This version: 28/02/2014

Chapter 4

Holomorphic and real analytic differential geometry

In this chapter we develop the basic theory of holomorphic and real analytic manifolds. We will be assuming that the reader has a solid background in basic smooth differential geometry such as one would get from an introductory graduate course on the subject, or from texts such as [Abraham, Marsden, and Ratiu 1988, Boothby 1986, Lee 2002, Warner 1983]. While a reader could, in principle, cover the basic of smooth differential geometry by replacing "holomorphic or real analytic" in our treatment with "smooth," we do not recommend doing so. Very often holomorphic differential geometry is included in texts on several complex variables. Such texts, and ones we will refer to, include [Fritzsche and Grauert 2002, Gunning and Rossi 1965, Hörmander 1973, Taylor 2002]. There is a decided paucity of literature on real analytic differential geometry. A good book on basic real analyticity is [Krantz and Parks 2002].

4.1 $\mathbb{C}$-linear algebra

Many of the constructions we shall make in complex differential geometry are done first on tangent spaces, and then made global by taking sections. In this section we collect together the constructions from $\mathbb{C}$-linear algebra that we shall use. Some of what we say is standard and can be found in a text on linear algebra [e.g., Axler 1997]. A good presentation of the not completely standard ideas can be found in the book of Huybrechts [2005]. In this section, since we will be dealing concurrently with $\mathbb{R}$- and $\mathbb{C}$-vector spaces and bases for these, we shall use the expressions "$\mathbb{R}$-basis" and "$\mathbb{C}$-basis" to discriminate which sort of basis we are talking about.

4.1.1 Linear complex structures

To study the structure of complex manifolds, it is convenient to first look at linear algebra.

Let us first consider $\mathbb{C}^n$ as a $\mathbb{R}$-vector space and see how the complex structure can be represented in a real way. The general feature we are after is the following.

4.1.1 Definition (Linear complex structure) A *linear complex structure* on a $\mathbb{R}$-vector space

V is an endomorphism $J \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$ such that $J \circ J = -\mathrm{id}_{\mathsf{V}}$. The ***conjugate linear complex structure*** associated to a linear complex structure J is the linear complex structure $-J$. •

Sometimes, for emphasis, if V is a $\mathbb{R}$-vector space with a linear complex structure J, we shall denote by $\overline{\mathsf{V}}$ the same vector space, but with the conjugate linear complex structure $-J$.

Let us give a useful normal form for linear complex structures.

4.1.2 Proposition (Normal form for linear complex structures) *If J is a linear complex structure on the n-dimensional $\mathbb{R}$-vector space V then n is even, say $n = 2m$, and there exists a $\mathbb{R}$-basis $(e_1, \dots, e_m, e_{m+1}, \dots, e_{2m})$ such that the matrix representative of J in this basis is*

$$\begin{bmatrix} \mathbf{0}_{m\times m} & -\mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0}_{m\times m} \end{bmatrix}.$$

Proof Suppose that λ is an eigenvalue for the complex structure J with eigenvector v. Then

$$J(v) = \lambda v \quad \Longrightarrow \quad -v = J \circ J(v) = \lambda^2 v$$

and so $-\lambda^2 = -1$ and thus the eigenvalues of J are $\pm\mathrm{i}$. Moreover, since $J^2 + \mathrm{id}_{\mathsf{V}} = 0$, the minimal polynomial of J is $\lambda^2 + 1$ and so J is diagonalisable over $\mathbb{C}$. An application of the real Jordan normal form theorem [Shilov 1977, §6.6] gives the existence of a $\mathbb{R}$-basis $(f_1, \dots, f_m, f_{m+1}, \dots, f_{2m})$ such that the matrix representative of J in this basis is

$$\begin{bmatrix} \boldsymbol{J}_2 & \cdots & \mathbf{0}_{2\times 2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & \boldsymbol{J}_2 \end{bmatrix},$$

where

$$\boldsymbol{J}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If we take $e_j = f_{2j-1}$ and $e_{j+m} = f_{2j}$, $j \in \{1, \dots, m\}$, the result follows. ■

Note that if $(\beta^1, \dots, \beta^m, \beta^{m+1}, \dots, \beta^{2m})$ is the $\mathbb{R}$-basis dual to a basis as in the preceding proposition, we have

$$J = \sum_{j=1}^{m} e_{m+j} \otimes \beta^j - \sum_{j=1}^{m} e_j \otimes \beta^{m+j}.$$

We shall call a $\mathbb{R}$-basis for V with this property a ***J-adapted basis***.

4.1.3 Examples (Linear complex structures and $\mathbb{C}$-vector spaces)

1. If we take the $\mathbb{R}$-vector space $\mathsf{V} = \mathbb{C}^m$, then the linear complex structure is defined by $J(v) = \mathrm{i}v$. A $\mathbb{R}$-basis for the $\mathbb{R}$-vector space $\mathbb{C}^n$ in which the matrix representative J takes the normal form of Proposition 4.1.2 is given by

$$e_1 = (1, \dots, 0), \dots, e_m = (0, \dots, 1),\ e_{m+1} = (\mathrm{i}, \dots, 0), \dots, e_{2m} = (0, \dots, \mathrm{i}).$$

Thus the linear complex structure in this case is given by

$$J(x^1 + \mathrm{i}y^1, \dots, x^m + \mathrm{i}y^m) = (-y^1 + \mathrm{i}x^1, \dots, -y^m + \mathrm{i}x^m),$$

which is, of course, just multiplication by i.

We can expand on this further.

2. On a finite-dimensional $\mathbb{R}$-vector space with a linear complex structure we can define a $\mathbb{C}$-linear structure as follows. If $a + \mathrm{i}b \in \mathbb{C}$ for $a, b \in \mathbb{R}$ and if $v \in \mathsf{V}$, we take

$$(a + \mathrm{i}b)v = av + bJ(v).$$

One readily verifies that this does indeed define the structure of a $\mathbb{C}$-vector space on V.

Conversely, if V is a $\mathbb{C}$-vector space, we can certainly think of it as a $\mathbb{R}$-vector space. We can then define $J \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$ by $J(v) = \mathrm{i}v$, and this certainly define a linear complex structure on V. Now one has two $\mathbb{C}$-vector space structures on V, the prescribed one and the one coming from J as in the preceding paragraph. This are very easily seen to agree. (But be careful, we shall shortly see cases of vector spaces with two $\mathbb{C}$-vector space structures that *do not* agree.)

The preceding discussion shows that there is, in fact, a natural correspondence between $\mathbb{R}$-vector spaces with linear complex structures and $\mathbb{C}$-vector spaces. This is a sometimes confusing fact. To overcome some of this confusion, we shall generally deal with *real* vector spaces and consider the $\mathbb{C}$-vector space structure as arising from a linear complex structure. •

Now we consider the complexification $\mathsf{V}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathsf{V}$ of V with $J_{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(\mathsf{V})$ the resulting endomorphism of $\mathsf{V}_{\mathbb{C}}$ defined by requiring that $J_{\mathbb{C}}(a \otimes v) = a \otimes J(v)$ for $a \in \mathbb{C}$ and $v \in \mathsf{V}$. Note that if $v \in \mathsf{V}_{\mathbb{C}}$ we can write $v = 1 \otimes v_1 + \mathrm{i}v_2$ for some $v_1, v_2 \in \mathsf{V}$. We can define ***complex conjugation*** in $\mathsf{V}_{\mathbb{C}}$ by

$$\overline{a \otimes v} = \bar{a} \otimes v, \qquad a \in \mathbb{C},\ v \in \mathsf{V}.$$

We make the following definition.

4.1.4 Definition (Holomorphic and antiholomorphic subspace) Let J be a linear complex structure on a finite-dimensional $\mathbb{R}$-vector space V.

(i) The ***holomorphic subspace*** for J is the $\mathbb{C}$-subspace $\mathsf{V}^{1,0}$ of $\mathsf{V}_{\mathbb{C}}$ given by

$$\mathsf{V}^{1,0} = \ker(J_{\mathbb{C}} - \mathrm{i}\,\mathrm{id}_{\mathsf{V}_{\mathbb{C}}}).$$

(ii) The ***antiholomorphic subspace*** for J is the $\mathbb{C}$-subspace $\mathsf{V}^{0,1}$ of $\mathsf{V}_{\mathbb{C}}$ given by

$$\mathsf{V}^{0,1} = \ker(J_{\mathbb{C}} + \mathrm{i}\,\mathrm{id}_{\mathsf{V}_{\mathbb{C}}}).$$ •

The "holomorphic" language here is a little unmotivated in the linear case, but will hopefully become clearer as we move on.

We then have the following properties of $\mathsf{V}_{\mathbb{C}}$, $\mathsf{V}^{1,0}$, and $\mathsf{V}^{0,1}$.

4.1.5 Proposition (Complexification of linear complex structures) *Let* J *be a linear complex structure on a finite-dimensional* $\mathbb{R}$*-vector space* V*. The following statements hold:*

(i) $\mathsf{V}_{\mathbb{C}} = \mathsf{V}^{1,0} \oplus \mathsf{V}^{0,1}$ *(direct sum of* $\mathbb{C}$*-vector spaces);*

(ii) $\{1 \otimes \mathrm{v} \mid \mathrm{v} \in \mathsf{V}\} = \{\mathrm{u} \in \mathsf{V}_{\mathbb{C}} \mid \bar{\mathrm{u}} = \mathrm{u}\}$*;*

(iii) $\mathsf{V}^{1,0} = \{\bar{\mathrm{v}} \in \mathsf{V}_{\mathbb{C}} \mid \mathrm{v} \in \mathsf{V}^{0,1}\}$*;*

(iv) the map $\sigma_+ \colon \mathsf{V} \to \mathsf{V}^{1,0}$ *defined by*

$$\sigma_+(\mathrm{v}) = \frac{1}{2}(1 \otimes \mathrm{v} - \mathrm{i} \otimes \mathrm{J}(\mathrm{v}))$$

is a isomorphism of $\mathbb{C}$*-vector spaces, meaning that*

$$\sigma_+(\mathrm{J}(\mathrm{v})) = \mathrm{i}\sigma_+(\mathrm{v});$$

(v) the map $\sigma_- \colon \overline{\mathsf{V}} \to \mathsf{V}^{0,1}$ *defined by*

$$\sigma_-(\mathrm{v}) = \frac{1}{2}(1 \otimes \mathrm{v} + \mathrm{i} \otimes \mathrm{J}(\mathrm{v}))$$

is an isomorphism of $\mathbb{C}$*-vector spaces, meaning that*

$$\sigma_-(-\mathrm{J}(\mathrm{v})) = \mathrm{i}\sigma_-(\mathrm{v});$$

Proof (i) Note that $J_{\mathbb{C}}$ is diagonalisable since its minimal polynomial has no repeated factors. Because $\mathsf{V}^{1,0}$ and $\mathsf{V}^{0,1}$ are the eigenspaces for the eigenvalues i and $-\mathrm{i}$, respectively, we have

$$\mathsf{V}_{\mathbb{C}} = \mathsf{V}^{1,0} \oplus \mathsf{V}^{0,1}.$$

(ii) For $v \in \mathsf{V}_{\mathbb{C}}$ we can write $v = 1 \otimes v_1 + \mathrm{i} \otimes v_2$ for $v_1, v_2 \in \mathsf{V}$. Then $\bar{v} = v$ if and only if $v_2 = 0$, and from this the result follows.

(iii) We compute

$$\begin{aligned}
\mathsf{V}^{1,0} &= \{v \in \mathsf{V}_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = \mathrm{i}v\} \\
&= \{1 \otimes v_1 + \mathrm{i} \otimes v_2 \in \mathsf{V}_{\mathbb{C}} \mid 1 \otimes J(v_1) + \mathrm{i} \otimes J(v_2) = \mathrm{i} \otimes v_1 - 1 \otimes v_2\} \\
&= \{1 \otimes v_1 + \mathrm{i} \otimes v_2 \in \mathsf{V}_{\mathbb{C}} \mid 1 \otimes J(v_1) - \mathrm{i} \otimes J(v_2) = -\mathrm{i} \otimes v_1 - 1 \otimes v_2\} \\
&= \{1 \otimes v_1 - \mathrm{i} \otimes v_2 \in \mathsf{V}_{\mathbb{C}} \mid 1 \otimes J(v_1) + \mathrm{i} \otimes J(v_2) = -\mathrm{i} \otimes v_1 + 1 \otimes v_2\} \\
&= \{\bar{v} \in \mathsf{V}_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = -\mathrm{i}v\} = \{\bar{v} \in \mathsf{V}_{\mathbb{C}} \mid v \in \mathsf{V}^{0,1}\},
\end{aligned}$$

as desired.

(iv) First of all, note that $u \in \mathrm{image}(\sigma_+)$ if and only if $J_{\mathbb{C}}(u) = \mathrm{i}u$, i.e., if and only if $u \in \mathsf{V}^{1,0}$. Thus σ_+ is well-defined and surjective. That it is an isomorphism follows from a dimension count. That σ_+ is a $\mathbb{C}$-isomorphism in the sense stated follows via direct verification.

(v) The proof here goes much like that for the preceding part of the proof. ■

As we warned in Example 4.1.3–2, we have in $\mathsf{V}_{\mathbb{C}}$ a case of a $\mathbb{R}$-vector space with two $\mathbb{C}$-vector space structures. The first comes from the fact that $\mathsf{V}_{\mathbb{C}}$ is the complexification of a $\mathbb{R}$-vector space, and is defined by $\mathrm{i}(a \otimes v) = (\mathrm{i}a) \otimes v$ for $a \in \mathbb{C}$ and $v \in \mathsf{V}$. The other comes from the fact that the *real* endomorphism $J_{\mathbb{C}}$ is a linear almost complex structure on $\mathsf{V}_{\mathbb{C}}$ and so defines a $\mathbb{C}$-vector space structure by $\mathrm{i}(a \otimes v) = a \otimes J(v)$ for $a \in \mathbb{C}$ and $v \in \mathsf{V}$. From Proposition 4.1.5 we see that these two $\mathbb{C}$-vector space structures agree on $\mathsf{V}^{1,0}$ but are conjugate on $\mathsf{V}^{0,1}$. Unless we say otherwise, the $\mathbb{C}$-vector space structure we use on $\mathsf{V}_{\mathbb{C}}$ will be that coming from the fact that $\mathsf{V}_{\mathbb{C}}$ is the complexification of the $\mathbb{R}$-vector space V.

We can use linear complex structures to characterise $\mathbb{C}$-linear maps between vector spaces with such structures.

4.1.6 Proposition (ℂ-linear maps between vector spaces with linear complex structures) *Let V_1 and V_2 be finite-dimensional $\mathbb{R}$-vector spaces with linear almost complex structures J_1 and J_2, respectively. If $\mathrm{A} \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}_1, \mathsf{V}_2)$ with $\mathrm{A}_{\mathbb{C}} \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{V}_{1,\mathbb{C}}; \mathsf{V}_{2,\mathbb{C}})$ the complexification of A defined by $\mathrm{A}_{\mathbb{C}}(\mathrm{a} \otimes \mathrm{v}) = \mathrm{a} \otimes \mathrm{A}(\mathrm{v})$ for $\mathrm{a} \in \mathbb{C}$ and $\mathrm{v} \in \mathsf{V}_1$. Then the following statements are equivalent:*

(i) $\mathrm{A} \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{V}_1, \mathsf{V}_2)$ *(using the $\mathbb{C}$-vector space structure on V_1 and V_2 defined by J_1 and J_2);*

(ii) the diagram

$$\begin{array}{ccc} \mathsf{V}_1 & \xrightarrow{\mathrm{J}_1} & \mathsf{V}_1 \\ {\scriptstyle \mathrm{A}}\downarrow & & \downarrow{\scriptstyle \mathrm{A}} \\ \mathsf{V}_2 & \xrightarrow[\mathrm{J}_2]{} & \mathsf{V}_2 \end{array}$$

commutes;

(iii) $\mathrm{A}_{\mathbb{C}}(\mathsf{V}_1^{1,0}) \subseteq \mathsf{V}_2^{1,0}$*;*

(iv) $\mathrm{A}_{\mathbb{C}}(\mathsf{V}_1^{0,1}) \subseteq \mathsf{V}_2^{0,1}$*.*

Proof (i) $\iff$ (ii) This follows since, by definition, multiplication by i in V_1 and V_2 is given by $\mathrm{i}v_k = J_k(v_k)$ for all $v_k \in \mathsf{V}_k$, $k \in \{1, 2\}$.

(iii) $\iff$ (iv) We have

$$\begin{aligned} & A_{\mathbb{C}}(v) \in \mathsf{V}_2^{1,0} \text{ for all } v \in \mathsf{V}_1^{1,0} \\ \iff \quad & \overline{A_{\mathbb{C}}(v)} \in \mathsf{V}_2^{0,1} \text{ for all } v \in \mathsf{V}_1^{1,0} \\ \iff \quad & A_{\mathbb{C}}(\bar{v}) \in \mathsf{V}_2^{0,1} \text{ for all } v \in \mathsf{V}_1^{1,0} \\ \iff \quad & A_{\mathbb{C}}(v) \in \mathsf{V}_2^{0,1} \text{ for all } v \in \mathsf{V}_1^{0,1}, \end{aligned}$$

using Proposition 4.1.5(iii) and the fact that $A_{\mathbb{C}}$ is the complexification of a $\mathbb{R}$-linear map.

(ii) $\implies$ (iii) Let $v \in \mathsf{V}^{1,0}$ so that $J_{1,\mathbb{C}}(v) = \mathrm{i}v$. Then

$$J_{2,\mathbb{C}} \circ A_{\mathbb{C}}(v) = A_{\mathbb{C}} \circ J_{1,\mathbb{C}}(v) = \mathrm{i}A_{\mathbb{C}}(v),$$

and so $A_{\mathbb{C}}(v) \in \mathsf{V}_2^{1,0}$.

(iii,iv) $\Longrightarrow$ (ii) Let $v \in \mathsf{V}_1^{1,0}$ so that $J_{1,\mathbb{C}}(v) = \mathrm{i}v$. Then

$$A_{\mathbb{C}} \circ J_{1,\mathbb{C}}(v) = \mathrm{i}A_{\mathbb{C}}(v) = J_{2,\mathbb{C}} \circ A_{\mathbb{C}}(v).$$

Similarly, $A_{\mathbb{C}} \circ J_{1,\mathbb{C}}(v) = J_{2,\mathbb{C}} \circ A_{\mathbb{C}}(v)$ for every $v \in \mathsf{V}_1^{1,0}$. Since $\mathsf{V}_{1,\mathbb{C}} = \mathsf{V}_1^{1,0} \oplus \mathsf{V}_2^{0,1}$ it follows that the diagram

$$\begin{array}{ccc} \mathsf{V}_{1,\mathbb{C}} & \xrightarrow{J_{1,\mathbb{C}}} & \mathsf{V}_{1,\mathbb{C}} \\ {\scriptstyle A_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle A_{\mathbb{C}}} \\ \mathsf{V}_{2,\mathbb{C}} & \xrightarrow[J_{2,\mathbb{C}}]{} & \mathsf{V}_{2,\mathbb{C}} \end{array}$$

commutes. Since $A_{\mathbb{C}}$, $J_{1,\mathbb{C}}$, and $J_{2,\mathbb{C}}$ are complexifications of $\mathbb{R}$-linear maps, this part of the proof is concluded. ■

Let us close this section by giving basis representations for the various constructions in this section.

4.1.7 Proposition (Basis representations for linear complex structures) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space with a linear complex structure* J*. Let* $(\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{m}, \mathrm{e}_{\mathrm{m}+1}, \dots, \mathrm{e}_{2\mathrm{m}})$ *be a* J*-adapted* $\mathbb{R}$*-basis for* V *with dual basis* $(\beta^1, \dots, \beta^\mathrm{m}, \beta^{\mathrm{m}+1}, \dots, \beta^{2\mathrm{m}})$*. Then the following statements hold:*

(i) the vectors $1 \otimes \mathrm{e}_\mathrm{j} + \mathrm{i} \otimes 0$, $\mathrm{j} \in \{1, \dots, 2\mathrm{m}\}$*, form a* $\mathbb{R}$*-basis for* $\mathsf{V} \subseteq \mathsf{V}_{\mathbb{C}}$*;*

(ii) the vectors $\frac{1}{2}(1 \otimes \mathrm{e}_\mathrm{j} - \mathrm{i} \otimes \mathrm{e}_{\mathrm{m}+\mathrm{j}})$, $\mathrm{j} \in \{1, \dots, \mathrm{m}\}$*, form a* $\mathbb{C}$*-basis for* $\mathsf{V}^{1,0}$*;*

(iii) the vectors $\frac{1}{2}(1 \otimes \mathrm{e}_\mathrm{j} + \mathrm{i} \otimes \mathrm{e}_{\mathrm{m}+\mathrm{j}})$, $\mathrm{j} \in \{1, \dots, \mathrm{m}\}$*, form a* $\mathbb{C}$*-basis for* $\mathsf{V}^{0,1}$*.*

Proof (i) This is clear since V is the subspace of $\mathsf{V}_{\mathbb{C}}$ given by the image of the map $v \mapsto 1 \otimes v$.

(ii) We compute

$$\begin{aligned} J_{\mathbb{C}}(1 \otimes e_j - \mathrm{i} \otimes e_{m+j}) - \mathrm{i}(1 \otimes e_j - \mathrm{i} \otimes e_{m+j}) &= 1 \otimes J(e_j) - \mathrm{i} \otimes J(e_{m+j}) - \mathrm{i} \otimes e_j - 1 \otimes e_{m+j} \\ &= 1 \otimes e_{m+j} + \mathrm{i} \otimes e_j - \mathrm{i} \otimes e_j - 1 \otimes e_{m+j} = 0, \end{aligned}$$

and so $\frac{1}{2}(1 \otimes e_j - \mathrm{i} \otimes e_{m+j}) \in \ker(J_{\mathbb{C}} - \mathrm{i}\,\mathrm{id}_{\mathsf{V}_{\mathbb{C}}}) = \mathsf{V}^{1,0}$. That the stated vector form a $\mathbb{C}$-basis for $\mathsf{V}^{1,0}$ follows from a dimension count.

(iii) This is a similar computation to the preceding. ■

Note that if $(e_1, \dots, e_m, e_{m+1}, \dots, e_{2m})$ is a J-adapted $\mathbb{R}$-basis for the vector space V with linear complex structure J, then $(\sigma_+(e_1), \dots, \sigma_+(e_m))$ and $(\sigma_-(e_1), \dots, \sigma_-(e_m))$ are $\mathbb{C}$-bases for $\mathsf{V}^{1,0}$ and $\mathsf{V}^{0,1}$, respectively.

4.1.2 Determinants of $\mathbb{C}$-linear maps

In the preceding section we saw that $\mathbb{C}$-vector spaces are realised as $\mathbb{R}$-vector spaces with a certain $\mathbb{R}$-endomorphism. We also saw in Proposition 4.1.6 that this real structure allows us to characterise $\mathbb{C}$-linear maps. Since a $\mathbb{C}$-linear map is also $\mathbb{R}$-linear, a $\mathbb{C}$-linear endomorphism has a real and complex determinant. In this section we establish the relationship between these two determinants. This will be useful to us in Section 4.1.6 when we look at orientations on $\mathbb{R}$-vector spaces with linear complex structures.

We first establish an interesting result of Silvester [2000]. To do so, let us set us the appropriate framework. We let F be a field and $\mathsf{F}^{r\times s}$ be the set of $r \times s$ matrices with values in F. Let $m, n \in \mathbb{Z}_{>0}$. An **(m, n)** ***block matrix*** is an element of $\mathbb{F}^{mn\times mn}$ for which we recognise the following block structure:

$$A = \begin{bmatrix} a_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \tag{4.1}$$

where $a_{jk} \in \mathbb{F}^{m\times m}$, $j,k \in \{1, \dots, n\}$. Now let $\mathsf{R} \subseteq \mathsf{F}^{m\times m}$ be a subring of $\mathsf{F}^{m\times m}$ and denote by $\mathsf{R}(n)$ the subset of (m, n) block matrices of the form (4.1) for which $a_{jk} \in \mathsf{R}$, $j,k \in \{1, \dots, n\}$. Note that elements of $\mathsf{R}(n)$ can be regarded naturally as elements of the set $\mathsf{R}^{n\times n}$, the set of $n \times n$ matrices with elements in R. Thus we have a bijection $\iota_k \colon \mathsf{R}(n) \to \mathsf{R}^{n\times n}$. We also have a few determinant functions floating around, and let us give distinct notation for these. For $k \in \mathbb{Z}_{>0}$ we denote by $\det^k_{\mathsf{F}} \colon \mathsf{F}^{k\times k} \to \mathsf{F}$ and $\det^k_{\mathsf{R}} \colon \mathsf{R}^{k\times k} \to \mathsf{R}$ the usual determinant functions. With this notation, we have the following result.

4.1.8 Lemma (Determinants for block matrices) *Let F be a field, let $\mathrm{m}, \mathrm{n} \in \mathbb{Z}_{>0}$, and let $\mathsf{R} \subseteq \mathbb{F}^{\mathrm{m}\times\mathrm{m}}$ be a commutative subring of matrices. We then have*

$$\det{}^{\mathrm{mn}}_{\mathsf{F}}(\mathbf{A}) = \det{}^{\mathrm{m}}_{\mathsf{F}}(\det{}^{\mathrm{n}}_{\mathsf{R}}(\iota_{\mathrm{n}}(\mathbf{A})))$$

for every $\mathbf{A} \in \mathsf{R}(\mathrm{n})$.

Proof The proof is by induction on n. For $n = 1$ the result is clear. Now let $A \in \mathsf{R}(n)$ for $n \geq 2$. Let us write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

for $a_{11} \in \mathsf{R}^{(n-1)\times(n-1)}$, $a_{12} \in \mathsf{R}^{(n-1)\times 1}$, $a_{21} \in \mathsf{R}^{1\times(n-1)}$, and $a_{22} \in \mathsf{R}$. A direct computation, using the fact that R is commutative, gives

$$A \begin{bmatrix} a_{22} I_{n-1} & \mathbf{0}_{(n-1)\times 1} \\ -a_{21} & 1_{\mathsf{R}} \end{bmatrix} = \begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{12} \\ \mathbf{0}_{1\times(n-1)} & a_{22} \end{bmatrix}.$$

We then have

$$(\det{}^n_{\mathsf{R}} A) a_{22}^{n-1} = (\det{}^{n-1}_{\mathsf{R}}(a_{22}a_{11} - a_{12}a_{21})) a_{22}$$

which gives

$$\det{}^m_{\mathsf{F}}(\det{}^n_{\mathsf{R}} A) \det{}^m_{\mathsf{F}}(a_{22})^{n-1} = \det{}^m_{\mathsf{F}}(\det{}^{n-1}_{\mathsf{R}}(a_{22}a_{11} - a_{12}a_{21})) \det{}^m_{\mathsf{F}}(a_{22}). \tag{4.2}$$

We also have

$$(\det{}^{mn}_{\mathsf{F}} A)(\det{}^m_{\mathsf{F}}(a_{22}))^{n-1} = (\det{}^{m(n-1)}_{\mathsf{F}}(a_{22}a_{11} - a_{12}a_{21})) \det{}^m_{\mathsf{F}}(a_{22}). \tag{4.3}$$

By the induction hypothesis,

$$\det{}^{m(n-1)}_{\mathsf{F}}(a_{22}\boldsymbol{a}_{11} - \boldsymbol{a}_{12}\boldsymbol{a}_{21}) = \det{}^{m}_{\mathsf{F}}(\det{}^{n-1}_{\mathsf{R}}(a_{22}\boldsymbol{a}_{11} - \boldsymbol{a}_{12}\boldsymbol{a}_{21})). \tag{4.4}$$

Combining equations (4.2)–(4.4) gives

$$(\det{}^{mn}_{\mathsf{F}}\boldsymbol{A} - \det{}^{m}_{\mathsf{F}}(\det{}^{n}_{\mathsf{R}}\boldsymbol{A}))(\det{}^{m}_{\mathsf{F}}(a_{22}))^{n-1} = 0_{\mathsf{F}}.$$

If $\det{}^{m}_{\mathsf{F}}(a_{22}) \neq 0_{\mathsf{F}}$ then the lemma follows. Otherwise, we make a small modification to the computations above by defining

$$\boldsymbol{A}_{\xi} = \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \xi\boldsymbol{I}_m + a_{22} \end{bmatrix}$$

for $\xi \in \mathbb{F}$. We think of this as being a matrix with entries in the polynomial ring $\mathsf{F}[\xi]$ or as a block matrix with blocks in the polynomial ring $\mathsf{R}[\xi]$. Upon doing so, the computations above may be carried out in the same way to give

$$(\det{}^{mn}_{\mathsf{F}[\xi]}\boldsymbol{A}_{\xi} - \det{}^{m}_{\mathsf{F}[\xi]}(\det{}^{n}_{\mathsf{R}[\xi]}\boldsymbol{A}_{\xi}))(\det{}^{m}_{\mathsf{F}[\xi]}(\xi\boldsymbol{I}_m + a_{22}))^{n-1} = 0_{\mathsf{F}[\xi]}.$$

Note that $(\det{}^{m}_{\mathsf{F}[\xi]}(\xi\boldsymbol{I}_m + a_{22}))^{n-1}$ is a monic polynomial and so we conclude that

$$\det{}^{mn}_{\mathsf{F}[\xi]}\boldsymbol{A}_{\xi} - \det{}^{m}_{\mathsf{F}[\xi]}(\det{}^{n}_{\mathsf{R}[\xi]}\boldsymbol{A}_{\xi}) = 0_{\mathsf{F}[\xi]}.$$

Evaluating this polynomial at $\xi = 0$ gives the result. ■

With this result we have the following.

4.1.9 Proposition (Real and complex determinants) *If* V *is a* $\mathbb{R}$*-vector space with a linear complex structure* J *and if* $\mathsf{A} \in \mathrm{End}_{\mathbb{C}}(\mathsf{V};\mathsf{V})$*, then* $\det_{\mathbb{R}}\mathsf{A} = |\det_{\mathbb{C}}\mathsf{A}|^2$.

Proof Let $(e_1,\dots,e_m,e_{m+1},\dots,e_{2m})$ is a J-adapted (real) basis for V, then $(e_1,\dots,e_m)$ is a $\mathbb{C}$-basis. We let $\boldsymbol{A}_{\mathbb{R}}$ be the real matrix representative of A with respect to the $\mathbb{R}$-basis $(e_1,e_{m+1},\dots,e_m,e_{2m})$. We also denote by $\boldsymbol{A}_{\mathbb{C}}$ the complex matrix representative of A with respect to the $\mathbb{C}$-basis $(e_1,\dots,e_m)$. We let $\mathsf{R} \subseteq \mathbb{R}^{2m\times 2m}$ denote the subring of matrices having the block form

$$\begin{bmatrix} \boldsymbol{a}_{11} & \cdots & \boldsymbol{a}_{1m} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_{m1} & \cdots & \boldsymbol{a}_{mm} \end{bmatrix},$$

where each of the 2×2 matrices $\boldsymbol{a}_{jk}$, $j,k \in \{1,\dots,m\}$, has the form

$$\boldsymbol{a}_z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

for $x,y \in \mathbb{R}$; associated to each such matrix is the complex number $z = x + \mathrm{i}y$. Obviously,

$$\det{}_{\mathbb{R}}\boldsymbol{a}_z = |z|^2. \tag{4.5}$$

One directly verifies that the map $z \mapsto \boldsymbol{a}_z$ is a ring isomorphism. Since complex multiplication is commutative, it follows that R is a commutative subring of $\mathbb{R}^{n\times n}$. Also note that $\boldsymbol{A}_{\mathbb{R}} \in \mathsf{R}$. Since $z \mapsto \boldsymbol{a}_z$ is a ring isomorphism, $\det_{\mathsf{R}}\boldsymbol{A}_{\mathbb{R}} = \boldsymbol{a}_{\det \boldsymbol{A}_{\mathbb{C}}}$. By Lemma 4.1.8 and (4.5) we have

$$\det \boldsymbol{A}_{\mathbb{R}} = \det{}_{\mathbb{R}} \det{}_{\mathsf{R}}\boldsymbol{A}_{\mathbb{R}} = |\det \boldsymbol{A}_{\mathbb{C}}|^2,$$

which is the result. ■

4.1.3 Duality and linear complex structures

Next we study dual spaces of $\mathbb{R}$-vector spaces with linear complex structures. Since we will consider both $\mathbb{R}$- and $\mathbb{C}$-vector space structures, we need to be careful with notation. Indeed, there are many ways to represent the dual of a complexification. . . or is it the complexification of a dual. . . By $(\mathsf{V}^*)_{\mathbb{C}}$ we denote the complexification of the $\mathbb{R}$-vector space V^*, i.e.,

$$(\mathsf{V}^*)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathsf{V}^*.$$

By $(\mathsf{V}_{\mathbb{C}})^*$ we denote the complex dual of $\mathsf{V}_{\mathbb{C}}$, i.e.,

$$(\mathsf{V}_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(\mathsf{V}_{\mathbb{C}};\mathbb{C}).$$

We also note that the set $\mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$ has a natural $\mathbb{C}$-vector space structure with scalar multiplication given by

$$(a\alpha)(v) = a(\alpha(v)), \qquad a \in \mathbb{C},\ \alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C}),\ v \in \mathsf{V}.$$

With this structure in mind, we have the following result.

4.1.10 Lemma (Complexification and duality) *For a $\mathbb{R}$-vector space V we have natural $\mathbb{R}$-vector space isomorphisms*

$$(\mathsf{V}^*)_{\mathbb{C}} \simeq (\mathsf{V}_{\mathbb{C}})^* \simeq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C}).$$

Proof We can write $\alpha \in (\mathsf{V}^*)_{\mathbb{C}}$ as $\alpha = 1 \otimes \alpha_1 + \mathrm{i} \otimes \alpha_2$ for $\alpha_1, \alpha_2 \in \mathsf{V}^*$. We then have the isomorphism

$$(\mathsf{V}^*)_{\mathbb{C}} \ni \alpha_1 + \mathrm{i}\alpha_2 \mapsto \alpha_1 + \mathrm{i}\alpha_2 \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C}).$$

The isomorphism from $\mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$ to $(\mathsf{V}_{\mathbb{C}})^*$ is given by assigning to $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$ the element $\bar{\alpha} \in (\mathsf{V}_{\mathbb{C}})^*$ defined by

$$\bar{\alpha}(a \otimes v) = a\alpha(v).$$

We leave to the reader the mundane chore of checking that these are well-defined isomorphisms of $\mathbb{R}$-vector spaces. ■

Because of the lemma we will simply write $\mathsf{V}^*_{\mathbb{C}}$ in place of either $(\mathsf{V}_{\mathbb{C}})^*$, $(\mathsf{V}^*)_{\mathbb{C}}$, or $\mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$. We shall most frequently think of $\mathsf{V}^*_{\mathbb{C}}$ as either $\mathbb{C} \otimes_{\mathbb{R}} \mathsf{V}^*$ or $\mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$. In the former situation, an element of $\mathsf{V}^*_{\mathbb{C}}$ is written as $\alpha = 1 \otimes \alpha_1 + \mathrm{i} \otimes \alpha_2$ and, in the latter, an element of $\mathsf{V}^*_{\mathbb{C}}$ is written as $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. These two representations are unambiguously related of course. But the reader should be warned that we shall use both, on occasion.

With respect to this notation, we have the following result.

4.1.11 Proposition (Linear complex structures and duality) *Let V be a finite-dimensional $\mathbb{R}$-vector space with a linear complex structure J. Then*

(i) J^* is a linear complex structure on V^* and,

(ii) with respect to the linear complex structure J^**, we have isomorphisms*

$$(\mathsf{V}^*)^{1,0} = \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \alpha(v) = 0 \text{ for every } v \in \mathsf{V}^{0,1}\} \simeq (\mathsf{V}^{1,0})^*$$

and

$$(\mathsf{V}^*)^{0,1} = \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \alpha(v) = 0 \text{ for every } v \in \mathsf{V}^{1,0}\} \simeq (\mathsf{V}^{0,1})^*;$$

(iii) thinking of $\mathsf{V}^*_{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$*,*

$$(\mathsf{V}^*)^{1,0} = \mathrm{Hom}_{\mathbb{C}}(\mathsf{V};\mathbb{C});$$

(iv) thinking of $\mathsf{V}^*_{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$*,*

$$(\mathsf{V}^*)^{0,1} = \mathrm{Hom}_{\mathbb{C}}(\overline{\mathsf{V}};\mathbb{C}).$$

Proof (i) We have, for $\alpha \in \mathsf{V}^*$ and $v \in \mathsf{V}$,

$$\langle J^* \circ J^*(\alpha); v\rangle = \langle J^*(\alpha); J(v)\rangle = \langle \alpha; J \circ J(v)\rangle = \langle -\alpha; v\rangle,$$

and so $J^* \circ J^*(\alpha) = -\alpha$, as desired.

(ii) We have

$$\begin{aligned}(\mathsf{V}^*)^{1,0} &= \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid J^*(\alpha) = \mathrm{i}\alpha\}\\ &= \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \langle J^*(\alpha); v\rangle = \langle \mathrm{i}\alpha; v\rangle \text{ for all } v \in \mathsf{V}\}\\ &= \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \langle \alpha; (J - \mathrm{i}\,\mathrm{id}_{\mathsf{V}_{\mathbb{C}}})(v)\rangle = 0 \text{ for all } v \in \mathsf{V}\}\\ &= \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \langle \alpha; v\rangle = 0 \text{ for all } v \in \mathsf{V}^{0,1}\},\end{aligned}$$

using the fact that $\mathrm{image}(J - \mathrm{i}\,\mathrm{id}_{\mathsf{V}_{\mathbb{C}}}) = \mathsf{V}^{0,1}$. One similarly proves that

$$(\mathsf{V}^*)^{0,1} = \{\alpha \in \mathsf{V}^*_{\mathbb{C}} \mid \langle \alpha; v\rangle = 0 \text{ for all } v \in \mathsf{V}^{1,0}\}.$$

The fact that $(\mathsf{V}^*)^{1,0} \simeq (\mathsf{V}^{0,1})^*$ and $(\mathsf{V}^*)^{0,1} \simeq (\mathsf{V}^{1,0})^*$ follows from the following fact whose easy proof we leave to the reader: If $\mathsf{U} = \mathsf{V} \oplus \mathsf{W}$, then

$$\mathsf{U}^* = \mathrm{ann}(\mathsf{W}) \oplus \mathrm{ann}(\mathsf{V}) \simeq \mathsf{V}^* \oplus \mathsf{W}^*.$$

(iii) We need to show that $\alpha \in (\mathsf{V}^*)^{1,0} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$ if and only if $\alpha(\mathrm{i}v) = \mathrm{i}\alpha(v)$ for every $v \in \mathsf{V}$. Suppose first that $\alpha \in (\mathsf{V}^*)^{1,0}$. Then

$$\langle \alpha; \mathrm{i}v\rangle = \langle \alpha; J(v)\rangle = \langle J^*(\alpha); v\rangle = \langle \mathrm{i}\alpha; v\rangle,$$

or, in different notation $\alpha(\mathrm{i}v) = \mathrm{i}\alpha(v)$, this holding for every $v \in \mathsf{V}$. Reversing the argument gives $\alpha \in (\mathsf{V}^*)^{1,0}$ if $\alpha(\mathrm{i}v) = \mathrm{i}\alpha(v)$ for every $v \in \mathsf{V}$.

(iv) As in the preceding part of the proof, we must show that $\alpha \in (\mathsf{V}^*)^{0,1} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{C})$ if and only if $\alpha(-\mathrm{i}v) = \mathrm{i}\alpha(v)$ for every $v \in \mathsf{V}$. And, still along the lines of the preceding part of the proof, this follows from the computation

$$\langle \alpha; -\mathrm{i}v\rangle = \langle \alpha; -J(v)\rangle = \langle -J^*(\alpha); v\rangle = \langle \mathrm{i}\alpha; v\rangle,$$

this holding if and only if $\alpha \in (\mathsf{V}^*)^{0,1}$. ■

Let us look at basis representations for duals.

4.1.12 Proposition (Dual basis representations for linear complex structures) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space with a linear complex structure* J*. Let* $(\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{m}, \mathrm{e}_{\mathrm{m}+1}, \mathrm{e}_{2\mathrm{m}})$ *be a* J*-adapted* $\mathbb{R}$*-basis for* V *with dual basis* $(\beta^1, \dots, \beta^\mathrm{m}, \beta^{\mathrm{m}+1}, \dots, \beta^{2\mathrm{m}})$*. Then the following statements hold:*

(i) the vectors $1 \otimes \beta^\mathrm{j} + \mathrm{i} \otimes \beta^{\mathrm{m}+\mathrm{j}}$, $\mathrm{j} \in \{1, \dots, \mathrm{m}\}$*, form a* $\mathbb{C}$*-basis for* $(\mathsf{V}^*)^{1,0}$*;*

(ii) the vectors $1 \otimes \beta^\mathrm{j} - \mathrm{i} \otimes \beta^{\mathrm{m}+\mathrm{j}}$, $\mathrm{j} \in \{1, \dots, \mathrm{m}\}$*, form a* $\mathbb{C}$*-basis for* $(\mathsf{V}^*)^{0,1}$*.*

Proof (i) Here we note that

$$\langle 1 \otimes \beta^j + \mathrm{i} \otimes \beta^{m+j}; 1 \otimes e_k + \mathrm{i} \otimes e_{m+k} \rangle = 0$$

for every $j, k \in \{1, \dots, m\}$. Thus the vectors $1 \otimes \beta^j + \mathrm{i} \otimes \beta^{m+j}$ are a $\mathbb{C}$-basis for the annihilator of $\mathsf{V}^{0,1}$, and the result follows from Proposition 4.1.11(ii).

(ii) This follows similarly to the preceding part of the proof. ■

4.1.4 Exterior algebra on vector spaces with linear complex structures

In Section F.3 we define the algebras $\bigwedge(\mathsf{V})$ and $\mathrm{T}\bigwedge(\mathsf{V})$ for a vector space V over an arbitrary field. These algebras are, in fact, isomorphic and the natural products on each space are in correspondence with one another by Corollary F.3.15. For this reason, we shall use the notation $\bigwedge(\mathsf{V})$, even if we think of the elements as being alternating tensors. We shall also denote the product by "$\wedge$."

Now let V be a finite-dimensional $\mathbb{R}$-vector space with linear complex structure J. Motivated by our constructions with duals from the preceding section, we denote

$$\textstyle\bigwedge^m(\mathsf{V}^*_\mathbb{C}) = \mathbb{C} \otimes_\mathbb{R} \bigwedge^m(\mathsf{V}^*).$$

Thus an element $\omega \in \bigwedge^m(\mathsf{V}^*_\mathbb{C})$ can be written as $\omega = \omega_1 + \mathrm{i}\omega_2$ for $\omega_1, \omega_2 \in \bigwedge^m(\mathsf{V}^*)$. More or less exactly as with duals in the preceding section, cf. Lemma 4.1.10, we have alternative characterisations of $\bigwedge^m(\mathsf{V}^*_\mathbb{C})$ as (1) the set of $\mathbb{R}$-multilinear alternating maps from V to $\mathbb{C}$ and (2) the $\mathbb{C}$-multilinear maps on the $\mathbb{C}$-vector space $\mathsf{V}_\mathbb{C}$. For concreteness and future reference, let us indicate how $\omega = 1 \otimes \omega_1 + \mathrm{i} \otimes \omega_2 \in \bigwedge^2(\mathsf{V}^*_\mathbb{C})$ is to be regarded as a $\mathbb{C}$-multlinear map on $\mathsf{V}_\mathbb{C}$. To do so, let $1 \otimes u_1 + \mathrm{i} \otimes u_2, 1 \otimes v_1 + \mathrm{i} \otimes v_2 \in \mathsf{V}_\mathbb{C}$ and note that

$$\begin{aligned}(1 \otimes \omega_1 + \mathrm{i} \otimes \omega_2)(1 \otimes u_1 + \mathrm{i} \otimes u_2, 1 \otimes v_1 + \mathrm{i} \otimes v_2) &= (\omega_1(u_1, v_1) - \omega_1(u_2, v_2)) \\ - (\omega_2(u_1, v_2) + \omega_2(u_2, v_1)) + \mathrm{i}((\omega_2(u_1, v_1) - \omega_2(u_2, v_2)) &+ (\omega_1(u_1, v_2) + \omega_1(u_2, v_1))), \quad (4.6)\end{aligned}$$

using $\mathbb{R}$-multilinearity and the definition of tensor product.

If $\omega \in \bigwedge^m(\mathsf{V}^*_\mathbb{C})$ we define $\bar{\omega} \in \bigwedge^m(\mathsf{V}^*_\mathbb{C})$ by

$$\bar{\omega}(v_1, \dots, v_m) = \overline{\omega(\bar{v}_1, \dots, \bar{v}_m)},$$

where we think of ω as a $\mathbb{C}$-multilinear map on $\mathsf{V}_\mathbb{C}$. An element $\omega \in \bigwedge^m(\mathsf{V}^*_\mathbb{C})$ is ***real*** if $\bar{\omega} = \omega$. We have the following characterisation of real forms.

4.1.13 Lemma (Characterisation of real exterior forms) *An exterior form $\omega \in \bigwedge^{\mathrm{m}}(\mathsf{V}^*_{\mathbb{C}})$ is real if and only if $\omega = 1 \otimes \omega'$ for some $\omega' \in \bigwedge^{\mathrm{m}}(\mathsf{V}^*)$.*

Proof Suppose that $\omega = 1 \otimes \omega_1 + \mathrm{i} \otimes \omega_2$ is real. Then, for any $v_1, \dots, v_m \in \mathsf{V}$, we have

$$\begin{aligned} & \bar{\omega} = \omega, \\ \Longrightarrow \quad & \bar{\omega}(1 \otimes v_1, \dots, 1 \otimes v_m) = \omega(1 \otimes v_1, \dots, 1 \otimes v_m), \\ \Longrightarrow \quad & \omega_1(v_1, \dots, v_m) - \mathrm{i}\omega_2(v_1, \dots, v_m) = \omega_1(v_1, \dots, v_m) + \mathrm{i}\omega_2(v_1, \dots, v_m). \end{aligned}$$

As this must hold for all $v_1, \dots, v_m \in \mathsf{V}$, we have $\omega_2 = 0$.

For the converse, suppose that $\omega = 1 \otimes \omega'$ for $\omega \in \bigwedge^m(\mathsf{V}^*)$. By $\mathbb{C}$-multilinearity of ω we have

$$\omega(a_1 \otimes v_1, \dots, a_m \otimes v_m) = a_1 \cdots a_m \omega'(v_1, \dots, v_m)$$

from which we immediately deduce that

$$\bar{\omega}(a_1 \otimes v_1, \dots, a_m \otimes v_m) = \overline{\bar{a}_1 \cdots \bar{a}_m} \omega'(v_1, \dots, v_m) = \omega(a_1 \otimes v_1, \dots, a_m \otimes v_m).$$

Universality of the tensor product gives the result. ■

We are interested in distinguished spaces of alternating tensors that are adapted to the linear complex structure.

4.1.14 Definition (Alternating tensors of bidegree (k, l)) Let V be a $\mathbb{R}$-vector space with linear complex structure J and let $k, l, m \in \mathbb{Z}_{\geq 0}$ satisfy $m = k + l$. An alternating tensor $\omega \in \bigwedge^m(\mathsf{V}^*_{\mathbb{C}})$ has ***bidegree* (k, l)** if

$$\omega(av_1, \dots, av_m) = a^k \bar{a}^l \omega(v_1, \dots, v_m)$$

for all $a \in \mathbb{C}$ and $v_1, \dots, v_m \in \mathsf{V}$. The set of alternating tensors with bidegree (k, l) is denoted by $\bigwedge^{k,l}(\mathsf{V}^*_{\mathbb{C}})$. By convention, $\bigwedge^{0,0}(\mathsf{V}^*_{\mathbb{C}}) = \mathbb{C}$. •

Let us state a few basic properties of such forms.

4.1.15 Proposition (Properties of alternating forms with bidegree) *Let V be a $\mathbb{R}$-vector space with linear complex structure J and let $\mathrm{k}, \mathrm{l}, \mathrm{k}', \mathrm{l}', \mathrm{m}, \mathrm{m}' \in \mathbb{Z}_{\geq 0}$ satisfy $\mathrm{m} = \mathrm{k} + \mathrm{l}$ and $\mathrm{m}' = \mathrm{k}' + \mathrm{l}'$. Then the following statements hold:*

(i) $\bigwedge^{\mathrm{k,l}}(\mathsf{V}^*_{\mathbb{C}}) \cap \bigwedge^{\mathrm{k',l'}}(\mathsf{V}^*_{\mathbb{C}}) = \{0\}$ *unless* $\mathrm{k} = \mathrm{k}'$ *and* $\mathrm{l} = \mathrm{l}'$;

(ii) $\bigwedge^{\mathrm{k,l}}(\mathsf{V}^*_{\mathbb{C}})$ *is a* $\mathbb{C}$*-subspace of* $\bigwedge^{\mathrm{m}}(\mathsf{V}^*_{\mathbb{C}})$;

(iii) *if* $\omega \in \bigwedge^{\mathrm{k,l}}(\mathsf{V}^*_{\mathbb{C}})$ *then* $\overline{\omega} \in \bigwedge^{\mathrm{l,k}}(\mathsf{V}^*_{\mathbb{C}})$;

(iv) *if* $\omega \in \bigwedge^{\mathrm{k,l}}(\mathsf{V}^*_{\mathbb{C}})$ *and* $\omega' \in \bigwedge^{\mathrm{k',l'}}(\mathsf{V}^*_{\mathbb{C}})$, *then* $\omega \wedge \omega' \in \bigwedge^{\mathrm{k+k',l+l'}}(\mathsf{V}^*_{\mathbb{C}})$;

(v) $\bigwedge^{\mathrm{m}}(\mathsf{V}^*_{\mathbb{C}}) = \bigoplus_{\substack{\mathrm{k,l} \\ \mathrm{k+l=m}}} \bigwedge^{\mathrm{k,l}}(\mathsf{V}^*_{\mathbb{C}})$.

Proof (i) We can obviously suppose that $k+l = k'+l'$. Suppose that $\omega \in \bigwedge^{k,l}(\mathsf{V}^*_{\mathbb{C}}) \cap \bigwedge^{k',l'}(\mathsf{V}^*_{\mathbb{C}})$ is nonzero and let $v_1, \dots, v_m \in \mathsf{V}$ be such that $\omega(v_1, \dots, v_m) \neq 0$. We then have

$$a^k \bar{a}^l \omega(v_1, \dots, v_m) = a^{k'} \bar{a}^{l'} \omega(v_1, \dots, v_m)$$

for every $a \in \mathbb{C}$. Taking $a = \mathrm{e}^{\mathrm{i}\theta}$ for $\theta \in \mathbb{R}$, we must have $\mathrm{e}^{\mathrm{i}\theta(k-l)} = \mathrm{e}^{\mathrm{i}\theta(k'-l')}$. This implies that $k - l - (k' - l')$ is an integer multiple of 2π, and so must be zero.

Proofs of parts (ii), (iii), and (iv) consist of simple verifications.

