

Jet bundles and algebro-geometric characterisations for controllability of affine systems

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Feedback invariance

- We will talk about control-affine systems:

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^m u^a(t) f_a(x(t)).$$

- Since trajectories are of most interest and since feedback transformations preserve trajectories, one's approach to control theory should be feedback invariant, i.e., identify systems that are feedback equivalent.
- Working with a specific f_0, f_1, \dots, f_m is like working with coordinates in differential geometry: Sometimes it is necessary, but it is best avoided if comprehension is what is sought.
- Two possible approaches:
 - 1 Choose f_0, f_1, \dots, f_m , then show results are independent of this choice.
 - 2 Develop an approach that does not involve this choice.

Feedback invariance

- Many useful controllability theorems are not “feedback invariant.”

Example

On $\mathbb{R}^m \times \mathbb{R}^{n-m}$ consider a system with governing equations

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{u}, \\ \dot{\mathbf{x}}_2 &= \mathbf{Q}(\mathbf{x}_1),\end{aligned}$$

where \mathbf{Q} is a \mathbb{R}^{n-m} -valued homogeneous polynomial of degree 2. Write

$$\mathbf{Q}(\mathbf{x}_1) = (\mathbf{B}_1(\mathbf{x}_1, \mathbf{x}_1), \dots, \mathbf{B}_{n-m}(\mathbf{x}_1, \mathbf{x}_1))$$

for quadratic forms $\mathbf{B}_1, \dots, \mathbf{B}_{n-m}$.

Feedback invariance

Example (cont'd)

- By the “generalised Hermes condition,” (Sussmann¹) the system is STLC from $(\mathbf{0}, \mathbf{0})$ if the diagonal entries in the matrices $\mathbf{B}_1, \dots, \mathbf{B}_{n-m}$ are zero.
- Is this condition necessary? No.
- Is this condition invariant under feedback transformations? No.
- However... TFAE:
 - ① STLC from $(\mathbf{0}, \mathbf{0})$;
 - ② there exists a feedback transformation turning the system into one in which the generalised Hermes condition applies;
 - ③ $\mathbf{0} \in \text{int}(\text{conv}(\text{image}(\mathbf{Q})))$.

¹SIAM J. Control Optim., **25**(1), 158–194

Feedback invariance

- Condition 2 represents the “choose generators and show independence on these” approach.
- Condition 3 represents “generator independent” approach.
- Generalisation of the first approach seems hopeless (and sort of disgusting).
- Generalisation of the second approach. . . how does one do it?
- Let us choose ignorance over hopelessness (and sort of disgust).

Affine systems

- Replace the data $\{f_0, f_1, \dots, f_m\}$ with a subset $A \subset TM$ such that, in a neighbourhood of any point $x_0 \in M$, there exist vector fields X_0, X_1, \dots, X_k such that

$$A_x \triangleq A \cap T_x M = \left\{ X_0(x) + \sum_{a=1}^k u^a X_a(x) \mid \mathbf{u} \in \mathbb{R}^k \right\}.$$

- The object A is a **locally finitely generated affine distribution** on M .
- The (not uniquely defined) vector fields X_0, X_1, \dots, X_k are **local generators**.

Affine systems

- Typically controls for a control-affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^m u^a(t) f_a(x(t))$$

are restricted to lie in a subset $U \subset \mathbb{R}^m$. We generalise this by...

- An **affine system** in an affine distribution A assigns to each point $x \in M$ a subset $\mathcal{A}(x) \subset A_x$.
 - ▶ Require the nondegeneracy condition that $\text{aff}(\mathcal{A}(x)) = A_x$ and
 - ▶ require some fussy smoothness conditions that I will not state here.
- A **trajectory** of \mathcal{A} is a locally absolutely continuous curve $\gamma: I \rightarrow M$ such that $\gamma'(t) \in \mathcal{A}(\gamma(t))$ for a.e. $t \in I$.
- An **\mathcal{A} -vector field** is a map $\xi: M \rightarrow TM$ such that $\xi(x) \in \mathcal{A}(x)$.

