Fundamental problems of geometric control theory

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13/12/2012

Introduction

- Work with graduate students David Tyner (Carleton University), César Aguilar (Naval Postgraduate School), and Pantelis Isaiah (Technion).
- Controllability and stabilisability of real analytic systems of the form, say,
$$\xi'(t) = F(\xi(t), \mu(t)),$$
where $F\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ has a few properties, including the property that $x \mapsto F(x, u)$ is real analytic for fixed $u \in \mathcal{C}$.
- These are not solved problems...
- Want a geometric "feedback-invariant" approach. Thus, systems with the same trajectories should be not be differentiated in any way.

Feedback-invariance

- The problem is that the description "$\xi'(t) = F(\xi(t), \mu(t))$" relies on an explicit parameterisation of the admissible velocities by the control set $\mathcal{C}$.
- Two alternative ways to proceed:
  1. Work with an given parameterisation and try to develop conditions that do not depend on this.
     cf. Developing coordinate-independent notions in coordinates in differential geometry.
  2. Develop an intrinsically parameterisation-independent framework.
     cf. The coordinate-free approach to differential geometry.
- The second approach is more elegant, but one needs analysis tools that are amenable to this approach. These do not really exist.

Controllability problems

- Of course, much has been written about controllability.
- Many controllability conditions are not feedback-invariant.
- For example, the test "A system is locally controllable if its linearisation is controllable" is not feedback-invariant.
- Benefits of a feedback-invariant approach:
  1. Failure of a system to satisfy conditions is not the result of the "wrong" choice of control parameterisation.
  2. You believe that controllability conditions are saying something fundamental about the system.
  3. You can really attack the gap between necessary and sufficient conditions for controllability. Feedback-dependent conditions simply cannot do this.
- César Aguilar has developed a feedback-invariant means of generating control variations.

Controllability problems

Theorem

For each $k, p \in \mathbb{Z}_{>0}$ *there exists a unique map*

$$\mathscr{T}_p^{\,k}(x_0) \in \mathsf{L}(\mathsf{S}^{\leq k}(\mathsf{J}_{x_0}^{k-1}\pi_{\mathsf{TM}}^p); \mathsf{L}(\mathsf{T}_{x_0}^{*k}\mathsf{M}; (\mathbb{R}^p)^{*k}))$$

such that

$$\mathscr{T}_p^{\,k}(x_0)(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0))) = j^k\Phi_{x_0}^{\boldsymbol{\xi}}(\mathbf{0}_p).$$

for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ *of vector fields. Moreover, the diagram*

$$\begin{tikzcd}
\Delta_1(\mathsf{J}_{x_0}^0\pi_{\mathsf{TM}}^p) \arrow[d, "\mathscr{T}_p^{\,1}(x_0)"'] & \Delta_2(\mathsf{J}_{x_0}^1\pi_{\mathsf{TM}}^p) \arrow[l] \arrow[d, "\mathscr{T}_p^{\,2}(x_0)"'] & \Delta_3(\mathsf{J}_{x_0}^2\pi_{\mathsf{TM}}^p) \arrow[l] \arrow[d, "\mathscr{T}_p^{\,3}(x_0)"'] & \cdots \arrow[l] \\
\mathsf{Hom}(\mathsf{T}_{x_0}^{*1}\mathsf{M}; (\mathbb{R}^p)^{*1}) & \mathsf{Hom}(\mathsf{T}_{x_0}^{*2}\mathsf{M}; (\mathbb{R}^p)^{*2}) \arrow[l] & \mathsf{Hom}(\mathsf{T}_{x_0}^{*3}\mathsf{M}; (\mathbb{R}^p)^{*3}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

commutes, where the horizontal arrows are the canonical projections.

Controllability problems

Theorem

Let $k \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, and denote $J = \{1, \dots, p\}$. Then there exists a unique map

$$\beta_p^k \colon \mathsf{J}^k_{(0,\mathbf{0}_p)}(\mathbb{R}; \mathbb{R}^p) \to L(J)$$

such that,

(i) for every manifold M *and every $x_0 \in$* M*, and*

(ii) for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields on M*, and every $\boldsymbol{\tau} \in \mathsf{ET}_p$ for which $\mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) \geq k$,*

it holds that

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) = \mathsf{Ev}_\phi(\beta_p^k(j^k\boldsymbol{\tau}))(x_0),$$

where $\phi \colon J \to \Gamma^\infty(\pi_{\mathsf{TM}})$ is defined by $\phi(j) = \xi_j, j \in \{1, \dots, p\}$.

Stabilisability problems

- Various results make the stabilisability problem equivalent to the existence of a control Lyapunov function.
- Control Lyapunov functions have advantages:
  1. They provide access to basic problems, e.g., robustness.
  2. When they are known for a concrete problem, they are useful for design.
- But...the general existence theory for control Lyapunov functions seems no easier than the problem of stabilisability.
- $\Longrightarrow$ Control Lyapunov functions are more to do with stabilisation than stabilisability.
- $\Longrightarrow$ Need a “geometric” rather than “Lyapunov” approach to stabilisability. This seems mostly undeveloped.

Controllability $\leftrightsquigarrow$ stabilisability
Fundamental problems

- One imagines that controllability from x_0 (the study of the states reachable from x_0) is related to stabilisability to x_0 (the study of the states controllable to x_0). This seems mostly undeveloped.
- Pantelis Isaiah:
  1. shows that the reachable set of a controllable system can be Lyapunov stabilised by piecewise analytic feedback;
  2. gives the existence for this feedback of a function rather like a Lyapunov function.
- The implication stabilisable $\rightsquigarrow$ controllable is more delicate, and generally false.
- For example, bilinear systems are *never* controllable to the origin but are sometimes stabilisable.

The rôle of singularities

- Bilinear control systems are “singular” at the origin: all system vector fields vanish there.
- Singularities can come up in a few ways:
  1. in control-affine systems, the rank of the input distribution may not be locally constant;
  2. orbits can change dimension.
- Singularities give rise to lots of interesting phenomenon:
  1. systems that are stabilisable but not controllable;
  2. the lack of feedback-invariance of the linear controllability test;
  3. systems that are controllable, but only for large enough control sets.
- This all seems mostly undeveloped.

Controllability $\leftrightsquigarrow$ stabilisability
Practical problems

- Perhaps the essentially complementary strengths of the theories of controllability and stabilisability can impact their application.
- For example:
  1. The detailed system structure developed in controllability theory may be useful in understanding the structural properties of stabilisability and stabilisation.
  2. The advantages of the Lyapunov approach in stabilisation may be useful for motion planning.

Summary

- A feedback-invariant approach is interesting, probably useful, and mostly an open area.
- A geometric approach to stabilisation is quite undeveloped, particularly in comparison to controllability.
- The study of the rôle of singularities is mostly an open area.
- The complementary strengths of controllability and stabilisation may have practical benefits.
- If you know anything about any of this, let me know!