(v) By part (i) it suffices to show that if $\omega \in \bigwedge^m(\mathsf{V}^*_{\mathbb{C}})$ then we can write

$$\omega = \sum_{\substack{k,l \\ k+l=m}} \omega^{k,l}$$

for some $\omega^{k,l} \in \bigwedge^{k,l}(\mathsf{V}^*_{\mathbb{C}})$. This we show using a basis for V. Thus we let $(e_1, \dots, e_n, e_{n+1}, \dots, e_{2n})$ be a J-adapted $\mathbb{R}$-basis for V with dual basis $(\beta^1, \dots, \beta^n, \beta^{n+1}, \dots, \beta^{2n})$. By Propositions 4.1.12 and F.3.5, the alternating forms

$$\begin{aligned}(1 \otimes \beta^{a_1} + \mathrm{i} \otimes \beta^{n+a_1}) \wedge \dots \wedge (1 \otimes \beta^{a_k} + \mathrm{i} \otimes \beta^{n+a_k})& \\ \wedge (1 \otimes \beta^{b_1} - \mathrm{i} \otimes \beta^{n+b_1}) \wedge \dots \wedge (1 \otimes \beta^{b_l} - \mathrm{i} \otimes \beta^{n+b_l}),& \\ 1 \leq a_1 < \dots < a_k \leq n,\ 1 \leq b_1 < \dots < b_l \leq n,\ k + l = m,&\end{aligned}$$

form a $\mathbb{R}$-basis for $\bigwedge^m(\mathsf{V}^*_{\mathbb{C}})$. Since the alternating forms in the preceding expression with k and l fixed have bidegree (k, l), the result follows. ■

An alternative and equivalent way to understand the subspaces $\bigwedge^{k,l}(\mathsf{V}^*_{\mathbb{C}})$ is by the formula

$$\textstyle\bigwedge^m(\mathsf{V}^*_{\mathbb{C}}) = \bigwedge^m((\mathsf{V}^*)^{1,0} \oplus (\mathsf{V}^*)^{0,1}) = \displaystyle\bigoplus_{\substack{k,l \\ k+l=m}} \textstyle\bigwedge^k((\mathsf{V}^*)^{1,0}) \otimes \bigwedge^l((\mathsf{V}^*)^{0,1}),$$

which is Lemma 1 from the proof of Proposition F.3.5. With this as backdrop, we can define

$$\textstyle\bigwedge^{k,l}(\mathsf{V}^*_{\mathbb{C}}) = \bigwedge^k((\mathsf{V}^*)^{1,0}) \otimes \bigwedge^l((\mathsf{V}^*)^{0,1}),$$

and this description is easily shown to be equivalent to the one we gave above; indeed, this is contained in the proof of the preceding proposition.

4.1.5 Hermitian forms and inner products

In this section we consider the structure of an inner product on a $\mathbb{C}$-vector space, or equivalently a $\mathbb{R}$-vector space with a linear complex structure. We shall adopt the usual terminology of referring to a symmetric real bilinear map as a bilinear form. In the complex case, the usual terminology is the following.

4.1.16 Definition (Hermitian form and Hermitian inner product) Let V be a finite-dimensional $\mathbb{C}$-vector space. A ***Hermitian form*** on V is a map $h\colon \mathsf{V}\times\mathsf{V}\to\mathbb{C}$ with the following properties:

(i) $h(v_2, v_1) = \overline{h(v_1, v_2)}$ for all $v_1, v_2 \in \mathsf{V}$;

(ii) $h(v_1 + v_2, u) = h(v_1, u) + h(v_2, u)$ for all $u, v_1, v_2 \in \mathsf{V}$;

(iii) $h(av_1, v_2) = a\,h(v_1, v_2)$ for all $v_1, v_2 \in \mathsf{V}$ and $a \in \mathbb{C}$.

A map $h\colon \mathsf{V}\times\mathsf{V}\to\mathbb{C}$ satisfying the above properties but with property (iii) replaced by

$$h(v_1, av_2) = a\,h(v_1, v_2), \qquad v_1, v_2 \in \mathsf{V},\ a \in \mathbb{C},$$

then h is a ***conjugate Hermitian form***.

If we have

(iv) $h(v, v) \geq 0$ for all $v \in \mathsf{V}$

then h is ***positive-semidefinite*** and if, additionally,

(v) $h(v, v) = 0$ then $v = 0$,

then h is a ***Hermitian inner product***. •

If h is a Hermitian form on a $\mathbb{C}$-vector space V, then we can define an associated conjugate Hermitian form $\bar{h}$ in the obvious way: $\bar{h}(u, v) = \overline{h(u, v)}$.

The following elementary result characterises Hermitian forms in a basis.

4.1.17 Lemma (Basis representations of Hermitian forms) *If* $(\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{n})$ *is a* $\mathbb{C}$*-basis for a* $\mathbb{C}$*-vector space* V*, then the following statements hold:*

(i) if h *is a Hermitian form on* V*, then the matrix* $\hat{\mathrm{h}} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ *defined by* $\mathrm{h}_{\mathrm{jk}} = \mathrm{h}(\mathrm{e}_\mathrm{j}, \mathrm{e}_\mathrm{k})$, $\mathrm{j}, \mathrm{k} \in \{1, \dots, \mathrm{n}\}$*, satisfies* $\bar{\mathbf{h}}^T = \mathbf{h}$*;*

(ii) conversely, if $\hat{\mathrm{h}} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ *satisfies* $\bar{\mathbf{h}}^T = \mathbf{h}$ *then the map* $\mathrm{h}\colon \mathsf{V}\times\mathsf{V}\to\mathbb{C}$ *defined by*

$$\mathrm{h}\Big(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{u}^\mathrm{j}\mathrm{e}_\mathrm{j}, \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{v}^\mathrm{j}\mathrm{e}_\mathrm{j}\Big) = \sum_{\mathrm{j},\mathrm{k}=1}^{\mathrm{n}} \mathrm{h}_{\mathrm{jk}}\mathrm{u}^\mathrm{j}\bar{\mathrm{v}}^\mathrm{k}$$

is a Hermitian form.

As a consequence of the lemma, let us introduce some notation. We let $(e_1, \dots, e_n)$ be a $\mathbb{C}$-basis for the $\mathbb{C}$-vector space V with dual basis $(\beta^1, \dots, \beta^n)$. For $j \in \{1, \dots, n\}$ define $\bar{\beta}^j \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}; \mathbb{C})$ by $\bar{\beta}^j(v) = \overline{\beta^j(v)}$. Note that $\bar{\beta}^j$ is antilinear and so is not an element of $\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}; \mathbb{C})$, but an element of $\mathrm{Hom}_{\mathbb{C}}(\overline{\mathsf{V}}; \mathbb{C})$. In any case, we can write

$$h = \sum_{j,k=1}^{n} h_{jk}\beta^j \otimes \bar{\beta}^k,$$

understanding that this means that

$$h(u,v) = \sum_{j,k=1}^{n} h_{jk}\beta^j(u)\bar{\beta}^k(v) = \sum_{j,k=1}^{n} h_{jk}u^j\bar{v}^k,$$

as desired.

The standard Gram–Schmidt procedure [Axler 1997, Theorem 6.20] shows that, given a Hermitian inner product h, there exists a $\mathbb{C}$-basis $(e_1,\dots,e_n)$ for a $\mathbb{C}$-vector space V for which

$$h(e_j,e_k) = \begin{cases} 1, & j=k, \\ 0, & j\neq k. \end{cases}$$

Such a basis is called ***orthonormal***.

As we saw in Section 4.1.1, there is a natural correspondence between $\mathbb{C}$-vector spaces and $\mathbb{R}$-vector spaces with linear complex structures. We shall study Hermitian forms in the context of a finite-dimensional $\mathbb{R}$-vector space V with linear complex structure J. In this case, a Hermitian form h, being $\mathbb{C}$-valued, can be written as

$$h(v_1,v_2) = g(v_1,v_2) - \mathrm{i}\omega(v_1,v_2),$$

where g and ω are $\mathbb{R}$-valued $\mathbb{R}$-bilinear maps on the $\mathbb{R}$-vector space V. (The minus sign is a convenient convention, as we shall see.) Let us examine the properties of g and ω.

4.1.18 Proposition (The real and imaginary parts of a Hermitian form) *Let V be a finite-dimensional $\mathbb{R}$-vector space with a linear almost complex structure J, and let $\mathrm{h} = \mathrm{g} - \mathrm{i}\omega$ be a Hermitian form on V. Then the following statements hold:*

- *(i) g is symmetric;*
- *(ii) ω is skew-symmetric;*
- *(iii) $\mathrm{g}(\mathrm{J}(\mathrm{v}_1),\mathrm{J}(\mathrm{v}_2)) = \mathrm{g}(\mathrm{v}_1,\mathrm{v}_2)$ for all $\mathrm{v}_1,\mathrm{v}_2\in\mathsf{V}$;*
- *(iv) $\omega(\mathrm{J}(\mathrm{v}_1),\mathrm{J}(\mathrm{v}_2)) = \omega(\mathrm{v}_1,\mathrm{v}_2)$ for all $\mathrm{v}_1,\mathrm{v}_2\in\mathsf{V}$;*
- *(v) $\omega(\mathrm{v}_1,\mathrm{v}_2) = \mathrm{g}(\mathrm{J}(\mathrm{v}_1),\mathrm{v}_2)$ for all $\mathrm{v}_1,\mathrm{v}_2\in\mathsf{V}$.*
- *(vi) $\mathrm{g}(\mathrm{v}_1,\mathrm{v}_2) = \omega(\mathrm{v}_1,\mathrm{J}(\mathrm{v}_2))$ for all $\mathrm{v}_1,\mathrm{v}_2\in\mathsf{V}$.*

Moreover, if $\mathbb{R}$-bilinear maps g and ω are given satisfying conditions (i)–(v), then the map $\mathrm{h}\colon \mathsf{V}\times\mathsf{V}\to\mathbb{C}$ defined by

$$\mathrm{h}(\mathrm{v}_1,\mathrm{v}_2) = \mathrm{g}(\mathrm{v}_1,\mathrm{v}_2) - \mathrm{i}\omega(\mathrm{v}_1,\mathrm{v}_2)$$

is a Hermitian form and is a Hermitian inner product if g is an inner product.

Proof (i) and (ii) For $v_1,v_2\in\mathsf{V}$ we have

$$g(v_2,v_1) - \mathrm{i}\omega(v_2,v_1) = h(v_2,v_1) = \overline{h(v_1,v_2)} = g(v_1,v_2) + \mathrm{i}\omega(v_1,v_2),$$

and the symmetry of g and skew-symmetry of ω follow by taking real and imaginary parts.

(iii) For $v \in \mathsf{V}$ we have

$$\begin{aligned} g(J(v), J(v)) &= g(J(v), J(v)) - \mathrm{i}\omega(J(v), J(v)) = h(J(v), J(v)) \\ &= h(\mathrm{i}v, \mathrm{i}v) = h(v, v) = g(v, v) - \mathrm{i}\omega(v, v) = g(v, v). \end{aligned}$$

Now, for $v_1, v_2 \in \mathsf{V}$ we have

$$\begin{aligned} g(J(v_1), J(v_2)) &= \tfrac{1}{2}g(J(v_1) + J(v_2), J(v_1) + J(v_2)) - \tfrac{1}{2}g(J(v_1), J(v_1)) - \tfrac{1}{2}g(J(v_2), J(v_2)) \\ &= \tfrac{1}{2}g(v_1 + v_2, v_1 + v_2) - \tfrac{1}{2}g(v_1, v_2) - \frac{1}{2}g(v_2, v_2) = g(v_1, v_2). \end{aligned}$$

(iv) We use part (v) proved below. Using this, we compute

$$\begin{aligned} \omega(J(v_1), J(v_2)) &= g(J^2(v_1), J(v_2) = -g(v_1, J(v_2)) \\ &= -g(J(v_2), v_1) = -\omega(v_2, v_1) = \omega(v_1, v_2). \end{aligned}$$

(v) and (vi) Here we have

$$h(\mathrm{i}u, v) = h(J(u), v) = g(J(u), v) - \mathrm{i}\omega(J(u), v)$$

Since h is Hermitian we have $h(\mathrm{i}u, v) = \mathrm{i}h(u, v)$ which gives

$$g(J(u), v) - \mathrm{i}\omega(J(u), v) = \mathrm{i}g(u, v) + \omega(u, v).$$

Matching real and imaginary parts gives $\omega(u, v) = g(J(u), v)$ and $g(u, v) = \omega(v_1, J(v_2))$, as desired.

For the final assertion, if is clear that h as defined is $\mathbb{R}$-bilinear, satisfies $h(v, u) = \overline{h(u, v)}$, and is positive-definite if g is positive-definite. To complete the proof it suffices to prove linearity with respect to scalar multiplication by i in the first entry. To this end we compute

$$\begin{aligned} h(\mathrm{i}u, v) &= h(J(u), v) = g(J(u), v) - \mathrm{i}\omega(J(u), v) \\ &= g(J \circ J(u), J(v)) + \mathrm{i}g(u, v) \\ &= \mathrm{i}(\mathrm{i}g(u, J(v)) + g(u, v)) \\ &= \mathrm{i}(g(u, v) + \mathrm{i}g(J(u), J \circ J(v))) \\ &= \mathrm{i}(g(u, v) - \mathrm{i}\omega(u, v) = \mathrm{i}h(u, v), \end{aligned}$$

as desired. ■

Motivated by the preceding result, we have the following definitions.

4.1.19 Definition (Compatible bilinear form, fundamental form) A real bilinear form g on a $\mathbb{R}$-vector space V with linear complex structure J is ***compatible*** with J if $g(J(v_1), J(v_2)) = g(v_1, v_2)$ for all $v_1, v_2 \in \mathsf{V}$. For a compatible bilinear form, the alternating two-form ω defined by $\omega(v_1, v_2) = g(J(v_1), v_2)$ is the ***fundamental form*** associated to g. •

Let us illustrate the preceding notions with a simple example.

4.1.20 Example ($\mathbb{C}^n$ as a Hermitian vector space) We consider the $\mathbb{C}$-vector space $\mathbb{C}^m$ with its standard linear complex structure as in Example 4.1.3–1. Denoting by $\mathbb{H}$ the standard Hermitian metric, we have

$$\mathbb{H}((x^1 + \mathrm{i}y^1, \dots, x^m + \mathrm{i}y^m), (u^1 + \mathrm{i}v^1, \dots, u^m + \mathrm{i}v^m)) = \sum_{j=1}^{m}(x^j u^j + y^j v^j) - \mathrm{i}\sum_{j=1}^{m}(x^j v^j - y^j u^j).$$

Thus, if we write $\mathbb{H} = \mathbb{G} - \mathrm{i}\Omega$ for a bilinear form $\mathbb{G}$ and an exterior two-form Ω, then

$$\mathbb{G} = \sum_{j=1}^{m}(\mathrm{d}x^j \otimes \mathrm{d}x^j + \mathrm{d}y^j \otimes \mathrm{d}y^j), \quad \Omega = \sum_{j=1}^{m} \mathrm{d}x^j \wedge \mathrm{d}y^j. \qquad \bullet$$

Note that the final assertion of the preceding result gives rise to a Hermitian form on V with respect to the $\mathbb{C}$-vector space structure associated to J. We also have the $\mathbb{C}$-vector space $\mathsf{V}_{\mathbb{C}}$, and we recall that we use by default the $\mathbb{C}$-vector space structure coming from the usual complexification, rather than from $J_{\mathbb{C}}$. With respect to this $\mathbb{C}$-vector space structure and a (real) bilinear form A on V, we can define a form $A_{\mathbb{C}}$ on $\mathsf{V}_{\mathbb{C}}$ by

$$A_{\mathbb{C}}(a \otimes u, b \otimes v) = a\bar{b}A(u, v). \tag{4.7}$$

Note that if A is symmetric, $A_{\mathbb{C}}$ is Hermitian.

The following result relates this complexified Hermitian form to the Hermitian form h constructed from g in Proposition 4.1.18.

4.1.21 Proposition (Properties of complexified forms) *Let V be a finite-dimensional $\mathbb{R}$-vector space with a linear complex structure J, compatible bilinear form g, and fundamental form ω. Then the following statements hold:*

(i) $\mathrm{g}_{\mathbb{C}}(\mathrm{u}, \mathrm{v}) = 0$ *for* $\mathrm{u} \in \mathsf{V}^{1,0}$ *and* $\mathrm{v} \in \mathsf{V}^{0,1}$*;*

(ii) $\sigma_+^*(\mathrm{g}_{\mathbb{C}}|\mathsf{V}^{1,0}) = \frac{1}{2}\mathrm{h}$*, where σ_+ is the isomorphism from Proposition 4.1.5(iv);*

(iii) $\sigma_-^*(\mathrm{g}_{\mathbb{C}}|\mathsf{V}^{0,1}) = \frac{1}{2}\overline{\mathrm{h}}$*, where σ_- is the isomorphism from Proposition 4.1.5(v);*

(iv) $\sigma_+^*(\omega_{\mathbb{C}}|\mathsf{V}^{1,0}) = \frac{\mathrm{i}}{2}\mathrm{h}$*, where σ_+ is the isomorphism from Proposition 4.1.5(iv);*

(v) $\sigma_-^*(\omega_{\mathbb{C}}|\mathsf{V}^{0,1}) = \overline{\frac{\mathrm{i}}{2}\mathrm{h}}$*, where σ_- is the isomorphism from Proposition 4.1.5(v);*

(vi) $1 \otimes \omega \in \bigwedge^2(\mathsf{V}_{\mathbb{C}}^*)$ *is real and of bidegree* $(1, 1)$*.*

Proof (i) By Proposition 4.1.5 we write elements of $\mathsf{V}^{1,0}$ and $\mathsf{V}^{0,1}$ as

$$1 \otimes u - \mathrm{i} \otimes J(u), \quad 1 \otimes v + \mathrm{i} \otimes J(v).$$

respectively, for $u, v \in \mathsf{V}$. A calculation, using compatibility of J and g, gives

$$\begin{aligned} g_{\mathbb{C}}(1 \otimes u &- \mathrm{i} \otimes J(u), 1 \otimes v + \mathrm{i} \otimes J(v)) \\ &= g_{\mathbb{C}}(1 \otimes u, 1 \otimes v) + g_{\mathbb{C}}(1 \otimes u, \mathrm{i} \otimes J(v)) - g_{\mathbb{C}}(\mathrm{i} \otimes J(u), 1 \otimes v) - g_{\mathbb{C}}(\mathrm{i} \otimes J(u), \mathrm{i} \otimes J(v)) \\ &= g(u, v) - \mathrm{i}g(u, J(v)) - \mathrm{i}g(J(u), v) - g(J(u), J(v)) = 0. \end{aligned}$$

(ii) As in the preceding part of the proof, this is a direct computation:

$$\begin{aligned}
g_{\mathbb{C}}(1\otimes u - \mathrm{i}\otimes J(u), 1\otimes v - \mathrm{i}\otimes J(v)) \\
&= g_{\mathbb{C}}(1\otimes u, 1\otimes v) - g_{\mathbb{C}}(1\otimes u, \mathrm{i}\otimes J(v)) - g_{\mathbb{C}}(\mathrm{i}\otimes J(u), 1\otimes v) + g_{\mathbb{C}}(\mathrm{i}\otimes J(u), \mathrm{i}\otimes J(v)) \\
&= g(u,v) + \mathrm{i}g(u,J(v)) - \mathrm{i}g(J(u),v) + g(J(u),J(v)) = 2h(u,v),
\end{aligned}$$

for $u, v \in \mathsf{V}$.

(iii) Here we compute

$$\begin{aligned}
g_{\mathbb{C}}(1\otimes u + \mathrm{i}\otimes J(u), 1\otimes v + \mathrm{i}\otimes J(v)) \\
&= g_{\mathbb{C}}(1\otimes u, 1\otimes v) + g_{\mathbb{C}}(1\otimes u, \mathrm{i}\otimes J(v)) + g_{\mathbb{C}}(\mathrm{i}\otimes J(u), 1\otimes v) + g_{\mathbb{C}}(\mathrm{i}\otimes J(u), \mathrm{i}\otimes J(v)) \\
&= g(u,v) - \mathrm{i}g(u,J(v)) + \mathrm{i}g(J(u),v) + g(J(u),J(v)) = 2\overline{h(u,v)},
\end{aligned}$$

for $u, v \in \mathsf{V}$.

(iv) Here we compute

$$\begin{aligned}
\omega_{\mathbb{C}}(1\otimes u - \mathrm{i}\otimes J(u), 1\otimes v - \mathrm{i}\otimes J(v)) \\
&= \omega_{\mathbb{C}}(1\otimes u, 1\otimes v) - \omega_{\mathbb{C}}(1\otimes u, \mathrm{i}\otimes J(v)) - \omega_{\mathbb{C}}(\mathrm{i}\otimes J(u), 1\otimes v) + \omega_{\mathbb{C}}(\mathrm{i}\otimes J(u), \mathrm{i}\otimes J(v)) \\
&= \omega(u,v) + \mathrm{i}\omega(u,J(v)) - \mathrm{i}\omega(J(u),v) + \omega(J(u),J(v)) = 2\mathrm{i}h(u,v),
\end{aligned}$$

for $u, v \in \mathsf{V}$.

(v) Here we compute

$$\begin{aligned}
\omega_{\mathbb{C}}(1\otimes u + \mathrm{i}\otimes J(u), 1\otimes v + \mathrm{i}\otimes J(v)) \\
&= \omega_{\mathbb{C}}(1\otimes u, 1\otimes v) + \omega_{\mathbb{C}}(1\otimes u, \mathrm{i}\otimes J(v)) + \omega_{\mathbb{C}}(\mathrm{i}\otimes J(u), 1\otimes v) + \omega_{\mathbb{C}}(\mathrm{i}\otimes J(u), \mathrm{i}\otimes J(v)) \\
&= \omega(u,v) - \mathrm{i}\omega(u,J(v)) + \mathrm{i}\omega(J(u),v) + \omega(J(u),J(v)) = \overline{2\mathrm{i}h(u,v)},
\end{aligned}$$

for $u, v \in \mathsf{V}$.

(vi) That $1\otimes\omega$ is real follows from Lemma 4.1.13. Let $a = a_1 + \mathrm{i}a_2 \in \mathbb{C}$ and calculate

$$\begin{aligned}
\omega((a_1+\mathrm{i}a_2)v_1, (a_1+\mathrm{i}a_2)v_2) &= \omega(a_1v_1, a_1v_2) + \omega(a_1v_1, a_2J(v_2)) \\
&\quad + \omega(a_2J(v_1), a_1v_2) + \omega(a_2J(v_1), a_2J(v_2)) \\
&= a_1^2\omega(v_1,v_2) + a_1a_2g(J(v_1), J(v_2)) \\
&\quad + a_1a_2g(J\circ J(v_1), v_2) + a_2^2g(J\circ J(v_1), J(v_2)) \\
&= (a_1^2 + a_2^2)\omega(v_1,v_2) = a\bar{a}\omega(v_1,v_2),
\end{aligned}$$

giving the result. ■

The following result establishes an important correspondence between Hermitian forms and their imaginary parts.

4.1.22 Proposition (Hermitian forms and real alternating forms of bidegree (1, 1)) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space with a linear complex structure* J*. For a Hermitian form* h *on* V *let* g *and* ω *be the real and imaginary parts of* h*, as above. Then the map* $\mathrm{h} \mapsto \omega$ *is an isomorphism between the* $\mathbb{R}$*-vector spaces of Hermitian forms and the real alternating forms of bidegree* $(1,1)$*.*

Proof The map $\phi\colon h \mapsto \omega$ is clearly $\mathbb{R}$-linear. To see that this map is injective, suppose that $\phi(h) = 0$. By Proposition 4.1.18(iv) it follows that the real part of h is also zero and so h is zero.

To prove surjectivity of ϕ, let ω be a real form of bidegree $(1,1)$. Define a map $g\colon \mathsf{V}\times\mathsf{V} \to \mathbb{R}$ by $g(v_1, v_2) = \omega(v_1, J(v_2))$. We claim that g is symmetric. Indeed,

$$\begin{aligned} g(v_2, v_1) = \omega(v_2, J(v_1)) = -\omega(J(v_1), v_2) = \mathrm{i}^2\omega(J(v_1), v_2) = -\mathrm{i}\bar{\mathrm{i}}\omega(J(v_1), v_2) \\ = -\omega(J^2(v_1), J(v_2)) = \omega(v_1, J(v_2)) = g(v_1, v_2), \end{aligned}$$

using the fact that ω has bidegree $(1,1)$. Now define $h\colon \mathsf{V}\times\mathsf{V} \to \mathbb{C}$ by

$$h(v_1, v_2) = g(v_1, v_2) - \mathrm{i}\omega(v_1, v_2).$$

We claim that h is Hermitian. Indeed, h is obviously $\mathbb{R}$-bilinear and satisfies $h(v_2, v_1) = \overline{h(v_1, v_2)}$. Moreover, we have

$$\begin{aligned} h(\mathrm{i}v_1, v_2) = g(J(v_1), v_2) - \mathrm{i}\omega(J(v_1), v_2) = \omega(J(v_1), J(v_2)) - \mathrm{i}\omega(J(v_1), v_2) \\ = -\mathrm{i}^2\omega(v_1, v_2) + \mathrm{i}\omega(v_2, J(v_1)) = \mathrm{i}(g(v_1, v_2) - \mathrm{i}\omega(v_1, v_2)) = \mathrm{i}h(v_1, v_2), \end{aligned}$$

as desired. ■

The correspondence between a Hermitian form and its imaginary part is often written as

$$h = -2\mathrm{i}\omega. \tag{4.8}$$

By Proposition 4.1.21(iv) this formula makes sense if ω is replaced with $\omega_{\mathbb{C}}$. By Proposition 4.1.23 below, particularly parts (i) and (iii), this formula makes sense for the components of h and ω with respect to appropriate bases. An heuristic verification of (4.8) can be given as follows:

$$h(u, v) = g(u, v) - \mathrm{i}\omega(u, v) = \omega(u, J(v)) - \mathrm{i}\omega(u, v) = \omega(u, \mathrm{i}v) - \mathrm{i}\omega(u, v) = -2\mathrm{i}\omega(u, v).$$

This computation stops short of making sense because the relation $\omega(u, \mathrm{i}v) = -\mathrm{i}\omega(u, v)$ does not make sense, unless ω is replaced with $\omega_{\mathbb{C}}$. In any case, the formula (4.8) is often used, but only makes sense upon interpretation.

Let us now give the basis representations for the various objects described above.

4.1.23 Proposition (Basis representations associated to Hermitian forms) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space with a linear complex structure* J*. Let* g *be a real bilinear form compatible with* J*, let* ω *be the fundamental form associated with* g*, and let* $\mathrm{h} = \mathrm{g} - \mathrm{i}\omega$ *be the associated Hermitian form. Let* $(\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{m}, \mathrm{e}_{\mathrm{m}+1}, \dots, \mathrm{e}_{2\mathrm{m}})$ *be a* J*-adapted* $\mathbb{R}$*-basis for* V *with dual basis* $(\beta^1, \dots, \beta^\mathrm{m}, \beta^{\mathrm{m}+1}, \dots, \beta^{2\mathrm{m}})$*, and define*

$$\mathrm{f}_\mathrm{j} = \tfrac{1}{2}(1\otimes\mathrm{e}_\mathrm{j} - \mathrm{i}\otimes\mathrm{e}_{\mathrm{m}+\mathrm{j}}), \quad \bar{\mathrm{f}}_\mathrm{j} = \tfrac{1}{2}(1\otimes\mathrm{e}_\mathrm{j} + \mathrm{i}\otimes\mathrm{e}_{\mathrm{m}+\mathrm{j}}), \qquad \mathrm{j}\in\{1,\dots,\mathrm{m}\},$$

and

$$\gamma^\mathrm{j} = 1\otimes\beta^\mathrm{j} + \mathrm{i}\otimes\beta^{\mathrm{m}+\mathrm{j}}, \quad \bar{\gamma}^\mathrm{j} = 1\otimes\beta^\mathrm{j} - \mathrm{i}\otimes\beta^{\mathrm{m}+\mathrm{j}}, \qquad \mathrm{j}\in\{1,\dots,\mathrm{m}\},$$

Define $\mathrm{h}_{\mathrm{jk}} \in \mathbb{C}$ *by* $\mathrm{h}_{\mathrm{jk}} = \mathrm{h}(\mathrm{e}_\mathrm{j}, \mathrm{e}_\mathrm{k})$*,* $\mathrm{j},\mathrm{k}\in\{1,\dots,\mathrm{m}\}$*. Then we have the following formulae:*

(i) $$\mathrm{h} = \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{h_{jk}}(\beta^{\mathrm{j}} \otimes \beta^{\mathrm{k}} + \beta^{\mathrm{m+j}} \otimes \beta^{\mathrm{m+k}}) - \mathrm{i} \sum_{\mathrm{j,k=1}}^{\mathrm{m}} (\mathrm{h_{jk}}\beta^{\mathrm{j}} \otimes \beta^{\mathrm{m+k}} - \bar{\mathrm{h}}_{\mathrm{jk}}\beta^{\mathrm{m+k}} \otimes \beta^{\mathrm{j}})$$
$$= \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{h_{jk}}\gamma^{\mathrm{j}} \otimes \bar{\gamma}^{\mathrm{k}};$$

(ii) $$\mathrm{g} = \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{Re}(\mathrm{h_{jk}})(\beta^{\mathrm{j}} \otimes \beta^{\mathrm{k}} + \beta^{\mathrm{m+j}} \otimes \beta^{\mathrm{m+k}}) + \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{Im}(\mathrm{h_{jk}})(\beta^{\mathrm{j}} \otimes \beta^{\mathrm{m+k}} + \beta^{\mathrm{m+k}} \otimes \beta^{\mathrm{j}})$$
$$= \frac{1}{2} \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{h_{jk}}(\gamma^{\mathrm{j}} \otimes \bar{\gamma}^{\mathrm{k}} + \bar{\gamma}^{\mathrm{k}} \otimes \gamma^{\mathrm{j}});$$

(iii) $$\omega = -\sum_{(\mathrm{j,k})=\{1,\dots,\mathrm{m}\}^2}' \mathrm{Im}(\mathrm{h_{jk}})(\beta^{\mathrm{j}} \wedge \beta^{\mathrm{k}} + \beta^{\mathrm{m+j}} \wedge \beta^{\mathrm{m+k}}) + \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{Re}(\mathrm{h_{jk}})\beta^{\mathrm{j}} \wedge \beta^{\mathrm{m+k}}$$
$$= -\frac{\mathrm{i}}{2} \sum_{\mathrm{j,k=1}}^{\mathrm{m}} \mathrm{h_{jk}}\gamma^{\mathrm{j}} \wedge \bar{\gamma}^{\mathrm{k}}.$$

Proof (i) We have $h(e_j, e_k) = h_{jk}$, $j, k \in \{1, \dots, m\}$, by definition. Since h is a Hermitian form on the $\mathbb{C}$-vector space V,

$$\begin{aligned} h(e_{m+j}, e_{m+k}) &= h(\mathrm{i}e_j, \mathrm{i}e_k) = h(e_j, e_k) = h_{jk}, \\ h(e_j, e_{m+k}) &= h(e_j, \mathrm{i}e_k) = -\mathrm{i}h(e_j, e_k) = -\mathrm{i}h_{jk}, \\ h(e_{m+k}, e_j) &= h(\mathrm{i}e_k, e_j) = \mathrm{i}h_{kj} = \mathrm{i}\bar{h}_{jk}. \end{aligned}$$

From these observations, the first formula in this part of the result follows. For the second, write $u, v \in$ V as

$$u = \sum_{j=1}^{m} u^j e_j, \quad v = \sum_{j=1}^{m} v^j e_j,$$

for $u^j, v^j \in \mathbb{C}$, $j \in \{1, \dots, m\}$. We then have

$$h(u, v) = \sum_{j,k=1}^{m} h_{jk} u^j \bar{v}^k.$$

Next we note that

$$\begin{aligned} \gamma^j(u) &= \gamma^j\Big(\sum_{j=1}^{m} u^j e_j\Big) = 1 \otimes \beta^j\Big(\sum_{j=1}^{m} u^j e_j\Big) + \mathrm{i} \otimes \beta^{m+j}\Big(\sum_{j=1}^{m} u^j e_j\Big) \\ &= 1 \otimes \beta^j\Big(\sum_{j=1}^{m}(\mathrm{Re}(u^j)e_j + \mathrm{Im}(u^j)e_{m+j})\Big) + \mathrm{i} \otimes \beta^{m+j}\Big(\sum_{j=1}^{m}(\mathrm{Re}(u^j)e_j + \mathrm{Im}(u^j)e_{m+j})\Big) \\ &= \mathrm{Re}(u_j) + \mathrm{i}\,\mathrm{Im}(u_j) = u_j \end{aligned}$$

and similarly $\bar{\gamma}^j(v) = \bar{v}^j$. We, therefore, have

$$\sum_{j,k=1}^{m} h_{jk}\gamma^j \otimes \bar{\gamma}^k(u,v) = \sum_{j,k=1}^{m} h_{jk}u^j\bar{v}^k.$$

This gives the second formula in this part of the result.

(ii) We have

$$\begin{aligned} g(e_j, e_k) &= \operatorname{Re}(h(e_j, e_k)), \\ g(e_{m+j}, e_{m+k}) &= g(J(e_j), J(e_k)) = g(e_j, e_k) = \operatorname{Re}(h(e_j, e_k)), \\ g(e_j, e_{m+k}) &= g(e_j, J(e_k)) = -\omega(e_j, e_k) = \operatorname{Im}(h(e_j, e_k)), \\ g(e_{m+k}, e_j) &= g(e_j, e_{m+k}) = \operatorname{Im}(h(e_j, e_k)), \end{aligned}$$

giving the first formula. For the second, we first write $u, v \in \mathsf{V}$ as

$$u = \sum_{j=1}^{m} u^j e_j, \quad v = \sum_{j=1}^{m} v^j e_j,$$

for $u^j, v^j \in \mathbb{C}$, $j \in \{1, \dots, m\}$. Then

$$g(u,v) = \operatorname{Re}(h(u,v)) = \frac{1}{2}(h(u,v) + \overline{h(u,v)}) = \frac{1}{2}\Big(\sum_{j,k=1}^{m} h_{jk}u^j\bar{v}^k + \bar{h}_{jk}\bar{u}^j v^k\Big).$$

From this we conclude that

$$\begin{aligned} g &= \frac{1}{2}\sum_{j,k=1}^{m}(h_{jk}\gamma^j \otimes \bar{\gamma}^k + \bar{h}_{jk}\bar{\gamma}^j \otimes \gamma^k) \\ &= \frac{1}{2}\sum_{j,k=1}^{m} h_{jk}(\gamma^j \otimes \bar{\gamma}^k + \bar{\gamma}^k \otimes \gamma^j), \end{aligned}$$

as desired.

(iii) Here we compute

$$\begin{aligned} \omega(e_j, e_k) &= -\operatorname{Im}(h(e_j, e_k)) = -\operatorname{Im}(h_{jk}), \\ \omega(e_{m+j}, e_{m+k}) &= g(J(e_{m+j}), e_{m+k}) = -g(e_j, J(e_{m+k})) = g(J(e_j), e_k) = \omega(e_j, e_k) = -\operatorname{Im}(h_{jk}), \\ \omega(e_j, e_{m+k}) &= g(J(e_j), e_{m+k}) = g(e_{m+j}, e_{m+j}) = \operatorname{Re}(h_{jk}), \end{aligned}$$

which is the first formula. For the second formula, as in part (ii) of the proof we compute

$$\omega(u,v) = \operatorname{Im}(h(u,v)) = -\frac{\mathrm{i}}{2}(h(u,v) - \overline{h(u,v)}) = -\frac{\mathrm{i}}{2}\Big(\sum_{j,k=1}^{m} h_{jk}u^j\bar{v}^k - \bar{h}_{jk}\bar{u}^j v^k\Big).$$

Thus

$$\begin{aligned} \omega &= -\frac{\mathrm{i}}{2}\sum_{j,k=1}^{m}(h_{jk}\gamma^j \otimes \bar{\gamma}^k - \bar{h}_{jk}\bar{\gamma}^j \otimes \gamma^k) = -\frac{\mathrm{i}}{2}\sum_{j,k=1}^{m}(h_{jk}\gamma^j \otimes \bar{\gamma}^k - h_{kj}\bar{\gamma}^j \otimes \gamma^k) \\ &= -\frac{\mathrm{i}}{2}\sum_{j,k=1}^{m} h_{jk}(\gamma^j \otimes \bar{\gamma}^k - \bar{\gamma}^k \otimes \gamma^j) = -\frac{\mathrm{i}}{2}\sum_{j,k} h_{jk}\gamma^j \wedge \bar{\gamma}^k, \end{aligned}$$

as claimed. ■

4.1.6 Volume forms on vector spaces with linear complex structures

Volume forms arise on vector spaces with linear complex structures in a natural manner. First of all, we let $(e_1,\dots,e_m,e_{m+1},e_{2m})$ be a J-adapted basis for a $\mathbb{R}$-vector space V with a linear complex structure J, and let $(\beta^1,\dots,\beta^m,\beta^{m+1},\dots,\beta^{2m})$ be the corresponding dual basis. We also denote, as usual,

$$\gamma^j = 1\otimes\beta^j + \mathrm{i}\otimes\beta^{m+j},\ \bar{\gamma}^j = 1\otimes\beta^j - \mathrm{i}\otimes\beta^{m+j},\qquad j\in\{1,\dots,m\}.$$

Then we have a volume form

$$\beta^1\wedge\beta^{m+1}\wedge\dots\wedge\beta^m\wedge\beta^{2m}$$

that satisfies

$$\left(\frac{\mathrm{i}}{2}\right)^m\gamma^1\wedge\bar{\gamma}^1\wedge\dots\wedge\gamma^m\wedge\bar{\gamma}^m = 1\otimes\beta^1\wedge\beta^{m+1}\wedge\dots\wedge\beta^m\wedge\beta^{2m}.$$

Now let $(f_1,\dots,f_m,f_{m+1},\dots,f_{2m})$ be another J-adapted basis with dual basis $(\alpha^1,\dots,\alpha^m,\alpha^{m+1},\dots,\alpha^{2m})$. Let $\boldsymbol{A}\in\mathbb{R}^{2m\times 2m}$ be defined by

$$f_j = \sum_{k=1}^{2m} A^k_j e_k,\qquad j\in\{1,\dots,2m\}.$$

By the change of basis formula we have $\boldsymbol{A}\boldsymbol{J}_2 = \boldsymbol{J}_1\boldsymbol{A}$ where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the matrix representatives of J in the bases $(e_1,\dots,e_m,e_{m+1},e_{2m})$ and $(f_1,\dots,f_m,f_{m+1},\dots,f_m)$, respectively. We also have

$$\boldsymbol{J}_1 = \boldsymbol{J}_2 = \begin{bmatrix}\boldsymbol{0}_{m\times m} & -\boldsymbol{I}_m\\ \boldsymbol{I}_m & \boldsymbol{0}_{m\times m}\end{bmatrix},$$

from which we deduce from Proposition 4.1.6 that $\boldsymbol{A}\colon\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ is $\mathbb{C}$-linear with respect to the standard linear complex structure on $\mathbb{R}^{2m}$. Thus

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{C}\\ -\boldsymbol{C} & \boldsymbol{B}\end{bmatrix},$$

where $\boldsymbol{B},\boldsymbol{C}\in\mathbb{R}^{m\times m}$. Since $\boldsymbol{A}$ is invertible, $\boldsymbol{B}$ is also invertible.

The above computations contribute to the following result.

4.1.24 Proposition (Volume forms on vector spaces with linear complex structures) *Let* V *be a* $\mathbb{R}$*-vector space with linear complex structure* J*, and let* $(\mathrm{e}_1,\dots,\mathrm{e}_m,\mathrm{e}_{m+1},\dots,\mathrm{e}_{2m})$ *be a* J*-adapted with dual basis* $(\beta^1,\dots,\beta^m,\beta^{m+1},\dots,\beta^{2m})$*. Let*

$$\gamma^j = 1\otimes\beta^j + \mathrm{i}\otimes\beta^{m+j},\ \bar{\gamma}^j = 1\otimes\beta^j - \mathrm{i}\otimes\beta^{m+j},\qquad j\in\{1,\dots,m\}.$$

Then V *possesses a canonical orientation for which the following statements regarding* $\nu\in\bigwedge^{2m}(\mathsf{V}^*)$ *are equivalent:*

(i) ν *is positively oriented;*

(ii) ν *is a positive multiple of*

$$\beta^1 \wedge \beta^{m+1} \wedge \cdots \wedge \beta^m \wedge \beta^{2m};$$

(iii) ν *is of bidegree* (m, m) *and* $1 \otimes \nu$ *is a positive multiple of*

$$\left(\frac{i}{2}\right)^m \gamma^1 \wedge \bar{\gamma}^1 \wedge \cdots \wedge \gamma^m \wedge \bar{\gamma}^m.$$

Proof Let us carry on using the notation preceding the statement of the proposition. Note that

$$\alpha^1 \wedge \alpha^{m+1} \wedge \cdots \wedge \alpha^m \wedge \alpha^{2m} = \lambda \beta^1 \wedge \beta^{m+1} \wedge \cdots \wedge \beta^m \wedge \beta^{2m}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. The result will follow from the computations preceding its statement provided we can show that, in fact, $\lambda \in \mathbb{R}_{>0}$. This will follow if we can show that the matrix $\boldsymbol{A}$ above has a positive determinant. This, however, follows from Proposition 4.1.9. ■

4.1.7 Totally real subspaces

The canonical finite-dimensional $\mathbb{C}$-vector space $\mathbb{C}^n$ features a natural $\mathbb{R}$-subspace of dimension n that we call the "real part" of $\mathbb{C}^n$, namely the subspace

$$\{\boldsymbol{x} + i\boldsymbol{0} \mid \boldsymbol{x} \in \mathbb{R}^n\}.$$

However, this subspace is not as natural as it seems. To wit, given a general finite-dimensional $\mathbb{R}$-vector space V with linear complex structure J, there is no natural choice for the "real part." Nonetheless, one can characterise the subspaces having the properties of $\mathbb{R}^n \subseteq \mathbb{C}^n$.

4.1.25 Definition (Totally real subspace) Let V be a finite-dimensional $\mathbb{R}$-vector space with linear complex structure J. A subspace $\mathsf{U} \subseteq \mathsf{V}$ (subspace as a $\mathbb{R}$-vector space) is ***totally real*** if $J(\mathsf{U}) \cap \mathsf{U} = \{0\}$. •

Let us characterise totally real subspaces in the case that we have an inner product compatible with the linear complex structure.

4.1.26 Proposition (Characterisation of totally real subspaces) *Let* V *be a finite-dimensional* $\mathbb{R}$*-vector space, let* J *be a linear complex structure on* V*, and let* g *be a (real) inner product on* V *compatible with* J*. Then, for a (real) subspace* U *of* V*, the following statements are equivalent:*

(i) U *is totally real;*

(ii) U *and* $J(\mathsf{U})$ *are* g*-orthogonal.*

Proof Since the second assertion clearly implies the first, we only prove the other implication. Since g is compatible with J,

$$g(J(v_1), J(v_2)) = g(v_1, v_2), \qquad v_1, v_2 \in \mathsf{V}.$$

Thus J is g-orthogonal. Since J is diagonalisable over $\mathbb{C}$ and has only eigenvalues $\pm\mathrm{i}$, there exists a g-orthogonal decomposition

$$\mathsf{V} = \mathsf{V}_1 \oplus \cdots \oplus \mathsf{V}_n$$

of V into J-invariant two-dimensional subspaces. If U is totally real it follows that $\mathsf{U} \cap \mathsf{V}_j$ is either one- or zero-dimensional for each $j \in \{1, \dots, n\}$. By relabelling if necessary, suppose that there exists $v_1, \dots, v_k \in \mathsf{V}$ such that $\mathsf{U} \cap \mathsf{V}_j = \operatorname{span}_{\mathbb{R}}(v_j)$, $j \in \{1, \dots, k\}$ and $\mathsf{U} \cap \mathsf{V}_j = \{0\}$ for $j \in \{k+1, \dots, n\}$. To prove that $J(\mathsf{U})$ and U are g-orthogonal, it then suffices to show that $g(J(v_j), v_j) = 0$ for each $j \in \{1, \dots, k\}$. This, however, follows easily. Indeed, for any $v \in \mathsf{V}$ we have

$$g(J(v), v) = g(J^2(v), J(v)) = -g(v, J(v)) = -g(J(v), v),$$

giving $g(J(v), v) = 0$, as desired. ■

This allows us to prove the following result, showing that bases for totally real subspaces can be extended to J-adapted bases.

4.1.27 Lemma (Extending bases for totally real subspaces) *Let V be a finite-dimensional $\mathbb{R}$-vector space with linear complex structure J, and let $\mathsf{U} \subseteq \mathsf{V}$ be a totally real subspace. If $(\mathrm{e}_1, \dots, \mathrm{e}_k)$ is a basis for U, then there exist linear independent vectors $\mathrm{e}_{k+1}, \dots, \mathrm{e}_n \in \mathsf{V}$ such*

$$(\mathrm{e}_1, \dots, \mathrm{e}_n, \mathrm{J}(\mathrm{e}_1), \dots, \mathrm{J}(\mathrm{e}_n))$$

is a J-adapted basis for V.

Proof We choose a J-compatible inner product g on V, e.g., the real part of a Hermitian inner product on V. As we saw in the proof of Proposition 4.1.26, there then exists a g-orthogonal decomposition

$$\mathsf{V} = \mathsf{V}_1 \oplus \cdots \oplus \mathsf{V}_n$$

such that $e_j \in \mathsf{V}_j$, $j \in \{1, \dots, k\}$. Then choose $e_j \in \mathsf{V}_j$ for $j \in \{k+1, \dots, n\}$. It is then immediate that $(e_1, \dots, e_n, J(e_1), \dots, J(e_n))$ is linearly independent, and so a basis. The fact that $J \circ J = -\mathrm{id}_{\mathsf{V}}$ easily shows that this basis is also J-adapted. ■

4.2 Holomorphic and real analytic manifolds, submanifolds, and mappings

In this section we simultaneously consider the basic ingredients of holomorphic and real analytic geometry. Only in a few places do we focus on specific properties of holomorphic manifolds.

4.2.1 Holomorphic and real analytic manifolds

We shall very quickly go through the motions of defining the basic objects of holomorphic and real analytic differential geometry, the holomorphic and real analytic manifolds. Almost all of these basic definitions go as in the smooth case.

4.2.1 Definition (Holomorphic or real analytic charts, atlases, and differentiable structures) Let S be a set, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. An $\mathbb{F}$-***chart*** for S is a pair $(\mathcal{U}, \phi)$ with

(i) $\mathcal{U}$ a subset of S, and

(ii) $\phi\colon \mathcal{U} \to \mathbb{F}^n$ an injection for which $\phi(\mathcal{U})$ is an open subset of $\mathbb{F}^n$.

A $\mathbf{C^r}$***-atlas*** for S is a family $\mathscr{A} = ((\mathcal{U}_a, \phi_a))_{a\in A}$ of $\mathbb{F}$-charts for S with the properties that $S = \cup_{a\in A}\mathcal{U}_a$, and that, whenever $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$, we have

(iii) $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ and $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$ are open subsets of $\mathbb{F}^n$, and

(iv) the ***overlap map*** $\phi_{ab} \triangleq \phi_b \circ \phi_a^{-1}|\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ is a C^r-diffeomorphism from $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ to $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$.

Two C^r-atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are ***equivalent*** if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also a C^r-atlas. A $\mathbf{C^r}$***-differentiable structure***, or a ***holomorphic differentiable structure***, on S is an equivalence class of atlases under this equivalence relation. A $\mathbf{C^r}$***-differentiable manifold***, or a $\mathbf{C^r}$***-manifold***, or a ***holomorphic manifold***, M is a set S with a C^r-differentiable structure. An ***admissible*** $\mathbb{F}$***-chart*** for a manifold M is a pair $(\mathcal{U}, \phi)$ that is an $\mathbb{F}$-chart for some atlas defining the differentiable structure. If all $\mathbb{F}$-charts take values in $\mathbb{F}^n$ for some fixed n, then n is the ***dimension*** of M, denoted by $\dim_{\mathbb{F}}(\mathsf{M})$. The ***manifold topology*** on a set S with a differentiable structure is the topology generated by the domains of the admissible $\mathbb{F}$-charts. •

In Figure 4.1 we illustrate how one should think about the overlap condition.

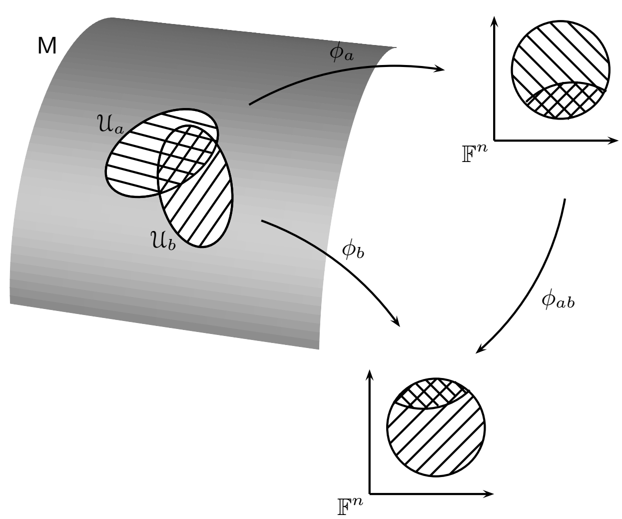

Figure 4.1 An interpretation of the overlap condition

Note that a holomorphic or real analytic manifold is immediately a smooth manifold, the latter assertion being trivial and the former since holomorphic maps from open subsets of $\mathbb{C}^n$ into $\mathbb{C}^m$ are infinitely differentiable as maps from open subsets of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ into $\mathbb{C}^m \simeq \mathbb{R}^{2m}$.

We shall very often consider manifolds whose topology has additional assumptions placed upon it. One we very often make is that of the manifold topology being Hausdorff. Manifolds whose topology is not Hausdorff exist, but are not regarded as being interesting.[1] Another set of common assumptions are those of second countability and paracompactness. These are not unrelated. For example, second countable Hausdorff manifolds are paracompact [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.5]. Also, connected paracompact manifolds are second countable [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.11].

Let us consider some elementary examples of holomorphic and real analytic manifolds.

4.2.2 Examples (Holomorphic and real analytic manifolds)

1. If $\mathcal{U} \subseteq \mathbb{F}^n$ is open then it is a holomorphic or real analytic manifold with the holomorphic or real analytic differentiable structure defined by the single chart $(\mathcal{U}, \mathrm{id}_{\mathcal{U}})$.
2. If $\mathcal{U} \subseteq \mathsf{M}$ is an open subset of a holomorphic or real analytic manifold, then it is itself a holomorphic or real analytic manifold. The holomorphic or real analytic differentiable structure is provided by the restriction to $\mathcal{U}$ of the admissible $\mathbb{F}$-charts for M.
3. Take $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ to be the unit n-sphere. We claim that $\mathbb{S}^n$ is an n-dimensional real analytic manifold. To see this, we shall provide an atlas for $\mathbb{S}^n$ consisting of two charts. The chart domains are

$$\mathcal{U}_+ = \mathbb{S}^n \setminus \{(0,\dots,0,1)\}, \quad \mathcal{U}_- = \mathbb{S}^n \setminus \{(0,\dots,0,-1)\}.$$

Define

$$\begin{aligned} \phi_+ \colon \mathcal{U}_+ &\to \mathbb{R}^n \\ (x_1,\dots,x_{n+1}) &\mapsto \Big(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}}\Big) \end{aligned}$$

and

$$\begin{aligned} \phi_- \colon \mathcal{U}_- &\to \mathbb{R}^n \\ (x_1,\dots,x_{n+1}) &\mapsto \Big(\frac{x_1}{1+x_{n+1}},\dots,\frac{x_n}{1+x_{n+1}}\Big). \end{aligned}$$

(See Figure 4.2 for $n = 1$.) One may verify that $\phi_+(\mathcal{U}_+) = \phi_-(\mathcal{U}_-) = \mathbb{R}^n \setminus \{\mathbf{0}\}$ and that the inverse of ϕ_+ is given by

$$\phi_+^{-1}(\boldsymbol{y}) = \Big(\frac{2y_1}{\|\boldsymbol{y}\|^2+1},\dots,\frac{2y_n}{\|\boldsymbol{y}\|^2+1},\frac{\|\boldsymbol{y}\|^2-1}{\|\boldsymbol{y}\|^2+1}\Big),$$

[1] Here is an example of a non-Hausdorff real analytic manifold. On the set $S = (\mathbb{R}\times\{0\}) \cup (\mathbb{R}\times\{1\})$ consider the equivalence relation $(x,0) \sim (y,1)$ if $x = y$ and $x, y \neq 0$. Let $\mathsf{M} = S/\sim$ be the set of equivalence classes and let $\pi\colon S \to \mathsf{M}$ be the canonical projection. Consider the charts $(\mathcal{U}_0, \phi_0)$ and $(\mathcal{U}_1, \phi_1)$ defined by $\mathcal{U}_0 = \pi(\mathbb{R}\times\{0\})$ and $\mathcal{U}_1 = \pi(\mathbb{R}\times\{1\})$, with chart maps $\phi_0([(x,0)]) = x$ and $\phi_1([x,1]) = x$. We leave it to the reader to show that these charts define a C^ω-differentiable structure on M for which the manifold topology is not Hausdorff. This manifold is called ***the line with two origins***.

