Tautological control systems

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What these ideas are and are not about

- There is nothing practical in this talk.
- The machinery in this talk is intended to provide a framework for studying *fundamental structural problems* in control theory, nothing more... but nothing less either.
- This talk is a mere sketch of a larger body of work:
  1. *Tautological Control Systems*, Springer-Verlag, 2014, 118pp+xii
  2. *Time-Varying Vector Fields and Their Flows* (with S. Jafarpour), Springer-Verlag, 2014, 119pp+viii
  3. Search `tautological control systems` on YouTube for the 17 hour version.

What is the "problem"?

- Why is a different framework needed from what is already out there?
- Let us consider the simplest possible illustration of this.
  - If one has a vector field X on a manifold M with an equilibrium point $x_0 \in \mathsf{M}$, the notions of "linearisation of X about x_0" and "linear stability of X at x_0" are unambiguous, i.e., understood in a coordinate-invariant way.
  - The same is not true of "linearisation of control systems," "linear controllability of control systems," and "linear stabilisability of control systems."

What is the “problem”? (cont’d)

Example

Consider the two control-affine systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), \end{aligned} \qquad \begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), \end{aligned}$$

- The systems are related by a simple feedback transformation and have the same trajectories.
- The system on the left has a linearisation that is neither controllable nor stabilisable and the linearisation on the right is controllable (and so stabilisable).

Conclusion: The standard notions of linearisation, linear controllability, and linear stabilisability are not feedback-invariant.

What is the “problem”? (cont’d)

- *Plea:* Do not try to “figure out” the example, but rather understand that it just says that the usual definitions have a lurking problem.
- Nonlinear control theory is filled with many rather complicated constructions and theorems for doing things like determining when a system is controllable or stabilisable, and for determining the conditions for optimality of an extremal.
- There are likely very few constructions in nonlinear control theory that are feedback-invariant.
- To be able to address fundamental structural problems in control theory, one needs to have a feedback-invariant approach, or else hypotheses and/or conclusions will change with different system representations.

What is the “problem”? (cont’d)

- There are at least two approaches:
  - Make constructions with a given representation, and verify that these are, in fact, feedback-invariant.
  - Develop a methodology that is representation independent.
- The former is rather like making a coordinate construction in differential geometry and showing it, in fact, does not depend on the choice of coordinates, e.g., the linearisation of a vector field about an equilibrium point using the Jacobian in a set of coordinates.
- This approach seems really hard, probably impossible, definitely extremely messy.
- The latter approach is like making constructions in differential geometry that are *a priori* independent of coordinates.
- This latter approach is what we use here. It seems more elegant, but has its own difficulties.

Warning!

We are interested in "feedback-invariance," not "feedback-invariants."

Why differential inclusions do not do what we want

- To eliminate dependence on control parameterisation, it seems natural to use differential inclusions, and my students and I thought about this seriously for a few years.
- While differential inclusions lose the undesirable structure of dependence on control parameterisation, they also lose some useful structure possessed by control systems.
  1. Notions of regularity such as are important in geometric control theory, e.g., smoothness and real analyticity, seem extremely difficult to reproduce in a differential inclusion framework.
  2. A trajectory for a differential inclusion is merely a curve, while for a control system a trajectory is an integral curve of a time-varying vector field which has a flow. Thus a trajectory carries with it variations of initial condition, etc.

Topologies for spaces of vector fields

- An essential ingredient for the framework are effective (meaning with explicit seminorms) locally convex topologies for spaces of vector fields.
- With Jafarpour,[1] we have done the following.
  1. For regularity class $\nu \in \{m, m + \mathrm{lip}, \infty, \omega, \mathrm{hol}\}$, $m \in \mathbb{Z}_{\geq 0}$, we have produced a useful description of a "compact-open type" topology for the set $\Gamma^\nu(\mathsf{TM})$ of C^ν-vector fields on a C^r-manifold M ($r \in \{\infty, \omega, \mathrm{hol}\}$ as required).
  2. For $\nu \neq \omega$ these topologies are the classical topologies that correspond to "uniform convergence of the required number of derivatives on compacta."
  3. For $\nu = \omega$ (an important case in control theory), the topology is not classical, but derived from work of Martineau,[2] Domanski,[3] and Vogt.[4]

[1] *Time-Varying Vector Fields and Their Flows*, Springer-Verlag, 2014

[2] Math. Ann., **163**(1), 62-88, 1966

[3] Cont. Math., **561**, 3-47, 2012

[4] ArXiv:1309.6292, 2013

Topologies for spaces of vector fields (cont'd)

4. With these topologies one can provide useful notions of
   a. continuity for maps from an arbitrary topological space,
   b. measurability for maps from a measurable space (preimages of Borel sets are measurable), and
   c. integrability for maps from a measure space (the classical Bochner integral).

   into $\Gamma^\nu(\mathsf{TM})$.

