Geodesic invariance and the symmetric product

Mechanics, control theory, and geometry

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The principal objects of interest

Let M be a smooth manifold and let ∇ be a smooth affine connection on M.

Definition

The ***symmetric product*** is the $\mathbb{R}$-bilinear operator $\langle \cdot : \cdot \rangle \colon \Gamma^\infty(\mathrm{TM}) \times \Gamma^\infty(\mathrm{TM}) \to \Gamma^\infty(\mathrm{TM})$ given by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

Further let $\mathrm{D} \subseteq \mathrm{TM}$ be a smooth distribution (constant rank).

Definition

The distribution D is ***geodesically invariant*** if, for every $x \in \mathrm{M}$, a geodesic $t \mapsto \gamma(t)$ satisfying $\gamma'(0) \in \mathrm{D}_x$ is such that $\gamma'(t) \in \mathrm{D}_{\gamma(t)}$ for all t.

The talk will be how these two notions arise in various ways in mechanics, control theory, and geometry.

Control theory (generalities)

- We consider a ***control-affine system***:

$$\xi'(t) = f_0 \circ \xi(t) + \sum_{j=1}^{m} \mu_j(t) f_j \circ \xi(t). \tag{1}$$

  Here we have:

  1. a ***state manifold*** M;
  2. the ***drift vector field*** $f_0 \in \Gamma^\infty(\mathsf{TM})$;
  3. ***control vector fields*** $f_1, \dots, f_m \in \Gamma^\infty(\mathsf{TM})$;
  4. a ***control*** $\boldsymbol{\mu} \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \mathbb{R}^m)$ defined on some interval $\mathbb{T} \subseteq \mathbb{R}$;
  5. a ***controlled trajectory*** $\xi \colon \mathbb{T} \to \mathsf{M}$.

- The ***reachable set*** from x_0 in time $T \in \mathbb{R}_{\geq 0}$ is

$$\mathcal{R}_T(x_0) = \{\xi(T) \mid \xi \text{ satisfies (1) with } \xi(0) = x_0 \text{ for some } \boldsymbol{\mu} \in \mathsf{L}^1([0,T]; \mathbb{R}^m)\}$$

  The ***reachable set*** from x_0 in time at most T is $\mathcal{R}_{\leq T}(x_0) = \cup_{t \in [0,T]} \mathcal{R}_t(x_0)$.

Control theory (generalities) (cont'd)

Accessibility theory

- For $x \in \mathsf{M}$, let

$$\mathsf{L}^{(\infty)}(f_0, f_1, \dots, f_m)_x = \operatorname{span}_{\mathbb{R}}([f_{j_1}[f_{j_2}, \cdots [f_{j_{k-1}}, f_{j_k}]]](x) \mid j_1, \dots, j_k \in \{0, 1, \dots, m\},\ k \in \mathbb{Z}_{>0});$$

$\mathsf{L}^{(\infty)}(f_0, f_1, \dots, f_m)$ is the smallest involutive distribution generated by the vector fields $f_0, f_1, \dots, f_m$.

Theorem (Sussmann/Jurdjevic[1])

1. *If* $\mathsf{L}^{(\infty)}(f_0, f_1, \dots, f_m)_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$*, then* $\operatorname{int}(\mathcal{R}_{\leq T}(x_0)) \neq \varnothing$ *for* $T \in \mathbb{R}_{>0}$.
2. *If* M *and* $f_0, f_1, \dots, f_m$ *are real analytic, then the converse is true.*

[1] *J. Differential Equations*, **12**(1), 95–116, 1972

Control theory (for mechanical systems)

- We work with a special class of control-affine systems. We let
  1. M be a smooth manifold,
  2. ∇ be a smooth affine connection on M, and
  3. $Y_1, \dots, Y_m \in \Gamma^\infty(\mathsf{TM})$.