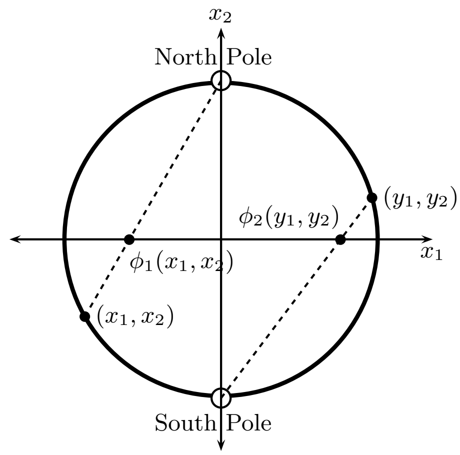

Figure 4.2 Stereographic coordinates for $\mathbb{S}^1$

and from this we determine that the overlap map is given by

$$(\phi_- \circ \phi_+^{-1})(\boldsymbol{y}) = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^2}$$

for $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$. This map is easily seen to be real analytic and, as $\mathbb{S}^n = \mathcal{U}_+ \cup \mathcal{U}_-$, this makes $\mathbb{S}^n$ into a C^ω-differentiable manifold.

We claim that this real analytic structure can be made into a holomorphic structure when $n = 2$. Indeed, in this case, denoting a point (y_1, y_2) in $\mathbb{R}^2 \simeq \mathbb{C}$ as $z = y_1 + \mathrm{i}y_2$ we have that the overlap map is

$$(\phi_- \circ \phi_+^{-1})(z) = \frac{1}{\bar{z}}.$$

Therefore, if we use the map

$$\bar{\phi}_-(x_1, x_2, x_3) = \Big(\frac{x_1}{1 + x_3}, -\frac{x_2}{1 + x_3}\Big)$$

in place of ϕ_-, we see that the overlap map satisfies

$$(\bar{\phi}_- \circ \phi_+^{-1})(z) = \frac{1}{z}.$$

and this is readily verified to be a holomorphic diffeomorphism from $\mathbb{C} \setminus \{0\}$ to itself, e.g., by verifying the Cauchy–Riemann equations.

4. A ***line*** in $\mathbb{F}^n$ is a subspace of $\mathbb{F}^n$ of $\mathbb{F}$-dimension 1. By $\mathbb{FP}^n$ we denote the set of lines in $\mathbb{F}^{n+1}$, which we call $\mathbb{F}$-***projective space***. If $(x_0, x_1, \ldots, x_n) \in \mathbb{F}^{n+1} \setminus \{\boldsymbol{0}\}$ let us denote the line through this point by $[x_0 : x_1 : \cdots : x_n]$. There are $n + 1$ natural $\mathbb{F}$-charts for $\mathbb{FP}^n$ that we denote by $(\mathcal{U}_j, \phi_j)$, $j \in \{0, 1, \ldots, n\}$. These are defined as follows:

$$\mathcal{U}_j = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{FP}^n \mid x_j \neq 0\},$$
$$\phi_j([x_0 : x_1, \cdots : x_n]) = \Big(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\Big).$$

The overlap map $\phi_j \circ \phi_k^{-1}$, $j < k$, is

$$\phi_j \circ \phi_k^{-1}(a_1, \dots, a_n) = \Big(\frac{a_1}{a_{j+1}}, \dots, \frac{a_j}{a_{j+1}}, \frac{a_{j+2}}{a_{j+1}}, \dots, \frac{a_k}{a_{j+1}}, \frac{1}{a_{j+1}}, \frac{a_{k+1}}{a_{j+1}}, \dots, \frac{a_n}{a_{j+1}}\Big),$$

which is real analytic or holomorphic, as appropriate. In the case of $n = 1$ the overlap condition is

$$\phi_0 \circ \phi_1^{-1}(a) = a^{-1},$$

and we conclude by referring to the preceding example that $\mathbb{RP}^1 \simeq \mathbb{S}^1$ and $\mathbb{CP}^1 \simeq \mathbb{S}^2$.

5. In the set $\mathbb{F}^n$ define an equivalence relation by $\boldsymbol{z} \sim \boldsymbol{w}$ if $\boldsymbol{z} - \boldsymbol{w} \in \Lambda^n$, where $\Lambda = \mathbb{Z}^n \subseteq \mathbb{R}^n$ in the case of $\mathbb{F} = \mathbb{R}$ and

$$\Lambda = \{\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^n\}$$

in the case of $\mathbb{F} = \mathbb{C}$. The set $\mathbb{T}^n_{\mathbb{F}} = \mathbb{F}^n / \sim$ is the $\mathbb{F}$-***torus*** of dimension n. Note that $\mathbb{T}^n_{\mathbb{C}} \simeq \mathbb{T}^{2n}_{\mathbb{R}}$. In particular, $\mathbb{T}^1_{\mathbb{C}}$ is identified with the standard 2-torus as depicted in Figure 4.3. •

Figure 4.3 A depiction of $\mathbb{T}^1_{\mathbb{C}} \simeq \mathbb{T}^2_{\mathbb{R}}$

4.2.2 Holomorphic and real analytic mappings

Now we turn to maps between manifolds.

4.2.3 Definition (Local representative of a map, holomorphic or real analytic map) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. Let M and N be C^r-manifolds and let $\Phi\colon \mathsf{M} \to \mathsf{N}$ be a map. Let $x \in \mathsf{M}$, let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-chart for which $\mathcal{U}$ is a neighbourhood of x, and let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-chart for which $\mathcal{V}$ is a neighbourhood of $\Phi(x)$, assuming that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$ (if Φ is continuous, $\mathcal{U}$ can always be made sufficiently small so that this holds). The ***local representative*** of Φ with respect to the $\mathbb{F}$-charts $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ is the map $\Phi_{\phi\psi}\colon \phi(\mathcal{U}) \to \psi(\mathcal{V})$ given by

$$\Phi_{\phi\psi}(\boldsymbol{x}) = \psi \circ \Phi \circ \phi^{-1}(\boldsymbol{x}).$$

With this notation we make the following definitions.

(i) We say that $\Phi\colon \mathsf{M} \to \mathsf{N}$ is of ***class* $\mathbf{C}^r$** or is ***holomorphic*** or ***real analytic***, if, for every point $x \in \mathsf{M}$ and every $\mathbb{F}$-chart $(\mathcal{V}, \psi)$ for N for which $\Phi(x) \in \mathcal{V}$, there exists a $\mathbb{C}$-chart $(\mathcal{U}, \phi)$ for M such that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$ and for which the local representative $\Phi_{\phi\psi}$ is of class C^r.

(ii) The set of class C^r maps from M to N is denoted by $\mathrm{C}^r(\mathsf{M}; \mathsf{N})$.

(iii) We denote by $\mathrm{C}^r(\mathsf{M}) = \mathrm{C}^r(\mathsf{M}; \mathbb{F})$ the set of holomorphic functions on M.

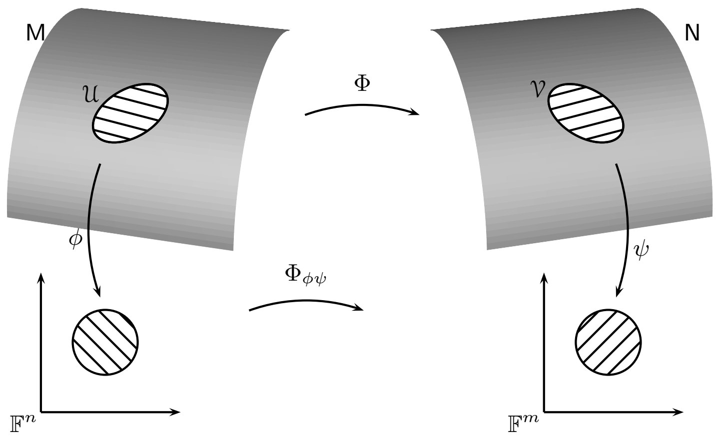

Figure 4.4 The local representative of a map

(iv) If Φ is a bijection of class C^r, and if Φ^{-1} is also of class C^r, then Φ is a $\mathbf{C}^r$-***diffeomorphism*** or a ***holomorphic diffeomorphism***. •

In Figure 4.4 we depict how one should think about the local representative.

Analogous to the situation for functions defined on subset of $\mathbb{F}^n$, if M is a holomorphic or real analytic manifold, if $A \subseteq \mathsf{M}$, and if $\boldsymbol{f}: A \to \mathbb{F}^m$ is continuous, we denote

$$\|\boldsymbol{f}\|_A = \sup\{\|\boldsymbol{f}(x)\| \mid x \in A\}. \tag{4.9}$$

As with smooth manifolds, we can define the pull-back of functions by mappings, and holomorphicity or real analyticity is preserved by Proposition 1.2.2.

4.2.4 Definition (Pull-back of a function) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r = \text{hol}$ if $\mathbb{F} = \mathbb{C}$. Let M and N be C^r-manifolds and let $\Phi: \mathsf{M} \to \mathsf{N}$ be a C^r-map. For $g \in C^r(\mathsf{N})$, the ***pull-back*** of g is the function $\Phi^*g \in C^r(\mathsf{M})$ given by $\Phi^*g(z) = g \circ \Phi(z)$. •

4.2.3 Holomorphic and real analytic functions and germs

We shall be much concerned with algebraic structure arising from holomorphic and real analytic functions. This will not be addressed systematically until Chapter GA2.1. For the moment, however, we shall need a small part of this development, and we give it here. The discussion of germs here resembles that given in Section 2.3.1, of course. We start by considering functions.

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r = \text{hol}$ if $\mathbb{F} = \mathbb{C}$. We let M be a C^r-manifold and note that $C^r(\mathsf{M})$ is a ring with the operations of pointwise addition and multiplication:

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x),$$

for $f, g \in C^r(\mathsf{M})$ and $x \in \mathsf{M}$. As we saw in Section 1.2.1, these operations preserve the

C^r-structure. Moreover, $C^r(M)$ additionally has the $\mathbb{F}$-vector space structure defined by $(af)(x) = a(f(x))$ for $a \in \mathbb{F}$, $f \in C^r(M)$, and $x \in M$. Thus $C^r(M)$ is a $\mathbb{F}$-algebra.

Holomorphic or real analytic functions on a manifold have the same restrictions on their global behaviour from local conditions as we saw with the Identity Theorem in $\mathbb{F}^n$, stated as Theorem 1.1.18.

4.2.5 Theorem (Identity Theorem on manifolds) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r = \omega$ if $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. If M is a connected manifold of class C^r, if $\mathcal{U} \subseteq M$ is a nonempty open set and if $f, g \in C^r(M)$ satisfy $f|\mathcal{U} = g|\mathcal{U}$, then $f = g$.*

Proof It suffices to show that if $f(x) = 0$ for every $x \in \mathcal{U}$ then f is the zero function. Let

$$\mathcal{C} = \{x \in M \mid f(x) = 0\},$$

and note that $\mathcal{C}$ is closed with $\mathrm{int}(\mathcal{C}) \neq \emptyset$ since it contains $\mathcal{U}$. We claim that $\mathrm{bd}(\mathrm{int}(\mathcal{C})) = \emptyset$. Indeed, suppose that $x \in \mathrm{bd}(\mathcal{C})$. Let $(\mathcal{V}, \psi)$ be a chart with $x \in \mathcal{V}$. By continuity of f and its derivatives, the Taylor series of $f \circ \psi^{-1}$ at $\psi(x)$ is zero. Since f is holomorphic or real analytic, this implies that $f \circ \psi^{-1}$ vanishes in a neighbourhood of $\psi(x)$, and so f vanishes in a neighbourhood of x, which is a contradiction. We now claim that $M \setminus \mathrm{int}(\mathcal{C})$ is open. Indeed, let $x \in M \setminus \mathrm{int}(\mathcal{C})$ be a point not in the interior of $M \setminus \mathrm{int}(\mathcal{C})$. Then every neighbourhood of x must intersect $\mathcal{C}$ and so $x \in \mathrm{bd}(\mathrm{int}(\mathcal{C})) = \emptyset$, and so $M \setminus \mathrm{int}(\mathcal{C})$ is open. Since M is now the union of the disjoint open sets $\mathrm{int}(\mathcal{C})$ and $M \setminus \mathrm{int}(C)$ and since the former is nonempty, we must have $M \setminus \mathrm{int}(\mathcal{C}) = \emptyset$, giving $\mathrm{int}(\mathcal{C}) = M$ and so $\mathcal{C} = M$. ■

The Identity Theorem, then, indicates that it may generally be difficult to extend locally defined holomorphic or real analytic functions to globally defined functions. To deal with this and for other reasons, we introduce germs.

Let $x_0 \in M$. We define as follows an equivalence relation on the set of ordered pairs $(f, \mathcal{U})$, where $\mathcal{U} \subseteq M$ is a neighbourhood of x_0 and $f \in C^r(\mathcal{U})$. We say that $(f_1, \mathcal{U}_1)$ and $(f_2, \mathcal{U}_2)$ are ***equivalent*** if there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ of x_0 such that $f_1|\mathcal{U}_1 = f_2|\mathcal{U}$. This notion of equivalence is readily verified to be an equivalence relation. We denote a typical equivalence class by $[(f, \mathcal{U})]_{x_0}$, or simply by $[f]_{x_0}$ if the domain of f is understood or immaterial. The set of equivalence classes we denote by $\mathscr{C}^r_{x_0,M}$, which we call the set of ***germs*** of holomorphic or real analytic functions at x_0, respectively. We make the set of germs into a ring by defining the following operations of addition and multiplication:

$$\begin{aligned} [(f_1, \mathcal{U}_1)]_{x_0} + [(f_2, \mathcal{U}_2)]_{x_0} &= [f_1|\mathcal{U}_1 \cap \mathcal{U}_2 + f_2|\mathcal{U}_1 \cap \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2]_{x_0} \\ [(f_1, \mathcal{U}_1)]_{x_0} \cdot [(f_2, \mathcal{U}_2)]_{x_0} &= [(f_1|\mathcal{U}_1 \cap \mathcal{U}_2)(f_2|\mathcal{U}_1 \cap \mathcal{U}_2), \mathcal{U}_1 \cap \mathcal{U}_2]_{x_0}. \end{aligned}$$

It is elementary to verify that these operations are well-defined, and indeed make the set of germs of holomorphic or real analytic functions into a ring. As with functions, germs of functions also have a $\mathbb{F}$-vector space structure: $a[(f, \mathcal{U})]_{x_0} = [(af, \mathcal{U})]_{x_0}$. Thus $\mathscr{C}^r_{x_0,M}$ is a $\mathbb{F}$-algebra.

Let us prove some results about this algebraic structure. The first more or less trivial observation is the following.

4.2.6 Proposition (Characterisation of holomorphic and real analytic function germs) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. For M a manifold of class C^{r} and for $\mathrm{x}_0 \in \mathsf{M}$, the ring $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{M}}$ is isomorphic to the ring $\hat{\mathbb{F}}[[\xi_1, \dots, \xi_{\mathrm{n}}]]$ of convergent power series in n indeterminates, where n is the dimension of the connected component of M containing x_0.*

Proof Let $(\mathcal{U}, \phi)$ be a $\mathbb{F}$-chart about x_0 such that $\phi(x_0) = \mathbf{0}$. We identify $\mathcal{U}$ with $\phi(\mathcal{U}) \subseteq \mathbb{F}^n$ and a function on $\mathcal{U}$ with its local representative. A function of class C^r, by definition, is one whose Taylor series converges in some neighbourhood of every point in its domain of definition, and which is equal to its Taylor series on that neighbourhood. Thus, if $[(f, \mathcal{V})]_{\mathbf{0}} \in \mathscr{C}^r_{\mathbf{0},\mathcal{U}}$, in some neighbourhood $\mathcal{V}'$ of $\mathbf{0}$ in $\mathcal{U}$ we have

$$f(x_1, \dots, x_n) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{\partial^{|I|} f}{\partial \boldsymbol{x}^I}(\mathbf{0}).$$

Thus $[(f, \mathcal{V})]_{\mathbf{0}}$ is determined by its Taylor series, which gives a surjective map from $\mathscr{C}^r_{\mathbf{0},\mathcal{U}}$ to $\hat{\mathbb{F}}[[\boldsymbol{\xi}]]$. That this map is also injective follows since two analytic functions having the same Taylor series at a point are obviously equal on some neighbourhood of that point. ■

Note that the isomorphism of $\mathscr{C}^r_{x_0,\mathsf{M}}$ with $\hat{\mathbb{F}}[[\boldsymbol{\xi}]]$ in the preceding result is not natural, but depends on a choice of coordinate chart. However, the key point is that if one chooses *any* coordinate chart, the isomorphism is induced. We shall often use this fact to reduce ourselves to the case where the manifold is $\mathbb{F}^n$. This simplifies things greatly. However, it is also interesting to have a coordinate independent way of thinking of the isomorphism of the preceding result, and this leads naturally to the construction of jet bundles as in Chapter 5.

For the moment, however, let us use the isomorphism from the preceding result to state some useful facts about the ring $\mathscr{C}^r_{x_0,\mathsf{M}}$.

4.2.7 Theorem (Algebraic properties of the ring of germs of holomorphic or real analytic functions) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. For M a manifold of class C^{r} and for $\mathrm{x}_0 \in \mathsf{M}$, the following statements hold:*

(i) $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{M}}$ is a local ring;

(ii) $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{M}}$ is a unique factorisation domain;

(iii) $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{M}}$ is a Noetherian ring.

Proof By Proposition 4.2.6 the ring $\mathscr{C}^r_{x_0,\mathsf{M}}$ is isomorphic to the ring $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}$ considered in Section 2.3. Thus $\mathscr{C}^r_{x_0,\mathsf{M}}$ will possess any isomorphism invariant properties possessed by the ring $\mathscr{C}^r_{\mathbf{0},\mathbb{F}^n}$. Since the properties of being a local ring, being a unique factorisation domain, and being a Noetherian ring are isomorphism invariant, the theorem follows from Theorems 2.3.1, 2.3.3, and 2.3.4. ■

Of course, the previous constructions apply equally well in the smooth case, and we shall occasionally access the notation $\mathscr{C}^\infty_{x,\mathsf{M}}$ for the ring of germs of smooth functions at x_0. Note, however, that while $\mathscr{C}^\infty_{x,\mathsf{M}}$ is a local ring by Proposition 2.3.5, it is neither a unique factorisation domain (by Proposition 2.3.6) nor a Noetherian ring (by Proposition 2.3.7). Another point of distinction with the smooth case and the holomorphic or real analytic cases has to do with global representatives for germs. In the smooth case the smooth Tietze Extension Theorem gives the following result (see also Proposition 5.6.4 below).

4.2.8 Proposition (Global representative of smooth germs) *If* M *is a smooth manifold, if* $x_0 \in \mathsf{M}$*, and if* $[(f, \mathcal{U})]_{x_0} \in \mathscr{C}^{\infty}_{x_0,\mathsf{M}}$*, then there exists* $g \in C^{\infty}(\mathsf{M})$ *such that* $[(g, \mathsf{M})]_{x_0} = [(f, \mathcal{U})]_{x_0}$.

Such a result as the preceding does not hold in the holomorphic or real analytic case, and we illustrate what can happen with two examples.

4.2.9 Example (A germ that has no globally defined representative) We take $\mathsf{M} = \mathsf{N} = \mathbb{F}$. We let $\epsilon \in \mathbb{R}_{>0}$, let $\mathcal{U}_\epsilon = \mathsf{D}^{\epsilon}(0, 1)$, and consider the function $f\colon \mathcal{U}_\epsilon \to \mathbb{F}$ defined by $f(x) = \frac{\epsilon^2}{\epsilon^2 - x^2}$. Note that f is holomorphic or real analytic on $\mathcal{U}_\epsilon$. However, there is no function $g \in C^r(\mathbb{F}; \mathbb{F})$ for which $[(g, \mathbb{F})]_0 = [(f, \mathcal{U}_\epsilon)]_0$. Indeed, by Theorem 1.1.18 it follows that any holomorphic or real analytic function agreeing with f on a neighbourhood of 0 must agree with f on any connected open set containing 0 on which it is defined. In particular, if $[(g, \mathbb{F})]_0 = [(f, \mathcal{U}_\epsilon)]_0$ then $g|\mathcal{U}_\epsilon = f$. Since there is no holomorphic or real analytic mapping on $\mathbb{F}$ agreeing with f on $\mathcal{U}_\epsilon$, our claim follows.

The example shows, in fact, that there can be no neighbourhood of a point on a holomorphic or real analytic manifold to which every germ can be extended. •

4.2.4 Some particular properties of holomorphic functions

Just as was the case in Section 1.1.7 with holomorphic functions defined on open subsets of $\mathbb{C}^n$, holomorphic functions on holomorphic manifolds have properties not shared by their real analytic brethren. Here we consider the most basic of these arising from the following result.

4.2.10 Theorem (Maximum Modulus Principle on holomorphic manifolds) *If* M *is a connected holomorphic manifold, if* $f \in C^{\mathrm{hol}}(\mathsf{M})$*, and if there exists* $z_0 \in \mathsf{M}$ *such that* $|f(z)| \le |f(z_0)|$ *for every* $z \in \mathsf{M}$*, then* f *is constant on* M.

Proof Let $g \in C^{\mathrm{hol}}(\mathsf{M})$ be defined by $g(z) = f(z_0)$. Let $(\mathcal{U}, \phi)$ be an $\mathbb{C}$-chart with $x_0 \in \mathcal{U}$. Note that $|f \circ \phi^{-1}(z)| \le |f \circ \phi^{-1}(\phi(z_0))|$ for every $z \in \phi(\mathcal{U})$. By Theorem 1.1.26 we have that $f \circ \phi^{-1}(z) = g \circ \phi^{-1}(z)$ for every $z \in \phi(\mathcal{U})$ and so f and g agree on $\mathcal{U}$. The result follows from the Identity Theorem in the form of Theorem 4.2.5. ■

There are a few immediate consequences of this.

4.2.11 Corollary (Holomorphic functions on compact holomorphic manifolds are locally constant) *If* M *is a compact holomorphic manifold and if* $f \in C^{\mathrm{hol}}(\mathsf{M})$*, then* f *is locally constant.*

Proof The function $z \mapsto |f(z)|$ is continuous, and so achieves its maximum on M. The result follows immediately from the Maximum Modulus Principle. ■

4.2.5 Holomorphic and real analytic submanifolds

Next we consider submanifolds.

4.2.12 Definition (Holomorphic or real analytic submanifold) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r = \text{hol}$ if $\mathbb{F} = \mathbb{C}$. A subset S of a C^r-manifold M is a $\mathbf{C^r}$***-submanifold*** or a ***holomorphic*** or ***real analytic*** submanifold if, for each point $x \in \mathsf{S}$, there is an admissible $\mathbb{F}$-chart $(\mathcal{U}, \phi)$ for M with $x \in \mathcal{U}$, and such that

(i) ϕ takes its values in a product $\mathbb{F}^k \times \mathbb{F}^{n-k}$, and

(ii) $\phi(\mathcal{U} \cap \mathsf{S}) = \phi(\mathcal{U}) \cap (\mathbb{F}^k \times \{\mathbf{0}\})$.

A $\mathbb{F}$-chart with these properties is a $\mathbb{F}$***-submanifold chart*** for S. •

In Figure 4.5 we illustrate how one should think about submanifolds.

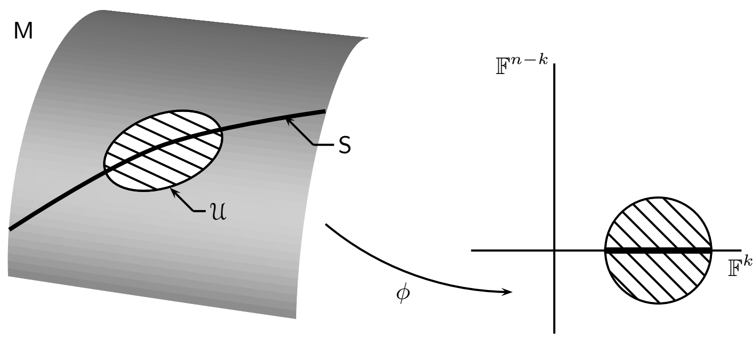

Figure 4.5 A submanifold chart

Let us look at some examples.

4.2.13 Examples (Holomorphic and real analytic submanifolds)

1. Any open subset of a holomorphic or real analytic manifold is a holomorphic or real analytic submanifold.
2. Note that $\mathbb{S}^n$ is a real analytic submanifold of $\mathbb{R}^{n+1}$. This follows from the real analytic Inverse Function Theorem since $\mathbb{S}^n = f^{-1}(0)$ for the real analytic function $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 - 1$ whose derivative does not vanish on $\mathbb{S}^n$.
3. If M is a compact holomorphic submanifold of $\mathbb{C}^n$ then $\dim_{\mathbb{C}}(\mathsf{M}) = 0$. Indeed, by Corollary 4.2.11, the coordinate functions on $\mathbb{C}^n$ restricted to M are holomorphic functions on M, and so must be constant.

   As a consequence, the holomorphic manifold $\mathbb{S}^2$ is not a submanifold of $\mathbb{C}^n$ for any n. •

4.3 Holomorphic and real analytic vector bundles

Vector bundles arise naturally in differential geometry in terms of tangent bundles and cotangent bundles. The more general notion of a vector bundle also comes up in a natural way in various contexts, e.g., with respect to jet bundles and with respect to various sorts of sheaves. In this section we shall give the very basic constructions involving vector bundles, noting that in subsequent chapters, especially Chapters 5 and GA2.1, we shall cover other aspects of the theory in a more comprehensive manner.

4.3.1 Local vector bundles, vector bundle structures

There are various ways to construct vector bundles, and the approach we take here is a direct one, more or less mirroring the way we construct manifolds. That is, we start locally, and then ask that local object obey appropriate transformation rules. The local models for vector bundles are as follows. Our initial definition allows for both $\mathbb{R}$- and $\mathbb{C}$-vector bundles with smooth or real analytic structures. After doing this, we can specialise to the holomorphic case.

4.3.1 Definition (Local $\mathbb{F}$-vector bundle) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $r \in \{\infty, \omega\}$.

(i) A ***local*** $\mathbb{F}$***-vector bundle over*** $\mathbb{F}$ is a product $\mathcal{U} \times \mathbb{F}^k$, where $\mathcal{U} \subseteq \mathbb{F}^n$ is an open subset.

(ii) If $\mathcal{U} \times \mathbb{F}^k$ and $\mathcal{V} \times \mathbb{F}^l$ are local $\mathbb{F}$-vector bundles, then a map $\boldsymbol{g}\colon \mathcal{U} \times \mathbb{F}^k \to \mathcal{V} \times \mathbb{F}^l$ is a $\mathbf{C}^{\mathbf{r}}$***-local*** $\mathbb{F}$***-vector bundle map*** if it has the form $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}) = (\boldsymbol{g}_1(\boldsymbol{x}), \boldsymbol{g}_2(\boldsymbol{z}) \cdot \boldsymbol{v})$, where $\boldsymbol{g}_1\colon \mathcal{U} \to \mathcal{V}$ and $\boldsymbol{g}_2\colon \mathcal{U} \to \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^k; \mathbb{F}^l)$ are of class C^r.

(iii) If, in part (ii), $\boldsymbol{g}_1$ is a C^r-diffeomorphism and $\boldsymbol{g}_2(\boldsymbol{x})$ is an isomorphism for each $\boldsymbol{x} \in \mathcal{U}$, then we say that $\boldsymbol{g}$ is a $\mathbf{C}^{\mathbf{r}}$***-local*** $\mathbb{F}$***-vector bundle isomorphism***. •

A vector bundle is constructed, just as was a manifold, by patching together local objects.

4.3.2 Definition ($\mathbb{F}$-vector bundle) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $r \in \{\infty, \omega\}$. A $\mathbf{C}^{\mathbf{r}}$***-vector bundle over*** $\mathbb{F}$ is a set S that has an atlas $\mathscr{A} = \{(\mathcal{U}_a, \phi_a)\}_{a \in A}$ where $\mathrm{image}(\phi_a)$ is a local $\mathbb{F}$-vector bundle, $a \in A$, and for which the overlap maps are C^r-local $\mathbb{F}$-vector bundle isomorphisms. Such an atlas is a $\mathbf{C}^{\mathbf{r}}$***-vector bundle atlas over*** $\mathbb{F}$. Two C^r-vector bundle atlases, $\mathscr{A}_1$ and $\mathscr{A}_2$, are ***equivalent*** if $\mathscr{A}_1 \cup \mathscr{A}_2$ is a C^r-vector bundle atlas. A $\mathbf{C}^{\mathbf{r}}$***-vector bundle structure over*** $\mathbb{F}$ is an equivalence class of such atlases. A chart in one of these atlases is called an ***admissible*** $\mathbb{F}$***-vector bundle chart***. A typical vector bundle will be denoted by E. •

The ***base space*** M of a vector bundle E is given by all points $e \in \mathsf{E}$ having the property that there exists an admissible $\mathbb{F}$-vector bundle chart $(\mathcal{V}, \psi)$ such that $\psi(v) = (\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{U} \times \mathbb{F}^k$. This definition may easily be shown to make sense, since the overlap maps are local vector bundle isomorphisms that map the zero vector in one local vector bundle to the zero vector in another. To any point $e \in \mathsf{E}$ we associate a point $x \in \mathsf{M}$ as follows. Let $(\mathcal{U}, \phi)$ be a $\mathbb{F}$-vector bundle chart for E around e. Thus $\psi(e) = (\boldsymbol{x}, \boldsymbol{v}) \in \mathcal{U}' \times \mathbb{F}^k$. Define $x = \psi^{-1}(\boldsymbol{x}, \boldsymbol{0})$. Once again, since the overlap maps are local vector bundle isomorphisms, this definition makes sense. We denote the resulting map from E to M by π and we call this the ***vector bundle projection***. Sometimes we will write a vector bundle as $\pi\colon \mathsf{E} \to \mathsf{M}$. The set $\mathsf{E}_x \triangleq \pi^{-1}(x)$ is the ***fibre*** of E over x. This has the structure of a $\mathbb{F}$-vector space induced from that for the local vector bundles, and the vector space operations are well-defined since the overlap maps are local vector bundle isomorphisms. The zero vector in E_x corresponds to the point x in the base space, and will sometimes be denoted by 0_x.

Suppose we have a vector bundle chart $(\mathcal{U}, \phi)$ for a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ mapping $\mathcal{U}$ bijectively onto the local vector bundle $\mathcal{U}' \times \mathbb{F}^k$. We define an induced chart $(\mathcal{U}_0, \phi_0)$ for M by asking that $\mathcal{U}_0 = \pi(\mathcal{U})$ and that $\phi(0_x) = (\phi_0(x), \mathbf{0})$.

4.3.3 Remark (On typical fibres being $\mathbb{F}^k$) In our construction of a vector bundle from local models, we supposed the fibres to be isomorphic to $\mathbb{F}^k$ for some $k \in \mathbb{Z}_{>0}$. As we shall see, situations can naturally arise where the typical fibre is not $\mathbb{F}^k$ but rather some other finite-dimensional $\mathbb{F}$-vector space. However, since all such vector spaces are isomorphic, even if not necessarily naturally so, to $\mathbb{F}^k$ for some $k \in \mathbb{Z}_{>0}$, we lose no generality by assuming the typical model for the fibre to be $\mathbb{F}^k$. That being said, there is something to be said for modelling fibres for vector bundles on finite-dimensional vector spaces rather than $\mathbb{F}^k$. But we trust the reader can navigate this on their own. •

Note that a $\mathbb{C}$-vector bundle is not just a $\mathbb{R}$-vector bundle with fibres being $\mathbb{C}$-vector spaces. This is because the linear part of the overlap maps are required to be $\mathbb{C}$-linear mappings, not $\mathbb{R}$-linear mappings. In the complex case, one can then further impose the structure of holomorphicity.

4.3.4 Definition (Holomorphic vector bundle) Let M be a holomorphic manifold. A ***holomorphic vector bundle***, or a $\mathrm{C}^{\mathrm{hol}}$-vector bundle, over M is a smooth $\mathbb{C}$-vector bundle over *manM* possessing a vector bundle atlas for which the maps g_1 and g_2 from Definition 4.3.1 associated with the overlap maps are holomorphic. •

The reader should understand carefully the hypotheses when reading some of our definitions and results on vector bundles. If we are working in the smooth or real analytic category, we will consider both $\mathbb{R}$- and $\mathbb{C}$-vector bundles. In the holomorphic category, vector bundles are, of course, always complex. One should be sure to distinguish between smooth $\mathbb{C}$-vector bundles and holomorphic vector bundles.

Let us give some examples of vector bundles.

4.3.5 Examples (Vector bundles)

1. An $\mathbb{F}$-vector bundle whose fibres are one-dimensional is called a ***line bundle***.
2. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let M be a manifold of class C^r and let $k \in \mathbb{Z}_{>0}$. By $\mathbb{F}^k_{\mathsf{M}}$ we denote the ***trivial vector bundle*** $\mathsf{M} \times \mathbb{F}^k$ which we regard as a vector bundle using the projection $\mathrm{pr}_1\colon \mathbb{F}^k_{\mathsf{M}} \to \mathsf{M}$ onto the first factor.
3. Not all vector bundles are trivial. Let us make two comments about this.
   (a) A vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ is ***trivialisable*** if there exists a vector bundle isomorphism (see Definition 4.3.9) $\Phi\colon \mathsf{E} \to \mathbb{F}^k_{\mathsf{M}}$. One can relatively easily show that if M is contractible,—i.e., if there exists $x_0 \in \mathsf{M}$ and a continuous map $h\colon \mathsf{M} \times [0,1] \to \mathsf{M}$ for which $p(x,0) = x$ for $p(x,1) = x_0$ for every $x \in \mathsf{M}$—then every vector bundle over M is trivialisable [Abraham, Marsden, and Ratiu 1988, Theorem 3.4.35].

(b) There exist vector bundles that are not trivialisable. The classic examples are the tangent bundles of even-dimensional spheres which are not trivialisable as smooth or real analytic vector bundles (see [Milnor 1978] for an elementary proof). This implies that $\mathsf{T}^{1,0}\mathbb{CP}^1$ is a nontrivialisable holomorphic vector bundle.

4. Holomorphic vector bundles are in some sense more difficult to come by, so let us give a collection of examples of these, namely the line bundles over $\mathbb{CP}^1$. To do this, we recall from Example 4.2.2–4 that $\mathbb{CP}^1 \simeq \mathbb{S}^2$ and we consider the charts $(\mathcal{U}_+, \phi_+)$ and $(\mathcal{U}_-, \phi_-)$ given by

$$\mathcal{U}_+ = \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad \mathcal{U}_- = \mathbb{S}^2 \setminus \{(0,0,-1)\}$$

and

$$\phi_+(x_1,x_2,x_3) = \frac{x_1}{1-x_3} + \mathrm{i}\frac{x_2}{1-x_3}, \quad \phi_-(x_1,x_2,x_3) = \frac{x_1}{1+x_3} - \mathrm{i}\frac{x_2}{1+x_3}.$$

We alter slightly the notation of Example 4.2.2–3 to our purposes. We denote coordinates in these charts by z_+ and z_-, respectively. The overlap map, as we have seen, is $\phi_- \circ \phi_+^{-1}(z_+) = z_+^{-1}$. We will construct holomorphic line bundles over $\mathbb{S}^2$ by considering defining them on the chart domains $\mathcal{U}_+$ and $\mathcal{U}_-$, and asking that on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ the local vector bundle structures be related by a holomorphic vector bundle isomorphism. The vector bundles we construct we will denote by $\mathsf{O}_{\mathbb{CP}^1}(k)$, and these will be indexed by $k \in \mathbb{Z}$. First of all, note that since $\mathcal{U}_+$ and $\mathcal{U}_-$ are both contractible (they are holomorphically diffeomorphic to $\mathbb{C}$) every vector bundle over these open sets is trivialisable, so we can without loss of generality suppose them to be trivial. That is, we let $k \in \mathbb{Z}$ and we consider the two local vector bundles

$$\mathsf{E}_+(k) = \mathcal{U}_+ \times \mathbb{C}, \quad \mathsf{E}_-(k) = \mathcal{U}_- \times \mathbb{C}.$$

We then have holomorphic diffeomorphisms

$$\mathcal{U}_+ \times \mathbb{C} \ni (\boldsymbol{x}, w_+) \mapsto (z_+ = \phi_+(\boldsymbol{x}), w_+) \in \mathbb{C} \times \mathbb{C}$$

and

$$\mathcal{U}_- \times \mathbb{C} \ni (\boldsymbol{x}, w_-) \mapsto (z_- = \phi_-(\boldsymbol{x}), w_-) \in \mathbb{C} \times \mathbb{C},$$

i.e., we denote local vector bundle coordinates by (z_+, w_+) and (z_-, w_-), respectively. To define the vector bundle $\mathsf{O}_{\mathbb{CP}^1}(k)$ we patch these local vector bundle charts together by the local vector bundle isomorphism

$$\begin{aligned} \psi_\pm \colon \mathbb{C} \times \mathbb{C} &\to \mathbb{C} \times \mathbb{C} \\ (z_+, w_+) &\mapsto (z_+^{-1}, z_+^{-k} w_+) \end{aligned}.$$

The resulting vector bundle we denote by $\mathsf{O}_{\mathbb{CP}^1}(k)$, which is the ***line bundle of degree k*** over $\mathbb{CP}^1$. Note that $\mathsf{O}_{\mathbb{CP}^1}(0)$ is the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}$ since the overlap map is the identity map in this case.

As we shall see as we go along, various of these line bundle $\mathsf{O}_{\mathbb{CP}^1}(k)$ are of particular interest. In particular,

(a) $\mathsf{O}_{\mathbb{CP}^1}(-1)$ is called the ***tautological line bundle*** (see the construction of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)$ in Section 4.4).
(b) $\mathsf{O}_{\mathbb{CP}^1}(1)$ is called the ***hyperplane line bundle*** (again, see the more general construction in Section 4.4),
(c) $\mathsf{O}_{\mathbb{CP}^1}(-k)$ is the dual of $\mathsf{O}_{\mathbb{CP}^1}(k)$ (see Example 4.3.11),
(d) $\mathsf{O}_{\mathbb{CP}^1}(2)$ is isomorphic to the holomorphic tangent bundle of $\mathbb{CP}^1$ (see Example 4.5.14), and
(e) $\mathsf{O}_{\mathbb{CP}^1}(-2)$ is isomorphic to the bundle of holomorphic one-forms on $\mathbb{CP}^1$ (see Example 4.6.9).

We shall study line bundles over general projective spaces in Section 4.4.

Those who study such things prove that the line bundles $\mathsf{O}_{\mathbb{CP}^1}(k)$ are the *only* line bundles over $\mathbb{CP}^1$ up to isomorphism. It is also shown that any vector bundle over $\mathbb{CP}^1$ is isomorphic to a direct sum of these line bundles, a fact that is no longer true for vector bundles over higher-dimensional complex projective spaces [Griffiths and Harris 1978, Section 1.1]. •

One may verify the following properties of vector bundles.

4.3.6 Proposition *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathrm{r} \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^{r}. Then*

(i) M is a C^{r}-submanifold of E, and

(ii) π is a surjective submersion of class C^{r}.

When we wish to think of the base space M as a submanifold of E, we shall call it the ***zero section*** and denote it by $Z(\mathsf{E})$. For $z \in \mathsf{M}$, the set $\pi^{-1}(z)$ is the ***fibre*** over z, and is often written E_z. One may verify that the operations of vector addition and $\mathbb{C}$-scalar multiplication defined on E_z in a fixed vector bundle chart are actually independent of the choice made for this chart. Thus E_z is indeed a vector space. We will sometimes denote the zero vector in E_z as 0_z. If $\mathsf{N} \subseteq \mathsf{M}$ is a holomorphic submanifold, we denote by $\mathsf{E}|\mathsf{N}$ the ***restriction*** of the vector bundle to N, and we note that this is a vector bundle with base space N.

4.3.2 Holomorphic and real analytic vector bundle mappings

Let us now consider mappings between vector bundles.

4.3.7 Definition (Holomorphic and real analytic vector bundle mapping) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{N}$ be C^r-vector bundles and let $\Phi\colon \mathsf{E} \to \mathsf{F}$ be a map.

(i) The map Φ is ***fibre-preserving*** if there exists a map $\Phi_0\colon \mathsf{M} \to \mathsf{N}$ such that the diagram

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{\Phi} & \mathsf{F} \\ {\scriptstyle \pi_{\mathsf{E}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{F}}} \\ \mathsf{M} & \xrightarrow[\Phi_0]{} & \mathsf{N} \end{array}$$

commutes.

(ii) The map Φ is a ***vector bundle mapping*** of class C^r if

(a) it is fibre-preserving,

(b) it is of class C^r,

(c) the induced map $\Phi_0 \colon \mathsf{M} \to \mathsf{N}$ is of class C^r, and

(d) the map $\Phi_x \triangleq \Phi|\mathsf{E}_x \colon \mathsf{E}_x \to \mathsf{F}_{\Phi(x)}$ is $\mathbb{F}$-linear. •

We will often encode the induced mapping Φ_0 associated to a vector bundle mapping Φ by saying that "Φ is a vector bundle mapping over Φ_0." Let us look at the local form of a vector bundle mapping. Thus we let $\Phi \colon \mathsf{E} \to \mathsf{F}$ be a vector bundle mapping over $\Phi_0 \colon \mathsf{M} \to \mathsf{N}$, let $x \in \mathsf{M}$, and let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-vector bundle chart for E such that $\mathsf{E}_x \subseteq \mathcal{U}$, and let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart for F such that $\mathsf{F}_{\Phi_0(x)} \subseteq \mathcal{V}$ and $\Phi(\mathcal{U}) \subseteq \mathcal{V}$. Let us denote

$$\phi(\mathcal{U}) = \mathcal{U}_0 \times \mathbb{F}^k, \quad \psi(\mathcal{V}) = \mathcal{V}_0 \times \mathbb{F}^l$$

for open sets $\mathcal{U}_0 \subseteq \mathbb{F}^n$ and $\mathcal{V}_0 \subseteq \mathbb{F}^m$. Then one directly verifies that the local representative of Φ is given by

$$\psi \circ \Phi \circ \phi^{-1}(\boldsymbol{x}, \boldsymbol{v}) = (\boldsymbol{F}_0(\boldsymbol{x}), \boldsymbol{F}_1(\boldsymbol{x}) \cdot \boldsymbol{v}),$$

where $\boldsymbol{F}_0 \colon \mathcal{U}_0 \to \mathcal{V}_0$ and $\boldsymbol{F}_1 \colon \mathcal{U}_0 \to \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^k; \mathbb{F}^l)$ are of class C^r.

Associated with any vector bundle mapping are standard algebraic constructions.

4.3.8 Definition (Kernel and image of vector bundle mapping) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be C^r-vector bundles and let $\Phi \colon \mathsf{E} \to \mathsf{F}$ be a C^r-vector bundle mapping over id_{M}.

(i) The ***kernel*** of Φ is the subset $\ker(\Phi)$ of E given by $\ker(\Phi) = \cup_{x \in \mathsf{M}} \ker(\Phi_x)$.

(ii) The ***image*** of Φ is the subset $\mathrm{image}(\Phi)$ of F given by $\mathrm{image}(\Phi) = \cup_{x \in \mathsf{M}} \mathrm{image}(\Phi_x)$.

•

There are then some naturally induced constructions one can introduce associated with sequences of vector bundle mappings.

4.3.9 Definition (Exact sequences of vector bundles) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$, $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$, and $\pi_{\mathsf{G}} \colon \mathsf{G} \to \mathsf{M}$ be C^r-vector bundles, and let $\Phi \colon \mathsf{E} \to \mathsf{F}$ and $\Psi \colon \mathsf{F} \to \mathsf{G}$ be C^r-vector bundle mappings over id_{M}.

(i) The sequence

$$\mathsf{E} \xrightarrow{\Phi} \mathsf{F} \xrightarrow{\Psi} \mathsf{G}$$

is ***exact*** if $\ker(\Psi) = \mathrm{image}(\Phi)$.

(ii) The vector bundle mapping Ψ is ***injective*** if the sequence

$$0 \longrightarrow \mathsf{F} \xrightarrow{\Psi} \mathsf{G}$$

is exact.

(iii) The vector bundle mapping Φ is ***surjective*** if the sequence

$$\mathsf{E} \xrightarrow{\Phi} \mathsf{F} \longrightarrow 0$$

is exact.

(iv) The vector bundle mapping Φ is an ***isomorphism*** if the sequence

$$0 \longrightarrow \mathsf{E} \xrightarrow{\Phi} \mathsf{F} \longrightarrow 0$$

is exact, i.e., if it is surjective and injective;

(v) A ***short exact sequence*** of vector bundles is a sequence

$$0 \longrightarrow \mathsf{E} \xrightarrow{\Phi} \mathsf{F} \xrightarrow{\Psi} \mathsf{G} \longrightarrow 0$$

for which each subsequence is exact. •

Let us show that vector bundle mappings themselves comprise the basis for a new vector bundle formed from existing vector bundles. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. To keep things simple, we consider C^r-vector bundles $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ over the same base and vector bundle mappings $\Phi \colon \mathsf{E} \to \mathsf{F}$ over id_{M}. Let us denote $\mathrm{Hom}_{\mathbb{F}}(\mathsf{E}; \mathsf{F})_x = \mathrm{Hom}_{\mathbb{F}}(\mathsf{E}_x, \mathsf{F}_x)$ and

$$\mathrm{Hom}_{\mathbb{F}}(\mathsf{E}; \mathsf{F}) = \mathop{\mathring{\bigcup}}_{x \in \mathsf{M}} \mathrm{Hom}_{\mathbb{F}}(\mathsf{E}; \mathsf{F})_x.$$

Suppose that we have $\mathbb{F}$-vector bundle charts $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ for E and F, respectively. We can suppose, without loss of generality, that the induced charts $(\mathcal{U}_0, \phi_0)$ and $(\mathcal{U}_0, \psi_0)$ for M are the same. Thus

$$\phi(e_x) = (\phi_0(x), \phi_1(x) \cdot e_x) \in \phi_0(\mathcal{U}_0) \times \mathbb{F}^k, \quad \psi(f_x) = (\phi_0(x), \psi_1(x) \cdot f_x) \in \phi_0(\mathcal{U}_0) \times \mathbb{F}^l$$

for $e_x \in \mathcal{U}$ and $f_x \in \mathcal{V}$, and where $\phi_1(x) \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{E}_x, \mathbb{F}^k)$ and $\psi_1(x) \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{F}_x; \mathbb{F}^l)$. We then define

$$\mathrm{Hom}_{\mathbb{F}}(\mathcal{U};\mathcal{V}) = \mathop{\mathring{\cup}}_{x\in\mathcal{U}_0} \mathrm{Hom}_{\mathbb{F}}(\mathsf{E};\mathsf{F})_x$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}}(\phi;\psi)\colon \mathrm{Hom}_{\mathbb{F}}(\mathcal{U};\mathcal{V}) &\to \phi_0(\mathcal{U}_0)\times \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^k;\mathbb{F}^l) \\ A_x &\mapsto \psi_1(x)\circ A_x \circ \phi_1(x)^{-1}. \end{aligned} \tag{4.10}$$

One readily verifies that $(\mathrm{Hom}_{\mathbb{F}}(\mathcal{U};\mathcal{V}), \mathrm{Hom}_{\mathbb{F}}(\phi,\psi))$ is a vector bundle chart and that two overlapping vector bundle charts satisfy the required overlap condition. Thus we endow $\mathrm{Hom}_{\mathbb{F}}(\mathsf{E};\mathsf{F})$ with the structure of a vector bundle.

Of special interest is the following.

4.3.10 Definition (Dual bundle) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. The ***dual bundle*** to E is the bundle $\mathsf{E}^* = \mathrm{Hom}_{\mathbb{F}}(\mathsf{E};\mathbb{F}_{\mathsf{M}})$. •

Let us consider some examples of holomorphic dual bundles.

4.3.11 Example (Dual bundles for line bundles over $\mathbb{CP}^1$) We consider the line bundles $\mathsf{O}_{\mathbb{CP}^1}(k)$, $k \in \mathbb{Z}$, introduced in Example 4.3.5–4. We claim that the dual of $\mathsf{O}_{\mathbb{CP}^1}(k)$ is isomorphic to $\mathsf{O}_{\mathbb{CP}^1}(-k)$ for every $k \in \mathbb{Z}$. To prove this we work with the local trivialisations

$$\mathsf{E}_+ = \mathcal{U}_+ \times \mathbb{C}, \quad \mathsf{E}_- = \mathcal{U}_- \times \mathbb{C}$$

for $\mathsf{O}_{\mathbb{CP}^1}(k)$ with the overlap map

$$(z_+, w_+) \mapsto (z_+^{-1}, z_+^{-k} w_+).$$

The corresponding overlap map for $\mathsf{O}^*_{\mathbb{CP}^1}(k)$ is then

$$(z_+, \alpha_+) \mapsto (z_+^{-1}, z_+^{k}\alpha_+)$$

cf. (4.10), and this establishes our claim. •

4.3.3 Sections and germs of sections of holomorphic and real analytic vector bundles

Just as with holomorphic and real analytic functions, one of the areas of departure holomorphic and real analytic geometry from smooth geometry concerns sections of vector bundles. In this section we merely give the basic definitions, reserving for later the difficult questions of existence of nontrivial objects we define.

4.3.12 Definition (Holomorphic and real analytic section) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. A ***C^r-section*** of E is a C^r-map $\xi\colon \mathsf{M} \to \mathsf{E}$ such that $\xi(x) \in \mathsf{E}_x$. The set of C^r-sections of E is denoted by $\Gamma^r(\mathsf{E})$. •

Of course, in both the real and complex cases, we can also consider smooth sections. These are denoted by $\Gamma^\infty(\mathsf{E})$ in the usual way.

Let us consider some simple examples of sections of vector bundles.

4.3.13 Examples (Sections of vector bundles)

1. A C^r-section of the trivial vector bundle $\mathbb{F}^k_{\mathsf{M}} = \mathsf{M} \times \mathbb{F}^k$ takes the form $x \mapsto (x, \boldsymbol{\xi}(x))$ for a C^r-map $\boldsymbol{\xi}\colon \mathsf{M} \to \mathbb{F}^k$. Thus we identify sections of the trivial vector bundle with $\mathbb{F}^k$-valued functions. In case $k = 1$ this means that we identify sections of $\mathbb{F}_{\mathsf{M}}$ with functions in the usual sense.
2. A section Φ of $\mathrm{Hom}_{\mathbb{F}}(\mathsf{E}; \mathsf{F})$ is simply a vector bundle map $\Phi\colon \mathsf{E} \to \mathsf{F}$ over id_{M}. •

Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a $\mathbb{F}$-vector bundle. Let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. We note that $\Gamma^r(\mathsf{E})$ is a module over the ring $\mathrm{C}^r(\mathsf{M})$ with the module structure defined in the obvious way:

$$(\xi + \eta)(x) = \xi(x) + \eta(x), \quad (f\xi)(x) = f(x)\xi(x),$$

where $\xi, \eta \in \Gamma^r(\mathsf{E})$ and $f \in \mathrm{C}^r(\mathsf{M})$. It is also the case that $\Gamma^r(\mathsf{E})$ has the structure of an $\mathbb{F}$-vector space if we defined scalar multiplication by $(a\xi)(x) = a(\xi(x))$.

Note that one can find many sections of smooth vector bundles, because locally defined section can be extended to globally defined sections using constructions involving the Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, §5.5]. However, for holomorphic and real analytic vector bundles, there is no *a priori* reason that there are sections other than the zero section. As a concrete instance of this phenomenon, note that holomorphic or real analytic sections of the trivial vector bundle $\mathsf{M} \times \mathbb{F}$ are merely holomorphic or real analytic functions. Thus, for example, if M is a compact holomorphic manifold, the only holomorphic sections of the trivial vector bundle $\mathsf{M} \times \mathbb{C}$ are constant sections.

Let us flesh out the preceding discussion by considering spaces of sections of line bundles over $\mathbb{CP}^1$.

4.3.14 Example (Sections of line bundles over $\mathbb{CP}^1$) We shall consider the line bundles $\mathsf{O}_{\mathbb{CP}^1}(k)$, $k \in \mathbb{Z}$, introduced in Example 4.3.5–4. We shall characterise the space of sections of these bundles. In doing so, we recall the following fact:

If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic, it admits a global power series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

that converges absolutely and uniformly on compact sets [Conway 1978, Proposition IV.3.3].

