Chapter 3

Domains of holomorphy and notions of convexity in $\mathbb{C}^n$

In this chapter we study an important concept in holomorphic analysis, having to do with the existence of extensions of holomorphic functions. The objects of interest here are open subsets possessing an holomorphic functions that cannot be extended to a larger domain. As we shall see, there is a great deal of difficult analysis wrapped up in the characterisations of these domains. The topic of this chapter is fundamental to the theory of complex analysis in several variables, and so is covered in any text on the subject, such as [Fritzsche and Grauert 2002, Gunning and Rossi 1965, Hörmander 1973, Krantz 1992, Laurent-Thiébaut 2011, Range 1986, Taylor 2002]. In all cases, there will be different points of emphasis, depending on which aspect of the theory is most important to a given author. However, the basic points are, by now, pretty well established.

Unlike Chapters 1 and 2 where the complex and real theory was developed, for the most part, side-by-side, here the development is exclusively complex, and only gets real for brief periods where some concepts and intuitions are best developed in $\mathbb{R}^2$.

3.1 Domains of holomorphy and holomorphic convexity

We begin our discussion of domains of holomorphy by examining their relationship with a notion of convexity, so-called holomorphic convexity. We begin by giving the definition of a domain of holomorphy and examining a few examples that introduce us to the special character of these domains.

3.1.1 Definitions and elementary properties

Let us give the definition of the domains of interest.

3.1.1 Definition (Domain of holomorphy) A domain $\Omega \subseteq \mathbb{C}^n$ is a ***domain of holomorphy*** if, for every connected open set $\mathcal{V} \subseteq \mathbb{C}^n$ for which $\mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$ and for every connected component $\mathcal{W}$ of $\Omega \cap \mathcal{V}$, there exists $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ such that $f|\mathcal{W}$ cannot be extended to a holomorphic function on $\mathcal{V}$. •

To understand the definition, let us observe that the set $\mathcal{W}$ in the definition shares boundary points with Ω.

3.1.2 Lemma (Boundary points of domains of holomorphy) *With Ω, $\mathcal{V}$, and $\mathcal{W}$ as in Definition 3.1.1,* $\mathrm{bd}(\mathcal{W}) \cap \mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$.

Proof Since $\mathcal{W}$ is a connected component of the open set $\Omega \cap \mathcal{V}$, $\mathcal{W}$ is an open subset of $\mathbb{C}^n$ and a closed subset of $\Omega \cap \mathcal{V}$. Since $\mathcal{V}$ is connected and $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{W}$ is not closed in $\mathcal{V}$. Thus $\mathrm{cl}_{\mathcal{V}}(\mathcal{W}) \neq \mathcal{W}$ and thus there exists $z \in (\mathrm{bd}(\mathcal{W}) \cap \mathcal{V}) - \mathcal{W}$. Suppose that $z \notin \mathrm{bd}(\Omega)$. Then there exists a neighbourhood $\mathcal{N}$ of z such that either (1) $\mathcal{N} \cap \Omega = \emptyset$ or (2) $\mathcal{N} \cap (\mathbb{C}^n \setminus \Omega) = \emptyset$. Since $\mathcal{W} \subseteq \Omega$, the first condition implies that $\mathcal{N} \cap \mathcal{W} = \emptyset$, contradicting the fact that $z \in \mathrm{bd}(\mathcal{W})$. Similarly, the second condition implies that

$$z \in \mathrm{int}(\mathbb{C}^n \setminus (\mathbb{C}^n \setminus \Omega)) = \Omega.$$

This again prohibits z from being in $\mathrm{bd}(\mathcal{W})$. Thus $z \in \mathrm{bd}(\Omega)$. ■

The idea, roughly, is that on a domain of holomorphy Ω there exists a holomorphic function on Ω that cannot be extended to a larger open set at any point on the boundary of Ω.

3.1.3 Examples (Domains of holomorphy)

1. If you cannot imagine what a domain of holomorphy will be like based on your experience with functions of a single complex variable, there is a reason for this. Indeed, a domain $\Omega \subseteq \mathbb{C}$ is a domain of holomorphy. To see this, let $\mathcal{V} \subseteq \mathbb{C}$ be a connected open set such that $\mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$ and let $\mathcal{W}$ be a connected component of $\Omega \cap \mathcal{V}$. Let $z_0 \in \mathrm{bd}\,\Omega \cap \mathcal{V}$ and define $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ by $f(z) = \frac{1}{z - z_0}$. Then we see that $f|\mathcal{W}$ cannot be extended to a holomorphic function on $\mathcal{V}$. Thus Ω is a domain of holomorphy.
2. If $\Omega \subseteq \mathbb{C}^n$ is convex (see Section B.2) we claim that Ω is a domain of holomorphy. Indeed, let $\mathcal{V} \subseteq \mathbb{C}^n$ be open, connected, and such that $\mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$. Let $\mathcal{W}$ be a connected component of $\Omega \cap \mathcal{V}$. For $z_0 \in \mathcal{V} \cap \mathrm{bd}(\Omega)$, by Corollary B.2.14 let $\phi\colon \mathbb{C}^n \to \mathbb{R}$ be a $\mathbb{R}$-affine function such that

$$\Omega \subseteq \{z \in \mathbb{C}^n \mid \phi(z > 0\}$$

and $\phi(z_0) = 0$. We can write

$$\phi(z) = \sum_{j=1}^{n} \lambda_j (z_j - z_{0j}) + \sum_{j=1}^{n} \bar{\lambda}_j (\bar{z}_j - \bar{z}_{0j})$$

for $\lambda \in \mathbb{C}^n$. Thus we can write $\phi(z) = \mathrm{Re}(g(z))$ where

$$g(z) = 2 \sum_{j=1}^{n} \lambda_j (z_j - z_{0j})$$

is holomorphic. Now note that the function $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ given by $f(z) = (g(z))^{-1}$ has the property that $f|\mathcal{W}$ is holomorphic but does not extend to a holomorphic function on $\mathcal{V}$.

3. Let $n \geq 2$. In this example we consider a region Ω in $\mathbb{C}^n \simeq \mathbb{C} \times \mathbb{C}^{n-1}$ defined as follows. Let $\mathcal{H}' \subseteq \mathcal{H}$ be open sets in $\mathbb{C}^{n-1}$ with $\mathcal{H}$ connected and let $r \in \mathbb{R}_{\geq 0}$ and $R \in \mathbb{R}_{>0}$ satisfy $r < R$. Define

$$\Omega = \{(z_0, \boldsymbol{z}_1) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z_0 \in \mathsf{D}^1(R, 0) \setminus \overline{\mathsf{D}}^1(r, 0),\ \boldsymbol{z}_1 \in \mathcal{H}\} \\ \cup \{(z_0, \boldsymbol{z}_1) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z_0 \in \mathsf{D}^1(R, \boldsymbol{0}),\ \boldsymbol{z}_1 \in \mathcal{H}'\}. \quad (3.1)$$

A set defined in this manner is called a ***Hartogs figure***. In Figure 3.1 we illustrate

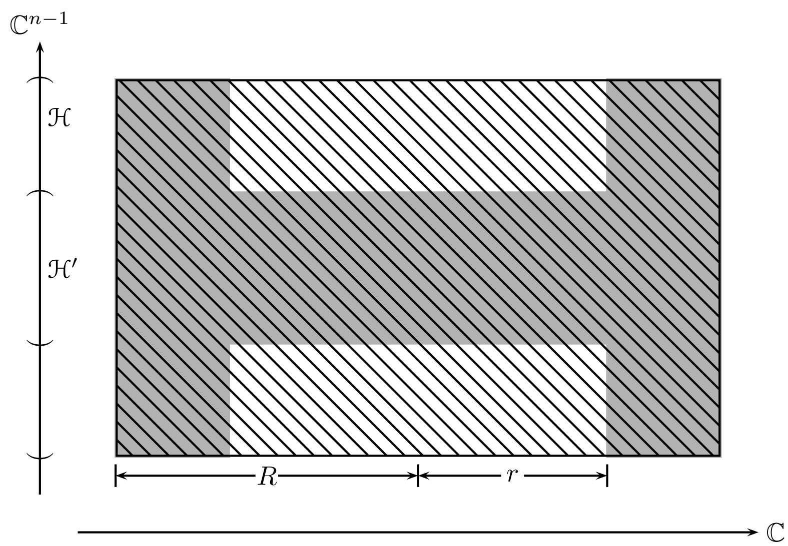

Figure 3.1 A depiction of a set that is not a domain of holomorphy

with the shaded region how one can think of Ω. We will show that Ω is not a domain of holomorphy. To do this, take

$$\mathcal{V} = \{(z_0, \boldsymbol{z}_1) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z_0 \in \mathsf{D}^1(R, \boldsymbol{0}),\ \boldsymbol{z}_1 \in \mathcal{H}\},$$

noting that $\mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$. In Figure 3.1 the hatched region depicts $\mathcal{V}$. Let

$$\mathcal{W} = \{(z_0, \boldsymbol{z}_1) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z_0 \in \mathsf{D}^1(R, \boldsymbol{0}),\ \boldsymbol{z}_1 \in \mathcal{H}'\}.$$

Now let $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$, let $(z_0, \boldsymbol{z}_1) \in \mathcal{V}$, and let $\rho \in \mathbb{R}_{>0}$ be such that $\max\{|z_0|, r\} < \rho < R$. Define $\hat{f}\colon \mathcal{V} \to \mathbb{C}$ by asking that

$$\hat{f}(z_0, \boldsymbol{z}_1) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta|=\rho} \frac{f(\zeta, \boldsymbol{z}_1)}{\zeta - z_0}\, \mathrm{d}\zeta.$$

By standard theorems on holomorphic dependence of integrals [Titchmarsh 1939, page 99], $\hat{f}$ is holomorphic on $\mathcal{V}$. By the Cauchy Integral Formula, $\hat{f}|\mathcal{W} = f|\mathcal{W}$. Therefore, since Ω is connected, $\hat{f}|\Omega = f$. Thus $\hat{f}$ is an extension of f to $\mathcal{V}$ and this prohibits Ω from being a domain of holomorphy.

4. The previous example can be used to produce a significant generalisation. Indeed, let M be a holomorphic manifold and let $\mathsf{S} \subseteq \mathsf{M}$ be a submanifold of codimension at least 2. We claim that any holomorphic function on $\mathsf{M} \setminus \mathsf{S}$ can be extended to a holomorphic function on M. It suffices to do this locally, so we suppose, by working with an appropriate submanifold chart, that $\mathsf{M} = \mathsf{D}^n(\hat{R}, \mathbf{0})$ and that
$$\mathsf{S} = \{z \in \mathsf{M} \mid z_1 = \cdots = z_k = 0\}$$
for some $k \geq 2$. In the notation of the preceding example we take $\mathcal{H} = \mathsf{D}^{n-1}(\hat{R}, \mathbf{0})$ and
$$\mathcal{H}' = \mathcal{H} \setminus \{z \in \mathcal{H} \mid z_2 = \cdots = z_k = 0\}$$
(taking points in $\mathsf{D}^{n-1}(\hat{R}, \mathbf{0})$ to be of the form $(z_2, \dots, z_n)$). We also take $r = 0$ so that, if $\Omega = \mathsf{M} \setminus \mathsf{S}$, then Ω is as in (3.1), with r, R, $\mathcal{H}'$, and $\mathcal{H}$ as just defined. We also define $\mathcal{V} = \mathsf{D}^n(\hat{R}, \mathbf{0})$. Thus any holomorphic function on Ω can be extended to Ω, as desired.
If we were to have given our definition of domains of holomorphy for manifolds, we would say at this time that $\mathsf{M} \setminus \mathsf{S}$ is not a domain of holomorphy.
5. A special case of the preceding example shows the following. Let $\Omega \subseteq \mathbb{C}^n$ is open and let $D \subseteq \Omega$ be a collection of points possessing no accumulation point in Ω. By our preceding result, any holomorphic function on $\Omega \setminus D$ extends to a holomorphic function on Ω. That is, holomorphic functions in n-variables, $n \geq 2$, cannot have isolated singularities. •

Let us take note of the examples above that are *not* domains of holomorphy. A key idea is that in an open set that is *not* a domain of holomorphy, there must necessarily be some constraints present on the holomorphic functions that can be defined on this domain. These constraints are imposed precisely by the requirement that all holomorphic functions can be extended to some larger open set. This prohibits, for example, functions that blow up all along the boundary of open set that is not a domain of holomorphy. We shall see this made precise in Section 3.1.3.

Our discussion of domains of holomorphy will focus mainly on various characterisations of these, upon which we now embark.

3.1.2 Holomorphic convexity

Our first seemingly crazy diversion to understand domains of holomorphy will be a form of convexity. We shall begin by first providing an alternative characterisation of convex sets in the usual sense. We use the notation
$$\sup_C(f) = \sup\{f(x) \mid x \in C\}$$
for a $\mathbb{R}$-valued function f defined on a set X with $C \subseteq X$.

3.1.4 Proposition (Alternative characterisation of a closed convex set) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space and let* $\mathscr{A}(\mathsf{V})$ *be the set of affine functions on* V*, i.e., functions of the form*

$$\mathrm{v} \mapsto \alpha(\mathrm{v}) + \mathrm{a}, \qquad \alpha \in \mathsf{V}^*,\ \mathrm{a} \in \mathbb{R}.$$

Then, for a subset $\mathrm{C} \subseteq \mathsf{V}$*, the following statements are equivalent:*

(i) C *is closed and convex;*

(ii) $\mathrm{C} = \{\mathrm{v} \in \mathsf{V} \mid \mathrm{f}(\mathrm{v}) \leq \sup_{\mathrm{C}}(\mathrm{f})$ *for every* $\mathrm{f} \in \mathscr{A}(\mathsf{V})\}$.

Proof We shall show that

$$\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C) \triangleq \{v \in \mathsf{V} \mid f(v) \leq \sup_C(f) \text{ for every } f \in \mathscr{A}(\mathsf{V})\} = \mathrm{cl}(\mathrm{conv}(C)),$$

and from this the result will follow.

It is clear that $C \subseteq \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$.

First we show that $\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$ is closed by showing that its complement is open. Let $v_0 \notin \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$. Thus there exists $f \in \mathscr{A}(\mathsf{V})$ such that $f(v_0) > \sup_C(f)$. By continuity, $f(v) > \sup_C(f)$ for v in a neighbourhood of v_0, and this gives openness of the complement of $\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$.

Next we show that $\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$ is convex. Let $v_1, v_2 \in \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$, let $s \in [0,1]$, and let $f \in \mathscr{A}(\mathsf{V})$ be given by $f(v) = \alpha(v) + a$. Then

$$\begin{aligned} f((1-s)v_1 + sv_2) &= ((1-s)\alpha(v_1) + (1-s)a) + (sv_s + sa) = (1-s)f(v_1) + sf(v_2) \\ &\leq (1-s)\sup_C(f) + s\sup_C(f) = \sup_C(f), \end{aligned}$$

showing that $(1-s)v_1 + sv_2 \in \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C)$, as desired.

Now we show that if $C_0 \subseteq \mathsf{V}$ is a closed convex set such that $C \subseteq C_0$, then $\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C) \subseteq C_0$. To show this, we first show that $\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C_0) = C_0$. Let $v_0 \notin C_0$ and, by Proposition B.1.3, let $u_0 \in \mathrm{bd}(C_0)$ be such that $\mathrm{dist}(v_0, C_0) = \|v_0 - u_0\|$. If $s_0 \in (0,1)$ then

$$w_0 \triangleq (1-s_0)u_0 + s_0 v_0 \notin C_0$$

and, by Corollary B.2.14, there exists $f \in \mathscr{A}(\mathsf{V})$ such that $f(w_0) = 0$ and $\sup_{C_0}(f) < 0$. Since the function

$$s \mapsto f((1-s)u_0 + sv_0)$$

is increasing on $[0,1]$, we conclude that $f(v_0) \geq 0$ and so $v_0 \notin \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C_0)$. Thus we indeed have $C_0 = \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C_0)$. Now, since $C \subseteq C_0$ we have

$$\mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C) \subseteq \mathrm{conv}_{\mathscr{A}(\mathsf{V})}(C_0) = C_0,$$

completing the proof. ■

The preceding result should be seen as motivation for the following definition. We refer the reader to (1.1) for the notation $\|\cdot\|_K$ used in the definition.

3.1.5 Definition (Holomorphically convex hull) If $\Omega \subseteq \mathbb{C}^n$ is a domain and if $K \subseteq \Omega$, the ***holomorphically convex hull*** of K is the set

$$\mathrm{hconv}_\Omega(K) = \{z \in \Omega \mid \; |f(z)| \leq \|f\|_K \text{ for all } f \in \mathsf{C}^{\mathrm{hol}}(\Omega)\}.$$

A set K is called $\mathbf{C^{hol}(\Omega)}$***-convex*** if $\mathrm{hconv}_\Omega(K) = K$. •

Let us give some elementary properties of the holomorphically convex hull.

3.1.6 Proposition (Properties of the holomorphically convex hull) *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $K, L \subseteq \Omega$. Then the following statements hold:*

(i) $K \subseteq \mathrm{hconv}_\Omega(K)$*;*

(ii) *if* $K \subseteq L$ *then* $\mathrm{hconv}_\Omega(K) \subseteq \mathrm{hconv}_\Omega(L)$*;*

(iii) $\mathrm{hconv}_\Omega(\mathrm{hconv}_\Omega(K)) = \mathrm{hconv}_\Omega(K)$*;*

(iv) $\mathrm{hconv}_\Omega(K)$ *is a closed subset in the relative topology of* Ω*;*

(v) *if* K *is bounded then* $\mathrm{hconv}_\Omega(K)$ *is bounded;*

(vi) $\mathrm{hconv}_\Omega(K) \subseteq \mathrm{cl}(\mathrm{conv}(K))$.

Proof (i) This is obvious.

(ii) This too is obvious.

(iii) By parts (i) and (ii) we have

$$\mathrm{hconv}_\Omega(K) \subseteq \mathrm{hconv}_\Omega(\mathrm{hconv}_\Omega(K)).$$

To prove the opposite inclusion, let $z \notin \mathrm{hconv}_\Omega(K)$. Then there exists $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$ such that $|f(z)| > \|f\|_K$. This implies, however, that $|f(z)| > |f(w)|$ for every $w \in \mathrm{hconv}_\Omega(K)$ (by definition of $\mathrm{hconv}_\Omega(K)$) and so $|f(z)| > \|f\|_{\mathrm{hconv}_\Omega(K)}$, showing that $z \notin \mathrm{hconv}_\Omega(\mathrm{hconv}_\Omega(K))$.

(iv) For $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$ note that

$$C_f \triangleq \{z \in \Omega \mid \; |f(z)| \leq \|f\|_K\}$$

is a closed subset of Ω. Moreover, one can see easily that

$$\mathrm{hconv}_\Omega(K) = \cap_{f \in \mathsf{C}^{\mathrm{hol}}(\Omega)} C_f,$$

giving closedness of $\mathrm{hconv}_\Omega(K)$ in Ω.

(v) Let $\zeta_1, \dots, \zeta_n \in \mathsf{C}^{\mathrm{hol}}(\Omega)$ be the coordinate functions: $\zeta_j(z) = z_j$, $j \in \{1, \dots, n\}$. Since K is bounded, the functions $\zeta_1, \dots, \zeta_n$ are bounded on K. Since $\zeta_j(z) \leq \|\zeta_j\|_K$, $j \in \{1, \dots, n\}$, for all $z \in \mathrm{hconv}_\Omega(K)$, it follows that $\zeta_1, \dots, \zeta_n$ are bounded on $\mathrm{hconv}_\Omega(K)$, and so $\mathrm{hconv}_\Omega(K)$ is also bounded.

(vi) Let $\hat{z} \notin \mathrm{cl}(\mathrm{conv}(K))$. Using Corollary B.2.14 and the computations of Example 3.1.3–2, there exists $\lambda \in \mathbb{C}^n$ and $z_0 \in \mathbb{C}^n$ such that

$$\mathrm{cl}(\mathrm{conv}(K)) \subseteq \{z \in \mathbb{C}^n \mid \; \mathrm{Re}(\langle z - z_0, \lambda \rangle) < 0\}, \quad \mathrm{Re}(\langle \hat{z} - z_0, \lambda \rangle) > 0.$$

Let $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$ be defined by $f(z) = \langle \lambda, z - z_0 \rangle$. Now, for any $z \in \mathrm{cl}(\mathrm{conv}(K))$ we have

$$|\exp \circ f(z)| = \exp(\mathrm{Re}(f(z))) < 1.$$

But, in a similar manner, we have $|\exp \circ f(\hat{z})| > 1$ which implies that $\hat{z} \notin \mathrm{hconv}_\Omega(K)$ since $\exp \circ f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$. ■

From the previous result, the holomorphically convex hull of a compact set is always bounded. The situation where it is also always closed (in $\mathbb{C}^n$!) is of interest to us.

3.1.7 Definition (Holomorphically convex domain in $\mathbb{C}^n$) A domain $\Omega \subseteq \mathbb{C}$ is ***holomorphically convex*** if $\mathrm{hconv}_\Omega(K)$ is compact for every compact $K \subseteq \Omega$. •

Let us give some examples of open sets that are and are not holomorphically convex.

3.1.8 Examples (Holomorphically convex sets)

1. As with domains of holomorphy, the notion of holomorphic convexity is not so interesting in dimension 1 (forgetting, for the moment, that we will later show that these two notions are equivalent). Indeed, we claim that if $\Omega \subseteq \mathbb{C}$ is a domain, then it is holomorphically convex. To see this, we first note that if $K \subseteq \Omega$ is compact then $\mathrm{hconv}_\Omega(K)$ is bounded from Proposition 3.1.6. We must show that $\mathrm{cl}(\mathrm{hconv}_\Omega(K)) \subseteq \Omega$. Suppose otherwise so that there exists $z_0 \in \mathrm{cl}(\mathrm{hconv}_\Omega(K)) - \Omega$. Since $\mathrm{hconv}_\Omega(K) \subseteq \Omega$ this implies that $z_0 \in \mathrm{bd}(\mathrm{hconv}_\Omega(K)) \cap \mathrm{bd}(\Omega)$. Define $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$ by $f(z) = \frac{1}{z-z_0}$. Let $(z_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{hconv}_\Omega(K)$ converging to z_0. Then $\lim_{j\to\infty}|f(z_j)| = \infty$ and so there exists $N \in \mathbb{Z}_{>0}$ such that $|f(z_N)| > \|f\|_K$ since the latter is finite. This is a contradiction.
2. Let $\Omega \subseteq \mathbb{C}^n$ be holomorphically convex and let $f_1, \dots, f_k \in \mathsf{C}^{\mathrm{hol}}(\Omega)$. We claim that the set
$$\Omega_f = \{\boldsymbol{z} \in \Omega \mid |f_j(\boldsymbol{z})| < 1\}$$
is then holomorphically convex. Indeed, let $K \subseteq \Omega_f$ be compact and let $r \in [0, 1)$ be such that $|f_j(\boldsymbol{z})| \leq r$ for each $j \in \{1, \dots, k\}$ and $\boldsymbol{z} \in K$. Thus $|f_j(\boldsymbol{z})| \leq r$ for each $j \in \{1, \dots, k\}$ and $\boldsymbol{z} \in \mathrm{hconv}_{\Omega_f}(K)$. Therefore,
$$\mathrm{hconv}_{\Omega_f}(K) \subseteq \{\boldsymbol{z} \in \Omega \mid |f_j(\boldsymbol{z})| \leq r\}.$$
Thus $\mathrm{hconv}_{\Omega_f}(K)$ is a closed subset of the compact (because Ω is holomorphically convex) set Ω, and so is compact [Runde 2005, Proposition 3.3.6].
3. Let $n \geq 2$ and take $\Omega = \mathbb{C}^n \setminus \{\boldsymbol{0}\}$. By Example 3.1.3–4 it follows that every holomorphic function on Ω can be extended to a holomorphic function on $\mathbb{C}^n$. Let us consider the compact set
$$K = \{\boldsymbol{z} \in \Omega \mid \|\boldsymbol{z}\| = 1\}.$$
We claim that $\mathrm{hconv}_\Omega(K) = \overline{\mathsf{B}}^n(1, \boldsymbol{0}) \setminus \{\boldsymbol{0}\}$. By Proposition 3.1.6(vi) it follows that $\mathrm{hconv}_\Omega(K) \subseteq \overline{\mathsf{B}}^n(1, \boldsymbol{0}) \setminus \{\boldsymbol{0}\}$. For the converse inclusion, let $f \in \mathsf{C}^{\mathrm{hol}}(\Omega)$. Then let $\hat{f} \in \mathsf{C}^{\mathrm{hol}}(\mathbb{C}^n)$ be the extension of f to $\mathbb{C}^n$. Note that $\hat{f}$ is bounded on $\overline{\mathsf{B}}^n(1, \boldsymbol{0})$ so that, by the Maximum Modulus Theorem, $|\hat{f}|\overline{\mathsf{B}}^n(1, \boldsymbol{0})|$ has its maximum on K. Therefore, for every $\boldsymbol{z} \in \overline{\mathsf{B}}^n(1, \boldsymbol{0}) \setminus \{\boldsymbol{0}\}$ we must have $|f(\boldsymbol{z})| \leq \|f\|_K$, as desired. This shows that Ω is not holomorphically convex since $\mathrm{hconv}_\Omega(K)$ is not compact. •

Let us give some properties of holomorphically convex sets.

3.1.9 Proposition (Basic properties of holomorphically convex sets) *The following statements hold:*

(i) *if domains $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are holomorphically convex, then so too is $\Omega \times \Delta$;*

(ii) *if $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex, there exists a sequence $(K_j)_{j\in\mathbb{Z}_{>0}}$ of compact subsets of Ω with the following properties:*

(a) $\mathrm{hconv}_\Omega(K_j) = K_j$;

(b) $K_j \subseteq \mathrm{int}(K_{j+1})$ *for* $j \in \mathbb{Z}_{>0}$;

(c) $\Omega = \cup_{j\in\mathbb{Z}_{>0}} K_j$.

Proof (i) Let $K \subseteq \Omega \times \Delta$ be compact and let $L \subseteq \Omega$ and $M \subseteq \Delta$ be compact subsets for which $K \subseteq L \times M$. Note that if $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ then, $\hat{f}(z, w) = f(z)$ defines $\hat{f} \in \mathrm{C}^{\mathrm{hol}}(\Omega \times \Delta)$. If $(z, w) \in \mathrm{hconv}_{\Omega\times\Delta}(L \times M)$ then

$$|\hat{f}(z, w)| = |f(z)| \leq \|f\|_L.$$

Thus $z \in \mathrm{hconv}_\Omega(L)$ and so

$$\mathrm{hconv}_{\Omega\times\Delta}(L \times M) \subseteq \mathrm{hconv}_\Omega(L) \times \Delta.$$

Similarly,

$$\mathrm{hconv}_{\Omega\times\Delta}(L \times M) \subseteq \Omega \times \mathrm{hconv}_\Delta(M)$$

and so

$$\mathrm{hconv}_{\Omega\times\Delta}(L \times M) \subseteq \mathrm{hconv}_\Omega(L) \times \mathrm{hconv}_\Delta(M).$$

By hypothesis, the set on the right is compact. Since

$$\mathrm{hconv}_{\Omega\times\Delta}(K) \subseteq \mathrm{hconv}_{\Omega\times\Delta}(L \times M),$$

we have that $\mathrm{hconv}_{\Omega\times\Delta}(K)$ is a closed subset of a compact set, and so is compact.

