Controllability: A brief revisionist tutorial

Andrew D. Lewis

Department of Mathematics and Statistics, Queen's University

08/05/2008

Control-affine systems

Background for the controllability-impaired.

- A ***control-affine system*** is a triple $\Sigma = (\mathsf{M}, \mathscr{F} = \{f_0, f_1, \ldots, f_m\}, U)$ where
  1. M is the state manifold,
  2. f_0 is the drift vector field and $f_1, \ldots, f_m$ are the control vector fields, and
  3. $U \subset \mathbb{R}^m$ is the control set.
- The governing equations are

$$\gamma'(t) = f_0(\gamma(t)) + \sum_{a=1}^{m} u^a(t) f_a(\gamma(t)),$$

  for $\boldsymbol{u}\colon I \to U$ locally integrable and $\gamma\colon I \to \mathsf{M}$ locally absolutely continuous.
- The pair $(\gamma, \boldsymbol{u})$ is a ***controlled trajectory***.

Controllability definitions for control-affine systems

- Denote

$$\mathcal{R}_\Sigma(x_0, T) = \{\gamma(T) \mid \ (\gamma, \boldsymbol{u}) \text{ is a controlled trajectory on } [0, T] \text{ such that } \gamma(0) = x_0\}$$

and $\mathcal{R}_\Sigma(x_0, \leq T) = \cup_{t\in[0,T]}\mathcal{R}_\Sigma(x_0, t)$.

- Two flavours of controllability from $x_0 \in$ M:
  1. ***Accessibility:*** $\mathrm{int}(\mathcal{R}_\Sigma(x_0, \leq T)) \neq \varnothing$;
  2. ***Small-time local controllability (STLC):*** $x_0 \in \mathrm{int}(\mathcal{R}_\Sigma(x_0, \leq T))$.
- Let us agree to sometimes call "small-time local controllability" just "controllability."

Accessibility of control-affine systems

- Define a distribution F:

$$\mathsf{F}_x = \operatorname{span}(f_0(x), f_1(x), \dots, f_m(x)).$$

- Define a sequence of distributions:

$$\begin{aligned}
\operatorname{Lie}^{(0)}(\mathsf{F})_x &= \mathsf{F}_x, \\
&\vdots \\
\operatorname{Lie}^{(k)}(\mathsf{F})_x &= \operatorname{Lie}^{(k-1)}(\mathsf{F})_x \\
&\quad + \operatorname{span}([f_a, X](x) \mid \; a \in \{0, \dots, m\},\; X \in \Gamma^\infty(\operatorname{Lie}^{(k-1)}(\mathsf{F}))).
\end{aligned}$$

- This defines

$$\operatorname{Lie}^{(0)}(\mathsf{F}) \subset \operatorname{Lie}^{(1)}(\mathsf{F}) \subset \cdots \subset \operatorname{Lie}^{(\infty)}(\mathsf{F}).$$

Accessibility of control-affine systems

- Sussmann and Jurdjevic[1] and Kreener:[2]

Theorem

An analytic Σ is accessible from x_0 if and only if $\mathrm{Lie}^{(\infty)}(\mathsf{F})_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$.

- The condition $\mathrm{Lie}^{(\infty)}(\mathsf{F})_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$ is called the ***local accessibility rank condition*** (***LARC***).
- Notes:
  - Algorithmically, accessibility is decidable using an algorithm that is polynomial in the "size" of the system. (Sontag[3]).
  - The LARC is thus sharp and computable.

[1] J. Differential Equations, **12**, 95–116, 1972
[2] J. Soc. Indust. Appl. Math. Ser. A Control, **12**, 43–52, 1974
[3] SIAM J. Control Optim., **26**(5), 1106–1118, 1988

Controllability of control-affine systems

- Sontag[4] and Kawski[5] show that deciding controllability is NP-hard in the “size” of the system.
- Hmmm... maybe we should just give up? Maybe not. We are after insight, not algorithms.