  For a control $\boldsymbol{\mu} \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \mathbb{R}^m)$, we have the differential equation

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{j=1}^{m} \mu_j(t) Y_j \circ \gamma(t).$$

- If we first-orderify, we have
  1. "$\mathsf{M} = \mathsf{TM}$,"
  2. $f_0 = Z_\nabla$ (the geodesic spray), and
  3. $f_j = Y_j^{\mathsf{V}}, j \in \{1, \dots, m\}$ (vertical lifts),

  and the differential equations are

$$\Upsilon'(t) = Z_\nabla \circ \Upsilon(t) + \sum_{j=1}^{m} \mu_j(t) Y_j^{\mathsf{V}} \circ \Upsilon(t).$$

Control theory (for mechanical systems) (cont'd)
Accessibility theory

- We want to compute $\mathsf{L}^{(\infty)}(Z_\nabla, Y_1^{\mathsf{v}}, \dots, Y_m^{\mathsf{v}})_{v_x}$ for $v_x \in \mathsf{TM}$.
- If possible, we want to replace Lie brackets of $Z_\nabla, Y_1^{\mathsf{v}}, \dots, Y_m^{\mathsf{v}}$ with lifts of constructions on M.
- Of particular interest is the case of $v_x = 0_x$, where many Lie brackets evaluate to zero.
- A key formula is a simple computation:

$$[Y_{j_1}^{\mathsf{v}}, [Z_\nabla, Y_{j_2}^{\mathsf{v}}]] = \langle Y_{j_1} : Y_{j_2} \rangle^{\mathsf{v}}.$$

- Other formulae, noting that $\mathsf{T}_{0_x}\mathsf{TM} \simeq \mathsf{T}_x\mathsf{M} \oplus \mathsf{T}_x\mathsf{M}$:[2]

$$[Z_\nabla, Y_j^{\mathsf{v}}](0_x) = Y_j(x) \oplus 0_x, \quad [[Z_\nabla, Y_{j_1}^{\mathsf{v}}], [Z_\nabla, Y_{j_2}^{\mathsf{v}}]]_{0_x} = [Y_{j_1}, Y_{j_2}] \oplus 0_x.$$

- Let $\mathscr{S}^{(\infty)}(Y_1, \dots, Y_m)$ be the set of all iterated symmetric products of the vector fields $Y_1, \dots, Y_m$.
- Let $\mathscr{L}^{(\infty)}(\mathscr{S}^{(\infty)}(Y_1, \dots, Y_m))$ be the set of all iterated Lie brackets of vector fields from $\mathscr{S}^{(\infty)}(Y_1, \dots, Y_m)$.
- For $x \in \mathsf{M}$, denote $\mathsf{S}^{(\infty)}(Y_1, \dots, Y_m)_x = \mathrm{span}_{\mathbb{R}}(X(x) \mid X \in \mathscr{S}^{(\infty)}(Y_1, \dots, Y_m))$.

[2] Order is horizontal⊕vertical.

Control theory (for mechanical systems) (cont'd)

Accessibility theory (cont'd)

Theorem (L/Murray[3])

Noting that $\mathsf{T}_{0_{x_0}}\mathsf{TM} \simeq \mathsf{T}_{x_0}\mathsf{M} \oplus \mathsf{T}_{x_0}\mathsf{M}$,

$$\mathsf{L}^{(\infty)}(Z_\nabla, Y_1^{\mathsf{v}}, \dots, Y_m^{\mathsf{v}})_{0_{x_0}} = \mathsf{L}^{(\infty)}(\mathscr{S}^\infty(Y_1, \dots, Y_m))_{x_0} \oplus \mathsf{S}^{(\infty)}(Y_1, \dots, Y_m)_{x_0}.$$

Proof.

Tedious induction. □

Corollary

1. *If* $\mathsf{S}^{(\infty)}(Y_1, \dots, Y_m)_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$, *then* $\operatorname{int}(\mathcal{R}_{\leq T}(0_{x_0})) \neq \varnothing$.
2. *If* $\mathsf{L}^{(\infty)}(\mathscr{S}^{(\infty)}Y_1, \dots, Y_m))_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$, *then* $\operatorname{int}(\pi_{\mathsf{TM}}(\mathcal{R}_{\leq T}(0_{x_0}))) \neq \varnothing$.
3. *If* M, ∇, *and* $Y_1, \dots, Y_m$ *are real analytic, then the converses are true.*

[3] *SIAM J. Control Optim.*, **35**(3), 766–790, 1997

Control theory (punchline)

The symmetric product features prominently in the control theory for mechanical systems.

