Mathematics coming from applied mathematics

Andrew D. Lewis

Department of Mathematics and Statistics, Queen's University

03/04/2008

What is applied mathematics?

Depends who you ask. For example, here are some extreme, i.e., stereotypical, responses.

- For someone who uses mathematics, but dislikes abstraction, applied mathematics is:
  - Useful mathematics
    Criticisms: Useful according to who?
    There are numerous instances of "useless" mathematics becoming useful, e.g., from the National Institute of Standards and Technology website:

    Unscheduled downtime of manufacturing equipment is one of the most important impediments to achieving cost-effective, timely production schedules.

    NIST recommends (but does not require) the following set of Elliptic Curves for Federal government use...

What is applied mathematics?

- Minimal mathematics, i.e., no “mathematics for the sake of mathematics”
  Criticisms: Sometimes “mathematics for the sake of mathematics” leads to something.
  The minimalist approach often ends up ignoring essential structure, leading to a continual unnecessary reinvention of ideas.

What is applied mathematics?

- For a mathematical snob, applied mathematics is:
  - Easy or shallow mathematics
    Criticisms: Easy or shallow mathematics is not applied mathematics. It is *non*-mathematics. It is to mathematics as:
    - accounting is to mathematics...
    - knowing what happened in 1066 is to being an historian...
    - having a brain is to being a psychologist...
  - Non-rigorous mathematics, where such things as
    - division by infinity,
    - freely swapping all mathematical operations,
    - treating all entities in an equation as if they are numbers

    are permissible.
    Criticism: This is just *bad* mathematics... nay... a bad lifestyle.

What is applied mathematics?

- The fact of the matter is that, as with mathematics, there are many sorts of applied mathematics.
- Perhaps it might be characterised
  - by its involving that which mathematicians do, in some form,
  - by its involving “interesting” mathematics in a nontrivial way, and
  - by its demanding knowledge, sometimes genuine expertise, in a field that is not mathematics.

What is the connection of mathematics to the non-mathematical world?

This is a difficult question to think about, but here are some organising points to mull over.

- Is there a connection between mathematics and the physical world? Physical theories with mathematical underpinnings are frequently overwritten by new theories.
- Is mathematics “out there,” or is it a human or sentient activity?
- Why does it seem that mathematics is often spectacularly successful at describing the physical world?[1]
  - Is it to do with mathematics? the physical world? human beings? a combination of some or all of these?

[1] E. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Comm. Pure Appl. Math., **13**(1), 1960.

What is the connection of mathematics to the non-mathematical world?

- If a race on a planet in a galaxy far far away existed,[2]
  - would they develop mathematics?
  - if so, what would their mathematics look like?

  That is, is mathematics a social construct?
- Does the non-mathematical world force mathematics on us, or do we force mathematics on it?
- *If you value the actual activity of mathematics or science, do not spend all of your time thinking about these questions.*

[2] R. W. Hamming, *Mathematics on a distant planet*, Amer. Math. Monthly, **105**(7), 1998.

Modes of applied mathematics

- What are some of the ways in which mathematical activities been valuable in non-mathematics?
- We shall consider some concrete instances of this.
- Note that “mathematical activity”$\neq$“use of mathematics.”

Modes of applied mathematics I
Non-mathematics turned into mathematics

- To some, mathematics is the activity of producing axioms and proving theorems which follow from those axioms. In this case...
  - "correct" axioms lead to interesting mathematics and
  - "incorrect" axioms lead to dull or stupid mathematics.
- The same activity can be conducted in applied mathematics, but to the above criterion for axioms we add...
  - axioms must capture the relevant non-mathematical phenomenon.
- When this is successful, it is a satisfying undertaking.
- Examples include:
  - classical mechanics;
  - continuum mechanics;
  - quantum mechanics;
  - relativistic mechanics;
  - control theory;
  - information theory;
  - branches of economics;
  - etc.