Example (Kawski)

Let $n \in \mathbb{Z}_{>0}$, let $r = \lceil \frac{1}{3}(4n-1) \rceil$, and let

$$\begin{aligned}
\gamma_j &= \lfloor (4n-2j)/(2j+1) \rfloor, \\
\alpha_j &= (2j+1)(\gamma_j+2) - (4n+1), \\
\beta_j &= (2j+1)(\gamma_j+2) - (4n+1)
\end{aligned}$$

for $j \in \{1, \dots, 2n\}$.

[4] SIAM J. Control Optim., **26**(5), 1106–1118, 1988

[5] Systems Control Lett., **15**(1), 9–14, 1990

Controllability of control-affine systems

Example (cont'd)

Consider the system on $\mathsf{M} = \mathbb{R}^{2n+r+2}$ with governing equations

$$\begin{aligned} \dot{x}_0 &= u, & & \\ \dot{x}_j &= x_{j-1}, & & j \in \{1, \dots, r\}, \\ \dot{y}_j &= x_{\gamma_j}^{\alpha_j} x_{\gamma_j+1}^{\beta_j}, & & j \in \{1, \dots, n-1, n+1, \dots, 2n\}, \\ \dot{y}_n &= x_0^2 x_1^{2n-1} x_2, & & \\ \dot{z} &= P(y), & & \end{aligned}$$

where P is a homogeneous polynomial.
"Standard" techniques easily show that the $(\boldsymbol{x}, \boldsymbol{y})$ subsystem is STLC from $\mathbf{0}$.
It is then more or less clear that the system is STLC if and only if P changes sign in any neighbourhood of $\mathbf{0}$: the decidability of this is NP-hard.

Controllability of control-affine systems

- The point is, even the decidability of STLC for a system whose controllability is intuitively clear is NP-complete. So, soldier on...
- A useful fact of, e.g., Grasse,[6] Sussmann,[7] Warga:[8]

Theorem

For $\Sigma = (\mathsf{M}, \mathscr{F}, U)$ *define* $\mathrm{cl}(\mathrm{conv}(\Sigma)) = (\mathsf{M}, \mathscr{F}, \mathrm{cl}(\mathrm{conv}(U)))$. *If* Σ *satisfies LARC and* $x_0 \in \mathsf{M}$, *then the following statements are equivalent:*

(i) Σ *is STLC from* x_0*;*

(ii) $\mathrm{cl}(\mathrm{conv}(\Sigma))$ *is STLC from* x_0*;*

(iii) Σ *is STLC from* x_0 *using piecewise constant controls;*

(iv) $\mathrm{cl}(\mathrm{conv}(\Sigma))$ *is STLC from* x_0 *using piecewise constant controls.*

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[6] Math. Control Signals Systems, **5**(1), 41–66, 1992
[7] SIAM J. Control Optim., **25**(1), 158–194
[8] J. Math. Anal. Appl., **4**, 111-128, 1962

Controllability of control-affine systems

- It thus suffices to show that for $T \in \mathbb{R}_{>0}$ we have

$$x_0 \in \operatorname{int}\Big(\{\Phi^{f_{\boldsymbol{u}_1}}_{t_1} \circ \cdots \circ \Phi^{f_{\boldsymbol{u}_p}}_{t_p}(x_0) \mid p \in \mathbb{Z}_{>0},\ \boldsymbol{u}_1, \dots, \boldsymbol{u}_p \in U,\ t_1 + \cdots + t_p \leq T\}\Big),$$

where $f_{\boldsymbol{u}} \triangleq f_0 + \sum_{a=1}^{m} u^a f_a$.

Example (Kawski)

Consider the system on $\mathsf{M} = \mathbb{R}^4$ with governing equations

$$\begin{aligned}
\dot{x}_1 &= u, \\
\dot{x}_2 &= x_1, \\
\dot{x}_3 &= x_1^3, \\
\dot{x}_4 &= x_3^2 - x_2^7.
\end{aligned}$$

Controllability of control-affine systems

Example (cont'd)

Kawski[a] shows that this system is STLC from $\mathbf{0}$ but that the number of switches p required to reach all points in a neighbourhood of $\mathbf{0}$ is unbounded.

