Mathematical models for geometric control systems

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What this talk is and is not about

- This talk is really disconnected from immediate applications to control theory.
- My interest is in developing a framework for effectively studying fundamental problems in control theory, e.g., necessary and sufficient conditions for controllability, stabilisability, optimality...
- To be effective, a control theoretic framework must not rely on extraneous structure that will lead to studying the extraneous structure, not control theory.
- My interest is in differential geometric ordinary differential equation models.
- We all understand by now (I guess) that in differential geometric models, a proper framework *must* be coordinate-invariant.

What this talk is and is not about (cont'd)

- Example: The equations coming from the PMP take the coordinate form

$$\left.\underbrace{\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{F}(\boldsymbol{x}(t),\boldsymbol{u}(t)),\\ \dot{\boldsymbol{\lambda}}(t) &= -\boldsymbol{DF}^T(\boldsymbol{x}(t),\boldsymbol{u}(t))\boldsymbol{\lambda}(t) + \text{cost terms}\end{aligned}}_{\text{“Adjoint equation”}}\right\}\ \textit{This is intrinsic}$$

- It can take self-discipline to really buy into this coordinate-invariance principle.
- In this talk, we will further extend demands on self-discipline by applying this "invariance" principle to controls.

“Control invariance”

- Here is a typically used class of control systems:

$$\xi'(t) = F(\xi(t), \mu(t)).$$

- Models such as this, where there is an explicit dependence on control, are natural in applications of control theory, where the control u is a “real” physical input like a force or a voltage.
- Mathematically, there is a drawback to these sorts of models, namely that the model comes with an explicit dependence on the control u.
- The problem is this: fundamental structural issues in control theory are often about trajectories, and it is possible that two different systems as above may give rise to the same trajectories, but have rather different control theoretic attributes.

"Control invariance" (cont'd)

- Here is an example. Consider the two systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), \end{aligned} \qquad \begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), \end{aligned}$$

  with $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(u_1, u_2) \in \mathbb{R}^2$.
  - These two systems have the same trajectories.
  - The left system has a linearisation at $(0, 0, 0)$ that is *not* controllable.
  - The right system has a linearisation at $(0, 0, 0)$ that *is* controllable.

“Control invariance” (cont’d)

- So what’s going wrong?
  - The notion of linearisation depends on how the system is parameterised by controls.
  - The notion of linearisation being controllable depends on how the systems is parameterised by controls.
- We do not like things that depend on how a system is parameterised by control. Two possible solutions:
  - think of how to do things in a way that can be shown to be independent of how one parameterises a system by controls;
  - eliminate the parameterisation by controls.
- Cf. Think about defining things in differential geometry:
  - make a definition in coordinates, and then show that it is independent of coordinates;
  - make a definition that is coordinate-independent at the outset.
- The latter is what we’ll do here.

The smooth and real analytic categories

- Much of geometric control theory works with smooth systems.
- This is okay: Many constructions in control theory are natural to the smooth category, and additional structure can be a distraction.
- However, one *must* be able to work in the real analytic category.
  1. Mathematical motivation: Real analyticity gives
     a. *Weaker hypotheses*, e.g., the "finite generation" condition in Frobenius's Theorem and/or
     b. *Stronger conclusions*, e.g., the LARC in accessibility theory.
  2. Nature is not replete with models that are smooth but not real analytic.
- We will develop a framework that works in the smooth and real analytic categories, meaning that all ideas developed can be developed in both cases.

Topologies for spaces of vector fields

- We want a rigorous framework where one can prove theorems. As we shall see, this requires us to topologise the space $\Gamma^r(\mathsf{TM})$, $r \in \{\infty, \omega\}$, of vector fields of class C^r.
- We will also talk about functions $\implies$ we may as well talk about vector bundles…
- Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle with $\Gamma^r(\mathsf{E})$ the $\mathbb{R}$-vector space of sections.
- For vector fields $\mathsf{E} = \mathsf{TM}$ and for functions $\mathsf{E} = \mathsf{M} \times \mathbb{R}$.
- We consider $r = \infty$ and $r = \omega$ separately.