Now suppose that we have a holomorphic section ξ of $\mathsf{O}_{\mathbb{CP}^1}(k)$, and let

$$z_+ \mapsto (z_+, \xi_+(z_+)), \quad z_- \mapsto (z_-, \xi_-(z_-))$$

denote the local representatives of this section in the trivialisations E_+ and E_-, cf. Example 4.3.5–4. Let us write

$$\xi_+(z_+) = \sum_{j=0}^{\infty} a_{+,j} z_+^j,$$

so that, applying the overlap map to the section ξ_+, we have

$$\psi_\pm(z_+, \xi_+(z_+)) = \Big(z_+^{-1}, \sum_{j=0}^{\infty} a_{+,j} z_+^{j-k}\Big) = \Big(z_-, \sum_{j=0}^{\infty} a_{+,j} z_-^{k-j}\Big).$$

Since

$$z_- \mapsto \sum_{j=0}^{\infty} a_{+,j} z_-^{k-j}$$

must be a holomorphic function if ξ is a holomorphic section. It must be the case, therefore, that $a_{+,j} = 0$ if $j > k$. From this we conclude that

$$\dim_{\mathbb{C}}(\Gamma^{\mathrm{hol}}(\mathsf{O}_{\mathbb{CP}^1}(k))) = \begin{cases} k+1, & k \in \mathbb{Z}_{\geq 0}, \\ 0, & k \in \mathbb{Z}_{<0}. \end{cases}$$

A few comments here are worth making.

1. In all cases, the dimension of the vector space of sections of these vector bundles is finite. This is not like what we are used to for smooth sections, where the vector space of smooth sections of a vector bundle over a manifold with a component of positive dimension is always infinite-dimensional. This immediately begs a few related questions.

   (a) Are there holomorphic vector bundles whose $\mathbb{C}$-vector space of holomorphic sections is infinite-dimensional?

   (b) Is the $\mathbb{R}$-vector space of real analytic sections of a real analytic vector bundle finite or infinite-dimensional?

   The answers to these questions are, "Yes, vector bundles over Stein manifolds," and, "It is infinite-dimensional if the base space has a component of positive dimension." To obtain these answers requires substantial effort, particularly the use of sheaf cohomology. This will all be revealed in time.

2. Keeping Example 4.3.11 in mind, we see that the dimension of $\Gamma^{\mathrm{hol}}(\mathsf{O}_{\mathbb{CP}^1}(k))$ and $\Gamma^{\mathrm{hol}}(\mathsf{O}_{\mathbb{CP}^1}(k)^*)$ agree only when $k = 0$. This again is contrary to the smooth case where the existence of a vector bundle metric provides an isomorphism between the space of sections of a vector bundle and its dual bundle. •

The Identity Theorem applies to sections.

4.3.15 Theorem (Identity Theorem for sections) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a C^{r}-vector bundle with M a connected manifold, if $\mathcal{U} \subseteq \mathsf{M}$ is a nonempty open set and if $\xi, \eta \in \Gamma^{\mathrm{r}}(\mathsf{E})$ satisfy $\xi|\mathcal{U} = \eta|\mathcal{U}$, then $\xi = \eta$.*

Proof It suffices to show that if $\xi(x) = 0$ for every $x \in \mathcal{U}$ then ξ is the zero function. Let

$$\mathcal{C} = \{x \in \mathsf{M} \mid \ \xi(x) = 0\},$$

and note that $\mathcal{C}$ is closed with $\mathrm{int}(\mathcal{C}) \neq \emptyset$ since it contains $\mathcal{U}$. We claim that $\mathrm{bd}(\mathrm{int}(\mathcal{C})) = \emptyset$. Indeed, suppose that $x \in \mathrm{bd}(\mathcal{C})$. Let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart with $x \in \mathcal{V}$. Let $\mathcal{V}_0 \subseteq \mathsf{M}$ be such that $\mathcal{V} = \pi^{-1}(\mathcal{V}_0)$. Also let $\psi_0\colon \mathcal{V}_0 \to \mathbb{F}^n$ be the chart map for M with domain $\mathcal{V}_0$. Thus ψ_0 is defined by asking that $\psi_0(x) = \psi(e)$ for every $e \in \mathsf{E}_x$. The local representative of ξ is then of the form

$$\psi \circ \xi \circ \psi_0^{-1}(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$$

for holomorphic or real analytic $\boldsymbol{\xi}\colon \psi_0(\mathcal{V}_0) \to \mathbb{F}^k$. By continuity of ξ and its derivatives, the Taylor series of $\boldsymbol{\xi}$ at $\psi_0(x)$ is zero. Since ξ is holomorphic or real analytic, this implies that $\boldsymbol{\xi}$ vanishes in a neighbourhood of $\psi_0(x)$, and so ξ vanishes in a neighbourhood of x, which is a contradiction. We now claim that $\mathsf{M} \setminus \mathrm{int}(\mathcal{C})$ is open. Indeed, let $x \in \mathsf{M} \setminus \mathrm{int}(\mathcal{C})$ be a point not in the interior of $\mathsf{M} \setminus \mathrm{int}(\mathcal{C})$. Then every neighbourhood of x must intersect $\mathcal{C}$ and so $x \in \mathrm{bd}(\mathrm{int}(\mathcal{C})) = \emptyset$, and so $\mathsf{M} \setminus \mathrm{int}(\mathcal{C})$ is open. Since M is now the union of the disjoint open sets $\mathrm{int}(\mathcal{C})$ and $\mathsf{M} \setminus \mathrm{int}(\mathsf{C})$ and since the former is nonempty, we must have $\mathsf{M} \setminus \mathrm{int}(\mathcal{C}) = \emptyset$, giving $\mathrm{int}(\mathcal{C}) = \mathsf{M}$ and so $\mathcal{C} = \mathsf{M}$. ■

Next we consider germs of sections as this will allow us to systematically discuss local constructions. This is done more or less exactly as was done for functions. Let $x_0 \in \mathsf{M}$. We define as follows an equivalence relation on the set of ordered pairs $(\xi, \mathcal{U})$, where $\mathcal{U} \subseteq \mathsf{M}$ is a neighbourhood of x_0 and $\xi \in \Gamma^r(\mathcal{U})$. We say that $(\xi_1, \mathcal{U}_1)$ and $(\xi_2, \mathcal{U}_2)$ are ***equivalent*** if there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ of x_0 such that $\xi_1 \mathcal{U}_1 = \xi_2|\mathcal{U}$. We denote a typical equivalence class by $[(\xi, \mathcal{U})]_{x_0}$, or simply by $[\xi]_{x_0}$ if the domain of ξ is understood or immaterial. The set of equivalence classes we denote by $\mathscr{G}^r_{x_0,\mathsf{E}}$, which we call the set of ***germs*** of holomorphic or real analytic sections at x_0, respectively. We make the set of germs into a module over $\mathscr{C}^r_{x_0,\mathsf{M}}$ by defining the following operations of addition and multiplication:

$$\begin{aligned}
[(\xi_1, \mathcal{U}_1)]_{x_0} + [(\xi_2, \mathcal{U}_2)]_{x_0} &= [\xi_1|\mathcal{U}_1 \cap \mathcal{U}_2 + \xi_2|\mathcal{U}_1 \cap \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2]_{x_0} \\
[(f, \mathcal{U}_1)]_{x_0} \cdot [(\xi, \mathcal{U}_2)]_{x_0} &= [(f|\mathcal{U}_1 \cap \mathcal{U}_2)(\xi|\mathcal{U}_1 \cap \mathcal{U}_2), \mathcal{U}_1 \cap \mathcal{U}_2]_{x_0}.
\end{aligned}$$

It is elementary to verify that these operations are well-defined, and indeed make the set of germs of holomorphic or real analytic sections into a module as asserted. As with sections, germs of sections also have an $\mathbb{F}$-vector space structure: $a[(f, \mathcal{U})]_{x_0} = [(af, \mathcal{U})]_{x_0}$.

Let us examine some of the algebraic properties of $\mathscr{C}^r_{x_0,\mathsf{M}}$. As we did when talking about the algebraic structure of germs of functions, we state a result which turns the problem into one about convergent formal power series.

4.3.16 Proposition (Characterisation of holomorphic and real analytic section germs) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. For $\pi\colon \mathsf{E} \to \mathsf{M}$ a vector bundle of class C^{r} and for $\mathrm{x}_0 \in \mathsf{M}$, the module $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{E}}$ is isomorphic to the ring $\hat{\mathbb{F}}[[\xi_1, \dots, \xi_{\mathrm{n}}]] \otimes \mathbb{F}^{\mathrm{m}}$ of $\mathbb{F}^{\mathrm{m}}$-valued convergent power series in n indeterminates, where n is the dimension of the connected component of M containing x_0 and $\mathrm{m} = \dim_{\mathbb{F}}(\mathsf{E}_{\mathrm{x}_0})$.*

Proof Let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart for E with $(\mathcal{U}, \phi)$ be the associated $\mathbb{F}$-chart about x_0 such that $\phi(x_0) = \mathbf{0}$. We identify $\mathcal{U}$ with $\phi(\mathcal{U}) \subseteq \mathbb{F}^n$ and a section of class C^r on $\mathcal{U}$ with its local representative. A section of class C^r, by definition, is one whose Taylor series converges in some neighbourhood of every point in its domain of definition, and which is equal to its Taylor series on that neighbourhood. Thus, if $[(\xi, \mathcal{V})]_{\mathbf{0}} \in \mathscr{G}^r_{\mathbf{0},\mathcal{U}}$, in some neighbourhood $\mathcal{V}'$ of $\mathbf{0}$ in $\mathcal{U}$ we have

$$\boldsymbol{\xi}(x_1, \dots, x_n) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{\partial^{|I|} \xi^a}{\partial \boldsymbol{x}^I}(\mathbf{0}) \boldsymbol{e}_a,$$

where $\xi(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$ and where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ is the standard basis for $\mathbb{F}^m$. Thus $[(\xi, \mathcal{V})]_{\mathbf{0}}$ is determined by its Taylor series, which gives a surjective map from $\mathscr{C}^r_{\mathbf{0},\mathcal{U}}$ to $\hat{\mathbb{F}}[[\boldsymbol{\xi}]] \otimes \mathbb{F}^m$. That this map is also injective follows since two analytic functions having the same Taylor series at a point are obviously equal on some neighbourhood of that point. ■

With this result we have the following structure of the module of holomorphic or real analytic section.

4.3.17 Theorem (The module of holomorphic or real analytic section germs is Noetherian) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. For $\pi\colon \mathsf{E} \to \mathsf{M}$ a vector bundle of class C^{r} and for $\mathrm{x}_0 \in \mathsf{M}$, the module $\mathscr{C}^{\mathrm{r}}_{\mathrm{x}_0,\mathsf{E}}$ is Noetherian.*

Proof As in the proof of Theorem 4.2.7, this follows from Proposition 4.3.16, along with Theorem 4.2.7 and Proposition 2.2.15. ■

Of course, the previous constructions apply equally well in the smooth case, and we shall occasionally access the notation $\mathscr{G}^{\infty}_{x,\mathsf{E}}$ for the module of germs of smooth sections at x_0. However, this module is not Noetherian as in the holomorphic and real analytic cases. As with germs of functions (see the end of Section 4.2.3), there is also an important distinction between germs of smooth sections and germs of holomorphic or real analytic sections in terms of the existence of global generators. In the smooth case, any germ of a smooth section has a globally defined representative. We state this as Proposition 5.6.5 below, also cf. Propositions 5.6.4 and 4.2.8.

4.3.4 Holomorphic and real analytic subbundles and quotients

Subbundles arise naturally in various ways when working with vector bundles. In this section we provide the necessary definitions.

4.3.18 Definition (Holomorphic and real analytic subbundle) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. A subset $\mathsf{F} \subseteq \mathsf{E}$ is a $\mathbf{C}^{\mathbf{r}}$***-subbundle*** of E if, for each $x \in \mathsf{M}$, there exists an $\mathbb{F}$-vector bundle chart $(\mathcal{U}, \phi)$ such that

(i) $\mathsf{E}_x \subseteq \mathcal{U}$,

(ii) $\text{image}(\phi) = \mathcal{U}_0 \times (\mathbb{F}^k \times \mathbb{F}^l)$, and

(iii) $\phi(\mathsf{F} \cap \mathcal{U}) = \mathcal{U}_0 \times (\mathbb{F}^k \times \{\mathbf{0}\})$.

A vector bundle chart for E as above is called ***adapted*** to the subbundle F. •

It is evident that a subbundle of a vector bundle is itself a vector bundle.

One of the places where subbundles can naturally arise is as in the following result.

4.3.19 Proposition (Kernel and image of a vector bundle mapping as subbundles) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be C^{r}-vector bundles, and let $\Phi \colon \mathsf{E} \to \mathsf{F}$ be a C^{r}-vector bundle mapping over id_{M}. The following statements are equivalent:*

(i) $\ker(\Phi)$ is a subbundle of E;

(ii) $\text{image}(\Phi)$ is a subbundle of F;

(iii) the map $\mathrm{x} \mapsto \text{rank}(\Phi_{\mathrm{x}})$ is locally constant.

Proof Let us first make some computations with the local representative of Φ about $x_0 \in \mathsf{M}$. Thus we let $(\mathcal{U}, \phi)$ be a vector bundle chart for E about x_0 and let $(\mathcal{V}, \psi)$ be a vector bundle chart for F about x_0. Let $(\mathcal{U}_0, \phi_0)$ and $(\mathcal{V}_0, \psi_0)$ be the induced charts for M. Without loss of generality (by composing one of these charts with the overlap map) we can suppose that these charts agree. Denote $\boldsymbol{x}_0 = \phi_0(x_0)$. Let us write

$$\phi(\mathcal{U}) = \phi_0(\mathcal{U}_0) \times \mathbb{F}^k, \quad \psi(\mathcal{V}) = \phi_0(\mathcal{U}_0) \times \mathbb{F}^l.$$

The local representative of Φ we denote by

$$(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{v}).$$

Let $\mathsf{K} = \ker(\boldsymbol{\Phi}(\boldsymbol{x}_0))$ and let $\mathsf{L} \subseteq \mathbb{F}^k$ be a complement to K. Let $\mathsf{R} = \text{image}(\boldsymbol{\Phi}(\boldsymbol{x}_0))$ and let $\mathsf{S} \subseteq \mathbb{F}^l$ be a complement to R. Then we represent $\boldsymbol{\Phi}(\boldsymbol{x})$ relative to the decompositions $\mathbb{F}^k = \mathsf{K} \oplus \mathsf{L}$ and $\mathbb{F}^l = \mathsf{R} \oplus \mathsf{S}$ by

$$\begin{bmatrix} \boldsymbol{\Phi}_{11}(\boldsymbol{x}) & \boldsymbol{\Phi}_{12}(\boldsymbol{x}) \\ \boldsymbol{\Phi}_{21}(\boldsymbol{x}) & \boldsymbol{\Phi}_{22}(\boldsymbol{x}) \end{bmatrix}.$$

Note that $\boldsymbol{\Phi}_{21}(\boldsymbol{x}_0)$ and $\boldsymbol{\Phi}_{22}(\boldsymbol{x}_0)$ are both zero since S is complementary to $\text{image}(\boldsymbol{\Phi}(\boldsymbol{x}_0))$. Since $\mathsf{K} = \ker(\boldsymbol{\Phi}(\boldsymbol{x}_0))$, $\boldsymbol{\Phi}_{11}(\boldsymbol{x}_0)$ is also zero. One then directly checks that $\boldsymbol{\Phi}_{12}(\boldsymbol{x}_0)$ is an isomorphism. By shrinking $\mathcal{U}_0$ if necessary, we can ensure that $\boldsymbol{\Phi}(\boldsymbol{x})$ is an isomorphism for $\boldsymbol{x} \in \phi_0(\mathcal{U}_0)$. Let us define an isomorphism $\boldsymbol{A}$ from $\mathsf{K} \oplus \mathsf{L}$ to $\mathsf{K} \oplus \mathsf{L}$ by

$$\begin{bmatrix} \mathrm{id}_{\mathsf{K}} & \mathbf{0} \\ -\boldsymbol{\Phi}_{12}^{-1}(\boldsymbol{x})\boldsymbol{\Phi}_{11}(\boldsymbol{x}) & \mathrm{id}_{\mathsf{L}} \end{bmatrix}.$$

We then compute

$$\boldsymbol{\Phi}\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\Phi}_{12}(\boldsymbol{x}) \\ \boldsymbol{\Phi}_{21}(\boldsymbol{x}) - \boldsymbol{\Phi}_{22}\boldsymbol{\Phi}_{12}^{-1}(\boldsymbol{x})\boldsymbol{\Phi}_{11}(\boldsymbol{x}) & \boldsymbol{\Phi}_{22}(\boldsymbol{x}) \end{bmatrix}. \tag{4.11}$$

Note that $(\mathcal{U}, (\mathrm{id}_{\phi_0(\mathcal{U}_0)} \times \boldsymbol{A}^{-1}) \circ \phi)$ is a vector bundle chart for E about x_0 and the local representative of Φ relative to this chart and the vector bundle chart $(\mathcal{V}, \psi)$ is (4.11).

With these computations we proceed with the proof.

(i) $\Longrightarrow$ (ii) Since $\ker(\Phi)$ is a subbundle, $\dim(\ker(\Phi_x))$ is constant in a sufficiently small neighbourhood of x_0. Thus, by the Rank–Nullity formula, $\mathrm{rank}(\Phi_x)$ is also constant in a sufficiently small neighbourhood of x_0. Using our local representative above, the map

$$\boldsymbol{v}_1 \oplus \boldsymbol{v}_2 \mapsto \boldsymbol{\Phi}_{11}(\boldsymbol{x}) \cdot \boldsymbol{v}_1 \oplus \boldsymbol{\Phi}_{12}(\boldsymbol{x}) \cdot \boldsymbol{v}_2$$

is onto R. Since $\mathrm{rank}(\Phi_x)$ can be taken to be constant on $\phi_0(\mathcal{U}_0)$ it follows that $\boldsymbol{\Phi}_{21}(\boldsymbol{x})$ and $\boldsymbol{\Phi}_{22}(\boldsymbol{x})$ are zero for $\boldsymbol{x} \in \phi_0(\mathcal{U}_0)$. Thus

$$\psi(\mathrm{image}(\Phi) \cap \mathcal{V}) = \mathcal{U}_0 \times (\mathsf{R} \oplus \boldsymbol{0}).$$

This shows that $\mathrm{image}(\Phi)$ is a subbundle.

(ii) $\Longrightarrow$ (iii) This follows directly.

(iii) $\Longrightarrow$ (i) As we saw in the proof that (i) $\Longrightarrow$ (ii), local constancy of $x \mapsto \mathrm{rank}(\Phi_x)$ implies that $\boldsymbol{\Phi}_{21}(\boldsymbol{x})$ and $\boldsymbol{\Phi}_{22}(\boldsymbol{x})$ are zero for $\boldsymbol{x} \in \phi_0(\mathcal{U}_0)$. Thus

$$\phi(\ker(\Phi) \cap \mathcal{U}) = \mathcal{U}_0 \times (\mathsf{K} \oplus \boldsymbol{0}).$$

This shows that $\ker(\Phi)$ is a subbundle. ■

Let us briefly consider quotients of vector bundles by subbundles. Thus we let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a holomorphic or real analytic vector bundle and let $\mathsf{F} \subseteq \mathsf{E}$ be a holomorphic or real analytic subbundle. Let

$$\mathsf{E}/\mathsf{F} = \mathop{\mathring{\cup}}_{x \in \mathsf{M}} \mathsf{E}_x/\mathsf{F}_x.$$

Suppose that $(\mathcal{U}, \phi)$ is a vector bundle chart for E adapted to F and let $(\mathcal{U}_0, \phi_0)$ be the corresponding chart for M. Thus ϕ takes values in $\mathbb{F}^k \times \mathbb{F}^l$ and $\phi(\mathsf{F} \cap \mathcal{U}) = \phi(\mathcal{U}_0) \times (\mathbb{F}^k \times \{\boldsymbol{0}\})$. For $x \in \mathcal{U}_0$ and $e_x \in \mathsf{E}_x$ we can write $e_x = \mathrm{pr}_1(e_x) + \mathrm{pr}_2(e_x)$, where

$$\phi(\mathrm{pr}_1(e_x)) = (\phi_0(x), \{\boldsymbol{v}_1\} \times \{\boldsymbol{0}\}), \quad \phi(\mathrm{pr}_2(e_x)) = (\phi_0(x), \{\boldsymbol{0}\} \times \{\boldsymbol{v}_2\})$$

for some appropriate $\boldsymbol{v}_1 \in \mathbb{F}^k$ and $\boldsymbol{v}_2 \in \mathbb{F}^l$. Let us define a vector bundle chart $(\hat{\mathcal{U}}, \hat{\phi})$ for E/F by asking that

$$\hat{\mathcal{U}} = \mathop{\mathring{\cup}}_{x \in \mathcal{U}_0} \mathsf{E}_x/\mathsf{F}_x$$

and that

$$\hat{\phi}(e_x + \mathsf{F}_x) = \phi(\mathrm{pr}_2(e_x)) \in \phi_0(\mathcal{U}_0) \times \{\boldsymbol{0}\} \times \mathbb{R}^l \simeq \phi_0(\mathcal{U}_0) \times \mathbb{R}^l.$$

One readily verifies that if $((\mathcal{U}_a, \phi_a))_{a \in A}$ is an atlas of vector bundle charts for E adapted to F, then $((\hat{\mathcal{U}}_a, \hat{\phi}_a))_{a \in A}$ is a vector bundle atlas for E/F, and so the latter is a vector bundle, called the ***quotient*** of E by F. We denote by $\pi_{\mathsf{E}/\mathsf{F}}\colon \mathsf{E} \to \mathsf{E}/\mathsf{F}$ the canonical projection. We evidently have a short exact sequence

$$0 \longrightarrow \mathsf{F} \xrightarrow{\iota_{\mathsf{F}}} \mathsf{E} \xrightarrow{\pi_{\mathsf{E}/\mathsf{F}}} \mathsf{E}/\mathsf{F} \longrightarrow 0$$

with ι_{F} being the inclusion.

4.3.5 Sums and tensor products of vector bundles

Most of the standard algebraic operations one performs with vector spaces can be performed fibre-wise for vector bundles to produce new vector bundles. We have already seen one such instance with quotients in the last section. In this section we consider two algebraic constructions, direct sums and tensor products. We shall simply provide the definitions for the vector bundle structure in these cases, leaving the verifications that the overlap maps are local vector bundle isomorphisms to the reader.

First let us consider direct sums. We consider the direct sum of two vector bundles, the extension to more than two factors being mere notation. Thus we let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$ be holomorphic or real analytic vector bundles over the same base. We denote

$$\mathsf{E} \oplus \mathsf{F} = \mathop{\mathring{\cup}}_{x \in \mathsf{M}} \mathsf{E}_x \oplus \mathsf{F}_x.$$

We let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be $\mathbb{F}$-vector bundle charts for E and F, respectively. We can suppose, without loss of generality, that the induced charts $(\mathcal{U}_0, \phi_0)$ and $(\mathcal{U}_0, \psi_0)$ for M are the same. Thus

$$\phi(e_x) = (\phi_0(x), \phi_1(x) \cdot e_x) \in \phi_0(\mathcal{U}_0) \times \mathbb{F}^k, \quad \psi(f_x) = (\phi_0(x), \psi_1(x) \cdot f_x) \in \phi_0(\mathcal{U}_0) \times \mathbb{F}^l$$

for $e_x \in \mathcal{U}$ and $f_x \in \mathcal{V}$, and where $\phi_1(x) \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{E}_x, \mathbb{F}^k)$ and $\psi_1(x) \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{F}_x; \mathbb{F}^l)$. In this case, we define a chart $(\mathcal{U} \oplus \mathcal{V}, \phi \oplus \psi)$ for $\mathsf{E} \oplus \mathsf{F}$ by

$$\mathcal{U} \oplus \mathcal{V} = \mathop{\mathring{\cup}}_{x \in \mathcal{U}_0} \mathsf{E}_x \oplus \mathsf{F}_x$$

and

$$\phi \oplus \psi(e_x \oplus f_x) = (\phi_0(x), (\phi_1(x) \cdot e_x) \oplus (\psi_1(x) \cdot f_x)) \in \mathcal{U} \times (\mathbb{F}^k \oplus \mathbb{F}^l).$$

The vector bundle $\mathsf{E} \oplus \mathsf{F}$ is sometimes called the ***Whitney sum*** of E and F. Sometimes it is convenient to use the fibred product representation of $\mathsf{E} \oplus \mathsf{F}$:

$$\mathsf{E} \oplus \mathsf{F} \simeq \{(e, f) \in \mathsf{E} \times \mathsf{F} \mid \pi_{\mathsf{E}}(e) = \pi_{\mathsf{F}}(f)\}.$$

Now let us consider tensor products. As above, we let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$ be holomorphic or real analytic vector bundles over the same base. We denote

$$\mathsf{E} \otimes \mathsf{F} = \mathop{\mathring{\cup}}_{x \in \mathsf{M}} \mathsf{E}_x \otimes \mathsf{F}_x.$$

We use the same sort of vector bundle charts as above, i.e., charts $(\mathcal{U}, \phi)$ and $(\mathcal{U}, \psi)$ with the same domain and inducing the same chart $(\mathcal{U}_0, \phi_0)$ for M. Using notation adopted from our direct sums above, we define a chart $(\mathcal{U} \otimes \mathcal{V}, \phi \otimes \psi)$ for $\mathsf{E} \otimes \mathsf{F}$ by

$$\mathcal{U} \otimes \mathcal{V} = \mathop{\mathring{\cup}}_{x \in \mathcal{U}_0} \mathsf{E}_x \otimes \mathsf{F}_x$$

and

$$\phi \otimes \psi(e_x \otimes f_x) = (\phi_0(x), (\phi_1(x) \cdot e_x) \otimes (\psi_1(x) \cdot f_x)) \in \mathcal{U} \times (\mathbb{F}^k \otimes \mathbb{F}^l).$$

Again, we leave to the reader the task of verifying that these charts obey vector bundle overlap conditions.

We have at our disposal some nice examples of tensor products of vector bundles.

4.3.20 Example (Tensor products of line bundles over $\mathbb{CP}^1$) We consider the line bundles $\mathsf{O}_{\mathbb{CP}^1}(k)$ over $\mathbb{CP}^1$ introduced in Example 4.3.5–4. It is easy to see that, because the overlap map for $\mathsf{O}_{\mathbb{CP}^1}(k_1 + k_2)$ is the product of the overlap map maps for $\mathsf{O}_{\mathbb{CP}^1}(k_1)$ and $\mathsf{O}_{\mathbb{CP}^1}(k_2)$, and since the tensor product in $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is multiplication, we have

$$\mathsf{O}_{\mathbb{CP}^1}(k_1 + k_2) \simeq \mathsf{O}_{\mathbb{CP}^1}(k_1) \otimes \mathsf{O}_{\mathbb{CP}^1}(k_2).$$

This implies, for example, that

$$\mathsf{O}_{\mathbb{CP}^1}(k) \simeq \otimes_{j=1}^k \mathsf{O}_{\mathbb{CP}^1}(1), \quad \mathsf{O}_{\mathbb{CP}^1}(-k) \simeq \otimes_{j=1}^k \mathsf{O}_{\mathbb{CP}^1}(-1),$$

if $k \in \mathbb{Z}_{>0}$. •

4.3.6 Pull-back bundles

In this section we provide a standard construction that induces a vector bundle on the domain of a mapping between manifolds, given a vector bundle on the codomain. The definition is as follows.

4.3.21 Definition (Pull-back bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M and N be C^r-manifolds, let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$, and let $\pi \colon \mathsf{E} \to \mathsf{N}$ be a $\mathbb{F}$-vector bundle of class C^r. The ***pull-back*** of E to M is given by

$$\Phi^*\pi \colon \Phi^*\mathsf{E} \to \mathsf{M},$$

where

$$\Phi^*\mathsf{E} = \{(e, x) \in \mathsf{E} \times \mathsf{M} \mid \pi(e) = \Phi(x)\}$$

and $\Phi^*\pi(e, x) = x$. •

The pull-back of a vector bundle has a natural vector bundle structure.

4.3.22 Proposition (The pull-back of a vector bundle is a vector bundle) *Let $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $\mathrm{r} \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $\mathrm{r} = \mathrm{hol}$. Let M and N be C^r-manifolds, let $\Phi \in \mathrm{C}^\mathrm{r}(\mathsf{M}; \mathsf{N})$, and let $\pi \colon \mathsf{E} \to \mathsf{N}$ be a $\mathbb{F}$-vector bundle of class C^r. Then $\Phi^* \colon \Phi^*\mathsf{E} \to \mathsf{M}$ has a natural vector bundle structure.*

Proof We shall construct vector bundle charts for $\Phi^*\mathsf{E}$. Let $(\mathcal{U}, \phi)$ be a chart for M and let $(\mathcal{V}, \psi)$ be a vector bundle chart for E so that $\Phi(\mathcal{U}) \subseteq \mathcal{W} \triangleq \pi(\mathcal{V})$. This defines a neighbourhood $\mathcal{V} \times \mathcal{U}$ in $\mathsf{E} \times \mathsf{M}$. Let us suppose that $\mathcal{U}$ is diffeomorphic to an open subset of $\mathbb{F}^n$, that $\mathcal{W}$

is diffeomorphic to an open subset of $\mathbb{F}^m$, and that the fibres of E over $\mathcal{W}$ have dimension k. Let $\psi_0 \colon \mathcal{W} \to \mathbb{F}^m$ be the induced chart map for N. We then have

$$\begin{aligned} \psi \times \phi \colon \mathcal{V} \times \mathcal{U} &\to (\mathbb{F}^m \times \mathbb{F}^k) \times \mathbb{F}^n \\ (e, x) &\mapsto (\psi(e), \phi(x)). \end{aligned}$$

Denote by $\bar{\Phi} \colon \phi(\mathcal{U}) \to \psi_0(\mathcal{W})$ the local representative of Φ. With this notation, locally the subset $\Phi^*\mathsf{E}$ of $\mathsf{E} \times \mathsf{M}$ is given by

$$\bar{\Phi}^*\mathsf{E} \triangleq \{((\boldsymbol{y}, \boldsymbol{v}), \boldsymbol{x}) \in \psi(\mathcal{V}) \times \phi(\mathcal{U}) \mid \ \boldsymbol{y} = \bar{\Phi}(\boldsymbol{x})\}.$$

Now define a map g from $\bar{\Phi}^*\mathsf{E}$ to $\phi(\mathcal{U}) \times \mathbb{F}^k$ by

$$g((\boldsymbol{y}, \boldsymbol{v}), \boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{v})$$

We claim that

$$\{((\mathcal{V} \times \mathcal{U})|\Phi^*\mathsf{E}, g \circ (\psi \times \phi|\Phi^*\mathsf{E})) \mid \\ (\mathcal{V}, \psi) \text{ is a vector bundle chart for } \mathsf{E} \text{ and } (\mathcal{U}, \phi) \text{ is a chart for } \mathsf{M}\}$$

is a vector bundle atlas for $\Phi^*\mathsf{E}$. We must verify the overlap conditions. We make things simpler by assuming another chart for M of the form $(\mathcal{U}, \phi')$ (i.e., the domain is the same as the chart $(\mathcal{U}, \phi)$) and a vector bundle chart for E of the form $(\mathcal{V}, \psi')$ (again the domain is the same). These simplifications can always be made by restriction if necessary. Since the charts $(\mathcal{U}, \phi)$ and $(\mathcal{U}, \phi')$ satisfy the overlap conditions we have that

$$\phi' \circ \phi^{-1} \colon \phi(\mathcal{U}) \to \phi'(\mathcal{U})$$

is a C^r-diffeomorphism. Similarly, since $(\mathcal{V}, \psi)$ and $(\mathcal{V}, \psi')$ are vector bundle charts,

$$\psi' \circ \psi^{-1}(\boldsymbol{y}, \boldsymbol{v}) = (\boldsymbol{\sigma}(\boldsymbol{y}), \boldsymbol{A}(\boldsymbol{y}) \cdot \boldsymbol{v}),$$

where $\boldsymbol{\sigma} \colon \psi_0(\mathcal{W}) \to \psi_0'(\mathcal{W})$ is a C^r-diffeomorphism and $\boldsymbol{A} \colon \psi_0(\mathcal{W}) \to GL(k; \mathbb{F})$ is of class C^r.

Now we consider the two charts

$$((\mathcal{V} \times \mathcal{U})|\Phi^*\mathsf{E}, g \circ (\psi \times \phi|\Phi^* E)) \quad \text{and} \quad ((\mathcal{V} \times \mathcal{U})|\Phi^*\mathsf{E}, g \circ (\psi' \times \phi'|\Phi^* E))$$

for $\Phi^* E$ and show that they satisfy the overlap conditions. Let $(e, x) \in (\mathcal{V} \times \mathcal{U})|\Phi^*\mathsf{E}$. We write

$$\psi \times \phi(e, x) = ((\bar{\Phi}(\boldsymbol{x}), \boldsymbol{v}), \boldsymbol{x}),$$

defining $\boldsymbol{x} \in \phi(\mathcal{U})$ and $\boldsymbol{v} \in \mathbb{F}^k$. If $\bar{\Phi}' \colon \phi'(\mathcal{U}) \to \psi_0'(\mathcal{W})$ is the local representative of Φ in the "primed" chart, we may write

$$\psi \times \phi(e, x) = ((\bar{\Phi}'(\boldsymbol{x}'), \boldsymbol{v}'), \boldsymbol{x}'),$$

defining $\boldsymbol{x}' \in \phi'(\mathcal{U})$ and $\boldsymbol{v}' \in \mathbb{F}^k$. Since $(\mathcal{U}, \phi)$, $(\mathcal{U}, \phi')$, $(\mathcal{V}, \psi)$, and $(\mathcal{V}, \psi')$ satisfy the overlap conditions we must have

$$\boldsymbol{x}' = \phi' \circ \phi^{-1}(\boldsymbol{x}), \quad \boldsymbol{v}' = (\boldsymbol{A} \circ \boldsymbol{\sigma}(\boldsymbol{x})) \cdot \boldsymbol{v}.$$

This shows that the overlap condition is indeed satisfied. ■

One directly verifies that the map $\hat{\Phi}\colon \Phi^*\mathsf{E} \to \mathsf{E}$ defined by $\hat{\Phi}(e, x) = e$ is a vector bundle mapping over Φ. That is, the diagram

$$\begin{array}{ccc} \Phi^*\mathsf{E} & \xrightarrow{\hat{\Phi}} & \mathsf{E} \\ {\scriptstyle \Phi^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathsf{M} & \xrightarrow[\Phi]{} & \mathsf{N} \end{array}$$

commutes and $\hat{\Phi}$ is $\mathbb{F}$-linear on fibres.

4.4 Line bundles over general projective spaces

In this section, as an illustration of some of the topics of interest in algebraic geometry connected with holomorphic differential geometry, we generalise and beautify our constructions presented in the course of Section 4.3 of line bundles over $\mathbb{CP}^1$ to general projective spaces. The constructions we perform are most naturally performed using general vector spaces over general fields, so we phrase things in this way so as to isolate the essential algebraic and geometric character. Some of the topics we discuss here are dealt with nicely in the book of Berger [1987].

4.4.1 Setup

We let F be a field and we let V be a finite-dimensional F-vector space. For $k \in \mathbb{Z}_{\geq 0}$, by $\mathrm{S}^k(\mathsf{V})$ we denote the degree k symmetric tensor algebra of V. For $k \in \mathbb{Z}_{>0}$ and $v \in \mathsf{V}$ we denote by $v^{\otimes k}$ the image of $v \otimes \cdots \otimes v$ under the projection $\mathsf{V}^{\otimes k} \triangleq \mathrm{T}^k(\mathsf{V})$ to $\mathrm{S}^k(\mathsf{V})$ i.e., using the notation of Section F.2.2,

$$v^{\otimes k} = v \otimes \cdots \otimes v + I_S(\mathsf{V}).$$

We begin our constructions by first defining the generalisation of our construction of $\mathbb{FP}^n$ to the general setting. This is easy. A ***line*** in V is a one-dimensional subspace, typically denoted by L. By $\mathbb{P}(\mathsf{V})$ we denote the set of lines in V. Equivalently, $\mathbb{P}(\mathsf{V})$ is the set of equivalence classes in $\mathsf{V} \setminus \{0\}$ under the equivalence relation $v_1 \sim v_2$ if $v_2 = av_1$ for $a \in \mathsf{F} \setminus \{0\}$. We call $\mathbb{P}(\mathsf{V})$ the ***projective space*** of V. If $v \in \mathsf{V} \setminus \{0\}$ we denote by $[v]$ the line generated by v. We can thus denote a point in $\mathbb{P}(\mathsf{V})$ in two ways: (1) by $[v]$ when we wish to emphasise that a line is a line through a point in V; (2) by L when we wish to emphasise that a line is a vector space.

We will study a family $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$, $d \in \mathbb{Z}$, of line bundles over $\mathbb{P}(\mathsf{V})$. We shall refer to the index d as the ***degree*** of the line bundles. The simplest of these line bundles occurs for $d = 0$, in which case we have the trivial bundle

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(0) = \mathbb{P}(\mathsf{V}) \times \mathsf{F}.$$

We have the obvious projection

$$\begin{aligned}\pi^{(0)}_{\mathbb{P}(\mathsf{V})}\colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(0) &\to \mathbb{P}(\mathsf{V})\\ ([v],a) &\mapsto [v].\end{aligned}$$

The study of the line bundles of nonzero degree in a comprehensive and elegant way requires some development of projective geometry.

In this section we shall frequently use differential geometric language such as "vector bundle" and "section," even though we are not in the setting of differential geometry. This should not cause confusion, as a quick mental translation into the case of $\mathsf{F} \in \{\mathbb{R}, \mathbb{C}\}$ should make all such statements seem reasonable.

4.4.2 The affine structure of projective space minus a projective hyperplane

In Definition 5.1.1, in preparation for our discussion of jet bundles, we define the notion of an affine space, and we will use this definition here. At a few points in this section we shall make use of a particular affine structure, and in this section we describe this. The discussion is initiated with the following lemma.

4.4.1 Lemma (The affine structure of $\mathbb{P}(\mathsf{V})$ with a projective hyperplane removed) *If* F *is a field, if* V *is an* F*-vector space, and if* $\mathsf{U} \subseteq \mathsf{V}$ *is a subspace of codimension* 1*, then the set* $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ *is an affine space modelled on* $\mathrm{Hom}_{\mathsf{F}}(\mathsf{V}/\mathsf{U};\mathsf{U})$.

Proof Let $\pi_{\mathsf{U}}\colon \mathsf{V} \to \mathsf{V}/\mathsf{U}$ be the canonical projection. For $v + \mathsf{U} \in \mathsf{V}/\mathsf{U}$, $\pi_{\mathsf{U}}^{-1}(v + \mathsf{U})$ is an affine subspace of the affine space V modelled on U, as is easily checked. Moreover, if L is a complement to U, then $\pi_{\mathsf{U}}|\mathsf{L}$ is an isomorphism. Now, if $v + \mathsf{U} \in \mathsf{V}/\mathsf{U}$ and if L_1 and L_2 are two complements to U, note that

$$(\pi_{\mathsf{U}}|\mathsf{L}_1)^{-1}(v + \mathsf{U}) - (\pi_{\mathsf{U}}|\mathsf{L}_2)^{-1}(v + \mathsf{U}) \in \mathsf{U}$$

since

$$\pi_{\mathsf{U}}((\pi_{\mathsf{U}}|\mathsf{L}_1)^{-1}(v + \mathsf{U}) - (\pi_{\mathsf{U}}|\mathsf{L}_2)^{-1}(v + \mathsf{U})) = (v + \mathsf{U}) - (v + \mathsf{U}) = 0.$$

Moreover, the map

$$\mathsf{V}/\mathsf{U} \ni v + \mathsf{U} \mapsto (\pi_{\mathsf{U}}|\mathsf{L}_1)^{-1}(v + \mathsf{U}) - (\pi_{\mathsf{U}}|\mathsf{L}_2)^{-1}(v + \mathsf{U}) \in \mathsf{U} \tag{4.12}$$

is in $\mathrm{Hom}_{\mathsf{F}}(\mathsf{V}/\mathsf{U};\mathsf{U})$. We, therefore, define the affine structure on $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ by defining subtraction of elements of $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ as elements of the model vector space by taking $\mathsf{L}_1 - \mathsf{L}_2$ to be the element of $\mathrm{Hom}_{\mathsf{F}}(\mathsf{V}/\mathsf{U};\mathsf{U})$ given in (4.12). It is now a simple exercise to verify that this gives the desired affine structure. ■

To make the lemma more concrete and to connect it to constructions we already have seen concerning projective spaces, in the setting of the lemma, we let $\mathsf{O} \in \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$, let $e_{\mathsf{O}} \in \mathsf{O} \setminus \{0\}$, and, for $v \in \mathsf{V}$, write $v = v_{\mathsf{O}} e_{\mathsf{O}} + v_{\mathsf{U}}$ for $v_{\mathsf{O}} \in \mathsf{F}$ and $v_{\mathsf{U}} \in \mathsf{U}$. With this notation, we have the following result.

4.4.2 Lemma (A concrete representation of $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$) *The map*

$$\begin{aligned}\phi_{\mathsf{U},\mathsf{O}} \colon \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U}) &\to \mathsf{U}\\ [\mathrm{v}] &\mapsto \mathrm{v}_{\mathsf{O}}^{-1}\mathrm{v}_{\mathsf{U}}\end{aligned}$$

is an affine space isomorphism mapping O *to zero.*

Proof Let $[v] \in \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ and write e_{O} in its $[v]$- and U-components:

$$e_{\mathsf{O}} = \alpha(v_{\mathsf{O}}e_{\mathsf{O}} + v_{\mathsf{U}}) + v'_{\mathsf{U}},$$

for $v'_{\mathsf{U}} \in \mathsf{U}$. Evidently, $\alpha = v_{\mathsf{O}}^{-1}$ and $v'_{\mathsf{U}} = v_{\mathsf{O}}^{-1}v_{\mathsf{U}}$. According to the proof of Lemma 4.4.1, if $w + \mathsf{U} \in \mathsf{V}/\mathsf{U}$, then

$$\begin{aligned}([v] - \mathsf{O})(w + \mathsf{U}) &= ([v_{\mathsf{O}}e_{\mathsf{O}} + v_{\mathsf{U}}] - [e_{\mathsf{O}}])(w_{\mathsf{O}}e_{\mathsf{O}} + \mathsf{U})\\ &= (w_{\mathsf{O}}e_{\mathsf{O}})_{[v]} - w_{\mathsf{O}}e_{\mathsf{O}}\\ &= w_{\mathsf{O}}(e_{\mathsf{O}} + v_{\mathsf{O}}^{-1}v_{\mathsf{U}}) - w_{\mathsf{O}}e_{\mathsf{O}}\\ &= w_{\mathsf{O}}v_{\mathsf{O}}^{-1}v_{\mathsf{U}},\end{aligned}$$

where $(w_{\mathsf{O}}e_{\mathsf{O}})_{[v]}$ denotes the $[v]$-component of $w_{\mathsf{O}}e_{\mathsf{O}}$.

We now verify that $\phi_{\mathsf{U},\mathsf{O}}$ is affine by using Proposition 5.1.7. Let $[v_1], [v_2] \in \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$. Then, for $w + \mathsf{U} \in \mathsf{V}/\mathsf{U}$,

$$(([v_1] - \mathsf{O}) + ([v_2] - \mathsf{O}))(w + \mathsf{U}) = w_{\mathsf{O}}(v_{1,\mathsf{O}}^{-1}v_{1,\mathsf{U}} - v_{2,\mathsf{O}}^{-1}(v_{2,\mathsf{U}}))$$

and so

$$\mathsf{O} + (([v_1] - \mathsf{O}) + ([v_2] - \mathsf{O})) = [e_{\mathsf{O}} + v_{1,\mathsf{O}}^{-1}v_{1,\mathsf{U}} + v_{2,\mathsf{O}}^{-1}v_{2,\mathsf{U}}].$$

Thus, using the vector space structure on $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ determined by the origin O,

$$\begin{aligned}\phi_{\mathsf{U},\mathsf{O}}([v_1] + [v_2]) &= \phi_{\mathsf{U},\mathsf{O}}(\mathsf{O} + (([v_1] - \mathsf{O}) + ([v_2] - \mathsf{O})))\\ &= v_{1,\mathsf{O}}^{-1}v_{1,\mathsf{U}} + v_{2,\mathsf{O}}^{-1}v_{2,\mathsf{U}}\\ &= \phi_{\mathsf{U},\mathsf{O}}([v_1]) + \phi_{\mathsf{U},\mathsf{O}}([v_2]).\end{aligned}$$

Also,

$$a([v] - \mathsf{O}) = w_{\mathsf{O}}av_{\mathsf{O}}^{-1}v_{\mathsf{U}}$$

which gives

$$\mathsf{O} + a([v] - \mathsf{O}) = [v_{\mathsf{O}}e_{\mathsf{O}} + av_{\mathsf{U}}].$$

Therefore,

$$\phi_{\mathsf{U},\mathsf{O}}(a[v]) = \phi_{\mathsf{U},\mathsf{O}}(\mathsf{O} + a([v] - \mathsf{O})) = av_{\mathsf{O}}^{-1}v_{\mathsf{U}} = a\phi_{\mathsf{U},\mathsf{O}}([v]),$$

showing that $\phi_{\mathsf{U},\mathsf{O}}$ is indeed a linear map from $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U})$ to U with origins O and 0, respectively.

Finally, we show that $\phi_{\mathsf{U},\mathsf{O}}$ is an isomorphism. Suppose that $\phi_{\mathsf{U},\mathsf{O}}([v]) = 0$, meaning that $v_{\mathsf{O}}^{-1}v_{\mathsf{U}} = 0$. This implies that $v_{\mathsf{U}} = 0$ and so $v \in \mathsf{O}$, showing that $\phi_{\mathsf{U},\mathsf{O}}$ is injective. Since the dimensions of the domain and codomain of $\phi_{\mathsf{U},\mathsf{O}}$ agree, the result follows. ■

Next let us see how, if we exclude two distinct hyperplanes, one can compare the two affine structures.

4.4.3 Lemma (Transition functions between affine subspaces) *Let* F *be a field, let* V *be a finite-dimensional* F*-vector space, and let* $\mathsf{U}_1, \mathsf{U}_2 \subseteq \mathsf{V}$ *be distinct codimension* 1 *subspaces, let* $\mathsf{O}_j \in \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U}_j)$, $j \in \{1, 2\}$, *let* $e_{\mathsf{O}_j} \in \mathsf{O}_j \setminus \{0\}$, *and let* $\phi_{\mathsf{U}_j,\mathsf{O}_j} \colon \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U}_j) \to \mathsf{U}_j$, $j \in \{1, 2\}$, *be the isomorphisms of Lemma 4.4.2. Then*

$$\phi_{\mathsf{U}_2,\mathsf{O}_2} \circ \phi^{-1}_{\mathsf{U}_1,\mathsf{O}_1}(u_1) = (e_{\mathsf{O}_1} + u_1)^{-1}_{\mathsf{O}_2}(e_{\mathsf{O}_1} + u_1)_{\mathsf{U}_2},$$

where $(e_{\mathsf{O}_1} + u_1)_{\mathsf{O}_2}$ *is the* O_2*-component and* $(e_{\mathsf{O}_1} + u_1)_{\mathsf{U}_2}$ *is the* U_2*-component, respectively, of* $e_{\mathsf{O}_1} + u_1$.

If, furthermore, $\mathsf{O}_1 \in \mathbb{P}(\mathsf{U}_2)$ *and* $\mathsf{O}_2 \in \mathbb{P}(\mathsf{U}_1)$, *then the formula simplifies to*

$$\phi_{\mathsf{U}_2,\mathsf{O}_2} \circ \phi^{-1}_{\mathsf{U}_1,\mathsf{O}_1}(u_1) = u^{-1}_{1,\mathsf{O}_2}(e_{\mathsf{O}_1} + u_{1,\mathsf{U}_2}), \qquad u_1 \in \phi_{\mathsf{U}_1,\mathsf{O}_1}(\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U}_2)),$$

where u_{1,O_2} *is the* O_2*-component and* u_{1,U_2} *is the* U_2*-component, respectively, of* u_1.

Proof This follows by direct computation using the definitions. ■

Let us consider an important special case of the preceding developments to make connections to the differentiable structure for projective space presented in Example 4.2.2–4.

4.4.4 Example (The canonical projective space) We let $\mathsf{V} = \mathsf{F}^{n+1}$ and denote a point in V by $(a_0, a_1, \dots, a_n)$. We follow the convention adopted in Example 4.2.2–4 and denote by $[a_0 : a_1 : \cdots : a_n]$ the line through $(a_0, a_1, \dots, a_n)$. For $j \in \{0, 1, \dots, n\}$ we denote by U_j the subspace

$$\mathsf{U}_j = \{(a_0, a_1, \dots, a_n) \in \mathsf{V} \mid a_j = 0\}.$$

Note that U_j is isomorphic to F^n in a natural way, and we make this identification without explicit mention. Note that the affine spaces $\mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\mathsf{U}_j)$ correspond to the chart domains $\mathcal{U}_j$, $j \in \{0, 1, \dots, n\}$ from Example 4.2.2–4, and so we denote these exactly by $\mathcal{U}_j$, for brevity. For each $j \in \{0, 1, \dots, n\}$ we denote $\mathsf{O}_j = \operatorname{span}_{\mathsf{F}}(e_j)$, $j \in \{0, 1, \dots, n\}$, where e_j is the jth (according to our numbering system starting with "0") standard basis vector for V. Note that $\mathsf{O}_j \in \mathsf{U}_k$ for $j \neq k$, as prescribed by the hypotheses of Lemma 4.4.3.

With this as buildup, we then have

$$\phi_{\mathsf{U}_j,\mathsf{O}_j}([a_0 : a_1 : \cdots : a_n]) = a_j^{-1}(a_0, a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n),$$

consistent with the chart maps from Example 4.2.2–4. We can also verify that, if $j, k \in \{0, 1, \dots, n\}$ satisfy $j < k$, then we have

$$\phi_{\mathsf{U}_j,\mathsf{O}_j} \circ \phi^{-1}_{\mathsf{U}_k,\mathsf{O}_k}(a_1, \dots, a_n) = \Big(\frac{a_1}{a_{j+1}}, \dots, \frac{a_j}{a_{j+1}}, \frac{a_{j+2}}{a_{j+1}}, \dots, \frac{a_k}{a_{j+1}}, \frac{1}{a_{j+1}}, \frac{a_{k+1}}{a_{j+1}}, \dots, \frac{a_n}{a_{j+1}}\Big),$$

which agrees with the overlap maps from Example 4.2.2–4. •

4.4.3 The affine structure of projective space with a point removed

In order to study below the line bundles $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ for $d \in \mathbb{Z}_{>0}$, we need an enjoyable linear algebra diversion. Our setup is the following. We let U be an F-vector space and let $\mathsf{W} \subseteq \mathsf{U}$ be a subspace. We then have a natural identification of $\mathbb{P}(\mathsf{W})$ with a subset of $\mathbb{P}(\mathsf{U})$ by considering lines in W as being lines in U. Note that we also have the canonical projection $\pi_{\mathsf{W}} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{U}; \mathsf{U}/\mathsf{W})$ and so an induced map

$$\begin{aligned} \mathbb{P}(\pi_{\mathsf{W}})\colon \mathbb{P}(\mathsf{U}) \setminus \mathbb{P}(\mathsf{W}) &\to \mathbb{P}(\mathsf{U}/\mathsf{W}) \\ \mathsf{L} &\mapsto (\mathsf{L} + \mathsf{W})/\mathsf{W} \subseteq \mathsf{U}/\mathsf{W}. \end{aligned}$$

Note that we do require that this map not be evaluated on points in $\mathbb{P}(\mathsf{W})$ since these will not project to a line in U/W. The same line of thinking allows one to conclude that $\mathbb{P}(\pi_{\mathsf{W}})$ is surjective. The following structure of this projection is of value.