(ii) Let $(L_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of Ω such that $L_j \subseteq \mathrm{int}(L_{j+1})$ and $\Omega = \cup_{j\in\mathbb{Z}_{>0}} L_j$ (using [Aliprantis and Border 2006, Lemma 2.76]). We let $K_1 = \mathrm{hconv}_\Omega(L_1)$. Now suppose that we have defined $K_1, \dots, K_m$ with the desired properties. Choose $N_m \geq m$ sufficiently large that $K_m \subseteq L_{N_m}$ and take $K_{m+1} = \mathrm{hconv}_\Omega(L_{N_m})$. One readily verifies, using Proposition 3.1.6, that the sequence $(K_j)_{j\in\mathbb{Z}_{>0}}$ has the asserted properties. ■

We close this section by proving that domains of holomorphy are holomorphically convex.

3.1.10 Theorem (Domains of holomorphy are holomorphically convex) *If a domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy, it is holomorphically convex.*

Proof We first prove a lemma.

1 Lemma *If* $\Omega \subseteq \mathbb{C}^n$ *is a domain of holomorphy,* $\mathrm{dist}(K, \mathrm{bd}(\Omega)) = \mathrm{dist}(\mathrm{hconv}_\Omega(K), \mathrm{bd}(\Omega))$.

Proof Let $r \in \mathbb{R}_{>0}$ be such that $\mathrm{dist}(K, \mathrm{bd}(\Omega)) > r$, this being possible by compactness of K. Now define

$$L = K + \overline{\mathsf{B}}^n(r, \mathbf{0}) = \{z + w \mid z \in K,\ w \in \overline{\mathsf{B}}^n(r, \mathbf{0})\},$$

and an easy argument shows that L is compact. Thus, if $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$, we have $M \triangleq \|f\|_L < \infty$. Thus, by Corollary 1.1.24 and Theorem 1.1.17, the Taylor series for f at every point in $z_0 \in K$ converges on $\mathsf{B}^n(r, z_0)$ and, moreover, the derivatives satisfy

$$|\boldsymbol{D}^I f(z_0)| \le I! M r^{-I}, \qquad I \in \mathbb{Z}^n_{\ge 0}.$$

Since $\boldsymbol{D}^I f$ is holomorphic, by definition of $\mathrm{hconv}_\Omega(K)$ we have

$$|\boldsymbol{D}^I f(z_0)| \le I! M r^{-I}, \qquad I \in \mathbb{Z}^n_{\ge 0}$$

for all $z_0 \in \mathrm{hconv}_\Omega(K)$. Thus the Taylor series for f at $z_0 \in \mathrm{hconv}_\Omega(K)$ converges on $\mathsf{B}^n(r, z_0)$ for every $z_0 \in \mathrm{hconv}_\Omega(K)$. Since Ω is a domain of holomorphy, this implies that $\mathsf{B}^n(r, z_0) \subseteq \Omega$ for every $z_0 \in \mathrm{hconv}_\Omega(K)$ (since, otherwise, this would give the holomorphic extension of every $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ across the boundary of Ω). Thus we have shown that

$$\mathrm{dist}(\mathrm{hconv}_\Omega(K), \mathrm{bd}(\Omega)) \le \mathrm{dist}(K, \mathrm{bd}(\Omega)).$$

As the opposite inequality follows from the fact that $K \subseteq \mathrm{hconv}_\Omega(K)$, the lemma follows. ▼

The theorem now follows since, if Ω is a domain of holomorphy and if $K \subseteq \Omega$ is compact, $\mathrm{hconv}_\Omega(K)$ is also compact since it is relatively closed and bounded (Proposition 3.1.6) and its absolute closure does not intersect $\mathrm{bd}(\Omega)$. ■

3.1.3 Singular functions

In this section we see that holomorphically convex sets possess holomorphic functions with particularly nasty behaviour at the boundary. The following definition captures the desired behaviour.

3.1.11 Definition (Singular function) Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$. The function f is ***singular*** if, for any connected open set $\mathcal{V} \subseteq \mathbb{C}^n$ for which $\mathcal{V} \cap \mathrm{bd}(\Omega) \neq \emptyset$ and for any connected component $\mathcal{W}$ of $\Omega \cap \mathcal{V}$, there does not exist $g \in \mathrm{C}^{\mathrm{hol}}(\mathcal{V})$ such that $g|\mathcal{W} = f|\mathcal{W}$. •

In our definition of a domain of holomorphy, we asked, essentially, that, around any point in $\mathrm{bd}(\Omega)$, there should be no holomorphic function that can be extended across the boundary in a neighbourhood of this point. A singular function cannot be extended across *any* point of the boundary. It is thus clear that a domain possessing a singular function is a domain of holomorphy. What is not clear, yet true, is that every domain of holomorphy possesses a singular function. We shall not prove this in this section; the fact will follow from what we prove in this section, along with Theorem 3.5.1. In this section we explore the relationship between holomorphic convexity and the existence of singular functions.

We first prove that we can construct functions that are unbounded on sequences of points without accumulation points.

3.1.12 Theorem (A characterisation of holomorphically convex sets) *A domain $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if and only if, for every sequence $(\mathbf{z}_j)_{j\in\mathbb{Z}_{>0}}$ in Ω possessing no accumulation points and for every sequence $(a_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{C}$, there exists $f \in C^{hol}(\Omega)$ such that $f(\mathbf{z}_j) = a_j$ for every $j \in \mathbb{Z}_{>0}$.*

Proof First suppose that Ω is holomorphically convex and, by Proposition 3.1.9(ii), let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact sets such that

(i) $\mathrm{hconv}_\Omega(K_j) = K_j$,

(ii) $K_j \subseteq \mathrm{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$, and

(iii) $\Omega = \cup_{j\in\mathbb{Z}_{>0}} K_j$.

We can suppose, without loss of generality, that either the sequence $(z_j)_{j\in\mathbb{Z}}$ is comprised of finitely or infinitely many distinct points. In the former case, if there are say N distinct points, we suppose that these points are $z_1, \dots, z_N$, again without loss of generality. In the latter case, since the sequence $(z_j)_{j\in\mathbb{Z}_{>0}}$ has no accumulation points, for each $j \in \mathbb{Z}_{>0}$ it must be the case that K_j contains only finitely many of these points. We thus suppose in this case that the first k_1 terms in the sequence are in K_1, the next k_2 terms in the sequence are in K_2, and so on.

For $j \in \mathbb{Z}_{>0}$ let $m_j \in \mathbb{Z}_{>0}$ be the unique integer with the property that $z_j \in K_{m_j+1} \setminus K_{m_j}$. Since $z_j \notin \mathrm{hconv}(K_{m_j})$, let $f_j \in C^{hol}(\Omega)$ be such that $|f_j(z_k)| > \|f_j\|_{K_{m_j}}$. By rescaling, suppose that $f_j(z_j) = 1$. We take $m_j = 1$ if $z_j \in K_1$ and just take f_j such that $f_j(z_j) = 1$ in this case. Let $p_j \in C^{hol}(\mathbb{C}^n)$ be a polynomial function satisfying

$$p_j(z_j) = 1, \quad p_j(z_1) = \cdots = p_j(z_{j-1}) = 0.$$

If we are in the case where there are only finitely many distinct points in the sequence $(z_j)_{j\in\mathbb{Z}_{>0}}$, we stop this construction after N steps with holomorphic functions $f_1, \dots, f_N$ and polynomials $p_1, \dots, p_N$. Otherwise, we construct sequences $(f_j)_{j\in\mathbb{Z}_{>0}}$ of holomorphic functions and $(p_j)_{j\in\mathbb{Z}_{>0}}$ of polynomials functions with the prescribed properties. In the first case we take $J = \{1, \dots, N\}$ and in the second case we rake $J = \mathbb{Z}_{>0}$.

Now we recursively define $\lambda_j \in \mathbb{C}$ and $r_j \in \mathbb{Z}_{>0}$, $j \in J$, as follows. We let $\lambda_1 = a_1$ and r_1 be such that

$$\|\lambda_1 p_1 f_1^{r_1}\|_{K_{m_1}} < \frac{1}{2},$$

provided that $m_1 > 1$. If $m_1 = 1$ we take $r_1 = 1$. If $\lambda_1, \dots, \lambda_{j-1}$ and $r_1, \dots, r_{j-1}$ have been defined, define λ_j and r_j to satisfy

$$\lambda_j = a_j - \sum_{l=1}^{j-1} \lambda_l p_l(z_j) f_l(z_j)^{r_l}, \qquad \|\lambda_j p_j f_j^{r_j}\|_{K_{m_j}} \le 2^{-j},$$

provided that $m_j > 1$. If $m_j = 1$ we take $r_j = 1$. Now define

$$f(z) = \sum_{j\in J} \lambda_j p_j(z) f_j(z)^{r_j}.$$

We claim that the sum converges uniformly on compact sets. We need only consider the case where $J = \mathbb{Z}_{>0}$. For simplicity, let us suppose that none of the points in the sequence

$(z_j)_{j\in\mathbb{Z}_{>0}}$ lie in K_1. This can be done without loss of generality since we can subtract from the sum for f the front end of the sum corresponding to the k_1 terms where $z_1,\dots,z_{k_1}\in K_1$. This assumption having been made, let $K\subseteq\Omega$ be compact. Note that, with the assumptions in effect, $\lim_{j\to\infty} m_j=\infty$. Thus we can choose k sufficiently large that $K\subseteq K_{m_j}$ for $j\geq k$. Let us also suppose that k is chosen sufficiently large that

$$\sum_{j=q+1}^{p} 2^{-j}<\epsilon$$

if $p,q\geq k$ are chosen such that $p>q$. (This is possible by convergence of $\sum_{j=1}^{\infty}2^{-j}$.) Then, for $p,q\geq k$ with $p>q$ we have

$$\Big\|\sum_{j=1}^{p}\lambda_j p_j f_j^{r_j}-\sum_{j=1}^{q}\lambda_j p_j f_j^{r_j}\Big\|_K\leq\sum_{j=q+1}^{p}\|\lambda_j p_j f_j^{r_j}\|_{K_{m_j}}<\epsilon,$$

giving the desired uniform convergence on K. Moreover, since $p_k(z_j)=0$ for $k>j$, we have

$$f(z_j)=\sum_{l=1}^{j}\lambda_l p_l(z_j)f_l(z_j)^{r_l}=\lambda_j+\sum_{l=1}^{j-1}\lambda_l p_l(z_j)f_l(z_j)^{r_l}=a_j,$$

giving the "only if" assertion.

For the "if" assertion, note that the hypotheses implies that, given a sequence $(z_j)_{j\in\mathbb{Z}_{>0}}$ comprised of distinct points and with no accumulation points, there exists $f\in\mathsf{C}^{\mathrm{hol}}(\Omega)$ such that $f(z_j)=j$. Let $K\subseteq\Omega$ be compact and let $(z_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{hconv}_\Omega(K)$. By definition of the holomorphically convex hull it follows that, for any $f\in\mathsf{C}^{\mathrm{hol}}(\Omega)$,

$$\sup\{|f(z_j)|\mid j\in\mathbb{Z}_{>0}\}\leq\|f\|_K<\infty.$$

In particular, there can be no $f\in\mathsf{C}^{\mathrm{hol}}(\Omega)$ such that $f(z_j)=j$ for every $j\in\mathbb{Z}_{>0}$. But our hypothesis implies that the sequence $(z_j)_{j\in\mathbb{Z}_{>0}}$ must have accumulation points, giving compactness of $\mathrm{hconv}_\Omega(K)$. ■

We can now prove the following theorem.

3.1.13 Theorem (Holomorphically convex sets and singular functions) *If Ω is holomorphically convex then there exists a singular function on Ω.*

Proof We first prove a technical lemma.

1 Lemma *Let $\Omega\subseteq\mathbb{R}^n$ be open and let $(\mathrm{K_j})_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact sets such that (1) $\mathrm{K_j}\subseteq\mathrm{int}(\mathrm{K_{j+1}})$ and (2) $\Omega=\cup_{j\in\mathbb{Z}_{>0}}\mathrm{K_j}$. Then there exists a sequence $(\mathrm{k_j})_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ and a sequence $(\mathbf{x}_j)_{j\in\mathbb{Z}_{>0}}$ in Ω such that*

(i) $\mathbf{x}_j\in\mathrm{K}_{k_{j+1}}\setminus\mathrm{K}_{k_j}$ *and*

(ii) for any $\mathbf{x}\in\mathrm{bd}(\Omega)$, any connected neighbourhood $\mathcal{V}$ of $\mathbf{x}$, and any connected component $\mathcal{W}$ of $\Omega\cap\mathcal{V}$, we have

$$\mathrm{card}(\{j\in\mathbb{Z}_{>0}\mid \mathbf{x}_j\in\mathcal{W}\})=\infty.$$

Proof Let $(\boldsymbol{y}_j)_{j\in\mathbb{Z}_{>0}}$ be an enumeration of $\Omega \cap \mathbb{Q}^n$. For each $j \in \mathbb{Z}_{>0}$, let $r_j = \mathrm{dist}(\boldsymbol{y}_j, \mathrm{bd}(\Omega))$ and denote $B_j = \mathsf{B}^n(r_j, \boldsymbol{y}_j)$. Denote by $(\beta_j)_{j\in\mathbb{Z}_{>0}}$ the sequence

$$\beta_1 = B_1,\ \beta_2 = B_1, \beta_3 = B_2,\ \beta_4 = B_1,\ \beta_5 = B_2,\ \beta_6 = B_3, \dots$$

Let $k_1 = 1$ and define $(k_j)_{j\in\mathbb{Z}_{>0}}$ and $(\boldsymbol{x}_j)_{j\in\mathbb{Z}_{>0}}$ inductively as follows. Suppose that $k_1, \dots, k_m$ and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{m-1}$ have been defined for $m \geq 2$ so that $\boldsymbol{x}_j \in K_{k_{j+1}} \setminus K_{k_j}$ for $j \in \{1, \dots, m-1\}$. Note that β_m is not contained in any compact subset of Ω since it shares a boundary point with Ω. Thus there exists $\boldsymbol{x}_m \in \beta_m \setminus K_{k_m}$. Let us choose k_{m+1} sufficiently large that $\boldsymbol{x}_m \in K_{k_{m+1}}$. Thus the sequences $(k_j)_{j\in\mathbb{Z}_{>0}}$ and $(\boldsymbol{x}_j)_{j\in\mathbb{Z}_{>0}}$ satisfy the first conclusion of the lemma.

For the second, with $\mathcal{V}$ and $\mathcal{W}$ as given in the second of the conclusions, let $\boldsymbol{x} \in \mathrm{bd}(\mathcal{W}) \cap \mathcal{V} \cap \mathrm{bd}(\Omega)$ by virtue of Lemma 3.1.2. We can then choose $\boldsymbol{y}_m$ sufficiently close to $\boldsymbol{x}$ that $B_m \subseteq \mathcal{W}$. Let

$$J = \{j \in \mathbb{Z}_{>0} \mid \beta_j = B_m\},$$

noting that $\mathrm{card}(J) = \infty$. Since $\boldsymbol{x}_j \in \beta_j = B_m$ for each $j \in J$, we conclude that $\mathcal{W}$ contains infinitely many points from $(\boldsymbol{x}_j)_{j\in\mathbb{Z}_{>0}}$. ▼

Now we proceed with the proof of the theorem. First of all, the theorem holds if $\Omega = \mathbb{C}^n$, so we suppose otherwise. Since Ω is holomorphically convex, by Proposition 3.1.9(ii) let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of convex sets such that

1. $\mathrm{hconv}_\Omega(K_j) = K_j$;
2. $K_j \subseteq \mathrm{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$;
3. $\Omega = \cup_{j\in\mathbb{Z}_{>0}} K_j$.

Let $(k_j)_{j\in\mathbb{Z}_{>0}}$ and $(\boldsymbol{z}_j)_{j\in\mathbb{Z}_{>0}}$ satisfy the conclusions of the lemma above. By Theorem 3.1.12, let $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ be such that $\lim_{j\to\infty}|f(\boldsymbol{z}_j)| = \infty$. Let us show that f is singular on Ω. Let $\boldsymbol{z} \in \mathrm{bd}(\Omega)$, let $\mathcal{V}$ be a connected neighbourhood of $\boldsymbol{z}$, and let $\mathcal{W}$ be a connected component of $\Omega \cap \mathcal{V}$. Suppose that there exists $g \in \mathrm{C}^{\mathrm{hol}}(\mathcal{V})$ such that $f|\mathcal{W} = g|\mathcal{W}$. Let $\mathcal{V}'$ be a relatively compact neighbourhood of $\boldsymbol{z}$ for which $\mathrm{cl}(\mathcal{V}') \subseteq \mathcal{V}$. Let $\mathcal{W}'$ be the connected component of $\mathcal{V}' \cap \Omega$ having intersection with $\mathcal{W}$. We then have

$$\|g\|_{\mathcal{W}'} \leq \|g\|_{\mathcal{V}'} < \infty$$

since g is holomorphic on $\mathcal{V}$. By the second of the conclusions from the lemma above the set $\mathcal{W}'$ contains infinitely many of the points $(\boldsymbol{z}_j)$ and $\lim_{j\to\infty}|g(\boldsymbol{z}_j)| = \infty$, and so we arrive at a contradiction. Thus there can be no such function g as asserted. ■

3.2 Harmonic, subharmonic, and plurisubharmonic functions

In this section we introduce an important analytical tool in the study of domains of holomorphy. This class of functions has its roots in the theory of functions of a single complex variable. This is where we begin our discussion.

3.2.1 Harmonic and subharmonic functions

One of the characterisations we shall use for domains of holomorphy involves a sort of peculiar class of functions called "plurisubharmonic." These shall be discussed in the next section. The "pluri" here is a reference to the fact that these are functions of more than one complex variable, and the "subharmonic" makes reference to the notion of subharmonic functions of a single complex variable. That is, plurisubharmonic functions are the generalisation to several complex variables of subharmonic functions of a single complex variable.

In this section we thus recall the basic facts about subharmonic functions of a single complex variable. As with the notion of holomorphic convexity, we can get some insight by thinking first about standard notions of convexity. A function $u\colon I \to \mathbb{R}$ defined on an interval I is ***convex*** if

$$u((1-s)x_1 + sx_2) \leq (1-s)u(x_1) + su(x_2)$$

for every distinct $x_1, x_2 \in I$ and for every $s \in (0,1)$. In Figure 3.2 we depict how

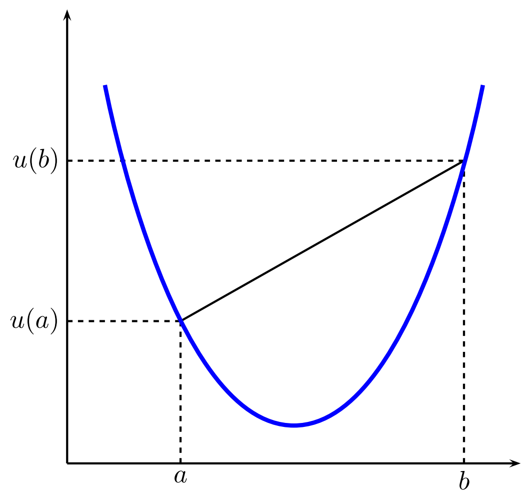

Figure 3.2 A convex function

the definition works. The idea—and one that relates to how we will think of how subharmonic relates to harmonic—is that if u agrees with a linear function at points a and b, then u does not exceed the linear function on (a,b). Said in this way, it is perhaps not unreasonable to think of "convex" functions as being "sublinear." Moreover, the linear functions can be thought of those twice differentiable functions with zero second derivative. It is a classical result that a function of class C^2 is convex if and only if $u''(x) \geq 0$ for every $x \in I$ [Webster 1994, Theorem 5.5.5]. We shall see in Theorem 3.2.2(x) below that a similar interpretation holds for subharmonic functions, but with "second derivative" being replaced with "Laplacian."

Now we turn to harmonic and subharmonic functions. We recall that a function $f\colon \mathcal{S} \to [-\infty, \infty)$ is ***upper semicontinuous*** if $f^{-1}([-\infty, \alpha))$ is open for every $\alpha \in \mathbb{R}$.[1]

[1] A function $f\colon \mathcal{S} \to [-\infty, \infty)$ is upper semicontinuous if and only if, for each $x_0 \in \mathcal{S}$ and each $\epsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{N}$ of x_0 such that $f(x) \leq f(x_0) + \epsilon$ for every $x \in \mathcal{N}$. From this, one can easily see, for example, that upper semicontinuous functions are locally bounded from above.

3.2.1 Definition (Harmonic function, subharmonic function) Let $\Omega \subseteq \mathbb{C}$ be open and let $u\colon \Omega \to [-\infty, \infty)$.

(i) The function u is ***harmonic*** if it is of class C^2 and if $\frac{\partial^2 u}{\partial z \partial \bar{z}}(z) = 0$ for every $z \in \Omega$.

(ii) The function u is ***subharmonic*** if

(a) it is upper semicontinuous;

(b) for every $r \in \mathbb{R}_{>0}$ and $z_0 \in \Omega$ for which $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$ and for every continuous $\sigma\colon \overline{\mathsf{D}}^1(r, z_0) \to \mathbb{R}$ such that (1) $\sigma|\mathsf{D}^1(r, z_0)$ is harmonic, and (2) $\sigma(z) \geq u(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$, we have $\sigma(z) \geq u(z)$ for every $z \in \overline{\mathsf{D}}^1(r, z_0)$. •

It will be convenient on occasion to use the notation

$$\Delta u(z) = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}(z),$$

this being the ***Laplacian*** of u.

Let us give some of the basic properties of harmonic and subharmonic functions.

3.2.2 Theorem (Properties of harmonic and subharmonic functions) *If $\Omega \subseteq \mathbb{C}$ is open, the following statements hold:*

(i) if $\Phi \in C^{\mathrm{hol}}(\Omega)$ then $\mathrm{Re}(\Phi)$ is harmonic;

(ii) if Ω is an open disk and if $\mathrm{u}\colon \Omega \to \mathbb{R}$ is harmonic, then there exists $\Phi \in C^{\mathrm{hol}}(\Omega)$ such that $\mathrm{u} = \mathrm{Re}(\Phi)$;

(iii) if $\mathrm{u}\colon \Omega \to \mathbb{R}$ is harmonic then, for each $\mathrm{z}_0 \in \Omega$, there exists $\rho \in \mathbb{R}_{>0}$ such that $\overline{\mathsf{D}}^1(\rho, \mathrm{z}_0) \subseteq \Omega$ and such that

$$\mathrm{u}(\mathrm{z}_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{u}(\mathrm{z}_0 + \mathrm{r}e^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

for every $\mathrm{r} \in (0, \rho]$;

(iv) if u is harmonic then it is subharmonic;

(v) if $(\mathrm{u}_j)_{j \in \mathbb{Z}_{>0}}$ is a sequence of subharmonic functions on Ω such that $\mathrm{u}_{j+1}(\mathrm{z}) \leq \mathrm{u}_j(\mathrm{z})$ for each $\mathrm{j} \in \mathbb{Z}_{>0}$ and $\mathrm{z} \in \Omega$, then the function u on Ω defined by $\mathrm{u}(\mathrm{z}) = \lim_{j \to \infty} \mathrm{u}_j(\mathrm{z})$ is subharmonic;

(vi) if $(\mathrm{u}_a)_{a \in A}$ is a family of subharmonic functions on Ω then the function u on Ω defined by

$$\mathrm{u}(\mathrm{z}) = \sup\{\mathrm{u}_a(\mathrm{z}) \mid a \in A\}$$

is subharmonic if it is upper semicontinuous and everywhere finite;

(vii) if $\mathrm{u}_1, \dots, \mathrm{u}_k\colon \Omega \to [-\infty, \infty)$ are subharmonic and if $\mathrm{F}\colon \mathbb{R}^k \to \mathbb{R}$ is continuous, convex, and nondecreasing in each component, and if we extend F to $\bar{\mathrm{F}}\colon ([-\infty, \infty))^k \to [-\infty, \infty)$ by continuity,[2] then the function

$$\mathrm{z} \mapsto \mathrm{F}(\mathrm{u}_1(\mathrm{z}), \dots, \mathrm{u}_k(\mathrm{z}))$$

[2] A little precisely, we define $\bar{\mathrm{F}}$ as follows. Suppose that we wish to evaluate $\bar{\mathrm{F}}$ at a point where $\mathrm{x}_{j_1} = \cdots = \mathrm{x}_{j_m} = -\infty$ for and only for some $j_1, \dots, j_m \in \{1, \dots, k\}$. We then let each of the coordinates $\mathrm{x}_{j_1}, \dots, \mathrm{x}_{j_m}$ tend together monotonically to ∞, while fixing the remaining coordinates at their desired values. The value of $\bar{\mathrm{F}}$ at this point is then the limit of the values of F.

is subharmonic;

(viii) *if* $\mathrm{u}\colon \Omega \to \mathbb{R}$ *is upper semicontinuous, then it is subharmonic if and only if, for each* $\mathrm{z}_0 \in \Omega$*, there exists* $\rho \in \mathbb{R}_{>0}$ *such that* $\overline{\mathsf{D}}^1(\rho, \mathrm{z}_0) \subseteq \Omega$ *and such that*

$$\mathrm{u}(\mathrm{z}_0) \le \frac{1}{2\pi}\int_0^{2\pi} \mathrm{u}(\mathrm{z}_0 + \mathrm{r}e^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

for every $\mathrm{r} \in (0, \rho]$*;*

(ix) *if* Ω *is connected and if* u *is subharmonic and has a global maximum in* Ω*, then* u *is constant;*

(x) *if* u *is of class* C^2*, then it is subharmonic if and only if* $\frac{\partial^2 \mathrm{u}}{\partial \mathrm{z}\partial\bar{\mathrm{z}}}(\mathrm{z}) \ge 0$ *for every* $\mathrm{z} \in \Omega$*.*

Proof (i) Write $\Phi(z) = u(z) + \mathrm{i}v(z)$ for $\mathbb{R}$-valued smooth functions u and v on Ω. Referring to (1.11) we have

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big).$$

The Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and from this we immediately see that $\frac{\partial^2 u}{\partial z\partial\bar{z}} = 0$ by equality of mixed partials.

(ii) We will define $v\colon \Omega \to \mathbb{R}$ such that $\Phi \triangleq u + \mathrm{i}v$ is holomorphic. Let $z_0 = x_0 + \mathrm{i}y_0 \in \Omega$ be the centre of the disk Ω and let $r \in \mathbb{R}_{>0}$ be the radius. For $z = x + \mathrm{i}y \in \Omega$ define

$$v(z) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, \eta)\,\mathrm{d}\eta - \int_{x_0}^{x} \frac{\partial u}{\partial y}(\xi, y_0)\,\mathrm{d}\xi.$$

One can verify by direct computation that $\Phi = u + \mathrm{i}v$ satisfies the Cauchy–Riemann equations, and so is holomorphic.

(iii) Let Φ be holomorphic in a neighbourhood of z containing $\overline{\mathsf{D}}^1(r, z)$. By the Cauchy integral formula,

$$\Phi(z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{bd}(\mathsf{D}^1(r,z))} \frac{\Phi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Letting $\zeta = z_0 + re^{\mathrm{i}\theta}$ and taking real parts gives the result.