- Here's a sample (nontrivial) result showing the power of the unified theory for topologising spaces of vector fields.

Theorem

For a time-varying vector field $\mathbb{R} \ni t \mapsto X_t \in \Gamma^\nu(\mathsf{TM})$, the following are equivalent:

1. *X is measurable;*
2. *for each $x \in \mathsf{M}$, the mapping $t \mapsto X_t(x)$ is measurable.*

Control systems

Definition

A $\mathbf{C}^{\nu}$***-control system*** is a triple $(\mathsf{M}, F, \mathcal{C})$ where

1. M is a C^r-manifold,
2. $\mathcal{C}$ is a topological space, and
3. $F\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ is such that
   a. the mapping
   $$\begin{aligned} F^u\colon \mathsf{M} &\to \mathsf{TM} \\ x &\mapsto F(x,u) \end{aligned}$$
   is a C^{ν}-vector field, and
   b. the mapping $\mathcal{C} \ni u \mapsto F^u \in \Gamma^{\nu}(\mathsf{TM})$ is continuous.

- The feature of being able to define all at once the notion of a control system for a long list of regularity classes is a feature of our methodology.

Control systems (cont'd)

- This definition is deceptively simple.

Remarks

1. The condition of being a "C^1-control system" corresponds *exactly* to assumptions common in the literature (that F and its derivative with respect to x be jointly continuous in (x, u)).
2. The condition of being a "C$^\infty$-control system" is rarely seen, but upon reflection is the natural definition for such a notion (it is equivalent to the condition that F and all x-derivatives be jointly continuous in (x, u)).
3. The condition of being a "C$^\omega$-control system" is new and gives, for the first time, a useful notion of what is meant by a "real analytic control system."
4. A control-affine system with C$^\nu$ drift and control vector fields is a C$^\nu$-control system.

Time-varying vector fields

Definition

A measurable time-varying vector field $X\colon t \mapsto X_t \in \Gamma^\nu(\mathsf{TM})$ is ***locally integrally*** $\mathbf{C}^\nu$***-bounded*** if it is locally integrable (in the Bochner sense).

- Again, this definition is deceptively simple.

Remarks

1. The condition of being "locally $\mathrm{C}^{\mathrm{lip}}$-bounded" corresponds *exactly* to the usual hypotheses of the Carathéodory existence and uniqueness theorem (that X be locally integrable and locally Lipschitz in x with local Lipschitz constant bounded by an integrable function).
2. The condition of being "locally C^∞-bounded" is (not obviously) the same as in the original "chronological calculus" paper of Agrachev and Gamkrelidze.[a]

[a] Math. USSR-Sb., **107**(4), 467-532, 1978

Time-varying vector fields (cont'd)

- Two theorems that suggest our definitions of time-varying vector fields and control systems are the "right" ones.

Theorem

If $\nu \geq \mathrm{lip}$ then the flow of a locally integrally C^ν-bounded vector field depends on initial conditions in a C^ν-manner.

- This holds in the real analytic case!

Theorem

For a C^ν-control system $(\mathsf{M}, F, \mathcal{C})$, if $t \mapsto \mu(t) \in \mathcal{C}$ is locally essentially bounded (in the relatively compact bornology), then the open-loop system $(t, x) \mapsto F(x, \mu(t))$ is locally integrally C^ν-bounded.

Tautological control systems (definition)

- The notion of a tautological control system has two main features.
  1. It replaces the parameterised set of vector fields $\{F^u \mid u \in \mathcal{C}\}$ for a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$ with an unparameterised set of vector fields.
  2. Data is defined only locally to systematically deal, e.g., with the fact that flows for control systems are only locally defined.

Definition

A ***C^ν-tautological control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where

1. M is a C^r-manifold and
2. $\mathscr{F}$ is a presheaf of sets of C^ν-vector fields, i.e., $\mathscr{F}$ assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U})$ of vector fields on $\mathcal{U}$ with the property that if $\mathcal{V} \subseteq \mathcal{U}$ then $X|\mathcal{V} \in \mathscr{F}(\mathcal{V})$ for every $X \in \mathscr{F}(\mathcal{U})$.

