Some interesting things about real analyticity

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Why should one care about (real) analyticity?

- Those of us who work with geometric things work with smooth manifolds and smooth objects defined on these smooth manifolds.
- In practice, however, physical models either (1) have very low differentiability, e.g., discontinuous, continuous but not differentiable, or (2) are analytic.
- Even in case (1) the models are typically “piecewise analytic” in some way.
- Models that are smooth but not analytic are mathematical constructs; the appearance of e^{-1/x^2} is a sure sign that there is something non-physical going on.

Why should one care about (real) analyticity?

- However, smooth differential geometry is a convenient framework because certain problems are quite easily dealt with in this framework.
- Moreover, the arguments from analytic geometry are not as transparent as their smooth counterparts.
- *Important observation:* Nonetheless, the stronger hypothesis of analyticity often gives rise to (1) simpler hypotheses because some things are always true in the analytic case that are not always true in the smooth case and/or (2) stronger conclusions.
- One thing that is not clear at the outset is what is possible and what is not possible in analytic geometry. This is the point of this talk.

What kinds of problems are we talking about?

Go for broke!

- *Extending functions:* Given a function f defined in a neighbourhood of a point x, does there exist a globally defined function $\bar{f}$ agreeing with f near x?
  1. In the smooth case, this is done easily using one of those non-physical bump functions.
  2. In the analytic case, this is generally not possible for a multitude of possible reasons.
  3. Analytic functions are just not that flexible.

Let's scale back what we want.

- *Prescribing finite jet data:* At a point x specify the value of a function along with finitely many of its derivatives. Does there exist a globally defined function having the prescribed data at x?
  1. In the smooth case, this is an easier version of the first problem.
  2. In the analytic case...

The holomorphic world

- In order to get some feeling for extending analytic functions, let us consider the holomorphic setting for a moment.
  1. If M is a compact connected holomorphic manifold, then $\dim_{\mathbb{C}}(C^{\mathsf{hol}}(\mathsf{M})) = 1$, i.e., all holomorphic functions are constant.
  2. In general, the dimension (over $\mathbb{C}$) of the holomorphic sections of a holomorphic vector bundle over a compact manifold is finite.
  3. There is a family $\mathsf{O}(k)$, $k \in \mathbb{Z}$, of one-dimensional vector bundles over the Riemann sphere with

$$\dim_{\mathbb{C}}(\Gamma^{\mathsf{hol}}(\mathsf{O}(k))) = \begin{cases} k+1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

  4. The $\mathbb{C}$-vector space of holomorphic vector fields on the Riemann sphere is 3-dimensional!
- This is rather unlike the smooth case, where the space of sections is always infinite-dimensional.

Is analytic like smooth or like holomorphic?

- Smooth objects are very flexible, while holomorphic objects can be very rigid.
- Which case does analytic resemble?
- It resembles a special class of holomorphic manifolds called *Stein manifolds*.
  1. After Karl Stein, 1913–2000.
  2. Precise definition is not immediately insightful...
  3. Stein manifolds have lots of holomorphic functions.
  4. Moreover, the holomorphic functions are quite flexible, e.g., one can uniformly approximate locally defined holomorphic functions on compact sets by globally defined holomorphic functions.
  5. Stein manifolds are precisely those holomorphic manifolds admitting a proper holomorphic embedding in $\mathbb{C}^N$ for $N \in \mathbb{Z}_{>0}$ sufficiently large (Reinhold Remmert, 1955).

Is analytic like smooth or like holomorphic?

- *Difficult theorem:* If M is an analytic manifold, then it possesses a complexification that is a Stein manifold $\implies$ an analytic manifold admits a proper analytic embedding in $\mathbb{R}^N$ for some $N \in \mathbb{Z}_{>0}$ (Hassler Whitney/François Bruhat, Hans Grauert, 1958).
- Quite a few existential questions can be answered using the Grauert embedding theorem.
  1. Analytic manifolds possess analytic Riemannian metrics (just restrict the Euclidean metric from the embedding).
  2. Globally defined analytic functions exist satisfying finite jet data at a point (just use a suitable polynomial in $\mathbb{R}^N$ restricted to M).
- Other natural problems require more sophisticated techniques.

