Differential geometry, control theory, and mechanics

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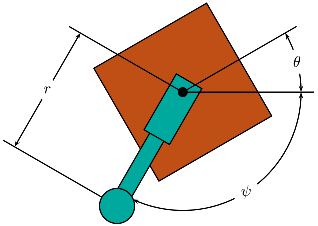
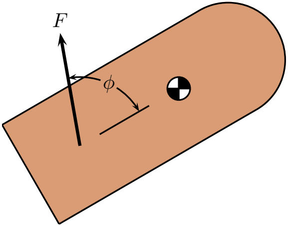
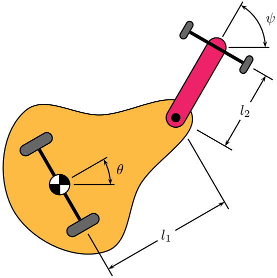
The objective of the talk

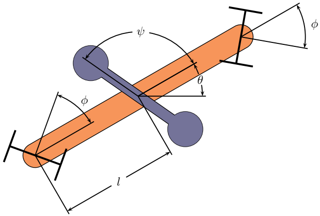
- To illustrate where some fairly sophisticated mathematics has been used to solve (hopefully somewhat interesting) problems that may be difficult, or impossible, to solve otherwise.
- There will be *no* details in the talk. However, details exist.
- *Collaborators:* Francesco Bullo, Bahman Gharesifard, Kevin Lynch, Richard Murray, David Tyner.
- *Relies on work by:* Suguru Arimoto, Guido Blankenstein, Anthony Bloch, Dong Eui Chang, Hubert Goldschmidt, Fabio Gómez-Estern, Velimir Jurdjevic, Naomi Leonard, Jerrold Marsden, Romeo Ortega, Mark Spong, Héctor Sussmann, Morikazu Takegaki, Arjan van der Schaft.

Some toy problems to keep in mind









Snakeboard gait: x Snakeboard gait: y Snakeboard gait: θ

Mechanical systems: mathematical modelling

- *Question:* What is the mathematical structure of the equations governing the motion of a mechanical system?
- We know that we *derive* these equations using Newton/Euler or Euler/Lagrange, but do the resulting equations have a useful unifying structure?
- We will use the Euler–Lagrange equations.
- We begin with the kinetic energy Lagrangian.
- Expressed in ("generalised") coordinates $(q^1, \dots, q^n)$ this Lagrangian is

$$L = \sum_{i,j=1}^{n} \frac{1}{2} \mathbb{G}_{ij}(q) \dot{q}^i \dot{q}^j.$$

- Here $\mathbb{G}_{ij}(q)$, $i, j = 1, \dots, n$, are the components of a symmetric $n \times n$ matrix which represents the inertial properties of the system.

Mechanical systems: mathematical modelling

- Mathematically, $\mathbb{G}_{ij}$, $i, j = 1, \dots, n$, are the components of a ***Riemannian metric*** on the configuration space of the system. Some call this the ***mass matrix***, ***inertia tensor***, etc. Let us call this the ***kinetic energy metric***.
- For a system with kinetic energy determined by the kinetic energy metric $\mathbb{G}$ and acted upon by no external forces, the following statements are equivalent for a curve $q(t)$ in configuration space:
  1. $q(t)$ satisfies the force and moment balance equations of Newton/Euler;
  2. $q(t)$ satisfies the Euler–Lagrange equations,

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \dots, n,$$

  where L is the kinetic energy Lagrangian.

Mechanical systems: mathematical modelling

- Let us do the computation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right)-\frac{\partial L}{\partial q^i} &= \sum_{j=1}^{n}\mathbb{G}_{ij}\left[\ddot{q}^j+\sum_{k=1}^{n}\mathbb{G}^{jk}\sum_{l,m=1}^{n}\left(\frac{\partial \mathbb{G}_{kl}}{\partial q^m}-\frac{1}{2}\frac{\partial \mathbb{G}_{lm}}{\partial q^k}\right)\dot{q}^l\dot{q}^m\right]\\ &= \sum_{j=1}^{n}\mathbb{G}_{ij}\left[\ddot{q}^j+\sum_{l,m=1}^{n}\overset{\mathbb{G}}{\Gamma}{}^j_{lm}\dot{q}^l\dot{q}^m\right],\end{aligned}$$

  where

$$\overset{\mathbb{G}}{\Gamma}{}^j_{lm}=\frac{1}{2}\mathbb{G}^{jk}\left(\frac{\partial \mathbb{G}_{kl}}{\partial q^m}+\frac{\partial \mathbb{G}_{km}}{\partial q^l}-\frac{\partial \mathbb{G}_{lm}}{\partial q^k}\right),$$

  and where $\mathbb{G}^{jk}$ are the components of the inverse of the matrix with components $\mathbb{G}_{ij}$.
- What's the stuff in red?