(iv) If u is harmonic it is continuous and so upper semicontinuous. By parts (iii) and (viii) below, it then follows that if u is harmonic it is subharmonic.

(v) By [Aliprantis and Border 2006, Lemma 2.41] we have that u is upper semicontinuous. Let $z_0 \in \Omega$ and $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$. Let σ be a continuous function on $\overline{\mathsf{D}}^1(r, z_0)$ that is harmonic on $\mathsf{D}^1(r, z_0)$ and is such that $\sigma(z) \ge u(z)$ for all $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. Let $\epsilon \in \mathbb{R}_{>0}$ and for $j \in \mathbb{Z}_{>0}$ define

$$K_{j,\epsilon} = \{z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0)) \mid u_j(z) \ge \sigma(z) + \epsilon\}.$$

Note that $K_{j,\epsilon}$ is compact, that $K_{j+1,\epsilon} \subseteq K_{j,\epsilon}$, and that $\cap_{j\in\mathbb{Z}_{>0}} K_{j,\epsilon} = \emptyset$, the latter since $\lim_{j\to\infty} u_j(x) = u(x) \le \sigma(x)$. It follows, since the intersection of a nested sequence of nonempty compact sets is nonempty [Rudin 1976, Corollary to Theorem 2.36], that there exists $N \in \mathbb{Z}_{>0}$ such that $K_{N,\epsilon} = \emptyset$. Thus, for $j \geq N$, $u_j(z) < \sigma(z) + \epsilon$ for every $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$ and so $u_j(z) < \sigma(z) + \epsilon$ for every $z \in \overline{\mathsf{D}}^1(r, z_0)$. It follows that $u(z) < \sigma(z) + \epsilon$ for every $z \in \overline{\mathsf{D}}^1(r, z_0)$, and so $u(z) \leq \sigma(z)$ for every $z \in \overline{\mathsf{D}}^1(r, z_0)$, as desired.

(vi) Let $z_0 \in \Omega$ and $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$. Let $\sigma \colon \overline{\mathsf{D}}^1(r, z_0) \to \mathbb{R}$ be continuous, harmonic on $\mathsf{D}^1(r, z_0)$, and satisfying $\sigma(z) \geq u(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. We then have $\sigma(z) \geq u_a(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$ and $a \in A$. It follows that $\sigma(z) \geq u_a(z)$ for $z \in \overline{\mathsf{D}}^1(r, z_0)$ and $a \in A$, and so $\sigma(z) \geq u(z)$ for $z \in \overline{\mathsf{D}}^1(r, z_0)$, as desired.

(vii) Let $(\phi_a)_{a \in A}$ be a family of affine functions $\phi_a \colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\{(\boldsymbol{x}, y) \in \mathbb{R}^k \times \mathbb{R} \mid y \geq F(\boldsymbol{x})\} = \cap_{a \in A}\{(\boldsymbol{x}, y) \in \mathbb{R}^k \times \mathbb{R} \mid y \geq \phi_a(\boldsymbol{x}) \text{ for all } a \in A\}$$

what (this is possible since the epigraph of a convex function is convex). Then we have

$$F(\boldsymbol{x}) = \sup\{\phi_a(\boldsymbol{x}) \mid a \in A\}$$

for every $\boldsymbol{x} \in \mathbb{R}^k$ [Webster 1994, Theorem 5.4.2]. If we write $\phi_a(\boldsymbol{x}) = \langle \boldsymbol{m}_a, \boldsymbol{x}\rangle + b_a$, $a \in A$, the fact that F is increasing implies that the components of $\boldsymbol{m}$ are nonnegative. By subharmonicity of $u_1, \dots, u_k$ and part (viii) below we thus have

$$\begin{aligned}\sum_{j=1}^k m_{a,j} u_j(z) + b_a &\leq \frac{1}{2\pi} \sum_{j=1}^k m_{a,j} \int_0^{2\pi} (u_j(z + r\mathrm{e}^{\mathrm{i}\theta}) + b_a)\,\mathrm{d}\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} F(u_1(z + r\mathrm{e}^{i\theta}), \dots, u_k(z + r\mathrm{e}^{\mathrm{i}\theta}))\,\mathrm{d}\theta\end{aligned}$$

for sufficiently small $r \in \mathbb{R}_{>0}$ and for all $a \in A$. This part of the result follows by taking the supremum over $a \in A$ and again applying part (viii) below.

(viii) First consider a general upper semicontinuous function $v \colon \Omega \to \mathbb{R}$ that satisfies

$$v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

for $r \in \mathbb{R}_{>0}$ and $z_0 \in \Omega$ such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$. A look through the proof of part (ix) below shows that this implies that v is constant on any connected component of Ω on which it attains its maximum.

Now suppose that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

for $r \in \mathbb{R}_{>0}$ and $z_0 \in \Omega$ such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$. Now let $r \in \mathbb{R}_{>0}$ and $z_0 \in \Omega$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$ and let $\sigma \colon \overline{\mathsf{D}}^1(r, z_0) \to \mathbb{R}$ be continuous, harmonic on $\mathsf{D}^1(r, z_0)$, and be such that $u(z) \leq \sigma(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. If we take $v = u - \sigma$ then we have, by hypothesis and harmonicity of σ,

$$v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

As mentioned in the preceding paragraph, this implies that v attains its maximum on $\mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. That is,

$$u(z) - \sigma(z) \leq \sup\{u(\zeta) - \sigma(\zeta) \mid \zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))\} \leq 0$$

for every $\zeta \in \overline{\mathsf{D}}^1(r, z_0)$. Thus u is subharmonic.

For the converse assertion, we use a lemma which makes reference to the so-called ***Poisson kernel***. This is a family of maps P_r defined for each $r \in \mathbb{R}_{>0}$ by

$$\begin{aligned} P_r \colon \mathsf{D}^1(r, 0) \times \mathrm{bd}(\mathsf{D}^1(r, 0)) &\to \mathbb{R} \\ (z, r\mathrm{e}^{\mathrm{i}\theta}) &\mapsto \frac{1}{2\pi} \frac{r^2 - |z|^2}{|z - r\mathrm{e}^{\mathrm{i}\theta}|^2}. \end{aligned}$$

We shall require the following fact that is rather important in its own right, as it is the solution to the so-called ***Dirichlet Problem*** for the unit disk.

1 **Lemma** *If* $\mathrm{u} \colon \mathrm{bd}(\overline{\mathsf{D}}^1(1, 0)) \to \mathbb{R}$ *is continuous, then the function* $\sigma \colon \overline{\mathsf{D}}^1(1, 0) \to \mathbb{R}$ *defined by*

$$\sigma(\mathrm{z}) = \begin{cases} \displaystyle\int_0^{2\pi} \mathrm{u}(\mathrm{e}^{\mathrm{i}\phi}) \mathrm{P}_1(\mathrm{z}, \mathrm{e}^{\mathrm{i}\phi}) \, \mathrm{d}\phi, & \mathrm{z} \in \mathsf{D}^1(1, 0), \\ \mathrm{u}(\mathrm{z}), & \mathrm{z} \in \mathrm{bd}(\mathsf{D}^1(1, 0)), \end{cases}$$

is continuous and harmonic on $\mathsf{D}^1(1, 0)$.

Proof First we prove that σ is continuous at points on $\mathrm{bd}(\overline{\mathsf{D}}^1(1, 0))$. Let $z_0 = \mathrm{e}^{\mathrm{i}\theta_0} \in \mathrm{bd}(\overline{\mathsf{D}}^1(1, 0))$. Denote

$$M = \sup\{|u(z)| \mid z \in \mathrm{bd}(\mathsf{D}^1(1, 0))\}.$$

Let $\epsilon \in \mathbb{R}_{>0}$ and use uniform continuity of u to choose $\delta \in \mathbb{R}_{>0}$ such that if $|a - b| < \delta$ then $|u(\mathrm{e}^{\mathrm{i}a}) - u(\mathrm{e}^{\mathrm{i}b})| < \frac{\epsilon}{2}$. Let $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathsf{D}^1(1, 0)$ be chosen sufficiently close to z_0 so that $|\theta - \theta_0| < \frac{\delta}{3}$ and $r \in [\frac{1}{2}, 1)$ and $1 - r < \frac{\delta^2 \epsilon}{100M}$. We then perform a couple of preliminary estimates.

First we note that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |r\mathrm{e}^{\mathrm{i}\theta}|^2}{|r\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\phi}|^2} \, \mathrm{d}\phi = 1. \tag{3.2}$$

(This can be proved by using the Poisson Integral Formula; I used Mathematica®.) We have

$$\Big| \frac{1}{2\pi} \int_{|\phi - \theta_0| < \delta} (u(\mathrm{e}^{\mathrm{i}\theta_0}) - u(\mathrm{e}^{\mathrm{i}\phi})) \frac{1 - r^2}{|r\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\phi}|^2} \, \mathrm{d}\phi \Big| \leq \frac{\epsilon}{2} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|r\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\phi}|^2} \, \mathrm{d}\phi \leq \frac{\epsilon}{2},$$

using (3.2) and the fact that

$$\frac{1 - r^2}{|r\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\phi}|^2} \geq 0.$$

By elementary computations we have

$$|r\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\phi}|^2 = |1 - r\mathrm{e}^{\mathrm{i}(\theta - \phi)}|^2 = 1 - 2r\cos(\theta - \phi) + r^2$$

Now we estimate

$$|re^{i\theta} - e^{i\phi}|^2 = (1-r)^2 + 2r(1-\cos(\theta-\phi))$$
$$\geq 2r(1-\cos(\theta-\phi)) \geq 2r\frac{(\theta-\phi)^2}{2} \geq \frac{(\theta-\phi)^2}{4},$$

using the Taylor expansion of cos for small angles and using the definition of r. Given that $|\theta - \theta_0| < \frac{\delta}{3}$ and if we take $|\phi - \theta_0| \geq \delta$ we have that $|\theta - \phi| \geq \frac{2\delta}{3}$. Thus we have

$$\left|\frac{1}{2\pi}\int_{|\phi-\theta_0|\geq\delta}(u(e^{i\theta_0}) - u(e^{i\phi}))\frac{1-r^2}{|re^{i\theta}-e^{i\phi}|^2}\,d\phi\right| \leq \frac{1}{2\pi}8M\int_{|\phi-\theta_0|\geq\delta}\frac{1-r^2}{(\theta-\phi)^2}\,d\phi$$
$$\leq \frac{1}{2\pi}\frac{72M}{4\delta^2}\int_0^{2\pi}(1+r)(1-r)\,d\phi$$
$$\leq \frac{1}{2\pi}\frac{72M}{4\delta^2}2\frac{\delta\epsilon}{100M} \leq \frac{\epsilon}{2}.$$

Now let us put the preceding estimates together. Using (3.2) we have

$$\sigma(z_0) - \sigma(z) = \frac{1}{2\pi}\int_0^{2\pi}(u(z_0) - u(z))\frac{1-|z^2|}{|z-e^{i\phi}|^2}\,d\phi.$$

Then

$$\left|\sigma(z_0) - \sigma(z)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}(u(z_0)-u(z))\frac{1-|z^2|}{|z-e^{i\phi}|^2}\,d\phi\right|$$
$$\leq \left|\frac{1}{2\pi}\int_{|\phi-\theta_0|\leq\delta}u(z_0)-u(z))\frac{1-|z^2|}{|z-e^{i\phi}|^2}\,d\phi\right|$$
$$+\left|\frac{1}{2\pi}\int_{|\phi-\theta_0|\geq\delta}u(z_0)-u(z))\frac{1-|z^2|}{|z-e^{i\phi}|^2}\,d\phi\right| \leq \epsilon,$$

giving continuity at boundary points, as desired.

Now we show that σ is harmonic on $\mathsf{D}^1(1,0)$. Here we use the directly verified identity

$$\frac{1-|z|^2}{|z-e^{i\phi}|^2} = \frac{e^{i\phi}}{e^{i\phi}-z} + \frac{e^{-i\phi}}{e^{-i\phi}-z} - 1.$$

Thus, for $z \in \mathsf{D}^1(1,0)$,

$$\sigma(z) = \frac{1}{2\pi}\int_0^{2\pi}u(e^{i\phi})\frac{e^{i\phi}}{e^{i\phi}-z}\,d\phi + \frac{1}{2\pi}\int_0^{2\pi}u(e^{i\phi})\frac{e^{-i\phi}}{e^{-i\phi}-z}\,d\phi - \frac{1}{2\pi}\int_0^{2\pi}u(e^{i\phi})\,d\phi.$$

The first term on the right is holomorphic in z. The second term can be verified to be harmonic (i.e., its real and imaginary parts are harmonic) by simply differentiating under the integral sign to verify the conditions for a harmonic function. The last term is constant and so harmonic. Since u is real, we can take real parts to see that the right-hand side, each of which will be harmonic, to see that u is harmonic. ▼

Now suppose that u is continuous and subharmonic. Let $z_0 \in \Omega$ and $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$. By the lemma (and an elementary change of variable to translate $1 \to r$ and $0 \to z_0$), if we define

$$\sigma(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(z - z_0, re^{i\theta})(u(z_0 + re^{i\theta}))\,d\theta,$$

then σ is harmonic on $\mathsf{D}^1(r, z_0)$ for each $\epsilon \in \mathbb{R}_{>0}$. Moreover, $\sigma(z) = u(z)$ for each $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. Since u is subharmonic, this implies that $u(z) \leq \sigma(z)$ for $z \in \mathsf{D}^1(r, z_0)$. This implies, for example, that

$$u(z_0) \leq \sigma(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

It remains to show that if u is upper semicontinuous and subharmonic, then

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

for $z_0 \in \Omega$ and for sufficiently small r. By [Aliprantis and Border 2006, Theorem 3.13] we let $(u_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of continuous functions converging pointwise from above to u on $\overline{\mathsf{D}}^1(r, z_0)$. Then we have

$$u(z_0) = \lim_{j\to\infty} u_j(z_0) \leq \lim_{j\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_j(z_0 + re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta,$$

as desired.

(ix) Let

$$M = \sup\{u(z) \mid z \in \Omega\},$$

noting that $M < \infty$ by hypothesis. Indeed, there is $z_0 \in \Omega$ such that $u(z_0) = M$. We first claim that u is constant in some neighbourhood of z_0. Suppose otherwise, and let $r \in \mathbb{R}_{>0}$ be such that, for some $z_1 \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$ we have $u(z_0) > u(z_1)$. Since u is upper semicontinuous, let $(v_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of continuous functions on $\mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$ converging pointwise to u and such that $v_j(z) \geq u(z)$ for every $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$ [Aliprantis and Border 2006, Theorem 3.13]. Choose N sufficiently large that $v_N(z_1) < M$. Then the function

$$\sigma(z) = \min\{v_N(z), M\}, \qquad z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0)),$$

is continuous. By the lemma above, we can extend σ to a harmonic function, which we also denote by σ, on $\overline{\mathsf{D}}^1(r, z_0)$. We then have, by part (iii) and noting that $\sigma(z_1) < M$,

$$\sigma(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(z_0 + re^{i\theta})\,d\theta < M = u(z_0),$$

contradicting the fact that u is subharmonic.

Thus u is constant in any neighbourhood of a point where it attains its maximum. Thus the set of points where u attains its maximum is open. As this set is clearly closed (its complement is $u^{-1}([-\infty, M))$ which is open since u is upper semicontinuous) and since Ω is connected, u is everywhere equal to M.

(x) We first prove a lemma known as ***Green's third formula***. We use the following vector calculus notation. If $I \subseteq \mathbb{R}$ is an interval, if $\gamma\colon I \to \mathbb{R}^2$ is a differentiable curve for which $\|\gamma'(s)\| = 1$ for each $s \in I$, and if $u\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable, we denote

$$\frac{\partial u}{\partial \boldsymbol{n}_\gamma}(s) = \operatorname{grad} u(\gamma(s)) \cdot \boldsymbol{n}_\gamma(s),$$

where $\boldsymbol{n}_\gamma(s) = (\gamma_2'(s), -\gamma_1'(s))$ is the normal vector to γ at s. With this notation, if I is compact we denote

$$\int_{\mathrm{image}(\gamma)} \frac{\partial u}{\partial \boldsymbol{n}}\, \mathrm{d}s \triangleq \int_I \frac{\partial u}{\partial \boldsymbol{n}_\gamma}(s)\, \mathrm{d}s.$$

With this notation we have the following result.

2 Lemma *Let $\Omega \subseteq \mathbb{C}$ be a connected open set for which $\mathrm{bd}(\Omega)$ is the image of a finite number of differentiable curves and let $\mathrm{z}_0 \in \Omega$. Let $\mathrm{u}\colon \mathrm{cl}(\Omega) \to \mathbb{R}$ be continuous on $\mathrm{cl}(\Omega)$, of class C^2 on Ω, and such that $\mathbf{D}\mathrm{u}$ extends to a continuous function on $\mathrm{bd}(\Omega)$. Let $\mathrm{v}\colon \mathrm{cl}(\Omega) \setminus \{\mathrm{z}_0\} \to \mathbb{R}$ be continuous, harmonic on $\Omega \setminus \{\mathrm{z}_0\}$, be such that $\mathbf{D}\mathrm{v}$ extends to a continuous function on $\mathrm{cl}(\Omega) \setminus \{\mathrm{z}_0\}$, and such that $\mathrm{z} \mapsto \mathrm{v}(\mathrm{z}) - \log(|\mathrm{z} - \mathrm{z}_0|^{-1})$ is harmonic in a neighbourhood of z_0. Then, denoting $\mathrm{z} = \mathrm{x} + \mathrm{iy}$,*

$$\mathrm{u}(\mathrm{z}_0) = -\frac{1}{2\pi}\int_\Omega \mathrm{v}(\mathrm{x},\mathrm{y})\Big(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}(\mathrm{x},\mathrm{y}) + \frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2}(\mathrm{x},\mathrm{y})\Big)\,\mathrm{d}\mathrm{x}\mathrm{d}\mathrm{y} - \frac{1}{2\pi}\int_{\mathrm{bd}(\Omega)}\Big(\mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathbf{n}} - \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathbf{n}}\Big)\,\mathrm{d}s. \tag{3.3}$$

Proof Let $\gamma_j\colon [0, L_j] \to \mathbb{C}$, $j \in \{1, \dots, k\}$, be differentiable curves for which (1) $\gamma_j|[0, L_j)$ is a injection into $\mathrm{bd}(\Omega)$ for each $j \in \{1, \dots, k\}$ and (2) $\mathrm{bd}(\Omega)$ is a disjoint union of $\gamma([0, L_j))$, $j \in \{1, \dots, k\}$. Let u be as in the statement of the lemma, and let σ also have the same properties. Then, using Green's Theorem [Lang 1987, Chapter XIV],

$$\begin{aligned}
\int_{\mathrm{bd}(\Omega)} u\frac{\partial\sigma}{\partial \boldsymbol{n}}\,\mathrm{d}s &= \sum_{j=1}^k \int_0^{L_j} u(\gamma_j(s))\Big(\frac{\partial\sigma}{\partial x}(\gamma_j(s))\gamma_{j,2}'(s) - \frac{\partial\sigma}{\partial y}(\gamma_j(s))\gamma_{j,1}'(s)\Big)\,\mathrm{d}s \\
&= \int_\Omega\Big(\frac{\partial u}{\partial x}(x,y)\frac{\partial\sigma}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y)\frac{\partial\sigma}{\partial y}(x,y) + u\frac{\partial^2 v}{\partial x^2}(x,y) + u\frac{\partial^2\sigma}{\partial y^2}(x,y)\Big)\,\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega (\operatorname{grad} u(x,y)\cdot\operatorname{grad}\sigma(x,y) + u(x,y)\Delta\sigma(x,y))\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

This is ***Green's first formula***. Swapping the rôles of u and σ and subtracting then gives

$$\int_{\mathrm{bd}(\Omega)}\Big(u\frac{\partial\sigma}{\partial\boldsymbol{n}} - \sigma\frac{\partial u}{\partial\boldsymbol{n}}\Big)\,\mathrm{d}s = \int_\Omega (u(x,y)\Delta\sigma(x,y) - \sigma(x,y)\Delta u(x,y))\,\mathrm{d}x\mathrm{d}y,$$

which is ***Green's second formula***.

Let u and v be as in the statement of the lemma and let σ satisfy the same conditions as u, plus the condition that σ is harmonic on Ω. By Green's second formula we then have

$$\int_{\mathrm{bd}(\Omega)}\Big(u\frac{\partial\sigma}{\partial\boldsymbol{n}} - \sigma\frac{\partial u}{\partial\boldsymbol{n}}\Big)\,\mathrm{d}s + \int_\Omega \sigma(x,y)\Delta u(x,y)\,\mathrm{d}x\mathrm{d}y = 0.$$

Thus, in the formula (3.3), we can add a harmonic function to v and the formula still holds. In particular, we can add to v the harmonic function $z \mapsto -v(z) + \log(|z - z_0|^{-1})$ to conclude that, without loss of generality, we may take $v(z) = \log(|z - z_0|^{-1})$. Thus, in the remainder of the proof, we take this as v. One easily verifies that v is harmonic on $\mathbb{C} \setminus \{z_0\}$. Indeed, if we write $(z - z_0)^{-1} = re^{i\theta}$,

$$\log((z - z_0)^{-1}) = \log(|z - z_0|^{-1}) + i\theta,$$

and so, in any neighbourhood of any point in $\mathbb{C} \setminus \{z_0\}$, v is the real part of a holomorphic function, and so is harmonic by part (i).

Now let $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$ and define $\Omega_r = \Omega \setminus \overline{\mathsf{D}}^1(r, z_0)$. One then applies Green's second formula on Ω_r:

$$\int_{\mathrm{bd}(\Omega)} \Big(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\Big)\,\mathrm{d}s - \int_{\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))} \Big(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\Big)\,\mathrm{d}s = -\int_{\Omega_r} v(x,y)\Delta u(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{3.4}$$

Note that the singularity of v at z_0 is integrable. Indeed, making the change of variables to polar coordinates,

$$\int_{\mathsf{D}^1(1,0)} \log(|z - z_0|^{-1})\,\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^1 \log(r^{-1})r\,\mathrm{d}r\mathrm{d}\theta.$$

Since $\lim_{r\to 0} r\log(r^{-1}) = 0$, the integral is finite. From this we conclude that the right-hand side of (3.4) tends to 0 as r tends to 0. Let us now turn to the other terms in (3.4). First we denote by M a bound for $\operatorname{grad} u$ in a neighbourhood of z_0 containing $\overline{\mathsf{D}}^1(r, z_0)$. Then we have

$$\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))} v\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s \le 2\pi r M \log(r^{-1}).$$

Thus

$$\lim_{r\to 0}\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))} v\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s = 0.$$

On $\mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$, writing $z = z_0 + re^{i\theta}$, we have

$$\frac{\partial v}{\partial \boldsymbol{n}} = \frac{\partial}{\partial r}\log(r^{-1}) = -r^{-1}$$

and $\mathrm{d}s = r\mathrm{d}\theta$. Thus

$$-\int_{\mathrm{bd}(\Omega)} u\frac{\partial v}{\partial \boldsymbol{n}}\,\mathrm{d}s = \int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta.$$

Now let $\epsilon \in \mathbb{R}_{>0}$ and choose r sufficiently small that $|u(z) - u(z_0)| < \frac{\epsilon}{2\pi}$ for $z \in \overline{\mathsf{D}}^1(r, z_0)$. Then we have

$$\Big|2\pi u(z_0) - \int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta\Big| \le \int_0^{2\pi} |u(z_0) - u(z_0 + re^{i\theta})|\,\mathrm{d}\theta < \epsilon.$$

Thus

$$\lim_{r\to 0}\Big(-\int_{\mathrm{bd}(\Omega)} u\frac{\partial v}{\partial \boldsymbol{n}}\,\mathrm{d}s\Big) = 2\pi u(z_0).$$

Putting this all together gives the lemma. ▼

Proceeding with the proof of this part of the result, let $r \in \mathbb{R}_{>0}$ and z_0 be such that $\overline{\mathsf{B}}^1(r, z_0) \subseteq \Omega$. By the lemma and the computations from the proof of the lemma we have

$$u(z_0) = -\frac{1}{2\pi}\int_{\overline{\mathsf{D}}^1(r,z_0)} \Delta u(x,y)\log(r|z-z_0|^{-1})\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{3.5}$$

Note that $\log(r|z-z_0|^{-1})$ is nonnegative on $\overline{\mathsf{D}}^1(r, z_0)$ and only zero on the boundary.

Now suppose that $\Delta u(z_0) < 0$ for some $z_0 \in \Omega$. We then choose $r \in \mathbb{R}_{>0}$ such that $\Delta u(z) < 0$ for all $z \in \overline{\mathsf{D}}^1(r, z_0)$, and we then see from (3.5) that

$$u(z_0) > \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

By part (viii) it follows that u is not subharmonic.

Conversely, suppose that u is not subharmonic. By part (viii) there exists $z_0 \in \Omega$ and $r \in \mathbb{R}_{>0}$ such that

$$u(z_0) > \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

By (3.5) we conclude that Δu must be negative at points in a neighbourhood of z_0. ■

3.2.3 Remarks (Harmonic and subharmonic functions)

1. Note that it is clear that the imaginary part of a holomorphic function is also a harmonic function (since multiplication of a holomorphic function by $-\mathrm{i}$ produces another holomorphic function). Given a harmonic function u and a holomorphic function Φ for which $u = \mathrm{Re}(\Phi)$, we say that $\mathrm{Im}(\Phi)$ is the ***harmonic conjugate*** of u.
2. Since a harmonic function is the real part of a holomorphic function, it follows that harmonic functions are infinitely differentiable, although their definition only requires them to be of class C^2.
3. The condition of upper semicontinuity for subharmonic functions might seem a little unmotivated. Many natural subharmonic functions are continuous (and in fact many authors assume continuity in their definitions of subharmonic functions.) However, it comes as a consequence of properties (v) and (vi) that upper semicontinuity can arise in limiting processes where continuity is present. •

The preceding result then allows us to construct some examples of harmonic and subharmonic functions.

3.2.4 Examples (Harmonic and subharmonic functions)

1. If $\Omega \subseteq \mathbb{C}$ is open and if $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ then the function

$$\Omega \ni z \mapsto \log(|f|)(z) \triangleq \begin{cases} \log(|f(z)|), & f(z) \neq 0, \\ -\infty, & f(z) = 0, \end{cases}$$

is harmonic on $\Omega \setminus f^{-1}(0)$ and subharmonic on Ω. To see that $\log(|f|)$ is harmonic on $\Omega \setminus f^{-1}(0)$, let $z_0 \in \Omega \setminus f^{-1}(0)$ and in a neighbourhood of z_0 write

$$\log(f(z)) = \log(|f(z)|) + \mathrm{i}\theta,$$

where $f(z) = re^{\mathrm{i}\theta}$. Thus $\log(|f|)$ is the real part of a holomorphic function, and so harmonic in a neighbourhood of z. If f is identically zero in a neighbourhood of $z \in \Omega$, it is immediate that $\log(|f|)$ is subharmonic on this neighbourhood. It remains to consider points $z_0 \in \Omega$ such that $f(z_0) = 0$ but, on any neighbourhood of z_0, f is not identically zero. Let $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^1(r, z_0) \subseteq \Omega$ and let $\sigma\colon \overline{\mathsf{D}}^1(r, z_0) \to \mathbb{R}$ be a continuous function, harmonic on $\mathsf{D}^1(r, z_0)$, such that $\log(|f(z)|) \leq \sigma(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. Note that we clearly have

$$\log(|f(z_0)|) \leq \int_0^{2\pi} \log(|f(z_0 + re^{\mathrm{i}\theta})|)\,\mathrm{d}\theta.$$

The same condition holds for $\log(|f|) - \sigma$. Referring to the proof of part (ix) of Theorem 3.2.2, we see that this implies that $\log(|f|) - \sigma$, not being a constant function, has the property that it must achieve its maximum on $\mathrm{bd}(\overline{\mathsf{D}}^1(r, z_0))$. This implies that $\log(|f(z)|) \leq \sigma(z)$ for $z \in \mathsf{D}^1(r, z_0)$, giving subharmonicity of $\log(|f|)$.