[a] Bull. Amer. Math. Soc. (N.S.), **18**(2), 149–152, 1988

- *Ech!* But we soldier on...
- Consider the case of two switches a little explicitly. If X_1 and X_2 are analytic vector fields then, for $t_1 + t_2$ small, we have

$$\Phi_{t_1}^{X_1} \circ \Phi_{t_2}^{X_2}(x_0) \text{``}=\text{''} \Phi_1^{\mathrm{BCH}(t_1 X_1, t_2 X_2)}(x_0),$$

where

$$\mathrm{BCH}(\xi_1, \xi_2) = \xi_1 + \xi_2 + \tfrac{1}{2}[\xi_1, \xi_2] + \tfrac{1}{12}([\xi_1, [\xi_1, \xi_2]] + [\xi_2, [\xi_2, \xi_1]]) + \cdots .$$

Controllability of control-affine systems

- The formal series $\mathrm{BCH}(\xi_1, \xi_2)$ is called the ***Baker–Campbell–Hausdorff formula***.
- It may be used inductively to arrive at

$$\Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_p}^{X_p}(x_0)\text{``} = \text{''}\Phi_1^{\mathrm{BCH}(t_1 X_1, \ldots, t_p X_p)}(x_0)$$

  for some ungodly horrible (but explicitly determinable) formal series $\mathrm{BCH}(\xi_1, \ldots, \xi_p)$.
- The point is that Lie brackets appear in a natural way for controllability using piecewise constant controls.
- $\implies$ Seek controllability conditions involving Lie bracket conditions on the vector fields $\{f_0, f_1, \ldots, f_m\}$.

Controllability of control-affine systems

- The literature on this approach is vast and varied, and is mostly characterised as follows:
  - Authors A_n of paper P_n give some conditions C_n, necessary or sufficient for STLC;
  - If the authors A_n of P_n possess good pedagogical instincts, intuition is given for the conditions C_n;
  - Either
    1. authors A_n of P_n give a counterexample showing that the intuitive description of their condition C_n is misleading or
    2. authors A_{n+1} of paper P_{n+1} begin their paper with such a counterexample, and proceed to give conditions C_{n+1} which apply to the counterexample;
  - Repeat...

Controllability of control-affine systems

- There is an inherent problem with this approach: conditions are given subject to a *specific choice of the vector fields* $f_0, f_1, \ldots, f_m$.

Example

On $\mathsf{M} = \mathbb{R}^m \times \mathbb{R}^{n-m}$ consider a system with governing equations

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \boldsymbol{u},\\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{Q}(\boldsymbol{x}_1),\end{aligned}$$

where $\boldsymbol{Q}$ is a $\mathbb{R}^{n-m}$-valued homogeneous polynomial of degree 2. Write

$$\boldsymbol{Q}(\boldsymbol{x}_1) = (Q_1(\boldsymbol{x}_1), \ldots, Q_{n-m}(\boldsymbol{x}_1))$$

for scalar-valued quadratic functions $Q_1, \ldots, Q_{n-m}$. Write

$$Q_j(\boldsymbol{x}_1) = \boldsymbol{x}_1^T \boldsymbol{B}_j \boldsymbol{x}_1$$

for a symmetric matrix $\boldsymbol{B}_j$.

Controllability of control-affine systems

Example (cont'd)

By the "generalised Hermes condition," (Sussmann[a]) the system is STLC from $(\mathbf{0}, \mathbf{0})$ if the diagonal entries in the matrices $\boldsymbol{B}_1, \dots, \boldsymbol{B}_{n-m}$ are zero.

Is this condition necessary? No.

Is this condition invariant under feedback transformations of the form $\boldsymbol{u} \mapsto \boldsymbol{P}\boldsymbol{u}$ for $\boldsymbol{P} \in GL(m; \mathbb{R})$? No.

