Chapter 2

The Weierstrass Preparation Theorem and applications

In this chapter we start by stating and proving the Weierstrass Preparation Theorem. This theorem has some deep consequences for the local structure of holomorphic and real analytic functions, and we begin the presentation of these ideas in this chapter.

2.1 The Weierstrass Preparation Theorem

We now turn to one of the most important structural results for holomorphic and real analytic functions, the Weierstrass Preparation Theorem. In this section we shall setup, state, and prove the result. In subsequent sections we shall how the Weierstrass Preparation Theorem lends a great deal of algebraic structure to holomorphic or real analytic functions.

As with Chapter 1, in this chapter we work simultaneously with real and complex functions, and so use the notation of the previous chapter to handle this.

2.1.1 The setup for the Weierstrass Preparation Theorem

The Weierstrass Preparation Theorem is concerned with the behaviour of holomorphic or real analytic functions in one of the variables of which they are a function. It is useful to have some notation for this. We let $\mathcal{U} \subseteq \mathbb{F}^n$ be a neighbourhood of $\mathbf{0}$ and $\mathcal{V} \subseteq \mathbb{F}$ be a neighbourhood of 0. A typical point in $\mathcal{U}$ we shall denote by $\boldsymbol{x}$ and a typical point in $\mathcal{V}$ we shall denote by y. We are interested in functions in $\mathrm{C}^{\mathrm{hol}}(\mathcal{U} \times \mathcal{V})$ (for $\mathbb{F} = \mathbb{C}$) or $\mathrm{C}^{\omega}(\mathcal{U} \times \mathcal{V})$ for $\mathbb{F} = \mathbb{R}$). If a holomorphic or real analytic function $f \colon \mathcal{U} \times \mathcal{V} \to \mathbb{F}$ admits a power series expansion valid on all of $\mathcal{U} \times \mathcal{V}$, then we will write this power series expansion as

$$f(\boldsymbol{x}, y) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} f_{I,j} \boldsymbol{x}^I y^j.$$

Let us define a particular class of real analytic function where its character in one of the variables is polynomial.

2.1.1 Definition (Weierstrass polynomial) Let $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{F}^n \times \mathbb{F}$ be a neighbourhood of $(\mathbf{0}, 0)$.

A holomorphic or real analytic function $W\colon \mathcal{U}\times\mathcal{V}\to\mathbb{F}$ is a ***Weierstrass polynomial*** of degree k if there exists holomorphic or real analytic functions $w_0, w_1, \dots, w_{k-1}\colon \mathcal{U}\to\mathbb{F}$ satisfying

(i) $w_j(\mathbf{0}) = 0$, $j\in\{0,1,\dots,k\}$,

(ii) $W(\boldsymbol{x}, y) = y^k + \cdots + w_1(\mathbf{0})y + w_0(\boldsymbol{x})$ for all $(\boldsymbol{x}, y)\in\mathcal{U}\times\mathcal{V}$. •

The following technical lemmata will be helpful when we subsequently work with the Weierstrass Preparation Theorem. We first provide a class of analytic functions satisfying the hypotheses of the Weierstrass Preparation Theorem.

2.1.2 Definition (Normalised function) For an open subset $\mathcal{U}\times\mathcal{V}\subseteq\mathbb{F}^n\times\mathbb{F}$, a holomorphic or real analytic function $f\colon\mathcal{U}\to\mathbb{F}$ is ***normalised*** if, in a neighbourhood of $(\mathbf{0}, 0)$,

$$f(\boldsymbol{x}, y) = \sum_{I\in\mathbb{Z}^n_{\geq 0}}\sum_{j=0}^{\infty}\alpha_{I,j}\boldsymbol{x}^I y^j,$$

where $\alpha_{\mathbf{0},0} = \alpha_{\mathbf{0},1} = \cdots = \alpha_{\mathbf{0},k-1} = 0$ and $\alpha_{\mathbf{0},k} = 1$ for some $k\in\mathbb{Z}_{>0}$. •

The following lemma says that many functions are, in fact, normalisable in a simple manner.

2.1.3 Lemma (Functions satisfying the hypotheses of the Weierstrass Preparation Theorem) *Let $\mathcal{U}\times\mathcal{V}\subseteq\mathbb{F}^n\times\mathbb{F}$ and let $f_1,\dots,f_k\colon\mathcal{U}\to\mathbb{F}$ be nonzero functions with the property that $f_j(\mathbf{0},0) = 0$, $j\in\{1,\dots,k\}$. Then there exists a unitary transformation $\psi\colon\mathbb{F}^{n+1}\to\mathbb{F}^{n+1}$ such that $\psi^*f_j \triangleq f_j\circ\psi$, $j\in\{1,\dots,k\}$, are normalised. Consequently, there exist $E_1,\dots,E_k$, analytic in a neighbourhood $\mathcal{U}'\times\mathcal{V}'$ of $(\mathbf{0},0)$ and nonzero at $(\mathbf{0},0)$, and Weierstrass polynomials $W_1,\dots,W_k$ analytic in $\mathcal{U}'$ such that $\psi^*f_j(\mathbf{x},y) = E_j(\mathbf{x},y)W_j(\mathbf{x},y)$ for all $(\mathbf{x},y)\in\mathcal{U}'\times\mathcal{V}'$.*

Proof We first claim that there exists an open subset of unit vectors $\boldsymbol{u}\in\mathbb{F}^{n+1}$ such that the functions $a\mapsto f_j(a\boldsymbol{u})$, $j\in\{1,\dots,k\}$, are not identically zero in a neighbourhood of 0. We prove this by induction on k. For $k=1$, suppose otherwise and let $\boldsymbol{u}\in\mathbb{F}^{n+1}$ be a unit vector. Then there exists a neighbourhood $\mathcal{V}$ of $\boldsymbol{u}$ in the unit sphere $\mathbb{S}^n\subseteq\mathbb{F}^{n+1}$ and $\epsilon\in\mathbb{R}_{>0}$ such that $f_1(a\boldsymbol{v}) = 0$ for $\boldsymbol{v}\in\mathcal{V}$ and $a\in(-\epsilon,\epsilon)$. By the Identity Theorem it follows that f_1 is identically zero, contradicting our hypotheses. Thus there is some $\boldsymbol{u}_1\in\mathbb{S}^n$ such that the function $a\mapsto f_1(a\boldsymbol{u}_1)$ is not identically zero in a neighbourhood of 0. Moreover, there is a neighbourhood $\mathcal{U}_1$ of $\boldsymbol{u}_1$ in $\mathbb{S}^n$ such that $a\mapsto f_1(a\boldsymbol{u})$ is not identically zero in a neighbourhood of 0 for every $\boldsymbol{u}\in\mathcal{U}_1$. Now suppose the claim holds for $k\in\{1,\dots,m\}$ and let $f_1,\dots,f_{m+1}$ satisfy the hypotheses of the lemma. By the induction hypothesis there exists an open set $\mathcal{U}_m$ of $\mathbb{S}^n$ such that the functions $a\mapsto f_j(a\boldsymbol{u})$, $j\in\{1,\dots,m\}$, are not identically zero in a neighbourhood of 0 for each $\boldsymbol{u}\in\mathcal{U}_m$. By our argument above for $k=1$, there can be no open subset of $\mathcal{U}_m$ such that $a\mapsto f_{m+1}(a\boldsymbol{u})$ is not identically zero in a neighbourhood of 0 for all $\boldsymbol{u}\in\mathcal{U}_m$. This gives our claim.

By our claim, let $\boldsymbol{u}\in\mathbb{F}^{n+1}$ be such that none of the functions $a\mapsto f_j(a\boldsymbol{u})$ are identically zero in a neighbourhood of 0. Now let $\psi\colon\mathbb{F}^{n+1}\to\mathbb{F}^{n+1}$ be a unitary transformation for which $\psi(\boldsymbol{e}_{n+1}) = \boldsymbol{u}$. Note that $\psi^*f_j(\mathbf{0},y) = f_j(y\psi(\boldsymbol{e}_{n+1})) = f_j(a\boldsymbol{u})$. Therefore, the function $y\mapsto\psi^*f(\mathbf{0},y)$ is not identically zero in a neighbourhood of 0 for each $j\in\{1,\dots,k\}$. One can readily see that this implies that the functions $\psi^* f_1, \dots, \psi^* f_k$ satisfy the same hypotheses as the function "A" from the Weierstrass Preparation Theorem. ■

2.1.4 Lemma (Products of real analytic functions involving Weierstrass polynomials) *Let $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{F}^n \times \mathbb{F}$ be a neighbourhood of $(\mathbf{0}, 0)$ and suppose that $f, g, W \in C^\omega(\mathcal{U} \times \mathcal{V})$. The following statements hold:*

(i) *if f is a polynomial in y i.e.,*

$$f(\mathbf{x}, y) = f_k(\mathbf{x})y^k + \cdots + f_1(\mathbf{x})y + f_0(\mathbf{x}) \tag{2.1}$$

for some holomorphic or real analytic functions $f_j \colon \mathcal{U} \to \mathbb{F}$, $j \in \{0, 1, \dots, k\}$, if W is a Weierstrass polynomial, and if $f = gW$, then g is a polynomial in y, i.e.,

$$g(\mathbf{x}, y) = g_m(\mathbf{x})y^m + \cdots + g_1(\mathbf{x})y + g_0(\mathbf{x}) \tag{2.2}$$

for holomorphic or real analytic functions $g_j \colon \mathcal{U} \to \mathbb{F}$, $j \in \{0, 1, \dots, m\}$;

(ii) *if W is a Weierstrass polynomial, if f and g are polynomials in y, i.e., f and g are as in (2.1) and (2.2), respectively, and if $W = fg$, then there exists holomorphic or real analytic functions $E, F \colon \mathcal{U} \to \mathbb{F}$ such that $E(\mathbf{0}) \neq 0$ and $F(\mathbf{0}) \neq 0$ and such that Ef and Fg are Weierstrass polynomials.*

Proof (i) Since the coefficient of the highest degree term of y in W is 1, i.e., a unit in $C^\omega(\mathcal{U})$, and since f is a polynomial in y, we can perform polynomial long division to write $f = QW + R$ where the degree of R as a polynomial in y is less than that of W and where Q is a polynomial in y [Hungerford 1980, Theorem III.6.2]. By the uniqueness assertion of the Weierstrass Preparation Theorem, since $f = gW$, we must have $g = Q$ and $R = 0$. In particular, g is a polynomial in y.

(ii) Let k and m be the degrees of f and g, i.e., f_k and g_m are nonzero in (2.1) and (2.2). Let r be the degree of W. Then we must have

$$y^r = W(\mathbf{0}, y) = f(\mathbf{0}, y)g(\mathbf{0}, y) = f_k(\mathbf{0})g_m(\mathbf{0})y^{k+m},$$

implying that $f_k(\mathbf{0})$ and $g_m(\mathbf{0})$ are nonzero, and so f_k and g_m are invertible in a neighbourhood of $\mathbf{0}$. Thus the result follows by taking $E(\mathbf{x}) = f_k(\mathbf{x})^{-1}$ and $F(\mathbf{x}) = g_m(\mathbf{x})^{-1}$. ■

2.1.2 The Weierstrass Preparation Theorem

With the previous section as...er...preparation, we can state the Weierstrass Preparation Theorem, following [Krantz and Parks 2002, Theorem 6.1.3].

