The local structure of affine systems

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05/06/2008

Algebra in geometry

- $\mathsf{D} \subset \mathsf{TM}$ a distribution, $x_0 \in \mathsf{M}$.
- $\pi_{\mathsf{D}_{x_0}} : \mathsf{T}_{x_0}\mathsf{M} \to \mathsf{T}_{x_0}\mathsf{M}/\mathsf{D}_{x_0}$ the canonical projection.
- Define $B_{\mathsf{D}}(x_0) \in \bigwedge^2(\mathsf{D}^*_{x_0}) \otimes \mathsf{T}_{x_0}\mathsf{M}/\mathsf{D}_{x_0}$ by

$$B_{\mathsf{D}}(x_0)(u, v) = \pi_{\mathsf{D}_{x_0}}([U, V](x_0)),$$

  where U and V are vector fields extending $u, v \in \mathsf{D}_{x_0}$.
- Frobenius: If D is analytic or smooth with constant rank, it is integrable if and only if $B_{\mathsf{D}}(x) = 0$ for all $x \in \mathsf{M}$.

Feedback invariance

- Many controllability theorems are not “feedback invariant.”

Example

On $\mathbb{R}^m \times \mathbb{R}^{n-m}$ consider a system with governing equations

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{u}, \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{Q}(\boldsymbol{x}_1), \end{aligned}$$

where $\boldsymbol{Q}$ is a $\mathbb{R}^{n-m}$-valued homogeneous polynomial of degree 2. Write

$$\boldsymbol{Q}(\boldsymbol{x}_1) = (\boldsymbol{B}_1(\boldsymbol{x}_1, \boldsymbol{x}_1), \ldots, \boldsymbol{B}_{n-m}(\boldsymbol{x}_1, \boldsymbol{x}_1))$$

for quadratic forms $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{n-m}$.

Feedback invariance

Example (cont'd)

- By the "generalised Hermes condition," (Sussmann[1]) the system is STLC from $(\mathbf{0}, \mathbf{0})$ if the diagonal entries in the matrices $\boldsymbol{B}_1, \dots, \boldsymbol{B}_{n-m}$ are zero.
- Is this condition necessary? No.
- Is this condition invariant under feedback transformations of the form $\boldsymbol{u} \mapsto \boldsymbol{P}\boldsymbol{u}$ for $\boldsymbol{P} \in GL(m; \mathbb{R})$? No.
- However. . . the system is STLC from $(\mathbf{0}, \mathbf{0})$ if and only if there exists $\boldsymbol{P} \in GL(m; \mathbb{R})$ such that the diagonal entries of the matrices $\boldsymbol{P}^T \boldsymbol{B}_1 \boldsymbol{P}, \dots, \boldsymbol{P}^T \boldsymbol{B}_{n-m} \boldsymbol{P}$ are zero.

[1] SIAM J. Control Optim., **25**(1), 158–194

Feedback invariance

- Control-affine systems

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)),$$
$$\dot{x}(t) = g_0(x(t)) + \sum_{b=1}^{n} v^b(t) g_b(x(t))$$

on a manifold M are ***feedback equivalent*** when they have the same trajectories (essentially).

- Have a relation like

$$g_0(x) = \sum_{a=0}^{m} \lambda^a(x) f_a(x), \quad g_b(x) = \sum_{a=1}^{m} \Lambda_b^a(x) f_a(x), \qquad b \in \{1, \dots, n\}.$$

Feedback invariance

- Since trajectories are of most interest, one's approach to control theory should be feedback invariant, i.e., identify systems that are feedback equivalent.
- Working with a specific $f_0, f_1, \ldots, f_m$ is like working with coordinates in differential geometry: Sometimes it is necessary, but it is best avoided if comprehension is what is sought.
- Two possible approaches:
  1. Choose $f_0, f_1, \ldots, f_m$, then show results are independent of this choice.
  2. Develop an approach that does not involve this choice.

Feedback invariance

Example (cont'd)

Consider again:

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{u},$$
$$\dot{\boldsymbol{x}}_2 = \boldsymbol{Q}(\boldsymbol{x}_1).$$

TFAE:

1. STLC from $(\mathbf{0}, \mathbf{0})$;
2. there exists $\boldsymbol{P} \in GL(m; \mathbb{R})$ such that the diagonal entries of the matrices $\boldsymbol{P}^T \boldsymbol{B}_1 \boldsymbol{P}, \dots, \boldsymbol{P}^T \boldsymbol{B}_{n-m} \boldsymbol{P}$ are zero;
3. $\mathbf{0} \in \operatorname{int}(\operatorname{conv}(\operatorname{image}(\boldsymbol{Q})))$.