However... the system is STLC from $(\mathbf{0}, \mathbf{0})$ if and only if there exists $\boldsymbol{P} \in GL(m; \mathbb{R})$ such that the diagonal entries of the matrices $\boldsymbol{P}^T \boldsymbol{B}_1 \boldsymbol{P}, \dots, \boldsymbol{P}^T \boldsymbol{B}_{n-m} \boldsymbol{P}$ are zero.

But what does this condition really mean?

[a] SIAM J. Control Optim., **25**(1), 158–194

Affine distributions

- Stating theorems stated in terms of specific $f_0, f_1, \ldots, f_m$ is rather like stating theorems in differential geometry that rely on a specific choice of coordinates.
- Note that in applications the vector fields $f_0, f_1, \ldots, f_m$ are often a part of the problem, and so it might seem absurd to adopt a point of view where the rôle of these vector fields is pushed aside.
- But we are interested in insight, not particular applications.
- Anyway, how do we *not* make this choice of drift and control vector fields?
- The geometric object that the vector fields $\mathscr{F} = \{f_0, f_1, \ldots, f_m\}$ *really* represent is the affine subbundle $\mathsf{A}_{\mathscr{F}}$ of TM defined by

$$\mathsf{A}_{\mathscr{F},x} = \left\{ f_0(x) + \sum_{a=1}^{m} u^a f_a(x) \;\middle|\; \boldsymbol{u} \in \mathbb{R}^m \right\}.$$

Affine distributions

- So why not simply replace the data $\{f_0, f_1, \dots, f_m\}$ with a subset $\mathsf{A} \subset \mathsf{TM}$ such that, in a neighbourhood of any point $x \in \mathsf{M}$, there exist vector fields $X_0, X_1, \dots, X_k$ such that

$$\mathsf{A}_x \triangleq \mathsf{A} \cap \mathsf{T}_x\mathsf{M} = \left\{ X_0(x) + \sum_{a=1}^{k} u^a X_a(x) \;\middle|\; \boldsymbol{u} \in \mathbb{R}^k \right\}.$$

- The object A is a ***locally finitely generated affine distribution*** on M.
- The (not uniquely defined) vector fields $X_0, X_1, \dots, X_k$ are ***local generators***.

Affine systems

- Let us see if we can develop a theory of systems and their controllability using our notion of an affine distribution as the starting point, rather than a set of vector fields.
- An ***affine system*** in an affine distribution A assigns to each point $x \in$ M a subset $\mathscr{A}(x) \subset$ A$_x$.
  - Require the nondegeneracy condition that $\mathrm{aff}(\mathscr{A}(x)) =$ A$_x$ and
  - require some fussy smoothness conditions that I will not state here.
- For a control-affine system $\Sigma = ($M$, \mathscr{F}, U)$ we have the affine system

$$\mathscr{A}_\Sigma(x) = \left\{ f_0(x) + \sum_{a=1}^m u^a f_a(x) \;\middle|\; \boldsymbol{u} \in U \right\}.$$

- A ***trajectory*** of $\mathscr{A}$ is a locally absolutely continuous curve $\gamma \colon I \to$ M such that $\gamma'(t) \in \mathscr{A}(\gamma(t))$ for a.e. $t \in I$.

Controllability definitions for affine systems

- Denote

$$\mathcal{R}_{\mathscr{A}}(x_0, T) = \{\gamma(T) \mid \gamma \text{ is a trajectory on } [0, T] \text{ such that } \gamma(0) = x_0\}$$

  and $\mathcal{R}_{\mathscr{A}}(x_0, \leq T) = \cup_{t \in [0,T]} \mathcal{R}_{\mathscr{A}}(x_0, t)$.
- Two flavours of controllability from $x_0 \in \mathsf{M}$:
  1. ***Accessibility:*** $\mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq T)) \neq \varnothing$;
  2. ***Small-time local controllability (STLC):*** $x_0 \in \mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq T))$.