2.1.5 Theorem (Weierstrass Preparation Theorem) *Let $\mathcal{U}_A \times \mathcal{V}_A \subseteq \mathbb{F}^n \times \mathbb{F}$ be a neighbourhood of $(\mathbf{x}, 0)$ and suppose that the holomorphic or real analytic function $A \colon \mathcal{U}_A \times \mathcal{V}_A \to$ is given by*

$$A(\mathbf{x}, y) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} A_{I,j}\mathbf{x}^I y^j, \qquad (\mathbf{x}, y) \in \mathcal{U}_A \times \mathcal{V}_A,$$

where

$$A_{\mathbf{0},0} = A_{\mathbf{0},1} = \cdots = A_{\mathbf{0},k-1} = 0, \qquad A_{\mathbf{0},k} = 1$$

for some $k \in \mathbb{Z}_{>0}$. Then the following statements hold:

(i) *if* $B\colon \mathcal{U}_B \times \mathcal{V}_B \to \mathbb{F}$ *has a convergent power series in a neighbourhood* $\mathcal{U}_B \times \mathcal{V}_B$ *of* $(\mathbf{0}, 0)$, *then there exist unique holomorphic or real analytic functions* $Q\colon \mathcal{U}_Q \times \mathcal{V}_Q \to \mathbb{F}$ *and* $R\colon \mathcal{U}_R \times \mathcal{V}_R \to \mathbb{F}$ *defined on neighbourhoods* $\mathcal{U}_Q \times \mathcal{V}_Q$ *and* $\mathcal{U}_R \times \mathcal{V}_R$, *respectively, of* $(\mathbf{0}, 0)$ *such that*

(a) Q *and* R *are represented by power series*

$$Q(\mathbf{x}, y) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} Q_{I,j} \mathbf{x}^I y^j, \qquad (\mathbf{x}, y) \in \mathcal{U}_Q \times \mathcal{V}_Q,$$

$$R(\mathbf{x}, y) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} R_{I,j} \mathbf{x}^I y^j, \qquad (\mathbf{x}, y) \in \mathcal{U}_R \times \mathcal{V}_R,$$

where $R_{I,j} = 0$ *for all* $I \in \mathbb{Z}^n_{\geq 0}$ *and* $j \geq k$ *and*

(b) $B(\mathbf{x}, y) = Q(\mathbf{x}, y)A(\mathbf{x}, y) + R(\mathbf{x}, y)$ *for all* $\mathbf{x} \in \mathcal{U} \subseteq \mathcal{U}_A \cap \mathcal{U}_B \cap \mathcal{U}_Q \cap \mathcal{U}_R$ *and* $y \in \mathcal{V} \subseteq \mathcal{V}_A \cap \mathcal{V}_B \cap \mathcal{V}_Q \cap \mathcal{V}_R$;

(ii) *there exist unique holomorphic or real analytic functions* $W\colon \mathcal{U}_W \times \mathcal{V}_W \to \mathbb{F}$ *and* $E\colon \mathcal{U}_E \times \mathcal{V}_E \to \mathbb{F}$ *defined on neighbourhoods* $\mathcal{U}_W \times \mathcal{V}_W$ *and* $\mathcal{U}_E \times \mathcal{V}_E$, *respectively, of* $(\mathbf{0}, 0)$ *such that*

(a) W *is a Weierstrass polynomial of degree* k,

(b) $E(\mathbf{0}, 0) \neq 0$, *and*

(c) $E(\mathbf{x}, y)A(\mathbf{x}, y) = W(\mathbf{x}, y)$ *for all* $\mathbf{x} \in \mathcal{U} \subseteq \mathcal{U}_A \cap \mathcal{U}_E \cap \mathcal{U}_W$ *and* $y \in \mathcal{V} \subseteq \mathcal{V}_A \cap \mathcal{V}_E \cap \mathcal{V}_W$.

Proof We store the following lemma for later use, using the notation that if $I, J \in \mathbb{Z}^n_{\geq 0}$ then $J \leq I$ if $j_k \leq i_k$ for each $k \in \{1, \dots, n\}$.

1 Lemma *Let* $a, b \in \mathbb{R}_{>0}$ *with* $b < a$ *and let* $I \in \mathbb{Z}^n_{\geq 0}$ *be such that* $i_k = 0$ *for some* $k \in \{1, \dots, n\}$. *Then*

(i) $\displaystyle\sum_{\substack{J \leq I \\ j_k < i_k}} \left(\frac{a}{b}\right)^{|J|} \leq \frac{ba^{n-1}}{(a-b)^n}\left(\frac{a}{b}\right)^{|I|}$ *and*

(ii) $\displaystyle\sum_{\substack{J \leq I \\ |J| < |I|}} \left(\frac{a}{b}\right)^{|J|} \leq \frac{nba^{n-1}}{(a-b)^n}\left(\frac{a}{b}\right)^{|I|}$.

Proof Let us prove the first statement. Note that for $\alpha \neq 1$ and $r \in \mathbb{Z}_{>0}$ we have

$$(\alpha - 1)\sum_{s=0}^{r} \alpha^s = \alpha^{r+1} - 1 \quad \Longrightarrow \quad \sum_{s=0}^{r} \alpha^s = \frac{\alpha^{r+1} - 1}{\alpha - 1}. \tag{2.3}$$

Using this fact we compute

$$\begin{aligned}\sum_{\substack{J\leq I\\ j_k<i_k}}\Big(\frac{a}{b}\Big)^{|J|} &= \Big(\sum_{j_k=0}^{i_k-1}\Big(\frac{a}{b}\Big)^{j_k}\Big)\Big(\prod_{\substack{l=1\\ l\neq k}}^{n}\Big(\sum_{j_l=0}^{i_l}\Big(\frac{a}{b}\Big)^{j_l}\Big)\Big) = \frac{(a/b)^{i_k}-1}{(a/b)-1}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{(a/b)^{i_l+1}-1}{(a/b)-1}\\ &= \frac{b^{|I|+n}}{b^{|I|+n}}\frac{(a/b)^{i_k}-1}{(a/b)-1}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{(a/b)^{i_l+1}-1}{(a/b)-1} = \frac{b}{b^{|I|}}\frac{a^{i_k}-b^{i_k}}{a-b}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{a^{i_l+1}-b^{i_l+1}}{a-b}\\ &\leq \frac{b}{b^{|I|}}\frac{a^{i_k}}{a-b}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{a^{i_l+1}}{a-b} = \frac{b}{b^{|I|}}\frac{a^{|I|+n-1}}{(a-b)^n} = \frac{ba^{n-1}}{(a-b)^n}\Big(\frac{a}{b}\Big)^{|I|},\end{aligned}$$

as desired.

The second assertion of the lemma follows from the first after noting that

$$\sum_{\substack{J\leq I\\ |J|<|I|}}\Big(\frac{a}{b}\Big)^{|J|} = \sum_{k=1}^{m}\sum_{\substack{J\leq I\\ j_k<i_k}}\Big(\frac{a}{b}\Big)^{|J|}.$$ ▼

Now we proceed with the proof of the first assertion of the theorem. Let us write

$$B(\boldsymbol{x},y) = \sum_{I\in\mathbb{Z}^n_{\geq 0}}\sum_{j=0}^{\infty}B_{I,j}\boldsymbol{x}^I y^j, \qquad (\boldsymbol{x},y)\in\mathcal{U}_B\times\mathcal{V}_B.$$

Let us first show that, at the level of formal power series, there exist unique formal power series in indeterminates $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n)$ and η

$$\sum_{I\in\mathbb{Z}^n_{\geq 0}}\sum_{j=0}^{\infty}Q_{I,j}\boldsymbol{\xi}^I\eta^j, \qquad \sum_{I\in\mathbb{Z}^n_{\geq 0}}\sum_{j=0}^{\infty}R_{I,j}\boldsymbol{\xi}^I\eta^j,$$

with $R_{I,j} = 0$ for $I\in\mathbb{Z}^n_{\geq 0}$ and $j\geq k$ and such that, at the level of formal power series, $B = QA + R$. Note that the formula $B = QA + R$ reads

$$B_{I,j} = \sum_{J\leq I}\sum_{m=0}^{j}Q_{J,m}A_{I-J,j-m} + R_{I,j}, \qquad I\in\mathbb{Z}^n_{\geq 0},\ j\in\mathbb{Z}_{\geq 0}. \tag{2.4}$$

For $j = k$, using the fact that $R_{I,k} = 0$ for all $I\in\mathbb{Z}^n_{\geq 0}$, the formula (2.4) reads

$$B_{I,k} = \sum_{J\leq I}\sum_{m=0}^{k}Q_{J,m}A_{I-J,k-m} = Q_{I,0} + \sum_{\substack{J\in I\\ |J|\leq|I|}}\sum_{m=0}^{k}Q_{J,m}A_{I-J,k-m},$$

since $A_{\boldsymbol{0},k} = 1$. Thus

$$Q_{I,0} = B_{I,k} - \sum_{\substack{J\in I\\ |J|\leq|I|}}\sum_{m=0}^{k}Q_{J,m}A_{I-J,k-m}, \qquad I\in\mathbb{Z}^n_{\geq 0}. \tag{2.5}$$

From this we infer that $Q_{I,0}$ is determined uniquely from A and B, and the set of $Q_{J,m}$, where $J \leq I$, $|J| < |I|$, and $m \in \{0, 1, \dots, k\}$. A particular case of the formula (2.5) is

$$Q_{\mathbf{0},0} = B_{\mathbf{0},k}. \tag{2.6}$$

Therefore, starting from (2.6), we can recursively and uniquely define $Q_{I,0}$ for all $I \in \mathbb{Z}^n_{\geq 0}$. Now take $j = k + l$ in (2.4) for $l \in \mathbb{Z}_{>0}$. Then we have, using the fact that $R_{I,j} = 0$ for $j \geq k$,

$$\begin{aligned} B_{I,k+l} &= \sum_{J \leq I} \sum_{m=0}^{k+l} Q_{J,m} A_{I-J,k+l-m} = \sum_{m=0}^{k+l} Q_{I,m} A_{\mathbf{0},k+l-m} + \sum_{\substack{J \leq I \\ |J|<|I|}} \sum_{m=0}^{k+l} Q_{J,m} A_{I-J,k+l-m} \\ &= Q_{I,l} + \sum_{m=0}^{l-1} Q_{I,m} A_{\mathbf{0},k+l-m} + \sum_{\substack{J \leq I \\ |J|<|I|}} \sum_{m=0}^{k+l} Q_{J,m} A_{I-J,k+l-m}, \end{aligned}$$

using the fact that $A_{\mathbf{0},j} = 0$ for $j \in \{0, 1, \dots, k\}$ and $A_{\mathbf{0},k} = 1$. Thus we have

$$Q_{I,l} = B_{I,k+l} - \sum_{m=0}^{l-1} Q_{I,m} A_{\mathbf{0},k+l-m} - \sum_{\substack{J \leq I \\ |J|<|I|}} \sum_{m=0}^{k+l} Q_{J,m} A_{I-J,k+l-m} \tag{2.7}$$

for $I \in \mathbb{Z}^n_{\geq 0}$ and $l \in \mathbb{Z}_{>0}$. From this we infer that we can solve uniquely for $Q_{I,l}$, $I \in \mathbb{Z}^n_{\geq 0}$, $l \in \mathbb{Z}_{>0}$, in terms of A and B, and the set of $Q_{J,m}$ with $J \leq I$, $|J| < |I|$, and $m \in \{0, 1, \dots, k\}$. When $I = \mathbf{0}$, in particular, the formula (2.7) reads

$$Q_{\mathbf{0},l} = B_{\mathbf{0},k+l} - \sum_{m=0}^{l-1} Q_{\mathbf{0},m} A_{\mathbf{0},k+l-m},$$

showing that we can recursively define $Q_{\mathbf{0},l}$ for $l \in \mathbb{Z}_{>0}$. Then (2.7) can be recursively applied to determine $Q_{I,l}$ for $I \in \mathbb{Z}^n_{\geq 0}$ and $l \in \mathbb{Z}_{>0}$. Finally, for $I \in \mathbb{Z}^n_{\geq 0}$ and $j \in \{0, 1, \dots, k-1\}$, we can directly apply (2.4) to obtain

$$\begin{aligned} R_{I,j} &= B_{I,j} - \sum_{J \leq I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m} = B_{I,j} - \sum_{m=0}^{j} Q_{I,m} A_{\mathbf{0},j-m} - \sum_{\substack{J \leq I \\ |J|<|I|}} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m} \\ &= B_{I,j} - \sum_{\substack{J \leq I \\ |J|<|I|}} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m}, \end{aligned} \tag{2.8}$$

using the fact that $A_{\mathbf{0},j} = 0$ for $j \in \{0, 1, \dots, k-1\}$ and $A_{\mathbf{0},k} = 1$. Thus $R_{I,j}$ is uniquely determined from those for A and B, and from the set of $Q_{J,m}$ with $J \leq I$, $|J| < |I|$, and $m \in \{0, 1, \dots, k-1\}$.