2. If $\Omega \subseteq \mathbb{C}$ is open, define the ***boundary distance*** function $\delta_\Omega\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}$ (here $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$) by

$$\delta_\Omega(z) = \sup\{r \in \mathbb{R}_{>0} \mid \mathsf{D}^1(r, z) \subseteq \Omega\}.$$

Note that if $\Omega \neq \mathbb{C}$ then

$$\delta_\Omega(z) = \mathrm{dist}(z, \mathbb{C} \setminus \Omega) = \mathrm{dist}(z, \mathrm{bd}(\Omega)).$$

From Proposition B.1.2 we know that δ_Ω is continuous. The function $-\log\delta_\Omega\colon \Omega \to [-\infty, \infty)$ defined by

$$-\log\delta_\Omega(z) = \begin{cases} -\log(\delta_\Omega(z)), & \delta_\Omega(z) \in \mathbb{R}, \\ -\infty, & \delta_\Omega(z) = \infty, \end{cases}$$

is subharmonic. If $\Omega = \mathbb{C}$ then the claim is trivial. In case $\Omega \subsetneq \mathbb{C}$, δ_Ω is continuous, finite, and positive-valued on Ω. Thus $-\log\delta_\Omega$ is continuous and $\mathbb{R}$-valued. If $w \in \mathrm{bd}(\Omega)$ define $\phi_w(z) = -\log(|z - w|)$, noting that this function is harmonic on Ω by our preceding example. Moreover, note that

$$-\log\delta_\Omega(z) = \sup\{\phi_w(z) \mid w \in \mathrm{bd}(\Omega)\},$$

which shows that $-\log\delta_\Omega$ is indeed subharmonic, it being continuous and the pointwise supremum of subharmonic functions. •

3.2.2 Plurisubharmonic functions

As mentioned at the beginning of the preceding section, the notion of a plurisubharmonic function generalises that of a subharmonic function to several variables. The notions one arrives at were introduced by Oka [1942a] and Lelong [1942], although the notion has shadows in the older work of Hartogs. One way that one might think to do the generalisation we are after is to use the $2n$-dimensional Laplacian for functions on $\mathbb{C}^n$,

$$\Delta u(z) = \sum_{j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_j^2}(z) + \frac{\partial^2 u}{\partial y_j^2}(z) \Big), \qquad z = x + \mathrm{i}y,$$

to define the class of harmonic functions on $\mathbb{C}^n$, and then define the notion of submarmonicity as in the dimension 1 case. This, however, will not give us what we want. For example, one loses the connections with holomorphic functions one has in dimension 1. Also, this sort of generalisation turns out not to be invariant under holomorphic diffeomorphisms, so limiting its usefulness in differential geometry. It turns out, in any case, that the correct generalisation is done as follows.

3.2.5 Definition (Plurisubharmonic function) If $\Omega \subseteq \mathbb{C}^n$ is open, an upper semicontinuous function $u\colon \Omega \to [-\infty, \infty)$ is ***plurisubharmonic*** if, for every $(z, w) \in \Omega \times \mathbb{C}^n$, the function

$$\mathbb{C} \ni \zeta \mapsto u(z + \zeta w) \in [-\infty, \infty)$$

is subharmonic on the connected component of

$$\{\zeta \in \mathbb{C} \mid z + \zeta w \in \Omega\}$$

containing 0. By $\mathrm{Psh}(\Omega)$ we denote the set of plurisubharmonic functions on Ω. •

Many of the basic properties of plurisubharmonic functions are derived from those for subharmonic functions. Let us catalogue these.

3.2.6 Proposition (Properties of plurisubharmonic functions) *If $\Omega \subseteq \mathbb{C}^n$ is open, the following statements hold:*

(i) if $(u_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathrm{Psh}(\Omega)$ such that $u_{j+1}(z) \le u_j(z)$ for each $j \in \mathbb{Z}_{>0}$ and $z \in \Omega$, then the function u on Ω defined by $u(z) = \lim_{j\to\infty} u_j(z)$ is plurisubharmonic;

(ii) if $(u_a)_{a\in A}$ is a family of functions in $\mathrm{Psh}(\Omega)$ then the function u on Ω defined by

$$u(z) = \sup\{u_a(z) \mid a \in A\}$$

is plurisubharmonic if it is upper semicontinuous and everywhere finite;

(iii) if Ω is connected and if u is plurisubharmonic and has a global maximum in Ω, then u is constant;

(iv) *if* $u_1, \dots, u_k \colon \Omega \to [-\infty, \infty)$ *are plurisubharmonic and if* $F \colon \mathbb{R}^k \to \mathbb{R}$ *is continuous, convex, and nondecreasing in each component, and if we extend* F *to* $\bar{F} \colon ([-\infty, \infty))^k \to [-\infty, \infty)$ *as in Theorem 3.2.2(vii), then the function*

$$z \mapsto F(u_1(z), \dots, u_k(z))$$

is plurisubharmonic.

Let us now turn to the analogue for plurisubharmonic functions of the second-derivative condition of Theorem 3.2.2(x) for subharmonic functions.

3.2.7 Definition (Levi form) If $\Omega \subseteq \mathbb{C}^n$ is open, if $u \in C^2(\Omega)$, and if $\boldsymbol{z} \in \Omega$, the ***Levi form*** of u at $\boldsymbol{z}$ is the quadratic form

$$\boldsymbol{w} \mapsto \operatorname{Lev}(u)(\boldsymbol{z}; \boldsymbol{w}) \triangleq \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(\boldsymbol{z}) w_j \bar{w}_k. \qquad \bullet$$

It is illustrative to consider a special case of the Levi form.

3.2.8 Example (The Levi form of a quadratic function) Let $\boldsymbol{B}$ be a real symmetric bilinear map on $\mathbb{C}^n$, i.e., an element of $S^2(\mathbb{C}^n)$, thinking of $\mathbb{C}^n$ as a $\mathbb{R}$-vector space. Let us denote a point in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ by $(\boldsymbol{x}, \boldsymbol{y})$. Let $f_{\boldsymbol{B}}$ be the corresponding quadratic function

$$f_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{B}((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{y})).$$

Let us write

$$f_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{B}_{11}(\boldsymbol{x}, \boldsymbol{x}) + \boldsymbol{B}_{12}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{B}_{21}(\boldsymbol{y}, \boldsymbol{x}) + \boldsymbol{B}_{22}(\boldsymbol{y}, \boldsymbol{y}).$$

Then, using (1.11) and denoting $\boldsymbol{w} = \boldsymbol{u} + i\boldsymbol{v}$, we directly compute

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 f_{\boldsymbol{B}}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k &= \tfrac{1}{2}(\boldsymbol{B}_{11}(\boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{B}_{12}(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{B}_{21}(\boldsymbol{v}, \boldsymbol{u}) + \boldsymbol{B}_{22}(\boldsymbol{v}, \boldsymbol{v})) \\ &\quad + \tfrac{1}{2}(\boldsymbol{B}_{11}(-\boldsymbol{v}, -\boldsymbol{v}) + \boldsymbol{B}_{12}(-\boldsymbol{v}, \boldsymbol{u}) + \boldsymbol{B}_{21}(\boldsymbol{u}, -\boldsymbol{v})).\end{aligned}$$

Noting that $-\boldsymbol{v} + i\boldsymbol{u} = i(\boldsymbol{u} + i\boldsymbol{v})$ and that $\boldsymbol{B} = \frac{1}{2}\operatorname{Hess} f_{\boldsymbol{B}}$ (Hess denoting the Hessian, i.e., the second derivative) we thus have

$$\operatorname{Lev}(f_{\boldsymbol{B}})(\boldsymbol{z}; \boldsymbol{w}) = \operatorname{Hess} f_{\boldsymbol{B}}(\boldsymbol{w}, \boldsymbol{w}) + \operatorname{Hess} f_{\boldsymbol{B}}(i\boldsymbol{w}, i\boldsymbol{w}).$$

That is, the Levi form in the direction of $\boldsymbol{w}$ is twice the “complex average” of the Hessian in the direction of $\boldsymbol{w}$.

Let us consider the special case when $\boldsymbol{B}$ is diagonal:

$$f_{\boldsymbol{B}}(\boldsymbol{z}) = \sum_{j=1}^{n} (\alpha_j x_j^2 + \beta_j y_j^2).$$

In this case we have

$$\mathrm{Lev}(f_B)(z;w) = \sum_{j=1}^{n}(\alpha_j x_j^2 + \beta_j y_j^2) + \sum_{j=1}^{n}(\alpha_j y_j^2 + \beta_j x_j^2) = \sum_{j=1}^{n}(\alpha_j + \beta_j)|z_j|^2.$$

Thus we see that f_B is plurisubharmonic if and only if $\alpha_j + \beta_j \geq 0$ for every $j \in \{1, \dots, n\}$, cf. Proposition 3.2.12 below. •

The preceding example, while special, immediate gives the following fact.

3.2.9 Lemma (The Levi form and the Hessian) *If $\Omega \subseteq \mathbb{C}^n$ is open and if $u \in C^2(\Omega)$, then*

$$\mathrm{Lev}(u)(\mathbf{z};\mathbf{w}) = \mathbf{D}^2u(\mathbf{z}) \cdot (\mathbf{w},\mathbf{w}) + \mathbf{D}^2u(\mathbf{z}) \cdot (i\mathbf{w}, i\mathbf{w}).$$

The following result relates the Levi form $\mathrm{Lev}(u)$ in a precise way to the Taylor expansion of u.

3.2.10 Lemma (The Levi form and the Taylor expansion) *Let $\Omega \subseteq \mathbb{C}^n$ be open, let $u \in C^2(\Omega)$, and define*

$$P(u)(\mathbf{z};\mathbf{w}) = -\Big(2\sum_{j=1}^{n}\frac{\partial u}{\partial z_j}(\mathbf{z})w_j + \sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j \partial z_k}(\mathbf{z})w_j w_k\Big).$$

Then the second-order Taylor polynomial for u at $\mathbf{z}$ is

$$\mathbf{w} \mapsto u(\mathbf{z}) - \mathrm{Re}(P(u)(\mathbf{z};\mathbf{w})) + \mathrm{Lev}(u)(\mathbf{z};\mathbf{w}).$$

Proof This is a mere direct, tedious computation. ■

The Levi form seems to resemble the Hessian from real multivariable calculus. However, it has some important differences, among them the following nice transformation law.

3.2.11 Lemma (The Levi form under holomorphic mappings) *If $\Omega \subseteq \mathbb{C}^n$ and $\mathcal{V} \subseteq \mathbb{C}^m$ are open, if $u \in C^2(\mathcal{V})$, and if $\mathbf{\Phi}\colon \Omega \to \mathcal{V}$ is holomorphic, then*

$$\mathrm{Lev}(u \circ \mathbf{\Phi})(\mathbf{z};\mathbf{w}) = \mathrm{Lev}(u)(\mathbf{\Phi}(\mathbf{z}); \mathbf{D}\mathbf{\Phi}(\mathbf{z}) \cdot \mathbf{w})$$

for every $\mathbf{z} \in \Omega$ and $\mathbf{w} \in \mathbb{C}^n$.

Proof This can be directly, if tediously, verified using the definition. ■

A consequence of the preceding lemma is the Levi form is a well-defined quadratic form under holomorphic changes of coordinate. This will be of importance when we come to discussing the generalisations of this section to manifolds.

Unsurprisingly, the Levi form is the appropriate device for proving the appropriate condition for plurisubharmonicity.

3.2.12 Proposition (Plurisubharmonic functions of class C^2) *If $\Omega \subseteq \mathbb{C}^n$ is open and if $u \in \mathrm{Psh}(\Omega) \cap C^2(\Omega)$, then u is plurisubharmonic if and only if $\mathrm{Lev}(u)(\mathbf{z}; \mathbf{w}) \geq 0$ for every $\mathbf{z} \in \Omega$ and $\mathbf{w} \in \mathbb{C}^n$.*

Proof Let $z \in \Omega$ and $w \in \mathbb{C}^n$. Define $\ell_{z,w} : \mathbb{C} \to \mathbb{C}^n$ by

$$\ell_{a,w}(\zeta) = z + \zeta w.$$

We must show that $u \circ \ell_{z,w}$ is subharmonic for every $z \in \Omega$ and $w \in \mathbb{C}^n$. Since $u \circ \ell_{z,w}$ is of class C^2, $u \circ \ell_{z,w}$ is subharmonic if and only if $\frac{\partial^2(u \circ \ell_{z,w})}{\partial\zeta\partial\bar{\zeta}}(\zeta) \geq 0$ for ζ in a neighbourhood of 0. Since

$$\ell_{z,w}(\zeta) = \ell_{z+\zeta w,w}(0),$$

it follows that $\frac{\partial^2(u \circ \ell_{z,w})}{\partial\zeta\partial\bar{\zeta}}(\zeta) \geq 0$ for every $z \in \Omega$, $w \in \mathbb{C}^n$, and ζ in a neighbourhood of 0 if and only if $\frac{\partial^2(u \circ \ell_{z,w})}{\partial\zeta\partial\bar{\zeta}}(0) \geq 0$ for every $z \in \Omega$ and $w \in \mathbb{C}^n$. But, using the Chain Rule, we compute

$$\frac{\partial^2(u \circ \ell_{z,w})}{\partial\zeta\partial\bar{\zeta}}(0) = \mathrm{Lev}(u \circ \ell_{z,w})(0, 1) = \mathrm{Lev}(u)(z; w),$$

and from this our result follows. ■

The preceding result can be extended to the case of functions that are not sufficiently smooth, provided that one is willing to interpret derivatives in a distributional sense, as in Section D.2.2. Let us be clear what we mean by this. We let $\Omega \subseteq \mathbb{C}^n$ be open and let $\theta \in \mathscr{D}'(\Omega; \mathbb{R})$. We then define, for each $j \in \{1, \dots, n\}$, the distributions $\frac{\partial\theta}{\partial z_j}$ and $\frac{\partial\theta}{\partial\bar{z}_j}$ by

$$\frac{\partial\theta}{\partial z_j}(\phi) = -\theta\Big(\frac{\partial\phi}{\partial z_j}\Big), \quad \frac{\partial\theta}{\partial\bar{z}_j}(\phi) = -\theta\Big(\frac{\partial\phi}{\partial\bar{z}_j}\Big)$$

for $\phi \in \mathscr{D}(\Omega; \mathbb{R})$. Thus we make the following definition.

3.2.13 Definition (Levi form for distributions) If $\Omega \subseteq \mathbb{C}^n$ is open and if $\theta \in \mathscr{D}'(\Omega; \mathbb{R})$, the ***Levi form*** of θ is the quadratic $\mathscr{D}'(\Omega; \mathbb{R})$-valued function $w \mapsto \mathrm{Lev}(\theta)(w)$ defined by

$$\mathrm{Lev}(\theta)(w) \cdot \phi = \theta(\mathrm{Lev}(\phi)(w)),$$

where, for $\phi \in \mathscr{D}(\Omega; \mathbb{R})$, $\mathrm{Lev}(\phi)(w) \in \mathscr{D}(\Omega; \mathbb{R})$ is defined by

$$z \mapsto \mathrm{Lev}(\phi)(z; w). \qquad \bullet$$

We should check that this generalised notion of the Levi form agrees with the usual definition in cases where both apply.

3.2.14 Proposition (Compatibility of two definitions of the Levi form) *If $\Omega \subseteq \mathbb{C}^n$ is open and if $u \in C^2(\Omega)$, then $\mathrm{Lev}(\theta_u(\mathbf{w})) = \theta_{\mathrm{Lev}(u)(\mathbf{w})}$ for every $\mathbf{w} \in \mathbb{C}^n$, where by $\mathrm{Lev}(u)(\mathbf{w})$ we denote the function on Ω given by $\mathbf{z} \mapsto \mathrm{Lev}(u)(\mathbf{z};\mathbf{w})$.*

Proof Let $\phi \in \mathscr{D}(\Omega;\mathbb{R})$ and compute

$$\mathrm{Lev}(\theta_u(\boldsymbol{w}))\cdot\phi = \int_\Omega u(\boldsymbol{z})\mathrm{Lev}(\phi)(\boldsymbol{z};\boldsymbol{w})\,\mathrm{d}\lambda(\boldsymbol{z}) = \int_\Omega \mathrm{Lev}(u)(\boldsymbol{z};\boldsymbol{w})\phi(\boldsymbol{z})\,\mathrm{d}\lambda(\boldsymbol{z}) = \theta_{\mathrm{Lev}(u)(\boldsymbol{w})}(\phi),$$

as desired, where we have used integration by parts. ■

Then we have the following result, making reference to the notion given in Section D.1.3 of a nonnegative distribution.

3.2.15 Proposition (Distributional derivative characterisation of plurisubharmonic function) *If $\Omega \subseteq \mathbb{C}^n$ is open, then $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$ is plurisubharmonic if and only if the distribution $\mathrm{Lev}(\theta_u)(\mathbf{w})$ is nonnegative for every $\mathbf{w} \in \mathbb{C}^n$.*

Proof Let $\rho \in C^\infty(\mathbb{C}^n)$ have the following properties:

1. $\rho(\boldsymbol{z}) \geq 0$;
2. $\mathrm{supp}(\rho) = \mathsf{B}^n(1,\boldsymbol{0})$;
3. $\rho(\boldsymbol{z}_1) = \rho(\boldsymbol{z}_2)$ whenever $\|\boldsymbol{z}_1\| = \|\boldsymbol{z}_2\|$;
4. $\int_{\mathbb{C}^n}\rho(\boldsymbol{z})\,\mathrm{d}\lambda(\boldsymbol{z}) = 1$.

Let $\rho_\epsilon(\boldsymbol{z}) = \epsilon^{-2n}\rho(\epsilon\boldsymbol{z})$ for $\epsilon \in \mathbb{R}_{>0}$. Let $K \subseteq \Omega$ be compact and let $\chi_K \in C^\infty(\Omega)$ have compact support and be such that $\chi_K(\boldsymbol{z}) = 1$ for $\boldsymbol{z}$ in a neighbourhood of K. Define $u_\epsilon \triangleq (\chi_K u) * \rho_\epsilon$. By standard results on approximation using convolution [Kecs 1982], the domain of u_ϵ is contained in Ω for $\epsilon \in \mathbb{R}_{>0}$ sufficiently small and $\lim_{\epsilon\to 0} u_\epsilon|K = u|K$, with convergence being uniform. Moreover, each of the functions u_ϵ is smooth.

We claim that

$$\mathrm{Lev}(\theta_{u_\epsilon})(\boldsymbol{w}) = \mathrm{Lev}(\theta_{\chi_K u})(\boldsymbol{w}) * \rho_\epsilon. \tag{3.6}$$

To see this, first let $\phi \in \mathscr{D}(\Omega;\mathbb{R})$ and compute

$$\begin{aligned}\mathrm{Lev}(\theta_{u_\epsilon})(\boldsymbol{w})\cdot\phi &= \theta_{u_\epsilon}\Big(\sum_{j,k=1}^n \frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}w_j\bar{w}_k\Big)\\ &= \sum_{j,k=1}^n w_j\bar{w}_k\int_{\mathbb{C}^n}\rho_\epsilon * (\chi_K u)(\boldsymbol{z})\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(\boldsymbol{z})\,\mathrm{d}\lambda(\boldsymbol{z})\\ &= \sum_{j,k=1}^n w_j\bar{w}_k\int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\rho_\epsilon(\boldsymbol{\zeta}-\boldsymbol{z})\chi_K(\boldsymbol{\zeta})u(\boldsymbol{\zeta})\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\lambda(\boldsymbol{z}).\end{aligned}$$

We also have

$$\begin{aligned}(\mathrm{Lev}(\theta_{\chi_K u})(\boldsymbol{w}) * \rho_\epsilon)\cdot\phi &= \langle \mathrm{Lev}(\theta_{\chi_K u})(\boldsymbol{w}) \otimes \rho_\epsilon; \Delta^*\phi\rangle\\ &= \sum_{j,k=1}^n w_j\bar{w}_k\int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\chi_K(\boldsymbol{z})u(\boldsymbol{z})\rho_\epsilon(\boldsymbol{\zeta})\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(\boldsymbol{z}+\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\lambda(\boldsymbol{z})\\ &= \sum_{j,k=1}^n w_j\bar{w}_k\int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\chi_K(\boldsymbol{z})u(\boldsymbol{z})\rho_\epsilon(\boldsymbol{\zeta}-\boldsymbol{z})\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(\boldsymbol{\zeta})\,\mathrm{d}\lambda(\boldsymbol{\zeta})\mathrm{d}\lambda(\boldsymbol{z}),\end{aligned}$$

giving (3.6). Since this holds for every compact set K we, in fact, have

$$\mathrm{Lev}(\theta_u * \rho_\epsilon)(\boldsymbol{w}) = \mathrm{Lev}(\theta_u)(\boldsymbol{w}) * \rho_\epsilon.$$

Using this fact, it is more or less straightforward to complete the proof.

First suppose that u is plurisubharmonic. As we will see in Sublemma 5 from the proof of Lemma GA2.7.1.4, $u * \rho_\epsilon$ is plurisubharmonic. Thus $\mathrm{Lev}(u * \rho_\epsilon)(\boldsymbol{w})$ is a nonnegative distribution by Propositions 3.2.12 and D.2.5. Since

$$\mathrm{Lev}(\theta_u)(\boldsymbol{w}) = \lim_{\epsilon\to 0} \mathrm{Lev}(\theta_u)(\boldsymbol{w}) * \rho_\epsilon = \lim_{\epsilon\to 0} \mathrm{Lev}(\theta_u * \rho_\epsilon)(\boldsymbol{w})$$

(limits being taken in $\mathscr{D}'(\Omega;\mathbb{R})$), it follows that $\mathrm{Lev}(\theta_u)(\boldsymbol{w})$ is also a nonnegative distribution.

Conversely, suppose that $\mathrm{Lev}(\theta_u)(\boldsymbol{w})$ is nonnegative for every $\boldsymbol{w} \in \mathbb{C}^n$. Then we have

$$\mathrm{Lev}(\theta_u * \rho_\epsilon)(\boldsymbol{w}) = \mathrm{Lev}(\theta_u)(\boldsymbol{w}) * \rho_\epsilon.$$

The distribution on the right is nonnegative being a convolution of two nonnegative distributions. Thus $\mathrm{Lev}(\theta_u * \rho_\epsilon)(\boldsymbol{w})$ is nonnegative and so $u * \rho_\epsilon$ is plurisubharmonic by Propositions 3.2.12 and D.2.5. From the proof of Sublemma 5 from the proof of Lemma GA2.7.1.4, we see that the family $(u * \rho_\epsilon)_{\epsilon\in\mathbb{R}_{>0}}$ is nonincreasing as $\epsilon \to 0$, and as a result u is plurisubharmonic by Proposition 3.2.6(i). ■

We shall make use of the preceding characterisation of plurisubharmonic functions in Section 6.1.2 when we approximate plurisubharmonic functions on manifolds with smooth plurisubharmonic functions.

The case where the inequality for the Levi form is strict is singled out.

3.2.16 Definition (Strictly plurisubharmonic function) If $\Omega \subseteq \mathbb{C}^n$ is open, a function $u \in \mathrm{C}^2(\Omega)$ is ***strictly plurisubharmonic*** if $\mathrm{Lev}(u)(\boldsymbol{z};\boldsymbol{w}) > 0$ for every $\boldsymbol{z} \in \Omega$ and every $\boldsymbol{w} \in \mathbb{C}^n \setminus \{\boldsymbol{0}\}$. •

With all of the above, we can give some examples of plurisubharmonic functions.

3.2.17 Examples (Plurisubharmonic functions)

1. We claim that if $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ then $\boldsymbol{z} \mapsto \log(|f(\boldsymbol{z})|)$ is plurisubharmonic. This follows from Example 3.2.4–1, noting that, for each $(\boldsymbol{z},\boldsymbol{w}) \in \Omega\times\mathbb{C}^n$,

$$\zeta \mapsto f(\boldsymbol{z} + \zeta\boldsymbol{w})$$

is holomorphic the connected component of its domain of definition containing 0.

2. Sometimes the function $-\log\delta_\Omega$ is not plurisubharmonic, even though in dimension 1 it always is, as we saw in Example 3.2.4–2. Let us consider an example of this. We let $\Omega = \mathbb{C}^n\setminus\{\boldsymbol{0}\}$, $n \geq 2$, and take $\boldsymbol{z} = \boldsymbol{e}_1$ and $\boldsymbol{w} = \boldsymbol{e}_2$, with $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)$ denoting the standard basis. We then have

$$-\log\delta_\Omega(\boldsymbol{z} + \zeta\boldsymbol{w}) = -\log(\|\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2\|) = -\log\sqrt{1 + |\zeta|^2}.$$

Note that the function

$$\zeta \mapsto -\log \sqrt{1+|\zeta|^2}$$

has a strict maximum at $\zeta = 0$ and so is not subharmonic. The reader will note that in Example 3.1.8–3 we showed that Ω is not holomorphically convex. It is not a coincidence that Ω is not holomorphically convex and also has the property that $-\log\delta_\Omega$ is not plurisubharmonic.

3. Let H be a $\mathbb{R}$-hyperplane in $\mathbb{C}^n$. As we saw in Example 3.1.3–2, this means that $\mathsf{H} = \phi^{-1}(0)$ for a function $\phi\colon \mathbb{C}^n \to \mathbb{R}$ of the form $\phi(z) = \mathrm{Re}(\langle z - z_0, \lambda\rangle)$. If we assume, without loss of generality, that $\|\lambda\| = 1$, then the distance $\delta_{\mathbb{C}^n\setminus\mathsf{H}}(z)$ from $z \notin \mathsf{H}$ to H is $|\phi(z)|$. On $\mathbb{C}^n \setminus \mathsf{H}$, $\delta_{\mathbb{C}^n\setminus\mathsf{H}}$ is smooth. Moreover, we compute

$$\frac{\partial\delta_{\mathbb{C}^n\setminus\mathsf{H}}}{\partial z_j}(z) = \frac{1}{2}\lambda_j, \quad \frac{\partial\delta_{\mathbb{C}^n\setminus\mathsf{H}}}{\partial \bar{z}_j}(z) = \frac{1}{2}\bar{\lambda}_j, \qquad j \in \{1,\dots,n\},$$

and

$$\frac{\partial^2\delta_{\mathbb{C}\setminus\mathsf{H}}}{\partial z_j\partial\bar{z}_k}(z) = 0, \qquad j,k \in \{1,\dots,n\},$$

as long as $\phi(z) > 0$. Thus, for such z,

$$\frac{\partial^2 \log\delta_{\mathbb{C}^n\setminus\mathsf{H}}}{\partial z_j\partial\bar{z}_k}(z) = -\frac{1}{4\delta_{\mathbb{C}^n\setminus\mathsf{H}}(z)^2}\lambda_j\bar{\lambda}_k.$$

Using this formula we readily deduce that $\mathrm{Lev}(-\log\delta_{\mathbb{C}^n\setminus\mathsf{H}})(z;\lambda) > 0$ and that $\mathrm{Lev}(-\log\delta_{\mathbb{C}^n\setminus\mathsf{H}})(z;w) = 0$ if w is orthogonal to λ. Thus $-\log\delta_{\mathbb{C}^n\setminus\mathsf{H}}$ is plurisubharmonic. •

The following plurisubharmonic characterisation of domains of holomorphy is important.