Problems for affine systems

- We aim to study the local properties of affine systems. To what end?
 - ① Understand controllability.
 - ② Understand stabilisability.
 - ③ Understand relationships between controllability and stabilisability.
- What geometric properties of affine distributions come into play to address these problems?
- We sort of think Lie brackets are involved, but how?
- Here: Understand this through controllability.

Controllability of affine systems

- Have the expected notions of controllability for an affine system \mathcal{A} .
- Denote

$$\mathcal{R}_{\mathcal{A}}(x_0, T) = \{\gamma(T) \mid \gamma \text{ is a trajectory on } [0, T] \text{ such that } \gamma(0) = x_0\}$$

and $\mathcal{R}_{\mathcal{A}}(x_0, \leq T) = \cup_{t \in [0, T]} \mathcal{R}_{\mathcal{A}}(x_0, t)$.

- Two flavours of controllability from $x_0 \in M$:
 - 1 **Accessibility**: $\text{int}(\mathcal{R}_{\mathcal{A}}(x_0, \leq T)) \neq \emptyset$;
 - 2 **Small-time local controllability (STLC)**: $x_0 \in \text{int}(\mathcal{R}_{\mathcal{A}}(x_0, \leq T))$.

Controllability of affine systems

- Bang bang type theorems \implies it “suffices” to consider piecewise constant controls.
- Thus need to consider trajectories of the form

$$\Phi_{t_1}^{\xi_1} \circ \dots \circ \Phi_{t_p}^{\xi_p}(x_0)$$

for \mathcal{A} -vector fields ξ_1, \dots, ξ_p and $t_1, \dots, t_p \in \mathbb{R}_{\geq 0}$.

- Baker–Campbell–Hausdorff: For $k \in \mathbb{Z}_{>0}$ there exists $\text{BCH}_k(\eta_1, \dots, \eta_p)$ such that

$$\Phi_{t_1}^{\xi_1} \circ \dots \circ \Phi_{t_p}^{\xi_p}(x_0) = \Phi_1^{\text{BCH}_k(t_1\xi_1, \dots, t_p\xi_p)}(x_0) + O((t_1 + \dots + t_p)^{k+1}).$$

- Have

$$\text{BCH}_k(\eta_1, \dots, \eta_p) = \eta_1 + \dots + \eta_p + \frac{1}{2} \sum_{\substack{j,k \in \{1, \dots, p\} \\ j < k}} [\eta_j, \eta_k] + \dots$$

Controllability of affine systems

- Standard technique in controllability: Find curves in the reachable set so the (possibly higher-order) tangent vector of this curve at x_0 is a “reachable direction.”
- Construction of such tangent vectors:
 - 1 Abbreviate $\Phi_{t_1}^{\xi_1} \circ \dots \circ \Phi_{t_p}^{\xi_p}(x_0) = \Phi_{x_0}^{\xi}(t_1, \dots, t_p)$.
 - 2 Consider the map $\mathbb{R}_{\geq 0}^p \ni \mathbf{t} \mapsto \Phi_{x_0}^{\xi}(\mathbf{t}) \in M$.
 - 3 Consider a curve $\mathbb{R}_{\geq 0} \ni s \mapsto \tau(s) \in \mathbb{R}_{\geq 0}^p$ with $\tau(0) = \mathbf{0}$.
 - 4 The composition $\Phi_{x_0}^{\xi} \circ \tau$ is a curve in the reachable set.
 - 5 Suppose the first $k - 1$ derivatives of $\Phi_{x_0}^{\xi} \circ \tau$ vanish.
 - 6 Then $\left. \frac{d^k}{ds^k} \right|_{s=0} \Phi_{x_0}^{\xi} \circ \tau(s)$ (assume nonzero) is a k th-order tangent vector to the reachable set.
 - 7 Denote $\text{ord}_{x_0}(\xi, \tau) = k$.

Controllability of affine systems

Example

Take $\xi = (-Y, -X, Y, X)$ and $\tau(s) = (s, s, s, s)$. Then $\text{ord}_{x_0}(\xi, \tau) = 2$ and

$$\left. \frac{d^2}{ds^2} \right|_{s=0} \Phi_{x_0}^{\xi} \circ \tau(s) = [X, Y](x_0).$$

- This is nothing new.