Feedback invariance

- Condition 2 represents the “choose generators and show independence on these” approach.
- Condition 3 represents “generator independent” approach.
- Generalisation of the first approach seems hopeless.
- Generalisation of the second approach. . . how does one do it?
- Let us choose ignorance over hopelessness.

Affine systems

- Replace the data $\{f_0, f_1, \dots, f_m\}$ with a subset $\mathsf{A} \subset \mathsf{TM}$ such that, in a neighbourhood of any point $x_0 \in \mathsf{M}$, there exist vector fields $X_0, X_1, \dots, X_k$ such that

$$\mathsf{A}_x \triangleq \mathsf{A} \cap \mathsf{T}_x\mathsf{M} = \left\{ X_0(x) + \sum_{a=1}^{k} u^a X_a(x) \;\middle|\; \boldsymbol{u} \in \mathbb{R}^k \right\}.$$

- The object A is a ***locally finitely generated affine distribution*** on M.
- The (not uniquely defined) vector fields $X_0, X_1, \dots, X_k$ are ***local generators***.

Affine systems

- Typically controls for a control-affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t))$$

  are restricted to lie in a subset $U \subset \mathbb{R}^m$. We generalise this by...
- An ***affine system*** in an affine distribution A assigns to each point $x \in \mathsf{M}$ a subset $\mathscr{A}(x) \subset \mathsf{A}_x$.
  - Require the nondegeneracy condition that $\mathrm{aff}(\mathscr{A}(x)) = \mathsf{A}_x$ and
  - require some fussy smoothness conditions that I will not state here.
- A ***trajectory*** of $\mathscr{A}$ is a locally absolutely continuous curve $\gamma \colon I \to \mathsf{M}$ such that $\gamma'(t) \in \mathscr{A}(\gamma(t))$ for a.e. $t \in I$.
- An ***$\mathscr{A}$-vector field*** is a map $\xi \colon \mathsf{M} \to \mathsf{TM}$ such that $\xi(x) \in \mathscr{A}(x)$.

Problems for affine systems

- We aim to study the local properties of affine systems. To what end?
  1. Understand controllability.
  2. Understand stabilisability.
  3. Understand relationships between controllability and stabilisability.
- What geometric properties of affine distributions come into play to address these problems?
- We sort of think Lie brackets are involved, but how?
- Today: Understand this through controllability.

Controllability of affine systems

- Have the expected notions of controllability for an affine system $\mathscr{A}$.
- Denote

$$\mathcal{R}_{\mathscr{A}}(x_0, T) = \{\gamma(T) \mid \gamma \text{ is a trajectory on } [0, T] \text{ such that } \gamma(0) = x_0\}$$

  and $\mathcal{R}_{\mathscr{A}}(x_0, \leq T) = \cup_{t \in [0,T]} \mathcal{R}_{\mathscr{A}}(x_0, t)$.
- Two flavours of controllability from $x_0 \in$ M:
  1. ***Accessibility:*** $\mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq T)) \neq \varnothing$;
  2. ***Small-time local controllability (STLC):*** $x_0 \in \mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq T))$.

Controllability of affine systems

- Bang bang type theorems $\implies$ it suffices to consider piecewise constant controls.
- Thus need to consider trajectories of the form

$$\Phi_{t_1}^{\xi_1} \circ \cdots \circ \Phi_{t_p}^{\xi_p}(x_0)$$

  for $\mathscr{A}$-vector fields $\xi_1, \ldots, \xi_p$ and $t_1, \ldots, t_p \in \mathbb{R}_{\geq 0}$.
- Baker–Campbell–Hausdorff: For $k \in \mathbb{Z}_{>0}$ there exists $\mathrm{BCH}_k(\eta_1, \ldots, \eta_p)$ such that

$$\Phi_{t_1}^{\xi_1} \circ \cdots \circ \Phi_{t_p}^{\xi_p}(x_0) = \Phi_1^{\mathrm{BCH}_k(t_1\xi_1, \ldots, t_p\xi_p)}(x_0) + O((t_1 + \cdots + t_p)^{k+1}).$$