3.2.18 Theorem (The log boundary distance for domains of holomorphy) *If $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy then $-\log\delta_\Omega$ is continuous and plurisubharmonic.*

Proof We prove a couple of technical lemmata.

1 Lemma *Let $\Omega \subseteq \mathbb{C}$ be open and let $u\colon \Omega \to [-\infty,\infty)$ be upper semicontinuous with the property that, if $D \subseteq \Omega$ is a closed disk and if $p \in C^{\mathrm{hol}}(\Omega)$ is a polynomial such that*

$$u(z) \le \mathrm{Re}(p(z)), \qquad z \in \mathrm{bd}(D),$$

then $u(z) \le \mathrm{Re}(p(z))$ for $z \in D$. Then u is subharmonic.

Proof Let $\overline{\mathsf{D}}^1(r,z_0) \subseteq \Omega$ be a closed disk and let $\sigma\colon \overline{\mathsf{D}}^1(r,z_0) \to \mathbb{R}$ be continuous, harmonic on $\mathsf{D}^1(r,z_0)$, and with the property that $\sigma(z) \ge u(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$. For $\lambda \in (0,1)$ the function $\sigma_\lambda(z) = \sigma(z_0 + \lambda(z - z_0))$ is harmonic on the open disk $\mathsf{D}^1(\frac{r}{\lambda},z_0)$ containing $\overline{\mathsf{D}}^1(r,z_0)$. Moreover, $\lim_{\lambda\to1}\sigma_\lambda = \sigma$ in the topology of uniform convergence on $\overline{\mathsf{D}}^1(r,z_0)$, as may be easily verified. By Theorem 3.2.2(ii), for each $\lambda \in (0,1)$ there is a holomorphic function g_λ defined on $\mathsf{D}^1(\frac{r}{\lambda},z_0)$ such that $\sigma_\lambda(z) = \mathrm{Re}(g_\lambda(z))$ for all $z \in \mathsf{D}^1(\frac{r}{\lambda},z_0)$.

Let $\epsilon \in \mathbb{R}_{>0}$. Let $\lambda \in (0,1)$ be sufficiently close to 1 that $\|\sigma - \sigma_\lambda\|_{\overline{\mathsf{D}}^1(r,z_0)} < \frac{\epsilon}{2}$, this by uniform convergence of $\sigma_\lambda \to \sigma$ as $\lambda \to 1$. Also, since the Taylor series for g_λ converges uniformly to g_λ on $\overline{\mathsf{D}}^1(r,z_0)$, let m be sufficiently large that $\|q - g_\lambda\|_{\overline{\mathsf{D}}^1(r,z_0)} < \frac{\epsilon}{4}$, where q is the degree m Taylor expansion for g_λ. Taking real parts, we also have $\|\mathrm{Re}(q) - \sigma_\lambda\|_{\overline{\mathsf{D}}^1(r,z_0)} < \frac{\epsilon}{4}$. Thus we have

$$\sigma(z) \leq \sigma_\lambda(z) + \tfrac{\epsilon}{2} \leq \sigma(z) + \epsilon$$

and

$$\sigma_\lambda(z) \leq \mathrm{Re}(q(z)) + \tfrac{\epsilon}{4} \leq \sigma_\lambda(z) + \tfrac{\epsilon}{2}$$

for every $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$, and so we deduce that

$$\sigma(z) \leq \mathrm{Re}(p(z)) \leq \sigma(z) + \epsilon$$

for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$, after taking $p = q + \frac{\epsilon}{4}$.

By our hypothesis and Theorem 3.2.2(iii), there exists $\rho \in \mathbb{R}_{>0}$ such that

$$u(z_0) \leq \mathrm{Re}(p(z_0)) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(p(z_0 + re^{\mathrm{i}\theta}))\,\mathrm{d}\theta \leq \int_0^{2\pi} \sigma(z_0 + re^{\mathrm{i}\theta})\,\mathrm{d}\theta + \epsilon$$

for $r \in (0,\rho]$. As $\epsilon \in \mathbb{R}_{>0}$ is arbitrary, we ascertain that

$$u(z_0) \leq \int_0^{2\pi} \sigma(z_0 + re^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

for every function σ that is continuous on $\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$ and satisfies $u(z) \leq \sigma(z)$ for $z \in \mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$. Recall now that the integral of the upper semicontinuous function $u|\,\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$ on the compact domain $\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$ is the infimum of the integral of those continuous functions on $\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$ bounding u from above on $\mathrm{bd}(\overline{\mathsf{D}}^1(r,z_0))$ cf. [Aliprantis and Border 2006, Theorem 3.13] and the Monotone Convergence Theorem. This implies that

$$u(z_0) \leq \int_0^{2\pi} u(z_0 + re^{\mathrm{i}\theta})\,\mathrm{d}\theta,$$

and the subharmonicity of u follows from Theorem 3.2.2(viii). ▼

2 Lemma *Let $\Omega \subseteq \mathbb{C}^n$ be a domain of holomorphy and let $K \subseteq \Omega$ be compact. If $f \in C^{\mathrm{hol}}(\Omega)$ satisfies $|f(\boldsymbol{z})| \leq \delta_\Omega(\boldsymbol{z})$ for all $\boldsymbol{z} \in K$, then $|f(\boldsymbol{z})| \leq \delta_\Omega(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$.*

Proof Let us perform a preliminary little construction. For $\boldsymbol{r} \in \mathbb{R}^n_{>0}$ and for $\boldsymbol{z} \in \Omega$, denote

$$\delta^{\boldsymbol{r}}_\Omega(\boldsymbol{z}) = \sup\{\rho \in \mathbb{R}_{>0} \mid \boldsymbol{z} + \rho\overline{\mathsf{D}}^n(\boldsymbol{r},\boldsymbol{0}) \subseteq \Omega\}.$$

We claim that

$$\mathrm{dist}(\boldsymbol{z},\mathrm{bd}(\Omega)) = \inf\{\delta^{\boldsymbol{r}}_\Omega(\boldsymbol{z}) \mid \boldsymbol{r} \in \mathbb{R}^n_{>0},\ \|\boldsymbol{r}\| = 1\}.$$

Certainly, for $\|\boldsymbol{r}\| = 1$ we have

$$(\boldsymbol{z} + \delta^{\boldsymbol{r}}_\Omega(\boldsymbol{z})\overline{\mathsf{D}}^n(\boldsymbol{r},\boldsymbol{0})) \cap \mathrm{bd}(\Omega) \neq \emptyset.$$

Let $\boldsymbol{w} \in \overline{\mathsf{D}}^n(\boldsymbol{r}, \boldsymbol{0})$ be such that

$$\boldsymbol{z} + \delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z})\boldsymbol{w} \in \mathrm{bd}(\Omega).$$

Then, since $\|\boldsymbol{w}\| \le 1$ (note that the points in $\overline{\mathsf{D}}^n(\boldsymbol{r}, \boldsymbol{0})$ furthest from $\boldsymbol{0}$ are the points where the radii in all coordinates are maximal, and these points are distance 1 from $\boldsymbol{0}$), we have

$$\mathrm{dist}(\boldsymbol{z}, \mathrm{bd}(\Omega)) \le \delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z})\|\boldsymbol{w}\| \le \delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z}) \tag{3.7}$$

for every $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ such that $\|\boldsymbol{r}\| = 1$. Now let $\hat{\boldsymbol{z}} \in \mathrm{bd}(\Omega)$ be such that $\mathrm{dist}(\boldsymbol{z}, \mathrm{bd}(\Omega)) = \|\boldsymbol{z} - \hat{\boldsymbol{z}}\|$, cf. Proposition B.1.3. Let $(\rho_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}_{>0}$ such that $\overline{\mathsf{B}}^n(\rho_j, \boldsymbol{z}) \subseteq \Omega$ and such that $\hat{\boldsymbol{z}} \in \mathrm{bd}(\cup_{j\in\mathbb{Z}_{>0}}\overline{\mathsf{B}}^n(\rho_j, \boldsymbol{z}))$. Thus there exists a sequence $(\boldsymbol{z}_j)_{j\in\mathbb{Z}_{>0}}$ converging to $\hat{\boldsymbol{z}}$ and such that $\boldsymbol{z}_j \in \mathrm{bd}(\overline{\mathsf{B}}^n(\rho_j, \boldsymbol{z}))$. Suppose that the sequence $(\boldsymbol{z}_j)_{j\in\mathbb{Z}_{>0}}$ is chosen so that none of the coordinates of the points is zero. Thus, if $\boldsymbol{z}_j = (z_{j,1}, \dots, z_{j,n})$, $r_{j,l} \triangleq |z_{j,l}| > 0$. Now let $\boldsymbol{r}_j = (r_{j,1}, \dots, r_{j,n}) \in \mathbb{R}_{>0}^n$ and note that

$$\overline{\mathsf{D}}^n(\boldsymbol{r}_j, \boldsymbol{z}) \subseteq \overline{\mathsf{B}}^n(\rho_j, \boldsymbol{z}) \subseteq \Omega.$$

Moreover, $\hat{\boldsymbol{z}} \in \mathrm{bd}(\cup_{j\in\mathbb{Z}_{>0}}\overline{\mathsf{D}}^n(\boldsymbol{r}_j, \boldsymbol{z}))$. Note that $\delta_\Omega^{\boldsymbol{r}_j}(\boldsymbol{z}) \le \|\boldsymbol{r}_j\|$ and $\lim_{j\to\infty}\|\boldsymbol{r}_j\| = \mathrm{dist}(\boldsymbol{z}, \Omega)$. Thus

$$\mathrm{dist}(\boldsymbol{z}, \mathrm{bd}(\Omega)) \ge \inf\{\delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z}) \mid \boldsymbol{r} \in \mathbb{R}_{>0}^n,\ \|\boldsymbol{r}\| = 1\},$$

which gives equality of both sides of this equation when combined with (3.7).

Let $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ be such that $|f(\boldsymbol{z})| \le \delta_\Omega(\boldsymbol{z})$ for every $\boldsymbol{z} \in K$. From our computations in the preceding paragraph, $|f(\boldsymbol{z})| \le \delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z})$ for every $\boldsymbol{z} \in K$ and every $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ satisfying $\|\boldsymbol{r}\| = 1$. Let us for the moment fix $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ with $\|\boldsymbol{r}\| = 1$. Let us also fix $t \in (0, 1)$. For such $\boldsymbol{r}$ and t, and for $\boldsymbol{z} \in K$, note that $\overline{\mathsf{D}}^n(t\boldsymbol{r}|f(\boldsymbol{z})|, \boldsymbol{z}) \subseteq \Omega$. Thus the closure of the set

$$C \triangleq \cup_{\boldsymbol{z}\in K}\overline{\mathsf{D}}^n(t\boldsymbol{r}|f(\boldsymbol{z})|, \boldsymbol{z})$$

is a compact subset of Ω. Using the Cauchy estimates of Corollary 1.1.24, for $g \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ we have

$$|\boldsymbol{D}^I g(\boldsymbol{z})| t^{|I|}\boldsymbol{r}^I |f(\boldsymbol{z})|^{|I|} \le I!\|g\|_C \tag{3.8}$$

for every $\boldsymbol{z} \in K$. Since

$$\boldsymbol{z} \mapsto \boldsymbol{D}^I g(\boldsymbol{z}) f(\boldsymbol{z})^{|I|}$$

is holomorphic on Ω, by definition of the holomorphically convex hull we conclude that the estimate (3.8) holds for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$. Since (3.8) holds for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$ and every $t \in (0, 1)$, we conclude that the Taylor series for every $g \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ converges on $\mathsf{D}^n(|f(\boldsymbol{z})|\boldsymbol{r}, \boldsymbol{z})$ for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$ and every $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ satisfying $\|\boldsymbol{r}\| = 1$. Since Ω is a domain of holomorphy, this implies that $\mathsf{D}^n(|f(\boldsymbol{z})|\boldsymbol{r}, \boldsymbol{z}) \subseteq \Omega$ for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$ and every $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ satisfying $\|\boldsymbol{r}\| = 1$. (Indeed, were this not the case, this would imply the existence of a holomorphic function on Ω that can be extended across a point on boundary of Ω.) Thus we have $|f(\boldsymbol{z})| \le \delta_\Omega^{\boldsymbol{r}}(\boldsymbol{z})$ for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$ and every $\boldsymbol{r} \in \mathbb{R}_{>0}^n$ satisfying $\|\boldsymbol{r}\| = 1$. Thus, by our constructions from the first paragraph, $|f(\boldsymbol{z})| \le \delta_\Omega(\boldsymbol{z})$ for every $\boldsymbol{z} \in \mathrm{hconv}_\Omega(K)$. ▼

Now we complete the proof of the theorem. Let $z_0 \in \Omega$ and let $w \in \mathbb{C}^n \setminus \{0\}$. Choose $r \in \mathbb{R}_{>0}$ sufficiently small that

$$D_r \triangleq \{z_0 + \zeta w \mid \zeta \in \overline{\mathsf{D}}^1(r, 0)\} \subseteq \Omega.$$

Let us denote

$$\partial D_r = \{z_0 + \zeta w \mid \zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0))\}$$

(noting that this is not the boundary of D_r when $n \geq 2$). Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial function satisfying

$$-\log \delta_\Omega(z_0 + \zeta w) \leq \mathrm{Re}(p(\zeta))$$

for every $\zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0))$. One directly verifies that there exists a polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ satisfying $P(z_0 + \zeta w) = p(\zeta)$ for every $\zeta \in \mathbb{C}$. We then have $|\exp(-P(z))| \leq \delta_\Omega(z)$ for every $z \in \partial D_r$.

Let $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ and note that the function

$$\zeta \mapsto f(z_0 + \zeta w)$$

is holomorphic on $\mathsf{D}^1(r, 0)$. Thus, by the Maximum Modulus Theorem, $|f(z)| \leq \|f\|_{\partial D_r}$ for every $z \in D_r$. Therefore, $D_r \subseteq \mathrm{hconv}_\Omega(\partial D_r)$. By the second lemma above, this implies that

$$|\exp(-P(z))| \leq \delta_\Omega(z), \qquad z \in D_r,$$

which then implies that

$$-\log \delta_\Omega(z_0 + \zeta w) \leq \mathrm{Re}(p(\zeta)), \qquad \zeta \in \overline{\mathsf{D}}^1(r, 0).$$

By the first lemma above, the function

$$\zeta \mapsto -\log \delta_\Omega(z_0 + \zeta w)$$

is subharmonic in a neighbourhood of 0, implying that δ_Ω is plurisubharmonic. ■

We also have the following result, although it will be seen that the proof relies on techniques from Section GA2.7.1.3.

3.2.19 Proposition (Plurisubharmonicity is invariant under holomorphic mappings) *If $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are open subsets, if $\mathbf{\Phi}\colon \Omega \to \Delta$ is holomorphic, and if $\mathrm{u} \in \mathrm{Psh}(\Delta)$, then $\mathrm{u} \circ \mathbf{\Phi} \in \mathrm{Psh}(\Omega)$.*

Proof We suppose without loss of generality that Δ is connected and that u is not identically $-\infty$ on Δ. Let

$$\Omega_j = \{z \in \Omega \mid \|z\| < j,\ \delta_\Omega(z) > \tfrac{1}{j}\}$$

and

$$\Delta_j = \{w \in \Delta \mid \|w\| < j,\ \delta_\Delta(w) > \tfrac{1}{j}\}.$$

Let $(u_j)_{j \in \mathbb{Z}_{>0}}$ be the sequence of smooth plurisubharmonic functions on Δ_j as in Sublemma 2 from the proof of Lemma GA2.7.1.4. Let $z \in \Omega$ and let j be sufficiently large that $z \in \Omega_j$. Let $k \in \mathbb{Z}_{>0}$ be sufficiently large that $\Omega_j \subseteq \mathbf{\Phi}^{-1}(\mathcal{V}_k)$. By Lemma 3.2.11 and Proposition 3.2.12 we have that $u_j \circ \mathbf{\Phi}|\Omega_j$ is plurisubharmonic. Moreover, for each $z \in \Omega$, $(u_j \circ \mathbf{\Phi}(z))_{j \in \mathbb{Z}_{>0}}$ decreases monotonically to $u \circ \mathbf{\Phi}(z)$ by Sublemma 3 from the proof of Lemma GA2.7.1.4. By Proposition 3.2.6(i) we have that u is plurisubharmonic. ■

3.3 Pseudoconvexity

In Section 3.1 we saw that domains of holomorphy are connected with one notion of convexity, namely convexity with respect to holomorphic functions. In the preceding section we introduced a special class of functions for which it is difficult to imagine their relationship with domains of holomorphy: the plurisubharmonic functions. In this section we begin to examine just this relationship.

3.3.1 Exhaustion functions

For our purposes, we are interested in plurisubharmonic functions having a special property, as prescribed by the following general definition.

3.3.1 Definition (Exhaustion function) If $\mathcal{S}$ is a topological space domain, a function $u\colon \mathcal{S} \to [-\infty, \infty)$ is an ***exhaustion function*** for $\mathcal{S}$ if the sublevel set $u^{-1}([-\infty, \alpha))$ is a relatively compact subset of $\mathcal{S}$ for every $\alpha \in \mathbb{R}$. •

The following elementary general result will be useful in a few places.

3.3.2 Lemma (Characterisation of exhaustion functions) *A continuous map* $\mathrm{u}\colon \mathcal{S} \to [-\infty, \infty)$ *is an exhaustion function for a topological space* $\mathcal{S}$ *if and only if, for any* $\alpha \in \mathbb{R}$*, there exists a compact set* $\mathrm{K} \subseteq \mathcal{S}$ *such that* $\mathrm{u}(\mathrm{x}) > \alpha$ *for every* $\mathrm{x} \in \mathcal{S} \setminus \mathrm{K}$.

Proof First suppose that u is an exhaustion function and let $\alpha \in \mathbb{R}_{>0}$. Since u is an exhaustion function, $u^{-1}([-\infty, \alpha])$ is compact. Moreover, $u(x) > \alpha$ for every $x \in \mathcal{S} \setminus u^{-1}([-\infty, \alpha])$.

For the converse, suppose that, for any $\alpha \in \mathbb{R}_{>0}$, there exists a compact set $K \subseteq \mathcal{S}$ such that $u(x) > \alpha$ for every $x \in \mathcal{S} \setminus K$. Let $\alpha \in \mathbb{R}$ and note that $u^{-1}([-\infty, \alpha])$ is compact, being a closed subset of a compact set [Runde 2005, Proposition 3.3.6], showing that u is an exhaustion function. ■

Next let us show that exhaustion functions often exist.

3.3.3 Lemma (Existence of exhaustion functions) *If* $\mathcal{S}$ *is a locally compact, second countable, regular topological space, then there exists a continuous exhaustion function on* $\mathcal{S}$*. Moreover, if* $\mathcal{S}$ *has a smooth differentiable structure, then there exists a smooth exhaustion function on* $\mathcal{S}$*.*

Proof Since $\mathcal{S}$ is locally compact, it can be covered by relatively compact sets. Since it is second countable and Hausdorff, it is paracompact [Dugundji 1966, Theorem VIII.6.5] and possesses a countable open cover by relatively compact open sets [Dugundji 1966, Theorem VIII.6.3]. We can thus take a partition of unity subordinate to such an open cover, so defining continuous compactly supported functions ρ_j, $j \in \mathbb{Z}_{>0}$, which sum to 1 [Dugundji 1966, Theorem VIII.4.2]. We claim that the function $u = \sum_{j=1}^{\infty} j\rho_j$ is an exhaustion function. Indeed, let $\alpha \in \mathbb{R}$ and let $N \in \mathbb{Z}_{>0}$ be such that $N > \alpha$. Let K be a compact set containing the supports of $\rho_1, \dots, \rho_N$. We claim that $u^{-1}((-\infty, \alpha]) \subseteq K$. Indeed, suppose that

$$x \notin K \quad \Longrightarrow \quad x \notin \cup_{j=1}^{N} \operatorname{supp}(\rho_j).$$

Then

$$\sum_{j=N+1}^{\infty} \rho_j(x) = \sum_{j=1}^{\infty} \rho_j(x) = 1$$

and so

$$u(x) = \sum_{j=1}^{\infty} j\rho_j(x) = \sum_{j=N+1}^{\infty} j\rho_j(x) \geq N \sum_{j=N+1}^{\infty} \rho_j(x) = N > \alpha,$$

as desired. By Lemma 3.3.2 it follows that u is indeed an exhaustion function.

The last statement follows by the existence of smooth partitions of unity in this case [Abraham, Marsden, and Ratiu 1988, Theorem 5.5.7]. ■

Our interest is primarily in exhaustion functions that are plurisubharmonic, so let us consider these for a moment. A couple of useful examples of plurisubharmonic exhaustion function are the following.

3.3.4 Examples (Exhaustion functions)

1. If $\Omega = \mathbb{C}^n$ we claim that the function u defined by $u(z) = \|z\|^2$ is a plurisubharmonic exhaustion function. Using the fact that

$$\|z\|^2 = \sum_{j=1}^{n} z_j \bar{z}_j,$$

we compute $\mathrm{Lev}(u)(z; w) = \|w\|^2$, and so u is strictly plurisubharmonic. It is clear that u is an exhaustion function for $\mathbb{C}^n$.

2. Let $\Omega \subseteq \mathbb{C}^n$ and define $\delta_\Omega \colon \Omega \to \mathbb{R}_{\geq 0}$ by $\delta_\Omega(z) = \mathrm{dist}(z, \mathbb{C}^n \setminus \{\Omega\})$, which we call the ***boundary distance*** function, as in the single variable case. By Proposition B.1.2 we have that δ_Ω is continuous. If Ω is bounded, then $-\log \delta_\Omega$ is an exhaustion function since $(-\log \delta_\Omega)^{-1}((-\infty, \alpha)$ has bounded closure. For general Ω, $-\log \delta_\Omega$ may not be an exhaustion function. For example, if

$$\Omega = \{z \in \mathbb{C} \mid \ \mathrm{Im}(z) > 0\},$$

then, for $\alpha \in \mathbb{R}_{>0}$,

$$(-\log \delta_\Omega)^{-1}((-\infty, \alpha) = \{z \in \Omega \mid \ \mathrm{Im}(z) < \mathrm{e}^{-\alpha}\},$$

which is not relatively compact.

3. As we just showed, $-\log \delta_\Omega$ is not an exhaustion function when Ω is unbounded, it is easy to modify it so as to produce an exhaustion function. Indeed, the function $u \colon \Omega \to \mathbb{R}$ defined by

$$u(z) = \max\{\|z\|^2, -\log \delta_\Omega\}$$

is readily verified to be a continuous exhaustion function. •

An important facet of the theory of general plurisubharmonic functions is their approximation by plurisubharmonic functions that have additional properties such as smoothness and strictness. In the case of domains in $\mathbb{C}^n$, the result we have is the following.

3.3.5 Theorem (The plurisubharmonic smoothing lemma) *If $\Omega \subseteq \mathbb{C}^n$ is a domain, if $u \in C^0(\Omega)$ is a plurisubharmonic exhaustion function for Ω, if $K \subseteq \Omega$ is compact, and if $\epsilon \in \mathbb{R}_{>0}$, there exists a smooth strictly plurisubharmonic exhaustion function v for Ω such that (1) $v(\mathbf{z}) \geq u(\mathbf{z})$ for all $\mathbf{z} \in \Omega$ and (2) $\|v - u\|_K < \epsilon$.*

Proof Our proof relies on the local constructions of Lemma GA2.7.1.4. Let

$$\mathcal{V}_j = \{z \in \Omega \mid u(z) < j\}, \qquad j \in \mathbb{Z}_{\geq 0}.$$

Let Ω_j, $j \in \mathbb{Z}_{>0}$ and $(u_j)_{j\in\mathbb{Z}_{>0}}$ be as in Sublemma 5 from the proof of Lemma GA2.7.1.4. Let us without loss of generality (by adding a suitable constant to u if necessary) suppose that $K \subseteq \mathcal{V}_0$. By Sublemma 5 from the proof of Lemma GA2.7.1.4, let $u_0 \in C^\infty(\Omega)$ be such that $u_0|\mathcal{V}_2$ is strictly plurisubharmonic and such that

$$u(z) < u_0(z) < u(z) + \epsilon, \qquad z \in \mathrm{cl}(\mathcal{V}_1).$$

Also by Sublemma 5 from the proof of Lemma GA2.7.1.4, for $j \geq 1$ let $u_j \in C^\infty(\Omega)$ be such that

$$u(z) < u_j(z) < u(z) + 1, \qquad z \in \mathcal{V}_j.$$

Given our definition of $\mathcal{V}_j$, $j \in \mathbb{Z}_{\geq 0}$, it follows that

$$\begin{aligned} u_j(z) + 1 - j &< 0, \qquad z \in \mathcal{V}_{j-2}, \\ u_j(z) + 1 - j &> 0, \qquad z \in \mathrm{cl}(\mathcal{V}_j) \setminus \mathcal{V}_{j-1}, \end{aligned}$$

for $j \geq 2$.

Now let $\beta \in C^\infty(\mathbb{R})$ have the property that $\beta(x) = 0$ for $x \leq 0$ and that $\beta^{(r)}(x) > 0$ for $x > 0$. Thus

$$\beta \circ u_j(z) + 1 - j \begin{cases} = 0, & z \in \mathcal{V}_{j-2}, \\ \geq 0, & \text{otherwise.} \end{cases}$$

For $\alpha \in C^\infty(\Omega)$ we have

$$\frac{\partial^2(\beta \circ \alpha)}{\partial z_j \partial \bar{z}_k}(z) = \beta^{(2)}(\alpha(z)) \frac{\partial \alpha}{\partial z_j}(z) \frac{\partial \alpha}{\partial \bar{z}_k}(z) + \beta^{(1)}(\alpha(z)) \frac{\partial^2 \alpha}{\partial z_j \partial \bar{z}_k}(z).$$

Therefore, using Proposition 3.2.12, we deduce that

$$z \mapsto \beta \circ u_j(z) + 1 - j$$

is plurisubharmonic on $\mathcal{V}_{j-1}$ and strictly plurisubharmonic and positive on $\mathrm{cl}(\mathcal{V}_j) \setminus \mathcal{V}_{j-1}$ for every $j \geq 2$. Therefore, we can choose N_1 sufficiently large that

$$z \mapsto v_1(z) \triangleq u_0(z) + N_1 \beta \circ u_1(z) + 1 - 1$$

is strictly plurisubharmonic $\mathcal{V}_1$. Proceeding inductively, for $j \in \mathbb{Z}_{>0}$ we choose $N_1, \dots, N_j$ such that

$$z \mapsto v_j(z) \triangleq u_0(z) + \sum_{k=1}^{j} N_j(\beta \circ u_j(z) + 1 - j)$$

is strictly plurisubharmonic on $\mathcal{V}_j$. By Proposition 3.2.6(i) we conclude that if $v(z) = \lim_{j\to\infty} v_j(z)$, then v has the required properties of the theorem. ■

3.3.2 Weak and strong pseudoconvexity

The rather detailed presentation of plurisubharmonic functions in the preceding section seems like an absurd excursion the first time one sees it. However, it turns out the plurisubharmonicity is exactly the notion needed to provide another of the several characterisations of domains of holomorphy that we shall provide.