Mechanical systems: mathematical modelling

- *Fact:* Associated to the kinetic energy metric $\mathbb{G}$ is a unique "affine connection," called the ***Levi-Civita connection***, satisfying certain properties.
- I will not say just what an affine connection is. However, in coordinates an affine connection is uniquely determined by n^3 coefficients, typically denoted by Γ^k_{lm}, $i, j, k = 1, \dots, n$, called the ***Christoffel symbols***. For the Levi-Civita connection, these Christoffel symbols are the $\overset{\mathbb{G}}{\Gamma}{}^j_{lm}$'s appearing on the previous slide.
- The differential equations

$$\ddot{q}^j + \sum_{l,m=1}^{n} \overset{\mathbb{G}}{\Gamma}{}^j_{lm} \dot{q}^l \dot{q}^m = 0, \qquad j = 1, \dots, n,$$

are the ***geodesic equations*** for the Levi-Civita connection.

Mechanical systems: mathematical modelling

- Now let's add forces. There is a rule for converting forces and moments in the world of Newton/Euler to a single quantity which is determined in coordinates by components $F_1, \dots, F_n$. These appear in the Euler–Lagrange equations according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = F_i, \qquad i = 1, \dots, n.$$

- Correspondingly, the geodesic equations are modified to be

$$\ddot{q}^j + \sum_{l,m=1}^{n} \overset{\mathbb{G}}{\Gamma}{}^j_{lm} \dot{q}^l \dot{q}^m = \sum_{k=1}^{n} \mathbb{G}^{jk} F_k, \qquad j = 1, \dots, n.$$

- These equations can be read: $\text{acceleration} = \text{mass}^{-1}\text{force}$.

Mechanical systems: mathematical modelling

- *(Almost) inviolable rule:* Thou shalt not separate

$$\ddot{q}^j + \sum_{l,m=1}^{n} \overset{\mathbb{G}}{\Gamma}{}^j_{lm} \dot{q}^l \dot{q}^m$$

  into its summands. It is *one thing*, and we denote it by $\overset{\mathbb{G}}{\nabla}_{\dot{q}} \dot{q}$.
- With this notation, we can slickly write the governing equations for any mechanical system as

$$\boxed{\overset{\mathbb{G}}{\nabla}_{\dot{q}} \dot{q} = \mathbb{G}^{-1} F.}$$

  Again, this is: $\text{acceleration} = \text{mass}^{-1} \text{force}$.

What have we done?

- We have a compact (and well-known) representation of the equations governing the motion of a mechanical system, and a prominent rôle is played by the "Levi-Civita connection associated with the kinetic energy metric."
- *So what?* We already know how to write equations of motion.
- *Question:* Can we do anything interesting with the structure in our representation of the equations of motion?
- *Answer:* I think so, in control theory.

Control theory for mechanical systems

- In control theory we have control over some of the external forces.
- Thus we write the external force F as

$$F = F_{\text{ext}} + \sum_{a=1}^{m} u^a F^a,$$

  where F_{ext} represents uncontrolled forces and the total control force is $\sum_{a=1}^{m} u^a F^a$, i.e., the control force is a linear combination of forces $F^1, \dots, F^m$.
- *Assumption:* $F^1, \dots, F^m$ depend only on configuration q, and not on time or velocity $\dot{q}$.