4.4.5 Lemma (The affine bundle structure of the complement of a subspace in projective space) *If F is a field, if U is an F-vector space, if $\mathsf{W} \subseteq \mathsf{U}$ is a subspace, and if $\mathsf{L} \in \mathbb{P}(\mathsf{U}/\mathsf{W})$ then $\mathbb{P}(\pi_{\mathsf{W}})^{-1}(\mathsf{L})$ is an affine space modelled on $\mathrm{Hom}_{\mathsf{F}}(\pi_{\mathsf{W}}^{-1}(\mathsf{L})/\mathsf{W}; \mathsf{W})$.*

Proof If $\mathsf{L} \subseteq \mathsf{U}/\mathsf{W}$ is a line, then there exists $u \in \mathsf{U} \setminus \mathsf{W}$ such that

$$\mathsf{L} = \{au + \mathsf{W} \mid a \in \mathsf{F}\} = \{au + w + \mathsf{W} \mid a \in \mathsf{F}\} = (\mathsf{M} + \mathsf{W})/\mathsf{W},$$

where $\mathsf{M} = [u]$ and so $\mathsf{M} \cap \mathsf{W} = \{0\}$. Therefore, we can denote

$$\mathsf{A}_{\mathsf{L}} = \{\mathsf{M} \in \mathbb{P}(\mathsf{U}) \setminus \mathbb{P}(\mathsf{W}) \mid (\mathsf{M} + \mathsf{W})/\mathsf{W} = \mathsf{L}\}.$$

We claim that

$$\mathsf{A}_{\mathsf{L}} = \{\mathsf{M} \in \mathbb{P}(\mathsf{U}) \setminus \mathbb{P}(\mathsf{W}) \mid \mathsf{M} + \mathsf{W} = \pi_{\mathsf{W}}^{-1}(\mathsf{L})\};$$

that is, A_{L} is the set of complements to W in $\pi_{\mathsf{W}}^{-1}(\mathsf{L})$. To see this, first note that any such complement will necessarily have dimension 1 by the Rank–Nullity Theorem. Next let M be such a complement. Then

$$\mathsf{L} = \pi_{\mathsf{W}}(\pi_{\mathsf{W}}^{-1}(\mathsf{L})) = \pi_{\mathsf{W}}(\mathsf{M} + \mathsf{W}),$$

which is exactly the condition $\mathsf{M} \in \mathsf{A}_{\mathsf{L}}$. Next suppose that $(\mathsf{M} + \mathsf{W})/\mathsf{W} = \mathsf{L}$. This means that

$$\pi_{\mathsf{W}}(\mathsf{M} + \mathsf{W}) = \mathsf{L}.$$

By the Rank–Nullity Theorem, it follows that M is a complement to W in $\pi_{\mathsf{W}}^{-1}(\mathsf{L})$. The result now follows from Lemma 4.4.1. ■

For us, the most important application of the preceding lemma is the following corollary.

4.4.6 Corollary (A vector bundle structure for $\mathbb{P}(\mathsf{F} \oplus \mathsf{V})$) *Let F be a field, let V be an F-vector space, and consider the map*

$$\mathbb{P}(\mathrm{pr}_2)\colon \mathbb{P}(\mathsf{F} \oplus \mathsf{V}) \setminus \mathbb{P}(\mathsf{F} \oplus 0) \to \mathbb{P}(\mathsf{V}).$$

For $\mathsf{L} \in \mathbb{P}(\mathsf{V})$, $\mathbb{P}(\mathrm{pr}_2)^{-1}(\mathsf{L})$ has a canonical identification with L^*.

Proof We apply the lemma in a particular setting. We take $\mathsf{U} = \mathsf{F} \oplus \mathsf{V}$ and $\mathsf{W} = \mathsf{F} \oplus 0$. We have a natural isomorphism $\iota_{\mathsf{V}}\colon \mathsf{V} \to \mathsf{U}/\mathsf{F}$ defined by $\iota_{\mathsf{V}}(v) = 0 \oplus v + \mathsf{F}$. If we let $\mathrm{pr}_2\colon \mathsf{U} \to \mathsf{V}$ be projection onto the second factor, then we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{W} & \longrightarrow & \mathsf{U} & \xrightarrow{\pi_{\mathsf{W}}} & \mathsf{U}/\mathsf{W} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \simeq} & & \| & & \downarrow{\scriptstyle \iota_{\mathsf{V}}} & & \\ 0 & \longrightarrow & \mathsf{F} & \longrightarrow & \mathsf{F} \oplus \mathsf{V} & \xrightarrow[\mathrm{pr}_2]{} & \mathsf{V} & \longrightarrow & 0 \end{array} \tag{4.13}$$

which is commutative with exact rows. Note that $\mathrm{pr}_2^{-1}(\mathsf{L}) = \mathsf{F} \oplus \mathsf{L} \subseteq \mathsf{F} \oplus \mathsf{V}$. Therefore, $\mathrm{pr}_2^{-1}(\mathsf{L})/\mathsf{F} \simeq \mathsf{L}$. By the lemma and by the commutative diagram (4.13), $\mathbb{P}(\mathrm{pr}_2)^{-1}(\mathsf{L})$ is an affine space modelled on

$$\mathrm{Hom}_{\mathsf{F}}(\mathrm{pr}_2^{-1}(\mathsf{L})/\mathsf{F}; \mathsf{F}) \simeq \mathrm{Hom}_{\mathsf{F}}(\mathsf{L}; \mathsf{F}) = \mathsf{L}^*.$$

Since $0 \oplus \mathsf{L} \in \mathbb{P}(\mathrm{pr}_2)^{-1}(\mathsf{L})$ for every $\mathsf{L} \in \mathbb{P}(\mathsf{V})$, the affine space $\mathbb{P}(\mathrm{pr}_2)^{-1}(\mathsf{L})$ has a natural distinguished origin, and so this establishes a natural identification of $\mathbb{P}(\mathrm{pr}_2)^{-1}(\mathsf{L})$ with L^*, as desired. Explicitly, this identification is given by assigning to $[a \oplus v] \in \mathbb{P}(\mathsf{F} \oplus \mathsf{V}) \setminus \mathbb{P}(\mathsf{F} \oplus 0)$ the element $\alpha \in [v]^*$ determined by $\alpha(v) = a$. ■

4.4.4 Functions and maps to and from projective spaces

In order to intelligently talk about objects defined on projective space, e.g., spaces of sections of line bundles over projective space, we need to have at hand a notion of regularity for such mappings. We shall discuss this only in the most elementary setting, as this is all we need here. We refer to any basic algebraic geometry text, e.g., [Harris 1992], for a more general discussion.

Caveat We do not follow some of the usual conventions in algebraic geometry because we do not work exclusively with algebraically closed fields. Thus some of our definitions are not standard. We do not care to be fussy about how we handle this. At points where it is appropriate, we point out where algebraic closedness leads to the usual definitions. •

Functions on vector spaces

First, let us talk about functions on a vector space V taking values in F. We wish to use polynomial functions as our starting point. A ***polynomial function*** of homogeneous degree d on V is a function of the form

$$v \mapsto A(v, \dots, v),$$

for $A \in \mathrm{T}^d(\mathsf{V}^*)$. By Sublemma 1 from the proof of Lemma F.2.15 we now have the following result.

4.4.7 Proposition (Symmetric multilinear maps and homogeneous polynomial mappings) *Let* F *be a field and let* V *be an* F*-vector space. Then, for a homogeneous polynomial function* $f\colon \mathsf{V}\to\mathsf{F}$ *of degree* d*, there exists a unique* $A \in \mathrm{S}^d(\mathsf{V}^*)$ *such that*

$$f(x) = A(x,\dots,x).$$

Moreover, for $x_1,\dots,x_d \in \mathsf{V}$ *we have*

$$A(x_1,\dots,x_d) = \frac{1}{d!}\sum_{l=1}^{d}\sum_{\{j_1,\dots,j_l\}\subseteq\{1,\dots,d\}}(-1)^{d-l}A(x_{j_1}+\dots+x_{j_l},\dots,x_{j_1}+\dots+x_{j_l}).$$

A general (i.e., not necessarily homogeneous) polynomial function is then a sum of its homogeneous components, and so identifiable with an element of $\mathrm{S}(\mathsf{V}^*)$. Any element of $\mathrm{S}(\mathsf{V}^*)$ can be written as $A_0 + A_1 + \dots + A_d$ where $d \in \mathbb{Z}_{\geq 0}$ and $A_j \in \mathrm{S}^j(\mathsf{V}^*)$, $j \in \{0,1,\dots,d\}$. Justified by the proposition, we shall sometimes abuse notation slightly and write "$f \in \mathrm{S}(\mathsf{V}^*)$" if f is a polynomial function. When we wish to be explicit about the relationship between the function and the tensor, we shall write f_A, where $A = A_0 + A_1 + \dots + A_d$. If we wish to consider general polynomial functions taking values in an F-vector space U, these will then be identifiable with elements of $\mathrm{S}(\mathsf{V}^*)\otimes\mathsf{U}$.

We will need to go beyond polynomial functions, and this we do as follows.

4.4.8 Definition (Regular function on vector space) Let F be a field, let U and V be finite-dimensional F-vector spaces, and let $S \subseteq \mathsf{V}$. A map $f\colon \mathsf{V}\to\mathsf{U}$ is ***regular*** on S if there exists $N \in \mathrm{S}(\mathsf{V}^*)\otimes\mathsf{U}$ and $D \in \mathrm{S}(\mathsf{V}^*)$ such that

(i) $\{v \in S \mid f_D(v) = 0\} = \emptyset$ and

(ii) $f(v) = \dfrac{f_N(v)}{f_D(v)}$ for all $v \in \mathsf{V}$.

If f is regular on V, we shall often say f is simply ***regular***. •

In some cases regular functions take a simpler form.

4.4.9 Proposition (Regular functions on vector spaces are sometimes polynomial) *If* F *is an algebraically closed field and if* U *and* V *are finite-dimensional* F*-vector spaces, then* $f\colon \mathsf{V}\to\mathsf{U}$ *is regular on* V *if and only if there exists* $A \in \mathrm{S}(\mathsf{V}^*)\otimes\mathsf{U}$ *such that* $f = f_A$.

Proof The "if" assertion is clear. For the "only if" assertion, it is sufficient to show that, in the definition of a regular function, D can be taken to have degree zero. To see this, we suppose that D has (not necessarily homogeneous) degree $d \in \mathbb{Z}_{>0}$ and show that $f_D(v) = 0$ for some nonzero v. Write $D = D_0 + D_1 + \dots + D_d$ where $D_k \in \mathrm{S}^k(\mathsf{V}^*)$ for $k \in \{0,1,\dots,d\}$. Let $(e_1,\dots,e_n)$ be a basis for V, fix $a_2,\dots,a_n \in \mathsf{F}\setminus\{0\}$, and consider the function

$$\mathsf{F} \ni a \mapsto f_D(ae_1 + a_2e_2 + \dots + a_ne_n) \in \mathsf{F}. \tag{4.14}$$

Note that

$$f_D(ae_1 + a_2e_2 + \cdots + a_ne_n) = \sum_{k=0}^{d}\sum_{j=0}^{k}\binom{k}{j}D_k(\underbrace{ae_1,\dots,ae_1}_{j \text{ times}},\underbrace{a_2e_2+\cdots+a_ne_n}_{k-j \text{ times}}),$$

and so the function (4.14) is a polynomial function of (not necessarily homogeneous) degree d. If

$$\sum_{k=0}^{d} D_k(a_2e_2+\cdots+a_ne_n,\dots,a_2e_2+\cdots+a_ne_n) = 0$$

then f_D is zero at the nonzero point $a_2e_2+\cdots+a_ne_n$ and our claim follows. Otherwise, the function (4.14) is a scalar polynomial function of positive degree with nonzero constant term. Since F is algebraically closed, there is a nonzero root a_1 of this function, and so f_D is zero at $a_1e_1 + a_2e_2 + \cdots + a_ne_n$, giving our assertion. ■

The following example shows that the assumption of algebraic closedness is essential in the lemma.

4.4.10 Example (A non-polynomial regular function) The function $x \mapsto \frac{1}{1+x^2}$ from $\mathbb{R}$ to $\mathbb{R}$ is a regular function that is not polynomial. •

Functions on projective space

Note that if $f \in \mathsf{S}^d(\mathsf{V}^*)$, then $f(\lambda v) = \lambda^d f(v)$, and so f will not generally give rise to a well-defined function on $\mathbb{P}(\mathsf{V})$ since its value on lines will not be constant. However, this does suggest the following definition.

4.4.11 Definition (Regular function on projective space) Let F be a field and let V and U be F-vector spaces. A map $f\colon \mathbb{P}(\mathsf{V}) \to \mathsf{U}$ is ***regular*** if there exists $d \in \mathbb{Z}_{\geq 0}$ and $N \in \mathsf{S}^d(\mathsf{V}^*)\otimes\mathsf{U}$ and $D \in \mathsf{S}^d(\mathsf{V}^*)$ such that

(i) $\{v \in \mathsf{V} \mid f_D(v) = 0\} = \{0\}$ and

(ii) $f([v]) = \dfrac{f_N(v)}{f_D(v)}$ for all $[v] \in \mathbb{P}(\mathsf{V})$. •

Let us characterise regular functions on projective space with a sort of general result and an example showing that "sort of general" cannot be converted to "general."

First the sort of general result.

4.4.12 Proposition (Regular functions on projective space are sometimes constant) *If* F *is an algebraically closed field, if* V *is a finite-dimensional* F*-vector spaces, and if* $\mathsf{f}\colon \mathbb{P}(\mathsf{V}) \to \mathsf{F}$ *is regular, then* $\hat{\mathsf{f}}$ *is a constant function on* $\mathbb{P}(\mathsf{V})$.

Proof Suppose that $f(v) = \frac{f_N(v)}{f_D(v)}$ for $N, D \in \mathsf{S}^d(\mathsf{V}^*)$ where f_D does not vanish on $\mathsf{V}\setminus\{0\}$. Since F is algebraically closed, the same argument as was used in the proof of Proposition 4.4.9 shows that f_D is constant, i.e., of degree 0. Thus f_D is a nonzero constant function. It follows that f_N is also a constant function since we have $N \in \mathsf{S}^0(\mathsf{V}^*)$, and so f is constant. ■

The following example shows that algebraic closedness of F is essential.

4.4.13 Example (A nonconstant regular function on $\mathbb{RP}^n$) We let $\mathsf{F} = \mathbb{R}$ and $\mathsf{V} = \mathbb{R}^{n+1}$, denoting a point in V by $(a_0, a_1, \dots, a_n)$. Of course, $\mathbb{P}(\mathsf{V}) = \mathbb{RP}^n$. We define a regular function f on $\mathbb{RP}^n$ by

$$f([a_0 : a_1 : \cdots : a_n]) = \frac{f_N(a_0, a_1, \dots, a_n)}{a_0^2 + a_1^2 + \cdots + a_n^2},$$

where f_N is a nonzero polynomial function of homogeneous degree 2, e.g.,

$$f_N(a_0, a_1, \dots, a_n) = a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n.$$

This gives a nonconstant regular function, as desired. •

Mappings between projective spaces

Next let us consider a natural class of maps between projective spaces.

4.4.14 Definition (Morphisms between projective spaces) Let F be a field and let U and V be finite-dimensional F-vector spaces. A ***morphism*** of the projective spaces $\mathbb{P}(\mathsf{V})$ and $\mathbb{P}(\mathsf{U})$ is a map $\Phi \colon \mathbb{P}(\mathsf{V}) \to \mathbb{P}(\mathsf{U})$ for which there exist $d_N, d_D \in \mathbb{Z}_{\geq 0}$, $N \in \mathsf{S}^{d_N}(\mathsf{V}^*)$, and $D \in \mathsf{S}^{d_D}(\mathsf{V}^*) \otimes \mathsf{U}$ such that

(i) $\{v \in \mathsf{V} \mid f_N(v) = 0\} = \{0\}$,

(ii) $\{v \in \mathsf{V} \mid f_D(v) = 0\} = \{0\}$,

(iii) $\Phi([v]) = \left[\dfrac{f_N(v)}{f_D(v)}\right]$ for all $[v] \in \mathbb{P}(\mathsf{V})$. •

Let us give a couple of examples of morphisms of projective space.

4.4.15 Examples (Projective space morphisms)

1. If $A \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{U})$ is a homomorphism of vector spaces, then the induced map $\mathbb{P}(A) \colon \mathbb{P}(\mathsf{V}) \to \mathbb{P}(\mathsf{U})$ given by $\mathbb{P}(A)([v]) = [A(v)]$ is well-defined if and only $\ker(A) = \{0\}$. If $\ker(A) \neq \{0\}$, then $\mathbb{P}(A)([v])$ can only be defined for $[v] \notin \ker(A)$, i.e., we have a map
$$\mathbb{P}(A) \colon \mathbb{P}(\mathsf{V}) \setminus \mathbb{P}(\ker(A)) \to \mathbb{P}(\mathsf{U}),$$
which puts us in a setting similar to that of Section 4.4.3.
2. Let V be an F-vector space. Let us consider the map
$$\mathsf{V} \ni v \mapsto v^{\otimes d} \in \mathsf{S}^d(\mathsf{V}).$$
This is a polynomial function of homogeneous degree d, i.e., an element of
$$\mathsf{S}^d(\mathsf{V}^*) \otimes \mathsf{S}^d(\mathsf{V}) \simeq \mathrm{End}_{\mathsf{F}}(\mathsf{S}^d(\mathsf{V}));$$

indeed, one sees that the mapping corresponds to the identity endomorphism. This mapping vanishes only at $v = 0$, and, therefore, we have an induced mapping

$$\begin{aligned} \vartheta_d \colon \mathbb{P}(\mathsf{V}) &\to \mathbb{P}(\mathsf{S}^d(\mathsf{V})) \\ [v] &\mapsto [v^{\otimes d}], \end{aligned}$$

which is called the ***Veronese embedding***.

3. Let U and V be F-vector spaces and consider the map $\hat{\sigma}_{\mathsf{U},\mathsf{V}} \colon \mathsf{U} \times \mathsf{V} \to \mathsf{U} \otimes \mathsf{V}$ defined by $\hat{\sigma}_{\mathsf{U},\mathsf{V}}(u, v) = u \otimes v$. Note that

$$\hat{\sigma}(\lambda u, \mu v) = (\lambda\mu)\hat{\sigma}(u, v),$$

and from this we deduce that the map

$$\begin{aligned} \sigma_{\mathsf{U},\mathsf{V}} \colon \mathbb{P}(\mathsf{U}) \times \mathbb{P}(\mathsf{V}) &\to \mathbb{P}(\mathsf{U} \otimes \mathsf{V}) \\ ([u], [v]) &\mapsto [u \otimes v] \end{aligned}$$

is well-defined. This is called the ***Segre embedding***. •

4.4.5 The tautological line bundle

Now we get to defining our various line bundles. In the case of $d = -1$, denote

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1) = \{([v], \mathsf{L}) \in \mathbb{P}(\mathsf{V}) \times \mathbb{P}(\mathsf{V}) \mid v \in \mathsf{L}\}$$

and

$$\begin{aligned} \pi^{(-1)}_{\mathbb{P}(\mathsf{V})} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1) &\to \mathbb{P}(\mathsf{V}) \\ ([v], \mathsf{L}) &\mapsto [v]. \end{aligned}$$

The way to think of $\pi^{(-1)}_{\mathbb{P}(\mathsf{V})} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1) \to \mathbb{P}(\mathsf{V})$ is as a line bundle over $\mathbb{P}(\mathsf{V})$ for which the fibre over $[v]$ is the line generated by v. This is the ***tautological line bundle*** over $\mathbb{P}(\mathsf{V})$. In the case that $\mathsf{F} = \mathbb{R}$, the result is the so-called ***Möbius vector bundle*** over $\mathbb{RP}^1 \simeq \mathbb{S}^1$. This is a vector bundle with a one-dimensional fibre, and a "twist" as depicted in Figure 4.6.

Figure 4.6 A depiction of the Möbius vector bundle (imagine the fibres extending to infinity in both directions)

For $[v] \in \mathbb{P}(\mathsf{V})$, let us denote $\mathsf{Q}_{\mathsf{V},[v]} = \mathsf{V}/[v]$ and take

$$\mathsf{Q}_{\mathsf{V}} = \mathop{\mathring{\bigcup}}_{[v] \in \mathbb{P}(\mathsf{V})} \mathsf{Q}_{\mathsf{V},[v]}.$$

We can think of Q_{V} as being a vector bundle formed by the quotient of the trivial vector bundle $\mathbb{P}(\mathsf{V}) \times \mathsf{V}$ by the tautological line bundle. Note that we have an exact sequence

$$0 \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1) \longrightarrow \mathbb{P}(\mathsf{V}) \times \mathsf{V} \longrightarrow \mathsf{Q}_{\mathsf{V}} \longrightarrow 0$$

where all arrows are canonical, and where this is done for "fibres" over a fixed $[v] \in \mathbb{P}(\mathsf{V})$, i.e., this is a sequence of "vector bundles." This is called the ***tautological sequence***.

4.4.6 The degree $-d$ line bundles, $d \in \mathbb{Z}_{>0}$

For $d \in \mathbb{Z}_{>0}$ we define

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) = \{([v],([A],\mathsf{L})) \in \mathbb{P}(\mathsf{V}) \times \mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(-1) \mid \vartheta_d([v]) = \pi^{(-1)}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}([A],\mathsf{L})\}$$

and

$$\begin{aligned}\pi^{(-d)}_{\mathbb{P}(\mathsf{V})}\colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) &\to \mathbb{P}(\mathsf{V})\\ ([v],([A],\mathsf{L})) &\mapsto [v].\end{aligned}$$

The best way to think of $\pi^{(-d)}_{\mathbb{P}(\mathsf{V})}\colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) \to \mathbb{P}(\mathsf{V})$ is as the pull-back of the tautological line bundle over $\mathbb{P}(\mathsf{S}^d(\mathsf{V}))$ to $\mathbb{P}(\mathsf{V})$ by the Veronese embedding. (See Section 4.3.6 for a discussion of pull-back bundles.) The condition $\vartheta_d([v]) = \pi^{(-1)}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}([A],\mathsf{L})$ is phrased to emphasise this pull-back bundle interpretation of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$, but is more succinctly expressed by the requirement that $[v^{\otimes d}] \in [A]$. In any case, $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ is to be regarded as a vector bundle over $\mathbb{P}(\mathsf{V})$ whose fibre over $[v]$ is $[v^{\otimes d}]$.

Let us give a useful interpretation of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$.

4.4.16 Proposition (Morphisms associated with $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$) *For every* $d \in \mathbb{Z}_{>0}$ *we have a canonical isomorphism*

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^{\otimes d}$$

and a canonical inclusion

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) \mapsto \mathbb{P}(\mathsf{V}) \times \mathsf{S}^d(\mathsf{V}),$$

both being vector bundle mappings over $\mathrm{id}_{\mathbb{P}(\mathsf{V})}$.

Proof For the isomorphism, consider the map

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^{\otimes d} \ni ([v],u^{\otimes d}) \mapsto ([v],([v^{\otimes d}],u^{\otimes d})) \in \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) \subseteq \mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(-1).$$

Since $u \in [v]$, $u^{\otimes d} \in [v^{\otimes d}]$ from which one readily verifies that this map is indeed an isomorphism of vector bundles over $\mathbb{P}(\mathsf{V})$.

If we take the d-fold symmetric tensor product of the left half of the tautological sequence, we get the sequence

$$0 \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^{\otimes d} \longrightarrow \mathbb{P}(\mathsf{V}) \times \mathsf{S}^d(\mathsf{V})$$

which gives the inclusion when combined with the isomorphism from the first part of the proof. ■

4.4.7 The hyperplane line bundle

We refer here to the constructions of Section 4.4.3. With these constructions in mind, let us define

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1) = \mathbb{P}(\mathsf{F} \oplus \mathsf{V}) \setminus \mathbb{P}(\mathsf{F} \oplus 0)$$

and take $\pi^{(1)}_{\mathbb{P}(\mathsf{V})} = \mathbb{P}(\mathrm{pr}_2)$, so that we have the vector bundle $\pi^{(1)}_{\mathbb{P}(\mathsf{V})} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1) \to \mathbb{P}(\mathsf{V})$ whose fibre over $\mathsf{L} \in \mathbb{P}(\mathsf{V})$ is canonically isomorphic to L^*. Thus the fibres of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$ are linear functions on the fibres of the tautological line bundle. We call $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$ the ***hyperplane line bundle*** of $\mathbb{P}(\mathsf{V})$.

We have the following important attribute of the hyperplane line bundle.

4.4.17 Proposition (A projection from a trivial bundle onto the hyperplane line bundle) *We have a surjective mapping*

$$\mathbb{P}(\mathsf{V}) \times \mathsf{V}^* \to \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1),$$

as a vector bundle map over $\mathrm{id}_{\mathbb{P}(\mathsf{V})}$.

Proof Let $([v], A) \in \mathbb{P}(\mathsf{V}) \times \mathsf{S}^d(\mathsf{V}^*)$ and consider $[A(v) \oplus v] \in \mathbb{P}(\mathsf{F} \oplus \mathsf{V}) \setminus \mathbb{P}(\mathsf{F} \oplus 0)$. Since

$$[A(av) \oplus (av)] = [A(v) \oplus v], \qquad a \in \mathsf{F},$$

it follows that $[A(v) \oplus v]$ is a well-defined function of $[v]$. Recalling from Lemma 4.4.2 that vector addition and scalar multiplication on $\mathbb{P}(\mathrm{pr}_2)^{-1}([v])$ (with the origin $[0 \oplus v]$) are given by

$$[a \oplus v] + [b \oplus v] = [(a+b) \oplus v], \quad \alpha[a \oplus v] = [(\alpha a) \oplus v], \tag{4.15}$$

respectively, we see that the mapping $([v], A) \mapsto [A(v) \oplus v]$ is a vector bundle mapping. To see that the mapping is surjective, we need only observe that, if $[a \oplus v] \in \mathbb{P}(\mathrm{pr}_2)^{-1}([v])$, then, if we take $A \in \mathsf{S}^d(\mathsf{V}^*)$ to satisfy $A(v) = a$, we have $[A(v) \oplus v] = [a \oplus v]$, giving surjectivity. ■

If, for $[v] \in \mathbb{P}(\mathsf{V})$ we denote by $\mathsf{K}_{\mathsf{V},[v]}$ the kernel of the projection from $\{[v]\} \times \mathsf{V}^*$ onto $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)_{[v]}$, we have the following exact sequence,

$$0 \longrightarrow \mathsf{K}_{\mathsf{V}} \longrightarrow \mathbb{P}(\mathsf{V}) \times \mathsf{V}^* \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1) \longrightarrow 0$$

which we call the ***hyperplane sequence***. Note that $\mathsf{K}_{\mathsf{V},[v]} = \mathrm{ann}([\mathsf{v}])$, where "ann" denotes the annihilator.

The following result gives an essential property of the hyperplane line bundle.

4.4.18 Proposition (The hyperplane line bundle is the dual of the tautological line bundle) *We have an isomorphism*

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^* \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$$

as a vector bundle map over $\mathrm{id}_{\mathbb{P}(\mathsf{V})}$.

Proof If we take the dual of the tautological sequence, we get the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{Q}^*_{\mathsf{V}} & \longrightarrow & \mathbb{P}(\mathsf{V}) \times \mathsf{V}^* & \longrightarrow & \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^* & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathsf{K}_{\mathsf{V}} & \longrightarrow & \mathbb{P}(\mathsf{V}) \times \mathsf{V}^* & \longrightarrow & \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1) & \longrightarrow & 0
\end{array}$$

thinking of each component as a vector bundle over $\mathbb{P}(\mathsf{V})$ and each arrow as a vector bundle mapping over the identity. The leftmost vertical arrow is the defined by the canonical isomorphism

$$\mathsf{Q}^*_{\mathsf{V},[v]} = (\mathsf{V}/[v])^* \simeq \mathrm{ann}\,[v] = \mathsf{K}_{\mathsf{V},[v]}.$$

The dashed vertical arrow is then defined by taking a preimage of $\alpha_{[v]} \in \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)^*$ in $\mathbb{P}(\mathsf{V}) \times \mathsf{V}^*$ then projecting this to $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$. A routine argument shows that this mapping is a well-defined isomorphism. ■

4.4.8 The degree d line bundles, $d \in \mathbb{Z}_{>0}$

For $d \in \mathbb{Z}_{>0}$ we define

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d) = \{([v], \mathsf{M}) \in \mathbb{P}(\mathsf{V}) \times \mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(1) \mid \vartheta_d([v]) = \pi^{(1)}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(\mathsf{M})\}$$

and

$$\begin{aligned} \pi^{(d)}_{\mathsf{O}_{\mathbb{P}(\mathsf{V})}} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(d) &\to \mathbb{P}(\mathsf{V}) \\ ([v], \mathsf{M}) &\mapsto [v]. \end{aligned}$$

As with the negative degree line bundles, we think of this as the pull-back of $\mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(1)$ to $\mathbb{P}(\mathsf{V})$ by the Veronese embedding. Note that the fibre over $\mathbb{L} \in \mathbb{P}(\mathsf{V})$ is canonically isomorphic to $(\mathsf{S}^d(\mathsf{L}))^* \simeq \mathsf{S}^d(\mathsf{L}^*)$. Thus the fibres of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ are polynomial functions of degree d on the fibres of the tautological line bundle. With this in mind, we have the following adaptation of Proposition 4.4.16.

4.4.19 Proposition (Morphisms associated with $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathrm{d})$) *For* $\mathrm{d} \in \mathbb{Z}_{>0}$ *we have a canonical isomorphism*

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathrm{d}) \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)^{\otimes \mathrm{d}}$$

and a canonical surjective mapping

$$\mathbb{P}(\mathsf{V}) \times \mathsf{S}^{\mathrm{d}}(\mathsf{V}^*) \to \mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathrm{d}),$$

both being vector bundle mappings over $\mathrm{id}_{\mathbb{P}(\mathsf{V})}$.

Proof Keeping in mind the vector bundle structure on $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$ given explicitly by (4.15), an element of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)^{\otimes d}$ can be written as $[a^d \oplus v]$ for $[v] \in \mathbb{P}(\mathsf{V})$ and $a \in \mathsf{F}$. Thus consider the mapping

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)^{\otimes d} \ni [a^d \oplus v] \mapsto ([v], [a^d \oplus v^{\otimes d}]) \in \mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(1).$$

Another application of (4.15) to $\mathsf{O}_{\mathbb{P}(\mathsf{S}^d(\mathsf{V}))}(1)$ shows that the preceding map is a vector bundle map, and it is also clearly an isomorphism.

Now we can take the dual of the inclusion

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d) \to \mathbb{P}(\mathsf{V}) \times \mathsf{S}^d(\mathsf{V})$$

from Proposition 4.4.16 to give the surjective mapping in the statement of the proposition. ■

4.4.9 The tangent bundle, the cotangent bundle, and the Euler sequence

To motivate our discussion of tangent vectors and the tangent bundle, we consider the case when $\mathsf{F} = \mathbb{R}$ and so V is a $\mathbb{R}$-vector space. In this case, we establish a lemma.

4.4.20 Lemma (Tangent vectors on real projective space) *If* V *is a* $\mathbb{R}$*-vector space, there exists a canonical isomorphism of* $\mathsf{T}_{[v]}\mathbb{P}(\mathsf{V})$ *with* $\mathrm{Hom}_{\mathbb{R}}([v];\mathsf{V}/[v])$ *for every* $[v]\in\mathbb{P}(\mathsf{V})$*.*

Proof For $\mathsf{L}\in\mathbb{P}(\mathsf{V})$, the tangent space $\mathsf{T}_{\mathsf{L}}\mathbb{P}(\mathsf{V})$ consists of tangent vectors to curves at L. We define a map $T_{\mathsf{L}}\in\mathrm{Hom}_{\mathbb{R}}(\mathsf{T}_{\mathsf{L}}\mathbb{P}(\mathsf{V});\mathrm{Hom}_{\mathbb{R}}(\mathsf{L};\mathsf{V}/\mathsf{L}))$ as follows. Let $v\in\mathsf{T}_{\mathsf{L}}\mathbb{P}(\mathsf{V})$, let $\gamma\colon I\to\mathbb{P}(\mathsf{V})$ be a smooth curve for which $\gamma'(0)=v$. Let $u\in\mathsf{L}$ and let $\sigma\colon I\to\mathsf{V}$ be a smooth curve for which $\sigma(0)=u$ and $\gamma(t)=[\sigma(t)]$, and define $T_{\mathsf{L}}(v)\in\mathrm{Hom}_{\mathbb{R}}(\mathsf{L};\mathsf{V}/\mathsf{L})$ by

$$T_{\mathsf{L}}(v)\cdot u=\sigma'(0)+\mathsf{L}.$$

To see that T_{L} is well-defined, let τ be another curve for which $\tau(t)=u$ and $\gamma(t)=[\tau(t)]$. Since $\tau(0)-\sigma(0)=0$ we can apply Lemma 1 from the proof of Proposition 4.5.4 below to write $\tau(t)-\sigma(t)=t\rho(t)$ where $\rho\colon I\to\mathsf{V}$ satisfies $\rho(t)\in\gamma(t)$. Therefore,

$$\tau'(0)=\sigma'(0)+\rho(0)+\mathsf{L}=\sigma'(0)+\mathsf{L},$$

showing that $T_{\mathsf{L}}(v)$ is indeed well-defined. To show that T_{L} is injective, suppose that $T_{\mathsf{L}}(v)=0$. Thus $T_{\mathsf{L}}(v)\cdot u=0$ for every $u\in\mathsf{L}$. Let γ be a smooth curve on $\mathbb{P}(\mathsf{V})$ for which $\gamma'(0)=v$, let $u\in\mathsf{L}$, and let σ be a curve on V for which $\sigma(0)=u$ and $\gamma(t)=[\sigma(t)]$. Then

$$0=T_{\mathsf{L}}(v)\cdot u=\sigma'(0)+\mathsf{L}\quad\Longrightarrow\quad\sigma'(0)\in\mathsf{L}.$$

Since $\gamma(t)$ is the projection of $\sigma(t)$ from $\mathsf{V}\setminus\{0\}$ to $\mathbb{P}(\mathsf{V})$, it follows that $\gamma'(0)$ is the derivative of this projection applied to $\sigma'(0)$. But since $\sigma'(0)\in\mathsf{L}$ and since L is the kernel of the derivative of the projection, this implies that $v=\gamma'(0)=0$. Since

$$\dim_{\mathbb{R}}(\mathsf{T}_{\mathsf{L}}\mathbb{P}(\mathsf{V}))=\dim_{\mathbb{R}}(\mathrm{Hom}_{\mathbb{R}}(\mathsf{L};\mathsf{V}/\mathsf{L})),$$

it follows that T_{L} is an isomorphism. ■

With the lemma as motivation, in the general algebraic setting we define the ***tangent space*** of $\mathbb{P}(\mathsf{V})$ at $[v]$ to be

$$\mathsf{T}_{[v]}\mathbb{P}(\mathsf{V})=[v]^*\otimes\mathsf{V}/[v].$$

The ***tangent bundle*** is then, as usual, $\mathsf{T}\mathbb{P}(\mathsf{V})=\mathring{\cup}_{[v]\in\mathbb{P}(\mathsf{V})}\mathsf{T}_{[v]}\mathbb{P}(\mathsf{V})$. Recalling the quotient vector bundle Q_{V} used in the construction of the tautological sequence and recalling the definition of the hyperplane line bundle, we clearly have

$$\mathsf{T}\mathbb{P}(\mathsf{V})=\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)\otimes\mathsf{Q}_{\mathsf{V}}.$$

We then also have the ***cotangent bundle***

$$\mathsf{T}^*\mathbb{P}(\mathsf{V})=\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)\otimes\mathsf{K}_{\mathsf{V}},$$

noting that $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)\simeq\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)^*$ and $\mathsf{Q}_{\mathsf{V}}^*=\mathsf{K}_{\mathsf{V}}$.

We have the following result.

4.4.21 Proposition (The Euler sequence) *We have a short exact sequence*

$$0 \longrightarrow \mathbb{P}(\mathsf{V}) \times \mathsf{F} \longrightarrow \mathbb{P}(\mathsf{V}) \times (\mathsf{V} \otimes \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)) \longrightarrow \mathsf{T}\mathbb{P}(\mathsf{V}) \longrightarrow 0$$

Proof This follows by taking the tensor product of the tautological sequence with $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$, noting that

$$\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1) \otimes \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1) \simeq \mathsf{F}$$

by the isomorphism $v \otimes \alpha \mapsto \alpha(v)$. This is indeed an isomorphism since the fibres of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)$ and its dual $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)$ are one-dimensional. ■

Sometimes the dual

$$0 \longrightarrow \mathsf{T}^*\mathbb{P}(\mathsf{V}) \longrightarrow \mathbb{P}(\mathsf{V}) \times (\mathsf{V}^* \otimes \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-1)) \longrightarrow \mathbb{P}(\mathsf{V}) \times \mathsf{F}^* \longrightarrow 0$$

of the Euler sequence is referred to as the Euler sequence. In the more usual presentation of the Euler sequence one has $\mathsf{V} = \mathsf{F}^{n+1}$ so the sequence reads

$$0 \longrightarrow \mathbb{P}(\mathsf{F}^{n+1}) \times \mathsf{F} \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{V})}(1)^{n+1} \longrightarrow \mathsf{T}\mathbb{P}(\mathsf{F}^{n+1}) \longrightarrow 0$$

It is difficult to imagine that the Euler sequence can be of much importance from the manner in which it is developed here. But it has significance, for example, in commutative algebra where it is related to the so-called Koszul sequence [Eisenbud 1995, §17.5].

In case $\dim(\mathsf{V}) = 2$, the tangent and cotangent bundles are line bundles, and have a simple representation in terms the line bundles we have introduced above.

4.4.22 Proposition (Tangent and cotangent bundles of one-dimensional projective spaces) *If* F *is a field and if* V *is a two-dimensional* F*-vector space, then we have isomorphisms*

$$\mathsf{T}\mathbb{P}(\mathsf{V}) \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(2), \qquad \mathsf{T}^*\mathbb{P}(\mathsf{V}) \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-2).$$

Proof By a choice of basis, we can and do assume that $\mathsf{V} = \mathsf{F}^2$. We closely examine the Euler sequence. To do this, we first closely examine the tautological sequence in this case. The sequence is

$$0 \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(-1) \xrightarrow{I_1} \mathbb{P}(\mathsf{F}^2) \times \mathsf{F}^2 \xrightarrow{P_1} \mathsf{Q}_{\mathsf{F}^2} \longrightarrow 0$$

and, explicitly, we have

$$I_1(([(x,y)]), a(x,y)) = ([(x,y)], (ax, ay)), \quad P_1([(x,y)], (u,v) + [(x,y)]).$$

The Euler sequence is obtained by taking the tensor product of this sequence with $\mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)$:

$$0 \longrightarrow \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(-1) \otimes \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1) \xrightarrow{I_1 \otimes \mathrm{id}} \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)^2 \xrightarrow{P_1 \otimes \mathrm{id}} \mathsf{T}\mathbb{P}(\mathsf{F}^2) \longrightarrow 0$$

with id denoting the identity map on $\mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)$. Explicitly we have

$$I_1 \otimes \mathrm{id}([(x,y)], (a(x,y)) \otimes \alpha) = I_1([(x,y)], (ax, ay)) \otimes \alpha = (ax\alpha) \oplus (ay\alpha).$$

Now let $[(x, y)] \in \mathbb{P}(\mathsf{F}^2)$ so that x and/or y is nonzero. Obviously (x, y) is a basis for $\mathsf{L} = [(x, y)]$. Let $(\xi_{(x,y)}, \eta_{(x,y)}) \in \mathsf{F}^2$ be such that $((x, y), (\xi_{(x,y)}, \eta_{(x,y)}))$ is a basis for F^2. For $(u, v) \in \mathsf{F}^2$ write

$$(u, v) = a_{(x,y)}(u, v)(x, y) + b_{(x,y)}(u, v)(\xi_{(x,y)}, \eta_{(x,y)}),$$

uniquely defining $a_{(x,y)}(u, v), b_{(x,y)}(u, v) \in \mathsf{F}$. Using this we write

$$P_1 \otimes \mathrm{id}([(x, y)], (u, v) \otimes \alpha) = ([(x, y)], (b_{(x,y)}(u, v)(\xi_{(x,y)}, \eta_{(x,y)}) + [(x, y)]) \otimes \alpha).$$

Now consider the map

$$\begin{aligned} \phi\colon\ & \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)^2 \to \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(2) \\ & ([(x, y)], \alpha \oplus \beta) \mapsto ([(x, y)], (\xi_{(x,y)}\alpha) \otimes (\eta_{(x,y)}\beta)). \end{aligned}$$

Making the identification $\mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(-1) \otimes \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1) \simeq \mathbb{P}(\mathsf{F}^2) \times \mathsf{F}$ as in the proof of Proposition 4.4.21, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{P}(\mathsf{F}^2) \times \mathsf{F} & \longrightarrow & \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)^2 & \longrightarrow & \mathsf{T}\mathbb{P}(\mathsf{F}^2) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{P}(\mathsf{F}^2) \times \mathsf{F} & \longrightarrow & \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(1)^2 & \longrightarrow & \mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(2) & \longrightarrow & 0 \end{array}$$

with exact rows. The dashed arrow is defined by taking a preimage of $v_{\mathsf{L}} \in \mathsf{T}_{\mathsf{L}}\mathbb{P}(\mathsf{F}^2)$ in $\mathsf{O}^2_{\mathbb{P}(\mathsf{F}^2)}$ and projecting this to $\mathsf{O}_{\mathbb{P}(\mathsf{F}^2)}(2)$. One verifies easily that this map is a well-defined isomorphism.

That $\mathsf{T}^*\mathbb{P}(\mathsf{V}) \simeq \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-2)$ follows from Propositions 4.4.18 and 4.4.19. ■

4.4.10 Global sections of the line bundles

Let us consider the global sections of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ for $d \in \mathbb{Z}$. The sections we consider are those that satisfy the sort of regularity conditions we introduced in Section 4.4.4. This takes a slightly different form, depending on the degree of the line bundle.

4.4.23 Definition (Regular sections of line bundles over projective space) Let F be a field, let V be a finite-dimensional F-vector space, and let $d \in \mathbb{Z}$. A ***section*** of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ is a map $\sigma\colon \mathbb{P}(\mathsf{V}) \to \mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ for which $\pi^{(d)}_{\mathbb{P}(\mathsf{V})} \circ \sigma = \mathrm{id}_{\mathbb{P}(\mathsf{V})}$. A section σ is ***regular*** if

(i) $d < 0$: $\hat{\sigma}\colon \mathbb{P}(\mathsf{V}) \to \mathsf{S}^d(\mathsf{V})$ is regular in the sense of Definition 4.4.11, where $\hat{\sigma}$ is defined by the requirement that

$$\sigma([v]) = ([v], ([v^{\otimes d}], \hat{\sigma}([v])));$$

(ii) $d = 0$: $\hat{\sigma}\colon \mathbb{P}(\mathsf{V}) \to \mathsf{F}$ is regular in the sense of Definition 4.4.11, where $\hat{\sigma}$ is defined by the requirement that

$$\sigma([v]) = ([v], \hat{\sigma}([v]));$$

(iii) $d > 0$: $\hat{\sigma}\colon \mathsf{V} \to \mathsf{F}$ is regular in the sense of Definition 4.4.8, where $\hat{\sigma}$ is defined by the requirement that

$$\sigma([v]) = ([v], [\hat{\sigma}(v) \oplus v^{\otimes d}]).$$

The set of regular sections of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ we denote by $\Gamma(\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d))$. •

With these definitions, we have the following result that gives a complete characterisation of the space of global sections in the algebraically closed case.

4.4.24 Proposition (Sections of line bundles over projective space) *If* F *is a field and if* V *is an* $(n+1)$*-dimensional* F*-vector space, for* $d \geq 0$ *we have*

$$\dim_{\mathsf{F}}(\Gamma(\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d))) \geq \binom{n+d}{n} = \frac{(n+d)!}{n!d!}.$$

Moreover, if F *is algebraically closed, then we have*

$$\dim_{\mathsf{F}}(\Gamma(\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d))) = \begin{cases} 0, & d < 0, \\ \binom{n+d}{n}, & d \geq 0. \end{cases}$$

Proof Let $d \geq 0$. If $A \in \mathsf{S}^d(\mathsf{V}^*)$ then there is a corresponding regular section σ_A of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ defined by

$$\sigma_A([v]) = ([v], [A(v^{\otimes d}), v^{\otimes d}]).$$

Thus we have a mapping from $\mathsf{S}^d(\mathsf{V}^*)$ to $\Gamma(\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d))$. We claim that this map is injective. Indeed, if $\sigma_A([v]) = 0$ for every $[v] \in \mathbb{P}(\mathsf{V})$. This means that $A(v^{\otimes d}) = 0$ for every $v \in \mathsf{V}$ and so $A = 0$. The first statement of the proposition now follows from Proposition F.2.9, also cf. Lemma 1.1.1.

For the remainder of the proof we suppose that F is algebraically closed.

Let us next consider the negative degree case. Let σ be a global section of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ with $\hat{\sigma}\colon \mathbb{P}(\mathsf{V}) \to \mathsf{S}^d(\mathsf{V})$ the induced map. Let $\alpha \in \mathsf{S}^d(\mathsf{V}^*)$ so that $\alpha \circ \hat{\sigma}$ is an F-valued regular function on $\mathbb{P}(\mathsf{V})$, and so is constant by Proposition 4.4.12. We claim that this implies that $\hat{\sigma}$ is constant. Suppose otherwise, and that $\hat{\sigma}([v_1]) \neq \hat{\sigma}([v_2])$ for distinct $[v_1], [v_2] \in \mathbb{P}(\mathsf{V})$. This implies that we can choose $\alpha \in \mathsf{S}^d(\mathsf{V}^*)$ such that $\alpha \circ \hat{\sigma}([v_1]) \neq \alpha \circ \hat{\sigma}([v_2])$. To see this, suppose first that only one of $\hat{\sigma}([v_1])$ and $\hat{\sigma}([v_2])$ are nonzero, say $\hat{\sigma}([v_1])$. Then we need only choose α so that $\hat{\sigma}([v_1]) \neq 0$. If both of $\hat{\sigma}([v_1])$ and $\hat{\sigma}([v_2])$ are nonzero, then they are either collinear (in which case our conclusion follows) or linearly independent (so one can certainly choose α so that $\alpha \circ \hat{\sigma}([v_1]) \neq \alpha \circ \hat{\sigma}([v_2])$). Thus we can indeed conclude that $\hat{\sigma}$ is constant. Note that, for $[v] \in \mathbb{P}(\mathsf{V})$ we have $\hat{\sigma}([v]) = a_{[v]} v^{\otimes d}$ for some $a_{[v]} \in \mathsf{F}$. That is to say, $\hat{\sigma}([v])$ is a point on the line $[v^{\otimes d}]$ for every $[v] \in \mathbb{P}(\mathsf{V})$. The only point in $\mathsf{S}^d(\mathsf{V})$ on every such line is zero, and so $\hat{\sigma}$ is the zero function.

For $d = 0$ the result follows from Proposition 4.4.12.

Now consider $d > 0$ and let σ be a regular section of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$ with $\hat{\sigma}\colon \mathbb{P}(\mathsf{V}) \to \mathsf{F}$ the corresponding function. In order that this provide a well-defined section of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)$, we must have

$$[\hat{\sigma}([\lambda v]) \oplus (\lambda v)^{\otimes d}] = [\hat{\sigma}([v]) \oplus v^{\otimes d}],$$

which means that

$$\hat{\sigma}([\lambda v]) \oplus (\lambda v)^{\otimes d} = \alpha([\hat{\sigma}([v]) \oplus v^{\otimes d}])$$

for some $\alpha \in \mathsf{F}$. Since $v \neq 0$, $v^{\otimes d} \neq 0$ and so we must have $\alpha = \lambda^d$, and so $\hat{\sigma}(\lambda[v]) = \lambda^d \hat{\sigma}([v])$. The requirement that $\hat{\sigma}$ be regular then ensures that $\hat{\sigma} = f_A$ for $A \in \mathsf{S}^d(\mathsf{V}^*)$, according to Proposition 4.4.9, since F is algebraically closed. ■

Let us observe that the conclusions of the proposition do not necessarily hold when the field is not algebraically closed.

4.4.25 Example (Sections of line bundles over $\mathbb{RP}^1$) We consider the simple example of line bundles over $\mathbb{RP}^1$. First let us show that there are nonzero regular sections of the tautological line bundle in this case. To define a section σ of $\mathsf{O}_{\mathbb{RP}^1}(-1)$, we prescribe $\hat{\sigma}\colon \mathbb{RP}^1 \to \mathbb{R}^2$, as in Definition 4.4.23(i). There are many possibilities here, and one way to prescribe a host of these is to take $\hat{\sigma}$ to be of the form

$$\hat{\sigma}([a_0 : a_1]) = \left(a_0 \frac{p(a_0, a_1)}{a_0^{2k} + a_1^{2k}}, a_1 \frac{p(a_0, a_1)}{a_0^{2k} + a_1^{2k}}\right)$$

for $k \in \mathbb{Z}_{>0}$ and where p is a polynomial function of homogeneous degree $2k - 1$. In Figure 4.7 we show the images of $\hat{\sigma}$ in a few cases, just for fun. Note that if σ is a section

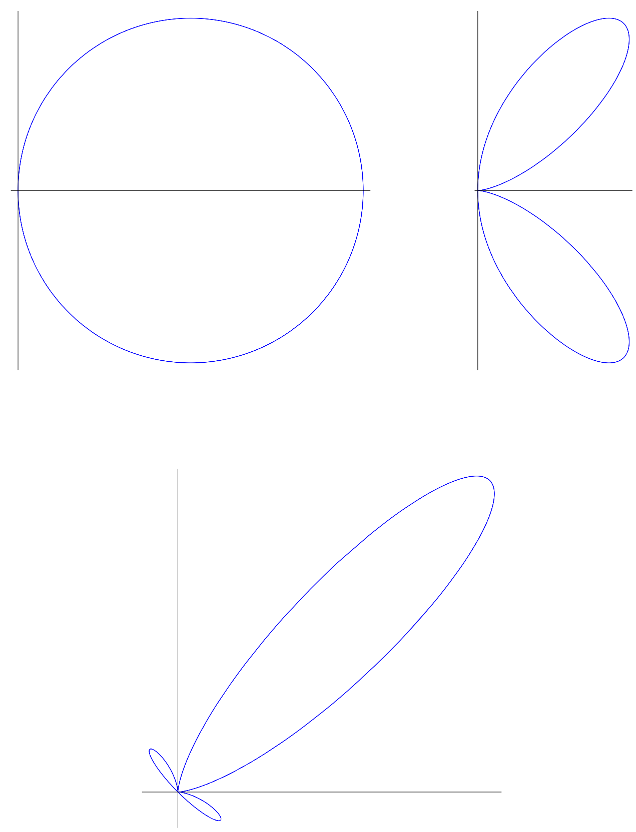

Figure 4.7 The image of $\hat{\sigma}$ for $k = 1$ and $p(a_0, a_1) = a0$ (top left), $k = 2$ and $p(a_0, a_1) = a_0^2 a_1$ (top right), and $k = 3$ and $p(a_0, a_1) = a_0^2 a_1^3 + a_0^3 a_1^2$ (bottom)

of $\mathsf{O}_{\mathbb{RP}^1}(-1)$ then $\sigma^{\otimes d}$ is a section of $\mathsf{O}_{\mathbb{RP}^1}(-d)$. In this way, we immediately deduce that $\mathsf{O}_{\mathbb{RP}^1}(-d)$ has nonzero regular sections for every $d \in \mathbb{Z}_{>0}$.

Of course, there are nonzero regular sections of $\mathsf{O}_{\mathbb{RP}^1}(0)$, as such sections are in correspondence with regular functions, cf. Example 4.4.10.

As for sections of $\mathsf{O}_{\mathbb{RP}^1}(d)$ for $d > 0$, it still follows from the proof of Proposition 4.4.24 that, if $A \in \mathsf{S}^d(\mathsf{V}^*)$, we have a corresponding regular section of $\mathsf{O}_{\mathbb{RP}^1}(d)$. However, there are many other global regular sections since, given a given a regular function f, there is the corresponding regular section fA. •

4.4.26 Remark (The case of real and complex projective spaces) Note that the preceding discussion regarding sections of line bundles reveals essential differences between the real and complex case that arise, at least in this algebraic setting, from the fact that $\mathbb{C}$ is algebraically closed, whereas $\mathbb{R}$ is not. These differences are also reflected in the geometric setting where, instead of regular sections, one wishes to consider holomorphic or real analytic sections. As we saw in Example 4.3.14, the restrictions for sections that we have seen in Proposition 4.4.24 in the algebraic case are also present in the holomorphic case. On the flip side of this, we see that even in the algebraic case, there are many sections of vector bundles over real projective space. This is, moreover, consistent with the fact that, in the geometric setting, real analytic vector bundles admit many real analytic sections, cf. Cartan's Theorem A in the real
ref analytic case. •

4.4.11 Coordinate representations

In this section, after working hard to this point to avoid the use of bases, we connect the developments above to the commonly seen transition function treatment of line bundles over projective space.