- Have

$$\mathrm{BCH}_k(\eta_1, \ldots, \eta_p) = \eta_1 + \cdots + \eta_p + \frac{1}{2} \sum_{\substack{j,k \in \{1,\ldots,p\} \\ j<k}} [\eta_j, \eta_k] + \cdots$$

Controllability of affine systems

- Standard technique in controllability: Find curves in the reachable set so the (possibly higher-order) tangent vector of this curve at x_0 is a "reachable direction."
- Construction of such tangent vectors:
  1. Abbreviate $\Phi^{\xi_1}_{t_1} \circ \cdots \circ \Phi^{\xi_p}_{t_p}(x_0) = \Phi^{\boldsymbol{\xi}}_{x_0}(t_1, \ldots, t_p)$.
  2. Consider the map $\mathbb{R}^p_{\geq 0} \ni \boldsymbol{t} \mapsto \Phi^{\boldsymbol{\xi}}_{x_0}(\boldsymbol{t}) \in \mathsf{M}$.
  3. Consider a curve $\mathbb{R}_{\geq 0} \ni s \mapsto \boldsymbol{\tau}(s) \in \mathbb{R}^p_{\geq 0}$ with $\boldsymbol{\tau}(0) = \mathbf{0}$.
  4. The composition $\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}$ is a curve in the reachable set.
  5. Suppose the first $k-1$ derivatives of $\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}$ vanish.
  6. Then $\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\right|_{s=0} \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s)$ (assume nonzero) is a kth-order tangent vector to the reachable set.
  7. Denote $\mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) = k$.

Controllability of affine systems

Example

Take $\boldsymbol{\xi} = (-Y, -X, Y, X)$ and $\boldsymbol{\tau}(s) = (s, s, s, s)$. Then $\mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) = 2$ and

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s) = [X, Y](x_0).$$

- This is nothing new.

Algebraic constructions using jets

- For analytic systems data is contained in the Taylor expansions at x_0, i.e., the infinite jets at x_0.
- Jet bundle notation and properties.
  1. For manifolds M and N, $\mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N})$ denotes the k-jets of maps for which $x \mapsto y$.
     - ★ $\mathsf{J}^k_{(x,0)}(\mathsf{M};\mathbb{R})$ is a $\mathbb{R}$-algebra.
     - ★ Elements of $\mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N})$ are homomorphisms of the $\mathbb{R}$-algebras $\mathsf{J}^k_{(y,0)}(\mathsf{N};\mathbb{R})$ and $\mathsf{J}^k_{(x,0)}(\mathsf{M};\mathbb{R})$.
  2. For a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, $\mathsf{J}^k_x\pi$ denotes the k-jets of sections of E at x.
     - ★ $\mathsf{J}^k_x\pi$ is a $\mathbb{R}$-vector space.

Algebraic constructions using jets

- If $\mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) = k$ we are interested in the k-jet of $s \mapsto \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s)$.
- We have $j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) = j^k\boldsymbol{\tau}(0) \circ j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0})$ (composition of algebra homomorphisms).
- $j^k\boldsymbol{\tau}(0) \in \mathsf{J}^k_{(0,\mathbf{0})}(\mathbb{R}; \mathbb{R}^p)$ is a rather canonical object.
- What about $j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0})$?

Algebraic constructions using jets

- Some notation:
  1. Denote $\mathsf{T}_x^{*k}\mathsf{M} = \mathsf{J}^k_{(x,0)}(\mathsf{M};\mathbb{R})$ and $(\mathbb{R}^p)^{*k} = \mathsf{J}^k_{(\mathbf{0},0)}(\mathbb{R}^p;\mathbb{R})$.
  2. $\pi^p_{\mathsf{TM}}\colon \mathsf{TM}^p \to \mathsf{M}$ denotes the p-fold Whitney sum.
  3. For a $\mathbb{R}$-vector space V, $\mathsf{S}^j(\mathsf{V})$ denotes the symmetric tensors of degree j.
  4. $\mathsf{S}^{\leq k}(\mathsf{V}) = \bigoplus_{j=1}^k \mathsf{S}^j(\mathsf{V})$.
  5. Define
$$\begin{aligned}\Delta_k\colon \mathsf{V} &\to \mathsf{S}^{\leq k}(\mathsf{V})\\ v &\mapsto v \oplus (v\otimes v) \oplus \cdots \oplus (v\otimes\cdots\otimes v).\end{aligned}$$
  6. $\mathsf{L}(\mathsf{U};\mathsf{V})$ denotes the linear maps between $\mathbb{R}$-vector spaces U and V.
  7. $\mathrm{Hom}(\mathsf{A};\mathsf{B})$ denotes the homomorphisms of $\mathbb{R}$-algebras A and B.