Control theory for mechanical systems

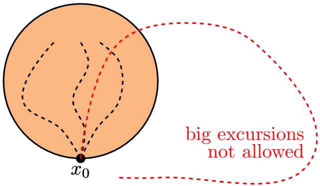
- The governing equations we consider are then

$$\boxed{\overset{\mathbb{G}}{\nabla}_{\dot{q}}\dot{q} = Y_{\mathrm{ext}} + \sum_{a=1}^{m} u^a Y_a,}$$

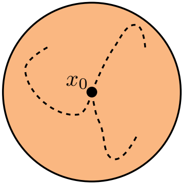
  where $Y_{\mathrm{ext}} = \mathbb{G}^{-1} F_{\mathrm{ext}}$ and $Y_a = \mathbb{G}^{-1} F^a$, $a = 1, \dots, m$.
- *Questions:*
  1. *Controllability:* Can a state x_2 be reached from a state x_1 by a suitable control u?
  2. *Stabilisability:* Can a state x_0 be made a stable equilibrium point for the system after a suitable control u has been prescribed?
  3. *Motion planning:* *Design* a control steering x_1 to x_2.
  4. *Stabilisation:* *Design* a control u that renders x_0 a stable equilibrium point.

(Local) controllability of mechanical systems

$$\overset{\mathbb{G}}{\nabla}_{\dot{q}}\dot{q} = Y_{\mathrm{ext}} + \sum_{a=1}^{m} u^a Y_a$$



accessibility

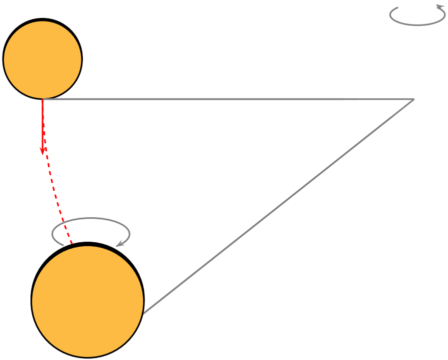


controllability

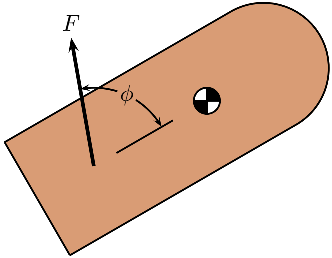
- Accessibility (does the set of points reachable from x_0 have a nonempty interior?) is easily decidable.
- Controllability (is x_0 in the interior of its own reachable set?) is very difficult to decide.

(Local) controllability of mechanical systems

- Controllability is only an interesting problem for underactuated systems; this excludes the “typical” robot. An example illustrates how controllability works.

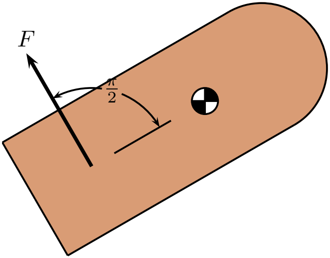


(Local) controllability of mechanical systems



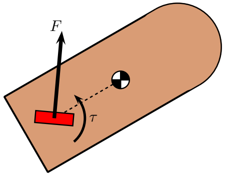
- Hovercraft system:
  1. *Question:* Is the system accessible?
  2. *Answer:* Yes (easy).
  3. *Question:* Is the system controllable?
  4. *Answer:* Yes (a little harder).

(Local) controllability of mechanical systems



- Now suppose that the fan cannot rotate.
  1. *Question:* Is the system accessible?
  2. *Answer:* Yes (easy).
  3. *Question:* Is the system controllable?
  4. *Answer:* No, at least not locally (nontrivial).

(Local) controllability of mechanical systems

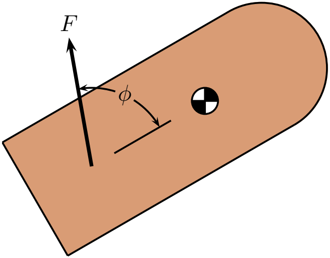


- Change the model by adding inertia to the fan.
  1. *Question:* Is the system accessible?
  2. *Answer:* Yes (easy).
  3. *Question:* Is the system controllable?
  4. *Answer:* No, at least not locally (getting really difficult now).

The punchline

- By slight alterations of the problem, a somewhat simple problem can be made very hard. To determine the answers to some of the controllability questions, difficult general theorems had to be proved.
- The proofs of the general theorems alluded to above in a specific context rely in an essential way on the Levi-Civita affine connection for the problem.
- *So what?* Can the affine connection actually be used to *solve* a problem?
- Let's look at the motion planning problem.

Motion planning



- Imagine trying to steer the hovercraft from one configuration at rest to another.
- We know this is possible (we answered the controllability question in the affirmative). But *how* can we do this?

