Characterisation of flows using locally convex topologies

Andrew D. Lewis

Department of Mathematics and Statistics
Queen's University, Kingston, ON, Canada

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Motivation

- Origins in long ago work on controllability and optimal control for mechanical systems.
- Extending this work in any generality necessitates thinking about controllability and optimality in a more general setting.

Punchline

Understanding controllability and optimality (and maybe stabilisability?) in a general way depend on the character of the reachable set.

> *We want to understand the reachable set as the image of some sort of exponential map.*

A typical theorem about differential equations

- Consider the time-varying, parameter-dependent differential equation
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), p)$$
for $(t, \boldsymbol{x}, p) \in \mathbb{T} \times \mathcal{U} \times \mathcal{M}$ with $\mathbb{T} \subseteq \mathbb{R}$ an interval, $\mathcal{U} \subseteq \mathbb{R}^n$ open, and $\mathcal{M}$ a metric space.
- Need conditions on $\boldsymbol{f}$ to ensure
  1. existence and uniqueness of solutions,
  2. standard semigroup properties of time-dependence,
  3. continuous dependence on parameters, and
  4. some sort of regular dependence on initial condition.

A typical theorem about differential equations (cont'd)

Hypotheses

1. $t \mapsto \boldsymbol{f}(t, \boldsymbol{x}, p)$ is measurable for each $\boldsymbol{x}$ and p;
2. for each compact $K \subseteq \mathcal{U}$ and bounded $B \subseteq \mathcal{M}$, there exists a compact interval $\mathbb{K} \subseteq \mathbb{T}$ and $g_0 \in \mathsf{L}^1(\mathbb{K}; \mathbb{R}_{\geq 0})$ such that
$$\|\boldsymbol{f}(t, \boldsymbol{x}, p)\| \leq g_0(t), \quad (t, \boldsymbol{x}, p) \in \mathbb{K} \times K \times B;$$
3. for each compact $K \subseteq \mathcal{U}$ and bounded $B \subseteq \mathcal{M}$, there exists a compact interval $\mathbb{K} \subseteq \mathbb{T}$ and $g_1 \in \mathsf{L}^1(\mathbb{K}; \mathbb{R}_{\geq 0})$ such that
$$\|\boldsymbol{f}(t, \boldsymbol{x}, p) - \boldsymbol{f}(t, \boldsymbol{y}, p)\| \leq g_1(t)\|\boldsymbol{x} - \boldsymbol{y}\|, \quad (t, (\boldsymbol{x}, \boldsymbol{y}), p) \in \mathbb{K} \times K^2 \times B;$$
4. for each compact $K \subseteq \mathcal{U}$, each compact interval $\mathbb{K} \subseteq \mathbb{T}$, and each $p_0 \in \mathcal{M}$,
$$\lim_{p \to p_0} \int_{\mathbb{K}} \|\boldsymbol{f}(t, \boldsymbol{x}, p) - \boldsymbol{f}(t, \boldsymbol{x}, p_0)\| \, \mathrm{d}t = 0, \qquad \boldsymbol{x} \in K.$$

A typical theorem about differential equations (cont'd)

Conclusions

There exists a maximal open set $D_f \subseteq \mathbb{T}^2 \times \mathcal{U} \times \mathcal{M}$ and a mapping $\Phi^f : D_f \to \mathcal{U}$ such that

1. $J_f(t_0, \boldsymbol{x}_0, p_0) \triangleq \{t \in \mathbb{T} \mid (t, t_0, \boldsymbol{x}_0, p_0) \in D_f\}$ is an interval;
2. $t \mapsto \Phi^f(t, t_0, \boldsymbol{x}_0, p_0)$ is locally absolutely continuous;
3. $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^f(t, t_0, \boldsymbol{x}_0, p_0) = \boldsymbol{f}(t, \Phi^f(t, t_0, \boldsymbol{x}_0, p_0), p_0)$;
4. $\Phi^f(t_0, t_0, \boldsymbol{x}_0, p_0) = \boldsymbol{x}_0$;
5. $\Phi^f(t_2, t_0, \boldsymbol{x}_0, p_0) = \Phi^f(t_2, t_1, \Phi^f(t_1, t_0, \boldsymbol{x}_0, p_0), p_0)$;
6. $\Phi^f(t_0, t_1, \Phi^f(t_1, t_0, \boldsymbol{x}, p), p) = \boldsymbol{x}$;
7. D_f is continuous;
8. $\boldsymbol{x} \mapsto \Phi^f(t, t_0, \boldsymbol{x}, p_0)$ is a bi-Lipschitz homeomorphism onto its image;

A typical theorem about differential equations (cont'd)

Conclusions (cont'd)

9. for $(t_0, \boldsymbol{x}_0, p_0) \in \mathbb{T} \times \mathcal{U} \times \mathcal{M}$ and for $\epsilon \in \mathbb{R}_{>0}$, there exists an open interval $t_0 \in \mathbb{T}' \subseteq \mathbb{T}$, a neighbourhood $\mathcal{V}$ of $\boldsymbol{x}_0$, and a neighbourhood $\mathcal{O}$ of p_0 such that

$$\begin{aligned} \sup J_{\boldsymbol{f}}(t, \boldsymbol{x}, p) &> \sup J_{\boldsymbol{f}}(t_0, \boldsymbol{x}_0, p_0) - \epsilon, \\ \inf J_{\boldsymbol{f}}(t, \boldsymbol{x}, p) &< \inf J_{\boldsymbol{f}}(t_0, \boldsymbol{x}_0, p_0) + \epsilon \end{aligned}$$

   for all $(t, \boldsymbol{x}, p) \in \mathbb{T}' \times \mathcal{U} \times \mathcal{O}$.