Algebraic constructions using jets

Theorem

For each $k, p \in \mathbb{Z}_{>0}$ there exists a unique map

$$\mathscr{T}_p^k(x_0) \in \mathsf{L}(\mathsf{S}^{\leq k}(\mathsf{J}_{x_0}^{k-1}\pi_{\mathsf{TM}}^p); \mathsf{L}(\mathsf{T}_{x_0}^{*k}\mathsf{M}; (\mathbb{R}^p)^{*k}))$$

such that

$$\mathscr{T}_p^k(x_0)(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0))) = j^k\Phi_{x_0}^{\boldsymbol{\xi}}(\mathbf{0})$$

for every family $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$ of C^∞-vector fields. Moreover, the diagram

$$\begin{tikzcd}
\Delta_1(\mathsf{J}_{x_0}^0\pi_{\mathsf{TM}}^p) \arrow[d, "\mathscr{T}_p^1(x_0)"'] & \Delta_2(\mathsf{J}_{x_0}^1\pi_{\mathsf{TM}}^p) \arrow[l] \arrow[d, "\mathscr{T}_p^2(x_0)"'] & \Delta_3(\mathsf{J}_{x_0}^2\pi_{\mathsf{TM}}^p) \arrow[l] \arrow[d, "\mathscr{T}_p^3(x_0)"'] & \cdots \arrow[l] \\
\mathrm{Hom}(\mathsf{T}_{x_0}^{*1}\mathsf{M}; (\mathbb{R}^p)^{*1}) & \mathrm{Hom}(\mathsf{T}_{x_0}^{*2}\mathsf{M}; (\mathbb{R}^p)^{*2}) \arrow[l] & \mathrm{Hom}(\mathsf{T}_{x_0}^{*3}\mathsf{M}; (\mathbb{R}^p)^{*3}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

commutes, where the horizontal arrows are the canonical projections.

Algebraic constructions using jets

- Important points:
  1. The diagram implies $\operatorname{proj}\lim_{k\to\infty}$ exists.
  2. The map $\mathscr{T}_p^k(x_0)$ can be computed (and the proof of its existence is made) using Baker–Campbell–Hausdorff.
  3. The map $\mathscr{T}_p^k(x_0)$ is system independent: It depends only on k, p, and $\dim(\mathsf{M})$.
- How is $\mathscr{T}_p^k(x_0)$ used for controllability?

Controllability of affine systems (again)

- Fix $k \geq 2$.
- *Neutralisation:* Find $\mathscr{A}$-vector fields $\xi_1, \dots, \xi_p$ and $\boldsymbol{\tau}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ such that
$$j^{k-1}\boldsymbol{\tau}(0) \circ \mathscr{T}_p^{k-1}(\Delta_{k-1}(j^{k-2}\boldsymbol{\xi}(x_0))) = 0.$$
- *Bad news:* This is an intractable nonlinear equation for $j^{k-1}\boldsymbol{\tau}(0)$ and $j^{k-2}\boldsymbol{\xi}(x_0)$.
- *Good news:* The equation is algebraic and it can be written down!
- *The point:* If $j^{k-1}\boldsymbol{\tau}(0)$ and $j^{k-2}\boldsymbol{\xi}(x_0)$ satisfy the neutralisability condition, then
$$j^{k}\boldsymbol{\tau}(0) \circ \mathscr{T}_p^{k}(\Delta_{k}(j^{k-1}\boldsymbol{\xi}(x_0)))$$
is a kth-order tangent vector to the reachable set.
- *Note:* The system only comes in through its jets.

Conclusions and future work

- *Conclusion:* The local structure of an analytic affine system about x_0 is captured by the canonical algebraic objects $\mathscr{T}_p^{\,k}(x_0)$ restricted to system data.
- *Future work:*
  1. *Must do examples* $\implies$ need symbolic software to do computations.
  2. See what kind of controllability theorems come from this framework.
  3. Think about stabilisability $\implies$ connection with Control Lyapunov Functions?
  4. $\vdots$