Topologies for spaces of smooth sections

- The ***smooth compact-open*** or **CO$^\infty$*-topology*** for $\Gamma^\infty(\mathsf{E})$ is that topology for which convergent sequences are described thusly: A sequence $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ξ if and only if $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $j_m\xi|K$ for every compact set K and every $m\in\mathbb{Z}_{\geq 0}$. (Here j_m means the m-jet, and so represents all derivatives up to order m.)
- With this topology, $\Gamma^\infty(\mathsf{E})$ has a property known as "nuclear." This means that compact sets are closed and bounded.
- *Closed:* A set $\mathscr{F}$ of sections is closed if it is closed under sequential convergence.
- *Bounded:* A set $\mathscr{F}$ of sections is bounded if, for every compact set K and every $m\in\mathbb{Z}_{\geq 0}$, $j_m\xi|K$ is bounded, uniformly for $\xi\in\mathscr{F}$.

Topologies for spaces of smooth sections (cont'd)

- It is more or less clear that the CO^∞-topology is defined by the seminorms
$$p_{K,m}(\xi) = \sup\{\|j_m\xi(x)\|_m \mid x \in K\}$$
for $K \subseteq \mathsf{M}$ compact and $m \in \mathbb{Z}_{\geq 0}$.
- For $\Gamma^\infty(\mathsf{TM})$, we have the following "weak" characterisation of the CO^∞-topology.

Theorem

The CO^∞-topology agrees with the locally convex topology defined by the seminorms

$$q_{K,m}(X) = \sup\{p_{K,m}(\mathscr{L}_X f) \mid p_{K,m+1}(f) = 1\}$$

for $K \subseteq \mathsf{M}$ compact and $m \in \mathbb{Z}_{\geq 0}$.

Topologies for spaces of real analytic sections

- The topology for $\Gamma^\omega(\mathsf{E})$ is harder to describe.
- First complexify to get a holomorphic vector bundle $\overline{\pi}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$.
- Let $\mathscr{N}_{\mathsf{M}}$ be the set of neighbourhoods of M in $\overline{\mathsf{M}}$.
- For $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$, topologise $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ with the compact-open topology, i.e., the topology of uniform convergence on compact sets.
- Have a mapping $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \to \Gamma^\omega(\mathsf{E})$ by restriction.
- Give $\Gamma^\omega(\mathsf{E})$ the direct limit topology...
- There is an alternative inverse limit topology that turns out to be equivalent by a hard theorem of Martineau.[3]
- We call this the ***C^ω-topology***.

[3] *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann., **163**, 62–88, 1966

Topologies for spaces of real analytic sections (cont'd)

- It is difficult to work with the C^ω-topology…at least for me…
- We are working on a real analytic version of the weak topology for smooth vector fields described above, and here is an interesting step along the way…

Theorem

If $\overline{\mathsf{M}}$ is a Stein manifold,[a] then the compact-open topology for $\Gamma^{\mathrm{hol}}(\mathsf{T}\overline{\mathsf{M}})$ agrees with the topology defined by the seminorms

$$q_K(\overline{X}) = \sup\{p_{K,0}(\mathscr{L}_{\overline{X}}\bar{f}) \mid p_{K,0}(\bar{f}) = 1\}$$

for $K \subseteq \mathsf{M}$ compact.

[a] Most concretely, this means that $\overline{\mathsf{M}}$ can be holomorphically embedded in $\mathbb{C}^N$.

Control systems (carefully)

- We will want to establish theorems of correspondence between various forms of systems, so let us precisely describe the "usual" notion of a control system, taking into account our general needs.
- A ***C^∞-control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$ where:
  1. M is a C^∞-manifold;
  2. $\mathcal{C}$ is a separable metric space;
  3. $F\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ has the properties:
     a. $F(x,u) \in \mathsf{T}_x\mathsf{M}$, i.e., $F^u\colon x \mapsto F(x,u)$ is a vector field;
     b. F^u is a smooth vector field;
     c. $(x,u) \mapsto j_m F^u(x)$ is continuous for every $m \in \mathbb{Z}_{\geq 0}$.
- In the smooth case, condition 3c ensures that the map $u \mapsto F^u$ is continuous.
- I do not know the right general definition for real analytic control systems.