Algebraic constructions using jets

- For analytic systems data is contained in the Taylor expansions at x_0 , i.e., the infinite jets at x_0 .
- Jet bundle notation and properties.
 - 1 For manifolds M and N , $J_{(x,y)}^k(M; N)$ denotes the k -jets of maps for which $x \mapsto y$.
 - ★ $J_{(x,0)}^k(M; \mathbb{R})$ is a \mathbb{R} -algebra.
 - ★ Elements of $J_{(x,y)}^k(M; N)$ are homomorphisms of the \mathbb{R} -algebras $J_{(y,0)}^k(N; \mathbb{R})$ and $J_{(x,0)}^k(M; \mathbb{R})$.
 - 2 For a vector bundle $\pi: E \rightarrow M$, $J_x^k \pi$ denotes the k -jets of sections of E at x .
 - ★ $J_x^k \pi$ is a \mathbb{R} -vector space.

Algebraic constructions using jets

- If $\text{ord}_{x_0}(\xi, \tau) = k$ we are interested in the k -jet of $s \mapsto \Phi_{x_0}^\xi \circ \tau(s)$.
- We have $j^k(\Phi_{x_0}^\xi \circ \tau)(0) = j^k\tau(0) \circ j^k\Phi_{x_0}^\xi(\mathbf{0})$ (composition of algebra homomorphisms).
- $j^k\tau(0) \in \mathbf{J}_{(0, \mathbf{0})}^k(\mathbb{R}; \mathbb{R}^p)$ is a rather canonical object.
- What about $j^k\Phi_{x_0}^\xi(\mathbf{0})$?

Algebraic constructions using jets

- Some notation:

① Denote $T_x^{*k}M = J_{(x,0)}^k(M; \mathbb{R})$ and $(\mathbb{R}^p)^{*k} = J_{(0,0)}^k(\mathbb{R}^p; \mathbb{R})$.

② $\pi_{TM}^p: TM^p \rightarrow M$ denotes the p -fold Whitney sum.

③ For a \mathbb{R} -vector space V , $S^j(V)$ denotes the symmetric tensors of degree j .

④ $S^{\leq k}(V) = \bigoplus_{j=1}^k S^j(V)$.

⑤ Define

$$\Delta_k: V \rightarrow S^{\leq k}(V)$$

$$v \mapsto v \oplus (v \otimes v) \oplus \cdots \oplus (v \otimes \cdots \otimes v).$$

⑥ $L(U; V)$ denotes the linear maps between \mathbb{R} -vector spaces U and V .

⑦ $\text{Hom}(A; B)$ denotes the homomorphisms of \mathbb{R} -algebras A and B .

Algebraic constructions using jets

Theorem

For each $k, p \in \mathbb{Z}_{>0}$ there exists a unique map

$$\mathcal{T}_p^k(x_0) \in \mathbf{L}(\mathbf{S}^{\leq k}(\mathbf{J}_{x_0}^{k-1} \pi_{\mathbf{T}\mathbf{M}}^p); \mathbf{L}(\mathbf{T}_{x_0}^{*k} \mathbf{M}; (\mathbb{R}^p)^{*k}))$$

such that

$$\mathcal{T}_p^k(x_0)(\Delta_k(j^{k-1} \xi(x_0))) = j^k \Phi_{x_0}^{\xi}(\mathbf{0})$$

for every family $\xi = (\xi_1, \dots, \xi_p)$ of C^∞ -vector fields. Moreover, the diagram