Coordinates for projective space

We fix a basis $(e_0, e_1, \dots, e_n)$ for V, giving an isomorphism

$$(x_0, x_1, \dots, x_n) \mapsto x_0 e_0 + x_1 e_1 + \cdots + x_n e_n$$

of F^{n+1} with V. We shall engage in a convenient abuse of notation and write

$$x = (x_0, x_1, \dots, x_n),$$

i.e., confound a vector with its components. The line

$$[x_0 e_0 + x_1 e_1 + \cdots + x_n e_n]$$

is represented by $[x_0 : x_1 : \cdots : x_n]$. Again, we shall often write

$$[x] = [x_0 : x_1 : \cdots : x_n],$$

confounding a line with its component representation. For $j \in \{0, 1, \dots, n\}$ we denote

$$\mathcal{U}_j = \{[x_0 : x_1 : \cdots : x_n] \mid x_j \neq 0\}$$

and note that $\mathbb{P}(\mathsf{V}) = \cup_{j=0}^{n} \mathcal{U}_j$. We let $\mathsf{O}_j = \operatorname{span}_{\mathsf{F}}(e_j)$, $j \in \{0, 1, \dots, n\}$. As per Lemma 4.4.2, the map

$$\begin{aligned} \phi_j \colon \mathcal{U}_j &\to \mathsf{F}^n \\ [x_0 : x_1 : \cdots : x_n] &\mapsto (x_j^{-1}x_0, x_j^{-1}x_1, \dots, x_j^{-1}x_{j-1}, x_j^{-1}x_{j+1}, \dots, x_j^{-1}x_n) \end{aligned}$$

is an affine isomorphism.

Coordinate representations for the negative degree line bundles

Let us consider the structure of our line bundles over $\mathbb{P}(\mathsf{V})$. We first consider the negative degree line bundles $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ for $d \in \mathbb{Z}_{>0}$. In doing this, we recall from Proposition 4.4.16 that $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ is a subset of the trivial bundle $\mathbb{P}(\mathsf{V}) \times \mathsf{S}^d(\mathsf{V})$. We will thus use coordinates

$$([x_0, x_1, \dots, x_n], A),$$

to denote a point in $([x], A) \in \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$, with the understanding that (1) this is a basis representation and (2) the requirement to be in $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ is that

$$[A] = [(x_0, x_1, \dots, x_n)^{\otimes d}].$$

The following lemma gives a local trivialisation of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ over the affine sets $\mathcal{U}_j$, $j \in \{0, 1, \dots, n\}$.

4.4.27 Lemma (Local trivialisation of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-\mathrm{d})$) *With all the above notation, for* $\mathrm{j} \in \{0, 1, \dots, \mathrm{n}\}$ *and* $\mathrm{d} \in \mathbb{Z}_{>0}$, *the map*

$$\begin{aligned} \tau_{\mathrm{j}}^{(-\mathrm{d})} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-\mathrm{d})|\mathcal{U}_{\mathrm{j}} &\to \mathcal{U}_{\mathrm{j}} \times \mathsf{F} \\ ([\mathrm{x}_0, \mathrm{x}_1 : \cdots : \mathrm{x}_\mathrm{n}], \mathrm{a}(\mathrm{x}_0, \mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n})^{\otimes \mathrm{d}}) &\mapsto ([\mathrm{x}_0 : \mathrm{x}_1 : \cdots : \mathrm{x}_\mathrm{n}], \mathrm{a}\mathrm{x}_{\mathrm{j}}^{\mathrm{d}}) \end{aligned}$$

is an isomorphism of vector bundles.

Proof Let us first show that $\tau_j^{(-d)}$ is well-defined. Suppose that $[x] \in \mathcal{U}_j$ is written as

$$[x] = [x_0 : x_1 : \cdots : x_n] = [y_0, y_1 : \cdots : y_n]$$

so that

$$x_j^{-1}(x_0, x_1, \dots, x_n) = y_j^{-1}(y_0, y_1, \dots, y_n).$$

If $v = (x_0, x_1, \dots, x_n)$ then we have

$$v = x_j y_j^{-1}(x_0, x_1, \dots, x_n)$$

and so

$$(x_0, x_1, \dots, x_n)^{\otimes d} = x_j^d y_j^{-d}(y_0, y_1, \dots, y_n)^{\otimes d}.$$

From this we deduce that

$$\begin{aligned}\tau_j^{(-d)}([x_0 : x_1 : \cdots : x_n], a(x_0, x_1, \dots, x_n)^{\otimes d}) &= ([x_0 : y_1 : \cdots : x_n], a x_j^d) \\ &= ([(x_j y_j^{-1}) y_0 : (x_j y_j^{-1}) y_1 : \cdots : (x_j y_j^{-1}) y_n], a x_j^d (y_j^d y_j^{-d})) \\ &= ([y_0 : y_1 : \cdots : y_n], a y_j^d (x_j^d y_j^{-d})) \\ &= \tau_j^{(-d)}([y_0 : y_1 : \cdots : y_n], a x_j^d y_j^{-d} (y_0, y_1, \dots, y_n)^{\otimes d}),\end{aligned}$$

and from this we see that $\tau_j^{(-d)}$ is well-defined. Clearly $\tau_j^{(-d)}$ is a vector bundle map. Moreover, since x_j is nonzero on $\mathcal{U}_j$, $\tau_j^{(-d)}$ is surjective, and so an isomorphism. ■

Now suppose that $[x] \in \mathcal{U}_j \cap \mathcal{U}_k$ and that $([x], A) \in \mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$. The following lemma relates the representations of $([x], A)$ in the two local trivialisations.

4.4.28 Lemma (Transition functions for $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$) *With all the above notation, if*

$$\tau_{\mathrm{j}}^{(-\mathrm{d})}([\mathrm{x}], \mathrm{A}) = ([\mathrm{x}_0 : \mathrm{x}_1 : \cdots : \mathrm{x}_\mathrm{n}], \mathrm{a}_\mathrm{j}), \quad \tau_{\mathrm{k}}^{(-\mathrm{d})}([\mathrm{x}], \mathrm{A}) = ([\mathrm{x}_0 : \mathrm{x}_1 : \cdots : \mathrm{x}_\mathrm{n}], \mathrm{a}_\mathrm{k}),$$

then $\mathrm{a}_\mathrm{k} = (\frac{\mathrm{x}_\mathrm{k}}{\mathrm{x}_\mathrm{j}})^\mathrm{d} \mathrm{a}_\mathrm{j}$.

Proof Note that

$$(\tau_j^{(-d)})^{-1}([x_0 : x_1 : \cdots : x_n], a) = ([x_0 : x_1 : \cdots : x_n], a x_j^{-d} (x_0, x_1, \dots, x_n)^{\otimes d})$$

and so

$$\begin{aligned}\tau_k^{(-d)} \circ (\tau_j^{(-d)})^{-1}([x_0 : x_1 : \cdots : x_n], a) &= \tau_k^{(-d)}([x_0 : x_1 : \cdots : x_n], a x_j^{-d} (x_0, x_1, \dots, x_n)^{\otimes d}) \\ &= ([x_0 : x_1 : \cdots : x_n], a x_k^d x_j^{-d}).\end{aligned}$$

We then compute

$$\begin{aligned}([x_0 : x_1 : \cdots : x_n], a_k) = \tau_k^{(-d)}([x], A) &= \tau_k^{(-d)} \circ (\tau_j^{(-d)})^{-1} \circ \tau_j^{(-d)}([x], A) \\ &= \tau_k^{(-d)} \circ (\tau_j^{(-d)})^{-1}([x_0 : x_1 : \cdots : x_n], a_j) \\ &= ([x_0 : x_1 : \cdots : x_n], a_j x_k^d x_j^{-d}),\end{aligned}$$

giving the desired conclusion. ■

Since the function

$$[x_0 : x_1 : \cdots : x_n] \mapsto \Big(\frac{x_k}{x_j}\Big)^d$$

is a regular function on $\mathcal{U}_j \cap \mathcal{U}_k$, we are finally justified in calling $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(-d)$ a vector bundle over $\mathbb{P}(\mathsf{V})$ since we have found local trivialisations which satisfy an appropriate overlap condition within our algebraic setting.

Coordinate representations for the positive degree line bundles

Next we turn to the positive degree line bundles. Here we have to consider sections of the bundle

$$\mathbb{P}(\mathsf{F} \oplus \mathsf{S}^d(\mathsf{V})) \setminus \mathbb{P}(\mathsf{F} \oplus 0),$$

so we establish some notation for this. We use the basis

$$1 \oplus 0, 0 \oplus e_1, \dots, 0 \oplus e_n$$

for $\mathsf{F} \oplus \mathsf{V}$ and denote a point

$$\mathsf{F} \oplus \mathsf{V} \ni (\xi, x) = \xi(1 \oplus 0) + x_0(0 \oplus e_0) + x_1(0 \oplus e_1) + \dots + x_n(0 \oplus e_n)$$

by $(\xi, (x_0, x_1, \dots, x_n)) \in \mathsf{F} \oplus \mathsf{F}^n$. The line $[(\xi, x)]$ is then denoted by $[\xi : [x_0 : x_1 : \dots : x_n]]$. We shall also need notation for lines in $\mathsf{S}^d(\mathsf{V})$ and $\mathsf{F} \oplus \mathsf{S}^d(\mathsf{V})$. For $x \in \mathsf{V} \setminus \{0\}$ we use the notation

$$[x_0 : x_1 : \dots : x_n]^{\otimes d}, \quad [\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}]$$

to denote the lines $[x^{\otimes d}]$ and $[\xi \oplus x^{\otimes d}]$, respectively.

We are now able to give the following local trivialisations for the positive degree line bundles.

4.4.29 Lemma (Local trivialisation of $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathrm{d})$) *With all the above notation, for* $\mathrm{j} \in \{0, 1, \dots, \mathrm{n}\}$ *and* $\mathrm{d} \in \mathbb{Z}_{>0}$*, the map*

$$\begin{aligned} \tau_{\mathrm{j}}^{(\mathrm{d})} \colon \mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathrm{d})|\mathcal{U}_{\mathrm{j}} &\to \mathcal{U}_{\mathrm{j}} \times \mathsf{F} \\ ([\mathrm{x}_0 : \mathrm{x}_1 : \dots : \mathrm{x}_\mathrm{n}], [\xi : [\mathrm{x}_0 : \mathrm{x}_1 : \dots : \mathrm{x}_\mathrm{n}]^{\otimes \mathrm{d}}]) &\mapsto ([\mathrm{x}_0 : \mathrm{x}_1 : \dots : \mathrm{x}_\mathrm{n}], \xi \mathrm{x}_\mathrm{j}^{-\mathrm{d}}) \end{aligned}$$

is an isomorphism of vector bundles.

Proof Suppose that

$$[x_0 : x_1 : \dots : x_n] = [y_0 : y_1 : \dots : y_n]$$

and

$$[\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}] = [\eta : [y_0 : y_1 : \dots : y_n]^{\otimes d}],$$

which implies that

$$x_j^{-1}(x_0, x_1, \dots, x_n) = y_j^{-1}(y_0, y_1, \dots, y_n)$$

and so $\xi x_j^{-d} = \eta y_j^{-d}$. From this we conclude that $\tau_j^{(d)}$ is well-defined. To verify that $\tau_j^{(d)}$ is linear, we recall from Lemma 4.4.2 that, with the origin $[0 : [x_0 : x_1 : \dots : x_n]^{\otimes d}]$, the operations of vector addition and scalar multiplication in $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(d)_{[x]}$ are given by

$$\begin{aligned} [\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}] + [\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}] &= [\xi + \eta : [x_0 : x_1 : \dots : x_n]^{\otimes d}], \\ \alpha[\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}] &= [\alpha\xi : [x_0 : x_1 : \dots : x_n]^{\otimes d}]. \end{aligned}$$

From this, the linearity of $\tau_j^{(d)}$ follows easily. It is also clear that $\tau_j^{(d)}$ is an isomorphism since x_j is nonzero on $\mathcal{U}_j$. ■

Finally, we can give the transition functions for the line bundles in this case. That is, we let $[x] \in \mathcal{U}_j \cap \mathcal{U}_k$ and consider the representation of $([x], [a \oplus x^{\otimes d}])$ in both trivialisations.

4.4.30 Lemma (Transition functions for $\mathsf{O}_{\mathbb{P}(\mathsf{V})}(\mathsf{d})$) *With all of the above notation, if*

$$\tau_j^{(d)}([x],[a\oplus x^{\otimes d}]) = ([x_0 : x_1 : \cdots : x_n], a_j),$$
$$\tau_k^{(d)}([x],[a\oplus x^{\otimes d}]) = ([x_0 : x_1 : \cdots : x_n], a_k),$$

then $a_k = (\frac{x_j}{x_k})^d a_j$.

Proof We have

$$(\tau_j^{(d)})^{-1}([x_0 : x_1 : \cdots : x_n], a) = ([x_0 : x_1 : \cdots : x_n], [ax_j^d : [x_0 : x_1 : \cdots : x_n]^{\otimes d}])$$

which gives

$$\begin{aligned}\tau_k^{(d)} \circ (\tau_j^{(d)})^{-1}([x_0 : x_1 : \cdots : x_n], a) &= \tau_k^{(d)}([x_0 : x_1 : \cdots : x_n], [ax_j^d : [x_0 : x_1 : \cdots : x_n]^{\otimes d}]) \\ &= ([x_0 : x_1 : \cdots : x_n], ax_j^d x_k^{-d}).\end{aligned}$$

Thus we compute

$$\begin{aligned}([x_0 : x_1 : \cdots : x_n], a_k) &= \tau_k^{(d)}([x];[a\oplus x^{\otimes d}]) \\ &= \tau_k^{(d)} \circ (\tau_j^{(d)})^{-1} \circ \tau_j^{(d)}([x];[a\oplus x^{\otimes d}]) \\ &= \tau_k^{(d)} \circ (\tau_j^{(d)})^{-1}([x_0 : x_1 : \cdots : x_n], a_j) \\ &= ([x_0 : x_1 : \cdots : x_n], a_j x_j^d x_k^{-d}),\end{aligned}$$

as desired. ■

4.5 Tangent bundles of holomorphic and real analytic manifolds

In this section we discuss tangent bundles of holomorphic and real analytic manifolds. This breaks into two parts. First we recall the basics of tangent bundles from smooth real differential geometry, but now applied to the real analytic case. There is really nothing new here, but we fix notation and conventions. In the holomorphic case, there is additional structure inherited from the fact that the real tangent spaces are, in fact, $\mathbb{C}$-vector spaces. This additional structure is considered in detail.

4.5.1 Real tangent vectors and the real tangent bundle

Note that a holomorphic manifold is a real analytic manifold and that a real analytic manifold is a smooth real manifold. Thus we can adopt from the smooth real setting the construction of tangent vectors. There are (at least) two equivalent definitions of tangent vector on a smooth manifold. The definition we use is a geometric definition.

4.5.1 Definition (Tangent vector, tangent space, tangent bundle) Let M be a smooth or real analytic manifold and let $x_0 \in \mathsf{M}$.

(i) A ***curve at*** $\mathbf{x_0}$ is a differentiable map $\gamma\colon I \to \mathsf{M}$ such that $I \subseteq \mathbb{R}$ is an interval, $0 \in \operatorname{int}(I)$, and $\gamma(0) = x_0$;

(ii) Two curves at x_0, $\gamma_1\colon I_1 \to \mathsf{M}$ and $\gamma_2\colon I_2 \to \mathsf{M}$, are ***equivalent*** if, for some $\mathbb{R}$-chart $(\mathcal{U}, \phi)$ with $x_0 \in \mathcal{U}$, we have

$$\boldsymbol{D}\phi \circ \gamma_1(0) = \boldsymbol{D}\phi \circ \gamma_2(0).$$

(iii) A ***tangent vector*** at x_0 is an equivalence class of curves under the preceding notion of equivalence. Thus a tangent vector may be denoted by $[\gamma]_{x_0}$ with γ being a curve at x_0.

(iv) The ***tangent space*** at x_0 is the set of all equivalence classes of curves, and is denoted by $\mathsf{T}_x\mathsf{M}$.

(v) The ***tangent bundle*** of M is the disjoint union of the tangent spaces, and is denoted by $\mathsf{TM} = \mathring{\cup}_{x\in\mathsf{M}} \mathsf{T}_x\mathsf{M}$.

(vi) The projection from TM to M is denoted by π_{TM}. •

In Figure 4.8 we depict the idea behind our definition of a tangent vector. One

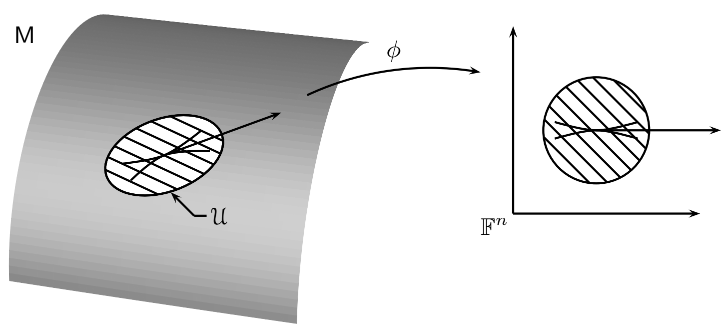

Figure 4.8 The idea behind equivalence of curves at x_0

can verify, using the Inverse Function Theorem and the fact that the overlap map is a diffeomorphism to verify that the definition of equivalence of curves does not depend on the particular choice of chart $(\mathcal{U}, \phi)$.

An alternative and equivalent characterisation of tangent vectors comes in terms of derivations.

4.5.2 Definition ($\mathbb{R}$-derivation at $\mathbf{x_0}$) Let M be a smooth or real analytic manifold and let $x_0 \in \mathsf{M}$. A $\mathbb{R}$-***derivation*** at x_0 is a $\mathbb{R}$-linear map $\theta\colon \mathrm{C}^\infty(\mathsf{M}) \to \mathbb{R}$ such that

$$\theta(fg) = \theta(f)g(x_0) + f(x_0)\theta(g).$$ •

We shall subsequently explore derivations in more detail in the holomorphic case, so let us say a few more things about their structure here.

4.5.3 Proposition (Derivations are local) *Let* M *be a smooth or real analytic manifold and let* $x_0 \in M$*. If* θ *is an* $\mathbb{R}$*-derivation at* $x_0 \in M$ *and if* $f, g \in C^\infty(M)$ *have the property that* $f|\mathcal{U} = g|\mathcal{U}$ *for some neighbourhood* $\mathcal{U}$ *of* x_0*, then* $\theta(f) = \theta(g)$*.*

Proof First let us suppose that g vanishes on $\mathcal{U}$. Let $\mathcal{V} \subseteq \mathcal{U}$ be a neighbourhood of x_0 such that $\mathrm{cl}(\mathcal{V}) \subseteq \mathcal{U}$ and, by the Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, §5.5], let $h \in C^\infty(M)$ be such that $h(x) = 0$ for $x \in \mathcal{V}$ and $h(x) = 1$ for $x \in M \setminus \mathcal{U}$. We then have $hf = f$ and so

$$\theta(f) = \theta(hf) = \theta(h)f(x_0) + h(x_0)\theta(f) = 0.$$

Now let g agree with f on $\mathcal{U}$. By our computation above and by the $\mathbb{R}$-linearity of derivations,

$$0 = \theta(f - g) = \theta(f) - \theta(g),$$

as desired. ■

The previous result has (at least) two important consequences. First of all, and somewhat pragmatically, it allows us to work locally in describing $\mathbb{R}$-derivations at x_0, and so we can work in the domain of a coordinate chart about x_0. Second of all, and of conceptual importance, we can as well think of a derivation θ as being a $\mathbb{R}$-linear map from the ring of germs $\mathscr{C}^\infty_{x_0,M}$ to $\mathbb{R}$ satisfying

$$\theta([f]_{x_0}[g]_{x_0}) = \theta([f]_{x_0})g(x_0) + f(x_0)\theta([g]_{x_0}). \tag{4.16}$$

This latter point will be crucial when we subsequently consider the holomorphic case in detail.

Let us go along the pragmatic lines suggested above to arrive at a concrete description of a derivation is a coordinate chart. Let us provide the notation first of all. We let M be a smooth or real analytic manifold, let $x_0 \in M$, and let $(\mathcal{U}, \phi)$ be a $\mathbb{R}$-chart about x_0. Let us denote the coordinates in the chart by $(x^1, \dots, x^n)$. For $j \in \{1, \dots, n\}$ define the $\mathbb{R}$-derivation $\frac{\partial}{\partial x^j}(x_0)$ at x_0 by asking that

$$\frac{\partial}{\partial x^j}(x_0)(f) = \frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(x_0)).$$

for $f \in C^\infty(M)$.

4.5.4 Proposition (Coordinate characterisation of derivations) *Let* M *be a smooth or real analytic manifold, let* $x_0 \in M$*, and let* $(\mathcal{U}, \phi)$ *be a* $\mathbb{R}$*-chart with* $x_0 \in \mathcal{U}$*. Then the following statements hold:*

(i) the set of $\mathbb{R}$*-derivations at* x_0 *has a natural* $\mathbb{R}$*-vector space structure;*

(ii) the $\mathbb{R}$*-derivations* $(\frac{\partial}{\partial x^1}(x_0), \dots, \frac{\partial}{\partial x^n}(x_0))$ *form a basis for the vector space of* $\mathbb{R}$*-derivations at* x_0*.*

Proof The $\mathbb{R}$-vector space structure for the set of $\mathbb{R}$-derivations is given by

$$(\theta_1 + \theta_2)(f) = \theta_1(f) + \theta_2(f), \quad (a\theta)(f) = a(\theta(f)),$$

for derivations $\theta, \theta_1, \theta_2$ and for $a \in \mathbb{R}$. The verification that these operations define a vector space structure is the usual tedious procedure.

To verify that the derivations $\frac{\partial}{\partial x^j}(x_0)$, $j \in \{1, \dots, n\}$, form a basis for the set of derivations. We first prove a lemma.

1 Lemma *Let* $f \in C^\infty(B^n(r, \mathbf{0}))$ *satisfy* $f(\mathbf{0}) = 0$. *Then*

$$f(\mathbf{x}) = \sum_{j=1}^{n} x^j g_j(\mathbf{x})$$

for $g_1, \dots, g_n \in C^\infty(B^n(r, \mathbf{0}))$.

Proof Let $\boldsymbol{x} \in B^n(r, \mathbf{0})$ and define

$$\gamma_{\boldsymbol{x}}(t) = f(t\boldsymbol{x}).$$

We calculate

$$\begin{aligned} f(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\mathbf{0}) &= \gamma_{\boldsymbol{x}}(1) - \gamma_{\boldsymbol{x}}(0) = \int_0^1 \gamma'_{\boldsymbol{x}}(t)\,\mathrm{d}t \\ &= \sum_{j=1}^{n} \int_0^1 x^j \frac{\partial f}{\partial x^j}(t\boldsymbol{x})\,\mathrm{d}t = \sum_{j=1}^{n} x^j g_j(\boldsymbol{x}), \end{aligned}$$

where

$$g_j(\boldsymbol{x}) = \int_0^1 \frac{\partial f}{\partial x^j}(t\boldsymbol{x})\,\mathrm{d}t.$$

The functions $g_1, \dots, g_n$ are smooth by standard theorems on parameter dependence of integrals. ▼

With this in mind, let us assume without loss of generality (by Proposition 4.5.3) that $(\mathcal{U}, \phi)$ is such that $\phi(\mathcal{U})$ is a ball about some point $\boldsymbol{x}_0 \in \mathbb{R}^n$. Let $\xi^1, \dots, \xi^n \in C^\infty(\mathcal{U})$ be defined by $\xi^j \circ \phi^{-1}(\boldsymbol{x}) = x^j$. By the lemma we have

$$f(x) = f(x_0) + \sum_{j=1}^{n} (\xi^j(x) - \xi^j(x_0)) g_j(x)$$

for smooth functions $g_1, \dots, g_n$ on $\mathcal{U}$. We claim that a $\mathbb{R}$-derivation θ at x_0 applied to a constant function vanishes. Let us first prove this for the constant function 1:

$$\theta(1) = \theta(1 \cdot 1) = \theta(1) \cdot 1 + 1 \cdot \theta(1) = 2\theta(1),$$

giving $\theta(1) = 0$. For a general constant function g taking the value α we have $g \cdot 1 = c \cdot 1$ and so, using $\mathbb{R}$-linearity of derivations,

$$\theta(g) = \theta(g \cdot 1) = \alpha\theta(1) = 0.$$

Now using the fact that derivations of constant functions are zero, we have

$$\theta(f) = \sum_{j=1}^{n} \theta(\xi^j - \xi^j(x_0)) g_j(x_0).$$

From the proof of the lemma we have $g_j(x_0) = \frac{\partial}{\partial x^j}(x_0)(f)$, giving the result. ■

Note that the proof of the preceding result also gives the coefficients of θ when written as a linear combination of the basis vectors:

$$\theta = \sum_{j=1}^{n} \theta(\xi^j) \frac{\partial}{\partial x^j}(x_0),$$

where $\xi^1, \dots, \xi^n$ are the coordinate functions.

Let us use this coordinate representation of a derivation to establish the correspondence between $\mathbb{R}$-derivations at x_0 and tangent vectors at x_0. Again, we stick to the real case to keep the notation simple. For $[\gamma]_{x_0} \in \mathsf{T}_{x_0}\mathsf{M}$ let us define a $\mathbb{R}$-derivation θ_γ at x_0 by

$$\theta_\gamma(f) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \gamma(s).$$

With this notation we have the following result.

4.5.5 Proposition (Tangent vectors and derivations) *If* M *is a smooth or real analytic manifold and if* $\mathsf{x}_0 \in \mathsf{M}$*, the map*

$$[\gamma]_{\mathsf{x}_0} \mapsto \theta_\gamma$$

is a bijection from $\mathsf{T}_{\mathsf{x}_0}\mathsf{M}$ *to the set of* $\mathbb{R}$*-derivations at* x_0*.*

Proof Let $(\mathcal{U}, \phi)$ be a chart about x_0. The derivation θ_γ is computed to be

$$\begin{aligned}\theta_\gamma(f) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \phi^{-1} \circ \phi \circ \gamma(s) \\ &= \sum_{j=1}^{n} \frac{\partial (f \circ \phi^{-1})}{\partial x^j}(\phi(x_0)) v^j,\end{aligned}$$

where $\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \phi \circ \gamma(s)$. From this expression we may directly verify the bijection asserted in the statement of the result. ■

Thus tangent vectors at x_0, in the geometric sense we have defined, are exactly the $\mathbb{R}$-derivations at x_0. We shall not distinguish these things.

4.5.6 Remark (Why are we using smooth objects on holomorphic or real analytic manifolds?) One might justifiably wonder whether the constructions we have made here are appropriate. Specifically, while we are considering smooth or real analytic manifolds, our curve definition of a tangent vector depended on only differentiable curves and the $\mathbb{R}$-derivation definition of a tangent vector depended on smooth functions. The reason this works is that tangent vectors are defined using only first derivatives of objects defined in a neighbourhood of the point where the tangent vector is anchored. For this reason, in the real analytic case one could as well define tangent vectors as equivalence classes of real analytic curves. In the derivation setting, we could as well use the definition of (4.16) to think of a $\mathbb{R}$-derivation at x as a $\mathbb{R}$-linear map $\theta\colon \mathscr{C}^{\omega}_{x,\mathsf{M}} \to \mathbb{R}$.

However, it is true that in the holomorphic case, there is additional structure to be gained by really working with holomorphic objects rather that with smooth real objects. We turn to this now. •

4.5.2 The complex structure of the tangent bundle of a holomorphic manifold

Let us now adapt the preceding discussion to manifolds. The discussion here is really a smooth one, so let us consider only this case for the moment.

4.5.7 Definition (Almost complex structure, complex structure) Let M be a smooth manifold.

(i) An ***almost complex structure*** on M is a smooth $(1,1)$-tensor field J on M such that $J(x) \in \mathrm{End}_{\mathbb{R}}(\mathsf{T}_x\mathsf{M})$ is a linear complex structure for every $x \in \mathsf{M}$.

(ii) An almost complex structure J is a ***complex structure*** if there exists an atlas $((\mathcal{U}_a, \phi_a))_{a\in A}$ for M such that the local representative of J with respect to each coordinate chart is constant and such that the derivatives of the overlap maps commute with the local representatives of J. •

First of all, let us be sure that we understand that complex structures arise naturally on holomorphic manifolds. We shall in Section 4.8 that, conversely, complex structures give rise to holomorphic manifolds.

4.5.8 Proposition (Holomorphic manifolds have complex structures) *A holomorphic manifold possesses a natural complex structure.*

Proof Suppose that M has the structure of a holomorphic manifold. Let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart and let us denote (real) coordinates by $(x^1, \dots, x^n, y^1, \dots, y^n)$. Let us define a $(1,1)$-tensor field J_ϕ on $\mathcal{U}$ as that whose local representative is

$$J_\phi = \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \otimes \mathrm{d}y^{m+j} - \sum_{j=1}^{n} \frac{\partial}{\partial x^{m+j}} \otimes \mathrm{d}y^j, \tag{4.17}$$

cf. Proposition 4.1.2. If we have another chart $(\mathcal{V}, \psi)$, then on the overlap $\mathcal{U} \cap \mathcal{V}$ the local representatives J_ϕ and J_ψ are related by the Jacobian of the overlap map. Since this Jacobian is $\mathbb{C}$-linear by virtue of the overlap map being holomorphic, it follows that multiplication by i is preserved by the Jacobian, and so agrees for the linear complex structures J_ϕ and J_ψ on each local representative of each tangent space. Thus we can use either J_ϕ or J_ψ to define a linear complex structure on the tangent spaces. In other words, M possesses a well-defined almost complex structure. However, we started the proof by showing that the natural holomorphic coordinates give a constant local representative for this almost complex structure. If $((\mathcal{U}_a, \phi_a))_{a\in A}$ is an atlas of $\mathbb{C}$-charts, one readily verifies that the derivatives of the overlap maps commute with the local representatives of J since the derivatives are $\mathbb{C}$-linear maps, cf. Proposition 4.1.6. Thus a holomorphic manifold possesses a natural complex structure. ■

It follows immediately that the constructions of Section 4.1.1 apply to each tangent space of a holomorphic manifold. Indeed, these constructions can be applied to each tangent space of a manifold with an almost complex structure, a fact that we will take advantage of in Section 4.8. But for now we just have the following definition.

4.5.9 Definition (Complex tangent bundle, holomorphic tangent bundle, antiholomorphic tangent bundle) Let M be a holomorphic manifold and denote by $\mathbb{C}_{\mathsf{M}} = \mathsf{M} \times \mathbb{C}$ the trivial $\mathbb{R}$-vector bundle.

(i) The ***complex tangent bundle*** is $\mathsf{T}^{\mathbb{C}}\mathsf{M} = \mathbb{C}_{\mathsf{M}} \otimes_{\mathbb{R}} \mathsf{TM}$.

(ii) The ***holomorphic tangent bundle***, denoted by $\mathsf{T}^{1,0}\mathsf{M}$, is the (real) subbundle of $\mathsf{T}^{\mathbb{C}}\mathsf{M}$ whose fibre at $x \in \mathsf{M}$ is $\mathsf{T}^{1,0}_x\mathsf{M} = (\mathsf{T}_x\mathsf{M})^{1,0}$.

(iii) The ***antiholomorphic tangent bundle***, denoted by $\mathsf{T}^{0,1}\mathsf{M}$, is the (real) subbundle of $\mathsf{T}^{\mathbb{C}}\mathsf{M}$ whose fibre at $x \in \mathsf{M}$ is $\mathsf{T}^{0,1}_x\mathsf{M} = (\mathsf{T}_x\mathsf{M})^{0,1}$. •

These are $\mathbb{R}$-vector bundles. As we shall see in Corollary 4.5.18, $\mathsf{T}^{1,0}\mathsf{M}$ has the structure of a $\mathbb{C}$-vector bundle.

Let us give a few alternative characterisations of the holomorphic tangent bundle that are insightful. To do this, we first make some definitions. We start with curves.

4.5.10 Definition (Complex curves) Let M be a holomorphic manifold and let $z_0 \in \mathsf{M}$.

(i) A ***complex curve*** at z_0 is a differentiable map $\gamma\colon \mathsf{D}^1(r, 0) \to \mathsf{M}$ such that $\gamma(0) = z_0$.

(ii) The ***tangent vector*** to a complex curve $\gamma\colon \mathsf{D}^1(r, 0) \to \mathsf{M}$ at z_0 is the element $\gamma'(0) \in \mathsf{T}^{\mathbb{C}}_{z_0}\mathsf{M}$ defined by

$$\gamma'(0) = \frac{1}{2}\Big(1 \otimes \frac{\partial\gamma}{\partial x}(0) - \mathrm{i} \otimes \frac{\partial\gamma}{\partial y}(0)\Big).$$

(iii) Two complex curves at z_0, $\gamma_1\colon \mathsf{D}^1(r_1, 0) \to \mathsf{M}$ and $\gamma_2\colon \mathsf{D}^1(r_2, 0) \to \mathsf{M}$, are ***equivalent*** if $\gamma_1'(0) = \gamma_2'(0)$. •

Of course, it is holomorphic curves that will be of most interest to us. But to make the setting have some context, we give general definitions.

Next we work with derivations.

4.5.11 Definition ($\mathbb{C}$-derivation at z_0) For a holomorphic manifold M and for $z_0 \in \mathsf{M}$, a $\mathbb{C}$-***derivation*** at z_0 is a $\mathbb{C}$-linear map $\theta\colon \mathscr{C}^{\mathrm{hol}}_{z_0,\mathsf{M}} \to \mathbb{C}$ such that

$$\theta([f]_{z_0}[g]_{z_0}) = \theta([f]_{z_0})g(z_0) + f(z_0)\theta([g]_{z_0}). \qquad \bullet$$

With these notions at hand, we have the following characterisations of holomorphic tangent vectors. As we see, the situation mirrors the smooth real case in a pleasing way.

4.5.12 Proposition (Characterisations of $\mathsf{T}^{1,0}\mathsf{M}$) *Let M be a holomorphic manifold and let $z_0 \in \mathsf{M}$. Then there exist natural $\mathbb{C}$-linear isomorphisms between the following vector spaces:*

(i) $\mathsf{T}^{1,0}_{z_0}\mathsf{M}$*;*

(ii) the set of equivalence classes of holomorphic curves at z_0;

(iii) the set of $\mathbb{C}$-derivations at z_0.

Proof Note that the tangent vector to a holomorphic curve at z_0 is simply the $\mathbb{C}$-derivative, which is a $\mathbb{C}$-linear map from $\mathbb{C}$ to $\mathsf{T}_{z_0}\mathsf{M}$, which we naturally identify with an element (the image of $1 \in \mathbb{C}$) of $\mathsf{T}_{z_0}\mathsf{M}$. Thus, by Proposition 4.1.5(iv), the set of equivalence classes of holomorphic curves at z_0 is isomorphic to $\mathsf{T}^{1,0}_{z_0}\mathsf{M}$.

Next let us consider derivations as represented in coordinates. We let $\theta\colon \mathscr{C}^{\text{hol}}_{z_0,\mathsf{M}} \to \mathbb{C}$ be a $\mathbb{C}$-derivation at z_0. Let $[(f,\mathcal{U})]_{z_0} \in \mathscr{C}^{\text{hol}}_{z_0,\mathsf{M}}$. Without loss of generality, we suppose that $\mathcal{U}$ is the domain of a $\mathbb{C}$-coordinate chart $(\mathcal{U},\phi)$ for which $\phi(\mathcal{U})$ is a ball about $\boldsymbol{z}_0 = \phi(z_0)$. Let us also suppose that the Taylor series for $f\circ\phi^{-1}$ at $\phi(z_0)$ converges uniformly on $\phi(\mathcal{U})$:

$$f\circ\phi^{-1}(\boldsymbol{z}) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \frac{1}{I!}\boldsymbol{D}^I(f\circ\phi^{-1})(\boldsymbol{z}_0)(\boldsymbol{z}-\boldsymbol{z}_0)^I, \qquad \boldsymbol{z}\in\phi(\mathcal{U}).$$

We can factor linear terms from each of the summands in the Taylor series to write

$$f(z) = f(z_0) + \sum_{j=1}^n (\zeta^j(z) - \zeta^j(z_0))g_j(z),$$

where $\zeta^j \in \mathrm{C}^{\text{hol}}(\mathcal{U})$, $j\in\{1,\dots,n\}$, are the coordinate functions and $g_j \in \mathrm{C}^{\text{hol}}(\mathcal{U})$, $j\in\{1,\dots,n\}$. As in the proof of Proposition 4.5.4, θ is zero applied to germs of constant functions. We thus have

$$\theta([f]_{z_0}) = \sum_{j=1}^n \theta([\zeta^j - \zeta^j(z_0)]_{z_0})g_j(z_0),$$

where

$$g_j(z_0) = \frac{1}{2}\Big(\frac{\partial}{\partial x^j}(z_0)([f]_{z_0}) - \mathrm{i}\frac{\partial}{\partial y^j}(z_0)([f]_{z_0})\Big), \qquad j\in\{1,\dots,n\}.$$

Knowing what a $\mathbb{C}$-derivation at z_0 looks like in coordinates, it is easy to verify in coordinates that, if we define a $\mathbb{C}$ derivation θ_γ at z_0 associated to an equivalence class $[\gamma]_{z_0}$ of holomorphic curves by

$$\theta_\gamma(f) = \frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}z}(0),$$

then the map $[\gamma]_{z_0} \mapsto \theta_\gamma$ is an isomorphism of $\mathbb{C}$-vector spaces. ■

Let us relate $\mathbb{R}$- and $\mathbb{C}$-derivations, a process begun in the preceding proof. First note that $\mathscr{C}^{\text{hol}}_{z_0,\mathsf{M}}$ is a $\mathbb{C}$-subspace of $\mathbb{C}\otimes\mathscr{C}^{\omega}_{z_0,\mathsf{M}}$. Thus we can write $[f]_{z_0} \in \mathscr{C}^{\text{hol}}_{z_0,\mathsf{M}}$ as

$$[f]_{z_0} = 1\otimes[g]_{z_0} + \mathrm{i}\otimes[h]_{z_0}$$

for $[g]_{z_0},[h]_{z_0} \in \mathscr{C}^{\omega}_{z_0,\mathsf{M}}$. We can then define a map $\theta \mapsto \hat{\theta}$ from the set $\mathsf{T}_{z_0}\mathsf{M}$ of $\mathbb{R}$-derivations at z_0 to the set of $\mathbb{C}$-derivations at z_0 by

$$\hat{\theta}(1\otimes[g]_{z_0} + \mathrm{i}\otimes[h]_{z_0}) = \theta([g]_{z_0}) + \mathrm{i}\theta([h]_{z_0}).$$

Now suppose that we have a $\mathbb{C}$-chart $(\mathcal{U},\phi)$ for M about z_0 with coordinates denoted by $z^j = x^j + \mathrm{i}y^j$, $j\in\{1,\dots,n\}$. Thus we have $\mathbb{R}$-derivations $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$, $j\in\{1,\dots,n\}$.

Associated with these $\mathbb{R}$-derivations (which are, by Proposition 4.5.5, elements of the $\mathbb{R}$-tangent space $\mathsf{T}_{z_0}\mathsf{M}$) are the elements of $\mathsf{T}^{\mathbb{C}}_{z_0}\mathsf{M}$

$$\frac{\partial}{\partial z^j}(z_0) = \frac{1}{2}\Big(1 \otimes \frac{\partial}{\partial x^j}(z_0) - \mathrm{i} \otimes \frac{\partial}{\partial y^j}(z_0)\Big), \qquad j \in \{1, \dots, n\},$$

and

$$\frac{\partial}{\partial \bar{z}^j}(z_0) = \frac{1}{2}\Big(1 \otimes \frac{\partial}{\partial x^j}(z_0) + \mathrm{i} \otimes \frac{\partial}{\partial y^j}(z_0)\Big), \qquad j \in \{1, \dots, n\}.$$

These form a basis for $\mathsf{T}^{\mathbb{C}}_{z_0}\mathsf{M}$, a fact we record as follows.

4.5.13 Proposition (Coordinate bases for $\mathsf{T}^{\mathbb{C}}_{z_0}\mathsf{M}$) *With the above notation,* $(\frac{\partial}{\partial z^1}(z_0), \dots, \frac{\partial}{\partial z^n}(z_0))$ *is a basis for* $\mathsf{T}^{1,0}_{z_0}\mathsf{M}$ *and* $(\frac{\partial}{\partial \bar{z}^1}(z_0), \dots, \frac{\partial}{\partial \bar{z}^n}(z_0))$ *is a basis for* $\mathsf{T}^{0,1}_{z_0}\mathsf{M}$.

Proof This follows from Proposition 4.1.7 since the local representative of the linear complex structure is

$$\sum_{j=1}^m \frac{\partial}{\partial y^j} \otimes \mathrm{d}x^j - \sum_{j=1}^m \frac{\partial}{\partial x^j} \otimes \mathrm{d}y^j. \qquad \blacksquare$$

Let us look at the tangent bundle of $\mathbb{CP}^1$.

4.5.14 Example (The tangent bundle of $\mathbb{CP}^1$) We again work with the one-dimensional complex projective space $\mathbb{CP}^1 \simeq \mathbb{S}^2$. To describe the holomorphic tangent bundle of $\mathbb{CP}^1$ we again start with our charts $(\mathcal{U}_+, \phi_+)$ and $(\mathcal{U}_-, \phi_-)$ as in Example 4.3.5–4. For our purposes here, it is advantageous to modify these charts slightly. We leave the domains unchanged, take $\psi_- = \phi_-$, and define $\psi_+ : \mathcal{U}_+ \to \mathbb{C}$ by $\psi_+(x) = -\phi_+(x)$. Since multiplication by -1 is a holomorphic map, the pair of charts $(\mathcal{U}_+, \psi_+)$ and $(\mathcal{U}_-, \psi_-)$ still provide a holomorphic atlas. The overlap condition in this case is $\psi_- \circ \psi_+^{-1}(z_+) = -z_+^{-1}$. As we have done previously, we let z_+ and z_- be the coordinates for the two charts, understanding that in the present setting these are not related as previously. The tangent bundle coordinates we denote by (z_+, w_+) and (z_-, w_-). Using the fact that holomorphic tangent vectors are equivalence classes of holomorphic curves, the representation of the tangent vector with local representative (z_+, w_+) in the chart $(\mathcal{U}_+, \psi_+)$ is given in the chart $(\mathcal{U}_-, \psi_-)$ by (z_-, w_-) with $z_- = -z_+^{-1}$ and

$$w_- = \frac{\partial z_-}{\partial z_+} w_+ = z_+^{-2} w_+.$$

This is precisely the overlap condition for $\mathsf{O}_{\mathbb{CP}^1}(2)$, and so we have a vector bundle atlas for $\mathsf{T}^{1,0}(\mathbb{CP}^1)$ which gives the same vector bundle structure as that for $\mathsf{O}_{\mathbb{CP}^1}(2)$.

For a more elevated presentation of the relationship between the tangent bundle and the degree 2 line bundle, we refer to Section 4.4.9. •

4.5.3 The derivative of a real analytic map

The notion of derivative for a real analytic map follows that for the smooth case, and in this section we recall this definition. In the next section we explore the additional structure possessed by the derivative when dealing with holomorphic manifolds.

First, let us give the basic definition. We deal with the smooth or real analytic case in the following definition.

4.5.15 Definition (Derivative of real analytic map) If M and N are smooth or real analytic maps and if $\Phi\colon \mathsf{M} \to \mathsf{N}$ is smooth or real analytic, the ***tangent map*** for Φ is the map $T\Phi\colon \mathsf{TM} \to \mathsf{TN}$ defined by $T\Phi([\gamma]_x) = [\Phi\circ\gamma]_{\Phi(x)}$, where $[\gamma]_x \in \mathsf{T}_x\mathsf{M}$ is an equivalence class of curves. The restriction of $T\Phi$ to $\mathsf{T}_x\mathsf{M}$ we denote by $T_x\Phi$. •

We leave to the reader the mundane chore of verifying that this definition of the tangent map is independent on the choice of representative for the tangent vector. We also leave to the reader the task of verifying that, if a tangent vector at x is thought of as a $\mathbb{R}$-derivation at x, then the tangent map is equivalently defined by asking that $T\Phi(\theta)$ is the $\mathbb{R}$-derivation at $\Phi(x)$ given by

$$T\Phi(\theta)\cdot g = \theta(\Phi^* g)$$

for $g \in \mathrm{C}^\infty(\mathsf{M})$. Equivalently, following Remark 4.5.6, in the real analytic setting, we can think of $T\Phi$ as acting on germs:

$$T\Phi(\theta)\cdot[(g,\mathcal{V})]_{\Phi(x)} = [(\Phi^* g, \Phi^{-1}(\mathcal{V}))]_x,$$

for $[(g,\mathcal{V})]_{\Phi(x)} \in \mathscr{C}^\omega_{\Phi(x),\mathsf{N}}$ and where $\Phi^* f$ is the pull-back of g from the neighbourhood $\mathcal{V}$ of $\Phi(x)$ to the neighbourhood $\Phi^{-1}(\mathcal{V})$ of x. The definition of $T\Phi$ using derivations shows that it is a $\mathbb{R}$-linear map.

The following coordinate characterisation of $T\Phi$ is easily proved using the above definitions.

4.5.16 Proposition (Local representative of the tangent map) *Let M and N be smooth or real analytic manifolds, let $\Phi\colon \mathsf{M} \to \mathsf{N}$ be smooth or real analytic, let $\mathrm{x}_0 \in \mathsf{M}$, and let $(\mathcal{U},\phi)$ and $(\mathcal{V},\psi)$ be charts about x_0 and $\Phi(\mathrm{x}_0)$, respectively. Then the components of $\mathsf{T}_{\mathrm{x}_0}\Phi$ with respect to the bases $(\frac{\partial}{\partial \mathrm{x}^1}(\mathrm{x}_0),\dots,\frac{\partial}{\partial \mathrm{x}^n}(\mathrm{x}_0))$ for $\mathsf{T}_{\mathrm{x}_0}\mathsf{M}$ and $(\frac{\partial}{\partial \mathrm{y}^1}(\Phi(\mathrm{x}_0)),\dots,\frac{\partial}{\partial \mathrm{y}^m}(\Phi(\mathrm{x}_0)))$ for $\mathsf{T}_{\Phi(\mathrm{x}_0)}\mathsf{N}$ are*

$$\frac{\partial \Phi^a}{\partial \mathrm{x}^j}(\phi(\mathrm{x}_0)), \qquad a \in \{1,\dots,m\},\ j \in \{1,\dots,n\}.$$

4.5.4 The derivative of a holomorphic map

Now we turn to the case of holomorphic maps. Thus we let M and N be holomorphic manifolds and let $\Phi\colon \mathsf{M} \to \mathsf{N}$ be a smooth map. For each $x \in \mathsf{M}$ the $\mathbb{R}$-linear map $T_x\Phi\colon \mathsf{T}_x\mathsf{M} \to \mathsf{T}_{\Phi(x)}\mathsf{N}$ extends to a $\mathbb{C}$-linear map

$$T_x^{\mathbb{C}}\Phi \triangleq (T_x\Phi)_{\mathbb{C}}\colon \mathsf{T}_x^{\mathbb{C}}\mathsf{M} \to \mathsf{T}^{\mathbb{C}}_{\Phi(x)}\mathsf{N}.$$

We can give the coordinate form for the tangent map of a holomorphic map. We let M and N be holomorphic manifolds, let $\Phi\colon \mathsf{M} \to \mathsf{N}$ be holomorphic, let $z_0 \in \mathsf{M}$, and let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be $\mathbb{C}$-charts about z_0 and $\Phi(z_0)$, respectively. Denote coordinates for M by $z^j = x^j + \mathrm{i}y^j$, $j \in \{1, \dots, n\}$, and for N by $w^a = u^a + \mathrm{i}v^a$, $a \in \{1, \dots, m\}$. As in Proposition 4.5.13, we have the basis vectors $\frac{\partial}{\partial z^j}(z_0)$ and $\frac{\partial}{\partial \bar{z}^j}(z_0)$, $j \in \{1, \dots, n\}$, for $\mathsf{T}^{\mathbb{C}}_{z_0}\mathsf{M}$. We have similar notation, of course, for a basis for $\mathsf{T}^{\mathbb{C}}_{\Phi(z_0)}\mathsf{N}$. Note that $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$, $j \in \{1, \dots, n\}$, are to be thought of as $\mathbb{R}$-derivations. The components of Φ relative to these coordinates are denoted by Φ^a, $a \in \{1, \dots, m\}$. We also define the partial derivative notation

$$\frac{\partial f}{\partial z^j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x^j} - \mathrm{i}\frac{\partial f}{\partial y^j}\Big), \qquad j \in \{1, \dots, n\},$$

and

$$\frac{\partial f}{\partial \bar{z}^j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x^j} + \mathrm{i}\frac{\partial f}{\partial y^j}\Big), \qquad j \in \{1, \dots, n\},$$

adapting that use in $\mathbb{C}^n$, cf. (1.11). With all this notation, we have the following result.

4.5.17 Proposition (Local representative of the holomorphic tangent map) *Let M and N be holomorphic manifolds, let $\Phi\colon \mathsf{M} \to \mathsf{N}$ be smooth, let $z_0 \in \mathsf{M}$, and let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be $\mathbb{C}$-charts about z_0 and $\Phi(z_0)$, respectively. Then the matrix of components of $T^{\mathbb{C}}_{z_0}\Phi$ with respect to the bases above is*

$$\begin{bmatrix} \frac{\partial \Phi^1}{\partial z^1} & \cdots & \frac{\partial \Phi^1}{\partial z^n} & \frac{\partial \Phi^1}{\partial \bar{z}^1} & \cdots & \frac{\partial \Phi^1}{\partial \bar{z}^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi^m}{\partial z^1} & \cdots & \frac{\partial \Phi^m}{\partial z^n} & \frac{\partial \Phi^m}{\partial \bar{z}^1} & \cdots & \frac{\partial \Phi^m}{\partial \bar{z}^n} \\ \frac{\partial \bar{\Phi}^1}{\partial z^1} & \cdots & \frac{\partial \bar{\Phi}^1}{\partial z^n} & \frac{\partial \bar{\Phi}^1}{\partial \bar{z}^1} & \cdots & \frac{\partial \bar{\Phi}^1}{\partial \bar{z}^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \bar{\Phi}^m}{\partial z^1} & \cdots & \frac{\partial \bar{\Phi}^m}{\partial z^n} & \frac{\partial \bar{\Phi}^m}{\partial \bar{z}^1} & \cdots & \frac{\partial \bar{\Phi}^m}{\partial \bar{z}^n} \end{bmatrix} (\phi(z_0))$$

This result has the following corollary which gives useful structure to the holomorphic tangent bundle.

4.5.18 Corollary (The holomorphic tangent bundle is a $\mathbb{C}$-vector bundle) *If M is a holomorphic manifold then $\mathsf{T}^{1,0}\mathsf{M}$, with the natural $\mathbb{R}$-vector bundle structure induced by the tangent bundle structure, is a $\mathbb{C}$-vector bundle.*

Proof If $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ are overlapping $\mathbb{C}$-charts, the resulting overlap map is holomorphic. The derivative of the overlap map is the $\mathbb{C}$-derivative of the overlap map, and this map is $\mathbb{C}$-linear. Thus, by the preceding proposition, the vector bundle structure for $\mathsf{T}^{\mathbb{C}}\mathsf{M}$, restricted to $\mathsf{T}^{1,0}\mathsf{M}$, is that of a $\mathbb{C}$-vector bundle. ■

We can then characterise holomorphicity of maps as follows.