Analytic distributions

- An *analytic distribution* on an analytic manifold M is a subset $\mathsf{D} \subset \mathsf{TM}$ such that $\mathsf{D}_x = \mathsf{D} \cap \mathsf{T}_x\mathsf{M}$ is a subspace and such that, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of x and analytic vector fields $(X_a)_{a \in A}$ on $\mathcal{N}$ such that

$$\mathsf{D}_x = \operatorname{span}_{\mathbb{R}}(X_a(x) |\; a \in A), \qquad x \in \mathcal{N}.$$

- Francesco said I should mention control theory, so here goes: Distributions are important in geometric control theory.

Analytic distributions

- *Important questions (related to control theory):*
  1. Does an analytic distribution possess "enough" global sections?
  2. How does one determine the involutive closure of an analytic distribution D? Can one, as one *always* does in practice, just choose locally defined analytic vector fields that generate D and iteratively compute their Lie brackets until nothing more happens, and then you have the involutive closure? (NB. This does not work in the smooth case.)
  3. Let X be a vector field. Are the following conditions equivalent?
     - ★ $[X, Y]$ is a D-valued vector field for every D-valued vector field Y.
     - ★ $(\Phi_t^X)^* Y$ is a D-valued vector field for every D-valued vector field Y.

Analytic distributions

- The answers to all questions are, “Yes.”
- The tools required to answer the questions rely on some form of local finite generation.
- Often this is attributed to the fact the following fact: The rings of germs of analytic functions is a Noetherian ring.
- However, this is a fairly simple fact (more or less) about power series.
- In fact, to really address these questions requires a far more subtle notion, that of “coherence,” which is a sort of “local Noetherian” property (as opposed to the pointwise property of germs).

The rôle of coherence

- Suppose that we have smooth, analytic, or holomorphic vector bundles $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ and a vector bundle map $\Phi \colon \mathsf{E} \to \mathsf{F}$ over id_{M}.
- Let η be a section of π_{F} and suppose that, about every $x \in \mathsf{M}$, there is a neighbourhood $\mathcal{N}_x$ and a section ξ_x of π_{E} over $\mathcal{N}_x$ satisfying $\Phi \circ \xi_x = \eta|\mathcal{N}_x$.
- Does there exist a global section ξ of π_{E} such that $\Phi \circ \xi = \eta$?
- Answers:
  1. Smooth: “Yes,” use a partition of unity argument.
  2. Holomorphic: “No,” in general, but, “yes,” on Stein manifolds.
  3. Analytic: “Yes.”
- In the Stein and analytic case, the tool one uses is *coherent analytic sheaves*, particularly Cartan’s Theorems A and B (Henri Cartan, 1951-2, 1957).

Differences between analytic and holomorphic

- *Topologies:*
  1. The appropriate topology for holomorphic functions on a holomorphic manifold is the compact-open topology: Convergent sequences are exactly those that converge uniformly on compact sets. This is a nice topology, e.g., it is metric.
  2. For analytic functions, the standard smooth topology—that of uniform convergence of functions and their derivatives on compact sets—is not suitable. The suitable topology is quite complicated.
- *Analytic sets:* An analytic set (real or complex) is locally the set of zeros of finitely many analytic or holomorphic functions. Given an analytic set, consider the ideal of functions that vanish on the set.
  1. In the holomorphic case, this ideal has the property of coherence.
  2. In the real analytic case, this is not necessarily true.

What is the point?

- *Bad news:* Analytic geometry is a lot different from smooth geometry.
- *Good news:* Analytic geometry is a lot different from general holomorphic geometry.
- *Good news:* While the techniques of smooth geometry do not apply, there are replacements in analytic geometry.
- *Bad news:* The techniques of analytic geometry are difficult to learn and to learn to apply.
- *Good news:* After the breaking in period, it is often quite easy to employ the machinery to do what you want: “partition of unity” arguments are replaced with “Cartan’s Theorem B” arguments.
- *The best news of all:* Analytic geometry is supercool.