$$\begin{array}{ccccccc}
 \Delta_1(\mathbf{J}_{x_0}^0 \pi_{\mathbf{T}\mathbf{M}}^p) & \longleftarrow & \Delta_2(\mathbf{J}_{x_0}^1 \pi_{\mathbf{T}\mathbf{M}}^p) & \longleftarrow & \Delta_3(\mathbf{J}_{x_0}^2 \pi_{\mathbf{T}\mathbf{M}}^p) & \longleftarrow & \dots \\
 \mathcal{T}_p^1(x_0) \downarrow & & \mathcal{T}_p^2(x_0) \downarrow & & \mathcal{T}_p^3(x_0) \downarrow & & \\
 \mathbf{Hom}(\mathbf{T}_{x_0}^{*1} \mathbf{M}; (\mathbb{R}^p)^{*1}) & \longleftarrow & \mathbf{Hom}(\mathbf{T}_{x_0}^{*2} \mathbf{M}; (\mathbb{R}^p)^{*2}) & \longleftarrow & \mathbf{Hom}(\mathbf{T}_{x_0}^{*3} \mathbf{M}; (\mathbb{R}^p)^{*3}) & \longleftarrow & \dots
 \end{array}$$

commutes, where the horizontal arrows are the canonical projections.

Algebraic constructions using jets

- Important points:
 - 1 The diagram implies $\text{proj} \lim_{k \rightarrow \infty}$ exists.
 - 2 The map $\mathcal{T}_p^k(x_0)$ can be computed (and the proof of its existence is made) using Baker–Campbell–Hausdorff.
 - 3 The map $\mathcal{T}_p^k(x_0)$ is system independent: It depends only on k , p , and $\dim(M)$.
- How is $\mathcal{T}_p^k(x_0)$ used for controllability?

Controllability of affine systems (again)

- Fix $k \geq 2$.
- *Neutralisation*: Find \mathcal{A} -vector fields ξ_1, \dots, ξ_p and $\tau: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}^p$ such that

$$j^{k-1}\tau(0) \circ \mathcal{F}_p^{k-1}(\Delta_{k-1}(j^{k-2}\xi(x_0))) = 0.$$

- *Bad news*: This is an intractable nonlinear equation for $j^{k-1}\tau(0)$ and $j^{k-2}\xi(x_0)$.
- *Good news*: The equation is algebraic and it can be written down!
- *The point*: If $j^{k-1}\tau(0)$ and $j^{k-2}\xi(x_0)$ satisfy the neutralisability condition, then

$$j^k\tau(0) \circ \mathcal{F}_p^k(\Delta_k(j^{k-1}\xi(x_0)))$$

is a k th-order tangent vector to the reachable set.

- *Note*: The system only comes in through its jets.

Controllability of affine systems (again)

- Denote by $\mathcal{V}_{\mathcal{A}}^k(x_0)$ the k th-order tangent vectors as above.
- Properties of $\mathcal{V}_{\mathcal{A}}^k(x_0)$:
 - 1 $\mathcal{V}_{\mathcal{A}}^k(x_0)$ is a convex cone;
 - 2 if $k_1, \dots, k_m \in \mathbb{Z}_{>0}$ then there exists $k \in \mathbb{Z}_{>0}$ such that $\mathcal{V}_{\mathcal{A}}^{k_r}(x_0) \subset \mathcal{V}_{\mathcal{A}}^k(x_0)$, $r \in \{1, \dots, m\}$.
- Let $\mathcal{L}_{\mathcal{A}}(x_0)$ be the \mathcal{A} -vector fields vanishing at x_0 .
- Note that $\mathcal{L}_{\mathcal{A}}(x_0)$ can be thought of as a subset of $\text{End}(T_{x_0}M)$.
- For a vector space V , $\mathcal{L} \subset \text{End}(V)$, and for $S \subset V$, let $\langle \mathcal{L}, S \rangle$ be the smallest subspace containing S which is invariant under all linear maps in \mathcal{L} .

Theorem

If S is a subspace of $\mathcal{V}_{\mathcal{A}}^k(x_0)$ and if $\langle \mathcal{L}_{\mathcal{A}}(x_0), S \rangle = T_{x_0}M$ then \mathcal{A} is STLC from x_0 .

Conclusions and future work

- *Conclusion:*

- 1 We have a means of producing feedback invariant controllability theorems.
- 2 The local structure of an analytic affine system about x_0 is captured by the canonical algebraic objects $\mathcal{T}_p^k(x_0)$ restricted to system data.

- *Future/other work:*

- 1 *Must do examples* \implies symbolic software to do computations.
- 2 See what kind of controllability theorems come from this framework.
- 3 Think about stabilisability.
- 4 \vdots