Generalised control systems

- For $r \in \{\infty, \omega\}$, a ***C^r-generalised control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where
  1. M is a C^r-manifold and
  2. $\mathscr{F}$ assigns to an open $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U}) \subseteq \Gamma^r(\mathsf{T}\mathcal{U})$ such that, if $\mathcal{V} \subseteq \mathcal{U}$, then the restrictions of elements of $\mathscr{F}(\mathcal{U})$ to $\mathcal{V}$ are in $\mathscr{F}(\mathcal{V})$ ($\mathscr{F}$ is a presheaf).
- The presheaf $\mathscr{F}$ is a ***sheaf*** if given $X_a \in \mathscr{F}(\mathcal{U}_a)$, $a \in A$, such that X_{a_1} and X_{a_2} agree on $\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}$, then there is a well-defined element of $\mathscr{F}(\cup_{a \in A} \mathcal{U}_a)$ agreeing with each X_a on $\mathcal{U}_a$.
- The presheaf $\mathscr{F}$ is ***globally generated*** if

$$\mathscr{F}(\mathcal{U}) = \{X|\mathcal{U} \mid X \in \mathscr{F}(\mathsf{M})\}.$$

  This is a typical case that one can keep in mind for safety.

Geometric system models (cont'd)

- What are trajectories? We work with the smooth case only; we do not yet understand the real analytic case.
- An ***open-loop system*** for a generalised control system $(\mathsf{M}, \mathscr{F})$ is a triple $\mathscr{O}_{\mathfrak{G}} = (\mathbb{T}, \mathcal{U}, X)$ where
  1. $\mathbb{T} \subseteq \mathbb{R}$ is an interval,
  2. $\mathcal{U} \subseteq \mathsf{M}$ is open, and
  3. $X \colon \mathbb{T} \to \mathscr{F}(\mathcal{U})$ has the properties
     a. $t \mapsto \mathscr{L}_{X_t} f(x)$ is measurable for every $x \in \mathsf{M}$ and $f \in C^\infty(\mathsf{M})$,
     b. for every $f \in C^\infty(\mathsf{M})$, $\mathscr{L}_{X_t} f \in C^\infty(\mathsf{M})$, and
     c. for each compact $K \subseteq \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in L^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{>0})$ such that
$$p_{K,m}(\mathscr{L}_{X_t} f) \leq g(t), \qquad t \in \mathbb{T},\ x \in K,$$
        for every $f \in C^\infty(\mathsf{M})$.
- Equivalently, $t \mapsto X_t$ is measurable and locally essentially bounded in the CO^∞-topology…
- A ***trajectory*** is an integral curve of some open-loop system.

Fun and games with generalised control systems

- Given a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$, we have the associated generalised control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$, where

$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \mid u \in \mathcal{C}\}.$$

- Given a differential inclusion $\mathscr{X} : \mathsf{M} \twoheadrightarrow \mathsf{TM}$, we have the associated generalised control system $\mathfrak{G}_{\mathscr{X}} = (\mathsf{M}, \mathscr{F}_{\mathscr{X}})$, where

$$\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \{X \in \Gamma^r(\mathsf{T}\mathcal{U}) \mid X(x) \in \mathscr{X}(x),\ x \in \mathcal{U}\}.$$

Fun and games with generalised control systems (cont'd)

- Given a smooth generalised control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ with $\mathscr{F}$ globally generated, we have the control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$, where
  1. $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ and
  2. $F(x, X) = X(x)$.
- (Stop having fun for a minute, and *prove* that this is a smooth control system according to our definition.)
- Given a generalised control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, we have a differential inclusion $\mathscr{X}_{\mathfrak{G}}$ given by

$$\mathscr{X}_{\mathfrak{G}}(x) = \{X(x) \mid [X]_x \in \mathscr{F}_x\}$$

($\mathscr{F}_x$ is the set of germs of vector fields from $\mathscr{F}$).