Motion planning

- For the general system

$$\overset{\mathbb{G}}{\nabla}_{\dot{q}}\dot{q} = \sum_{a=1}^{m} u^a Y_a$$

  (i.e., with no uncontrolled external forces) one can pose a natural question: What are those vector fields whose integral curves we can follow with an arbitrary parameterisation?
- This question has a very elegant answer expressed by using the affine connection.
- The answer rests on some deep connections with controllability as described above.
- In examples, the answer to this question can sometimes lead directly to strikingly simple motion control algorithms.

Motion planning (movies)

- For the planar body:
  Planar body motion 1
  Planar body motion 2
  Planar body motion plan

Another flavour of motion planner
Yet another flavour of motion planner

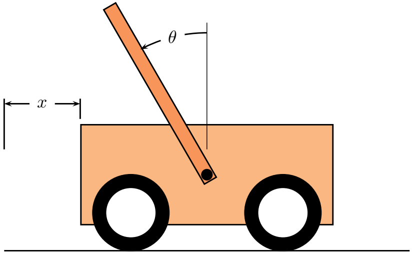
- For the snakeboard:
  Snakeboard motion plan 1
  Snakeboard motion plan 2

Stabilisation using energy shaping

- We are now thinking about mechanical systems for which the external force is solely provided by means of a potential function.
- We are interested here in the stabilisation problem. For systems with potential forces, equilibria are points where the derivative of the potential function is zero. An equilibrium is stable if it is a minimum of the potential function and unstable if it is a maximum of the potential function:

Stabilisation using energy shaping

- *Problem:* Using control, can we take a system with an unstable equilibrium and make it stable by altering the potential function to have a minimum at the desired point?
- For example, one can imagine the classical problem of stabilising the cart/pendulum system with the pendulum up:



The input is a horizontal force applied to the cart.

Stabilisation using energy shaping

- *Problem restatement:* Can we determine the set of potential functions that are achievable by using controls?
- If we only use control to alter the potential energy, it is possible to completely characterise the set of achievable potential functions. The set is often too small to be useful, e.g., for the pendulum/cart system, no stable potential is achievable in this way.
- *Question:* What if we allow not only the potential function to change, but also the kinetic energy metric?
- *Answer:* The set of achievable potential functions is then larger, e.g., for the pendulum/cart system there is now a stable potential achieved in this way.
- *Caveat:* To solve this problem requires solving a set of (generally overdetermined) nonlinear partial differential equations. . . gulp.

Stabilisation using energy shaping

- Nonetheless, maybe we can answer the question of when a given system is stabilisable using this “energy shaping” strategy.
- Studying the partial differential equations is complicated. Here is a simple paradigm for understanding what is going on.
  Problem: In Euclidean 3-space, given a vector field X, find a function f so that $\operatorname{grad} f = X$.
  Answer (from vector calculus): There is a solution if and only if $\operatorname{curl} X = 0$.
- The condition $\operatorname{curl} X = 0$ is called a ***compatibility condition***; it places the appropriate restrictions on the problem data to ensure that a solution exists.
- We have found the compatibility conditions for the energy shaping partial differential equations.

Stabilisation using energy shaping

- This is really not trivial: it involves lots of Riemannian geometry and enough homological algebra to, for one thing, make sense of the following exact and commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathsf{S}^2(\mathsf{T}^*\mathsf{M}) \arrow[r, "\sigma_1(\Phi)"] \arrow[d] & \mathsf{T}^*\mathsf{M} \otimes \mathsf{F} \arrow[r, "\tau"] \arrow[d] & \mathsf{K} \arrow[r] & 0 \\
0 \arrow[r] & \rho_1(\mathsf{R}) \arrow[r] \arrow[d] & J^2(\mathbb{R}, \mathsf{E}) \arrow[r, "\rho_1(\Phi)"] \arrow[d] & J^1(\mathbb{R}, \mathsf{F}) \arrow[d] & & \\
0 \arrow[r] & \mathsf{R} \arrow[r] & J^1(\mathbb{R}, \mathsf{E}) \arrow[r, "\Phi"] \arrow[d] & \mathsf{F} \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

  which is used to construct the compatibility operator as a map from the bottom left corner to the top right corner.

Summary

- Mathematical tools can very often provide concise elegant models for physical systems.
- Sometimes these mathematical tools can provide solutions to problems that may not be solvable were the problems not formulated in the "right" way.