Tautological control systems (attributes)

Definition

A C^{ν}-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***globally generated*** if there exists a family $\mathscr{X}$ of globally defined vector fields such that

$$\mathscr{F}(\mathcal{U}) = \{X|\mathcal{U} \mid X \in \mathscr{X}\}.$$

Definition

A C^{ν}-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***complete*** if $\mathscr{F}$ is a sheaf of sets of C^{ν}-vector fields, i.e., if, for every open $\mathcal{U} \subseteq \mathsf{M}$, every open cover $(\mathcal{U}_a)_{a\in A}$ of $\mathcal{U}$, and every family $(X_a)_{a\in A}$ of C^{ν}-vector fields on the open sets $\mathcal{U}_a$, $a \in A$, satisfying

$$X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b,$$

there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X|\mathcal{U}_a = X_a$.

Tautological control systems (examples)

Examples

1. The presheaf $\mathscr{F}(\mathcal{U}) = \Gamma^\nu(\mathsf{T}\mathcal{U})$ is the ***tangent sheaf*** denoted by $\mathscr{G}^\nu_{\mathsf{TM}}$. This presheaf is complete and not globally generated.
2. Consider a C^ν-control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$, and define a C^ν-tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ by
$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \mid u \in \mathcal{C}\}.$$
This system is obviously globally generated. It is seldom complete.
3. Let $(\mathcal{U}_a)_{a\in A}$ be an open cover of M with $X_a \in \Gamma^\nu(\mathsf{T}\mathcal{U}_a)$, $a \in A$. Define
$$\mathscr{F}(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \mathcal{U} \subseteq \mathcal{U}_a,\ X = X_a|\mathcal{U} \text{ for some } a \in A\}.$$
This system is generally neither globally generated nor complete.

Tautological control systems (examples) (cont'd)

Examples (cont'd)

4. Let $\mathsf{D} \subseteq \mathsf{TM}$ be a C^ν-distribution and define a C^ν-tautological control system $\mathfrak{G}_{\mathsf{D}} = (\mathsf{M}, \mathscr{F}_{\mathsf{D}})$ by

$$\mathscr{F}_{\mathsf{D}}(\mathcal{U}) = \{\mathsf{D}\text{-valued vector fields on } \mathcal{U} \text{ of class } \mathsf{C}^\nu\}.$$

   This system is complete and not globally generated.

5. Given a globally defined tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ define an "ordinary" control system $\Sigma_{\mathfrak{G}}$ with control set $\mathcal{C} = \mathscr{F}(\mathsf{M})$ and dynamics

$$\underbrace{F(x, X) = X(x)}_{\text{This is the tautology!}}$$

Tautological control systems (system correspondence)

- Note that we can go from a control system to a tautological control system back to a control system.
- Note that we can go from a globally defined tautological control system to a control system back to a tautological control system.

Proposition

Given a globally defined tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ and a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$:

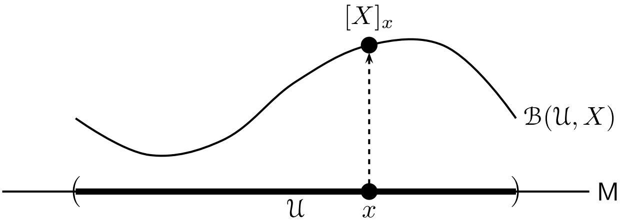
1. $\mathfrak{G}_{\Sigma_{\mathfrak{G}}} = \mathfrak{G}$;
2. $\Sigma_{\mathfrak{G}_{\Sigma}} = \Sigma$ *if the map $u \mapsto F^u$ is an homeomorphism onto its image.*

- We see here the first suggestion that topologies for spaces of vector fields are required in this framework. On this, much more to come.