Let us define the concept we shall use.

3.3.6 Definition (Weakly pseudoconvex, strongly pseudoconvex) A domain $\Omega \subseteq \mathbb{C}^n$ is ***weakly pseudoconvex*** (resp. ***strongly pseudoconvex***) if there exists a continuous, plurisubharmonic (resp. strictly plurisubharmonic) exhaustion function on Ω. •

We shall see that, for subsets of $\mathbb{C}^n$, weak and strong pseudoconvexity are indistinguishable. However, for manifolds, this is no longer true, and the distinction between weak and strong pseudoconvexity is important. To this end, the following result is helpful.

3.3.7 Lemma (Smoothness and strictness of exhaustion functions) *If $\Omega \subseteq \mathbb{C}^n$ is a domain, then Ω is weakly pseudoconvex if and only if there exists a smooth, strictly plurisubharmonic exhaustion function on Ω. In particular, Ω is weakly pseudoconvex if and only if it is strongly pseudoconvex.*

Proof The "if" assertions are obvious. So suppose that $u \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)$ is an exhaustion function. By Theorem 3.3.5 there exists a smooth, strictly plurisubharmonic exhaustion function. This proves both assertions of the lemma. ■

By virtue of the lemma, it is not uncommon for the definition of pseudoconvexity require that the plurisubharmonic exhaustion function be smooth and strictly plurisubharmonic, and we shall freely make these assumptions on our exhaustion functions, without loss of generality. We shall also sometimes simply say "pseudoconvex," rather than "weakly pseudoconvex" or "strongly pseudoconvex," when referring to open subsets of $\mathbb{C}^n$.

The following result gives an important property of domains of holomorphy.

3.3.8 Theorem (Domains of holomorphy are strongly pseudoconvex) *If $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy, then it is strongly pseudoconvex.*

Proof From Theorem 3.2.18, $-\log \delta_\Omega$ is plurisubharmonic. From Example 3.3.4–1 we know that $z \mapsto \|z\|^2$ is plurisubharmonic. Therefore, the function

$$u(z) \triangleq \max\{\|z\|^2, -\log \delta_\Omega(z)\}$$

is continuous and plurisubharmonic, the latter by Proposition 3.2.6(ii). Moreover, for $\alpha \in \mathbb{R}$, the set $u^{-1}((-\infty, \alpha])$ obviously closed and bounded, and thus u is a continuous plurisubharmonic exhaustion function. The result now follows from Lemma 3.3.7. ■

As with domains of holomorphy (see Theorem 3.2.18), it is possible to relate pseudoconvex domains with the log boundary distance function.

3.3.9 Theorem (The log boundary distance for pseudoconvex domains) *If $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex, then $-\log \delta_\Omega$ is continuous and plurisubharmonic.*

Proof As Ω is pseudoconvex, let u be a continuous plurisubharmonic exhaustion function on Ω.

The idea of the proof mirrors that for Theorem 3.2.18. Thus we let $z_0 \in \Omega$ and $w \in \mathbb{C}^n \setminus \{0\}$, and choose $r \in \mathbb{R}_{>0}$ sufficiently small that

$$D_r \triangleq \{z_0 + \zeta w \mid \zeta \in \overline{\mathsf{D}}^1(r, 0)\} \subseteq \Omega.$$

As in the proof of Theorem 3.2.18, we denote

$$\partial D_r = \{z_0 + \zeta w \mid \zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0))\}.$$

Also as in the proof of Theorem 3.2.18, let $p \colon \mathbb{C} \to \mathbb{C}$ be a polynomial function satisfying

$$-\log \delta_\Omega(z_0 + \zeta w) \le \mathrm{Re}(p(\zeta))$$

for every $\zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0))$. Equivalently,

$$|\exp(-p(\zeta))| \le \delta_\Omega(z_0 + \zeta w) \tag{3.9}$$

for every $\zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0))$. By Lemma 1 from the proof of Theorem 3.2.18, and using the same argument as in the last part of the proof of Theorem 3.2.18, it suffices to show that (3.9) holds for $\zeta \in \overline{\mathsf{D}}^1(r, 0)$.

To this end we prove a lemma, making use of the following definition for $K \subseteq \Omega$ compact:

$$\mathrm{pconv}_\Omega(K) = \{z \in \Omega \mid u(z) \le \sup_K u,\ u \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)\}.$$

This is the ***plurisubharmonic convex hull***, and for it we have the following lemma.

1 Lemma *If $\Omega \subseteq \mathbb{C}^n$ is weakly pseudoconvex and if $K \subseteq \Omega$ is compact, then $\mathrm{pconv}_\Omega(K)$ is compact.*

Proof Let u be a continuous plurisubharmonic exhaustion function on Ω and let $\alpha = \sup_K u$. It follows that $u(z) \le \sup_K u$ for all $z \in \mathrm{pconv}_\Omega(K)$ and so

$$\begin{aligned}\mathrm{pconv}_\Omega(K) &= \{z \in \Omega \mid v(z) \le \sup_K v,\ v \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)\} \\ &\subseteq \{z \in \Omega \mid u(z) \le \sup_K v\} \subseteq u^{-1}((-\infty, \alpha]).\end{aligned}$$

Since $\mathrm{pconv}_\Omega(K)$ is closed (cf. the proof in Proposition 3.1.6 if the same fact for $\mathrm{hconv}_\Omega(K)$), it follows that $\mathrm{cl}(\mathrm{pconv}_\Omega(K)) \subseteq \Omega$, as claimed. ▼

Next, for $a \in \mathbb{C}^n$, define a map

$$\begin{aligned} g_a \colon \overline{\mathsf{D}}^1(r, 0) &\to \mathbb{C}^n \\ \zeta &\mapsto z_0 + \zeta w + a \exp(-p(\zeta)). \end{aligned}$$

We claim that $\mathrm{image}(g_a) \subseteq \Omega$ if $\|a\| \le 1$. To see this, let us denote

$$A = \{\alpha \in [0, 1] \mid \mathrm{image}(g_a) \subseteq \Omega \text{ whenever } \|a\| \le \alpha\}.$$

First of all, $\mathrm{image}(\boldsymbol{g}_0) = D_r$ and so $0 \in A$. Moreover, A is clearly open and so it suffices to show that A is closed. Let

$$K = \{z_0 + \zeta \boldsymbol{w} + \boldsymbol{a} \exp(-p(\zeta)) \mid \zeta \in \mathrm{bd}(\overline{\mathsf{D}}^1(r, 0)),\ \boldsymbol{a} \in \overline{\mathsf{B}}^n(1, \boldsymbol{0})\},$$

noting that K is compact and contained in Ω. Let $u \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)$ and note that, by Proposition 3.2.19, the function

$$\zeta \mapsto u(z_0 + \zeta \boldsymbol{w} + \boldsymbol{a} \exp(-p(\zeta)))$$

is subharmonic on a neighbourhood of $\mathsf{D}^1(r, 0)$ whenever $\|\boldsymbol{a}\| \in A$. By the Maximum Principle for subharmonic functions, this implies that

$$u(z_0 + \zeta \boldsymbol{w} + \boldsymbol{a} \exp(-p(\zeta))) \leq \sup_K u$$

for every $\zeta \in \overline{\mathsf{D}}^1(r, 0)$ whenever $\|\boldsymbol{a}\| \in A$. By the definition of the plurisubharmonic hull, $\mathrm{image}(\boldsymbol{g}_{\boldsymbol{a}}) \subseteq \mathrm{pconv}_\Omega(K)$ if $\|\boldsymbol{a}\| \in A$. Suppose now that A is not closed. Since it is open and contains 0, this implies that there exists $\alpha_0 \in [0, 1)$ such that $[0, \alpha_0) \subseteq A$ but $\alpha_0 \notin A$. Let $(\alpha_j)_{j \in \mathbb{Z}_{>0}}$ be a strictly increasing sequence in $[0, \alpha_0)$ converging to α_0. Since $\alpha_0 \notin A$, there exists $\zeta \in \overline{\mathsf{D}}^1(r, 0)$ and $\boldsymbol{a}_0 \in \mathbb{C}^n$ such that $\|\boldsymbol{a}_0\| = \alpha_0$ and

$$z_0 + \zeta \boldsymbol{w} + \boldsymbol{a}_0 \exp(-p(\zeta)) \notin \Omega.$$

Let $(\boldsymbol{a}_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence in $\mathsf{B}^n(1, \boldsymbol{0})$ converging to $\boldsymbol{a}_0$ and such that $\|\boldsymbol{a}_j\| = \alpha_j$, $j \in \mathbb{Z}_{>0}$. Since $\alpha_j \in A$ for each $j \in \mathbb{Z}_{>0}$, our computations above give

$$z_0 + \zeta \boldsymbol{w} + \boldsymbol{a}_j \exp(-p(\zeta)) \in \mathrm{pconv}_\Omega(K), \qquad j \in \mathbb{Z}_{>0}.$$

Since $\mathrm{pconv}_\Omega(K)$ is compact by Lemma 1,

$$z_0 + \zeta \boldsymbol{w} + \boldsymbol{a}_0 \exp(-p(\zeta)) = \lim_{j \to \infty} z_0 + \zeta \boldsymbol{w} + \boldsymbol{a}_j \exp(-p(\zeta)) \in \mathrm{pconv}_\Omega(K) \subseteq \Omega,$$

and the resulting contradiction allows us to conclude that A is closed, and so $A = [0, 1]$. Thus (3.9) holds for $\zeta \in \overline{\mathsf{D}}^1(r, 0)$, and the reorganisation of this to

$$-\log \delta_\Omega(z_0 + \zeta \boldsymbol{w}) \leq \mathrm{Re}(p(\zeta))$$

for $\zeta \in \overline{\mathsf{D}}^1(r, 0)$ proves the theorem. ■

Using the above properties of pseudoconvex domains, it is relatively easy to prove some basic facts about these.

3.3.10 Proposition (Basic properties of pseudoconvex sets) *The following statements hold:*

- *(i) if $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are weakly pseudoconvex then $\Omega \times \Delta$ is weakly pseudoconvex;*
- *(ii) if $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are weakly pseudoconvex and if $\boldsymbol{\Phi}\colon \Omega \to \mathbb{C}^m$ is holomorphic, then $\boldsymbol{\Phi}^{-1}(\Delta)$ is weakly pseudoconvex;*
- *(iii) if $(\Omega_a)_{a \in A}$ is a family of weakly pseudoconvex sets in $\mathbb{C}^n$ for which $\mathrm{int}(\cap_{a \in A} \Omega_a) \neq \emptyset$, then each connected component of $\mathrm{int}(\cap_{a \in A} \Omega_a) \neq \emptyset$ is weakly pseudoconvex;*

(iv) *if* $(\Omega_j)_{j\in\mathbb{Z}_{>0}}$ *is a sequence of weakly pseudoconvex sets in* $\mathbb{C}^n$ *for which* $\Omega_j \subseteq \Omega_{j+1}$, $j \in \mathbb{Z}_{>0}$, *then* $\cup_{j\in\mathbb{Z}}\Omega_j$ *is weakly pseudoconvex.*

Proof (i) Let $u \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)$ and $v \in \mathrm{Psh}(\Delta) \cap \mathrm{C}^0(\Delta)$ be exhaustion functions. Let $\hat{u}\colon \Omega \times \Delta \to \mathbb{R}$ and $\hat{v}\colon \Omega \times \Delta \to \mathbb{R}$ be defined by

$$\hat{u}(z, w) = u(z), \quad \hat{v}(z, w) = v(w).$$

We claim that both $\hat{u}$ and $\hat{v}$ are plurisubharmonic. To see this for $\hat{u}$, let $\mathrm{pr}_1\colon \Omega \times \Delta \to \Omega$ be the projection onto the first factor, which is a holomorphic map. By Proposition 3.2.19 it follows that $\hat{u} = u \circ \mathrm{pr}_1$ is plurisubharmonic. Similarly, $\hat{v}$ is plurisubharmonic. Now define

$$\sigma(z, w) = \max\{u(z), v(w)\} = \max\{\hat{u}(z, w), \hat{v}(z, w)\}.$$

The function σ is continuous (obvious) and plurisubharmonic (by Proposition 3.2.6(ii)). Since

$$\sigma^{-1}((-\infty, \alpha)) \subseteq u^{-1}((-\infty, \alpha)) \times v^{-1}((-\infty, \alpha)),$$

it follows that σ is also an exhaustion function.

(ii) Let $u \in \mathrm{Psh}(\Omega) \cap \mathrm{C}^0(\Omega)$ and $v \in \mathrm{Psh}(\Delta) \cap \mathrm{C}^0(\Delta)$ be exhaustion functions. We define

$$\sigma(z) = \max\{u(z), v \circ \boldsymbol{\Phi}(z)\},$$

noting that σ is continuous (obvious) and plurisubharmonic (by Proposition 3.2.19 and Proposition 3.2.6(ii)). Moreover,

$$\sigma^{-1}((-\infty, \alpha)) \subseteq u^{-1}((-\infty, \alpha)) \cap (v \circ \boldsymbol{\Phi})^{-1}((-\infty, \alpha)),$$

showing that σ is also an exhaustion function.

(iii) Let $\mathcal{V}$ be a connected component of $\cap_{a\in A}\Omega_a$. Note that we obviously have

$$\delta_{\mathcal{V}}(z) \le \inf\{\delta_{\Omega_a}(z) \mid a \in A\}.$$

For the converse inequality, if $\hat{z} \in \mathrm{bd}(\mathcal{V})$ there is then a sequence of $(a_j)_{j\in\mathbb{Z}_{>0}}$ in A and $(z_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{C}^n$ such that $z_j \in \mathrm{bd}(\Omega_j)$ and $\hat{z} = \lim_{j\to\infty} z_j$. Thus, for $z \in \mathcal{V}$, $\|z - \hat{z}\| = \lim_{j\to\infty}\|z - z_j\|$. Since this is particularly true if $\hat{z}$ satisfies $\delta_{\mathcal{V}}(z) = \|z - \hat{z}\|$, it follows that

$$\delta_{\mathcal{V}}(z) = \inf\{\delta_{\Omega_a}(z) \mid a \in A\}.$$

Thus

$$-\log \delta_{\mathcal{V}}(z) = \sup\{-\log \delta_{\Omega_a}(z) \mid a \in A\}.$$

By Proposition 3.2.6(ii) it follows that $-\log \delta_{\mathcal{V}}$ is plurisubharmonic. Its continuity follows from Proposition B.1.2, and so follows the weak pseudoconvexity of $\mathcal{V}$.

(iv) Let $\Omega = \cup_{j\in\mathbb{Z}_{>0}}\Omega_j$. For $j \in \mathbb{Z}_{>0}$ define $u_j\colon \Omega \to [-\infty, \infty)$ by

$$u_j(z) = \begin{cases} -\log \delta_{\mathcal{U}_j}(z), & z \in \Omega_j, \\ -\infty, & z \notin \Omega_j. \end{cases}$$

One directly verifies (for example using the characterisation of part (vii) of Theorem 3.2.2 of subharmonic functions) that $u_j \in \mathrm{Psh}(\Omega)$. We claim that $-\log \delta_{\Omega}(z) = \lim_{j\to\infty} u_j(x)$. We clearly have $-\log \delta_\Omega(z) \geq \lim_{j\to\infty} u_j(x)$. For the opposite inequality, if $z \in \Omega$ then there exists $N \in \mathbb{Z}_{>0}$ such that $z \in \Omega_j$ for all $j \geq N$. If $\hat{z} \in \mathrm{bd}(\Omega)$ is such that $\delta_\Omega(z) = \|z - \hat{z}\|$, then there exists a sequence $(z_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{C}^n$ such that $z_j \in \mathrm{bd}(\Omega_j)$ and such that $\hat{z} = \lim_{j\to\infty} z_j$. We then have

$$-\log \delta_\Omega(z) \leq \lim_{j\to\infty} u_j(z),$$

with the limit being nonincreasing for $j \geq N$. Note that by Proposition 3.2.6(i) this argument also implies that $-\log \delta_\Omega$ is plurisubharmonic in some neighbourhood of any point in Ω, and so is plurisubharmonic. Since $-\log \delta_\Omega$ is continuous by Proposition B.1.2, we conclude that Ω is pseudoconvex. ■

Using the preceding developments, it is then possible to give examples of pseudoconvex open sets.

3.3.11 Examples (Pseudoconvex domains in $\mathbb{C}^n$)

1. By Theorem 3.3.8 and Example 3.2.4–2 we have that every open subset of $\mathbb{C}$ is strongly pseudoconvex.
2. Let $\phi\colon \mathbb{C}^n \to \mathbb{R}$ be defined by $\phi(z) = \mathrm{Re}(\langle z - z_0, \boldsymbol{\lambda}\rangle)$ and take

$$\Omega = \{z \in \mathbb{C}^n \mid \phi(z) > 0\}.$$

We note that Ω is strongly pseudoconvex by Example 3.2.17–3.

3. We claim that if $\Omega \subseteq \mathbb{C}^n$ is a convex open set, then Ω is strongly pseudoconvex. We can show this as follows. Let $w \in \mathrm{bd}(\Omega)$ and let H_w be a $\mathbb{R}$-hyperplane passing through w not intersecting Ω. Thus, for $z \in \Omega$, $\|z-w\| = \mathrm{dist}(z, \mathsf{H}_w)$. Let $u_w\colon \Omega \to \mathbb{R}$ be given by $u_w(z) = \mathrm{dist}(z, \mathsf{H}_w)$. Note that

$$\delta_\Omega(z) = \inf\{u_w(z) \mid w \in \mathrm{bd}(\Omega)\}.$$

Thus

$$-\log \delta_\Omega(z) = \sup\{-\log \phi_w(z) \mid w \in \mathrm{bd}(\Omega)\}.$$

As we saw in Example 3.2.17–3, $-\log \phi_w$ is plurisubharmonic for each $w \in \mathrm{bd}(\Omega)$. By Proposition 3.2.6(ii), $-\log \delta_\Omega$ is also plurisubharmonic, and so Ω is convex.

Alternatively, one could use the fact that a convex set is the intersection of its bounding half-spaces, and each of these half-spaces is strongly pseudoconvex as in the preceding example. Strong pseudoconvexity of the intersection then follows from Proposition 3.3.10(iii). •

3.3.3 Open sets with regular boundaries

The next part of our discussion regarding domains of holomorphy concerns characterising these by considerations of only their boundary. This that should be possible is suggested by the initial definition of a domain of holomorphy. In this section we consider open subsets of $\mathbb{C}^n$ having nice boundaries. These, historically, have played a crucial rôle in the study of domains of holomorphy, beginning with the work of Levi [1910].

We begin with a definition in $\mathbb{R}^n$; later we specialise to $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

3.3.12 Definition (Regular boundary of class C^k) Let $k \in \mathbb{Z}_{>0} \cup \{\infty\}$. An open set $\mathcal{U} \subseteq \mathbb{R}^n$ has a ***regular boundary of class*** $\mathbf{C}^\mathbf{k}$ if, for every $\hat{\boldsymbol{x}} \in \mathrm{bd}(\mathcal{U})$ there exists a neighbourhood $\mathcal{V}$ of $\hat{\boldsymbol{x}}$ and a function $r \in \mathrm{C}^k(\mathcal{V})$ such that $\mathrm{d}r(\boldsymbol{x}) \neq 0$ for every $\boldsymbol{x} \in \mathcal{V}$ and such that

$$\mathcal{U} \cap \mathcal{V} = \{\boldsymbol{x} \in \mathcal{V} \mid r(\boldsymbol{x}) < 0\}.$$

The function r is called a $\mathbf{C}^\mathbf{k}$***-local defining function*** for $\mathcal{U}$. •

If r is a local defining function for $\mathcal{U} \subseteq \mathbb{R}^n$ defined on $\mathcal{V}$, $r^{-1}(0) = \mathrm{bd}(\mathcal{U}) \cap \mathcal{V}$, and the hypothesis that the derivative of r does not vanish ensures that $\mathrm{bd}(\mathcal{U}) \cap \mathcal{V}$ is a submanifold of $\mathbb{R}^n$ of dimension $n-1$. Thus, if $\mathcal{U}$ has a regular boundary of class C^k, $\mathrm{bd}(\mathcal{U})$ is a submanifold of dimension $n-1$.

It is possible to make some simplifying assumptions about local defining functions. The following one is particularly useful.

3.3.13 Lemma (The gradient of local defining functions) *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set with regular boundary of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$ and let r_1 and r_2 be local defining functions defined on a neighbourhood $\mathcal{V}$ of $\mathbf{x}_0 \in \mathrm{bd}(\mathcal{U})$. Then there exists a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of $\mathbf{x}_0$ and $\psi \in \mathrm{C}^{k-1}(\mathcal{V}')$ such that*

(i) $\psi(\mathbf{x}) \in \mathbb{R}_{>0}$ *for every* $\mathbf{x} \in \mathcal{V}'$,

(ii) $r_1|\mathcal{V}' = \psi r_2|\mathcal{V}'$, *and*

(iii) $\mathrm{grad}\, r_1(\mathbf{x}) = \psi(\mathbf{x})\, \mathrm{grad}\, r_2(\mathbf{x})$ *for every* $\mathbf{x} \in \mathrm{bd}(\mathcal{U}) \cap \mathcal{V}'$.

In particular, given a local defining function r defined on $\mathcal{V}$, there exists another local defining function $\hat{r}$ defined on a smaller neighbourhood $\mathcal{V}'$ such that $\|\mathrm{grad}\, \hat{r}(\mathbf{x})\| = 1$ for $\mathbf{x} \in \mathrm{bd}(\mathcal{U}) \cap \mathcal{V}'$.

Proof Using the Implicit Function Theorem, we make a change of coordinates so that $r_2(\boldsymbol{x}) = x_n$. For simplicity, we also make a change of coordinates so that $\boldsymbol{x}_0 = \mathbf{0}$. Suppose that these coordinates are defined on $\mathcal{V}' = \mathsf{B}^{n-1}(\epsilon, \mathbf{0}) \times \mathsf{B}^1(\epsilon, 0)$ for some $\epsilon \in \mathbb{R}_{>0}$ sufficiently small. We denote a point in $\mathcal{V}'$ by $(\boldsymbol{x}', x_n)$. Then compute

$$\begin{aligned} r_1(\boldsymbol{x}', x_n) = r_1(\boldsymbol{x}', x_n) - r_1(\boldsymbol{x}', 0) &= \int_0^{x_n} \frac{\partial r_1}{\partial x_n}(\boldsymbol{x}', \xi)\, \mathrm{d}\xi \\ &= x_n \int_0^1 \frac{\partial r_1}{\partial x_n}(s x_n)\, \mathrm{d}s = r_2(\boldsymbol{x}', x_n)\psi(\boldsymbol{x}', x_n), \end{aligned}$$

where

$$\psi(\boldsymbol{x}', x_n) = \int_0^1 \frac{\partial r_1}{\partial x_n}(s x_n)\, \mathrm{d}s.$$

It follows from standard theorems on parameter dependence of integrals that $\psi \in \mathrm{C}^{k-1}(\mathcal{V}')$. Since r_2 vanishes on $\mathrm{bd}(\mathcal{U}) \cap \mathcal{V}'$, condition (iii) holds. Note that $\mathrm{grad}\, r_1$ and $\mathrm{grad}\, r_2$ are collinear on $\mathrm{bd}(\mathcal{U}) \cap \mathcal{V}'$, and in fact are *positive* multiples of one another. Thus $\psi(\boldsymbol{x}_0) > 0$, and this implies the positivity of ψ, possibly after shrinking ϵ.

For the final assertion of the lemma, as above we suppose that $r(\boldsymbol{x}', x_n) = x_n$ and that $\mathcal{V}'$ is a product of balls. We note that for $(\boldsymbol{x}', 0) \in \mathrm{bd}(\mathcal{U}) \cap \mathcal{V}'$ can write $\boldsymbol{n}(\boldsymbol{x}', 0) = \psi(\boldsymbol{x}', 0)\, \mathrm{grad}\, r(\boldsymbol{x}', 0)$ for some $\psi(\boldsymbol{x}', 0) \in \mathbb{R}_{>0}$, where $\boldsymbol{n}(\boldsymbol{x}', 0)$ is the unit outward pointing normal at $(\boldsymbol{x}', 0)$. Now, if we define $\psi(\boldsymbol{x}', x_n) = \psi(\boldsymbol{x}', 0)$ and $\hat{r} = \psi r$, we can verify that $\hat{r}$ is a local defining function with the appropriate properties. ■

Let us exhibit a particular and useful local defining function. For $\mathcal{U} \subseteq \mathbb{R}^n$ open let us adapt our definition of $\delta_{\mathcal{U}}\colon \boldsymbol{x} \mapsto \operatorname{dist}(\boldsymbol{x}, \operatorname{bd}(\mathcal{U}))$ by

$$\mathrm{d}_{\mathcal{U}}(\boldsymbol{x}) = \begin{cases} -\delta_{\mathcal{U}}(\boldsymbol{x}), & \boldsymbol{x} \in \mathcal{U}, \\ 0, & \boldsymbol{x} \in \operatorname{bd}(\mathcal{U}), \\ \delta_{\mathbb{C}^n \setminus \mathcal{U}}(\boldsymbol{x}), & \boldsymbol{x} \notin \operatorname{cl}(\mathcal{U}). \end{cases}$$

For this function, we have the following result.

3.3.14 Lemma (A natural local defining function) *If* $\mathrm{k} \in \mathbb{Z}_{>0} \cup \{\infty\}$, *if* $\mathcal{U} \subseteq \mathbb{R}^{\mathrm{n}}$ *is open with regular boundary of class* C^{k}, *and if* $\hat{\mathbf{x}} \in \operatorname{bd}(\mathcal{U})$, *then there exists a neighbourhood* $\mathcal{V}$ *of* $\hat{\mathbf{x}}$ *such that* $\mathrm{d}_{\mathcal{U}}|\mathcal{V}$ *is a* C^{k}*-local defining function for* $\mathcal{U}$.

Proof We clearly have

$$\mathcal{U} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathrm{d}_{\mathcal{U}}(\boldsymbol{x}) < 0\}.$$

Thus we need to show that $\mathrm{d}_{\mathcal{U}}$ is of class C^k and has nonvanishing derivative in a neighbourhood of every point in $\operatorname{bd}(\mathcal{U})$.