Precise correspondences

- For systems, we have the following correspondences:
  1. if $\mathscr{F}$ is globally generated, $\mathfrak{G}_{\Sigma_{\mathfrak{G}}} = \mathfrak{G}$ (tautological);
  2. $\Sigma_{\mathfrak{G}_{\Sigma}} = \Sigma$ (tautological);
  3. if $\mathscr{F}$ is a sheaf, $\mathfrak{G}_{\mathscr{X}_{\mathfrak{G}}} = \mathfrak{G}$ (sheafily tautological);
  4. $\mathscr{X}_{\mathfrak{G}_{\mathscr{X}}} \subseteq \mathscr{X}$.
- The last inclusion will virtually never be an equality for all kinds of reasons, including these two.
  1. $\mathscr{X}$ may not possess enough local regularity to section it regularly, even locally.
  2. In the real analytic case, it is unlikely that a differential inclusion will be compatible with the following fact: If one knows the Taylor series of a real analytic vector field at a point x, this uniquely determines X on the connected component of M containing x.
- Seems to me. . . differential inclusions satisfying $\mathscr{X}_{\mathfrak{G}_{\mathscr{X}}} = \mathscr{X}$ are the ones of interest in geometric control theory.

Precise correspondences (cont'd)

- Mostly we are interested in correspondence of trajectories. This requires hypotheses.
- Given $\Sigma = (\mathsf{M}, F, \mathcal{C})$:
  1. $\mathrm{Traj}(\Sigma) \subseteq \mathrm{Traj}(\mathfrak{G}_\Sigma)$ (easy);
  2. if F is proper, then $\mathrm{Traj}(\mathfrak{G}_\Sigma) \subseteq \mathrm{Traj}(\Sigma)$ (very hard);
- Given $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$:
  1. if $\mathscr{F}$ is globally generated and if $\mathscr{F}(\mathsf{M})$ is compact in the CO^∞-topology, then $\mathrm{Traj}(\Sigma_{\mathfrak{G}}) = \mathrm{Traj}(\mathfrak{G})$ (relatively easy);
  2. $\mathrm{Traj}(\mathfrak{G}) \subseteq \mathrm{Traj}(\mathscr{X}_{\mathfrak{G}})$ (easy);
- Given a differential inclusion $\mathscr{X}$:
  1. $\mathrm{Traj}(\mathfrak{G}_{\mathscr{X}}) \subseteq \mathrm{Traj}(\mathscr{X})$ (easy).
- Given $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$:
  1. $\mathrm{Traj}(\mathfrak{G}) \subseteq \mathrm{Traj}(\mathscr{X}_{\mathfrak{G}})$ (easy);
  2. if $\mathscr{F}$ is globally generated and if $\mathscr{F}(\mathsf{M})$ is compact in the CO^∞-topology, then $\mathrm{Traj}(\mathscr{X}_{\mathfrak{G}}) \subseteq \mathrm{Traj}(\mathfrak{G})$ (follows easily from the hard $\mathrm{Traj}(\mathfrak{G}_\Sigma) \subseteq \mathrm{Traj}(\Sigma)$).

Linearisation of generalised control systems

- To illustrate that generalised control systems possible give you something, let's look at linearisation.
- For a vector field X on M, let X^T denote the ***tangent lift*** of X, which is the vector field on TM defined by

$$X^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x\Phi_t^X(v_x),$$

  i.e., the flow of X^T is the linearised flow of X.
- For a vector field X on M, define a vector field $\mathrm{vlft}(X)$ on TM by

$$\mathrm{vlft}(X)(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (v_x + sX(x)).$$

Linearisation of generalised control systems (cont'd)

- We let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a smooth generalised control system.
- To construct the linearisation of $\mathfrak{G}$, we use...Jacobian linearisation like we teach undergraduates! But in a rather atypical setup...
- Assume that $\mathscr{F}$ is globally generated and let $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ be as above. Thus $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ and $F(x, X) = X(x)$.
- To do Jacobian linearisation at (x, X), we perturb x in the direction of $v_x \in \mathsf{T}_x\mathsf{M}$ and X in the direction of $Y \in \mathscr{F}(\mathsf{M})$:
  1. let $I \subseteq \mathbb{R}$ be an open interval containing 0;
  2. let $\gamma \colon I \to \mathsf{M}$ satisfy $\gamma'(0) = v_x$;
  3. consider the curve $s \mapsto X + sY$ in $\mathscr{F}(\mathsf{M})$.