Tautological control systems (étale space)

- For a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, by $\mathscr{F}_x$ denote the stalk of the presheaf $\mathscr{F}$, i.e., the germs of the locally defined vector fields for the system. For $\mathcal{U} \subseteq \mathsf{M}$ open and $X \in \mathscr{F}(\mathcal{U})$, denote by $[X]_x$ the germ of X.
- By $\mathrm{Et}(\mathscr{F}) = \mathring{\cup}_{x \in \mathsf{M}} \mathscr{F}_x$ denote the ***étale space*** of the presheaf $\mathscr{F}$.
- The ***étale topology*** for $\mathrm{Et}(\mathscr{F})$ has as basis

$$\mathcal{B}(\mathcal{U}, X) = \{[X]_x \mid x \in \mathcal{U}\}, \qquad \mathcal{U} \subseteq \mathsf{M} \text{ open}, \; X \in \mathscr{F}(\mathcal{U}).$$



Tautological control systems (stalk topology)

- For $x \in \mathsf{M}$ and $\mathcal{U}$ a neighbourhood of x, let $r_{\mathcal{U},x} \colon \mathscr{G}^{\nu}_{\mathsf{TM}}(\mathcal{U}) \to \mathscr{G}^{\nu}_{x,\mathsf{TM}}$ be given by $r_{\mathcal{U},x}(X) = [X]_x$.
- $\mathscr{G}^{\nu}_{\mathsf{TM}}(\mathcal{U}) = \Gamma^{\nu}(\mathsf{T}\mathcal{U})$ has the topology mentioned earlier.
- The ***C$^{\nu}$-stalk topology*** for $\mathscr{G}^{\nu}_{x,\mathsf{TM}}$ is the finest locally convex topology such that all maps $r_{\mathcal{U},x}$ are continuous.
- For a C$^{\nu}$-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, the stalk $\mathscr{F}_x \subseteq \mathscr{G}^{\nu}_{x,\mathsf{TM}}$ has the relative topology, which is the ***C$^{\nu}$-stalk topology*** for $\mathscr{F}_x$.
- The étale and stalk topologies are generally not Hausdorff unless $\nu = \omega$.

A “sheafy” generalisation of time-varying vector fields

Definition

Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. For $\mathcal{W} \subseteq \mathbb{R} \times \mathsf{M}$ open, a ***locally integrally C^ν-bounded étale vector field*** for $\mathfrak{G}$ is a map $\mathscr{X} : \mathcal{W} \to \mathrm{Et}(\mathscr{F})$ such that

1. $\mathscr{X}(t, x) \in \mathscr{F}_x$ for every $(t, x) \in \mathcal{W}$,
2. for fixed t, the map $x \mapsto \mathscr{X}(t, x)$ is continuous with respect to the étale topology, and
3. for fixed x, the map $t \mapsto \mathscr{X}(t, x)$ is locally (Bochner) integrable with respect to the stalk topology.

- The definition is equivalent to (a precise version of) the following: For each $(t, x) \in \mathcal{W}$ there exists an interval $\mathbb{T} \subseteq \mathbb{R}$, an open set $\mathcal{U} \subseteq \mathsf{M}$, and a locally integrally C^ν-bounded vector field X on $\mathbb{T} \times \mathcal{U}$ such that $\mathbb{T} \times \mathcal{U} \subseteq \mathcal{W}$ and $\mathscr{X}(t, x) = [X_t]_x$.

Tautological control systems (trajectories)

- For "family of vector field" models such as ours, one normally only has trajectories as concatenations of integral curves, i.e., piecewise constant controls.
- Because we have topologies with respect to which we can define integrability, we are able to extend the notion of integral curves to an analog of locally integrable controls.
- By $\mathrm{ev}_x \colon \mathscr{F}_x \to \mathsf{T}_x\mathsf{M}$ we denote the map $\mathrm{ev}_x([X]_x) = X(x)$.

Definition

A ***trajectory*** for a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is an absolutely continuous curve $t \mapsto \xi(t)$ for which there exists an open $\mathcal{W} \subseteq \mathbb{R} \times \mathsf{M}$ and a locally integrally C^ν-bounded étale vector field $\mathscr{X}$ for $\mathfrak{G}$ defined on $\mathcal{W}$ such that $\xi'(t) = \mathrm{ev}_x \circ \mathscr{X}(t, \xi(t))$ for almost every t.

Tautological control systems (trajectory correspondence)

- We can compare the trajectories of control systems and their tautological control systems, and vice versa.

Theorem

If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ is an ordinary control system with $\mathfrak{G}_\Sigma$ the associated tautological control system, then:

1. *trajectories of Σ are trajectories of $\mathfrak{G}_\Sigma$;*
2. *if $u \mapsto F^u$ is continuous, injective, and proper, then trajectories of $\mathfrak{G}_\Sigma$ are trajectories of Σ;*
3. *if $\mathcal{C}$ is a Suslin space and if F is continuous and proper, then trajectories of $\mathfrak{G}_\Sigma$ are trajectories of Σ.*

Tautological control systems (trajectory correspondence) (cont'd)

Theorem

If $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is a globally generated tautological control system with $\Sigma_{\mathfrak{G}}$ the associated ordinary control system, then trajectories of $\mathfrak{G}$ and $\Sigma_{\mathfrak{G}}$ agree.