The preceding computations show that formal power series exist for Q and R such that (2.4) holds. It remains to show that the resulting power series for Q and R converge. By Theorem 1.1.17 there exists $b, c \in \mathbb{R}_{>0}$ such that

$$\max\{|A_{I,j}|, |B_{I,j}|\} \leq bc^{|I|+j}, \qquad I \in \mathbb{Z}^n_{\geq 0},\ j \in \mathbb{Z}_{\geq 0}.$$

Let $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$ be chosen so that $\alpha > b$, $\beta, \gamma > c$, and

$$\frac{bc^k}{\alpha} < \frac{1}{3}, \qquad \frac{bc^{k+1}}{\beta - c} < \frac{1}{3}, \qquad \frac{nb\beta^{k+1}}{c^{k-1}(\beta - c)} \frac{\gamma^{n-1}}{(\gamma - c)^n} < \frac{1}{3}.$$

We claim that

$$|Q_{I,j}| \le \alpha\beta^j\gamma^{|I|}, \qquad I \in \mathbb{Z}^n_{\ge 0},\ j \in \mathbb{Z}_{\ge 0}. \tag{2.9}$$

We prove this by induction on $|I| + j$. By (2.6) we have

$$|Q_{\mathbf{0},0}| = |B_{\mathbf{0},k}| \le b < \alpha,$$

giving (2.9) for $|I| + j = 0$. Now assume that (2.9) holds for $I \in \mathbb{Z}_{\ge 0}$ and $j \in \mathbb{Z}_{\ge 0}$ such that $|I| + j = r - 1$. Then let $I \in \mathbb{Z}^n_{\ge 0}$ and $j \in \mathbb{Z}_{\ge 0}$ be such that $|I| + j = r$. By (2.7) we have

$$\begin{aligned} |Q_{I,j}| &\le bc^{|I|+k+j} + \sum_{m=0}^{l-1} \alpha\beta^m\gamma^{|I|}bc^{k+j-m} + \sum_{\substack{J \le I \\ |J|<|I|}} \sum_{m=0}^{k+j} \alpha\beta^m\gamma^{|J|}bc^{|I|-|J|+k+j-m} \\ &= \alpha\beta^j\gamma^{|I|}\Big(\frac{bc^k}{\alpha}\Big(\frac{c}{\beta}\Big)^j\Big(\frac{c}{\gamma}\Big)^{|I|} + \Big(\frac{c}{\beta}\Big)^j bc^k \sum_{m=0}^{j-1}\Big(\frac{\beta}{c}\Big)^m + \Big(\frac{c}{\beta}\Big)^j\Big(\frac{c}{\gamma}\Big)^{|I|} b\Big(\sum_{m=0}^{k+j}\Big(\frac{\beta}{c}\Big)^m\Big)\Big(\sum_{\substack{J \le I \\ |J|<|I|}}\Big(\frac{\gamma}{c}\Big)^{|J|}\Big)\Big). \end{aligned}$$

By definition of α and since $\beta, \gamma > c$,

$$\frac{bc^k}{\alpha}\Big(\frac{c}{\beta}\Big)^j\Big(\frac{c}{\gamma}\Big)^{|I|} < \frac{1}{3}.$$

Using (2.3) we compute

$$\begin{aligned} \Big(\frac{c}{\beta}\Big)^j bc^k \sum_{m=0}^{j-1}\Big(\frac{\beta}{c}\Big)^m &= \Big(\frac{c}{\beta}\Big)^j bc^k \frac{(\beta/c)^j - 1}{(\beta/c) - 1} = bc^{k+1}\frac{1}{\beta^j}\frac{\beta^j - c^j}{\beta - c} \\ &\le bc^{k+1}\frac{1}{\beta^j}\frac{\beta^j}{\beta - c} \le \frac{bc^{k+1}}{\beta - c} < \frac{1}{3}. \end{aligned}$$

By (2.3) and the lemma above we compute

$$\begin{aligned} \Big(\frac{c}{\beta}\Big)^j\Big(\frac{c}{\gamma}\Big)^{|I|} b\Big(\sum_{m=0}^{k+j}\Big(\frac{\beta}{c}\Big)^m\Big)\Big(\sum_{\substack{J \le I \\ |J|<|I|}}\Big(\frac{\gamma}{c}\Big)^{|J|}\Big) &\le \Big(\frac{c}{\beta}\Big)^j\Big(\frac{c}{\gamma}\Big)^{|I|} b\frac{(\beta/c)^{k+j+1} - 1}{(\beta/c) - 1}\frac{nc\gamma^{n-1}}{(\gamma - c)^n}\Big(\frac{\gamma}{c}\Big)^{|I|} \\ &= \frac{\beta^{k+1}}{\beta^{k+j+1}c^k} b\frac{\beta^{k+j+1} - c^{k+j+1}}{\beta - c}\frac{nc\gamma^{n-1}}{(\gamma - c)^n} \\ &\le \frac{\beta^{k+1}}{c^k(\beta - c)} b\frac{nc\gamma^{n-1}}{(\gamma - c)^n} < \frac{1}{3}. \end{aligned}$$

Combining the previous three estimates we obtain

$$|Q_{I,j}| \le \alpha\beta^j\gamma^{|I|},$$

as in (2.9).

Now, given that (2.9) holds, we claim that $Q_{I,j}$, $I \in \mathbb{Z}^n_{\geq 0}$, $j \in \mathbb{Z}_{\geq 0}$, defines a convergent power series. To see this, let $\lambda \in (0,1)$ and let $r, \rho \in \mathbb{R}_{>0}$ be chosen so that $r\beta, \rho\gamma = \lambda$. If $(\boldsymbol{x}, y) \in \mathbb{F}^n \times \mathbb{F}$ satisfy $|x_j| < r$, $j \in \{1, \ldots, n\}$, and $|y| < \rho$, then

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} |Q_{I,j}||\boldsymbol{x}|^I |y|^j \leq \sum_{m=0}^{\infty} \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I|=m}} \sum_{j=0}^{\infty} \alpha(r\beta)^j (\rho\gamma)^m = \sum_{m=0}^{\infty} \sum_{\substack{J \in \mathbb{Z}^{n+1}_{\geq 0} \\ |J|=m}} \alpha\lambda^m = \sum_{m=0}^{\infty} \binom{n+m}{n} \alpha\lambda^m.$$

This last series converges by the ratio test. Therefore, the series

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{j=0}^{\infty} Q_{I,j} \boldsymbol{x}^I y^j$$

converges for $(\boldsymbol{x}, y)$ satisfying $|x_j| < r$, $j \in \{1, \ldots, n\}$, and $|y| < \rho$. Since Q is analytic, $R = B - QA$ is also real analytic, and so the power series for R also converges. This gives the first part of the theorem.

For the second part of the theorem, define $B(\boldsymbol{x}, y) = y^k$. Then apply the first part of the theorem to give Q and R such that $B = QA + R$ in a neighbourhood of $(\boldsymbol{0}, 0)$. Then define $W = B - R$ and $E = Q$. Clearly $W = EA$. Since $B_{\boldsymbol{0},k} = 1$, by (2.6) we have $Q(\boldsymbol{0}, 0) = Q_{\boldsymbol{0},0} = 1 \neq 0$. If we apply (2.8) with $I = \boldsymbol{0}$ we have $R_{\boldsymbol{0},j} = B_{\boldsymbol{0},j}$ for $j \in \{1, \ldots, n\}$, giving $W_{\boldsymbol{0},j} = 0$ for $j \in \{1, \ldots, n\}$. Therefore, noting that $R_{I,j} = 0$ for $j \geq k$.

$$W(\boldsymbol{x}, y) = \sum_{j=0}^{k} \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I|>0}} (B_{I,j} - R_{I,j}) \boldsymbol{x}^I y^j,$$

which shows that W is a Weierstrass polynomial. ■

2.2 Review of relevant algebra

In this section we provide an overview of the definitions and theorems from algebra of which we shall make use. A good reference for much of this is [Hungerford 1980].

2.2.1 Unique factorisation domains

In this section we review some facts about unique factorisation domains. The key notion is the definition is the following.

2.2.1 Definition (Irreducible element) In an integral domain R, a element r is ***irreducible*** if it is a nonzero nonunit, and if it is not a product of two nonunits. •

We can now give the definition of a unique factorisation domain.

2.2.2 Definition (Unique factorisation domain) A ***unique factorisation domain*** is an integral domain R such that:

(i) if $r \in \mathsf{R}$ is nonzero and not a unit, then there exists irreducible elements $f_1, \dots, f_k \in \mathsf{R}$ such that $r = f_1 \cdots f_k$;

(ii) if, for irreducible elements $f_1, \dots, f_k, g_1, \dots, g_l \in \mathsf{R}$, we have $f_1 \cdots f_k = g_1 \cdots g_l$, then $k = l$ and there exists $\sigma \in \mathfrak{S}_k$ such that $f_j | g_{\sigma(j)}$ and $g_{\sigma(j)} | f_j$ for each $j \in \{1, \dots, k\}$. •

The first part of the definition tells us that every nonzero element of R that is not a unit is expressible as a product of irreducibles. The second part of the definition tells us that the expression as a product of irreducibles is unique up to order and the factors differing by a unit. It is often convenient to eliminate the ambiguity of knowing the irreducible factors only up to multiplication by units. Let us denote by $\mathscr{I}_{\mathsf{R}}$ the set of irreducible elements of a commutative unit ring. In $\mathscr{I}_{\mathsf{R}}$ define a relation by $p_1 \sim p_2$ if $p_2 = up_1$ for some unit u. The following definition develops some terminology associated to this.

2.2.3 Definition (Selection of irreducibles) Let R be a commutative unit ring and let $\mathscr{I}_{\mathsf{R}}$ be the set of irreducible elements in R, with $\sim$ the equivalence relation described above. A ***selection of irreducibles*** is a map $P\colon (\mathscr{I}_{\mathsf{R}}/\sim) \to \mathscr{I}_{\mathsf{R}}$ such that $P([p]) \in [p]$. We shall denote a selection of irreducibles P by $(p_a)_{a \in A_{\mathsf{R}}}$ where $A_{\mathsf{R}} = \mathscr{I}_{\mathsf{R}}/\sim$ and where $p_a = P(a)$. •

The following result encapsulates why the notion of a selection of irreducibles is valuable.

2.2.4 Proposition (Unique factorisation determined by selection of irreducibles) *If* R *is a unique factorisation domain and if* $(\mathrm{p_a})_{\mathrm{a} \in \mathrm{A}_{\mathsf{R}}}$ *is a selection of irreducibles, then, given a nonzero nonunit* $\mathrm{r} \in \mathsf{R}$, *there exists unique* $\mathrm{p}_{\mathrm{a}_1}, \dots, \mathrm{p}_{\mathrm{a}_k} \in (\mathrm{p_a})_{\mathrm{a} \in \mathrm{A}_{\mathsf{R}}}$ *and a unique unit* $\mathrm{u} \in \mathsf{R}$ *such that* $\mathrm{r} = \mathrm{u}\mathrm{p}_{\mathrm{a}_1} \cdots \mathrm{p}_{\mathrm{a}_k}$.

Proof Let $f_1, \dots, f_k$ be irreducibles such that $r = f_1 \cdots f_k$. For $j \in \{1, \dots, k\}$, define $p_{a_j} \in \{p_a \mid a \in A_{\mathsf{R}}\}$ so that $p_{a_j} \in [f_j]$. Then $f_j = u_j p_{a_j}$ for some unit $u_j \in \mathsf{R}$, $j \in \{1, \dots, k\}$. Then we have

$$r = u_1 \cdots u_k p_{a_1} \cdots p_{a_k},$$

giving the existence part of the result. Now suppose that

$$r = up_{a_1} \cdots p_{a_k} = u' p_{a'_1} \cdots p_{a'_{k'}} \tag{2.10}$$

are two representations of the desired form. Since $(up_{a_1})p_{a_2} \cdots p_{a_k}$ and $(u' p_{a'_1})p_{a_2} \cdots p_{a'_{k'}}$ are two factorisations by irreducibles, we immediately conclude that $k' = k$. For convenience, let us define $f_1 = up_{a_1}$, $f_j = p_{a_j}$, $j \in \{2, \dots, k\}$, and $f'_1 = u' p_{a'_1}$, $f'_j = p_{a'_j}$, $j \in \{2, \dots, k\}$. We can then assert the existence of $\sigma \in \mathfrak{S}_k$ such that $f'_j = v_j f_{\sigma(j)}$, $j \in \{1, \dots, k\}$, for units $v_1, \dots, v_k$. By the definition of a selection of irreducibles, for each $j \in \{1, \dots, k\}$ we have $f'_j = v_j f_{\sigma(j)} = w_j p_{b_j}$ for a unit w_j and $p_{b_j} \in (p_a)_{a \in A_{\mathsf{R}}}$. Now we have a few cases.

1. $f_1' = u'p_{a_1'} = v_1 f_1 = v_1 u p_{a_1}$: In this case we have
$$u'p_{a_1'} = uv_1 p_{a_1} = w_1 p_{b_1}.$$
We conclude that $p_{a_1'} \sim p_{a_1} \sim p_{b_1}$, implying that $p_{a_1'} = p_{a_1} = p_{b_1}$. We also conclude that $u' = uv_1 = w_1$.
2. $f_1' = u'p_{a_1'} = f_{\sigma(1)} = p_{a_{\sigma(1)}}$ for $\sigma(1) \neq 1$: Here we have
$$u'p_{a_1'} = v_1 p_{a_{\sigma(1)}} = w_1 p_{b_1},$$
and as above we conclude that $p_{a_1'} = p_{a_{\sigma(1)}} = p_{b_1}$ and $u' = v_1 = w_1$.
3. $f_j' = p_{a_j'} = v_j f_1 = v_j u p_1 = w_j p_{b_j}$ for $j \neq 1$: Here we conclude that $p_{a_j'} = p_1 = p_{b_j}$ and $1_{\mathsf{R}} = v_j u = w_j$.
4. $f_j' = p_{a_j'} = v_j f_{\sigma(j)} = v_j p_{a_j} = w_j p_{b_j}$ for $j \neq 1$ and $\sigma(j) \neq 1$: In this case we conclude that $p_{a_j'} = p_{a_j} = p_{b_j}$ and $1_{\mathsf{R}} = v_j = w_j$.

We then see that $p_{a_1'} \cdots p_{a_k'} = p_{a_{\sigma(1)}} \cdots p_{a_{\sigma(k)}}$, and we immediately conclude from (2.10) that $u' = u$, and this completes the proof of uniqueness. ■

For unique factorisation domains we have the following valuable characterisation of primes and irreducibles.

2.2.5 Proposition (Primes and irreducibles in unique factorisation domains) *If* R *is be a unique factorisation domain, then the following three statements concerning* $\mathrm{p} \in \mathsf{R}$ *are equivalent:*

(i) p *is prime;*

(ii) p *is irreducible;*

(iii) if d|p *then either* d *is a unit or* d = up *for a unit* u.