Modes of applied mathematics I
Non-mathematics turned into mathematics

- The axiomisation of physical theories was Hilbert's Sixth Problem.
- Much of the activity resembles the axiom $\rightsquigarrow$ theorem activity that some see as mathematics.
- But there are significant differences with "pure" mathematics.
  - The natural questions important for the non-mathematics may not always agree with the natural mathematical questions.
  - The generalisations and specialisations important to mathematics may be irrelevant to the non-mathematics.
- One should respect the non-mathematics.

Modes of applied mathematics I
Non-mathematics turned into mathematics

Let us consider a concrete example: Continuum mechanics.

- This is the study of continuous (as opposed to discrete) deformable (as opposed to rigid) media, e.g., elasticity and fluid mechanics.
- For the first ~ 150 years, the subject was disjointed, consisting of many essentially separate theories depending on the context, e.g., fluids, metallic materials, ceramic materials, polymeric materials, etc.
- In 1965 Truesdell and Noll published *The Non-Linear Field Theories of Mechanics*.
  - The formulation was axiomatic and very general.
  - All previous *ad hoc* work was subsumed and clarified.
  - The formulation made clear some of the real problems in continuum mechanics.
  - The work influenced everything in the area that followed.

Modes of applied mathematics I
Non-mathematics turned into mathematics

Other similar seminal works:

- Kolmogorov's 1933 book *Foundations of the Theory of Probability*;
- von Neumann and Morgenstern's 1944 book *Theory of Games and Economic Behavior*;
- Shannon's 1948 paper *A mathematical theory of communication*;
- von Neumann's 1955 book *Mathematical Foundations of Quantum Mechanics*;
- Kalman's 1963 paper *Mathematical description of linear dynamical systems*;
- Abraham and Marsden's 1967 book *Foundations of Mechanics*.

Modes of applied mathematics II
Going to town on nontrivial applied problems

- To some, mathematics is the activity of contributing to the solution of particular, often very difficult problems, e.g., the Riemann Hypothesis. In this case the problems should be chosen...
  - so that they give rise to independently interesting mathematics and
  - so that their solution has an impact in other areas of mathematics.
- The same activity can be conducted in applied mathematics, but the above criterion can be augmented or replaced by
  - the problem should be relevant to the application area.
- We look at a couple of instances of this.

Modes of applied mathematics II
Going to town on nontrivial applied problems
Fluid mechanics

- From the principles of continuum mechanics one derives the Navier–Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nu \Delta u - \operatorname{grad} p + f,$$
$$\operatorname{div} u = 0$$

- Fundamental questions are:
  1. Do the Navier–Stokes equations possess solutions?
  2. If so, are solutions uniquely defined for initial/boundary data?
  3. On what domain are solutions defined?
  4. What is the character of solutions?
- The above problem is one of the Clay Institute Millennium Problems.

- An important observation was made by V. I. Arnol'd in 1966.[3] Let us be slightly precise about the setup:
  - inviscid flow, i.e., $\nu = 0$;
  - no body forces, i.e., $f = 0$;
  - suppose that fluid occupies a compact region $R \subset \mathbb{R}^3$ with nonempty interior and smooth boundary.
- The inviscid assumption means energy is conserved.
- Let $\mathrm{Diff}(R)$ be the collection of diffeomorphisms of R; note that a motion of the fluid defines a curve in $\mathrm{Diff}(R)$.

[3] *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble), **16**(1).

Modes of applied mathematics II
Going to town on nontrivial applied problems
Fluid mechanics

- The problem possesses the "particle relabelling symmetry."
  - Let $t \mapsto \psi_t$ be a curve in $\mathrm{Diff}(R)$ be an admissible fluid motion.
  - Let $\phi \in \mathrm{Diff}(R)$.
  - Then $t \mapsto \psi_t \circ \phi$ is a curve in $\mathrm{Diff}(R)$ that describes an admissible fluid motion.
- Non sequitor... think about the motion of a rigid body in $\mathbb{R}^3$ with no external forces. Let $\mathrm{Isom}(\mathbb{R}^3)$ be the isometries of $\mathbb{R}^3$.
  - A motion corresponds to a curve in $\mathrm{Isom}(\mathbb{R}^3)$.
  - Let $t \mapsto \psi_t$ be a curve in $\mathrm{Isom}(\mathbb{R}^3)$ that describes an admissible body motion.
  - Let $\phi \in \mathrm{Isom}(\mathbb{R}^3)$.
  - Then $t \mapsto \phi \circ \psi_t$ is a curve in $\mathrm{Isom}(\mathbb{R}^3)$ that describes an admissible body motion.