Corollary

1. *Trajectories of control-affine systems correspond to trajectories of the corresponding tautological control system.*
2. *If Σ is a control system with compact control set, trajectories of Σ correspond to trajectories of the corresponding tautological control system.*

A "sheafy" take on flows

- By $\mathscr{D}\!\mathit{iff}^{\nu}_{\mathsf{M}}$ we denote the groupoid of germs of local diffeomorphisms of M. If $[\Phi]_x$ is the germ of a local diffeomorphism, then $\mathrm{src}([\Phi]_x) = x$ and $\mathrm{tgt}([\Phi]_x) = \Phi(x)$.
- Denote by $\mathscr{D}\!\mathit{iff}^{\nu}_{x,\mathsf{M}} = \mathrm{src}^{-1}(x)$ the ***stalk*** of the groupoid at x.
- We can give $\mathscr{D}\!\mathit{iff}^{\nu}_{\mathsf{M}}$ the étale topology in much the same way as we did for presheaves of sets of vector fields.
- If M and N are C^r-manifolds, we can topologise $C^{\nu}(\mathsf{M};\mathsf{N})$ in a manner generalising our locally convex topologies for spaces of vector fields.
- For $\mathcal{U}$ a neighbourhood of $x \in \mathsf{M}$, we have the restriction map

$$\begin{aligned} r_{\mathcal{U},x} \colon \mathscr{D}\!\mathit{iff}^{\nu}_{\mathsf{M}}(\mathcal{U}) &\to \mathscr{D}\!\mathit{iff}^{\nu}_{x,\mathsf{M}} \\ \Phi &\mapsto [\Phi]_x. \end{aligned}$$

- The ***$\mathbf{C}^{\nu}$-stalk topology*** for $\mathscr{D}\!\mathit{iff}^{\nu}_{x,\mathsf{M}}$ is the finest topology for which all maps $r_{\mathcal{U},x}$ are continuous.

A “sheafy” take on flows (cont’d)

- Let $\nu \geq \mathrm{lip}$.
- Let $\mathcal{W} \subseteq \mathbb{R} \times \mathsf{M}$ be open and let $\mathscr{X}$ be a locally integrally C^ν-bounded étale vector field over $\mathcal{W}$.
- There exists an open $D_{\mathscr{X}} \subseteq \mathbb{R} \times \mathbb{R} \times \mathsf{M}$ and a map $\Phi^{\mathscr{X}} : D_{\mathscr{X}} \to \mathsf{M}$ with the following properties (and others).
  1. For fixed $(t_0, x_0) \in \mathbb{R} \times \mathsf{M}$, the set $\{t \in \mathbb{R} \mid (t, t_0, x_0) \in D_{\mathscr{X}}\}$ is an interval.
  2. For fixed $(t_0, x_0) \in \mathbb{R} \times \mathsf{M}$, $t \mapsto \Phi^{\mathscr{X}}(t, t_0, x_0)$ is an integral curve for $\mathscr{X}$.
  3. For fixed $(t, t_0, x_0) \in D_{\mathscr{X}}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $\mathcal{U} \ni x \mapsto \Phi^{\mathscr{X}}(t, t_0, x) \in \mathsf{M}$ is a C^ν-diffeomorphism onto its image.

A “sheafy” take on flows (cont’d)

- A tautological control system, then, gives rise to a family of local diffeomorphisms, and the union of the germs of these gives an open subset of $\mathscr{D}\mathit{iff}^{\nu}_{\mathsf{M}}$.
- This leads one to a generalisation of control systems.

Definition

Let $\mathsf{G} \rightrightarrows \mathsf{M}$ be a C^{ν}-étale Lie groupoid. A ***control system*** in G is an open submanifold $\Sigma \subseteq \mathsf{G}$.

- It seems as if some properties of control systems can be studied in this very general setting.

What has been done?

- Apart from the basic constructions reported here:
  1. a study of transformations of tautological control systems;
  2. a theory of linearisation (harder than you might think);
  3. the Orbit Theorem for tautological control systems (S. Jafarpour);
  4. the beginning of optimal control theory (weirder than you might think, e.g., cost functions are sheaf morphisms).

What remains to be done?

Almost everything…