Geometry (affine connections and distributions)

- The preceding constructions beg the following question: *What is the meaning of a distribution being closed under symmetric product?*

We make some definitions, including one we had previously.

Definition

Let M be a smooth manifold, let ∇ be a smooth affine connection, and let D be a smooth distribution (constant rank).

1. D is ***geodesically invariant*** if, for every $x \in$ M, a geodesic $t \mapsto \gamma(t)$ satisfying $\gamma'(0) \in \mathsf{D}_x$ is such that $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ for all t.
2. D is ***totally geodesic*** if it is integrable and geodesically invariant.

Suppose that D has a complement D′ with P and P' the projections onto D and D′.

Definition

The ***second fundamental form*** for D is the section of $\mathsf{T}^2(\mathsf{D}) \otimes \mathsf{D}'$ defined by

$$S_{\mathsf{D}}(X, Y) = -(\nabla_X P')(Y).$$

Geometry (affine connections and distributions) (cont'd)

Theorem (L[4])

TFAE:

1. D *is geodesically invariant;*
2. D *is closed under* $\langle \cdot : \cdot \rangle$*;*
3. S_{D} *is skew-symmetric.*

Theorem

TFAE:

1. D *is totally geodesic;*
2. D *is closed under* $\langle \cdot : \cdot \rangle$ *and* $[\cdot, \cdot]$*.*

If ∇ *is torsion-free, these conditions are equivalent to:*

3. $S_{\mathsf{D}} = 0$*;*
4. D *is closed under* ∇*.*

[4] *Rep. Math. Phys.*, **42**(1/2), 135–164, 1998

Geometry (infinitesimal characterisation of symmetric product)

- For the Lie bracket, we have the following infinitesimal formula:

$$[X, Y](x) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \Phi^Y_{-t} \circ \Phi^X_{-t} \circ \Phi^Y_t \circ \Phi^X_t(x).$$

Letting X^{h} be the horizontal lift, we have the following infinitesimal characterisation of the symmetric product.

Theorem (Barbero-Liñán/L[5])

$$\langle X : Y \rangle^{\mathrm{v}}(v_x) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \Phi^{Y^{\mathrm{v}}}_{-t} \circ \Phi^{X^{\mathrm{h}}}_{-t} \circ \Phi^{Y^{\mathrm{v}}}_t \circ \Phi^{X^{\mathrm{h}}}_t \circ \Phi^{X^{\mathrm{v}}}_{-t} \circ \Phi^{Y^{\mathrm{h}}}_{-t} \circ \Phi^{X^{\mathrm{v}}}_t \circ \Phi^{Y^{\mathrm{h}}}_t(v_x)$$

Proof.

Understand compositions of flows. □

[5] *Int. J. Geom. Methods Mod. Phys.*, **9**(8), 1250073, 2012

Geometry (infinitesimal characterisation of symmetric product) (cont'd)

- The formula from the theorem can be modified in various ways.
- For a curve $\gamma\colon \mathbb{T} \to \mathsf{M}$, let $\tau_\gamma^{(t,s)}\colon \mathsf{T}_{\gamma(s)}\mathsf{M} \to \mathsf{T}_{\gamma(t)}\mathsf{M}$, $s, t \in \mathbb{T}$, be parallel transport.
- For an affine connection ∇, let

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T_\nabla(X, Y)$$

define the torsion-free affine connection with the same geodesics as ∇.

- Let $\overline{X}^{\mathsf{h}}$ and $\overline{\tau}_\gamma^{(t,s)}$ denote horizontal lift and parallel transport for $\overline{\nabla}$, respectively.