Proof The first two assertions we prove are valid for general integral domains.

(i) $\Longrightarrow$ (ii) Suppose that p is prime and that $p = ab$. Then $p|(ab)$ and since p is prime, without loss of generality we can assert that $p|a$. Thus $a = qp$ for some $q \in \mathsf{R}$. Then $p = ab = aqp$ which implies that $aq = 1_{\mathsf{R}}$. Thus a is a unit, and so p is irreducible.

(ii) $\Longrightarrow$ (iii) Let p be irreducible and suppose that $d|p$ so that $(p) \subseteq (d)$. Therefore, since p is irreducible, we have $(p) = (d)$ or $(d) = \mathsf{R}$. In the first case we have $p|d$, and so $p = ud$ for some unit u. In the second case, d is a unit.

(iii) $\Longrightarrow$ (i). Suppose that $p|(ab)$ and that p satisfies (iii). Using the properties of a unique factorisation domain, write $p = f_1 \dots f_k$ for irreducibles $f_1, \dots, f_k$. It follows from (iii) that $k = 1$ since irreducibles are not units. Now write
$$a = g_1 \dots g_l, \quad b = h_1 \dots h_m$$
for irreducibles $g_1, \dots, g_l, h_1, \dots, h_m$. Then there exists $r \in \mathsf{R}$ such that
$$rf_1 = g_1 \dots g_l h_1 \dots h_m.$$
Now write r as a product of irreducibles: $r = s_1 \cdots s_q$. Then
$$s_1 \cdots s_q f_1 = g_1 \dots g_l h_1 \dots h_m.$$

Using the definition of a unique factorisation domain we conclude that, for some $a \in \{g_1, \dots, g_l, h_1, \dots, h_m\}$, we have $f_1|a$ and $a|f_1$. This allows us to conclude that either $p|a$ or $p|b$. Thus p is prime. ■

The following property of unique factorisation domains will be useful for us in Proposition 2.3.6 where we show that a certain ring is *not* a unique factorisation domain.

2.2.6 Proposition (Ascending principal ideals in unique factorisation domains) *For a unique factorisation domain* R *and for a sequence* $(\mathsf{I}_j)_{j\in\mathbb{Z}_{>0}}$ *of principal ideals such that* $\mathsf{I}_j \subseteq \mathsf{I}_{j+1}$*,* $j \in \mathbb{Z}_{>0}$*, there exists* $N \in \mathbb{Z}_{>0}$ *such that* $\mathsf{I}_j = \mathsf{I}_N$ *for all* $j \geq N$*.*

Proof Let $(p_a)_{a\in A_\mathsf{R}}$ be a selection of irreducibles. Let $\mathsf{I}_j = (r_j)$ so that $r_{j+1}|r_j$ for each $j \in \mathbb{Z}_{>0}$. Then the factors from $(p_a)_{a\in A_\mathsf{R}}$ for r_j are also factors for r_1. Since R is a unique factorisation domain, r_1 has finitely many factors from $(p_a)_{a\in A_\mathsf{R}}$. This means that the factors from $(p_a)_{a\in A_\mathsf{R}}$ of r_j must agree with those of r_{j+1} for all sufficiently large j. Thus there exists $N \in \mathbb{Z}_{>0}$ such that $r_j = u_j r_N$ for $j \geq N$ and where u_j, $j \geq N$, are units. Thus $\mathsf{I}_j = \mathsf{I}_N$ for $j \geq N$. ■

Let us now examine polynomial rings over unique factorisation domains.

2.2.7 Definition (Primitive polynomial) Let R be a unique factorisation domain and let $A = \sum_{j=0}^{k} a_j \xi^j \in \mathsf{R}[\xi]$.

(i) A ***content*** of A is a greatest common divisor of $\{a_0, a_1, \dots, a_k\}$.

(ii) A is ***primitive*** if it has a content that is a unit in R. •

We now record some results about polynomials over unique factorisation domains. We need a little warm up before getting to the main statement.

2.2.8 Definition (Fraction field) Let R be an integral domain and define an equivalence relation $\sim$ in $\mathsf{R} \times (\mathsf{R} \setminus \{0_\mathsf{R}\})$ by

$$(r, s) \sim (r', s') \quad \Longleftrightarrow \quad rs' - r's = 0_\mathsf{R}$$

The set of equivalence classes under this equivalence relation is the ***fraction field*** of R, and is denoted by F_R. The equivalence class of (r, s) is denoted by $\frac{r}{s}$. •

The statement of the next result concerning polynomials over a unique factorisation domain relies on the fact that the polynomial ring over an integral domain R is naturally a subset of the polynomial ring over the fraction field F_R.

2.2.9 Proposition (Properties of polynomials over unique factorisation domains) *Let* R *be a unique factorisation domain with* F_R *its fraction field. Then the following statements hold for polynomials* $A, B \in \mathsf{R}[\xi] \subseteq \mathsf{F}_\mathsf{R}[\xi]$*.*

(i) $A = c_A A'$ *where* c_A *is a content of* A *and* $A' \in \mathsf{R}[\xi]$ *is primitive;*

(ii) if c_A *and* c_B *are contents of* A *and* B*, respectively, then* $c_A c_B$ *is a content of* $A \cdot B$*;*

(iii) if A *and* B *are primitive, then* $A \cdot B$ *is primitive;*

(iv) if A *and* B *are primitive, then* $A|B$ *and* $B|A$ *in* $\mathsf{R}[\xi]$ *if and only if* $A|B$ *and* $B|A$ *in* $\mathsf{F}_\mathsf{R}[\xi]$*;*

(v) *if* A *is primitive and if* $\deg(A) > 0$, *then* A *is irreducible in* $\mathsf{R}[\xi]$ *if and only if it is irreducible in* $\mathsf{F}_{\mathsf{R}}[\xi]$.

Proof (i) Write $A = \sum_{j=0}^{k} a_j\xi^j$ and write $a_j = c_A a'_j$ for $j \in \{0,1,\dots,k\}$. Then the result follows by taking $A' = \sum_{j=0}^{k} a'_j\xi^j$.

(ii) By part (i), write $A = c_A A'$ and $B = c_B B'$ for A' and B' primitive. If c' is a content for $A' \cdot B'$, it is easy to see that $c_A c_B c'$ is a content for $A \cdot B = (c_A A') \cdot (c_B B')$. Thus it suffices to show that c' is a unit, i.e., that $A' \cdot B'$ is primitive. Suppose that $A' \cdot B'$ is not primitive and write $C = A' \cdot B' = (c_k = \sum_{j=0}^{k} a'_j b'_{k-j})_{k\in\mathbb{Z}_{\geq 0}}$, where $A' = (a'_j)_{j\in\mathbb{Z}_{\geq 0}}$ and $B' = (b'_j)_{j\in\mathbb{Z}_{\geq 0}}$. Suppose that $p \in \mathsf{R}$ is irreducible and that $p|c_j$ for all j. If $c_{A'}$ is a content for A' we have $p \nmid c_{A'}$ since $c_{A'}$ is a unit. Similarly, $p \nmid c_{B'}$ where $c_{B'}$ is a content for B'. Now define

$$\begin{aligned} n_{A'} &= \inf\{l \in \{0,1,\dots,\deg(A)\} \mid p|a'_j,\ j \in \{0,1,\dots,l\},\ p \nmid a'_l\}, \\ n_{B'} &= \inf\{l \in \{0,1,\dots,\deg(B)\} \mid p|b'_j,\ j \in \{0,1,\dots,l\},\ p \nmid b'_l\}. \end{aligned}$$

Note that $p|c_{n_{A'}+n_{B'}}$, and since

$$c_{n_{A'}+n_{B'}} = a'_0 b'_{n_{A'}+n_{B'}} + \dots + a'_{n_{A'}-1} b'_{n_{B'}+1} + a'_{n_{A'}} b'_{n_{A'}} + a'_{n_{A'}+1} b'_{n_{B'}-1} + \dots + a'_{n_{A'}+n_{B'}} b'_0,$$

$p|a'_{n_{A'}} b'_{n_{A'}}$, which implies that $p|a'_{n_{A'}}$ or $p|b'_{n_{A'}}$ since irreducibles are prime in unique factorisation domains (Proposition 2.2.5). This implies that either A' or B' is not primitive.

(iii) This follows directly from part (ii) since the product of units is again a unit.

(iv) Since $\mathsf{R} \subseteq \mathsf{F}_{\mathsf{R}}$, it is clear that if $A|B$ and $B|A$ in R, then $A|B$ and $B|A$ in F_{R}. Now suppose that $B|A$ in F_{R}. Then $A = U \cdot B$ where $U \in \mathsf{F}_{\mathsf{R}}[\xi]$ is a unit. This means that $U = u$ for some $u \in \mathsf{F}_{\mathsf{R}}$, and ;et us write $u = \frac{a}{b}$ for $a, b \in \mathsf{R}$ with $b \neq 0_{\mathsf{R}}$. We thus have $bA = aB$. Since A and B are primitive, if c_A and c_B are contents for A and B, respectively, these must be units. Therefore, both b and bc_A are contents for bA and both a and ac_B are contents for aB. This means that $a = bv$ for a unit $v \in \mathsf{R}$ so that $bA = bvB$. Since $\mathsf{R}[\xi]$ is an integral domain, this implies that $A = vB$ for a unit $v \in \mathsf{R}$. Now, $A|B$ and $B|A$ in $\mathsf{R}[\xi]$ since v is also a unit in $\mathsf{R}[\xi]$.

(v) Suppose that A is not irreducible in $\mathsf{F}_{\mathsf{R}}[\xi]$ and write $A = B \cdot C$ for $\mathsf{F}_{\mathsf{R}}[\xi]$ both nonunits. Ee must therefore have $\deg(B), \deg(C) \geq 1$. Write

$$B = \sum_{j=0}^{k} \frac{a_j}{b_j}\xi^j, \quad C = \sum_{j=0}^{l} \frac{c_j}{d_j}\xi^j$$

for $a_j, b_j \in \mathsf{R}$ with $b_j \neq 0_{\mathsf{R}}$ for $j \in \{0,1,\dots,k\}$ and for $c_j, d_j \in \mathsf{R}$ with $d_j \neq 0_{\mathsf{R}}$ for $j \in \{0,1,\dots,l\}$. Write $b = b_0 b_1 \cdots b_k$ and for $j \in \{0,1,\dots,k\}$ define

$$\hat{b}_j = b_b b_1 \cdots b_{j-1} b_{j+1} \cdots b_k.$$

Define $B' = \sum_{j=0}^{k} a_j \hat{b}_j \xi^j \in \mathsf{R}[\xi]$ and write $B' = c_{B'} B''$ where $c_{B'}$ is a content of B' and where B'' is primitive, by part (i). A direct computation then shows that $B = \frac{1_{\mathsf{R}}}{b} B' = \frac{c_{B'}}{b} B''$. An entirely similar computation gives $C = \frac{c_{C'}}{d} C''$ where $c_{C'} \in \mathsf{R}$ and $C'' \in \mathsf{R}[\xi]$ is primitive. Therefore, since $A = B \cdot C$, we have $bdA = c_{B'} c_{C'} B'' \cdot C''$. Since A and $B'' \cdot C''$ are primitive, the latter by part (ii), it follows that both bd and $c_{B'} c_{C'}$ are contents for A. Thus $bd = u c_{B'} c_{C'}$

for a unit $u \in \mathsf{R}$. Thus $bdA = bduB'' \cdot C''$, or $A = uB'' \cdot C''$. Since $\deg(B'') = \deg(B) \geq 1$ and $\deg(C'') = \deg(C) \geq 1$, this implies that A is not irreducible in $\mathsf{R}[\xi]$.

Now suppose that A is irreducible in $\mathsf{F}_{\mathsf{R}}[\xi]$ and write $A = B \cdot C$ for $B, C \in \mathsf{R}[\xi] \subseteq \mathsf{F}_{\mathsf{R}}[\xi]$. Thus either B or C must be a unit in $\mathsf{F}_{\mathsf{R}}[\xi]$, and so we must have either $\deg(B) = 0$ or $\deg(C) = 0$. Suppose, without loss of generality, that $\deg(B) = 0$ so that $B = b_0 \in \mathsf{R} \setminus \{0\}$. Then, if c_C is a content for C, $b_0 c_C$ is a content for $A = B \cdot C$. Since A is primitive, $b_0 c_C$ must be a unit, and so, in particular, b_0 must be a unit in R. Thus B is a unit in $\mathsf{R}[\xi]$, showing that A is irreducible in $\mathsf{R}[\xi]$. ■

Now, using the proposition, we can prove our main result concerning the factorisation properties of polynomials over unique factorisation domains.