Let us now make a series of observations.

Rigid body motion:	$\mathrm{Isom}(\mathbb{R}^3)$ is a group.
Fluid motion:	$\mathrm{Diff}(R)$ is a group.
Rigid body motion:	The equations of motion possess $\mathrm{Isom}(\mathbb{R}^3)$ as a symmetry group.
Fluid motion:	The equations of motion possess $\mathrm{Diff}(R)$ as a symmetry group.
Rigid body motion:	Fact: Equations of motion = conservation of momentum
Fluid motion:	Fact: Equations of motion = Kelvin's circulation theorem

Modes of applied mathematics II
Going to town on nontrivial applied problems
Fluid mechanics

- The point is that there is a large framework in which inviscid fluid mechanics and rigid body mechanics are “the same.”
- Making this approach to fluid mechanics rigorous takes great care.[4]
- But you get some results:
  - existence and uniqueness of solutions for short times;
  - regularity of solutions;
  - continuous dependence on initial conditions.
- The Ebin/Marsden paper is one of the early papers in the area of so-called geometric analysis.

[4] D. G. Ebin and J. E. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid*, Ann. of Math. (2), **92**, 1970.

Modes of applied mathematics II
Going to town on nontrivial applied problems
Celestial mechanics

- The question is: Is the solar system stable?
- The answer is obtained by looking at the differential equations for N (maybe $N = 10$) point masses whose interactions are governed by Newton's inverse square law.
- For $N = 2$ one can obtain closed-form solutions and show stability "by hand."
- For $N \geq 3$ many big shots took a whack at the problem: Newton, Laplace, Lagrange.

- The first era of activity was crippled by the very way in which differential equations were viewed.
  - A differential equation was something for which one obtains solutions.
  - For example, $\ddot{x} + x = 0$ has solutions $x(t) = A\cos(t) + B\sin(t)$.
- *A "closed form solution" is one of those things like a "happy marriage"—a vague sort of goal to be attained. One little affair 20 years in the past, or using one little hypergeometric function: it's sort of up to the people involved how badly they want to feel about it.*

Modes of applied mathematics II
Going to town on nontrivial applied problems
Celestial mechanics

- In 1887 King Oscar II of Sweden announced a prize for a solution of the problem of the stability of the solar system.
- Poincaré claimed the prize in 1889, although he did not actually solve the problem.
- In some sense Poincaré did something more significant.
  - He introduced the notion of *qualitative* dynamics where one is interested, not in closed-form solutions, but important characteristics of these solutions.
  - By letting go of the insistence on closed-form solutions, Poincaré opened the Pandora's Box where simple differential equations can have complicated solutions not usefully described in closed-form.
  - Poincaré invented the modern field of dynamical systems (and along the way, topology).

Modes of applied mathematics III
Producing mathematical models for the non-mathematical world

- There seems to be no analogue of this activity in "pure" mathematics.
- Is this activity mathematics?
- Let us point out some historically prominent examples of this kind of activity. Ask yourself, "Which of these are mathematics?"
  - Newton's mechanics.
  - Derivation of fluid equations by Navier and Stokes.
  - Maxwell's equations for electromagnetism.
  - Anything done by Einstein, including general relativity.
  - Development of quantum mechanics.
  - Predator-prey models (e.g., by Lotka and Volterra).
  - Formulation of pricing equilibria as fixed point problems.
  - String theory.

Conclusions

- There are none.