbb{R}_{>0}$, there exists $s_0 \in (0, \tau_x)$ such that

$$L(x) = \ell \circ \Phi(s_0, x) \notin \mathrm{ChRec}(\Phi^{\mathrm{d},1}).$$

Now we have the following logical implications:

$$\begin{aligned} &\big(x \in \mathcal{X} \setminus \mathrm{ChRec}(\Phi) \implies L(x) \notin \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))\big) \\ \iff &\big(L(x) \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1})) \implies x \in \mathrm{ChRec}(\Phi)\big) \\ \iff &\big(L^{-1}(\alpha) \subseteq \mathrm{ChRec}(\Phi),\ \alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))\big) \\ \implies &\ell^{-1}(\alpha) = L^{-1}(\alpha),\ \alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1})), \end{aligned}$$

where the last implication was proved above. This shows that the partition of $\mathrm{ChRec}(\Phi) = \mathrm{ChRec}(\Phi^{\mathrm{d},1})$ by level sets $\ell^{-1}(\alpha)$, $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$, is the same as the partition of $\mathrm{ChRec}(\Phi)$ by the level sets $L^{-1}(\alpha)$, $\alpha \in \ell(\mathrm{ChRec}(\Phi^{\mathrm{d},1}))$. This shows that $L(C) = L(C')$ for chain components $C$ and $C'$ if and only if $C = C'$. ■

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