2.2.10 Theorem (Polynomial rings over unique factorisation domains are unique factorisation domains) *If* R *is a unique factorisation domain, then* $\mathsf{R}[\xi]$ *is a unique factorisation domain.*

Proof Let $A \in \mathsf{R}[\xi]$ be a nonzero nonunit. If $\deg(A) = 0$ then A is an element of R under the natural inclusion of R in $\mathsf{R}[\xi]$. In this case, A possesses a factorisation as a product of irreducibles since R is a unique factorisation domain. Now suppose that $\deg(A) \geq 1$, and by Proposition 2.2.9(i) write $A = c_A A'$ where $c_A \in \mathsf{R}$ is a content of A and where A' is primitive. If c_A is not a unit then write $c_A = c_{A,1} \cdots c_{A,l}$ where $c_{A,j} \in \mathsf{R}$, $j \in \{1, \dots, l\}$, are irreducible, this being possible since R is a unique factorisation domain. Note that the elements $c_{A,j}$, $j \in \{1, \dots, l\}$, are also irreducible thought of as elements of $\mathsf{R}[\xi]$ (why?). Now, since F_{R} is a unique factorisation domain since it is a Euclidean domain, write $A' = P'_1 \cdots P'_k$ where $P'_1, \dots, P'_k \in \mathsf{F}_{\mathsf{R}}[\xi]$ are irreducible. Now proceed as in the proof of Proposition 2.2.9(v) to show that, for $j \in \{1, \dots, k\}$, $P'_j = \frac{a_j}{b_j} P_j$ for $a_j, b_j \in \mathsf{R}$ with $b_j \neq 0_{\mathsf{R}}$ and with $P_j \in \mathsf{R}[\xi]$ primitive. Since $\frac{a_j}{b_j}$ is a unit in F_{R} and so in $\mathsf{F}_{\mathsf{R}}[\xi]$, it follows that P_j is irreducible in $\mathsf{F}_{\mathsf{R}}[\xi]$, and so in $\mathsf{R}[\xi]$ by Proposition 2.2.9(v). Writing $a = a_1 \cdots a_k$ and $b = b_1 \cdots b_k$, we have $A' = \frac{a}{b} P_1 \cdots P_k$, or $bA' = aP_1 \cdots P_k$. Since A' and $P_1 \cdots P_k$ are primitive (the latter by Proposition 2.2.9(iii)), it follows that $a = ub$ for u a unit in R. Therefore, if c_A is not a unit, we have

$$A = c_A A' = c_{A,1} \cdots c_{A,l} (uP_1) P_2 \cdots P_k,$$

where $c_{A,1}, \dots, c_{A,l} \in \mathsf{R} \subseteq \mathsf{R}[\xi]$ and $uP_1, P_2, \dots, P_k \in \mathsf{R}[\xi]$ are all irreducible in $\mathsf{R}[\xi]$. If c_A is a unit, then A is primitive already, and we can directly write

$$A = (uP_1) P_2 \cdots P_k,$$

where $(uP_1, P_2, \dots, P_k \in \mathsf{R}[\xi]$ are irreducible. This gives part (i) of Definition 2.2.2.

Now we verify part (ii) of Definition 2.2.2. We begin with a lemma. We already know from above that every element of $\mathsf{R}[\xi]$ possesses a factorisation as a product of irreducible. The lemma guarantees that the factorisation is of a certain form.

1 **Lemma** *If* R *is a unique factorisation domain and if* $\mathrm{A} \in \mathsf{R}[\xi]$ *is written as a product of irreducibles,* $\mathrm{A} = \mathrm{F}_1 \cdots \mathrm{F}_\mathrm{m}$*, then there exists irreducibles* $\mathrm{c}_1, \dots, \mathrm{c}_\mathrm{l} \in \mathsf{R}$ *and irreducibles* $\mathrm{P}_1, \dots, \mathrm{P}_\mathrm{k} \in \mathsf{R}[\xi]$ *such that* $\mathrm{l} + \mathrm{k} = \mathrm{m}$ *and such that* $\mathrm{F}_{\mathrm{j}_\mathrm{r}} = \mathrm{c}_\mathrm{r}$, $\mathrm{r} \in \{1, \dots, \mathrm{l}\}$*, and* $\mathrm{F}_{\mathrm{j}_{\mathrm{l+s}}} = \mathrm{P}_\mathrm{s}$, $\mathrm{s} \in \{1, \dots, \mathrm{k}\}$*, where* $\{1, \dots, \mathrm{m}\} = \{\mathrm{j}_1, \dots, \mathrm{j}_\mathrm{m}\}$.

Proof From the first part of the proof of the theorem we can write $A = c_{A,1} \cdots c_{A,l} P_1 \cdots P_k$ for irreducibles $c_{A,1}, \dots, c_{A,l} \in \mathsf{R}$ and irreducibles $P'_1, \dots, P'_k \in \mathsf{R}[\xi]$. We thus have

$$F_1 \cdots F_m = c_{A,1} \cdots c_{A,l} P'_1 \cdots P'_k.$$

Let $\{j_1, \dots, j_{l'}\}$ be the indices from $\{1, \dots, m\}$ such that $\deg(F_j) = 0$ if and only if $j \in \{1, \dots, j_{l'}\}$. Denote by $\{j_{l'+1}, \dots, j_m\}$ the remaining indices, so that $\deg(F_j) \geq 1$ if and only if $j \in \{j_{l'+1}, \dots, j_m\}$. Since the polynomials $F_{j_{l'+1}}, \dots, F_{j_m}$ are irreducible, they are primitive, so that $c_{A,1} \cdots c_{A,l}$ and $F_{j_1} \cdots F_{j_{l'}}$ are both contents for $P'_1 \cdots P'_k$. Thus there exists a unit $u \in \mathsf{R}$ such that $F_{j_1} \cdots F_{j_{l'}} = u c_{A,1} \cdots c_{A,l}$. By unique factorisation in R, $l' = l$ and there exists $\sigma \in \mathfrak{S}_l$ such that $F_{j_r} = u_{\sigma(r)} c_{A,\sigma(r)}$ for $r \in \{1, \dots, l\}$, and where $u_1, \dots, u_l$ are units in R. The result now follows by taking $c_r = u_{\sigma(r)} c_{A,\sigma(r)}$, $r \in \{1, \dots, l\}$ and $P_s = F_{j_{l+s}}$, $s \in \{1, \dots, k\}$. ▼

Now, using the lemma, let $c_1 \cdots c_l P_1 \cdots P_k$ and $c'_1 \cdots c'_{l'} P'_1 \cdots P'_{k'}$ be two factorisations of A by irreducibles, where $c_1, \dots, c_l, c'_1, \dots, c'_{l'} \in \mathsf{R}$ are irreducible and $P_1, \dots, P_k, P'_1, \dots, P'_{k'} \in \mathsf{R}[\xi]$ are irreducible. Since $P_1 \cdots P_k$ and $P'_1, \dots, P'_{k'}$ are primitive, $c_1 \cdots c_l$ and $c'_1 \cdots c'_{l'}$ are contents for A, and so there exists a unit $u \in \mathsf{R}$ such that $c_1 \cdots c_l = u c'_1 \cdots c'_{l'}$. Since R is a unique factorisation domain, $l = l'$ and there exists a permutation $\sigma \in \mathfrak{S}_l$ such that $c'_{\sigma(j)} = u_j c_j$ for $j \in \{1, \dots, l\}$, and for some set $u_1, \dots, u_l$ of units. Since $P_1 \cdots P_k$ and $P'_1 \cdots P'_{k'}$ have the same content, up to multiplication by a unit, it follows that $P'_1 \cdots P'_k = U P_1 \cdots P_k$ where $U \in \mathsf{R}[\xi]$ is a unit. Thus $U = v$ where $v \in \mathsf{R}$ is a unit. Therefore, since $\mathsf{F}_{\mathsf{R}}[\xi]$ is a unique factorisation domain, $k = k'$ and there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $P'_{\sigma(j)} = v_j P_j$ for $j \in \{1, \dots, k\}$, and where v_j is a unit in F_{R}. Thus, in $\mathsf{F}_{\mathsf{R}}[\xi]$, we have $P'_{\sigma(j)} | P_j$ and $P_j | P'_{\sigma(j)}$, $j \in \{1, \dots, k\}$. By Proposition 2.2.9(iv) we then have, in $\mathsf{R}[\xi]$, $P'_{\sigma(j)} | P_j$ and $P_j | P'_{\sigma(j)}$, $j \in \{1, \dots, k\}$. Therefore, there exists units $u_1, \dots, u_k \in \mathsf{R}$ such that $P'_{\sigma(j)} = u_j P_j$, $j \in \{1, \dots, k\}$. This then gives the uniqueness, up to units, of factorisation in $\mathsf{R}[\xi]$. ■

2.2.2 Noetherian rings and modules

The next bit of background in algebra we consider is part of what is commonly known as "commutative algebra." The results here can be found in §VIII.1 of [Hungerford 1980], for example.

2.2.11 Definition (Noetherian module, Noetherian ring) Let R be a commutative unit ring and let A be a unitary R-module.

(i) The module A is ***Noetherian*** if, for every family $(\mathsf{B}_j)_{j \in \mathbb{Z}_{>0}}$ of submodules of A satisfying $\mathsf{B}_j \subseteq \mathsf{B}_{j+1}$, $j \in \mathbb{Z}_{>0}$, there exists $k \in \mathbb{Z}_{>0}$ such that $\mathsf{B}_j = \mathsf{B}_k$ for $j \geq k$.

(ii) The ring R is ***Noetherian*** if, for every family $(\mathsf{I}_j)_{j \in \mathbb{Z}_{>0}}$ of ideals of R satisfying $\mathsf{I}_j \subseteq \mathsf{I}_{j+1}$, $j \in \mathbb{Z}_{>0}$, there exists $k \in \mathbb{Z}_{>0}$ such that $\mathsf{I}_j = \mathsf{I}_k$ for $j \geq k$. •

For us, the following result will be key.

2.2.12 Proposition (Finitely generated property of submodules of Noetherian modules) *Let A be a unitary module over a commutative unit ring R. Then A is Noetherian and if and only if every submodule of A is finitely generated.*

Proof Suppose that A is Noetherian and let $\mathsf{B} \subseteq \mathsf{A}$ be a submodule. Let $P(\mathsf{B})$ be the set of finitely generated submodules of B which we partially order by $\mathsf{C}_1 \preceq \mathsf{C}_2$ if $\mathsf{C}_1 \subseteq \mathsf{C}_2$ for $\mathsf{C}_1, \mathsf{C}_2 \in P(\mathsf{B})$. Note that $P(\mathsf{B})$ is nonempty as it contains, for example, the trivial submodule.

We claim that $P(\mathsf{B})$ contains a maximal element under the partial ordering. Suppose not so that, for each $\mathsf{C} \in P(\mathsf{B})$, there exists $\mathsf{C}' \in P(\mathsf{B})$ such that $\mathsf{C} \subsetneq \mathsf{C}'$. Use the Axiom of Choice to define $\phi\colon P(\mathsf{B}) \to P(\mathsf{B})$ such that $\phi(\mathsf{C}) \subsetneq \mathsf{C}$. Recursively define $\psi\colon \mathbb{Z}_{\geq 0} \to P(\mathsf{B})$ by $\psi(0) = \{0\}$ and $\psi(j+1) = \phi(\psi(j))$, $j \in \mathbb{Z}_{\geq 0}$. This gives a sequence $(\psi(j))_{j\in\mathbb{Z}_{\geq 0}}$ of finitely generated submodules of B such that $\psi(j) \subsetneq \psi(j+1)$, $j \in \mathbb{Z}_{>0}$, contradicting the fact that A is Noetherian. Thus $P(\mathsf{B})$ contains a maximal element C.

We claim that $\mathsf{C} = \mathsf{B}$. Let $(c_1, \dots, c_k)$ be generators for C. For $b \in \mathsf{B}$ let D_b be the module generated by $(b, c_1, \dots, c_k)$. Thus $\mathsf{D}_b \in P(\mathsf{B})$ and $\mathsf{C} \subseteq \mathsf{D}_b$. Since C is maximal in $P(\mathsf{B})$ it follows that $\mathsf{C} = \mathsf{D}_b$ for every $b \in \mathsf{B}$. This implies that $\mathsf{B} \subseteq \mathsf{C}$ and so $\mathsf{C} = \mathsf{B}$ since we obviously have $\mathsf{C} \subseteq \mathsf{B}$. Thus B is finitely generated.

Now suppose that every submodule of A is finitely generated. Let $(B_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of submodules of A satisfying $\mathsf{B}_j \subseteq \mathsf{B}_{j+1}$ for $j \in \mathbb{Z}_{>0}$. Then take $\mathsf{B} = \cup_{j\in\mathbb{Z}_{>0}}\mathsf{B}_j$. It is easy to verify that B is a submodule, and so is finitely generated. Therefore, there exists $b_1, \dots, b_m \in \mathsf{B}$ which generate B. By definition of B, $b_l \in \mathsf{B}_{j_l}$ for some $j_l \in \mathbb{Z}_{>0}$. Let $k = \max\{j_1, \dots, j_m\}$. Then $b_1, \dots, b_m \in \mathsf{B}_k$ and so $\mathsf{B} \subseteq \mathsf{B}_k$. Therefore, $\mathsf{B}_j = \mathsf{B} = \mathsf{B}_k$ for $j \geq k$. Thus B is Noetherian. ■

The following lemma about Noetherian modules is useful.