We let $\hat{\boldsymbol{x}} \in \operatorname{bd}(\mathcal{U})$ and let r be a C^k-local defining function for $\mathcal{U}$ defined on a neighbourhood $\mathcal{V}$ of $\hat{\boldsymbol{x}}$. By Lemma 3.3.13 we assume, without loss of generality, that $\operatorname{grad} r(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x})$ for $\boldsymbol{x} \in \operatorname{bd}(\mathcal{U}) \cap \mathcal{V}$, possibly after shrinking $\mathcal{V}$. By relabelling the coordinates we can ensure that $\frac{\partial r}{\partial x_n}(\hat{\boldsymbol{x}}) \neq 0$. By the Implicit Function Theorem, let $\mathcal{V}' \times \mathcal{V}'' \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$ and $\phi\colon \mathcal{V}' \to \mathcal{V}''$ be such that

$$\operatorname{bd}(\mathcal{U}) \cap (\mathcal{V}' \times \mathcal{V}'') = \{(\boldsymbol{x}', \phi(\boldsymbol{x}')) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \boldsymbol{x}' \in \mathcal{V}'\}.$$

An exercise in multivariable minimisation subject to constraints gives that, for $\boldsymbol{x}$ sufficiently near to $\hat{\boldsymbol{x}}$, the distance from $\boldsymbol{x}$ to $\operatorname{bd}(\mathcal{U})$ is given by $\|\boldsymbol{x} - \boldsymbol{b}(\boldsymbol{x})\|$ where $\boldsymbol{b}(\boldsymbol{x})$ lies on a line segment through $\boldsymbol{x}$ which intersects $\operatorname{bd}(\mathcal{U})$ orthogonally at $\boldsymbol{b}(\boldsymbol{x})$. Let us suppose that $\mathcal{V}'$ is chosen sufficiently small that this holds for all $\boldsymbol{x} \in \mathcal{V}' \times \mathcal{V}''$. We can thus write

$$\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{x}) + \mathrm{d}_{\mathcal{U}}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}),$$

and this expression is unique provided that $\boldsymbol{x}$ is sufficiently close to $\operatorname{bd}(\mathcal{U})$. We can define a map

$$\begin{aligned} \boldsymbol{\Phi}\colon \mathcal{V}' \times \mathbb{R} &\to \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \\ (\boldsymbol{x}', \xi) &\mapsto (\boldsymbol{x}', \phi(\boldsymbol{x}')) + \xi \boldsymbol{n}(\boldsymbol{x}'). \end{aligned}$$

The Jacobian of this mapping is

$$\boldsymbol{D\Phi}(\boldsymbol{x}', \xi) = \begin{bmatrix} \boldsymbol{I}_{n-1} + \xi \boldsymbol{A}(\boldsymbol{x}') & \boldsymbol{D}_1 r(\boldsymbol{x}', \phi(\boldsymbol{x}'))^T \\ \operatorname{grad} \phi(\boldsymbol{x}') + \xi \boldsymbol{b}(\boldsymbol{x}') & \frac{\partial r}{\partial x_n}(\boldsymbol{x}', \phi(\boldsymbol{x}')) \end{bmatrix}$$

for some $(n-1) \times (n-1)$-matrix $\boldsymbol{A}(\boldsymbol{x}')$ and $(n-1)$-vector $\boldsymbol{b}(\boldsymbol{x}')$. Differentiating $r(\boldsymbol{x}', \phi(\boldsymbol{x}') = 0$ for $\boldsymbol{x}' \in \mathcal{V}'$ gives

$$\boldsymbol{D\Phi}(\boldsymbol{x}', \xi) = \begin{bmatrix} \boldsymbol{I}_{n-1} + \xi \boldsymbol{A}(\boldsymbol{x}') & -\frac{\partial r}{\partial x_n}(\boldsymbol{x}', \phi(\boldsymbol{x}')) \operatorname{grad} \phi(\boldsymbol{x}')^T \\ \operatorname{grad} \phi(\boldsymbol{x}') + \xi \boldsymbol{b}(\boldsymbol{x}') & \frac{\partial r}{\partial x_n}(\boldsymbol{x}', \phi(\boldsymbol{x}')) \end{bmatrix}.$$

We wish to compute the determinant of this Jacobian at $(\boldsymbol{x}',0)$. We note that properties of determinant with respect to multiplication of a column with respect to a constant gives

$$\det \boldsymbol{D\Phi}(\boldsymbol{x}',0) = \frac{\partial r}{\partial x_n}(\boldsymbol{x}',\phi(\boldsymbol{x}')) \det \begin{bmatrix} \boldsymbol{I}_{n-1} & -\operatorname{grad}\phi(\boldsymbol{x}')^T \\ \operatorname{grad}\phi(\boldsymbol{x}') & 1 \end{bmatrix}.$$

Now, by performing row operations which leave the determinant unchanged, we have

$$\begin{aligned} \det \boldsymbol{D\Phi}(\boldsymbol{x}',0) &= \frac{\partial r}{\partial x_n}(\boldsymbol{x}',\phi(\boldsymbol{x}')) \det \begin{bmatrix} \boldsymbol{I}_{n-1} & -\operatorname{grad}\phi(\boldsymbol{x}')^T \\ 0 & 1+\|\operatorname{grad} r(\boldsymbol{x}')\|^2 \end{bmatrix} \\ &= \frac{\partial r}{\partial x_n}(\boldsymbol{x}',\phi(\boldsymbol{x}'))(1+\|\operatorname{grad} r(\boldsymbol{x}')\|^2). \end{aligned}$$

Thus, by the Inverse Function Theorem, for some sufficiently small $\epsilon \in \mathbb{R}_{>0}$ and by shrinking $\mathcal{V}'$, $\boldsymbol{\Phi}$ is a diffeomorphism of class C^k from $\mathcal{V}' \times \mathsf{B}^1(\epsilon,0)$ onto a neighbourhood $\mathcal{V}$ of $\hat{\boldsymbol{x}}$. Moreover, $\boldsymbol{\Phi}(\mathcal{V}' \times \{\boldsymbol{0}\}) = \operatorname{bd}(\mathcal{U}) \cap \mathcal{V}$ and $-\mathrm{d}_{\mathcal{U}}(\boldsymbol{x})$ is the nth-component of $\boldsymbol{\Phi}^{-1}(\boldsymbol{x})$, and so we conclude that $\mathrm{d}_{\mathcal{U}}$ is of class C^k on $\mathcal{V}$. Now define

$$\begin{aligned} \boldsymbol{\pi}'\colon \mathcal{V}' \times \mathsf{B}^1(\epsilon,0) &\to \mathcal{V}' \times \mathsf{B}^1(\epsilon,0) \\ (\boldsymbol{x}',\xi) &\mapsto (\boldsymbol{x}',0) \end{aligned}$$

and denote $\boldsymbol{\pi} = \boldsymbol{\pi}' \circ \boldsymbol{\Phi}$. Note that

$$-\mathrm{d}_{\mathcal{U}}(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})\|, & \boldsymbol{x} \in \mathcal{U} \cap \mathcal{V}, \\ -\|\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})\|, & \boldsymbol{x} \in \mathcal{V} - \operatorname{cl}(\mathcal{V}). \end{cases}$$

If $\boldsymbol{x} \notin \operatorname{bd}(\mathcal{U})$, for a suitable $\alpha \in \{-1,1\}$ we directly compute

$$-\operatorname{grad} \mathrm{d}_{\mathcal{U}}(\boldsymbol{x}) = \frac{\alpha}{\|\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})\|}(\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x}) - \boldsymbol{D\pi}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x}))).$$

Since $r \circ \boldsymbol{\pi}(\boldsymbol{x}) = 0$ and since $\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})$ is by definition a multiple of $\operatorname{grad} r(\boldsymbol{\pi}(\boldsymbol{x}))$, the Chain Rule gives

$$\boldsymbol{D\pi}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})) = 0.$$

Thus we have

$$-\operatorname{grad} \mathrm{d}_{\mathcal{U}}(\boldsymbol{x}) = \alpha\frac{\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})}{\|\boldsymbol{x}-\boldsymbol{\pi}(\boldsymbol{x})\|}$$

for $\boldsymbol{x} \notin \operatorname{bd}(\mathcal{U})$. Thus, in the limit as $\boldsymbol{x}$ approaches $\operatorname{bd}(\mathcal{U})$ the gradient will not vanish, and so does not vanish on $\mathcal{V}$. ■

Now we turn to the case of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Of course, our notion of an open set with regular boundary of class C^k and of a local defining function of class C^k still apply. What is new is that the tangent space to $\operatorname{bd}(\mathcal{U})$ is a $\mathbb{C}$-vector space, and so we can wonder about its *complex* subspaces.

3.3.15 Definition (Complex tangent space to a boundary) Let $\mathcal{U} \subseteq \mathbb{C}^n$ be an open set with regular boundary of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$. The ***complex tangent space*** to $\operatorname{bd}(\mathcal{U})$ at $\hat{\boldsymbol{z}} \in \operatorname{bd}(\mathcal{U})$, denoted by $\mathsf{T}^{\mathbb{C}}_{\hat{\boldsymbol{z}}}\operatorname{bd}(\mathcal{U})$, is the largest $\mathbb{C}$-subspace of $\mathbb{C}^n$ contained in the (real) tangent space to $\operatorname{bd}(\mathcal{U})$ at $\hat{\boldsymbol{z}}$. •

We have the following concrete description of the complex tangent space to a boundary.

3.3.16 Lemma (A characterisation of the complex tangent space to a boundary) *Let* $\mathrm{k} \in \mathbb{Z}_{>0} \cup \{\infty\}$*, let* $\mathcal{U} \subseteq \mathbb{C}^{\mathrm{n}}$ *be an open set with regular boundary of class* C^{k}*, let* $\hat{\mathbf{z}} \in \mathrm{bd}(\mathcal{U})$*, and let* r *be a local defining function of class* C^{k} *defined on a neighbourhood* $\mathcal{V}$ *of* $\hat{\mathbf{z}}$*. Then*

$$\mathsf{T}^{\mathbb{C}}_{\hat{\mathbf{z}}}\,\mathrm{bd}(\mathcal{U}) = \Big\{\mathbf{w} \in \mathbb{C}^{\mathrm{n}} \;\Big|\; \sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\partial \mathrm{r}}{\partial \mathrm{z}_{\mathrm{j}}}(\hat{\mathbf{z}})\mathrm{w}_{\mathrm{j}} = 0\Big\}.$$

Proof For brevity denote

$$\ker(\partial r)(\hat{z}) = \Big\{w \in \mathbb{C}^n \;\Big|\; \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(\hat{z})w_j = 0\Big\}.$$

The real tangent space to $\mathrm{bd}(\mathcal{U})$ at $\hat{z}$ is

$$\mathsf{T}_{\hat{z}}\,\mathrm{bd}(\mathcal{U}) = \Big\{w = u + \mathrm{i}v \in \mathbb{C}^n \;\Big|\; \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(\hat{z})u_j + \sum_{j=1}^{n} \frac{\partial r}{\partial y_j}(\hat{z})v_j = 0\Big\}, \tag{3.10}$$

cf. [Abraham, Marsden, and Ratiu 1988, Theorem 3.5.4]. Since

$$2\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(\hat{z})w_j = \sum_{j=1}^{n}\Big(\frac{\partial r}{\partial x_j}(\hat{z})u_j + \frac{\partial r}{\partial y_j}(\hat{z})v_j\Big) + \mathrm{i}\sum_{j=1}^{n}\Big(\frac{\partial r}{\partial x_j}(\hat{z})v_j - \frac{\partial r}{\partial y_j}(\hat{z})u_j\Big),$$

it follows that

$$\ker(\partial r)(\hat{z}) \subseteq \mathsf{T}_{\hat{z}}\,\mathrm{bd}(\mathcal{U}).$$

One also directly verifies that

$$w \in \ker(\partial r)(\hat{z}) \quad\implies\quad \mathrm{i}w \in \ker(\partial r)(\hat{z}),$$

showing is a $\mathbb{C}$-subspace. Moreover, if $\mathsf{W} \subseteq \mathsf{T}_{\hat{z}}\,\mathrm{bd}(\mathcal{U})$ is a $\mathbb{C}$-subspace, it must be closed under multiplication by i, and this along with (3.10) implies that, if $u + \mathrm{i}v \in \mathsf{W}$, then

$$\sum_{j=1}^{n}\Big(\frac{\partial r}{\partial x_j}(\hat{z})v_j - \frac{\partial r}{\partial y_j}(\hat{z})u_j\Big) = 0,$$

implying that $\mathsf{W} \subseteq \ker(\partial r)(\hat{z})$. Thus $\ker(\partial r)(\hat{z})$ is indeed the largest $\mathbb{C}$-subspace in $\mathsf{T}_{\hat{z}}\,\mathrm{bd}(\mathcal{U})$. ■

3.3.4 Levi pseudoconvexity

Now we use the constructions of the preceding section to give what is an alternative characterisation of weakly pseudoconvex sets. To motivate the constructions, we recall the following characterisation of a convex set.

3.3.17 Proposition (Convex sets with regular boundary) *For a domain $\Omega \subseteq \mathbb{R}^n$ with regular boundary of class C^2, the following are equivalent:*

(i) Ω is convex;

(ii) for any $\hat{\mathbf{x}} \in \mathrm{bd}(\Omega)$ and any local defining function r defined on a neighbourhood $\mathcal{V}$ of $\hat{\mathbf{x}}$, $\mathbf{D}^2\mathrm{r}(\hat{\mathbf{x}}) \cdot (\mathbf{v}, \mathbf{v}) \geq 0$ for every $\mathbf{v} \in \mathsf{T}_{\hat{\mathbf{x}}}\,\mathrm{bd}(\Omega)$.

Proof First of all, we comment that a straightforward computation, using Lemma 3.3.13 and the fact that $\mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega)$ is given by the kernel of the derivative of any local defining function (as we saw in the proof of Lemma 3.3.16), one easily shows that if the condition $\boldsymbol{D}^2 r(\hat{x}) \cdot (v, v) \geq 0$ holds for one local defining function defined near $\hat{x}$, it holds for any local defining function defined near $\hat{x}$. We shall, therefore, use a specific local defining function to whose construction we now turn.

By the Implicit Function Theorem, $\mathrm{bd}(\Omega)$ is the graph of a function defined on $\mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega)$. More specifically and precisely, there exists $\epsilon \in \mathbb{R}_{>0}$, a convex neighbourhood $\mathcal{V}$ of $\hat{x}$, and a function $f \colon \mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega) \to \mathbb{R}$ such that

$$\mathrm{bd}(\Omega) \cap \mathcal{V} = \{\hat{x} + \boldsymbol{u} + f(\boldsymbol{u})\boldsymbol{n} \mid \boldsymbol{u} \in \mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega) \cap \mathsf{B}^n(\epsilon, \boldsymbol{0})\},$$

where $\boldsymbol{n} \in \mathbb{R}^n$ is the unit outward normal to $\mathrm{bd}(\Omega)$ at $\hat{x}$. (The reader may wish to draw the picture associated with this.) By the Implicit Function Theorem, f is of class C^2. For $\boldsymbol{x} \in \mathcal{V}$ we write

$$\boldsymbol{x} = \hat{x} + \boldsymbol{u}(\boldsymbol{x}) + \alpha(\boldsymbol{x})\boldsymbol{n}$$

for some uniquely defined $\boldsymbol{u}(\boldsymbol{x}) \in \mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega)$ and $\alpha(\boldsymbol{x}) \in \mathbb{R}$. Explicitly, $\alpha(\boldsymbol{x}) = \langle \boldsymbol{n}, \boldsymbol{x} - \hat{x} \rangle$. We then define

$$r(\boldsymbol{x}) = \langle \boldsymbol{n}, \boldsymbol{x} - \hat{x} \rangle - f(\boldsymbol{u}(\boldsymbol{x})),$$

and note that r is a local defining function on $\mathcal{V}$.

Now suppose that Ω is convex. This implies that

$$\Omega \cap \mathcal{V} \subseteq \{\boldsymbol{x} \in \mathcal{V} \mid \langle \boldsymbol{n}, \boldsymbol{x} - \hat{x} \rangle \leq 0\}.$$

This, in turn, implies that $\boldsymbol{0}$ is a local maximum for the function f defined above. Thus $\boldsymbol{D}^2 f(\boldsymbol{0}) \cdot (v, v) \leq 0$ for every $v \in \mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega)$. Using the definition of r we see that $\boldsymbol{D}^2 r(\hat{x}) \cdot (v, v) \geq 0$ for every $v \in \mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega)$.

Now we prove the converse. This means that, with our construction of r at the beginning of the proof, $\boldsymbol{D}^2 r(\boldsymbol{x}) \cdot (v, v) \geq 0$ for every $\boldsymbol{x} \in \mathrm{bd}(\Omega) \cap \mathcal{V}$ and every $v \in \mathsf{T}_{\boldsymbol{x}}\,\mathrm{bd}(\Omega)$. Using the definition of r this implies that $\boldsymbol{D}^2 f(\boldsymbol{u}) \cdot (v, v) \leq 0$ for every $\boldsymbol{u} \in \mathsf{B}^n(\epsilon, \boldsymbol{0})$ and every $v \in \mathsf{T}_{\boldsymbol{x}}\,\mathrm{bd}(\Omega)$. But this implies that the function f is convex by [Webster 1994, Theorem 5.5.5] and, by [Webster 1994, Theorem 5.4.1], the epigraph of f (i.e., the region "above" the graph of f) is convex as a subset of $\mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega) \times \mathbb{R}$ and lies above the tangent space to the graph at every point. At $\hat{x}$ the tangent space to the graph of f is $\mathsf{T}_{\hat{x}}\,\mathrm{bd}(\Omega) \times \{0\}$ which implies that $\boldsymbol{0}$ is a local minimum for f. This, however, implies that

$$\Omega \cap \mathcal{V} \subseteq \{\boldsymbol{x} \in \mathcal{V} \mid \langle \boldsymbol{n}, \boldsymbol{x} - \hat{x} \rangle \leq 0\}.$$

Since this holds for every $\hat{x} \in \mathrm{bd}(\Omega)$, we conclude that Ω is convex by the Krein–Milman Theorem [Webster 1994, Theorem 2.6.16]. ■

Now, returning to our notion of pseudoconvexity, we begin with the following result, which reinforces the fact that it is really the boundary of a set that determines whether it is weakly pseudoconvex.

3.3.18 Theorem (Pseudoconvexity is a local property of the boundary) *If $\Omega \subseteq \mathbb{C}^n$ is a domain, the following statements are equivalent:*

(i) Ω is weakly pseudoconvex;

(ii) for any $\mathbf{z} \in \mathrm{bd}(\Omega)$ there exists a neighbourhood $\mathcal{V}$ of $\mathbf{z}$ such that $\Omega \cap \mathcal{V}$ is weakly pseudoconvex.

Proof (i) $\Longrightarrow$ (ii) Let $z \in \mathrm{bd}(\Omega)$ and let $\mathcal{V}$ be a convex neighbourhood of z. By Example 3.3.11–(3), $\mathcal{V}$ is weakly pseudoconvex and then, by Proposition 3.3.10(iii), the connected component of $\Omega \cap \mathcal{V}$ containing z is weakly pseudoconvex.

(ii) $\Longrightarrow$ (i) First we consider the case where Ω is bounded. In this case, let $z \in \mathrm{bd}(\Omega))$ and let $\mathcal{V}$ be a neighbourhood of z for which $\Omega \cap \mathcal{V}$ is weakly pseudoconvex. By Theorem 3.3.9, $-\log \delta_{\Omega\cap\mathcal{V}}$ is continuous and plurisubharmonic. Note that $\delta_{\Omega\cap\mathcal{V}}(w) = \delta_\Omega(w)$ for w close enough to z. That is to say, there exists a neighbourhood $\mathcal{W}_z$ of z such that $-\log\delta_\Omega$ is continuous and plurisubharmonic on $\Omega \cap \mathcal{W}_z$. Since $\mathrm{bd}(\Omega)$ is compact, finitely many of the sets $\mathcal{W}_z$, $z \in \mathrm{bd}(\Omega)$ (say $\mathcal{W}_{z_1}, \dots, \mathcal{W}_{z_k}$) will cover $\mathrm{bd}(\Omega)$, and $-\log\delta_\Omega$ will be continuous and plurisubharmonic on $\Omega \cap \mathcal{W} = \cup_{j=1}^k \Omega \cap \mathcal{W}_{z_j}$. Let $M = \|-\log\delta_\Omega\|_{\Omega-\mathcal{W}}$, noting that $M < \infty$ since Ω is bounded and $\mathrm{bd}(\Omega) \subseteq \mathcal{W}$. Now note that

$$z \mapsto \max\{-\log\delta_\Omega(z), \|z\|^2 + M + 1\}$$

is continuous (obvious) and plurisubharmonic (by Proposition 3.2.6(ii) and since $z \mapsto \|z\|^2 + M + 1$ is plurisubharmonic by Example 3.3.41).

Now, if Ω is unbounded, define $\Omega_j = \Omega \cap \mathsf{B}^n(j, \mathbf{0})$ for $j \in \mathbb{Z}_{>0}$. Note that, for each $z \in \mathrm{bd}(\Omega_j)$ there exists a neighbourhood $\mathcal{V}$ of z such that $\Omega_j \cap \mathcal{V}$ is weakly pseudoconvex, since we are assuming this property of Ω and since $\mathsf{B}^n(j, \mathbf{0})$ is weakly pseudoconvex, according to Example 3.3.11–(3). By the paragraph preceding, Ω_j is then weakly pseudoconvex. Since $\Omega = \cup_{j\in\mathbb{Z}_{>0}} \Omega_j$, weak pseudoconvexity of Ω follows from Proposition 3.3.10. ■

Next we consider the case where $\mathrm{bd}(\Omega)$ has some regularity as described in Section 3.3.3. The following definition introduces the key notion in this case.

3.3.19 Definition (Levi pseudoconvex) An open set $\Omega \subseteq \mathbb{C}^n$ with regular boundary of class C^2 is ***Levi pseudoconvex*** if, for any $\hat{z} \in \mathrm{bd}(\Omega)$ and any local defining function r of class C^2 defined on a neighbourhood $\mathcal{V}$ of $\hat{z}$, it holds that $\mathrm{Lev}(r)(z; w) \geq 0$ for all $w \in \mathsf{T}_z^{\mathbb{C}}\,\mathrm{bd}(\Omega)$ for all $z \in \mathrm{bd}(\Omega) \cap \mathcal{V}$. •

With this notion at hand, the main result is the following.

3.3.20 Theorem (Weak pseudoconvexity and Levi pseudoconvexity are equivalent) *If $\Omega \subseteq \mathbb{C}^n$ is a domain with a regular boundary of class C^2, then it is weakly pseudoconvex if and only if it is Levi pseudoconvex.*

Proof First suppose that Ω is weakly pseudoconvex. As we saw in Lemma 3.3.14, d_Ω is a local defining function in some neighbourhood of any point in $\mathrm{bd}(\Omega)$. From Theorem 3.3.9 we know that $-\log\delta_\Omega$ is plurisubharmonic, and using Proposition 3.2.12 we directly compute

$$\mathrm{Lev}(-\log\delta_\Omega)(z; w) = \sum_{j,k=1}^n \Big(-\frac{1}{\delta_\Omega(z)}\frac{\partial^2\delta_\Omega}{\partial z_j \partial\bar{z}_k}(z) + \frac{1}{\delta_\Omega(z)^2}\frac{\partial\delta_\Omega}{\partial z_j}(z)\frac{\partial\delta_\Omega}{\partial\bar{z}_k}(z)\Big) w_j \bar{w}_k \geq 0$$

for every $\boldsymbol{z} \in \Omega$ sufficiently close to $\mathrm{bd}(\Omega)$ and every $\boldsymbol{w} \in \mathbb{C}^n$. For all such $\boldsymbol{z}$ and for $\boldsymbol{w} \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \mathrm{d}_\Omega}{\partial z_j}(\boldsymbol{z}) w_j = 0, \tag{3.11}$$

this implies that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \mathrm{d}_\Omega}{\partial z_j \partial \bar{z}_k}(\boldsymbol{z}) w_j \bar{w}_k \geq 0,$$

and taking the limit as $\boldsymbol{z}$ approaches the boundary shows that this same inequality holds for $\boldsymbol{z} \in \mathrm{bd}(\Omega)$ for vectors $\boldsymbol{w}$ satisfying (3.11).

This shows that the condition on the Levi form for Levi pseudoconvexity holds for the defining function d_Ω. Let r be some other defining function defined on a neighbourhood of a point $\hat{\boldsymbol{z}} \in \mathrm{bd}(\Omega)$. By Lemma 3.3.13 it follows that $r = \psi \mathrm{d}_\Omega$ for some positive function ψ of class C^2. A straightforward computation using Lemma 3.3.16 shows that

$$\mathrm{Lev}(r)(\hat{\boldsymbol{z}}; \boldsymbol{w}) = \psi(\hat{\boldsymbol{z}}) \mathrm{Lev}(\mathrm{d}_\Omega)(\hat{\boldsymbol{z}}; \boldsymbol{w})$$

for every $\boldsymbol{w} \in \mathsf{T}^{\mathbb{C}}_{\hat{\boldsymbol{z}}}\Omega$. From this we conclude that the condition on the Levi form for Levi pseudoconvexity holds for any defining function if it holds for d_Ω, and so this proves that Ω is Levi pseudoconvex.