Linearisation of generalised control systems (cont'd)

- Denote
$$\Upsilon(t,s) = \Phi_t^{X+sY}(\gamma(s)).$$
- Now differentiate at $s = 0$ then $t = 0$:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\Upsilon \circ \delta(t,s) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t^X(\gamma(s)) +_2 I_{\mathsf{M}}\big(\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^{X+sY}(x)\big)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}T_x\Phi_t^X(v_x) +_2 I_{\mathsf{M}}\big(\frac{\mathrm{d}}{\mathrm{d}s}(X(x) + sY(x))\big)\\ &= X^T(v_x) +_1 \mathrm{vlft}(Y)(v_x),\end{aligned}$$

  where I_{M} is the double tangent bundle involution, and $+_1$ and $+_2$ refer to the "primary" and "secondary" vector bundle structures for TTM:
$$\pi_{\mathsf{TTM}} \colon \mathsf{TTM} \to \mathsf{TM}, \quad T\pi_{\mathsf{TM}} \colon \mathsf{TTM} \to \mathsf{TM}.$$

Linearisation of generalised control systems (cont'd)

- The ***linearisation*** of a generalised control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is the C^r-geometric system model $\mathscr{F}^T$ on TM defined by

$$\mathscr{F}^T(\mathsf{T}\mathcal{U}) = \{X^T + \mathrm{vlft}(Y) \mid X, Y \in \mathscr{F}(\mathcal{U})\}.$$

- The linearisation is a generalised control system on TM...niiice...
- Let's look at something familiar: linearisation about a trajectory.
  - Let $t \mapsto \xi_{\mathrm{ref}}(t) \in \mathcal{U}$ be a trajectory, i.e., $\xi'_{\mathrm{ref}}(t) = X_t(\xi_{\mathrm{ref}}(t))$.
  - Note that there may be many vector fields $t \mapsto X_t \in \mathscr{F}(\mathcal{U})$ having ξ_{ref} as an integral curve.
  - A ***trajectory for the linearisation about ξ_{ref}*** is a trajectory $t \mapsto \Xi(t)$ for $\mathscr{F}^T(\mathsf{T}\mathcal{U})$ for which $\pi_{\mathsf{TM}} \circ \Xi = \xi_{\mathrm{ref}}$.
  - In coordinates: If $t \mapsto \Xi(t)$ is represented by $t \mapsto (\boldsymbol{x}(t), \boldsymbol{v}(t))$ then

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{X}_t(\boldsymbol{x}(t)), \\ \dot{\boldsymbol{v}}(t) &= \underbrace{\boldsymbol{D}\boldsymbol{X}_t(\boldsymbol{x}(t)) \cdot \boldsymbol{v}(t)}_{\text{“}A(t)x(t)\text{”}} + \underbrace{\boldsymbol{Y}_t(\boldsymbol{x}(t))}_{\text{“}B(t)u(t)\text{”}} \end{aligned}\right\}$$

Alarm! Alarm!

Linearisation of generalised control systems (cont'd)

- One can also linearise about a flow, not just a trajectory.
  - Let $\mathcal{U} \subseteq \mathsf{M}$ be open.
  - Let $t \mapsto X_{\mathrm{ref},t} \in \mathscr{F}(\mathcal{U})$ be a vector field for the system.
  - A ***trajectory for the linearisation about X_{ref}*** is a trajectory $t \mapsto \Xi(t)$ for $\mathscr{F}^T$ for which $\pi_{\mathsf{TM}} \circ \Xi$ is an integral curve for $X_{\mathrm{ref},t}$.
  - In coordinates: If $t \mapsto \Xi(t)$ is represented by $t \mapsto (\boldsymbol{x}(t), \boldsymbol{v}(t))$ then