2.2.13 Lemma (Noetherian modules and exact sequences) *Let* R *be a commutative unit ring and let* A, B, *and* C *be unitary* R*-modules such that we have a short exact sequence*

$$0 \longrightarrow \mathsf{A} \xrightarrow{\phi} \mathsf{B} \xrightarrow{\psi} \mathsf{C} \longrightarrow 0$$

Then the following statements are equivalent:

(i) B *is Noetherian;*

(ii) A *and* C *are Noetherian.*

Proof (i) $\Longrightarrow$ (ii) Let B be Noetherian.

If $\mathsf{A}' \subseteq \mathsf{A}$ is a submodule, then $\phi(\mathsf{A}')$ is a submodule of B, and is isomorphic to A' since ϕ is injective. Let $(\mathsf{A}'_j)_{j\in\mathbb{Z}_{>0}}$ be a family of submodules of A satisfying $\mathsf{A}'_j \subseteq \mathsf{A}'_{j+1}$, $j \in \mathbb{Z}_{>0}$. Since B is Noetherian there exists $k \in \mathbb{Z}_{>0}$ such that $\phi(\mathsf{A}'_j) = \phi(\mathsf{A}'_k)$ for $j \geq k$. Since ϕ is an R-module monomorphism this means that $\mathsf{A}'_j = \mathsf{A}'_k$ for $j \geq k$. Thus A is Noetherian.

If $\mathsf{C}' \subseteq \mathsf{C}$ is a submodule, then $\psi^{-1}(\mathsf{C}')$ is a submodule of B. Let $(\mathsf{C}'_j)_{j\in\mathbb{Z}_{>0}}$ be a family of submodules of C satisfying $\mathsf{C}'_j \subseteq \mathsf{C}'_{j+1}$, $j \in \mathbb{Z}_{>0}$. Since B is Noetherian there exists $k \in \mathbb{Z}_{>0}$ such that $\psi^{-1}(\mathsf{C}'_j) = \psi^{-1}(\mathsf{C}'_k)$ for $j \geq k$. Surjectivity of ψ then implies that $\mathsf{C}'_j = \mathsf{C}'_k$ for $j \geq k$.

(ii) $\Longrightarrow$ (i) First we claim that if $\mathsf{B}' \subseteq \mathsf{B}'' \subseteq \mathsf{B}$ are submodules satisfying

$$\mathsf{B}' \cap \mathsf{A} = \mathsf{B}'' \cap \mathsf{A}, \quad (\mathsf{B}' + \mathsf{A})/\mathsf{A} = (\mathsf{B}'' + \mathsf{A})/\mathsf{A}$$

then $\mathsf{B}' = \mathsf{B}''$. Indeed, let $b'' \in \mathsf{B}''$. Then there exists $b' \in \mathsf{B}'$ and $a \in \mathsf{A}$ such that $b' + a + \mathsf{A} = b'' + \mathsf{A}$. Thus $b' - b'' \in \mathsf{A}$. Since $\mathsf{B}' \subseteq \mathsf{B}''$ we also have $b' - b'' \in \mathsf{B}''$ and so $b' - b'' \in \mathsf{B}'' \cap \mathsf{A} = \mathsf{B}' \cap \mathsf{A}$. Therefore $b' - b'' \in \mathsf{B}'$ and so $b'' \in \mathsf{B}'$, as claimed.

Now let $(\mathsf{B}'_j)_{j\in\mathbb{Z}_{>0}}$ such that $\mathsf{B}'_j \subseteq \mathsf{B}'_{j+1}$, $j \in \mathbb{Z}_{>0}$. Since A is Noetherian there exists $l \in \mathbb{Z}_{>0}$ such that $\mathsf{B}'_j \cap \mathsf{A} = \mathsf{B}'_l \cap \mathsf{A}$ for $j \geq l$. Since C is Noetherian, and noting that $\mathsf{C} \simeq \mathsf{B}/\mathsf{A}$, there exists $m \in \mathbb{Z}_{>0}$ such that

$$(\mathsf{B}'_j + \mathsf{A})/\mathsf{A} = (\mathsf{B}'_k + \mathsf{A})/\mathsf{A}$$

for $j \geq k$. Letting $k = \max\{l, m\}$ we see that

$$\mathsf{B}'_j \cap \mathsf{A} = \mathsf{B}'_l \cap \mathsf{A}, \quad (\mathsf{B}'_j + \mathsf{A})/\mathsf{A} = (\mathsf{B}'_k + \mathsf{A})/\mathsf{A}$$

for $j \geq k$. Our claim at the beginning of this part of the proof then gives $\mathsf{B}'_j = \mathsf{B}'_k$ for $j \geq k$. ■

The following consequence of the lemma will be useful.

2.2.14 Corollary (Finite direct sums of Noetherian modules are Noetherian) *If* R *is a commutative unit ring and if* $\mathsf{A}_1, \dots, \mathsf{A}_k$ *are unitary modules over* R*, then* $\oplus_{j=1}^{k} \mathsf{A}_j$ *is Noetherian if* $\mathsf{A}_1, \dots, \mathsf{A}_k$ *are Noetherian.*

Proof We prove this by induction on k, the case of $k = 1$ being vacuous. Suppose the lemma holds for $k \in \{1, \dots, m\}$ and let $\mathsf{A}_1, \dots, \mathsf{A}_{m+1}$ be Noetherian modules. Consider the exact sequence

$$0 \longrightarrow \mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_m \longrightarrow \mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_m \oplus \mathsf{A}_{m+1} \longrightarrow \mathsf{A}_{m+1} \longrightarrow 0$$

where the second arrow is the inclusion and the third arrow is the projection. By Lemma 2.2.13, $\mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_m \oplus \mathsf{A}_{m+1}$ is Noetherian since A_{m+1} and $\mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_m$ are Noetherian, the latter by the induction hypothesis. ■

Noetherian rings give rise to Noetherian modules.

2.2.15 Proposition (Modules over Noetherian rings are Noetherian) *If* A *is a finitely generated unitary module over a Noetherian ring* R*, then* A *is Noetherian.*

Proof Let $a_1, \dots, a_k$ be generators for A and define $\phi\colon \mathsf{R}^k \to \mathsf{A}$ by

$$\phi(r_1, \dots, r_k) = r_1 a_1 + \cdots + r_k a_k.$$

Then ϕ is an R-module epimorphism and so $\mathsf{A} \simeq \mathsf{R}^k/\ker(\phi)$ by the first isomorphism theorem [Hungerford 1980, Theorem IV.1.7]. Consider now the exact sequence

$$0 \longrightarrow \ker(\phi) \longrightarrow \mathsf{R}^k \longrightarrow \mathsf{A} \simeq \mathsf{R}^k/\ker(\phi) \longrightarrow 0$$

where the second arrow is the inclusion and the third arrow is the projection. By Corollary 2.2.14, R^k is Noetherian. By Lemma 2.2.13 it follows that A is Noetherian. ■

Our next result gives an interesting class of Noetherian rings.

2.2.16 Theorem (Hilbert Basis Theorem) *If* R *is a Noetherian ring then the polynomial ring* $\mathsf{R}[\xi_1, \dots, \xi_\mathrm{n}]$ *is also a Noetherian ring.*

Proof Since $\mathsf{R}[\xi_1, \dots, \xi_n] \simeq \mathsf{R}[\xi_1, \dots, \xi_{n-1}][\xi_n]$, it suffices by induction and Corollary 2.2.14 to prove the theorem when $n = 1$.

First some terminology. If $P \in \mathsf{R}[\xi]$ is given by

$$P = a_k\xi^k + \cdots + a_1\xi + a_0$$

with $a_k \neq 0$, then call a_k the *initial coefficient* of P.

Let I be an ideal in $\mathsf{R}[\xi]$. We will show I is finitely generated, so that $\mathsf{R}[\xi]$ is Noetherian by Proposition 2.2.12, recalling that submodules of R are precisely the ideals of R. We define a sequence $(P_k)_{k\in\mathbb{Z}_{>0}}$ in $\mathsf{R}[\xi]$ as follows. Let $P_0 \in \mathsf{R}[\xi]$ be chosen so that

$$\deg(P_0) = \min\{\deg(P) \mid P \in \mathsf{I}\}.$$

Then, if $P_0, P_1, \dots, P_k$ have been chosen then choose P_{k+1} so that

$$\deg(P_{k+1}) = \min\{\deg(P) \mid P \in \mathsf{I} \setminus (P_0, P_1, \dots, P_k)\}.$$

Let a_k be the initial coefficient of P_k, and consider the ideal J of R generated by the family $(a_k)_{k\in\mathbb{Z}_{>0}}$ of initial coefficients. Since R is Noetherian, by Proposition 2.2.12 there exists $m \in \mathbb{Z}_{>0}$ such that J is generated by $(a_0, a_1, \dots, a_m)$. We claim that I is generated by $P_0, P_1, \dots, P_m$. Indeed, suppose otherwise. Then, possibly by choosing m larger if necessary, $P_{m+1} \in \mathsf{I}\setminus(P_0, P_1, \dots, P_m)$. Since $a_0, a_1, \dots, a_m$ generate the ideal J, $a_{m+1} = \sum_{k=0}^{m} r_k a_k$ for some $r_1, \dots, r_m \in \mathsf{R}$. Let $Q = \sum_{k=0}^{m} r_k P_k \xi^{d_k}$ where $d_k = \deg(P_{m+1}) - \deg(P_k)$. One sees that $\deg(Q) = m + 1$ and the coefficient of ξ^{m+1} is a_{m+1}. Thus $\deg(P_{m+1} - Q) < \deg(P_{m+1})$ and also $P_{m+1} - Q \in \mathsf{I}$. Since $Q \in \mathsf{I}$ it follows that $P_{m+1} - Q \notin (P_0, P_1, \dots, P_m)$ since $P_{m+1} \notin \mathsf{I}$. But this contradicts the definition of $\deg(P_{m+1})$, and so we conclude I is finitely generated by $P_0, P_1, \dots, P_m$. ■

2.2.3 Local rings

In this section we introduce another notion from commutative algebra, that of a local ring.

2.2.17 Definition (Local ring) A commutative unit ring R is a ***local ring*** if it possesses a unique maximal ideal. •

The following result will also be interesting for us in Proposition 2.3.7 for showing that a certain ring is *not* Noetherian. We shall be mainly interested in the corollary following the theorem. The proof we give here comes from [Perdry 2004].

2.2.18 Theorem (Krull Intersection Theorem) *If* R *is Noetherian and* $\mathsf{I} \subseteq \mathsf{R}$ *is an ideal, then there exists* $\mathrm{a} \in \mathsf{I}$ *such that*

$$(1 - \mathrm{a}) \bigcap_{\mathrm{j}\in\mathbb{Z}_{>0}} \mathsf{I}^\mathrm{j} = \{0\}.$$

Proof By Proposition 2.2.12 there exists $a_1, \dots, a_n \in \mathsf{I}$ generating I. Let $b \in \cap_{j \in \mathbb{Z}_{>0}} \mathsf{I}^j$ and note that, since $b \in \mathsf{I}^m$ for each $m \in \mathbb{Z}_{>0}$, we can write

$$b = p_{m1} a_1^m + \cdots + p_{mn} a_n^m$$

for some $p_{ml} \in \mathsf{R}$, $l \in \{1, \dots, n\}$. Let us define

$$P_m = p_{m1} \xi_1^m + \cdots + p_{mn} \xi_n^m \in \mathsf{R}[\xi_1, \dots, \xi_n].$$

For $m \in \mathbb{Z}_{>0}$ define an ideal $\mathsf{J}_m \subseteq \mathsf{R}[\xi_1, \dots, \xi_n]$ as being generated by $P_1, \dots, P_m$. Then we clearly have $algJ_m \subseteq \mathsf{J}_{m+1}$ for $m \in \mathbb{Z}_{>0}$. By the Hilbert Basis Theorem (i.e., the lemma above), there exists $k \in \mathbb{Z}_{>0}$ such that $\mathsf{J}_m = \mathsf{J}_k$ for $m \geq k$. Thus $P_{k+1} \in \mathsf{J}_m$ and so there exists $Q_1, \dots, Q_k \in \mathsf{R}[\xi_1, \dots, \xi_n]$ such that

$$P_{k+1} = Q_1 P_k + \cdots + Q_n P_1,$$

and we may, moreover, without loss of generality assume that Q_j is homogeneous of degree j for each $j \in \{1, \dots, n\}$. By definition of P_j, $j \in \{1, \dots, k+1\}$, if we evaluate the above equality at $\xi_l = a_l$, $l \in \{1, \dots, n\}$, we have

$$b = (Q_1(a_1, \dots, a_n) + \cdots + Q_k(a_1, \dots, a_n))b,$$

and the coefficient of b on the right is in I, being a linear combination of powers of $a_1, \dots, a_n$. Thus $b \in b\mathsf{I}$, this holding for any $b \in \cap_{j \in \mathbb{Z}_{>0}} \mathsf{I}^j$. Therefore, for any $b \in \cap_{j \in \mathbb{Z}_{>0}}$ we have $(1 - a_b)b = 0$ for some $a_b \in \cap_{j \in \mathbb{Z}_{>0}} \mathsf{I}^j$. Now, by Proposition 2.2.12, let $b_1, \dots, b_d$ generate $\cap_{j \in \mathbb{Z}_{>0}} \mathsf{I}^j$, and let $a_1, \dots, a_d \in \mathsf{I}$ satisfy $(1 - a_j)b_j = 0$, $j \in \{1, \dots, d\}$. If $b \in \cap_{j \in \mathbb{Z}_{>0}} \mathsf{I}^j$ then we write $b = r_1 b_1 + \cdots + r_d b_d$ for $r_1, \dots, r_d \in \mathsf{R}$, and determine that

$$(1 - a_1) \cdots (1 - a_d) b = (1 - a_1) \cdots (1 - a_d)(r_1 b_1 + \cdots + r_d b_d) = 0.$$

We then let $a \in \mathsf{R}$ be such that $1 - a = (1 - a_1) \cdots (1 - a_d)$, and note that, actually, $a \in \mathsf{R}$. This gives the theorem. ■

For us, it is the following corollary that will be of immediate value. We recall that an ideal I in a ring R is ***maximal*** if $\mathsf{I} \neq \mathsf{R}$ and if $\mathsf{J} \subseteq \mathsf{R}$ is an ideal for which $\mathsf{I} \subseteq \mathsf{J}$, then either $\mathsf{J} = \mathsf{I}$ or $\mathsf{J} = \mathsf{R}$. A ***local ring*** is a ring possessing a unique maximal ideal.