Next suppose that Ω is not weakly pseudoconvex. By Theorem 3.3.18 this means that there is a point in $\mathrm{bd}(\Omega)$ for which every neighbourhood is not weakly pseudoconvex. By Theorem 3.3.9 and Lemma 3.3.14 this means that there exists an open set $\mathcal{V}$ intersecting $\mathrm{bd}(\Omega)$ and $\boldsymbol{z} \in \mathcal{V}$ such that $-\log \delta_\Omega$ is not plurisubharmonic at $\boldsymbol{z}$. By Proposition 3.2.12 this means that the derivative

$$\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \log \delta_\Omega(\boldsymbol{z} + \zeta \boldsymbol{w}) > 0$$

when evaluated at $\zeta = 0$ and for some $\boldsymbol{w} \in \mathbb{C}^n$. By Lemma 3.2.10 we have

$$\log \delta_\Omega(\boldsymbol{z} + \zeta \boldsymbol{w}) = \log \delta_\Omega(\boldsymbol{z}) + \mathrm{Re}(\alpha \zeta + \beta \zeta^2) + \lambda |\zeta|^2 + o(|\zeta|^2), \tag{3.12}$$

for some $\alpha, \beta \in \mathbb{C}$ and where λ is the derivative

$$\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \log \delta_\Omega(\boldsymbol{z} + \zeta \boldsymbol{w})$$

evaluated at $\zeta = 0$. By Proposition B.1.3 let $\boldsymbol{u} \in \mathbb{C}^n$ be such that $\|\boldsymbol{u}\| = \delta_\Omega(\boldsymbol{z})$ and $\boldsymbol{z} + \boldsymbol{u} \in \mathrm{bd}(\Omega)$. For $s \in (0, 1]$ denote

$$\boldsymbol{\gamma}_s(\zeta) = \boldsymbol{z} + \zeta \boldsymbol{w} + s \exp(\alpha \zeta + \beta \zeta^2) \boldsymbol{u}.$$

Let $\hat{\boldsymbol{z}} \in \mathrm{bd}(\Omega)$ and compute

$$\begin{aligned}
\|\boldsymbol{\gamma}_s(\zeta) - \hat{\boldsymbol{z}}\| &= \|\boldsymbol{z} + \zeta \boldsymbol{w} - \hat{\boldsymbol{z}} - (-s \exp(\alpha \zeta + \beta \zeta) \boldsymbol{u})\| \\
&\geq \big| \|\boldsymbol{z} + \zeta \boldsymbol{w} - \hat{\boldsymbol{z}}\| - |s \exp(\alpha \zeta + \beta \zeta)| \|\boldsymbol{u}\| \big| \\
&\geq \|\boldsymbol{z} + \zeta \boldsymbol{w} - \hat{\boldsymbol{z}}\| - |s \exp(\alpha \zeta + \beta \zeta)| \|\boldsymbol{u}\| \\
&\geq \delta_\Omega(\boldsymbol{z} + \zeta \boldsymbol{w}) - |s \exp(\alpha \zeta + \beta \zeta)| \|\boldsymbol{u}\|.
\end{aligned}$$

From this we conclude that

$$\operatorname{dist}(\boldsymbol{\gamma}_s(\zeta), \operatorname{bd}(\Omega)) \geq \delta_\Omega(\boldsymbol{z} + \zeta\boldsymbol{w}) - |s\exp(\alpha\zeta + \beta\zeta)|\|\boldsymbol{u}\|.$$

From (3.12) we have

$$\delta_\Omega(\boldsymbol{z} + \zeta\boldsymbol{w}) = \delta_\Omega(\boldsymbol{z})|s\exp(\alpha\zeta + \beta\zeta)|\exp(\tfrac{1}{2}\lambda|\zeta|^2)\exp(\tfrac{1}{2}\lambda|\zeta|^2 + o(|\zeta|^2)).$$

Choosing $\epsilon \in \mathbb{R}_{>0}$ such that $\frac{1}{2}\lambda|\zeta|^2 + o(|\zeta|^2) \geq 0$ for $|\zeta| \leq \epsilon$ we have that

$$\delta_\Omega(\boldsymbol{z} + \zeta\boldsymbol{w}) \geq \delta_\Omega(\boldsymbol{z})|s\exp(\alpha\zeta + \beta\zeta)|\exp(\tfrac{1}{2}\lambda|\zeta|^2)$$

for $s \in (0,1]$ and $|\zeta| \leq \epsilon$. This then gives

$$\operatorname{dist}(\boldsymbol{\gamma}_s(\zeta), \operatorname{bd}(\Omega)) \geq \delta_\Omega(\boldsymbol{z})(\exp(\tfrac{1}{2}\lambda|\zeta|^2) - s)|s\exp(\alpha\zeta + \beta\zeta)|$$

for $s \in (0,1]$ and $|\zeta| \leq \epsilon$. This in particular implies that $\operatorname{dist}(\boldsymbol{\gamma}_s(\zeta), \operatorname{bd}(\Omega)) > 0$ for $s \in (0,1)$ and $|\zeta| \leq \epsilon$. Since $\boldsymbol{\gamma}_s(0) = \boldsymbol{z} + s\boldsymbol{u}$ we have $\boldsymbol{\gamma}_s(0) \in \Omega$ for any $s \in (0,1)$ by definition of $\boldsymbol{u}$. From this we conclude that $\boldsymbol{\gamma}_s(\zeta) \in \Omega$ for every $s \in (0,1)$ and $|\zeta| \leq \epsilon$ (were this not the case, we would have to have $\operatorname{dist}(\boldsymbol{\gamma}_s(\zeta), \operatorname{bd}(\Omega)) = 0$ for some $s \in (0,1)$ and some ζ such that $|\zeta| \leq \epsilon$). Thus $\boldsymbol{\gamma}_1(\zeta) \in \operatorname{cl}(\Omega)$ for every $|\zeta| \leq \epsilon$. Moreover, possible after shrinking ϵ, we have $\boldsymbol{\gamma}_1(\zeta) \in \operatorname{cl}(\Omega) \cap \mathcal{V}$ for every $|\zeta| \leq \epsilon$. It therefore follows that $\mathrm{d}_\Omega(\boldsymbol{\gamma}_1(\zeta)) = -\delta_\Omega(\boldsymbol{\gamma}_1(\zeta))$, and so we have

$$-\mathrm{d}_\Omega(\boldsymbol{\gamma}_1(\zeta)) \geq \delta_\Omega(\boldsymbol{z})(\exp(\tfrac{1}{2}\lambda|\zeta|^2) - 1)|\exp(\alpha\zeta + \beta\zeta)| \tag{3.13}$$

The function

$$\zeta \mapsto \delta_\Omega(\boldsymbol{z})(\exp(\tfrac{1}{2}\lambda|\zeta|^2) - 1)|\exp(\alpha\zeta + \beta\zeta)|$$

is easily seen to be strictly convex in a neighbourhood of $0 \in \mathbb{C}$, and is zero at $\zeta = 0$. Since the function is positive, $\zeta = 0$ is a strict local minimum. Now note that $\mathrm{d}_\Omega \circ \boldsymbol{\gamma}_1(0) = 0$ since $\boldsymbol{\gamma}_1(0) \in \operatorname{bd}(\Omega)$. The inequality (3.13) then implies that 0 is a strict local maximum for $\mathrm{d}_\Omega \circ \boldsymbol{\gamma}_1$. This implies, in particular, that

$$\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\Big|_{\zeta=0}\delta_\Omega \circ \boldsymbol{\gamma}_1(\zeta) > 0, \quad \frac{\partial}{\partial\zeta}\Big|_{\zeta=0}\mathrm{d}_\Omega \circ \boldsymbol{\gamma}_1(\zeta) = 0.$$

Using the fact that $\frac{\partial\gamma_1}{\partial\bar{\zeta}} = 0$ since $\boldsymbol{\gamma}_1$ is holomorphic, we deduce that

$$\sum_{j,k=1}^{n} \frac{\partial^2\mathrm{d}_\Omega}{\partial z_j\partial\bar{z}_k}(\boldsymbol{z} + \boldsymbol{u})w_j\bar{w}_k < 0, \quad \sum_{j=1}^{n} \frac{\partial\mathrm{d}_\Omega}{\partial z_j}(\boldsymbol{z} + \boldsymbol{u})w_j = 0,$$

which implies that Ω is not Levi pseudoconvex since $\boldsymbol{z} + \boldsymbol{u} \in \operatorname{bd}(\Omega)$. ■

Let us consider a few examples of Levi pseudoconvex sets.

3.3.21 Examples (Levi pseudoconvex domains)

1. As with all of the properties for open sets we have introduced in this chapter, the situation in one-dimension is straightforward as concerns Levi pseudoconvexity. Indeed, if $\Omega \subseteq \mathbb{C}$ is open with regular boundary of class C^2, we have $\mathsf{T}^{\mathbb{C}}_{\hat{z}}\,\mathrm{bd}(\Omega) = \{0\}$ for every $\hat{z} \in \mathrm{bd}(\Omega)$. Thus, in this case the Levi form vanishes on the complex tangent space to the boundary, and so is necessarily positive-semidefinite. Thus Ω is Levi pseudoconvex.
2. Also as we have seen with the other concepts in this chapter, there is a connection between ordinary convexity and Levi pseudoconvexity. Indeed, by Proposition 3.3.17 we see that if $\Omega \subseteq \mathbb{C}^n$ is convex, it is immediately Levi pseudoconvex.
3. There are, of course, domains that are weakly pseudoconvex but not Levi pseudoconvex, since there are no restrictions on the regularity of the boundary of a general weakly pseudoconvex set. For example, the domain

$$\Omega = \{z \in \mathbb{C} \mid \ \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$$

is weakly pseudoconvex, but not Levi pseudoconvex. •

The precise relationship between convexity and pseudoconvexity is quite rich. To start, by the Riemann Mapping Theorem, note simply connected domains of holomorphy in $\mathbb{C}$ that are strict subsets of $\mathbb{C}$ are holomorphically diffeomorphic to the convex open unit disk. To flesh out the situation when $n \geq 2$, we make the following definition.

3.3.22 Definition (Strictly pseudoconvex) A bounded domain $\Omega \subseteq \mathbb{C}^n$ is ***strictly pseudoconvex*** if there exists a neighbourhood $\mathcal{V}$ of $\mathrm{bd}(\Omega)$ and a function $r \in C^2(\mathcal{V})$ with the following properties:

(i) $\Omega \cap \mathcal{V} = \{z \in \mathcal{V} \mid \ r(z) < 0\}$;

(ii) $\mathrm{d}r(z) \neq 0$ for every $z \in \mathcal{V}$;

(iii) $\mathrm{Lev}(r)(\hat{z}; w) > 0$ for every $\hat{z} \in \mathrm{bd}(\Omega)$ and every nonzero $w \in \mathsf{T}^{\mathbb{C}}_{\hat{z}}\,\mathrm{bd}(\Omega)$. •

Thus strict pseudoconvexity places two restrictions on Levi pseudoconvexity. First, it considers only bounded domains, and second it requires that the Levi form of the defining function is *strictly* positive-definite on the complex tangent space to the boundary. For a strictly pseudoconvex domain Ω, it can be shown that for every $\hat{z} \in \mathrm{bd}(\Omega)$ there is a neighbourhood $\mathcal{V}$ of $\hat{z}$ such that $\Omega \cap \mathcal{V}$ is the image of an open convex set under a holomorphic diffeomorphism [Laurent-Thiébaut 2011, Corollary 3.24]. Thus strictly pseudoconvex sets are, up to holomorphic diffeomorphism, locally convex. This is not true if one removes the condition of strictness, and Kohn and Nirenberg [1973] give a counterexample.

3.4 The Levi problem

We have discussed various notions relating to domains of holomorphy and explored connections between these. Indeed, let us list the following properties of a domain Ω:

1. Ω is a domain of holomorphy;
2. Ω is holomorphically convex;
3. Ω possesses a singular function;
4. Ω is weakly pseudoconvex;
5. Ω is strongly pseudoconvex;
6. $-\log \delta_\Omega$ is plurisubharmonic;
7. for each $z \in \mathrm{bd}(\Omega)$, there exists a neighbourhood $\mathcal{V}$ of z such that $\Omega \cap \mathcal{V}$ is weakly pseudoconvex;
8. Ω is Levi pseudoconvex (when it has a regular boundary of class C^2).

A review of our work so far in this chapter reveals that we have proved the implications shown in Figure 3.3. One easily sees that the loop is not closed, and that to close the

$$\begin{tikzcd}
& 1 \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] & 2 \arrow[l, Rightarrow] \arrow[r, Leftrightarrow] & 3 \\
6 & & & \\
& 4 \arrow[ul, Rightarrow] \arrow[r, Leftrightarrow] & 5 \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & 7 \\
& & 8 &
\end{tikzcd}$$

Figure 3.3 Implications proved in this chapter so far

loop we need to show that a weakly pseudoconvex domain is a domain of holomorphy. This is known as the ***Levi problem***. In this section we summarise the history of this problem (following Lieb [2008]) and some of the techniques that have been used to prove the desired implication.

3.4.1 The history of the Levi problem

The Levi problem played a central rôle in the development of complex analysis during the first half of the twentieth century. The fact that there are domains in $\mathbb{C}^n$,

$n \geq 2$, such that every holomorphic function can be extended to some larger domain was recognised by Hartogs [1906], and marked the beginning of the modern theory of several complex variables. The finding of domains for which no holomorphic function can be extended to a larger domain was then understood as a fundamental one. The problem was first recognised by Levi [1910] who recognised that domains of holomorphy could be characterised by a sort of convexity condition on the boundary. Levi then conjectures that his boundary condition is sufficient for a domain to be a domain of holomorphy. There was not attention paid to this problem for some years. The recognition of the Levi problem was renewed by the activity in complex analysis of Heinrich Behnke and Henri Cartan in the late 1920's. The notion of holomorphic convexity was introduced by Cartan [1931], and the equivalence of holomorphically convex domains with domains of holomorphy was proved by Cartan and Thullen [1932]. Moreover, the Levi problem was solved by Behnke and Thullen [1934] in special cases. In this work, Behnke and Thullen also enumerated a number of important problems in complex analysis, among them the Levi problem.

The young Japanese mathematician Kiyoshi Oka became familiar with the work being done in France and Germany on complex analysis while visiting Paris in 1929. Upon his return to Japan, Oka became familiar with the problems outlined by Behnke and Thullen, and devoted approximately the next twenty years (with the expected hiatus during the war) to solving these problems. The work of Oka forever altered the state of complex analysis, systematically in nine fundamental papers published from 1936–1953. In the sixth of these papers, Oka [1942b] solves the Levi problem in two dimensions, and in the last of these papers [Oka 1953] solves the Levi problem in generality. Around the same time, the Levi problem was solved independently by Bremermann [1953] and Norguet [1954]. The collected works of [Oka 1984] gather together his contributions.

The Levi problem for holomorphic manifolds was solved by Grauert [1958]. Comments on the Levi problem and matters arising from it can be found in [Siu 1978].

3.4.2 Comments on solutions of the Levi problem

There is no "easy" solution to the Levi problem. All known solutions involve baggage of some sort. The solution of Grauert [1958] is a combination of sheaf theory and functional analysis. Another popular solution of the Levi problem originates in the work of Hörmander [1965] on the $\bar{\partial}$-operator (see also the treatment in [Hörmander 1994]). We sketch the idea of this approach here for the Levi problem for domains in $\mathbb{C}^n$. In Section 6.3.2 we give a detailed proof of the existence theorem for the $\bar{\partial}$-problem on manifolds. Here we merely sketch the central ideas for open subsets of $\mathbb{C}^n$, and indicate how this gives rise to a solution of the Levi problem.

The problem Hörmander addresses is the existence of solutions to the generalised Cauchy–Riemann equations. Let us provide some notation for this. Let $\Omega \subseteq \mathbb{C}^n$ be open and let f be a form of bidegree $(0, r)$ with coefficients being functions (of some sort) defined on Ω. (We will discuss differential forms in Section 4.6.) If

$$f(z) = \sum_{j_1<\dots<j_r} f_{j_1\cdots j_r}(z)\mathrm{d}\bar{z}_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{j_r},$$

we denote

$$\bar{\partial} f(z) = \sum_{j=1}^{n} \sum_{j_1<\dots<j_r} \frac{\partial f_{j_1\cdots j_r}}{\partial \bar{z}_j}(z)\mathrm{d}\bar{z}_j \wedge \mathrm{d}\bar{z}_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{j_r},$$

which we regard as a form of bidegree $(0, r+1)$. We wish to study solutions of the equation $\bar{\partial}u = f$, where f is given and we wish to determine u. We will consider the case where the unknown is the form u of bidegree $(0, r)$, and, therefore, f is a form of bidegree $(0, r+1)$. Just like the exterior derivative for regular differential forms, the operator $\bar{\partial}$ satisfies $\bar{\partial} \circ \bar{\partial} = 0$, using symmetry of partial derivatives as usual. If $\bar{\partial}u = f$ this then imposes the necessary closedness condition $\bar{\partial}f = 0$ on the one-form f. The problem can then be stated as follows.

3.4.1 Problem (The $\bar{\partial}$-problem) For what open sets Ω is the condition $\bar{\partial}f = 0$ sufficient for there to exist a solution to the equation $\bar{\partial}u = f$? •

It is this problem that Hörmander solves. He does this by first considering data in $\mathsf{L}^2_{\mathrm{loc}}(\Omega; \mathbb{C})$. In this case, one can use Hilbert space methods to prove the following result.

3.4.2 Theorem (The solution to the $\bar{\partial}$-problem in L^2) *If $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex and if f is a form of bidegree $(0, r+1)$ with coefficients in $\mathsf{L}^2_{\mathrm{loc}}(\Omega; \mathbb{C})$ and for which $\bar{\partial}f = 0$, then there exists a form u of bidegree $(0, r)$ on Ω such that $\bar{\partial}u = f$ (with derivatives understood in the distributional sense).*

The key element in this theorem, as concerns its relationship to the Levi problem, is that it uses pseudoconvexity of the domain as the essential condition on the domain Ω. The rôle played by pseudoconvexity is not so easy to imagine, but it amounts to using the plurisubharmonic exhaustion function on the pseudoconvex domain to establish bounds on Hilbert space operators.

Next Hörmander shows that the existence of solutions to the $\bar{\partial}$-problem in the L^2-case implies existence of solutions in certain Sobolev spaces. For $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ we denote by $\mathsf{H}^s(\Omega; \mathbb{C})$ the $\mathbb{C}$-valued distributions on Ω whose derivatives up to order s are in $\mathsf{L}^2(\Omega; \mathbb{C})$. By $\mathsf{H}^s_{\mathrm{loc}}(\Omega; \mathbb{C})$ we denote the space of such distributions having this property on compact subsets of Ω. Then Hörmander proves the following result.

3.4.3 Theorem (The solution to the $\bar{\partial}$-problem in H^s) *If $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex and if f is a form of bidegree $(0, r+1)$ with coefficients in $\mathsf{H}^s_{\mathrm{loc}}(\Omega; \mathbb{C})$ and for which $\bar{\partial}f = 0$, then there exists a form u of bidegree $(0, r)$ on Ω such that $\bar{\partial}u = f$ (with derivatives understood in the distributional sense).*

One now uses the Sobolev embeddings $\mathsf{H}^{s+2n}_{\mathrm{loc}}(\Omega; \mathbb{C}) \subseteq \mathsf{C}^s(\Omega; \mathbb{C})$ for $\Omega \subseteq \mathbb{C}^n$ to arrive at the following result.

3.4.4 Corollary (The solution to the $\bar{\partial}$-problem in C^∞) *If $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex and if f is a form of bidegree $(0, r+1)$ with coefficients in $C^\infty(\Omega;\mathbb{C})$ and for which $\bar{\partial}f = 0$, then there exists a form u of bidegree $(0,r)$ on Ω such that $\bar{\partial}u = f$.*

This last version of the solution to the $\bar{\partial}$-problem can be used to solve the Levi problem as follows.

3.4.5 Theorem (Solution to the Levi problem) *Let $\Omega \subseteq \mathbb{C}^n$ be a domain such that, for every form f of bidegree $(0, r+1)$ with coefficients in $C^\infty(\Omega;\mathbb{C})$ and for which $\bar{\partial}f = 0$, there exists a form u of bidegree $(0,r)$ on Ω such that $\bar{\partial}u = f$. Then Ω is a domain of holomorphy.*

In particular, if Ω is pseudoconvex, it is a domain of holomorphy.

Proof The statement holds with $n = 1$ since every open set is both a domain of holomorphy and pseudoconvex. Suppose that the theorem holds for $n \in \{1, \dots, k-1\}$ and let $\Omega \subseteq \mathbb{C}^k$ satisfy the hypotheses of the theorem.

Let $C \subseteq \Omega$ be an open convex set such that $\mathrm{bd}(C) \cap \mathrm{bd}(\Omega) \neq \emptyset$. Let $\hat{z} \in \mathrm{bd}(C) \cap \mathrm{bd}(\Omega)$ and, by making an elementary change of coordinates, suppose that $\hat{z} = \mathbf{0}$ and that

$$C_0 = \{z \in C \mid z_k = 0\} \neq \emptyset.$$

Since C is convex, $\mathbf{0} \in \mathrm{bd}(C_0)$ and so $\mathbf{0} \in \mathrm{bd}(\Omega_0)$, where

$$\Omega_0 = \{z \in \Omega \mid z_k = 0\} \neq \emptyset.$$

Let $\iota\colon \Omega_0 \to \Omega$ be the inclusion and let $\pi\colon \mathbb{C}^k \to \mathbb{C}^{k-1}$ be the projection onto the first $k-1$ components. We can thus naturally identify Ω_0 with $\pi(\Omega_0) \subseteq \mathbb{C}^{k-1}$. Denote

$$\mathcal{A} = \{z \in \Omega \mid \pi(z) \notin \Omega_0\},$$

and note that $\mathcal{A}$ is closed in Ω by continuity of π. Note that Ω_0 and $\mathcal{A}$ are thus disjoint closed subsets of Ω and, by the Tietze Extension Theorem, we can find $\phi \in C^\infty(\Omega)$ such that $\phi(z) = 1$ for z in a neighbourhood of Ω_0 and $\phi(z) = 0$ for z in a neighbourhood of $\mathcal{A}$. For a form f_0 of bidegree $(0,r)$ on Ω_0 we easily and directly verify that $\iota^*(\phi\pi^* f_0) = f_0$ since ϕ is equal to 1 on Ω_0. Now suppose that f_0 satisfies $\bar{\partial} f_0 = 0$. Consider the form of bidegree $(0, r+1)$ on Ω defined by

$$z \mapsto z_k^{-1}\bar{\partial}\phi(z) \wedge \pi^* f_0(z),$$

which is well-defined since $\bar{\partial}\phi$ vanishes on Ω_0 as ϕ is constant on Ω_0. One immediately verifies that applying $\bar{\partial}$ to this form gives zero since $\bar{\partial} f_0 = 0$ and since $\bar{\partial} \circ \bar{\partial} = 0$. Therefore, by the hypotheses of the theorem, there exists a form α of bidegree $(0,r)$ on Ω such that

$$\bar{\partial}\alpha(z) = z_k^{-1}\bar{\partial}\phi(z) \wedge \pi^* f_0(z).$$

If we take

$$f(z) = \phi(z)\pi^* f_0(z) - z_k\alpha(z),$$

we then see that $\bar{\partial} f = 0$. We also immediately have $\iota^* f = f_0$.

We have proved that, given a form f_0 of bidegree $(0,r)$ on Ω_0 satisfying $\bar{\partial} f_0 = 0$, there exists a form f of bidegree $(0,r)$ on Ω such that $\bar{\partial} f = 0$ and $\iota^* f = f_0$. By the hypotheses of the theorem, let u be a form of bidegree $(0, r-1)$ on Ω be such that $\bar{\partial} u = f$. Then, if $u_0 = \iota^* u$,

$$\bar{\partial} u_0 = \iota^* \bar{\partial} u = \iota^* f = f_0.$$

By the induction hypothesis, Ω_0 is a domain of holomorphy. Thus there exists $g_0 \in \mathrm{C}^{\mathrm{hol}}(\Omega_0)$ that cannot be extended to any neighbourhood of $\mathrm{cl}(C_0)$. Note that $\bar{\partial} g_0 = 0$. By our constructions above, there exists $g \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ be such that $\iota^* g = g_0$, and note that g cannot be extended to any neighbourhood of C. This shows that Ω is indeed a domain of holomorphy.

The final assertion of the theorem follows from Corollary 3.4.4. ■

3.5 A summary of domains of holomorphy

In this section we pull together the previous results in this chapter and summarise their interconnections and consequences.

3.5.1 Characterisations of domains of holomorphy

Thus far in this section we have presented many different properties of domains in $\mathbb{C}^n$, and we have explored some connections between them. In this section we show that all of these properties are, in fact, equivalent. There are many other equivalent statements, and indeed some of these are contained in the preceding sections, but we do not clutter our statement here with all possible characterisations, but only the ones that are somehow important. Krantz [1992] gives twenty-six properties of an open subset of $\mathbb{C}^n$ equivalent to the set being a domain of holomorphy.

We state eight such equivalent conditions.

3.5.1 Theorem (Equivalent characterisations of domains of holomorphy) *For a domain $\Omega \subseteq \mathbb{C}^n$, the following statements are equivalent:*

(i) Ω is a domain of holomorphy;

(ii) Ω is holomorphically convex (the ***Cartan–Thullen Theorem****);*

(iii) Ω possesses a singular function;

(iv) Ω is weakly pseudoconvex;

(v) Ω is strongly pseudoconvex;

(vi) $-\log \delta_\Omega$ is plurisubharmonic;

(vii) for each $\mathbf{z} \in \mathrm{bd}(\Omega)$, there exists a neighbourhood $\mathcal{V}$ of $\mathbf{z}$ such that $\Omega \cap \mathcal{V}$ is a domain of holomorphy.

Moreover, if Ω has a regular boundary of class C^2-boundary, then the preceding seven statements are equivalent to the following one:

(viii) Ω is Levi pseudoconvex.

Proof (i) $\Longrightarrow$ (ii) This is Theorem 3.1.10.

(ii) $\Longrightarrow$ (iii) This is Theorem 3.1.13.

(iii) $\Longrightarrow$ (i) This follows from the definitions.

(i) $\Longrightarrow$ (iv) This is Theorem 3.3.8.

(iv) $\Longrightarrow$ (v) This follows from Lemma 3.3.7.

(v) $\Longrightarrow$ (i) This follows from the fact that the Levi problem has been solved.

(i) $\implies$ (vi) This is Theorem 3.2.18.

(vi) $\implies$ (iv) This follows since, in the proof of Theorem 3.3.8, we showed that $z \mapsto \max\{\|z\|^2, -\log \delta_\Omega(z)\}$ was a continuous plurisubharmonic exhaustion function if $-\log \delta_\Omega$ was plurisubharmonic.

(vii) $\iff$ (iv) This is Theorem 3.3.18, given that being a domain of holomorphy is equivalent to weak pseudoconvexity.

(iv) $\iff$ (viii) This is Theorem 3.3.20. ■

The equivalence of an open set being a domain of holomorphy and holomorphically convex was proved first by Cartan and Thullen [1932]. As we have done, they proved this by showing that a holomorphically convex set possesses a singular function. The equivalence of an open set being a domain of holomorphy and being pseudoconvex was first proved in full generality by Oka [1953], Bremermann [1953], and Norguet [1954].

3.5.2 Properties of domains of holomorphy

In this section we merely collect properties for domains of holomorphy that follow from our some of our various characterisations of domains of holomorphy, and facts we proved that derive from these characterisations.

3.5.2 Proposition (Basic properties of domains of holomorphy) *The following statements hold:*

(i) if $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are domains of holomorphy, then $\Omega \times \Delta \subseteq \mathbb{C}^n \times \mathbb{C}^m$ is a domain of holomorphy;

(ii) if $\Omega \subseteq \mathbb{C}^n$ and $\Delta \subseteq \mathbb{C}^m$ are domains of holomorphy and if $\mathbf{\Phi}\colon \Omega \to \mathbb{C}^m$ is holomorphic, then $\mathbf{\Phi}^{-1}(\Delta)$ is domain of holomorphy;

(iii) if $(\Omega_a)_{a\in A}$ is a family of domains of holomorphy in $\mathbb{C}^n$ for which $\mathrm{int}(\cap_{a\in A}\Omega_a) \neq \emptyset$, then each connected component of $\mathrm{int}(\cap_{a\in A}\Omega_a) \neq \emptyset$ is a domain of holomorphy;

*(iv) if $(\Omega_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence of domains of holomorphy in $\mathbb{C}^n$ for which $\Omega_j \subseteq \Omega_{j+1}$, $j \in \mathbb{Z}_{>0}$, then $\cup_{j\in\mathbb{Z}}\Omega_j$ is a domain of holomorphy (the **Behnke–Stein Theorem**).*

Proof (i) This follows from Proposition 3.3.10(i) along with Theorem 3.5.1.

(ii) This follows from Proposition 3.3.10(ii) along with Theorem 3.5.1.

(iii) This follows from Proposition 3.1.9(iii) along with Theorem 3.5.1.

(iv) This follows from Proposition 3.3.10(iv) along with Theorem 3.5.1. ■

Part (iv) of the preceding result was proved by Behnke and Stein [1939].

The following result is one of the first instances we see of how domains of holomorphy are so important in practice. The main point is that, in domains of holomorphy, one has a great deal of freedom in choosing holomorphic functions.

3.5.3 Theorem (Interpolation in domains of holomorphy) *If $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy, if $(\mathbf{z}_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence with no accumulation points, and if $(a_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathbb{C}$, then there exists $f \in \mathrm{C}^{\mathrm{hol}}(\Omega)$ such that $f(\mathbf{z}_j) = a_j$ for every $j \in \mathbb{Z}_{>0}$.*

Proof This follows from Theorem 3.1.12, along with Theorem 3.5.1. ■

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