Fundamental problems of geometric control theory

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13/12/2012

Introduction

- Work with graduate students David Tyner (Carleton University), César Aguilar (Naval Postgraduate School), and Pantelis Isaiah (Technion).
- Controllability and stabilisability of real analytic systems of the form, say,

$$\xi'(t) = F(\xi(t), \mu(t)),$$

where $F: M \times C \to TM$ has a few properties, including the property that $x \mapsto F(x, u)$ is real analytic for fixed $u \in C$.

- These are not solved problems...
- Want a geometric "feedback-invariant" approach. Thus, systems with the same trajectories should be not be differentiated in any way.

Feedback-invariance

- The problem is that the description " $\xi'(t) = F(\xi(t), \mu(t))$ " relies on an explicit parameterisation of the admissible velocities by the control set \mathcal{C} .
- Two alternative ways to proceed:
 - Work with an given parameterisation and try to develop conditions that do not depend on this.

cf. Developing coordinate-independent notions in coordinates in differential geometry.

- Develop an intrinsically parameterisation-independent framework. cf. The coordinate-free approach to differential geometry.
- The second approach is more elegant, but one needs analysis tools that are amenable to this approach. These do not really exist.

Controllability problems

- Of course, much has been written about controllability.
- Many controllability conditions are not feedback-invariant.
- For example, the test "A system is locally controllable if its linearisation is controllable" is not feedback-invariant.
- Benefits of a feedback-invariant approach:
 - Failure of a system to satisfy conditions is not the result of the "wrong" choice of control parameterisation.
 - You believe that controllability conditions are saying something fundamental about the system.
 - You can really attack the gap between necessary and sufficient conditions for controllability. Feedback-dependent conditions simply cannot do this.
- César Aguilar has developed a feedback-invariant means of generating control variations.

Controllability problems

Theorem

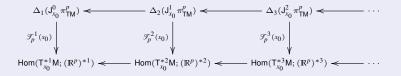
For each $k, p \in \mathbb{Z}_{>0}$ there exists a unique map

$$\mathscr{T}_p^k(x_0) \in \mathsf{L}(\mathsf{S}^{\leq k}(\mathsf{J}_{x_0}^{k-1}\pi_{\mathsf{TM}}^p);\mathsf{L}(\mathsf{T}_{x_0}^{*k}\mathsf{M};(\mathbb{R}^p)^{*k}))$$

such that

$$\mathscr{T}_p^k(x_0)(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0)))=j^k\Phi_{x_0}^{\boldsymbol{\xi}}(\boldsymbol{0}_p).$$

for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of vector fields. Moreover, the diagram



commutes, where the horizontal arrows are the canonical projections.

Controllability problems

Theorem

Let $k \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, and denote $J = \{1, \dots, p\}$. Then there exists a unique map

$$\beta_p^k \colon \mathsf{J}^k_{(0,\mathbf{0}_p)}(\mathbb{R};\mathbb{R}^p) \to L(J)$$

such that,

- (i) for every manifold M and every $x_0 \in M$, and
- (ii) for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^{∞} -vector fields on M, and every $\boldsymbol{\tau} \in \mathsf{ET}_p$ for which $\mathsf{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) \ge k$,

it holds that

$$j^{k}(\Phi_{x_{0}}^{\boldsymbol{\xi}}\circ\boldsymbol{\tau})(0)=\mathsf{Ev}_{\phi}(\beta_{p}^{k}(j^{k}\boldsymbol{\tau}))(x_{0}),$$

where $\phi: J \to \Gamma^{\infty}(\pi_{\mathsf{TM}})$ is defined by $\phi(j) = \xi_j, j \in \{1, \dots, p\}$.

Stabilisability problems

- Various results make the stabilisability problem equivalent to the existence of a control Lyapunov function.
- Control Lyapunov functions have advantages:
 - They provide access to basic problems, e.g., robustness.
 - When they are known for a concrete problem, they are useful for design.
- But... the general existence theory for control Lyapunov functions seems no easier than the problem of stabilisability.
- —> Control Lyapunov functions are more to do with stabilisation than stabilisability.
- Need a "geometric" rather than "Lyapunov" approach to stabilisability. This seems mostly undeveloped.

Controllability $\leftrightarrow stabilisability Fundamental problems$

- One imagines that controllability from x_0 (the study of the states reachable from x_0) is related to stabilisability to x_0 (the study of the states controllable to x_0). This seems mostly undeveloped.
- Pantelis Isaiah:
 - shows that the reachable set of a controllable system can be Lyapunov stabilised by piecewise analytic feedback;
 - gives the existence for this feedback of a function rather like a Lyapunov function.
- The implication stabilisable ~>> controllable is more delicate, and generally false.
- For example, bilinear systems are *never* controllable to the origin but are sometimes stabilisable.

The rôle of singularities

- Bilinear control systems are "singular" at the origin: all system vector fields vanish there.
- Singularities can come up in a few ways:
 - in control-affine systems, the rank of the input distribution may not be locally constant;
 - 2 orbits can change dimension.
- Singularities give rise to lots of interesting phenomenon:
 - systems that are stabilisable but not controllable;
 - the lack of feedback-invariance of the linear controllability test;
 - systems that are controllable, but only for large enough control sets.
- This all seems mostly undeveloped.

Controllability 🚧 stabilisability Practical problems

- Perhaps the essentially complementary strengths of the theories of controllability and stabilisability can impact their application.
- For example:
 - The detailed system structure developed in controllability theory may be useful in understanding the structural properties of stabilisability and stabilisation.
 - The advantages of the Lyapunov approach in stabilisation may be useful for motion planning.

Summary

- A feedback-invariant approach is interesting, probably useful, and mostly an open area.
- A geometric approach to stabilisation is quite undeveloped, particularly in comparison to controllability.
- The study of the rôle of singularities is mostly an open area.
- The complementary strengths of controllability and stabilisation may have practical benefits.
- If you know anything about any of this, let me know!

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