2.2.19 Corollary (Krull Intersection Theorem for local rings) *If R is a Noetherian local ring with unique maximal ideal $\mathfrak{m}$, then $\cap_{j \in \mathbb{Z}_{>0}} \mathfrak{m}^j = \{0\}$.*

Proof We first claim that, for a general local ring, $\mathfrak{m}$ consists of all of the nonunits of R. Indeed, if $a \in \mathsf{R}$ is a nonunit then the ideal (a) generated by a is not equal to R, and so, therefore, we must have $(a) \subseteq \mathfrak{m}$. In particular, $a \in \mathfrak{m}$. Conversely, if $a \in \mathfrak{m}$ then $(a) \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ is maximal $\mathfrak{m} \neq \mathsf{R}$ and so $(a) \neq \mathsf{R}$. Thus a is not a unit.

Now we claim that if $a \in \mathfrak{m}$ then $1 - a$ is a unit. Indeed, if $1 - a$ were not a unit, then our argument above gives $1 - a \in \mathfrak{m}$ and so gives $1 \in \mathfrak{m}$. This, however, $\mathfrak{m} = \mathsf{R}$ and so contradicts the maximality of $\mathfrak{m}$.

Now, according to the Krull Intersection Theorem, let $a \in \mathfrak{m}$ be such that $(1 - a) \cap_{j \in \mathbb{Z}_{>0}} \mathfrak{m}^j = \{0\}$. Thus, if $b \in \cap_{j \in \mathbb{Z}_{>0}} \mathfrak{m}^j$, we have $(1 - a)b = 0$. By our assertion of the previous paragraph, $(1 - a)$ is a unit. Therefore, $b = 0$, as claimed. ■

2.3 Algebraic properties of germs of holomorphic or real analytic functions

In this section we use the Weierstrass Preparation Theorem to prove some important results about the "infinitesimal" character of real analytic functions. First we must characterise what we mean by "infinitesimal."

2.3.1 Ring of germs of holomorphic or real analytic functions

We shall study, not functions, but rather germs of functions which are designed to capture the local behaviour of functions. We subsequently shall develop germs in greater generality in Sections 4.2.3, 4.3.3, and 5.6.1. Here we consider a special case that forms the foundation for the general definitions that follow. Indeed, general characterisations of germs on manifolds amount to choosing local coordinates and then using the constructions we give here.

We define as follows an equivalence relation on the set of ordered pairs $(f, \mathcal{U})$, where $\mathcal{U} \subseteq \mathbb{F}^n$ is a neighbourhood of $\mathbf{0} \in \mathbb{F}^n$ and $f\colon \mathcal{U} \to \mathbb{F}$ is a holomorphic or real analytic function. We say that $(f_1, \mathcal{U}_1)$ and $(f_1, \mathcal{U}_2)$ are ***equivalent*** if there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ of $\mathbf{0}$ such that $f_1|\mathcal{U}_1 = f_2|\mathcal{U}$. This notion of equivalence is readily verified to be an equivalence relation. We denote a typical equivalence class by $[(f, \mathcal{U})]_{\mathbf{0}}$, or simply by $[f]_{\mathbf{0}}$ if the domain of f is understood or immaterial. The set of equivalence classes we denote by $\mathscr{C}^{\mathrm{hol}}_{\mathbf{0},\mathbb{C}^n}$ if $\mathbb{F} = \mathbb{C}$ or by $\mathscr{C}^{\omega}_{\mathbf{0},\mathbb{R}^n}$ if $\mathbb{F} = \mathbb{R}$, which we call the set of ***germs*** of holomorphic or real analytic functions at $\mathbf{0}$, respectively. We make the set of germs into a ring by defining the following operations of addition and multiplication:

$$[(f_1, \mathcal{U}_1)]_{\mathbf{0}} + [(f_2, \mathcal{U}_2)]_{\mathbf{0}} = [f_1|\mathcal{U}_1 \cap \mathcal{U}_2 + f_2|\mathcal{U}_1 \cap \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2]_{\mathbf{0}}$$
$$[(f_1, \mathcal{U}_1)]_{\mathbf{0}} \cdot [(f_2, \mathcal{U}_2)]_{\mathbf{0}} = [(f_1|\mathcal{U}_1 \cap \mathcal{U}_2)(f_2|\mathcal{U}_1 \cap \mathcal{U}_2), \mathcal{U}_1 \cap \mathcal{U}_2]_{\mathbf{0}}.$$

It is elementary to verify that these operations are well-defined, and indeed make the set of germs of holomorphic or real analytic functions into a ring. We shall study the algebraic properties of this ring in this section.

2.3.2 Algebraic structure of germs

Our first "serious" algebraic result concerning the structure of germs of real analytic functions is the following.

2.3.1 Theorem (The ring of germs of holomorphic or real analytic functions is a local ring) *The ring of germs of holomorphic or real analytic functions at* $\mathbf{0}$ *is a local ring with unique maximal ideal given by*

$$\mathfrak{m} = \{[\mathrm{f}]_{\mathbf{0}} \mid \mathrm{f}(\mathbf{0}) = 0\}.$$

Proof Let $r \in \{\mathrm{hol}, \omega\}$ so that we can use $\mathscr{C}^{r}_{\mathbf{0},\mathbb{F}^n}$ to denote the ring of germs of holomorphic or real analytic functions.

First of all, note that $\mathfrak{m}$ is indeed an ideal.

Now suppose that $\mathsf{J} \subseteq \mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}$ is a nonzero proper ideal and define

$$k = \inf\Big\{r \in \mathbb{Z}_{\geq 0} \;\Big|\; \text{there exists} \sum_{I\in\mathbb{Z}^n_{\geq 0}} \alpha_I \boldsymbol{x}^I \in \mathsf{J} \text{ such that } \alpha_I \neq 0 \text{ for } |I| = r\Big\}.$$

We claim that $k \in \mathbb{Z}_{>0}$. Indeed, if $k = 0$ then J contains a unit and so J is not proper. If $k = \infty$ then all elements of J are zero. Thus k is indeed nonzero and finite. Now let $[f]_{\mathbf{0}} \in \mathsf{J}$. By definition of k we can write

$$f(\boldsymbol{x}) = \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq k}} \alpha_I \boldsymbol{x}^I$$

in a neighbourhood of $\mathbf{0}$. Thus $[f]_{\mathbf{0}} \in \mathfrak{m}$ and so $\mathsf{J} \subseteq \mathfrak{m}$. ■

The following obvious corollary will be used repeatedly.

2.3.2 Corollary (Units in the ring of germs of holomorphic or real analytic functions) *We have that a holomorphic or real analytic germ $[\mathrm{f}]_{\mathbf{0}}$ is a unit if and only if $[\mathrm{f}]_{\mathbf{0}} \notin \mathfrak{m}$, i.e., if and only if $\mathrm{f}(\mathbf{0}) \neq 0$.*

Proof In the proof of Corollary 2.2.19 we showed that in a local ring the set of units is precisely the complement of the unique maximal ideal. ■

We also have the following rather useful property of the ring of germs of real analytic functions.

2.3.3 Theorem (The ring of germs of holomorphic or real analytic functions is a unique factorisation domain) *The ring of germs of holomorphic or real analytic functions at $\mathbf{0}$ is a unique factorisation domain.*

Proof We let $r \in \{\mathrm{hol}, \omega\}$. For simplicity of notation let us denote a germ by $[\cdot]$ rather than by $[\cdot]_{\mathbf{0}}$. We prove the theorem by induction on n.

The following lemma will be helpful. We denote by $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}[\eta]$ the polynomial ring over the ring $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}$. We think of $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}[\eta]$ as a subset of $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$ by asking that the polynomial

$$P = [f_k]\eta^k + \cdots + [f_1]\eta + f_0$$

be mapped to the germ of the function

$$(\boldsymbol{x}, y) \mapsto f_k(\boldsymbol{x})y^k + \cdots + f_1(\boldsymbol{x})y + f_0(\boldsymbol{x}).$$

With this identification, we state the following.

1 Lemma *If $[\mathrm{P}] \in \mathscr{C}^{\mathrm{r}}_{\mathbf{0},\mathbb{F}^{\mathrm{n}}}[\eta]$ is a nonzero nonunit, then the following two statements are equivalent:*

(i) *$[\mathrm{P}]$ is irreducible in the ring $\mathscr{C}^{\mathrm{r}}_{(\mathbf{0},0),\mathbb{F}^{\mathrm{n}+1}}$;*

(ii) *$[\mathrm{P}]$ is irreducible in the ring $\mathscr{C}^{\mathrm{r}}_{\mathbf{0},\mathbb{F}^{\mathrm{n}}}[\eta]$.*

Proof (i) $\Longrightarrow$ (ii): Suppose that P is not irreducible as an element of $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}[\eta]$. Then $[P] = [P_1][P_2]$ for nonzero nonunits $[P_1], [P_2] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}[\eta]$. One readily checks that $[P_1]$ and $[P_2]$ are also nonzero nonunits in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$. Thus $[P]$ is irreducible in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$.

(ii) $\implies$ (i): We first prove this part of the lemma in the case that P is equal to a Weierstrass polynomial W. Thus we assume that $[W]$ is not irreducible in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$ so that $[W] = [f_1][f_2]$ for nonzero nonunits $[f_1],[f_2] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$. Note that $W(\mathbf{0},y) = f_1(\mathbf{0},y)f_2(\mathbf{0},y)$ so that neither of the functions $y \mapsto f_1(\mathbf{0},y)$ nor $y \mapsto f_2(\mathbf{0},y)$ is identically zero in a neighbourhood of 0. Thus f_1 and f_2 are normalised and so, by the Weierstrass Preparation Theorem, we write $[f_1] = [E_1][W_1]$ and $[f_2] = [E_2][W_2]$ for units $[E_1],[E_2] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$ and Weierstrass polynomials W_1 and W_2. Thus $[W] = [E_1][E_2][W_1][W_2]$ Since $[W_1][W_2]$ is a Weierstrass polynomial, it follows from the uniqueness of the second part of the Weierstrass Preparation Theorem that $[E_1][E_2]$ is the identity and $[W_1][W_2] = [W]$. Therefore, $[W]$ is not irreducible as an element of $\mathrm{C}^r(\mathbf{0})\mathbb{F}^n[\eta]$.

Now suppose that $[P]$ is not irreducible in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$. As an element of $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$ note that P is normalised. Thus we can write $[P] = [E][W]$ for a unit $[E] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^{n+1}}$ and for a Weierstrass polynomial W. Since $[P]$ is not irreducible in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$ it follows that $[W]$ is also not irreducible in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$. As we showed in the previous paragraph, this means that $[W] = [W_1][W_2]$ for Weierstrass polynomials W_1 and W_2. Therefore, $[P] = [E][W_1][W_2]$. Since $[P] \in \mathrm{C}^r(\mathbf{0})\mathbb{F}^n[\eta]$ and since $[W_1]$ is a Weierstrass polynomial, by Lemma 2.1.4(i) it follows that $[E][W_1] \in \mathrm{C}^r(\mathbf{0})\mathbb{F}^n[\eta]$, showing that $[P]$ is not irreducible in $\mathrm{C}^r(\mathbf{0})\mathbb{F}^n[\eta]$. ▼

We now prove the theorem by induction on n. For $n = 1$, let us first identify the irreducibles in $\mathscr{C}^r_{0,\mathbb{F}}$. If $[f]$ is irreducible then the ideal generated by $[f]$ is maximal [Hungerford 1980, Theorem III.3.4]. Thus the ideal generated by $[f]$ is the unique maximal ideal $\mathfrak{m}$ of germs of functions vanishing at 0. In particular $f(0) = 0$ and so the Taylor series for f in some neighbourhood of 0 is

$$f(x) = \sum_{j=k}^{\infty} \alpha_j x^j = x^k \sum_{j=0}^{\infty} \alpha_{k+j} x^j$$

for some $k \in \mathbb{Z}_{>0}$, and where $\alpha_k \neq 0$. In order that $[f]$ be irreducible, we must have $k = 1$ since, otherwise, $[f]$ is a product of two nonzero nonunits. Thus $[f]$ is irreducible if and only if $f(x) = xg(x)$ for a C^r-function g defined on some neighbourhood of 0 for which $g(0) \neq 0$.