4.5.19 Proposition (Characterisation of holomorphic maps) *If M and N are holomorphic manifolds and if $\Phi\colon \mathsf{M} \to \mathsf{N}$ is smooth, then the following statements are equivalent:*

(i) Φ is holomorphic;

(ii) *for each* $x \in M$, $T_x\Phi$ *is* $\mathbb{C}$*-linear with respect to the linear complex structures on* T_xM *and* $T_{\Phi(x)}N$*, cf. Proposition 4.5.8;*

(iii) *for each* $x \in M$, $T_x^{\mathbb{C}}\Phi(T_x^{1,0}M) \subseteq T_{\Phi(x)}^{1,0}N$*;*

(iv) *for each* $x \in M$, $T_x^{\mathbb{C}}\Phi(T_x^{0,1}M) \subseteq T_{\Phi(x)}^{0,1}N$.

Proof By Corollary 1.1.22 it follows that Φ is holomorphic if and only if

$$\frac{\partial \Phi^a}{\partial \bar{z}^j} = \frac{\partial \bar{\Phi}^a}{\partial \bar{z}^j} = 0, \qquad a \in \{1, \dots, m\},\ j \in \{1, \dots, n\}.$$

The result then follows from Proposition 4.5.17, along with the characterisations of $\mathbb{C}$-linear maps in Proposition 4.1.6. ■

4.5.5 Vector fields on real analytic manifolds

Let us first recall basic constructions for vector fields on smooth or real analytic manifolds. Thus we let M be a smooth or real analytic manifold. Recall that a ***C^r-vector field***, $r \in \{\infty, \omega\}$, is a C^r-section of the $\mathbb{R}$-vector bundle $\pi_{TM}\colon TM \to M$. A vector field defines a map $\mathscr{L}_X\colon C^r(M) \to C^r(M)$ by

$$\mathscr{L}_X f(x) = X(x)(f),$$

noting that tangent vectors at x are $\mathbb{R}$-derivations at x. Note that $\mathscr{L}_X$ is a derivation, by which we mean that it is $\mathbb{R}$-linear and satisfies

$$\mathscr{L}_X(fg) = (\mathscr{L}_X f)g + f(\mathscr{L}_X g).$$

It is advantageous to have this notion of derivation be localised, i.e., given on germs of functions. That is, as in (4.16), we think of a vector field X as an assignment to each $x \in M$ a $\mathbb{R}$-derivation $\mathscr{L}_{X,x}\colon \mathscr{C}^\infty_{x,M} \to \mathscr{C}^\infty_{x,M}$. In this definition of a vector field, one has to worry about how one prescribes that X be smooth or real analytic. This is taken care of using the language of sheaf theory by asking that the assignment $x \mapsto \mathscr{L}_{X,x}$ be continuous in an appropriate sense. However, we do not worry about this here. Instead, we use this characterisation of vector fields to define the ***Lie bracket*** of two C^r-vector fields X and Y by requiring that

$$\mathscr{L}_{[X,Y],x}[f]_x = \mathscr{L}_{X,x}\mathscr{L}_{Y,x}[f]_x - \mathscr{L}_{Y,x}\mathscr{L}_{X,x}[f]_x.$$

Using this definition one readily verifies that, in a chart $(\mathcal{U}, \phi)$ with coordinates $(x^1, \dots, x^n)$, the local representative of X is

$$[X, Y]|\mathcal{U} = \sum_{j=1}^n \Big(\sum_{k=1}^n \frac{\partial Y^j}{\partial x^k} X^k - \sum_{k=1}^n \frac{\partial X^j}{\partial x^k} Y^k\Big) \frac{\partial}{\partial x^j},$$

where

$$X|\mathcal{U} = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}, \quad Y|\mathcal{U} = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}.$$

4.5.6 Vector fields on holomorphic manifolds

Next we let M be a holomorphic manifold. We have the $\mathbb{R}$-vector bundles TM, $\mathsf{T}^{\mathbb{C}}\mathsf{M}$, $\mathsf{T}^{1,0}\mathsf{M}$, and $\mathsf{T}^{0,1}\mathsf{M}$. Thus these vector bundles possess sets of C^r-sections for $r \in \{\infty, \omega\}$, and we denote these sets of sections by $\Gamma^r(\mathsf{TM})$, $\Gamma^r(\mathsf{T}^{\mathbb{C}}\mathsf{M})$, $\Gamma^r(\mathsf{T}^{1,0}\mathsf{M})$, and $\mathsf{T}^{0,1}_r\mathsf{TM}$, respectively. Since $\mathsf{T}^{\mathbb{C}}\mathsf{M} = \mathbb{C}_{\mathsf{M}} \otimes_{\mathbb{R}} \mathsf{TM}$, a C^r-section Z of $\mathsf{T}^{\mathbb{C}}\mathsf{M}$, $r \in \{\infty, \omega\}$, can be written

$$Z = 1 \otimes U + \mathrm{i} \otimes V, \qquad U, V \in \Gamma^r(\mathsf{TM}).$$

Note that by Corollary 4.5.18 we have that $\mathsf{T}^{1,0}\mathsf{M}$ is a $\mathbb{C}$-vector bundle. A ***holomorphic vector field*** is a holomorphic section of $\mathsf{T}^{1,0}\mathsf{M}$. By $\Gamma^{\mathrm{hol}}(\mathsf{T}^{1,0}\mathsf{M})$ we denote the set of holomorphic vector fields.

As contrasted with the smooth case, the set of holomorphic vector fields can be quite small, as the following example shows.

4.5.20 Example (Holomorphic vector fields on $\mathbb{CP}^1$) According to Examples 4.3.14 and 4.5.14, the dimension of the $\mathbb{C}$-vector space of holomorphic vector fields is 3. •

Let us adapt our notion of Lie derivative to the complex setting. Thus let M be a holomorphic manifold, let $r \in \{\infty, \omega\}$, let $Z \in \Gamma^r(\mathsf{T}^{\mathbb{C}}\mathsf{M})$, and let $f \in \mathrm{C}^r(\mathsf{M}; \mathbb{C})$. We write

$$Z = 1 \otimes U + \mathrm{i} \otimes V, \quad f = g + \mathrm{i}h$$

for $U, V \in \Gamma^r(\mathsf{TM})$ and $g, h \in \mathrm{C}^r(\mathsf{M})$. We define the ***Lie derivative*** of f with respect to Z by extension:

$$\mathscr{L}_Z f = (\mathscr{L}_U g - \mathscr{L}_V h) + \mathrm{i}(\mathscr{L}_U h + \mathscr{L}_V g).$$

Let us see how these various sections are represented in coordinates. Thus we let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart with coordinates $z^j = x^j + \mathrm{i}y^j$, $j \in \{1, \dots, n\}$. We have the basis vector fields $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$, $j \in \{1, \dots, n\}$, for TM and $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$, $j \in \{1, \dots, n\}$, for $\mathsf{T}^{\mathbb{C}}\mathsf{M}$. Then the local representative of a section X of TM is given by

$$X|\mathcal{U} = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j} + \sum_{j=1}^n Y^j \frac{\partial}{\partial y^j}$$

for $X^j, Y^j \in \mathrm{C}^r(\mathcal{U})$, $j \in \{1, \dots, n\}$, $r \in \{\infty, \omega\}$. The local representative of a section Z of $\mathsf{T}^{\mathbb{C}}\mathsf{M}$, $\mathsf{T}^{1,0}\mathsf{M}$, or $\mathsf{T}^{0,1}\mathsf{M}$ is given by

$$Z|\mathcal{U} = \sum_{j=1}^n U^j \frac{\partial}{\partial z^j} + \sum_{j=1}^n V^j \frac{\partial}{\partial \bar{z}^j},$$

$$Z|\mathcal{U} = \sum_{j=1}^n U^j \frac{\partial}{\partial z^j}, \tag{4.18}$$

or

$$Z|\mathcal{U} = \sum_{j=1}^n V^j \frac{\partial}{\partial \bar{z}^j},$$

respectively, for $U^j, V^j \in \mathrm{C}^r(\mathcal{U};\mathbb{C})$, $j \in \{1,\dots,n\}$, $r \in \{\infty,\omega,\mathrm{hol}\}$. Thus a holomorphic vector field has the form (4.18), where $U^j \in \mathrm{C}^{\mathrm{hol}}(\mathcal{U};\mathbb{C})$.

Now let us define the notion of Lie bracket for vector fields on holomorphic manifolds. Let $r \in \{\infty,\omega\}$ and let

$$Z_a = 1 \otimes X_a + \mathrm{i} \otimes Y_a \in \Gamma^r(\mathsf{T}^{\mathbb{C}}\mathsf{M}), \qquad X_a, Y_a \in \Gamma^r(\mathsf{T}^*\mathsf{M}),\ a \in \{1,2\}.$$

The ***complex Lie bracket*** of Z_1 and Z_2 is

$$[Z_1,Z_2]_{\mathbb{C}} = 1 \otimes ([X_1,X_2] - [Y_1,Y_2]) + \mathrm{i} \otimes ([Y_1,X_2] + [X_1,Y_2]). \tag{4.19}$$

The following lemma is fundamental to the structure of a holomorphic manifold.

4.5.21 Lemma (Property of the complex Lie bracket) *If* M *is a holomorphic manifold then*

(i) $[\mathrm{Z}_1,\mathrm{Z}_2]_{\mathbb{C}} \in \Gamma^\infty(\mathsf{T}^{1,0}\mathsf{M})$ *for every* $\mathrm{Z}_1,\mathrm{Z}_2 \in \Gamma^\infty(\mathsf{T}^{1,0}\mathsf{M})$ *and*

(ii) $[\mathrm{Z}_1,\mathrm{Z}_2]_{\mathbb{C}} \in \Gamma^\infty(\mathsf{T}^{0,1}\mathsf{M})$ *for every* $\mathrm{Z}_1,\mathrm{Z}_2 \in \Gamma^\infty(\mathsf{T}^{0,1}\mathsf{M})$.

Proof We prove the first assertion, the second following in a similar manner.

If $Z_1, Z_2 \in \Gamma^\infty(\mathsf{T}^{1,0}\mathsf{M})$ then, by Proposition 4.1.5(iv), we write

$$Z_a = 1 \otimes X_a - \mathrm{i} \otimes J(X_a), \qquad a \in \{1,2\},$$

for some $X_a \in \Gamma^\infty(\mathsf{TM})$, $a \in \{1,2\}$. We then have

$$[Z_1,Z_2]_{\mathbb{C}} = 1 \otimes ([X_1,X_2] - [J(X_1),J(X_2)]) - \mathrm{i} \otimes ([J(X_1),X_2] + [X_1,J(X_2)]).$$

A direct slightly messy computation then gives

$$J_{\mathbb{C}}([Z_1,Z_2]_{\mathbb{C}}) - \mathrm{i}[Z_1,Z_2]_{\mathbb{C}} = -1 \otimes J \circ N_J(X_1,X_2)) + \mathrm{i} \otimes N_J(X_1,X_2),$$

where N_J is the Nijenhuis tensor of Section 4.8.1. By the easy part of Theorem 4.8.4 we then have

$$[Z_1,Z_2]_{\mathbb{C}} \in \Gamma^\infty((\ker(J_{\mathbb{C}} - \mathrm{i}\ \mathrm{id}_{\mathsf{T}^{\mathbb{C}}\mathsf{M}}))),$$

and so the lemma follows by definition of $\mathsf{T}^{1,0}\mathsf{M}$. ■

Let us consider the above constructions specialised to holomorphic vector fields, i.e., holomorphic sections of $\mathsf{T}^{1,0}\mathsf{M}$. First of all, let us show that holomorphic vector fields give rise to $\mathbb{C}$-derivations.

4.5.22 Proposition (Holomorphic vector fields are $\mathbb{C}$-derivations) *Let* M *be a holomorphic manifold and let* $\mathrm{Z} \in \Gamma^{\mathrm{hol}}(\mathsf{T}^{1,0}\mathsf{M})$. *Then the map*

$$\begin{aligned} \mathscr{L}_{\mathrm{Z}} \colon \mathrm{C}^{\mathrm{hol}}(\mathsf{M}) &\to \mathrm{C}^{\mathrm{hol}}(\mathsf{M}) \\ \mathrm{f} &\mapsto \mathscr{L}_{\mathrm{Z}}\mathrm{f} \end{aligned}$$

is a derivation of the $\mathbb{C}$*-algebra* $\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$, *i.e., the map is* $\mathbb{C}$*-linear and satisfies*

$$\mathscr{L}_{\mathrm{Z}}(\mathrm{fg}) = (\mathscr{L}_{\mathrm{Z}}\mathrm{f})\mathrm{g} + \mathrm{f}(\mathscr{L}_{\mathrm{Z}}\mathrm{g}).$$

Moreover, if $\mathrm{Z}_1,\mathrm{Z}_2 \in \Gamma^{\mathrm{hol}}(\mathsf{T}^{1,0}\mathsf{M})$ *and if* $\mathrm{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$, *then*

$$\mathscr{L}_{[\mathrm{Z}_1,\mathrm{Z}_2]_{\mathbb{C}}}\mathrm{f} = \mathscr{L}_{\mathrm{Z}_1}\mathscr{L}_{\mathrm{Z}_2}\mathrm{f} - \mathscr{L}_{\mathrm{Z}_2}\mathscr{L}_{\mathrm{Z}_1}\mathrm{f}.$$

Proof Let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart with coordinates $(z^1, \dots, z^n)$. A direct computation in coordinates shows that if the local representative of Z is

$$Z|\mathcal{U} = \sum_{j=1}^{n} Z^j \frac{\partial}{\partial z^j},$$

then the local representative of $\mathscr{L}_Z f$ is

$$\mathscr{L}_Z f|\mathcal{U} = \sum_{j=1}^{n} Z^j \frac{\partial f}{\partial z^j},$$

and the first part of the result follows from this, along with standard rules for $\mathbb{C}$-differentiation.

The second part of the result follows again from a direct computation in coordinates. ■

As in the real analytic case, this can all be localised by considering vector fields to be derivations on germs of functions. That is, we can think of a holomorphic vector field Z as an assignment to each $z \in \mathsf{M}$ a derivation $\mathscr{L}_{Z,z} \colon \mathscr{C}^{\text{hol}}_{z,\mathsf{M}} \to \mathscr{C}^{\text{hol}}_{z,\mathsf{M}}$. We can then define the Lie bracket of holomorphic vector fields as

$$\mathscr{L}_{[Z,W]_{\mathbb{C}},z}[f]_z = \mathscr{L}_{Z,z}\mathscr{L}_{W,z}[f]_z - \mathscr{L}_{W,z}\mathscr{L}_{Z,z}[f]_z,$$

and verify that, in coordinates $(z^1, \dots, z^n)$ in a $\mathbb{C}$-chart, we have

$$[Z, W]|\mathcal{U} = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \frac{\partial W^j}{\partial z^k} Z^k - \sum_{k=1}^{n} \frac{\partial Z^j}{\partial z^k} W^k \Big) \frac{\partial}{\partial z^j},$$

where

$$Z|\mathcal{U} = \sum_{j=1}^{n} Z^j \frac{\partial}{\partial z^j}, \quad W|\mathcal{U} = \sum_{j=1}^{n} W^j \frac{\partial}{\partial z^j}.$$

4.6 Differential forms on holomorphic and real analytic manifolds

In this section we turn to the study of differential forms on holomorphic and real analytic manifolds. In the real analytic case, our presentation will be along the lines of a review for readers familiar with smooth differential geometry; the constructions are the same and the main results are the same. In the holomorphic case, however, the complex structure has important interplay with the algebraic and analytical structure of differential forms, and we spend some time understanding this.

4.6.1 Differential forms on real analytic manifolds

Let us provide the notation we shall use for differential forms. First let M be a real analytic manifold. By $\pi_{\mathsf{T}^*\mathsf{M}} \colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$ we denote the cotangent bundle. By

$\bigwedge^k(\mathsf{T}^*\mathsf{M})$ we denote the vector bundle of alternating k-forms. A ***differential* k*-form*** of class C^r, $r \in \{\infty, \omega\}$, is a C^r-section of $\bigwedge^k(\mathsf{T}^*\mathsf{M})$, and we denote the set of all such sections by $\Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M}))$. Let us recall the usual notation for locally representing differential forms. We let $(\mathcal{U}, \phi)$ be a $\mathbb{R}$-chart with coordinates $(x^1, \dots, x^n)$. In the usual manner, cf. Proposition 4.5.4, we have the basis

$$\Big\{\frac{\partial}{\partial x^1}(x), \dots, \frac{\partial}{\partial x^n}(x)\Big\}$$

for $\mathsf{T}_x\mathsf{M}$ for each $x \in \mathcal{U}$. The dual basis for $\mathsf{T}_x^*\mathsf{M}$ is denoted by

$$\{\mathrm{d}x^1(x), \dots, \mathrm{d}x^n(x)\}.$$

Given $\alpha \in \Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M}))$ we can write

$$\alpha(x) = \sum_{\substack{j_1,\dots,j_k \in \{1,\dots,n\} \\ j_1<\cdots<j_k}} \alpha_{j_1\cdots j_k}(x)\mathrm{d}x^{j_1}(x) \wedge \cdots \wedge \mathrm{d}x^{j_k}(x), \qquad x \in \mathcal{U}, \tag{4.20}$$

where

$$\alpha_{j_1\cdots j_k}(x) = \alpha\Big(\frac{\partial}{\partial x^{j_1}}(x), \dots, \frac{\partial}{\partial x^{j_k}}(x)\Big).$$

We shall frequently write

$$\alpha(x) = \sum_{I \in \boldsymbol{n}^k}{}' \alpha_I \mathrm{d}\boldsymbol{x}^I$$

as a shorthand for expressions like (4.20), where $\boldsymbol{n} = \{1, \dots, n\}$, where

$$\alpha_I = \alpha_{i_1,\cdots,i_k}, \quad \mathrm{d}\boldsymbol{x}^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

if $I = (1_1, \dots, i_k)$, and where $\sum'$ denotes the sum over increasing multi-indices.

The ***exterior derivative*** is then a map $\mathrm{d}\colon \Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M})) \to \Gamma^r(\bigwedge^{k+1}(\mathsf{T}^*\mathsf{M}))$ of such a differential form is given by

$$\begin{aligned}\mathrm{d}\alpha(X_0, X_1, \dots, X_k) = &\sum_{j=0}^{k} (-1)^j \mathscr{L}_{X_j}(\alpha(X_0, X_1, \dots, \hat{X}_j, \dots, X_k)) \\ &+ \sum_{\substack{j,l \in \{0,1,\dots,k\} \\ j<l}} (-1)^{j+l} \alpha([X_j, X_l], X_0, \dots, \hat{X}_j, \dots, \hat{X}_l, \dots, X_k),\end{aligned} \tag{4.21}$$

where the terms involving a $\hat{}$ mean that that term is omitted from the argument. One can verify that, in local coordinates,

$$\mathrm{d}\alpha(x) = \sum_{j=1}^{n} \sum_{J \in \boldsymbol{n}^k}{}' \frac{\partial \alpha_J}{\partial x^j}(x)\, \mathrm{d}x^j(x) \wedge \mathrm{d}\boldsymbol{x}^I(x), \qquad x \in \mathcal{U}.$$

Alternatively, in terms of components,

$$\mathrm{d}\alpha_{j_1\cdots j_k j_{k+1}} = \sum_{m=1}^{k}(-1)^{m-1}\frac{\partial}{\partial x^{j_m}}\alpha_{j_1\cdots j_{m-1}j_{m+1}\cdots j_{k+1}} + (-1)^k\frac{\partial}{\partial x^{j_{k+1}}}\alpha_{j_1\cdots j_k}. \tag{4.22}$$

The exterior derivative obeys the following rules whose standard proofs can be found in [Abraham, Marsden, and Ratiu 1988, §6.4].

4.6.1 Proposition (Properties of exterior derivative) *Let* $\mathrm{r} \in \{\infty, \omega\}$ *and let* M *be a smooth or real analytic manifold. The exterior derivative has the following properties:*

(i) the map d *is well-defined, i.e., the expression (4.21) defines* $\mathrm{d}\alpha$ *as a differential* $(\mathrm{k}+1)$*-form;*

(ii) the map d *is* $\mathbb{R}$*-linear;*

(iii) $\mathrm{d}\circ\mathrm{d} = 0$*;*

(iv) if $\mathcal{U} \subseteq \mathsf{M}$ *is open and if* $\alpha \in \Gamma^{\mathrm{r}}(\bigwedge^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}))$*, then* $(\mathrm{d}\alpha)|\mathcal{U} = \mathrm{d}(\alpha|\mathcal{U})$*;*

(v) if $\alpha \in \Gamma^{\mathrm{r}}(\bigwedge^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}))$ *and* $\beta \in \Gamma^{\mathrm{r}}(\bigwedge^{1}(\mathsf{T}^*\mathsf{M}))$*, then*

$$\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^{\mathrm{k}}\alpha\wedge\mathrm{d}\beta.$$

The fact that $\mathrm{d}\circ\mathrm{d} = 0$ has important consequences, and the starting point for this is the following definition.

4.6.2 Definition (Closed/exact differential form) On a real analytic manifold M and for $r \in \{\infty, \omega\}$, a differential form $\alpha \in \Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M}))$ is ***closed*** if $\mathrm{d}\alpha = 0$ and is ***exact*** if there exists $\beta \in \Gamma^r(\bigwedge^{k-1}(\mathsf{T}^*\mathsf{M}))$ such that $\alpha = \mathrm{d}\beta$. •

Evidently, exact forms are closed, and so we have the ***de Rham complex***:

$$0 \longrightarrow C^r(\mathsf{M}) \xrightarrow{\mathrm{d}} \Gamma^r(\textstyle\bigwedge^1(\mathsf{T}^*\mathsf{M})) \xrightarrow{\mathrm{d}} - - - \xrightarrow{\mathrm{d}} \Gamma^r(\textstyle\bigwedge^n(\mathsf{T}^*\mathsf{M})) \longrightarrow 0$$

where n is the dimension of M. Let us denote by $\mathrm{Z}^k_{\mathrm{d}}(\mathsf{M})$ to be the kernel of $\mathrm{d}\colon \Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M})) \to \Gamma^r(\bigwedge^{k+1}(\mathsf{T}^*\mathsf{M}))$ and $\mathrm{B}^k_{\mathrm{d}}(\mathsf{M})$ to be the image of $\mathrm{d}\colon \Gamma^r(\bigwedge^{k-1}(\mathsf{T}^*\mathsf{M})) \to \Gamma^r(\bigwedge^k(\mathsf{T}^*\mathsf{M}))$. Since $\mathrm{B}^k_{\mathrm{d}}(\mathsf{M}) \subseteq \mathrm{Z}^k_{\mathrm{d}}(\mathsf{M})$, we can define the **k*th de Rham cohomology group*** to be

$$\mathrm{H}^k_{\mathrm{d}}(\mathsf{M}) = \frac{\mathrm{Z}^k_{\mathrm{d}}(\mathsf{M})}{\mathrm{B}^k_{\mathrm{d}}(\mathsf{M})}.$$

Note that $\mathrm{H}^0_{\mathrm{d}}(\mathsf{M}) \simeq \mathbb{R}^c$, where c is the number of connected components of M. This follows from the fact that smooth functions whose exterior derivative vanishes are locally constant, i.e., constant on connected components of M.

The following result says that closed forms are locally exact.

4.6.3 Theorem (Poincaré Lemma) *If* M *is a smooth manifold and if* $\alpha \in \Gamma^\infty(\bigwedge^k(T^*M))$ *is closed, then, for any* $x \in M$*, there exists a neighbourhood* $\mathcal{U}$ *of* x *and* $\beta \in \Gamma^\infty(\bigwedge^{k-1}(T^*\mathcal{U}))$ *such that* $\alpha|\mathcal{U} = d\beta$*.*

Proof As the result is local, we assume that M is a ball centred at $x = \mathbf{0} \in \mathbb{R}^n$. Let X be the time-varying vector field on $M \setminus \{\mathbf{0}\}$ given by $X(t, \boldsymbol{x}) = t^{-1}\boldsymbol{x}$. The flow of X starting at $t = 1$ at $\boldsymbol{x}$ is $\Phi^X_{1,t}(\boldsymbol{x}) = t\boldsymbol{x}$. Using Cartan's magic formula [Abraham, Marsden, and Ratiu 1988, Theorem 6.4.8(v)] we have

$$\frac{d}{dt}(\Phi^X_{1,t})^*\alpha = (\Phi^X_{1,t})^*\mathscr{L}_X\alpha = (\Phi^X_{1,t})^* dX \lrcorner \alpha = d((\Phi^X_{1,t})^* X \lrcorner \alpha).$$

If $t \in (0, 1]$ we have

$$\alpha - (\Phi^X_{1,t})^*\alpha = d\Big(\int_t^1 ((\Phi^X_{1,s})^* X \lrcorner \alpha) ds\Big).$$

For $\boldsymbol{y} \in M$ and for $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in T_{\boldsymbol{y}}M$ we have

$$(\Phi^X_{1,t})^*\alpha(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) = \alpha(T_{\boldsymbol{y}}\Phi^X_{1,t} \cdot \boldsymbol{v}_1, \dots, T_{\boldsymbol{y}}\Phi^X_{1,t} \cdot \boldsymbol{v}_1) = t^k\alpha(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k).$$

Thus

$$\lim_{t \to 0}(\Phi^X_{1,t})^*\alpha = 0$$

and so $\alpha = d\beta$ where

$$\beta = \int_0^1 ((\Phi^X_{1,s})^* X \lrcorner \alpha) ds,$$

and this gives the result. ■

Note that if $df = 0$ for $f \in C^\infty(M)$, then f is locally constant. Thus, for each $x \in M$, we have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{C}^\infty_{x,M} \xrightarrow{d} \mathscr{G}^\infty_{x,\bigwedge^1(T^*M)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{G}^\infty_{x,\bigwedge^n(T^*M)} \longrightarrow 0 \tag{4.23}$$

of $\mathbb{R}$-vector spaces, where by "$\mathbb{R}$" we mean the germs of functions that are constant in a neighbourhood of x (this will be made more clear and put into some context when we talk about the constant sheaf and its sheafification in Section GA2.1.1). In we shall extend this exact sequence from individual stalks to sheaves, and shall as a consequence say that this sequence is a soft resolution of the constant sheaf taking values in $\mathbb{R}$. what

4.6.2 Differential forms on holomorphic manifolds

Let us now adapt and extend the preceding discussion to holomorphic manifolds. Thus we let M be a holomorphic manifold. By

$$T^{*\mathbb{C}}_z M = (T^*_z M)_{\mathbb{C}} \simeq (T^{\mathbb{C}}_z M)^* \simeq \mathrm{Hom}_{\mathbb{R}}(T_z M; \mathbb{C}),$$

we denote the complex dual of $\mathsf{T}_z^{\mathbb{C}}\mathsf{M}$, the isomorphisms arising by virtue of Lemma 4.1.10. Let $(\mathcal{U},\phi)$ be a $\mathbb{C}$-chart for M with coordinates given by

$$(z^1 = x^1 + \mathrm{i}y^1, \dots, z^n = x^n + \mathrm{i}y^n).$$

By Proposition 4.5.13 we have the basis vectors $\frac{\partial}{\partial z^j}(z)$ and $\frac{\partial}{\partial \bar{z}^j}(z)$, $j \in \{1,\dots,n\}$, for $\mathsf{T}_z^{\mathbb{C}}\mathsf{M}$, $z \in \mathcal{U}$. According to Proposition 4.1.12, we have the basis vectors

$$\mathrm{d}z^j(z) = 1 \otimes \mathrm{d}x^j(z) + \mathrm{i} \otimes \mathrm{d}y^j(z), \quad \mathrm{d}\bar{z}^j(z) = 1 \otimes \mathrm{d}x^j(z) - \mathrm{i} \otimes \mathrm{d}y^j(z), \qquad j \in \{1,\dots,n\},$$

for $\mathsf{T}_z^{*\mathbb{C}}\mathsf{M}$, $z \in \mathcal{U}$.

Let $\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ be the subbundle of $\bigwedge^{k+l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ consisting of those alternating tensors of bidegree (k,l). By $\Gamma^r(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ we denote the set of C^r-sections, $r \in \{\infty,\omega,\mathrm{hol}\}$, of this subbundle, which we call the ***complex differential forms of bidegree* (k, l)**. Note that

$$\Gamma^r(\textstyle\bigwedge^{0,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) = \mathrm{C}^r(\mathsf{M};\mathbb{C}).$$

Let us represent these differential forms in a $\mathbb{C}$-chart $(\mathcal{U},\phi)$ with coordinates $(z^1,\dots,z^n)$, as above. As we saw at the end of Section 4.1.4, the local representative of $\alpha \in \Gamma^r(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ is given by

$$\alpha|\mathcal{U} = \sum_{\substack{i_1,\dots,i_k\in\{1,\dots,n\}\\ i_1<\dots<i_k}} \sum_{\substack{j_1,\dots,j_l\in\{1,\dots,n\}\\ j_1<\dots<j_l}} \alpha_{i_1,\dots,i_k,j_1,\dots,j_l} \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_k} \wedge \mathrm{d}\bar{z}^{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{j_l}, \tag{4.24}$$

where

$$\alpha_{i_1,\dots,i_k,j_1,\dots,j_l} = \alpha\Big(\frac{\partial}{\partial z^{i_1}},\dots,\frac{\partial}{\partial z^{i_k}},\frac{\partial}{\partial \bar{z}^{j_1}},\dots,\frac{\partial}{\partial \bar{z}^{j_l}}\Big)$$

are coefficients in $\mathrm{C}^r(\mathcal{U};\mathbb{C})$. We shall frequently abbreviate expression like (4.24) to

$$\alpha|\mathcal{U} = \sum_{I\in\boldsymbol{n}^k}{}' \sum_{J\in\boldsymbol{n}^l}{}' \alpha_{I,J} \mathrm{d}\boldsymbol{z}^I \wedge \mathrm{d}\bar{\boldsymbol{z}}^J,$$

with $\sum'$ denoting a sum over increasing multi-indices.

Let us now consider how the exterior derivative is adapted to complex differential forms. First we note that

$$\textstyle\bigwedge^m(\mathsf{T}_z^{*\mathbb{C}}\mathsf{M}) \simeq \mathbb{C} \otimes_{\mathbb{R}} \bigwedge^m(\mathsf{T}_z^*\mathsf{M}),$$

from which we deduce that

$$\Gamma^r(\textstyle\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \simeq \mathbb{C} \otimes_{\mathbb{R}} \Gamma^r(\bigwedge^m(\mathsf{T}^*\mathsf{M})).$$

Thus $\mathrm{d}\colon \Gamma^r(\bigwedge^m(\mathsf{T}^*\mathsf{M})) \to \Gamma^r(\bigwedge^{m+1}(\mathsf{T}^*\mathsf{M}))$ extends to a $\mathbb{C}$-linear map $\mathrm{d}_{\mathbb{C}}\colon \Gamma^r(\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \Gamma^r(\bigwedge^{m+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ which we call the ***complex exterior derivative***. The basic properties of the exterior derivative carry over to the complex exterior derivative.

4.6.4 Proposition (Properties of complex exterior derivative) *Let* M *be a holomorphic manifold. The complex exterior derivative has the following properties:*

(i) *the map* $\mathrm{d}_{\mathbb{C}}$ *is* $\mathbb{C}$*-linear;*

(ii) $\mathrm{d}_{\mathbb{C}} \circ \mathrm{d}_{\mathbb{C}} = 0$*;*

(iii) *if* $\mathcal{U} \subseteq \mathsf{M}$ *is open and if* $\alpha \in \Gamma^\infty(\bigwedge^{\mathrm{k}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$*, then* $(\mathrm{d}_{\mathbb{C}}\alpha)|\mathcal{U} = \mathrm{d}_{\mathbb{C}}(\alpha|\mathcal{U})$*;*

(iv) *if* $\alpha \in \Gamma^\infty(\bigwedge^{\mathrm{k}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *then* $\mathrm{d}_{\mathbb{C}}\bar{\alpha} = \overline{\mathrm{d}_{\mathbb{C}}\alpha}$*;*

(v) *if* $\alpha \in \Gamma^\infty(\bigwedge^{\mathrm{k}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *and* $\beta \in \Gamma^\infty(\bigwedge^{\mathrm{l}}(\mathsf{T}^{*}\mathsf{M}))$*, then*

$$\mathrm{d}_{\mathbb{C}}(\alpha \wedge \beta) = \mathrm{d}_{\mathbb{C}}\alpha \wedge \beta + (-1)^{\mathrm{k}}\alpha \wedge \mathrm{d}_{\mathbb{C}}\beta.$$

The following lemma is key to studying complex differential forms.

4.6.5 Lemma (Decomposition of the complex exterior derivative) *Let* M *be a holomorphic manifold, let* $\mathrm{m} \in \mathbb{Z}_{\geq 0}$*, and let* $\alpha \in \Gamma^\infty(\bigwedge^{\mathrm{k,l}}(\mathsf{T}^{*}\mathsf{M}))$*. Then*

$$\mathrm{d}_{\mathbb{C}}\alpha \in \Gamma^\infty(\textstyle\bigwedge^{\mathrm{k+1,l}}(\mathsf{T}^{*}\mathsf{M})) \oplus \Gamma^\infty(\textstyle\bigwedge^{\mathrm{k,l+1}}(\mathsf{T}^{*}\mathsf{M})).$$

Proof We adapt the formula (4.21) to the complex setting:

$$\begin{aligned}\mathrm{d}_{\mathbb{C}}\alpha(X_0, X_1, \dots, X_{k+l}) = &\sum_{r=0}^{k+l}(-1)^r \mathscr{L}_{X_r}(\alpha(X_0, X_1, \dots, \hat{X}_r, \dots, X_{k+l})) \\ &+ \sum_{\substack{r,s\in\{0,1,\dots,k+l\}\\ r<s}} (-1)^{r+s}\alpha([X_r, X_s]_{\mathbb{C}}, X_0, \dots, \hat{X}_r, \dots, \hat{X}_s, \dots, X_{k+l}),\end{aligned} \tag{4.25}$$

where $X_0, X_1, \dots, X_{k+l} \in \Gamma^\infty(\mathsf{T}^{\mathbb{C}}\mathsf{M})$. Note that the coordinate expression (4.24) implies that if we evaluate $\alpha(x)$ on $k+l$ tangent vectors of which more than k are in $\mathsf{T}^{1,0}_k\mathsf{M}$ or more than l are in $\mathsf{T}^{0,1}_x\mathsf{M}$, the result is necessarily zero. Taking advantage of Lemma 4.5.21, it follows that if we evaluate $\mathrm{d}_{\mathbb{C}}\alpha(x)$ on $k+l+1$ tangent vectors of which more than $k+1$ are in $\mathsf{T}^{1,0}_k\mathsf{M}$ or more than $l+1$ are in $\mathsf{T}^{0,1}_x\mathsf{M}$, the result is zero. Again by the coordinate expression (4.24), it follows that $\mathrm{d}_{\mathbb{C}}\alpha$ is a sum of differential forms of bidegree $(k+1, l)$ and $(k, l+1)$. ■

Given the lemma, for $\alpha \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*}\mathsf{M}))$, we can write

$$\mathrm{d}_{\mathbb{C}}\alpha = \partial\alpha + \bar{\partial}\alpha,$$

for some unique

$$\partial\alpha \in \Gamma^\infty(\textstyle\bigwedge^{k+1,l}(\mathsf{T}^{*}\mathsf{M})), \quad \bar{\partial}\alpha \in \Gamma^\infty(\textstyle\bigwedge^{k,l+1}(\mathsf{T}^{*}\mathsf{M})).$$

It is straightforward to give the local representatives of $\partial\alpha$ and $\bar{\partial}\alpha$.

4.6.6 Lemma (The local representative of the complex exterior derivative) *Let* M *be a holomorphic manifold, let* $m \in \mathbb{Z}_{\geq 0}$*, let* $\alpha \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^*\mathsf{M}))$*, and let* $(\mathcal{U}, \phi)$ *be a* $\mathbb{C}$*-chart for* M*. If the local representative of* α *is*

$$\alpha|\mathcal{U} = \sum_{\mathrm{I}\in\mathbf{n}^k}{}' \sum_{\mathrm{J}\in\mathbf{n}^l}{}' \alpha_{\mathrm{I},\mathrm{J}} \mathrm{d}\mathbf{z}^{\mathrm{I}} \wedge \mathrm{d}\bar{\mathbf{z}}^{\mathrm{J}},$$

then the local representatives of $\partial\alpha$ *and* $\bar{\partial}\alpha$ *are*

$$\partial\alpha|\mathcal{U} = \sum_{\mathrm{I}\in\mathbf{n}^k}{}' \sum_{\mathrm{J}\in\mathbf{n}^l}{}' \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\partial \alpha_{\mathrm{I},\mathrm{J}}}{\partial \mathrm{z}^{\mathrm{i}}} \mathrm{d}\mathrm{z}^{\mathrm{i}} \wedge \mathrm{d}\mathbf{z}^{\mathrm{I}} \wedge \mathrm{d}\bar{\mathbf{z}}^{\mathrm{J}}$$

and

$$\bar{\partial}\alpha|\mathcal{U} = \sum_{\mathrm{I}\in\mathbf{n}^k}{}' \sum_{\mathrm{J}\in\mathbf{n}^l}{}' \sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\partial \alpha_{\mathrm{I},\mathrm{J}}}{\partial \bar{\mathrm{z}}^{\mathrm{j}}} \mathrm{d}\bar{\mathrm{z}}^{\mathrm{j}} \wedge \mathrm{d}\mathbf{z}^{\mathrm{I}} \wedge \mathrm{d}\bar{\mathbf{z}}^{\mathrm{J}}.$$

Proof Let $f \in \mathrm{C}^\infty(\mathsf{M}; \mathbb{C})$. A direct computation gives the local representative $\mathrm{d}_{\mathbb{C}} f$ as

$$\mathrm{d}_{\mathbb{C}} f|\mathcal{U} = \sum_{i=1}^{n} \frac{\partial f}{\partial z^i} \mathrm{d}z^i + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}^j} \mathrm{d}\bar{z}^j.$$

The lemma then follows from Proposition 4.6.4(v). ■

We may alternatively express the local forms for ∂ and $\bar{\partial}$ by using components:

$$(\partial\alpha)_{i_1\cdots i_k i_{k+1}, j_1\cdots j_l} = \sum_{m=1}^{k} (-1)^{m-1} \frac{\partial}{\partial z^{j_m}} \alpha_{i_1\cdots i_{m-1} i_{m+1}\cdots i_{k+1}, j_1\cdots j_l} + (-1)^k \frac{\partial}{\partial z^{j_{k+1}}} \alpha_{i_1\cdots i_k, j_1\cdots j_l}$$

and

$$(\bar{\partial}\alpha)_{i_1\cdots i_k, j_1\cdots j_l j_{l+1}} = \sum_{m=1}^{l} (-1)^{m-1} \frac{\partial}{\partial \bar{z}^{j_m}} \alpha_{i_1\cdots i_k, j_1\cdots j_{m-1} j_{m+1}\cdots j_{l+1}} + (-1)^k \frac{\partial}{\partial \bar{z}^{j_{l+1}}} \alpha_{i_1\cdots i_k, j_1\cdots j_l} \tag{4.26}$$

Using the fact that

$$\Gamma^\infty(\textstyle\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) = \displaystyle\bigoplus_{\substack{k,l\\ k+l=m}} \Gamma^\infty(\textstyle\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})),$$

we can extend the maps ∂ and $\bar{\partial}$ to be defined on complex differential m-forms.

Applying Proposition 4.6.4 to the decomposition $\mathrm{d}_{\mathbb{C}} = \partial + \bar{\partial}$ gives the following result.

4.6.7 Proposition (Properties of ∂ and $\bar{\partial}$) *Let* M *be a holomorphic manifold. The following statements hold:*

(i) *the maps ∂ and $\bar{\partial}$ are $\mathbb{C}$-linear;*

(ii) $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$, *and* $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$;

(iii) *if* $\mathcal{U} \subseteq \mathsf{M}$ *is open and if* $\alpha \in \Gamma^\infty(\bigwedge^k(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, *then* $(\partial\alpha)|\mathcal{U} = \partial(\alpha|\mathcal{U})$ *and* $(\bar{\partial}\alpha)|\mathcal{U} = \bar{\partial}(\alpha|\mathcal{U})$;

(iv) *if* $\alpha \in \Gamma^\infty(\bigwedge^k(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *then* $\partial\bar{\alpha} = \bar{\partial}\alpha$ *and* $\bar{\partial}\bar{\alpha} = \overline{\partial\alpha}$;

(v) *if* $\alpha \in \Gamma^\infty(\bigwedge^k(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *and* $\beta \in \Gamma^\infty(\bigwedge^l(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, *then*

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta$$

and

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \bar{\partial}\beta.$$

Similarly to vector fields, one uses our decompositions of complex differential forms to define holomorphic differential forms.

4.6.8 Definition (Holomorphic differential form) If M is a holomorphic manifold and if $m \in \mathbb{Z}_{\geq 0}$, a ***holomorphic differential* m*-form*** is a complex differential form α of bidegree $(m, 0)$ for which $\bar{\partial}\alpha = 0$. By $\Gamma^{\mathrm{hol}}(\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ we denote the set of holomorphic differential m-forms. •

Note that Lemma 4.6.6 shows that the components of a holomorphic differential m-form are holomorphic functions of the local coordinates.

We have an interesting example of a bundle of holomorphic forms.

4.6.9 Example (Holomorphic forms on $\mathbb{CP}^1$) The holomorphic zero-forms on $\mathbb{CP}^1$ are precisely the holomorphic functions, and from Corollary 4.2.11 we know that such functions must be constant on $\mathbb{CP}^1$. Combining Examples 4.3.11 and 4.5.14 we see that $\bigwedge^1(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1) \simeq \mathsf{O}_{\mathbb{CP}^1}(-2)$. Moreover, from Example 4.3.14 we see that there are no nonzero holomorphic one-forms on $\mathbb{CP}^1$. •

4.6.3 The Dolbeault complex

In this section we study a homological construction associated with complex differential forms, a construction that is the holomorphic analogue of the de Rham complex for smooth differential forms on a smooth manifold.

The following definitions are analogous to those for the exterior derivative.

4.6.10 Definition ($\bar{\partial}$-closed/exact differential form) On a holomorphic manifold M, a differential form $\alpha \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ is ***$\bar{\partial}$-closed*** if $\bar{\partial}\alpha = 0$ and is ***$\bar{\partial}$-exact*** if there exists $\beta \in \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ such that $\alpha = \bar{\partial}\beta$. •

Analogously to the de Rham complex we have the ***Dolbeault complex***:

$$0 \longrightarrow \Gamma^\infty(\textstyle\bigwedge^{k,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \xrightarrow{\bar{\partial}} \Gamma^\infty(\bigwedge^{k,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Gamma^\infty(\bigwedge^{k,n}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \longrightarrow 0$$

if M has $\mathbb{C}$-dimension n. Let us denote by $\mathsf{Z}^{k,l}_{\bar{\partial}}(\mathsf{M})$ to be the kernel of $\bar{\partial}\colon \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \Gamma^\infty(\bigwedge^{k,l+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and $\mathsf{B}^{k,l}_{\bar{\partial}}(\mathsf{M})$ to be the image of $\bar{\partial}\colon \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Since $\mathsf{B}^{k,l}_{\bar{\partial}}(\mathsf{M}) \subseteq \mathsf{Z}^{k,l}_{\bar{\partial}}(\mathsf{M})$, we can define the **l*th Dolbeault cohomology group*** to be

$$\mathsf{H}^{k,l}_{\bar{\partial}}(\mathsf{M}) = \frac{\mathsf{Z}^{k,l}_{\bar{\partial}}(\mathsf{M})}{\mathsf{B}^{k,l}_{\bar{\partial}}(\mathsf{M})}.$$

Note that $\mathsf{H}^{k,0}_{\bar{\partial}}(\mathsf{M}) = \Gamma^{\mathrm{hol}}(\bigwedge^{k,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ since a smooth complex differential form α of bidegree $(k,0)$ on M is holomorphic if and only if $\bar{\partial}\alpha = 0$.

Evidently, $\bar{\partial}$-exact forms are $\bar{\partial}$-closed. The converse, however, is not true.

4.6.11 Examples ($\bar{\partial}$-closed and exact forms)

1. Let $\mathcal{U} \subseteq \mathbb{C}$ be open and let $\alpha \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ be $\bar{\partial}$-closed. Since $\mathcal{U}$ is pseudoconvex by Example 3.3.11–1, it follows from Corollary 3.4.4 (or using a deeper result that we will subsequently prove, Theorem 6.2.15) that there exists $\beta \in \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ such that $\alpha = \bar{\partial}\beta$. Thus $\bar{\partial}$-closed forms defined on open subsets of $\mathbb{C}$ are $\bar{\partial}$-exact, provided that $l \in \mathbb{Z}_{>0}$.
2. Let us consider $\mathsf{M} = \mathbb{CP}^1$. We shall determine $\mathsf{H}^{k,l}_{\bar{\partial}}(\mathbb{CP}^1)$ for $k, l \in \{0,1\}$, all other cohomology groups vanishing since $\dim_{\mathbb{C}}(\mathbb{CP}^1) = 1$.
   (a) $\mathsf{H}^{0,0}_{\bar{\partial}}(\mathbb{CP}^1)$: By Corollary 4.2.11, holomorphic functions on $\mathbb{CP}^1$ are constant. Thus
   $$\mathsf{H}^{0,0}_{\bar{\partial}}(\mathbb{CP}^1) \simeq \mathbb{C}.$$
   (b) $\mathsf{H}^{1,0}_{\bar{\partial}}(\mathbb{CP}^1)$: Suppose that $\alpha \in \mathsf{Z}^{1,0}_{\bar{\partial}}(\mathbb{CP}^1)$. Since $\partial\alpha = 0$ it follows that $\mathrm{d}_{\mathbb{C}}\alpha = 0$. Since $\mathsf{H}^1_{\mathrm{d}}(\mathbb{S}^2) = \{0\}$ (see [Bott and Tu 1982, Exercise I.4.3]), it follows that $\alpha = \mathrm{d}_{\mathbb{C}}f$ for some $f \in \mathsf{C}^\infty(\mathbb{CP}^1;\mathbb{C})$. Note that α is a holomorphic one-form. Thus $\mathrm{d}_{\mathbb{C}}f$ is holomorphic and so, in coordinates, $\mathrm{d}_{\mathbb{C}}f = \frac{\partial f}{\partial z}\mathrm{d}z$. Thus $\frac{\partial f}{\partial \bar{z}} = 0$ and so f must be holomorphic, and, therefore, constant by Corollary 4.2.11. Thus $\alpha = 0$ and so
   $$\mathsf{H}^{1,0}_{\bar{\partial}}(\mathbb{CP}^1) \simeq \{0\}.$$
   Thus there are no nonzero holomorphic one-forms on $\mathbb{CP}^1$, as we have already see in Example 4.6.9.
   (c) $\mathsf{H}^{0,1}_{\bar{\partial}}(\mathbb{CP}^1)$: Here we recall from Example 4.2.2–4 that we have two charts $(\mathcal{U}_1, \phi_1)$ and $(\mathcal{U}_2, \phi_2)$ for $\mathbb{CP}^1$ for which
   $$\phi_1(\mathcal{U}_1) = \phi_2(\mathcal{U}_2) = \mathbb{C}$$
   and
   $$\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = \mathbb{C}^* \triangleq \mathbb{C} \setminus \{0\}.$$

Let us denote the respective coordinates by z_1 and z_2. The overlap maps are then

$$\phi_2 \circ \phi_1^{-1}(z_1) = \frac{1}{z_1}, \quad \phi_1 \circ \phi_2^{-1}(z_2) = \frac{1}{z_2}.$$

We then compute the change of basis relation for the basis one-forms to be

$$\mathrm{d}z_2 = -\frac{1}{z_1^2}\mathrm{d}z_1, \quad \mathrm{d}\bar{z}_2 = -\frac{1}{\bar{z}_1^2}\mathrm{d}\bar{z}_1.$$

Now let $\alpha \in \mathrm{Z}_{\bar\partial}^{0,1}(\mathbb{CP}^1)$ have local representatives

$$\alpha|\mathcal{U}_1 = \alpha_1(z_1)\mathrm{d}\bar{z}_1, \quad \alpha|\mathcal{U}_2 = \alpha_2(z_2)\mathrm{d}\bar{z}_2.$$

Let us define $f \in \mathrm{C}^\infty(\mathbb{CP}^1;\mathbb{C})$ by requiring that the local representative of f in the chart $(\mathcal{U}_1, \phi_1)$ is

$$f \circ \phi_1^{-1}(z_1) = \int_{z_0}^{\bar{z}_1} \alpha_1(\zeta_1)\,\mathrm{d}\bar{\zeta}_1$$

for some $z_0 \neq 0$. We then directly compute $\frac{\partial(f \circ \phi_1^{-1})}{\partial \bar{z}_1} = \alpha_1$, using the fact that $\bar\partial\alpha = 0$. That is, on $\mathcal{U}_1$ we have $\bar\partial f = \alpha$.

We now need to show that f extends to a well-defined function on $\mathcal{U}_2$ and that the extended function also satisfies $\bar\partial f = \alpha$ on $\mathcal{U}_2$. First we note that

$$\bar{z}_2^2\alpha_2(z_2) = -\alpha_1(z_2^{-1}), \qquad z_2 \in \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2).$$

Thus for $z_2 \neq 0$ we have

$$f \circ \phi_2^{-1}(z_2) = \int_{z_0}^{1/\bar{z}_2} \alpha_1(\zeta_1)\,\mathrm{d}\bar{\zeta}_1 = -\int_{1/z_0}^{\bar{z}_2} \frac{\alpha_1(\zeta_2^{-1})}{\bar{\zeta}_2^2}\,\mathrm{d}\bar{\zeta}_2 = \int_{1/z_0}^{\bar{z}_2} \alpha_2(\zeta_2)\,\mathrm{d}\bar{\zeta}_2,$$

and from this we conclude that f indeed extends to a well-defined function on $\mathcal{U}_2$ satisfying $\bar\partial f = \alpha$.

From the above computations we conclude that

$$\mathrm{H}_{\bar\partial}^{0,1}(\mathbb{CP}^1) = \{0\}.$$

(d) $\mathrm{H}_{\bar\partial}^{1,1}(\mathbb{CP}^1)$: Note that if $\alpha \in \Gamma^\infty(\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1))$ then $\bar\partial\alpha = \mathrm{d}_{\mathbb{C}}\alpha$. Also, if $\alpha \in \Gamma^\infty(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1))$ then $\bar\partial\alpha = 0$. Thus

$$\bar\partial(\Gamma^\infty(\textstyle\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1))) = \mathrm{d}_{\mathbb{C}}(\Gamma^\infty(\bigwedge^{1}(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1))) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{d}(\Gamma^\infty(\bigwedge^{1}(\mathsf{T}^{*}\mathbb{CP}^1)))$$

$$\mathrm{Z}_{\bar\partial}^{1,1}(\mathbb{CP}^1) = \Gamma^\infty(\textstyle\bigwedge^{1,1}(\mathsf{T}^{*\mathbb{C}}\mathbb{CP}^1)) = \mathbb{C} \otimes_{\mathbb{R}} \Gamma^\infty(\bigwedge^{2}(\mathsf{T}^{*}\mathbb{CP}^1)).$$

Therefore, using the fact that $\mathrm{H}_{\mathrm{d}}^2(\mathbb{S}^2) \simeq \mathbb{R}$ (see [Bott and Tu 1982, Exercise I.4.3]), we have

$$\mathrm{H}_{\bar\partial}^{1,1}(\mathbb{CP}^1) \simeq \mathbb{C}.$$

3. Let M be a compact connected holomorphic manifold of dimension 1. Following the lines above for $\mathbb{CP}^1$, we see that

$$\mathrm{H}^{0,0}_{\bar{\partial}}(\mathsf{M}) = \mathbb{C},$$

and

$$\mathrm{H}^{1,1}_{\bar{\partial}}(\mathsf{M}) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}^2_{\mathrm{d}}(\mathsf{M}) \simeq \mathbb{C},$$

since $\mathrm{H}^n_{\mathrm{d}}(\mathsf{M}) = \mathrm{H}^0_{\mathrm{d}}(\mathsf{M})$ for a connected, compact, oriented manifold of dimension n [Bott and Tu 1982, Corollary 5.8]. •

From the examples we can see that the nonvanishing of Dolbeault cohomology groups provides a combination of information regarding the global topology and the global complex structure. We shall flesh out this vague observation more in . For now we state the following result, known as the ***∂̄-Poincaré Lemma*** or ***Dolbeault's Lemma***, the latter name after Dolbeault [1956, 1957]. The result says that locally the Dolbeault complex is exact.

what

4.6.12 Theorem (Dolbeault's Lemma) *If M is a holomorphic manifold and if* $\alpha \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *is* $\bar{\partial}$*-closed, then, for each* $z \in \mathsf{M}$*, there exists a neighbourhood* $\mathcal{U}$ *of* z *and* $\beta \in \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *such that* $\alpha|\mathcal{U} = \bar{\partial}(\beta|\mathcal{U})$*.*

Proof We begin by proving a generalisation of the Cauchy Integral Formula.