1. By definition of parallel transport, if γ and η are the integral curves of X and Y through x:

$$\langle X : Y\rangle^{\mathsf{v}}(v_x) = \frac{1}{2} \frac{\mathsf{d}^2}{\mathsf{d}t^2}\bigg|_{t=0} \Phi_{-t}^{Y^{\mathsf{v}}} \circ \tau_\gamma^{(0,t)} \circ \Phi_t^{Y^{\mathsf{v}}} \circ \tau_\gamma^{(t,0)} \circ \Phi_{-t}^{X^{\mathsf{v}}} \circ \tau_\eta^{(0,t)} \circ \Phi_t^{X^{\mathsf{v}}} \circ \tau_\gamma^{(t,0)}(v_x)$$

Geometry (infinitesimal characterisation of symmetric product) (cont'd)

2. By understanding the relationship between $\tau_\gamma^{(t,s)}$ and $\overline{\tau}_\gamma^{(t,s)}$, one shows that

$$\langle X : Y\rangle^{\mathsf{v}}(v_x) = \frac{1}{2}\,\frac{\mathsf{d}^2}{\mathsf{d}t^2}\bigg|_{t=0} \Phi_{-t}^{Y^{\mathsf{v}}} \circ \overline{\tau}_\gamma^{(0,t)} \circ \Phi_t^{Y^{\mathsf{v}}} \circ \overline{\tau}_\gamma^{(t,0)} \circ \Phi_{-t}^{X^{\mathsf{v}}} \circ \overline{\tau}_\eta^{(0,t)} \circ \Phi_t^{X^{\mathsf{v}}} \circ \overline{\tau}_\eta^{(t,0)}(v_x).$$

3. Therefore, again using the definition of horizontal lift,

$$\langle X : Y\rangle^{\mathsf{v}}(v_x) = \frac{1}{2}\,\frac{\mathsf{d}^2}{\mathsf{d}t^2}\bigg|_{t=0} \Phi_{-t}^{Y^{\mathsf{v}}} \circ \Phi_{-t}^{\overline{X}^{\mathsf{h}}} \circ \Phi_t^{Y^{\mathsf{v}}} \circ \Phi_t^{\overline{X}^{\mathsf{h}}} \circ \Phi_{-t}^{X^{\mathsf{v}}} \circ \Phi_{-t}^{\overline{Y}^{\mathsf{h}}} \circ \Phi_t^{X^{\mathsf{v}}} \circ \Phi_t^{\overline{Y}^{\mathsf{h}}}(v_x).$$

4. In the parallel transport formula above, γ and η can be replaced with the geodesics γ_X and γ_Y (of $\overline{\nabla}$, and so also of ∇) with initial condition $X(x)$ and $Y(x)$:

$$\langle X : Y\rangle^{\mathsf{v}}(v_x) = \frac{1}{2}\,\frac{\mathsf{d}^2}{\mathsf{d}t^2}\bigg|_{t=0} \Phi_{-t}^{Y^{\mathsf{v}}} \circ \overline{\tau}_{\gamma_X}^{(0,t)} \circ \Phi_t^{Y^{\mathsf{v}}} \circ \overline{\tau}_{\gamma_X}^{(t,0)} \circ \Phi_{-t}^{X^{\mathsf{v}}} \circ \overline{\tau}_{\gamma_Y}^{(0,t)} \circ \Phi_t^{X^{\mathsf{v}}} \circ \overline{\tau}_{\gamma_Y}^{(t,0)}(v_x).$$

Geometry (infinitesimal characterisation of symmetric product) (cont'd)

Proof of geodesically invariant $\iff$ closed under $\langle \cdot : \cdot \rangle$.

Key is the equivalence of the following:

1. D is geodesically invariant;
2. Z_∇ is tangent to D;
3. X^{v} is tangent to D for all $X \in \Gamma^\infty(\mathsf{D})$;
4. X^{h} is tangent to D for all $X \in \Gamma^\infty(\mathsf{D})$.

Barbero-Liñán/L give intrinsic proofs of these formulae. One then uses the formula

$$\langle X : Y \rangle^{\mathrm{v}}(v_x) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \Phi^{Y^{\mathrm{v}}}_{-t} \circ \tau^{(0,t)}_{\gamma_X} \circ \Phi^{Y^{\mathrm{v}}}_{t} \circ \tau^{(t,0)}_{\gamma_X} \circ \Phi^{X^{\mathrm{v}}}_{-t} \circ \tau^{(0,t)}_{\gamma_Y} \circ \Phi^{X^{\mathrm{v}}}_{t} \circ \tau^{(t,0)}_{\gamma_Y}(v_x)$$

along with a suitable characterisation of what it means for a vector field on the total space of a vector bundle to be tangent to a subbundle. □

Geometry (punchline)

To understand and make use of the symmetric product and geodesic invariance, composition of flows is important.