Now let $[f]$ be a nonzero nonunit. Since $[f]$ is a nonunit,

$$f(x) = \sum_{j=k}^{\infty} \alpha_j x^j = x^k \sum_{j=0}^{\infty} \alpha_{k+j} x^j$$

for some $k \in \mathbb{Z}_{>0}$, and where $\alpha_k \neq 0$. Thus f is a product of k irreducibles and the unit $\sum_{j=0}^{\infty} \alpha_{k+j} x^j$. This gives the theorem for $n = 1$.

Now suppose that the theorem holds for $n \in \{1,\dots,m\}$ and let $[f] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^{m+1}}$ be a nonzero nonunit. Note that the induction hypothesis is that $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}$ is a unique factorisation domain. Therefore, by Theorem 2.2.10, $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$ is a unique factorisation domain. By Lemma 2.1.3 there exists an orthogonal transformation ψ of $\mathbb{F}^{m+1}$ such that $\psi^* f$ is normalised. Thus there exists a Weierstrass polynomial W of degree (say) k and an analytic function E not vanishing at $(\mathbf{0},0)$ such that $[\psi^* f] = [E][W]$. Note that $[W]$ is a nonzero nonunit in $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$. Since $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$ is a unique factorisation domain, $[W] = [P_1]\cdots[P_k]$ for some irreducible

$[P_1], \dots, [P_k] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$. As functions in $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$, we note that $[P_1], \dots, [P_k]$ are normalised as a consequence of their being nonzero nonunits. Thus, by the Weierstrass Preparation Theorem, we can write $[P_j] = [E_j][W_j]$ for units $[E_j] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$ and for Weierstrass polynomials W_j, $j \in \{1, \dots, k\}$. Thus $[W] = [E'][W_1] \cdots [W_k]$ for a unit $[E] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$, and so

$$[\psi^* f] = [F][W_1] \cdots [W_k],$$

where $[F] = [E][E']$. By the lemma above, $[W_1], \dots, [W_k]$ are irreducible as elements of $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$. Since $[g] \mapsto [\psi_* g]$ is an isomorphism of $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$, it follows that $[\psi_* W_1], \dots, [\psi_* W_k]$ are irreducible. Thus $[f]$ is a finite product of irreducibles, namely

$$[f] = [\psi_* F][\psi_* W_1] \cdots [\psi_* W_k].$$

Now we show that the representation of $[f]$ as a product of irreducibles is unique, up to order and multiplication by units. Suppose that $[f] = [f_1] \cdots [f_l]$ for irreducibles $[f_1], \dots, [f_l] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$. Then

$$[\psi^* f] = [\psi^* f_1] \cdots [\psi^* f_l].$$

Now apply the second part of the Weierstrass Preparation Theorem to write, for each $j \in \{1, \dots, l\}$, $[\psi^* f_j] = [E_j][W'_j]$ for a Weierstrass polynomial W'_j and for $[E_j] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$ a unit. Thus

$$[\psi^* f] = [F'][W'_1] \cdots [W'_l]$$

for some unit $[F'] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$ and for Weierstrass polynomials $W'_1, \dots, W'_l$. By the uniqueness from the second part of Weierstrass Preparation Theorem, we must have $[F] = [F']$ and $[W'_1] \cdots [W'_l] = [W_1] \cdots [W_k]$. Since $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$ is a unique factorisation domain, we must have $k = l$ and $[W'_{\sigma(j)}] = [E_j][W_j]$, $j \in \{1, \dots, k\}$, for some $\sigma \in \mathfrak{S}_k$ and for units $[E_j] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$, $j \in \{1, \dots, k\}$. Since

$$[f] = [\psi_* F'][\psi_* W'_1] \cdots [\psi_* W'_l],$$

the uniqueness of the product or irreducibles follows. ■

The next result will be of particular interest to us, especially its consequences in the next section.

2.3.4 Theorem (The ring of germs of holomorphic or real analytic functions is a Noetherian ring) *The ring of germs of holomorphic or real analytic functions at* $\mathbf{0}$ *is a Noetherian ring.*

Proof We let $r \in \{\mathrm{hol}, \omega\}$. For simplicity of notation let us denote a germ by $[\cdot]$ rather than by $[\cdot]_{\mathbf{0}}$. We prove the theorem by induction on n.

For $n = 1$, we claim that all ideals in $\mathscr{C}^r_{0,\mathbb{F}}$ are principal. Indeed, let $\mathsf{I} \subseteq \mathscr{C}^r_{0,\mathbb{F}}$ be a nonzero ideal and note by Theorem 2.3.1 that $\mathsf{I} \subseteq \mathfrak{m}$. Therefore, the ideal generated by $x \mapsto x^k$ is contained in I for some least $k \in \mathbb{Z}_{>0}$. Moreover, if $[g] \in \mathsf{I}$ then $g(x) = x^m g'(x)$ for some $m \geq k$ and where $g'(x) \neq 0$. Therefore, $[g]$ is contained in the ideal generated by $x \mapsto x^k$ and so I is equal to the ideal generated by $x \mapsto x^k$. Thus I is finitely generated.

Now suppose that the theorem holds for $n \in \{1, \dots, m\}$ and let $\mathsf{I} \subseteq \mathscr{C}^r_{\mathbf{0},\mathbb{F}^{m+1}}$ be a nonzero ideal. By Lemma 2.1.3 we let ψ be an orthogonal transformation of $\mathbb{F}^{m+1}$ such that I

contains an element $[f]$ for which $\psi^* f$ is normalised. By the Weierstrass Preparation Theorem, write $[\psi^* f] = [E][W]$ for a unit $[E] \in \mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{m+1}}$ and a Weierstrass polynomial W. Suppose that the degree of W is k. Denote

$$\psi^*\mathsf{I} = \{[\psi^* g] \mid [g] \in \mathsf{I}\},$$

and note that, since $[g] \mapsto [\psi^* g]$ is an isomorphism of $\mathscr{C}^r_{(\mathbf{0},0),\mathbb{F}^{n+1}}$, $\psi^*\mathsf{I}$ is finitely generated if and only if I is finitely generated. Define

$$\mathsf{A} = \{[h] \in \psi^*\mathsf{I} \cap \mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta] \mid \text{ degree of } h \text{ is less than } k\}$$

to be the elements of $\psi^*\mathsf{I}$ that are germs of polynomial functions in y with degree less than that of W. Note that A is a module over $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}$, and as such is a submodule of $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$. By the induction hypothesis, $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}$ is a Noetherian ring. By Theorem 2.2.16, $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$ is also a Noetherian ring and so is finitely generated by Proposition 2.2.12. Let $[h_1], \dots, [h_r]$ be generators for the $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}$ module A.

Now let $[g] \in \psi^*\mathsf{I}$ and write $[g] = [\psi^* f][W] + [R]$ as in the Weierstrass Preparation Theorem. Note that the Weierstrass Preparation Theorem gives $[R] \in \mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}[\eta]$ and the polynomial degree of $[R]$ is less than k. Therefore, $[R]$ is in the $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^m}$ module A, and from this we see that $[h_1], \dots, [h_r], [\psi^* f]$ generate $\psi^*\mathsf{I}$, giving the theorem. ■

2.3.3 Comparison with smooth functions

The central topic of this book is holomorphic and real analytic geometry. However, there are times when it is useful to contrast properties of this sort of geometry with smooth geometry. In this section we do this for the algebraic structure for holomorphic and real analytic functions described above.

Just as we did above for holomorphic and real analytic functions, we can define the ring of germs of smooth functions at $\mathbf{0} \in \mathbb{R}^n$. This ring we denote by $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$. Let us describe the algebraic structure of this ring.

We first see that the property of being a local ring is as in the holomorphic and real analytic case.

2.3.5 Proposition (The ring of smooth functions is a local ring) *The ring of germs of smooth functions at $\mathbf{0} \in \mathbb{R}^n$ is a local ring with unique maximal ideal given by*

$$\mathfrak{m} = \{[f]_{\mathbf{0}} \mid f(\mathbf{0}) = 0\}.$$

Proof We denote germs by $[f]$ rather than $[f]_{\mathbf{0}}$.

We need only show that, if $\mathsf{J} \subseteq \mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ is a proper ideal, then $\mathsf{J} \subseteq \mathfrak{m}$. So let J be such an ideal and let $[f] \in \mathsf{J}$ be nonzero. If $f(\mathbf{0}) \neq 0$ then one easily sees that $[f]$ is a unit in $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ and, since J is proper, this implies that $f(\mathbf{0}) = 0$. Thus $\mathsf{J} \subseteq \mathfrak{m}$, as desired. ■

The local ring property, then, seems to be possessed by many rings of germs of functions. Indeed, one easily shows that rings of germs of continuous or finitely differentiable functions are also local, the proof following that above for smooth functions. However, things are different for the other two properties of holomorphic and real analytic germs. First we determine whether $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ is a unique factorisation domain.

2.3.6 Proposition (The ring of germs of smooth functions is not a unique factorisation domain) *The ring of germs of smooth functions at* $\mathbf{0} \in \mathbb{R}^{\mathrm{n}}$ *is not a unique factorisation domain.*

Proof We denote germs by $[f]$ rather than $[f]_{\mathbf{0}}$.

For $j \in \mathbb{Z}_{>0}$ define $f_j \in \mathrm{C}^\infty(\mathbb{R}^n)$ by

$$f_j(\boldsymbol{x}) = \begin{cases} x_1^{-j+1}\mathrm{e}^{-1/x_1^2}, & x_1 \neq 0, \\ 0, & x_1 = 0. \end{cases}$$

Note that $f_j(\boldsymbol{x}) = x_1 f_{j+1}(\boldsymbol{x})$ and so we have $([f_{j+1}]) \subsetneq ([f_j])$ for each $j \in \mathbb{Z}_{>0}$. By Proposition 2.2.6 it follows that $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ is not a unique factorisation domain. ■

Next we consider the Noetherian property.

2.3.7 Proposition (The ring of germs of smooth functions is not Noetherian) *The ring of germs of smooth functions at* $\mathbf{0} \in \mathbb{R}^{\mathrm{n}}$ *is not Noetherian.*

Proof We denote germs by $[f]$ rather than $[f]_{\mathbf{0}}$.

Let us first consider the case $n = 1$. As in the lemma from the proof of Proposition 4.5.4 below, the maximal ideal $\mathfrak{m}$ is generated by $[\mathrm{id}_{\mathbb{R}}]$. For $k \in \mathbb{Z}_{>0}$, $\mathfrak{m}^k$ is the set of all germs of functions in $\mathscr{C}^\infty_{0,\mathbb{R}}$ whose derivatives of order $0, 1, \dots, k-1$ vanish at 0, and so $\cap_{k\in\mathbb{Z}_{>0}}\mathfrak{m}^k$ is the set of germs of functions in $\mathscr{C}^\infty_{0,\mathbb{R}}$, all of whose derivatives vanish at 0. Note that $\cap_{k\in\mathbb{Z}_{\geq 0}}\mathfrak{m}^k \neq \{[0]\}$ since, for example, if

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then $[f] \in \cap_{k\in\mathbb{Z}_{\geq 0}}\mathfrak{m}^k$, but $[f]$ is not the zero germ. We now can immediately conclude from the Krull Intersection Theorem (Theorem 2.2.18) that $\mathscr{C}^\infty_{0,\mathbb{R}}$ is not Noetherian.

Now we consider the case of general n. Let $\mathcal{U} \subseteq \mathbb{R}$ be a neighbourhood of $0 \in \mathbb{R}$ so that $\mathcal{U}^n$ is a neighbourhood of $\mathbf{0} \in \mathbb{R}^n$. Given $f \in \mathrm{C}^\infty(\mathcal{U})$ define $\hat{f} \in \mathrm{C}^\infty(\mathcal{U}^n)$ by

$$\hat{f}(x_1, \dots, x_n) = f(x_1).$$

Now define $\psi\colon \mathscr{C}^\infty_{0,\mathbb{R}} \to \mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ by $\psi([f]) = [\hat{f}]$. It is immediate from the definition that ψ is an injective ring homomorphism. Thus ψ maps ideals of $\mathscr{C}^\infty_{0,\mathbb{R}}$ isomorphically to ideals of $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$. In particular, since $\mathscr{C}^\infty_{0,\mathbb{R}}$ contains an ideal that is not finitely generated, so too does $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$. ■

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