Controllability definitions for affine systems

- If one is interested in *geometry*, then our controllability conditions should be on A, not on $\mathscr{A}$.
- Say $\mathscr{A}$ is ***proper*** at x_0 if $0_{x_0} \in \operatorname{int}_{\mathsf{A}_{x_0}}(\operatorname{conv}(\mathscr{A}(x_0)))$.
- Say A is ***properly small-time locally controllable*** (***PSTLC***) from x_0 if $\mathscr{A}$ is STLC from x_0 whenever $\mathscr{A}$ is proper at x_0.
- Say A is ***small-time locally uncontrollable*** (***STLUC***) from x_0 if $\mathscr{A}$ is not STLC from x_0 whenever $\mathscr{A}(x_0)$ is compact.
- Say A is ***conditionally small-time locally uncontrollable*** (***CSTLC***) from x_0 if it is neither PSTLC nor STLUC from x_0.
- This characterisation partitions the set of affine distributions on M.

Accessibility of affine systems

- Accessibility is characterised much as for control-affine systems. Let $\mathrm{Lie}^{(0)}(\mathsf{A})_x = \mathrm{span}(\mathsf{A}_x)$ and inductively define

$$\mathrm{Lie}^{(k)}(\mathsf{A})_x = \mathrm{Lie}^{(k-1)}(\mathsf{A})_x + \mathrm{span}([X,Y](x) \mid X \in \Gamma^\infty(\mathsf{A}),\ Y \in \Gamma^\infty(\mathrm{Lie}^{(k-1)}(\mathsf{A}))).$$

- Analytic data $\implies$ accessible from x_0 if and only if $\mathrm{Lie}^{(\infty)}(\mathsf{A})_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$.

Controllability of affine systems

- One would like to come up with conditions for controllability that are independent of generators.
- There are two possible approaches:
  1. provide conditions that simply do not involve generators;
  2. give conditions using generators, and then show that these conditions do not actually depend on the choice.
- Problem: No one knows what the first approach means and the second approach seems hopeless.
- Choose "undefined" over "hopeless."

Controllability of affine systems

Example (cont'd)

Consider again the system with governing equations

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{u},$$
$$\dot{\boldsymbol{x}}_2 = \boldsymbol{Q}(\boldsymbol{x}_1),$$

where $\boldsymbol{Q}$ is a $\mathbb{R}^{n-m}$-valued homogeneous polynomial of degree 2. One may show that the system is STLC from $(\boldsymbol{0}, \boldsymbol{0})$ if and only if $\boldsymbol{0}$ is in the interior of the convex hull of $\mathrm{image}(\boldsymbol{Q})$. This, then, is the geometric version of the "generalised Hermes condition" we saw applied to this example above.

The verifiability of the convex hull condition, incidentally, is NP-complete.

Controllability of affine systems

- The point of the example is that existing conditions for controllability are not generator independent, but should properly be stated as, "If there exists a set of generators such that C_n holds. . . "
- By adding the prefix, "If there exists a set of generators," one can take a computable condition (like the generalised Hermes condition) and turn it into one that is not computable (like the convex hull condition in the example).
- But we expect this since we have already asserted that controllability is computationally difficult.
- So we soldier on. . .

Controllability of affine systems

- Now, at last, we have a clearly defined vague direction to head:

Problem

Give conditions for controllability of an affine system $\mathscr{A}$ in an affine distribution A *that do not involve generators for* A.

- Fine. . . where do we start?
- Apart from generators, what structure does an affine distribution A possess? Here are some facts:
  1. A is a subset of TM;
  2. sections of A are vector fields;
  3. Baker–Campbell–Hausdorff suggests that iterated Lie brackets of A-valued vector fields are important;
  4. an iterated Lie bracket of degree k of vector fields involves derivatives of those vector fields up to order $k-1$.

Controllability of affine systems

- $\implies$ We are interested in spaces which parameterise vector fields and their derivatives.
- These spaces are called “jet bundles,” and these bundles have a very detailed algebraic structure.
- $\implies$ “Jet bundles and algebro-geometric conditions for controllability of affine systems”
- Cesar?