Questions

1. Can the hypotheses be stated compactly?
2. Can the conclusions be stated compactly?
3. Can the results be extended beyond Lipschitz regularity?

The main point of this talk

Answer

All of these questions, and more, can be answered by taking a wide diversion that unifies and clarifies the meaning of "time-varying, parameter-dependent vector field" and "flow of same".

Topologies for spaces of vector fields

- An essential ingredient for the framework are effective (meaning with explicit seminorms) locally convex topologies for spaces of vector fields, cf. the "chronological calculus" of Agrachev, et al.[1]
- With Jafarpour,[2] we have done the following.
  1. For regularity class $\nu \in \{m, m + \mathrm{lip}, \infty, \omega, \mathrm{hol}\}$, $m \in \mathbb{Z}_{\geq 0}$, we have produced a useful description of a "compact-open type" topology for the set $\Gamma^\nu(\mathsf{TM})$ of C^ν-vector fields on a C^r-manifold M ($r \in \{\infty, \omega, \mathrm{hol}\}$ as required).
  2. For $\nu \neq \omega$ these topologies are the classical topologies that correspond to "uniform convergence of the required number of derivatives on compacta."
  3. For $\nu = \omega$, the topology is not classical, but derived from work of Martineau,[3] Domanski,[4] and Vogt.[5]

[1] e.g., Math. USSR-Sb., **107**(4), 467-532, 1978

[2] *Time-Varying Vector Fields and Their Flows*, Springer-Verlag, 2014

[3] Math. Ann., **163**(1), 62-88, 1966

[4] Cont. Math., **561**, 3-47, 2012

[5] ArXiv:1309.6292, 2013

Topologies for spaces of vector fields (cont'd)

4. With these topologies one can provide useful notions of
   a. continuity for maps from an arbitrary topological space,
   b. measurability for maps from a measurable space (preimages of Borel sets are measurable), and
   c. integrability for maps from a measure space (the classical Bochner integral).

   into $\Gamma^\nu(\mathsf{TM})$.

- A time-varying vector field $X\colon \mathbb{T}\times\mathsf{M}\to\mathsf{TM}$ that is class C^ν for t fixed defines a mapping $X\colon\mathbb{T}\to\Gamma^\nu(\mathsf{TM})$ by

$$X(t)(x) = X(t,x) \quad \text{(abusing notation).}$$

Theorem

For a time-varying vector field $X\colon\mathbb{T}\to\Gamma^\nu(\mathsf{TM})$, the following are equivalent:

1. *X is measurable;*
2. *for each $x\in\mathsf{M}$, the mapping $t\mapsto X(t)(x)$ is measurable.*

Topologies for spaces of vector fields (cont'd)

Proof.

Relevant facts:

1. $\Gamma^\nu(\mathsf{TM})$ is a Suslin space;
2. the family of functions $\mathrm{ev}_{\alpha_x} \colon \Gamma^\nu(\mathsf{TM}) \to \mathbb{R}$ given by $\mathrm{ev}_{\alpha_x}(X) = \langle \alpha_x; X(x) \rangle$, $\alpha_x \in \mathsf{T}^*\mathsf{M}$, is point separating.

Now use a result of Thomas on integration in Suslin spaces.[a] $\square$

[a] Trans. Amer. Math. Soc. **212**, 61–81, 1975

Time-varying vector fields

Definition

A ***time-varying vector field of class*** $\mathbf{C}^{\nu}$ is a locally Bochner integrable mapping $X \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^{\nu}(\mathsf{TM}))$.

Remarks

This definition is deceptively simple.

1. The condition $X \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^{\mathrm{lip}}(\mathsf{TM}))$ is (not obviously) the same as the usual hypotheses of the Carathéodory existence and uniqueness theorem.
2. The condition $X \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^{\infty}(\mathsf{TM}))$ is (not obviously) the same as in the original "chronological calculus" paper of Agrachev and Gamkrelidze.[a]

[a] Math. USSR-Sb., **107**(4), 467-532, 1978

Time-varying vector fields (cont'd)

- The following theorem suggests that our definition of time-varying vector fields is the "right" one.