Mechanics (constrained connection)

- Let $(\mathsf{M}, \mathbb{G})$ be a Riemannian manifold so $v_x \mapsto \frac{1}{2}\mathbb{G}(v_x, v_x)$ is the ***kinetic energy function***.
- The motion $t \mapsto \gamma(t)$ of the mechanical system with this kinetic energy is

$$\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) = 0.$$[6]

- We wish to subject the system to a ***nonholonomic constraint***, by which we mean that we require that $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ for all t, where D is the ***constraint distribution***.

Physics

The force that maintains the constraint $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ does no work on admissible motions.

Mathematics

There exists $t \mapsto \lambda(t) \in \mathsf{D}^{\perp}_{\gamma(t)}$ such that

$$\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) = \lambda(t),$$

$$P^{\perp}_{\mathsf{D}} \circ \gamma'(t) = 0.$$

[6] Potential energy can be inhcluded in all of this, but I am omitting it for simplicity.

Mechanics (constrained connection) (cont'd)

A little calculation:

$$\begin{aligned}&\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) = \lambda(t),\\ &P^{\perp}_{\mathsf{D}} \circ \gamma'(t) = 0\end{aligned} \quad \Longrightarrow \quad \begin{aligned}&P^{\perp}_{\mathsf{D}}(\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t)) = \lambda(t),\\ &(\nabla^{\mathbb{G}}_{\gamma'(t)}P^{\perp}_{\mathsf{D}})(\gamma'(t)) + P^{\perp}_{\mathsf{D}}(\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t)) = 0\end{aligned}$$

$$\Longrightarrow \underbrace{\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) + (\nabla^{\mathbb{G}}_{\gamma'(t)}P^{\perp}_{\mathsf{D}})(\gamma'(t))}_{\nabla^{\mathsf{D}}_{\gamma'(t)}\gamma'(t)} = 0$$

Theorem (L[7])

TFAE:

1. γ *satisfies the constrained equations of motion;*
2. γ *is a geodesic for the* ***constrained connection***

$$\nabla^{\mathsf{D}}_X Y = \nabla^{\mathbb{G}}_X Y + (\nabla^{\mathbb{G}}_X P^{\perp}_{\mathsf{D}})(Y)$$

with initial condition in D.

[7] *Rep. Math. Phys.*, **42**(1/2), 135–164, 1998

Mechanics (constrained connection) (cont'd)

- Note:
  1. D is geodesically invariant for ∇^{D};
  2. $\Gamma^\infty(\mathsf{TM}) \times \Gamma^\infty(\mathsf{D}) \ni (X, Y) \mapsto \nabla^{\mathsf{D}}_X Y \in \Gamma^\infty(\mathsf{D})$ is a vector bundle connection in D (this is stronger than geodesic invariance);
  3. for $Y \in \Gamma^\infty(\mathsf{D})$, $\nabla^{\mathsf{D}}_X Y = P_{\mathsf{D}}(\nabla^{\mathbb{G}}_X Y)$.

Another little calculation:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathbb{G}(\gamma'(t), \gamma'(t)) \\
&= \frac{1}{2}(\nabla^{\mathsf{D}}_{\gamma'(t)}\mathbb{G})(\gamma'(t), \gamma'(t)) + \mathbb{G}(\nabla^{\mathsf{D}}_{\gamma'(t)}\gamma'(t), \gamma'(t)) \\
&= -\mathbb{G}((\nabla^{\mathbb{G}}_{\gamma'(t)}P^{\perp}_{\mathsf{D}})(\gamma'(t)), \gamma'(t)) + \mathbb{G}(\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t), \gamma'(t)) + \mathbb{G}((\nabla^{\mathbb{G}}_{\gamma'(t)}P^{\perp}_{\mathsf{D}})(\gamma'(t)), \gamma'(t)) \\
&= 0.
\end{aligned}
$$

Energy is conserved!

- One can play a variety of mechanical games in this affine connection framework.