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{X}_{\mathrm{ref},t}(\boldsymbol{x}(t)), \\ \dot{\boldsymbol{v}}(t) &= \underbrace{\boldsymbol{D}\boldsymbol{X}_{\mathrm{ref},t}(\boldsymbol{x}(t)) \cdot \boldsymbol{v}(t)}_{\cancel{\text{“}A(t)x(t)\text{”}}} + \underbrace{\boldsymbol{Y}_t(\boldsymbol{x}(t))}_{\cancel{\text{“}B(t)u(t)\text{”}}} \end{aligned}\right\} \quad \begin{array}{l} \text{Alarm! (again)} \\ \text{Alarm! (again)} \end{array}$$

- Generally, the two sorts of linearisations are not the same. To see why, we consider stationary reference trajectories.

Linearisation of generalised control systems (cont'd)

- We look carefully at the linearisation at an equilibrium point.
  - A point x_0 is an ***equilibrium point*** if, for some (and so any sufficiently small) neighbourhood $\mathcal{U}$ of x_0, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X(x_0) = 0_{x_0}$.
  - If $X(x_0) = 0_{x_0}$ then $X^T(v_x) = A_{X,x_0}(v_x)$ for some $A_{X,x_0} \in \mathrm{End}_{\mathbb{R}}(\mathsf{T}_{x_0}\mathsf{M})$.
  - A trajectory for the linearisation about the trivial reference trajectory $t \mapsto x_0$ is then a curve $t \mapsto v(t)$ in $\mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\dot{v}(t) = A_{X_t,x_0}(v(t)) + b(t),$$

No alarm! No alarm!

for

1. $t \mapsto X_t \in \mathscr{F}(\mathcal{U})$ such that $X_t(x_0) = 0_{x_0}$ and
2. a measurable essentially bounded curve $t \mapsto b(t)$ taking values in

$$\{Y(x_0) \mid [Y]_{x_0} \in \mathscr{F}_{x_0}\},$$

where $\mathscr{F}_{x_0}$ is the stalk of $\mathscr{F}$ at x_0.

Linearisation of generalised control systems (cont'd)

- Um…linearisation at an equilibrium point x_0 is a *time-dependent* linear system on $\mathsf{T}_{x_0}\mathsf{M}$? Yes, in general!
- If we instead fix a time-invariant reference vector field $X_{\mathrm{ref}} \in \mathscr{F}(\mathcal{U})$ for which x_0 is an equilibrium point, then the trajectories for the linearisation about X_{ref} with initial condition in $\mathsf{T}_{x_0}\mathsf{M}$ satisfy

$$\dot{v}(t) = A_{X_{\mathrm{ref}},x_0}(v(t)) + b(t).$$

- This is more or less what we expect when linearising at an equilibrium point.
- There are cases, however, where the linearisation is always time-invariant.

Linearisation of generalised control systems (cont'd)

- For $x \in \mathsf{M}$ define a distribution

$$\mathsf{D}(\mathscr{F})_x = \operatorname{span}_{\mathbb{R}}(X(x) |\ [X]_x \in \mathscr{F}_x).$$

- If x_0 is an equilibrium point for $\mathscr{F}$ and *a regular point for* $\mathsf{D}(\mathscr{F})$, then a trajectory for the linearisation about the trivial reference trajectory $t \mapsto x_0$ is a curve $t \mapsto v(t)$ in $\mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\dot{v}(t) = A_{X,x_0}(v(t)) + b(t),$$

for

1. some $X \in \mathscr{F}(\mathcal{U})$ for which $X(x_0) = 0_{x_0}$ and
2. a measurable essentially bounded curve $t \mapsto b(t)$ taking values in

$$\{Y(x_0) \mid [Y]_{x_0} \in \mathscr{F}_{x_0}\}.$$

Summary

- We have a mathematical framework for a class of control systems that:
    1. is coordinate-invariant;
    2. is "representation-invariant";
    3. has the capacity to handle real analytic systems naturally;
    4. isolates local properties and constructions in a natural way.
- The most elementary of constructions in control theory, linearisation, suggests that the new framework illuminates hitherto unnoticed system structure.
- Future work. . . frighteningly vast. . .