Theorem

If $X \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{TM}))$*,* $\nu \geq \mathrm{lip}$*, then, one gets all of the conditions for a flow from the introduction, plus... the flow of depends on initial conditions in a* C^ν*-manner!*

Punchline

1. The correct class of time-varying vector fields with C^ν-dependence on state is $\mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{TM}))$.
2. In the case $\nu = \mathrm{lip}$, this reduces to the hypotheses listed in the introduction.
3. *But this works for all regularity classes!*

Time-varying, parameter-dependent vector fields

- First topologise $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{TM}))$ using seminorms

$$p^\nu_{\mathbb{K}}(X) = \int_{\mathbb{K}} p^\nu \circ X(t)\,\mathrm{d}t,$$

  where p^ν is a seminorm for $\Gamma^\nu(\mathsf{TM})$ and where $\mathbb{K} \subseteq \mathbb{T}$ is compact.
- Let $\mathcal{P}$ be an arbitrary (!) topological space.
- A time-varying, parameter-dependent vector field $X\colon \mathbb{T} \times \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ which is in $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{TM}))$ for p fixed defines a mapping $X\colon \mathcal{P} \to \mathsf{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{TM}))$ by

$$X(p)(t,x) = X(t,x,p) \ \ \text{(abusing notation).}$$

Time-varying, parameter-dependent vector fields (cont'd)

Definition

A ***time-varying, parameter-dependent vector field of class*** $\mathbf{C}^{\nu}$ is a continuous mapping $X \in \mathrm{C}^0(\mathcal{P}; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(\mathsf{TM})))$.

Remarks

This definition is deceptively simple.

1. The condition that $X \in \mathrm{C}^0(\mathcal{M}; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\mathrm{lip}}(\mathsf{TM})))$ corresponds *exactly* to the usual hypotheses of the existence of flows with parameter dependence (from the introduction).
2. The conditions for regularity $\nu > \mathrm{lip}$ are seldom produced and look complicated when written in a concrete form.

Time-varying, parameteter-dependent vector fields (cont'd)

- The following theorem suggests that our definition of time-varying, parameter-dependent vector field is the “right” one.

Theorem

If $X \in \mathrm{C}^0(\mathcal{P}; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{TM})))$*,* $\nu \geq \mathrm{lip}$*, then, one gets all of the conditions for a flow from the introduction (including continuous dependence of flow on parameter), plus... the flow of depends on initial conditions in a* C^ν*-manner!*

Punchline

1. The correct class of time-varying, parameter-dependent vector fields with C^ν-dependence on state is $\mathrm{C}^0(\mathcal{P}; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{TM})))$.
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The problem of the exponential map

- The machinery allows one to define an exponential map for the set of time-varying, parameter-dependent vector fields.
- In a Panglossian universe:

$$\begin{aligned}\exp\colon \{&\text{time-varying, parameter-dependent vector fields}\}\\ &\to \{\text{parameter-dependent diffeomorphisms}\}\end{aligned}$$

  defined by

$$\exp(X)(x,p) = \Phi^X(1,0,x,p).$$

- No such map exists dues to lack of completeness of flows.
- Kludges...
  1. Assume completeness 🤨
  2. Use cutoff function to force compact supported 🤮

The problem of the exponential map (cont'd)

- One can overcome this by "localising" everything, using sheaves.
- Let $\mathscr{G}^{\nu}_{\mathsf{CLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})$ be the sheaf over $\mathbb{T}\times\mathsf{M}\times\mathcal{P}$ whose stalk at (t,x,p) is the set of germs of time-varying, parameter-dependent vector fields about (t,x,p).
- Let $\mathscr{LocFlow}^{\nu}(\mathbb{T};\mathsf{M};\mathcal{P})$ be the "sheaf of local flows" whose stalk at (t,x,p) is the set of germs of parameter-dependent local flows about (t,x,p) (work not shown).
- Then...
  1. topologise everything in sight,
  2. use the standard existence of local flows,
  3. pass to the appropriate direct limit, then
  4. define the "stalk exponential map"

$$\exp_{(t,x,p)}\colon \mathscr{G}^{\nu}_{\mathsf{CLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})_{(t,x,p)}\to\mathscr{LocFlow}^{\nu}(\mathbb{T};\mathsf{M};\mathcal{P})_{(t,x,p)}.$$

  5. Gives the exponential mapping as a sheaf morphism $\exp\colon\mathscr{G}^{\nu}_{\mathsf{CLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})\to\mathscr{LocFlow}^{\nu}(\mathbb{T};\mathsf{M};\mathcal{P})$.

The problem of the exponential map (cont'd)

- What we know right now: $\exp$ is well defined and is a mapping of topological sheaves.
- What is likely true: $\exp$ is an isomorphism of topological sheaves.

Punchline

One can replace the lengthy hypotheses and conclusions of the introduction with the concise and more general statement:

> $\exp\colon \mathscr{G}^{\nu}_{\mathsf{CLI}}(\mathbb{T};\mathsf{TM};\mathcal{P}) \to \mathscr{LocFlow}^{\nu}(\mathbb{T};\mathsf{M};\mathcal{P})$ *is an isomorphism of topological sheaves.*

The drawback is that the definition of all symbols involved is difficult and a little complicated. But…the constructions are quite natural.

So what?

Application

A control system defines a subsheaf $\mathscr{F}$ of $\mathscr{G}^{\nu}_{\mathrm{CLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$. The local structure of the reachable set is described by $\exp|\mathscr{F}$. Many interesting structural properties of the system are contained in this local structure, e.g., controllability, stabilisability, optimality.