Mechanics (calculus of variations with constraints)

- The geodesics of the constrained connection are *not* extremals for the natural constrained variational problem. But it is interesting to compare these two things.[8]
- Action for $\gamma \in \mathsf{H}^1([t_0, t_1]; \mathsf{M}; x_0, x_1)$ is

$$A_{\mathbb{G}}(\gamma) = \int_{t_0}^{t_1} \tfrac{1}{2}\mathbb{G}(\gamma'(t), \gamma'(t))\,\mathrm{d}t.$$

Action restricted to curves satisfying the constraint is $A_{\mathbb{G},\mathsf{D}}$.

Problem (Nonholonomic (N))

Find $\gamma \in \mathsf{H}^1([t_0, t_1]; \mathsf{M}; \mathsf{D}; x_0, x_1)$ *such that*

$$\langle \mathrm{d}A_{\mathbb{G}}; \delta\rangle = 0,$$

$$\delta \in \mathsf{H}^1([t_0, t_1]; \gamma^*\mathsf{D}; x_0, x_1).$$

Problem (Variational (V))

Find $\gamma \in \mathsf{H}^1([t_0, t_1]; \mathsf{M}; \mathsf{D}; x_0, x_1)$ *such that*

$$\langle \mathrm{d}A_{\mathbb{G},\mathsf{D}}; \delta\sigma(0)\rangle = 0,$$

$$\sigma\colon (-\epsilon, \epsilon) \to \mathsf{H}^1([t_0, t_1]; \mathsf{M}; \mathsf{D}; x_0, x_1).$$

[8] L, *J. Geom. Mech.*, **12**(2), 165–308, 2020

Mechanics (calculus of variations with constraints) (cont'd)

- Some notation:
  1. ***Fröbenius curvature:*** $F_{\mathsf{D}}(X,Y) = P_{\mathsf{D}}^{\perp}([X,Y]) \quad (X,Y \in \Gamma^r(\mathsf{D}))$
  2. ***geodesic curvature:*** $G_{\mathsf{D}}(X,Y) = P_{\mathsf{D}}^{\perp}(\langle X : Y\rangle) \quad (X,Y \in \Gamma^r(\mathsf{D}))$

Problem (N) is equivalent to:

$$\nabla^{\mathsf{D}}_{\gamma'}\gamma' = 0$$

Problem (V) is (sort of) equivalent to:

$$\begin{aligned} \nabla^{\mathsf{D}}_{\gamma'}\gamma' &= F^{*}_{\mathsf{D}}(\gamma')(\lambda), \\ \nabla^{\mathsf{D}^{\perp}}_{\gamma'}\lambda &= \frac{1}{2}G_{\mathsf{D}}(\gamma',\gamma') + \frac{1}{2}G^{\star}_{\mathsf{D}^{\perp}}(\gamma')(\lambda) + \frac{1}{2}F^{\star}_{\mathsf{D}^{\perp}}(\gamma')(\lambda). \end{aligned} \tag{2}$$

Problem

Given a physical motion $t \mapsto \gamma(t)$ satisfying Problem (N), find all (if any) initial conditions for λ so that the resulting solution to (2) satisfies $F^{*}_{\mathsf{D}}(\gamma')(\lambda) = 0$.

Mechanics (calculus of variations with constraints) (cont'd)

When D and $\mathsf{D}^\perp$ are geodesically invariant for $\nabla^{\mathbb{G}}$, then these simplify to

$$\nabla^{\mathsf{D}}_{\gamma'}\gamma' = 0$$

and

$$\nabla^{\mathsf{D}}_{\gamma'}\gamma' = F^*_{\mathsf{D}}(\gamma')(\lambda),$$
$$\nabla^{\mathsf{D}^\perp}_{\gamma'}\lambda = \frac{1}{2}F^\star_{\mathsf{D}^\perp}(\gamma')(\lambda).$$

Questions

1. What is the significance of geodesic invariance?
2. Can one simply characterise the equivalence of Problems (N) and (V) in this case?

Mechanics (punchline)

The symmetric product and geodesic invariance show up, sometimes in not understood ways, in nonholonomic mechanics.

In closing...

THE END! THANK YOU!