1 Lemma *Let* $\mathcal{U} \subseteq \mathbb{C}$ *be open, let* $z \in \mathcal{U}$*, let* $r \in \mathbb{R}_{>0}$ *be such that* $\overline{\mathsf{D}}^1(r, z) \subseteq \mathcal{U}$*, and let* $f \in C^\infty(\mathcal{U}; \mathbb{C})$*. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(z,r))} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\overline{\mathsf{D}}^1(z,r)\setminus\{z\}} \frac{\partial f}{\partial\bar{\zeta}}(\zeta)\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta - z}.$$

Proof Let $\epsilon \in (0, r)$. On $\mathsf{D}^1(r, z) \setminus \overline{\mathsf{D}}^1(\epsilon, z)$ consider the $(1, 0)$-form α defined by

$$\alpha(\zeta) = \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

By Stokes Theorem,

$$\int_{\mathsf{D}^1(r,z)\setminus\overline{\mathsf{D}}^1(\epsilon,z)} \mathrm{d}\alpha = \int_{\mathrm{bd}(\overline{\mathsf{D}}^1(r,z))}\alpha - \int_{\mathrm{bd}(\overline{\mathsf{D}}^1(\epsilon,z))}\alpha.$$

Substituting the specific expression for α we have

$$\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(\epsilon,z))} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \int_{\mathrm{bd}(\overline{\mathsf{D}}^1(r,z))} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta + \int_{\mathsf{D}^1(r,z)\setminus\overline{\mathsf{D}}^1(\epsilon,z)} \frac{\partial f}{\partial\bar{\zeta}}(\zeta)\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta - z} \tag{4.27}$$

Using $\zeta = z + \epsilon e^{i\theta}$ for the first integral,

$$\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(\epsilon,z))} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = i\int_0^{2\pi} f(z + \epsilon e^{i\theta})\,\mathrm{d}\theta.$$

By the Dominated Convergence Theorem applied to the integral on the right,

$$\lim_{\epsilon\to 0}\int_{\mathrm{bd}(\overline{\mathsf{D}}^1(\epsilon,z))}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=2\pi\mathrm{i}f(z). \tag{4.28}$$

Now let us consider the following integral:

$$\int_{\overline{\mathsf{D}}^1(\epsilon,z)\setminus\{z\}}\frac{\partial f}{\partial\bar{\zeta}}(\zeta)\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta-z}$$

We first make the change of variable $w=\zeta-z$ so that the integral in question becomes

$$\int_{\overline{\mathsf{D}}^1(\epsilon,0)\setminus\{0\}}\frac{\partial f}{\partial\bar{\zeta}}(z+w)\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{w}$$

A direct computation shows that

$$\mathrm{d}w\wedge\mathrm{d}\bar{w}=2\mathrm{i}\mathrm{d}x\wedge\mathrm{d}y=2\mathrm{i}r\mathrm{d}r\wedge\mathrm{d}\theta.$$

Making the change to polar coordinates then gives

$$\int_{\overline{\mathsf{D}}^1(\epsilon,z)\setminus\{z\}}\frac{\partial f}{\partial\bar{\zeta}}(\zeta)\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta-z}=2\mathrm{i}\int_0^{\epsilon}\int_0^{2\pi}\frac{\partial f}{\partial\bar{\zeta}}(z+r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\mathrm{d}\theta.$$

Thus

$$\lim_{\epsilon\to 0}\int_{\overline{\mathsf{D}}^1(\epsilon,z)\setminus\{z\}}\frac{\partial f}{\partial\bar{\zeta}}(\zeta)\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta-z}=0.$$

Substituting the preceding expression and (4.28) into (4.27) gives the lemma. ▼

Now we prove the theorem. We note that the result is local so we assume that $\mathsf{M}=\mathcal{U}$ is an open subset of $\mathbb{C}^n$. First we consider the case of $n=1$. Thus let $\mathcal{U}\subseteq\mathbb{C}$ be an open set and let $\mathcal{D}$ be an open disk with $\mathrm{cl}(\mathcal{D})\subseteq\mathcal{U}$. Let $z_0\in\mathcal{D}$ and let ϵ be small enough that $\overline{\mathsf{D}}^1(2\epsilon,z_0)\subseteq\mathcal{D}$. Let $\alpha(z)=g(z)\mathrm{d}\bar{z}$ be a smooth $(0,1)$-form on $\mathcal{U}$ and note that $\bar{\partial}\alpha=0$. Define

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}}\frac{g(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}.$$

Let $\psi\in\mathrm{C}^\infty(\mathcal{U})$ be equal to the constant function with value 1 on $\overline{\mathsf{D}}^1(\epsilon,z_0)$ and 0 outside $\overline{\mathsf{D}}^1(2\epsilon,z_0)$. Let $g_1=\psi g$ and $g_2=(1-\psi)g$ so $g=g_1+g_2$. For $z\in\mathsf{D}^1(\epsilon,z_0)$ define

$$f_1(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}}\frac{g_1(\zeta)}{\zeta-z}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$$

and

$$f_2(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}}\frac{g_2(\zeta)}{\zeta-z}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$$

so that $f(z) = f_1(z) + f_2(z)$. Using the fact that g_2 vanishes on $\mathsf{D}^1(\epsilon, z_0)$ and using holomorphicity of $\zeta \mapsto \frac{1}{\zeta - z}$, we have $\frac{\partial f_2}{\partial \bar{z}} = 0$ on $\mathsf{D}^1(\epsilon, z_0)$. Also, using a change to polar coordinates as in the proof of the lemma above,

$$\begin{aligned}\frac{\partial f_1}{\partial \bar{z}}(z) &= \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial \bar{z}}\int_{\mathcal{D}}\frac{g_1(\zeta)}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} = \frac{1}{\pi}\int_{\mathcal{D}}\frac{\partial g_1}{\partial \bar{z}}(z + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}r \wedge \mathrm{d}\theta \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}}\frac{\partial g_1}{\partial \bar{z}}(\zeta)\frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{\zeta - z} = g_1(z),\end{aligned}$$

using the lemma above since g_1 vanishes $\mathrm{bd}(\mathcal{D})$. Thus

$$\frac{\partial f}{\partial \bar{z}}(z) = \frac{\partial f_1}{\partial \bar{z}}(z) + \frac{\partial f_2}{\partial \bar{z}}(z) = g_1(z) + g_2(z) = g(z)$$

for $z \in \mathsf{D}^1(\epsilon, z_0)$. Thus $\alpha = \bar{\partial} f$ on $\mathcal{D}$. Thus we have the theorem for $n = 1$ and $(k, l) = (0, 1)$.

Now we prove the theorem, in its local form, for general n. Thus we suppose that $\mathsf{M} = \mathcal{U}$ is an open subset of $\mathbb{C}^n$ and that $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ is an open polydisk such that $\mathrm{cl}(\mathcal{D}) \subseteq \mathcal{U}$. We let α be a smooth (k, l)-form on $\mathcal{U}$ satisfying $\bar{\partial}\alpha = 0$. We will prove the theorem in this case by induction on r, with r being the smallest nonnegative integer such that the coordinate expression for α does not involve $\mathrm{d}\bar{z}^{r+1}, \dots, \mathrm{d}\bar{z}^n$. For $r = 0$ the result is vacuously true since we must have $l \in \mathbb{Z}_{>0}$, and so r being zero implies that α is zero. Also, if the assertion holds $r = n$, then this is what we wish to prove.

So, we suppose that the theorem holds for $r \in \{0, 1, \dots, m-1\}$ and suppose that

$$\alpha = \mathrm{d}\bar{z}^m \wedge \omega + \theta \tag{4.29}$$

for $\omega \in \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathcal{D}))$, $\theta \in \Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathcal{D}))$, and where the coordinate expressions for both ω and θ do not involve $\mathrm{d}\bar{z}^m, \dots, \mathrm{d}\bar{z}^n$. Let us write

$$\omega = \sideset{}{'}\sum_{I \in \boldsymbol{n}^k}\sideset{}{'}\sum_{J \in \boldsymbol{n}^{l-1}}\omega_{I,J}\mathrm{d}\boldsymbol{z}^I \wedge \mathrm{d}\bar{\boldsymbol{z}}^J.$$

For the moment, let us fix increasing multi-indices $I \in \boldsymbol{n}^k$ and $J \in \boldsymbol{n}^{l-1}$. Since $\bar{\partial}\alpha = 0$ we have, by (4.29) and keeping in mind the attributes of ω and θ,

$$\frac{\partial}{\partial \bar{z}^j}\omega_{I,J} = 0, \qquad j \in \{m+1, \dots, n\}. \tag{4.30}$$

Thus ω is a holomorphic function of $z^{m+1}, \dots, z^n$.

Define

$$\Omega_{I,J}(\boldsymbol{z}) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}_m}\frac{1}{\zeta - z^k}\omega_{I,J}(z^1, \dots, z^{m-1}, \zeta, z^{m+1}, \dots, z^n)\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

By our conclusions from the case $n = 1$ above we have

$$\frac{\partial}{\partial \bar{z}^m}\Omega_{I,J} = \omega_{I,J}$$

in $\mathcal{D}$. By differentiating the expression for $\Omega_{I,J}$ under the integral sign and using (4.30) we get

$$\frac{\partial}{\partial \bar{z}^j}\Omega_{I,J} = 0, \qquad j \in \{m+1,\dots,n\}.$$

Now define

$$\Omega = \sum_{I\in \boldsymbol{n}^k}{}' \sum_{J\in \boldsymbol{n}^{l-1}}{}' \Omega_{I,J}\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J.$$

A direct computation using the properties of $\Omega_{I,J}$ above gives

$$\bar{\partial}\Omega = \mathrm{d}\bar{z}^m \wedge \omega + \theta'$$

on $\mathcal{D}''$, where θ' does not involve $\mathrm{d}\bar{z}^m,\dots,\mathrm{d}\bar{z}^n$ in its coordinate expression. Therefore, $\theta - \theta' = \alpha - \bar{\partial}\Omega$ does not involve $\mathrm{d}\bar{z}^m,\dots,\mathrm{d}\bar{z}^n$ in its coordinate expression. By the induction hypothesis, there exists $\beta' \in \Gamma^\infty(\bigwedge^{k,l-1}(\mathsf{T}^{*\mathbb{C}}\mathcal{D}'))$ such that

$$\bar{\partial}\beta' = \alpha - \bar{\partial}\Omega$$

in $\mathcal{D}'$. The result follows by taking $\beta = \beta' + \Omega$. ∎

The theorem gives rise to a resolution of the module of germs of holomorphic m-forms at $z \in \mathsf{M}$, i.e., an exact sequence

$$0 \longrightarrow \mathscr{G}^{\mathrm{hol}}_{z,\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \longrightarrow \mathscr{G}^{\infty}_{z,\bigwedge^{m,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \xrightarrow{\bar{\partial}} \mathscr{G}^{\infty}_{z,\bigwedge^{m,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{G}^{\infty}_{z,\bigwedge^{m,n}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \longrightarrow 0 \tag{4.31}$$

of $\mathbb{C}$-vector spaces, noting that there is a natural inclusion of $\mathscr{G}^{\mathrm{hol}}_{z,\bigwedge^m(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}$ into the module $\mathscr{G}^{\infty}_{z,\bigwedge^{m,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}$.

4.6.4 Differential forms with power series coefficients

In this section we consider the cases of real analytic and holomorphic differential forms in a little more detail by considering differential forms with power series coefficients. In this case, we shall see that the classical Poincaré Lemma restricts nicely to the real analytic and holomorphic cases. We do this by first considering an algebraic setting for differential forms.

Let F be a field and let V be a finite-dimensional F-vector space. For $k, l \in \mathbb{Z}_{\geq 0}$ we will find it illuminating to think of the vector space $\mathrm{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$ as being "differential l-forms with coefficients being homogeneous polynomials of degree k." To do this, we first establish the relationship between $\mathrm{S}^k(\mathsf{V}^*)$ and homogeneous polynomial functions. Let $\mathsf{F}_k[\mathsf{V}]$ denote the polynomial functions of degree k on V i.e., the F-valued functions of the form

$$f_A(x) = A(x,\dots,x), \qquad x \in \mathsf{V},\ A \in \mathrm{T}^k(\mathsf{V}^*).$$

With the identification of such functions with $\mathrm{S}^k(\mathsf{V}^*)$ as in Proposition 4.4.7, let us explicitly demonstrate the desired identification of $\mathrm{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$ with differential l-forms with polynomial coefficients of degree k.

4.6.13 Proposition (Characterisation of $\mathsf{S}^{\mathrm{k}}(\mathsf{V}^*) \otimes \bigwedge^{\mathrm{l}}(\mathsf{V}^*)$) *Let* $\mathrm{k}, \mathrm{l} \in \mathbb{Z}_{\geq 0}$. *The map* $\phi_{\mathrm{k},\mathrm{l}}$ *from* $\mathsf{S}^{\mathrm{k}}(\mathsf{V}^*) \otimes \bigwedge^{\mathrm{l}}(\mathsf{V}^*)$ *to* $\mathsf{F}_{\mathrm{k}}[\mathsf{V}] \otimes \bigwedge^{\mathrm{l}}(\mathsf{V}^*)$ *defined by*

$$\phi_{\mathrm{k},\mathrm{l}}(\mathrm{A} \otimes \alpha)(\mathrm{x})(\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{l}}) = \mathrm{A}(\mathrm{x}, \dots, \mathrm{x})\alpha(\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{l}}), \qquad \mathrm{x}, \mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{l}} \in \mathsf{V},$$

is a monomorphism of F*-vector spaces.*

Proof The map is clearly linear. To see that it is injective, suppose that

$$\phi_{k,l}(\alpha_1 \otimes A_1 + \dots + \alpha_m \otimes A_m) = 0.$$

This means that

$$A_1(x, \dots, x)\alpha_1(v_1, \dots, v_l) + \dots + A_k(x, \dots, x)\alpha_k(v_1, \dots, v_l) = 0$$

for every $x, v_1, \dots, v_l \in \mathsf{V}$. Now let $u_1, \dots, u_k, v_1, \dots, v_l \in \mathsf{V}$. By Proposition 4.4.7 we have

$$A_j(u_1, \dots, u_k) = \frac{1}{k!} \sum_{l=1}^{k} \sum_{\{j_1, \dots, j_l\} \subseteq \{1, \dots, k\}} (-1)^{k-l} A(u_{j_1} + \dots + u_{j_l}, \dots, u_{j_1} + \dots + u_{j_l}).$$

We may then write

$$\sum_{j=1}^{m} A_j(u_1, \dots, u_k)\alpha_j(v_1, \dots, v_l)$$
$$= \sum_{j=1}^{m} \frac{1}{k!} \sum_{l=1}^{k} \sum_{\{j_1, \dots, j_l\} \subseteq \{1, \dots, k\}} (-1)^{k-l} A(u_{j_1} + \dots + u_{j_l}, \dots, u_{j_1} + \dots + u_{j_l})\alpha_j(v_1, \dots, v_l).$$

The terms on the right vanish since $A_1 \otimes \alpha_1 + \dots + A_m \otimes \alpha_m \in \ker(\phi_{k,l})$, and so we have $A_1 \otimes \alpha_1 + \dots + A_m \otimes \alpha_m = 0$, giving injectivity of $\phi_{k,l}$. ■

We can formally perform the usual operations we perform with differential forms on these differential forms with polynomial coefficients. In particular, we will be interested in the exterior derivative and wedge product. To this end, if $f_1 \otimes \alpha_1 \in \mathsf{F}_{k_1}[\mathsf{V}] \otimes \bigwedge^{l_1}(\mathsf{V}^*)$ and if $f_2 \otimes \alpha_2 \in \mathsf{F}_{k_2}[\mathsf{V}] \otimes \bigwedge^{l_2}(\mathsf{V}^*)$, then we define

$$(f_1 \otimes \alpha_1) \wedge (f_2 \otimes \alpha_2) = (f_1 f_2) \otimes (\alpha_1 \wedge \alpha_2),$$

where $f_1 f_2$ denotes the usual product of functions, i.e.,

$$(f_1 f_2)(x, \dots, x) = f_1(x, \dots, x) f_2(x, \dots, x).$$

We can also define exterior differentiation as follows. If $f \otimes \alpha \in \mathsf{F}_k[\mathsf{V}] \otimes \bigwedge^l(\mathsf{V}^*)$ then we define $\mathrm{d}_{k,l}(f \otimes \alpha) \in \mathsf{F}_{k-1}[\mathsf{V}] \otimes \bigwedge^{l+1}(\mathsf{V}^*)$ by

$$\mathrm{d}_{k,l}(f \otimes \alpha)(x) \cdot (v_1, \dots, v_{l+1}) = \sum_{j=1}^{l+1} (-1)^{j+1} k A(v_j, x, \dots, x)\alpha(v_1, \dots, \hat{v}_j, \dots, v_{l+1}),$$

where $A \in \mathsf{S}^k(\mathsf{V}^*)$ is such that

$$f(x) = A(x, \dots, x), \qquad x \in \mathsf{V}.$$

A comparison with (4.22) shows that this definition agrees with the usual definition if one takes the case $\mathsf{F} = \mathbb{R}$ and thinks of V as being a differentiable manifold with coordinates chosen according to a basis for V, cf. [Abraham, Marsden, and Ratiu 1988, Corollary 6.4.2].

These operations translate into operations on $\mathsf{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$. To this end, if $A_1 \otimes \alpha_1 \in \mathsf{S}^{k_1}(\mathsf{V}^*) \otimes \bigwedge^{l_1}(\mathsf{V}^*)$ and $A_2 \otimes \alpha_2 \in \mathsf{S}^{k_2}(\mathsf{V}^*) \otimes \bigwedge^{l_2}(\mathsf{V}^*)$, define

$$(A_1 \otimes \alpha_1) \wedge (A_2 \otimes \alpha_2) = \frac{k_1!k_2!}{(k_1+k_2)!}(A_1 \odot A_2) \otimes (\alpha_1 \wedge \alpha_2) \in \mathsf{S}^{k_1+k_2}(\mathsf{V}^*) \otimes {\textstyle\bigwedge}^{l_1+l_2}(\mathsf{V}^*).$$

For $A \otimes \alpha \in \mathsf{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$, define $\delta_{k,l}(A \otimes \alpha) \in \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*)$ by

$$(\delta_{k,l}(A \otimes \alpha))(u_1, \dots, u_{r-1}, v_1, \dots, v_{l+1}) = \sum_{j=1}^{l+1} (-1)^{j+1} k A(v_j, u_1, \dots, u_{r-1}) \alpha(v_1, \dots, \hat{v}_j, \dots, v_{l+1}),$$

with the understanding that $\mathsf{S}^{-j}(\mathsf{V}^*) = \{0\}$ for $j \in \mathbb{Z}_{>0}$.

The following result shows that these algebraic operations agree with their differential counterparts under the map $\phi_{k,l}$.

4.6.14 Proposition (Wedge product and exterior derivative) *For* $A \otimes \alpha \in \bigwedge^l(\mathsf{V}^*) \otimes \mathsf{S}^k(\mathsf{V}^*)$, $A_1 \otimes \alpha_1 \in \bigwedge^{l_1}(\mathsf{V}^*) \otimes \mathsf{S}^{k_1}(\mathsf{V}^*)$, *and* $A_2 \otimes \alpha_2 \in \bigwedge^{l_2}(\mathsf{V}^*) \otimes \mathsf{S}^{k_2}(\mathsf{V}^*)$, *the following statements hold:*

(i) $\phi_{k_1+k_2, l_1+l_2}((A_1 \otimes \alpha_1) \wedge (A_2 \otimes \alpha_2)) = (\phi_{k_1,l_1}(A_1 \otimes \alpha_1)) \wedge (\phi_{k_2,l_2}(A_2 \otimes \alpha_2))$;

(ii) $\phi_{k-1,l+1} \circ \delta_{k,l}(A \otimes \alpha) = \mathrm{d}_{k,l} \circ \phi_{k,l}(A \otimes \alpha)$.

Proof (i) We have

$$\begin{aligned}(\phi_{k_1,l_1}(A_1 \otimes \alpha_1)(x)) \wedge (\phi_{k_2,l_2}(A_2 \otimes \alpha_2)(x)) &= A_1(x, \dots, x) A_2(x, \dots, x) \alpha_1 \wedge \alpha_2 \\ &= (A_1 \otimes A_2)(x, \dots, x) \alpha_1 \wedge \alpha_2 \\ &= \operatorname{Sym}(A_1 \otimes A_2)(x, \dots, x) \alpha_1 \wedge \alpha_2 \\ &= \frac{k_1!k_2!}{(k_1+k_2)!}(A_1 \odot A_2)(x, \dots, x) \alpha_1 \wedge \alpha_2 \\ &= \phi_{k_1+k_2, l_1+l_2}((\alpha_1 \otimes A_1) \wedge (\alpha_2 \otimes A_2))(x).\end{aligned}$$

(ii) Let us first consider the case when $l = 0$. In this case we have

$$\mathrm{d}_{k,0} \circ \phi_{k,0}(A)(x) \cdot v = kA(v, x, \dots, x) = \phi_{k-1,1} \circ \delta_{k,0}(A)(x)(v),$$

so proving the result when $l = 0$.

For $l > 0$ we first prove a lemma.

1 Lemma *If* $\mathrm{f}_1 \otimes \alpha_1 \in \mathsf{F}_{\mathrm{k}_1}[\mathsf{V}] \otimes \bigwedge^{\mathrm{l}_1}(\mathsf{V}^*)$ *and* $\mathrm{f}_2 \otimes \alpha_2 \in \mathsf{F}_{\mathrm{k}_2}[\mathsf{V}] \otimes \bigwedge^{\mathrm{l}_2}(\mathsf{V}^*)$*, then*

$$\begin{aligned} &\mathrm{d}_{\mathrm{k}_1+\mathrm{k}_2,\mathrm{l}_1+\mathrm{l}_2}((\mathrm{f}_1 \otimes \alpha_1) \wedge (\mathrm{f}_2 \otimes \alpha_2)) \\ &\qquad = (\mathrm{d}_{\mathrm{k}_1,\mathrm{l}_1}(\mathrm{f}_1 \otimes \alpha_1)) \wedge (\mathrm{f}_2 \otimes \alpha_2) + (-1)^{\mathrm{l}_1}(\mathrm{f}_1 \otimes \alpha_1) \wedge (\mathrm{d}_{\mathrm{k}_2,\mathrm{l}_2}(\mathrm{f}_2 \otimes \alpha_2)). \end{aligned}$$

Proof Let $A_1 \in \mathrm{S}^{k_1}(\mathsf{V}^*)$ and $A_2 \in \mathrm{S}^{k_2}(\mathsf{V}^*)$ be such that

$$f_a(x) = A_a(x, \dots, x), \qquad a \in \{1, 2\},\ x \in \mathsf{V}.$$

We compute

$$\begin{aligned} &\mathrm{d}_{k_1+k_2,l_1+l_2}((f_1 \otimes \alpha_1) \wedge (f_2 \otimes \alpha_2))(x)(v_1, \dots, v_{l_1+l+2}) \\ &= \mathrm{d}_{k_1+k_2,l_1+l_2}((f_1 f_2) \otimes (\alpha_1 \wedge \alpha_2))(x)(v_1, \dots, v_{l_1+l_2+1}) \\ &= \sum_{j=1}^{l_1+l_2+1} (-1)^{j+1} \operatorname{Sym}_{k_1+k_2}(A_1 \otimes A_2)(v_j, x, \dots, x)\alpha_1 \wedge \alpha_2(v_1, \dots, \hat{v}_j, \dots, v_{l_1+l_2+1}) \\ &= \sum_{j=1}^{l_1+1} (-1)^{j+1} k_1 A_1(v_j, x, \dots, x)\alpha_1(v_1, \dots, \hat{v}_j, \dots, v_{l_1+1}) A_2(x, \dots, x)\alpha_2(v_{l_1+2}, \dots, v_{l_1+l_2+1}) \\ &\quad + \sum_{j=l_1+2}^{l_1+l_2+1} (-1)^{j+1} A_1(x, \dots, x)\alpha_1(v_1, \dots, v_{l_1}) k_2 A_2(v_j, x, \dots, x)\alpha(v_{l_1+1}, \dots, \hat{v}_j, \dots, v_{l_1+l_2+1}) \\ &= (\mathrm{d}_{k_1,l_1}(f_1 \otimes \alpha_1)) \wedge (f_2 \otimes \alpha_2)(x)(v_1, \dots, v_{l_1+l_2+1}) \\ &\quad + (-1)^{l_1}(f_1 \otimes \alpha_1) \wedge (\mathrm{d}_{k_2,l_2}(f_2 \otimes \alpha_2))(x)(v_1, \dots, v_{l_1+l_2+1}), \end{aligned}$$

as desired. ▼

We now note that an arbitrary element in $\operatorname{image}(\phi_{k,l})$ is a finite linear combination of elements of the form $f_A \otimes \alpha$ for $A \in \mathrm{S}^k(\mathsf{V}^*)$ and $\alpha \in \bigwedge^l(\mathsf{V}^*)$. By the lemma we can write $f_A \otimes \alpha = f_A \wedge \alpha$, thinking of f_A as an element of $\mathsf{F}_k[\mathsf{V}] \otimes \bigwedge^0(\mathsf{V}^*)$. Using the fact that $\mathrm{d}_{0,l}\alpha = 0$ by definition of $\mathrm{d}_{0,l}$, we have

$$\begin{aligned} \mathrm{d}_{k,l} \circ \phi_{k,l}(A \otimes \alpha)(x) \cdot (v_1, \dots, v_{l+1}) &= (\mathrm{d}_{0,0} f_A \wedge \alpha)(x)(v_1, \dots, v_{l+1}) \\ &= \sum_{\sigma} \operatorname{sign}(\sigma) \mathrm{d}_{0,0} f_A(x)(v_{\sigma(1)})\, \alpha(v_{\sigma(2)}, \dots, v_{\sigma(l+1)}), \end{aligned}$$

where the sum is over all permutations σ of $\{1, \dots, l+1\}$ which satisfy

$$\sigma(2) < \sigma(3) < \cdots < \sigma(l+1).$$

This amounts to

$$\begin{aligned} \mathrm{d}_{k,l} \circ \phi_{k,l}(A \otimes \alpha)(x) \cdot (v_1, \dots, v_{l+1}) &= \sum_{j=1}^{l+1} (-1)^{j+1} \mathrm{d}_{0,0} f_A(x)(v_j)\, \alpha(v_1, \dots, \hat{v}_j, \dots, v_{l+1}) \\ &= \sum_{j=1}^{l+1} (-1)^{j+1} k A(v_j, x, \dots, x)\alpha(v_1, \dots, \hat{v}_j, \dots, v_{l+1}), \end{aligned}$$

where $\hat{}$ means the term is omitted from the argument. Thus

$$\mathrm{d}_{k,l} \circ \phi_{k,l}(A \otimes \alpha)(x) \cdot (v_1, \dots, v_{l+1}) = \phi_{k-1,l+1} \circ \delta_{k,l}(A \otimes \alpha)(x)(v_1, \dots, v_{l+1}),$$

as desired. ■

It is now relatively straightforward to extend the preceding discussion from differential forms with coefficients being homogeneous polynomial functions to differential forms with coefficients being power series. We begin by denoting $\mathsf{F}[[\mathsf{V}]] = \prod_{k=0}^{\infty} \mathsf{F}_k[\mathsf{V}]$, which we call the ***formal power series*** on V. Intuitively, a formal power series consists of linear combinations of homogeneous polynomial functions of all orders, just like an infinite Taylor series, but we do not bother ourselves with convergence, the setting here being purely algebraic. In this case, $\mathsf{F}[[\mathsf{V}]] \otimes \bigwedge^l(\mathsf{V}^*)$ is to be thought of as differential l-forms with power series coefficients. The isomorphisms

$$\phi_{k,l} \colon \mathsf{S}^k(\mathsf{V}^*) \otimes \textstyle\bigwedge^l(\mathsf{V}^*) \to \mathsf{F}_k[\mathsf{V}] \otimes \textstyle\bigwedge^l(\mathsf{V}^*)$$

extend component-wise over the direct product to give isomorphisms

$$\phi_l \colon \prod_{k=0}^{\infty} \mathsf{S}^k(\mathsf{V}^*) \otimes \textstyle\bigwedge^l(\mathsf{V}^*) \to \mathsf{F}[[\mathsf{V}]] \otimes \textstyle\bigwedge^l(\mathsf{V}^*).$$

Moreover, the maps

$$\delta_{k,l} \colon \mathsf{S}^k(\mathsf{V}^*) \otimes \textstyle\bigwedge^l(\mathsf{V}^*) \to \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \textstyle\bigwedge^{l+1}(\mathsf{V}^*)$$

similarly extend component-wise to maps

$$\delta_l \colon \prod_{k=0}^{\infty} \mathsf{S}^k(\mathsf{V}^*) \otimes \textstyle\bigwedge^l(\mathsf{V}^*) \to \prod_{k=0}^{\infty} \mathsf{S}^k(\mathsf{V}^*) \otimes \textstyle\bigwedge^{l+1}(\mathsf{V}^*)$$

and so by composition to a map, which we denote by the same symbol,

$$\delta_l \colon \mathsf{F}[[\mathsf{V}]] \otimes \textstyle\bigwedge^l(\mathsf{V}^*) \to \mathsf{F}[[\mathsf{V}]] \otimes \textstyle\bigwedge^{l+1}(\mathsf{V}^*).$$

We may now state our main result in this section.

4.6.15 Theorem (Formal Poincaré Lemma) *For a field* F *and an* n*-dimensional* F*-vector space* V*, the sequence*

$$0 \longrightarrow \mathsf{F}[[\mathsf{V}]] \xrightarrow{\delta_0} \mathsf{F}[[\mathsf{V}]] \otimes \mathsf{V}^* \xrightarrow{\delta_1} \mathsf{F}[[\mathsf{V}]] \otimes \textstyle\bigwedge^2(\mathsf{V}^*) \xrightarrow{\delta_2} \cdots$$

$$\cdots \xrightarrow{\delta_{\mathrm{n}-1}} \mathsf{F}[[\mathsf{V}]] \otimes \textstyle\bigwedge^{\mathrm{n}}(\mathsf{V}^*) \longrightarrow 0$$

is exact.

Proof It suffices to show that the sequence

$$0 \longrightarrow \mathsf{S}^k(\mathsf{V}^*) \xrightarrow{\delta_{k,0}} \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \mathsf{V}^* \xrightarrow{\delta_{k-1,1}} \mathsf{S}^{k-2}(\mathsf{V}^*) \otimes \textstyle\bigwedge^2(\mathsf{V}^*) \xrightarrow{\delta_{k-2,2}} \cdots$$
$$\cdots \xrightarrow{\delta_{k-n+1,n-1}} \mathsf{S}^{k-n}(\mathsf{V}^*) \otimes \textstyle\bigwedge^n(\mathsf{V}^*) \longrightarrow 0$$

is exact for each $k \in \mathbb{Z}_{>0}$.

First of all, if $f \otimes \alpha \in \mathsf{F}_k \otimes \bigwedge^l(\mathsf{V}^*)$, we compute

$$\begin{aligned}\mathrm{d}_{k-1,l+1} \circ \mathrm{d}_{k,l}(f \otimes \alpha)(x)(v_1, \dots, v_{l+2}) \\ = \sum_{j=1}^{l+1} (-1)^{j+1} \sum_{i=1}^{l+2} (-1)^{i+1} A(v_i, v_j, x, \dots, x)\alpha(v_1, \dots, \hat{v}_{r(i,j)}, \dots, \hat{v}_{s(i,j)}, \dots, v_{l+2}),\end{aligned}$$

where

$$r(i,j) = \begin{cases} i, & i < j, \\ j, & i > j, \end{cases} \qquad s(i,j) = \begin{cases} j, & i < j, \\ i, & i > j. \end{cases}$$

Note that

$$A(v_i, v_j, x, \dots, x)\alpha(v_1, \dots, \hat{v}_{r(i,j)}, \dots, \hat{v}_{s(i,j)}, \dots, v_{l+2}) = 0$$

since A is symmetric and α is skew-symmetric. Thus we have $\mathrm{image}(\mathrm{d}_{k,l}) \subseteq \ker(\mathrm{d}_{k-1,l+1})$.

For each $k, l \in \mathbb{Z}_{>0}$ we shall define a map $H_{k,l} \colon \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*) \to \mathsf{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$ with the property that $H_{k-1,l+1} \circ \delta_{k-1,l+1} + \delta_{k,l} \circ H_{k,l}$ is the identity map on $\mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*)$. For $B \otimes \beta \in \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*)$ define

$$H_{k,l}(B \otimes \beta)(u, \dots, u, v_1, \dots, v_l) = \frac{1}{k+l} B(u, \dots, u)\beta(u, v_1, \dots, v_l),$$

for $u, v_1, \dots, v_l \in \mathsf{V}$, noting that this uniquely defines $H_{k,l}(B \otimes \beta B) \in \mathsf{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$ by Proposition 4.4.7. Using the definition of $\delta_{k,l}$ we compute

$$\begin{aligned}\delta_{k,l} \circ H_{k,l}(B \otimes \beta)(u, \dots, u, v_1, \dots, v_{l+1}) \\ = \frac{1}{k+l}\Big(\sum_{j=1}^{l+1} (-1)^{j+1}(k-1)B(v_j, u, \dots, u)\beta(u, v_1, \dots, \hat{v}_j, \dots, v_{l+1}) \\ + \sum_{j=1}^{l+1} (-1)^{j+1} B(u, \dots, u)\beta(v_j, v_1, \dots, \hat{v}_j, \dots, v_{l+1})\Big),\end{aligned} \tag{4.32}$$

for $u, v_1, \dots, v_{l+1} \in \mathsf{V}$. Using the definition of $\delta_{k-1,l+1}$ we have

$$\begin{aligned}\delta_{k-1,l+1}(B \otimes \beta)(u, \dots, u, v_1, \dots, v_{l+2}) \\ = \sum_{j=1}^{l+2} (-1)^{j+1}(k-1)B(v_j, u, \dots, u)\beta(v_1, \dots, \hat{v}_j, \dots, v_{l+2})\end{aligned}$$

for $u, v_1, \dots, v_{l+2} \in \mathsf{V}$. Therefore, using the definition of $H_{k-1,l+1}$,

$$\begin{aligned} H_{k-1,l+1} \circ \delta_{k-1,l+1}(B \otimes \beta)(u, \dots, u, v_1, \dots, v_{l+1}) \\ = \frac{1}{k+l}\Big((k-1)B(u,\dots,u)\beta(v_1,\dots,v_{l+1}) \\ + \sum_{j=1}^{l+1}(-1)^j(k-1)B(v_j,u,\dots,u)\beta(u,v_1,\dots,\hat{v}_j,\dots,v_{l+1})\Big) \end{aligned} \tag{4.33}$$

for $u, v_1, \dots, v_{l+1} \in \mathsf{V}$. Combining (4.32) and (4.33) we arrive at

$$\begin{aligned} (H_{k-1,l+1} \circ \delta_{k-1,l+1} + \delta_{k,l} \circ H_{k,l})(B \otimes \beta)(v_1, \dots, v_{l+1}, u, \dots, u) \\ = \frac{1}{k+l}\Big((k-1)B(u,\dots,u)\beta(v_1,\dots,v_{k+1}) \\ + \sum_{j=1}^{k+1}(-1)^{j+1}B(u,\dots,u)\beta(v_j,v_1,\dots,\hat{v}_j,v_{l+1})\Big) \\ = \frac{1}{k+l}\Big((k-1)B(u,\dots,u)\beta(v_1,\dots,v_{l+1}) \\ + (l+1)B(u,\dots,u)\beta(v_1,\dots,v_{l+1})\Big) \\ = B(u,\dots,u)\beta(v_1,\dots,v_{l+1}) \\ = B \otimes \beta(u,\dots,u,v_1,\dots,v_{l+1}) \end{aligned}$$

for $u, v_1, \dots, v_{l+1} \in \mathsf{V}$. By extending the above computations using linearity and by using Proposition 4.4.7, it follows that $H_{k-1,l+1} \circ \delta_{k-1,l+1} + \delta_{k,l} \circ H_{k,l}$ is the identity on $\mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*)$.

Now, if $\delta_{k-1,l+1}(B_1 \otimes \beta_1 + \dots + B_m \otimes \beta_m) = 0$ for $\beta_j \otimes B_j \in \mathsf{S}^{k-1}(\mathsf{V}^*) \otimes \bigwedge^{l+1}(\mathsf{V}^*)$, $j \in \{1, \dots, m\}$, then we define $A_j \otimes \alpha_j \in \mathsf{S}^k(\mathsf{V}^*) \otimes \bigwedge^l(\mathsf{V}^*)$ by $A_j \otimes \alpha_j = H_{s,r}(B_j \otimes \beta_j)$, $j \in \{1, \dots, m\}$. Then

$$\begin{aligned} \delta_{k,l}\Big(\sum_{j=1}^m A_j \otimes \alpha_j\Big) &= \delta_{k,l} \circ H_{k,l}\Big(\sum_{j=1}^m B_j \otimes \beta_j\Big) \\ &= (\delta_{k,l} \circ H_{k,l} + H_{k-1,l+1} \circ \delta_{k-1,l+1})\Big(\sum_{j=1}^m B_j \otimes \beta_j\Big) = \sum_{j=1}^m B_j \otimes \beta_j, \end{aligned}$$

showing that $\operatorname{image}(\delta_{k,l}) = \ker(\delta_{k-1,l+1})$ as desired. ■

The preceding algebraic constructions are certainly entertaining. But they also give rise to the following important corollaries.

4.6.16 Corollary (The Poincaré Lemma in the real analytic case) *Let* M *be a manifold of class* C^ω. *If* $\alpha \in \Gamma^\omega(\bigwedge^1(\mathsf{T}^*\mathsf{M}))$ *is closed, then, for each* $x \in \mathsf{M}$, *there exists a neighbourhood* $\mathcal{U}$ *of* x *and* $\beta \in \Gamma^\omega(\bigwedge^{1-1}(\mathsf{T}^*\mathcal{U}))$ *such that* $\alpha|\mathcal{U} = \mathrm{d}\beta$.

Proof Since the result is local, we let M be a neighbourhood of $\mathbf{0} \in \mathbb{R}^n$. Since α is real analytic, in a neighbourhood of $\mathbf{0}$ we have the Taylor series expansion that we write in the general form

$$\alpha(\boldsymbol{x}) = \sum_{k=0}^{\infty}\sum_{r=1}^{m_k} \alpha_{kr}(\boldsymbol{x})\omega_{kr},$$

where α_{kr} are homogeneous polynomial functions of degree k for each $r \in \{1, \dots, m_k\}$ and where ω_{kr} are differential l-forms with constant coefficients, e.g., wedge products of the coordinate one-forms. Using the mappings $H_{k,l}$ from the proof of Theorem 4.6.15, define

$$H_l(\alpha) = \sum_{k=0}^{\infty}\sum_{r=1}^{m_r} H_{k,l}(\alpha_{kr} \otimes \omega_{kr}).$$

A moment's consideration of the definition of $H_{k,l}$ leads one to the formula

$$H_l(\alpha)(\boldsymbol{x})(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{l-1}) = \int_0^1 t^{l-1}\beta(t\boldsymbol{x})(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{l-1}),$$

which holds for $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{l-1} \in \mathbb{R}^n$. From this formula we see that $H_l(\alpha)$ is analytic. In the proof of Theorem 4.6.15 we showed that $H_l(\alpha)$ satisfies $\alpha = \mathrm{d}H_l(\alpha)$ in a neighbourhood of $\boldsymbol{0}$, and this gives the result. ■

Thus, for each $x \in \mathsf{M}$, we have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{C}^{\omega}_{x,\mathsf{M}} \xrightarrow{\mathrm{d}} \mathscr{G}^{\omega}_{x,\bigwedge^1(\mathsf{T}^*\mathsf{M})} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathscr{G}^{\omega}_{x,\bigwedge^n(\mathsf{T}^*\mathsf{M})} \longrightarrow 0 \qquad (4.34)$$

of $\mathbb{R}$-vector spaces, where by "$\mathbb{R}$" we mean the germs of functions that are constant in a neighbourhood of x (this will be made more clear and put into some context when we talk about the constant sheaf and its sheafification in Section GA2.1.1). In making this translation, we are tacitly using Proposition 5.6.6 below, where the (more or less obvious) correspondence between germs at a point and power series is established.

4.6.17 Corollary (The Poincaré Lemma in the holomorphic case) *Let* M *be a holomorphic manifold. If* $\alpha \in \Gamma^{\mathrm{hol}}(\bigwedge^1(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *is* ∂*-closed, i.e.,* $\partial\alpha = 0$*, then, for each* $z \in \mathsf{M}$*, there exists a neighbourhood* $\mathcal{U}$ *of* z *and* $\beta \in \Gamma^{\mathrm{hol}}(\bigwedge^{1-1}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ *such that* $\alpha|\mathcal{U} = \partial\beta$*.*

Proof The proof of Corollary 4.6.16 applies, replacing $\mathbb{R}$ with $\mathbb{C}$. ■

Thus, for each $z \in \mathsf{M}$, we have an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{C}^{\mathrm{hol}}_{z,\mathsf{M}} \xrightarrow{\partial} \mathscr{G}^{\mathrm{hol}}_{z,\bigwedge^1(\mathsf{T}^*\mathsf{M})} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathscr{G}^{\mathrm{hol}}_{z,\bigwedge^n(\mathsf{T}^*\mathsf{M})} \longrightarrow 0 \qquad (4.35)$$

of $\mathbb{C}$-vector spaces, where by "$\mathbb{C}$" we mean the germs of functions that are constant in a neighbourhood of x (this will be made more clear and put into some context when we talk about the constant sheaf and its sheafification in Section GA2.1.1).

4.6.18 Remark (Connection to the usual proof of the Poincaré Lemma) We can see from the proof of Corollary 4.6.16 that our proof of the Formal Poincaré Lemma is an adaptation to the algebraic setting of one of the usual proofs of the lemma involving the construction of a homotopy operator; see the proof of Theorem 4.6.3 above. •

4.7 Connections in vector bundles

In Section 4.3 we introduced the basic constructions associated with vector bundles. In this section we introduce one of the important constructions with vector bundles, namely connections in vector bundles. This is a venerable subject in differential geometry, and we do not attempt to do it anything like full justice here. Rather, we just provide the initial definitions and most elementary results. Readers interested in a deeper treatment of the theory of connections can refer to the two volumes of Kobayashi and Nomizu [1963].

As always, when we are talking about objects of class C^{hol}, we assume all relevant operations are over $\mathbb{C}$. In order to allow us to simultaneously consider the real and complex cases, we shall denote by $\mathsf{T}\mathsf{M}$ the holomorphic tangent bundle of a holomorphic manifold M and by $\mathsf{T}^*\mathsf{M}$ the dual bundle. That is, if M is a holomorphic manifold, we denote $\mathsf{T}\mathsf{M} = \mathsf{T}^{1,0}\mathsf{M}$ and $\mathsf{T}^*\mathsf{M} = \bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$.

4.7.1 The vertical lift for a vector bundle

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a $\mathbb{F}$-vector bundle of class C^r. The ***vertical subbundle*** of TE is $\mathsf{VE} = \ker(T\pi)$. Denote $\nu_{\mathsf{E}} = \pi_{\mathsf{TE}}|\mathsf{VE}$. Note that the fibres of $\nu_{\mathsf{E}}\colon \mathsf{VE} \to \mathsf{E}$ are the tangent spaces to the fibres of $\pi\colon \mathsf{E} \to \mathsf{M}$ thought of as $\mathbb{F}$-submanifolds of E. Since these fibres are vector spaces, this implies that the fibre $\mathsf{V}_{e_x}\mathsf{E}$ is naturally $\mathbb{F}$-isomorphic to E_x. We shall talk about curves in the fibre for both the real and complex settings. Specifically, for $e_x, f_x \in \mathsf{E}_x$ we have the curve

$$\mathbb{F} \ni t \mapsto e_x + t f_x \in \mathsf{E}_x.$$

The $\mathbb{F}$-derivative of this map at $t = 0$ is an element of $\mathsf{E}_x \simeq \mathsf{V}_{e_x}\mathsf{E}$. With this in mind, we define the ***vertical lift*** of f_x at e_x by

$$\mathrm{vlft}_{e_x}(f_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(e_x + t f_x) \in \mathsf{V}_{e_x}\mathsf{E}.$$

The following result gives a useful characterisation of the vertical lift.

4.7.1 Proposition (Characterisation of vlft) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. For an* $\mathbb{F}$*-vector bundle* $\pi\colon \mathsf{E} \to \mathsf{M}$ *of class* C^{r}*, the map* $\mathrm{vlft}\colon \pi^*\mathsf{E} \to \mathsf{VE}$

defined by $\mathrm{vlft}(e_x, f_x) = \mathrm{vlft}_{e_x}(f_x)$ *is an isomorphism of* $\mathbb{F}$*-vector bundles for which the diagram*

$$\begin{array}{ccc} \pi^*\mathsf{E} & \xrightarrow{\mathrm{vlft}} & \mathsf{VE} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \nu_{\mathsf{E}}} \\ \mathsf{E} & = & \mathsf{E} \end{array}$$

commutes, where $\mathrm{pr}_1\colon \pi^*\mathsf{E} \to \mathsf{E}$ *is the restriction of the projection from* $\mathsf{E} \times \mathsf{E}$ *onto the first factor to the submanifold* $\pi^*\mathsf{E} \subseteq \mathsf{E} \times \mathsf{E}$*.*

Proof Let $e_x \in \mathsf{E}_x$. The restriction of the map vlft in the statement of the result to the fibre of the vector bundle $\mathrm{pr}_1\colon \pi^*\mathsf{E} \to \mathsf{E}$ over e_x is given by $w_x \mapsto \mathrm{vlft}_{v_x}(w_x)$. We claim that this map is surjective. Indeed, let $X_{e_x} \in \mathsf{V}_{e_x}\mathsf{E}$. From our discussion preceding the statement of the result, there exists $f_x \in \mathsf{E}_x$ corresponding to X_{e_x} under the natural $\mathbb{F}$-isomorphism $\mathsf{V}_{e_x}\mathsf{E} \simeq \mathsf{E}_x$. Moreover, by definition of vlft, $\mathrm{vlft}_{e_x}(f_x) = X_{e_x}$, giving the desired surjectivity. Since the fibres in the top row of the diagram have the same dimension, the map vlft is a fibrewise $\mathbb{F}$-isomorphism. Moreover, the diagram is readily checked to commute, and so the result follows. ■

The preceding constructions can be further refined. Let us also denote

$$\mathsf{E} \times_\pi \mathsf{E} = \{(e, e') \in \mathsf{E} \times \mathsf{E} \mid \pi(e) = \pi(e')\}.$$

As a set, $\mathsf{E} \times_\pi \mathsf{E}$ is the same as $\pi^*\mathsf{E}$. However, we wish to endow it with the structure of a fibre bundle over M by the projection $\pi \times_\pi \pi\colon \mathsf{E} \times_\pi \mathsf{E} \to \mathsf{M}$ defined by $\pi \times_\pi \pi(e, e') = \pi(e)$. This fibre bundle is, in fact, an $\mathbb{F}$-vector bundle, and the vector bundle operations are given by

$$(e_x, f_x) + (e'_x, f'_x) = (e_x + e'_x, f_x + f'_x), \quad a(e_x, f_x) = (ae_x, af_x).$$

Given the vector bundle structure, let us denote $\mathsf{E} \oplus \mathsf{E} = \mathsf{E} \times_\pi \mathsf{E}$ and $\pi \oplus \pi = \pi \times_\pi \pi$. This vector bundle is known as the ***Whitney sum*** of E with itself.

Let us also denote by $\nu_{\mathsf{M}}\colon \mathsf{VE} \to \mathsf{M}$ the composition of the projections $\nu_{\mathsf{E}}\colon \mathsf{VE} \to \mathsf{E}$ and $\pi\colon \mathsf{E} \to \mathsf{M}$. Note that the map $\mathrm{vlft}\colon \mathsf{E} \oplus \mathsf{E} \to \mathsf{VE}$ defined by $\mathrm{vlft}(e_x, f_x) = \mathrm{vlft}_{e_x}(f_x)$ is an $\mathbb{F}$-diffeomorphism (by Proposition 4.7.1) and that the diagram

$$\begin{array}{ccc} \mathsf{E} \oplus \mathsf{E} & \xrightarrow{\mathrm{vlft}} & \mathsf{VE} \\ {\scriptstyle \pi\oplus\pi}\downarrow & & \downarrow{\scriptstyle \nu_{\mathsf{M}}} \\ \mathsf{M} & = & \mathsf{M} \end{array}$$

commutes. Therefore, the map vlft induces the vector bundle structure of $\mathsf{E} \oplus \mathsf{E}$ on VE, rendering the latter bundle a vector bundle over M in a natural way.

4.7.2 Linear vector fields

In this section we develop the notion of a linear vector field on a vector bundle.

4.7.2 Definition (Linear vector field) Let $r \in \{\infty, \omega, \text{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \text{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an $\mathbb{F}$-vector bundle of class C^r. A vector field $X \in \Gamma^r(\mathsf{TE})$ of class C^r is ***linear*** if

(i) X is π-projectable, i.e., there exists a vector field $\pi X \in \Gamma^r(\mathsf{TM})$ such that $T_{e_x}\pi(X(e_x)) = \pi X(x)$ for every $x \in \mathsf{M}$ and $e_x \in \mathsf{E}_x$, and

(ii) X is a vector bundle mapping for which the diagram

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{X} & \mathsf{TE} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ \mathsf{M} & \xrightarrow[\pi X]{} & \mathsf{TM} \end{array}$$

commutes. •

Let us now provide a flow interpretation for linear vector fields. To do so, we note that if X is a vector field of class C^r, $r \in \{\infty, \omega, \text{hol}\}$, then the notion of integral curves extends from the usual notion for $r \in \{\infty, \omega\}$ to a similar notion for $r = \text{hol}$ [Ilyashenko and Yakovenko 2008]. Thus we denote by t a point in $\mathbb{F}$ and by I a connected open subset of $\mathbb{F}$ containing 0. An ***integral curve*** through $x \in \mathsf{M}$ is then a map $\gamma\colon I \to \mathsf{M}$ of class C^r with the property that $\gamma'(t) = X(\gamma(t))$ for every $t \in I$, where $\gamma'(t)$ denotes the $\mathbb{F}$-derivative of γ, i.e., $\gamma'(t) = T_t^{\mathbb{C}}\gamma \cdot 1$. We then use the usual notation Φ_t^X to denote the flow of X in both the real and complex settings. Thus $t \mapsto \Phi_t^X(x)$ is the integral curve of X through $x \in \mathsf{M}$. By $\mathrm{Dom}(X)$ we denote the set of points in $\mathbb{F} \times \mathsf{M}$ for which $\Phi_t^X(x)$ makes sense.

The following property of linear vector fields explains their importance.

4.7.3 Proposition (Flows of linear vector fields) *Let* $\mathrm{r} \in \{\infty, \omega, \text{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \text{hol}$. *Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be an* $\mathbb{F}$*-vector bundle of class* C^{r}. *If* $\mathrm{X} \in \Gamma^{\mathrm{r}}(\mathsf{TE})$ *is a linear vector field projecting to* $\pi\mathrm{X}$, *then* $\Phi_{\mathrm{t}}^{\mathrm{X}}|\mathsf{E}_{\mathrm{x}}\colon \mathsf{E}_{\mathrm{x}} \to \mathsf{E}_{\Phi_{\mathrm{t}}^{\pi\mathrm{X}}(\mathrm{x})}$ *is an isomorphism of* $\mathbb{F}$*-vector spaces for every* $(\mathrm{t}, \mathrm{x}) \in \mathrm{Dom}(\pi\mathrm{X})$.

Moreover, let $\mathrm{I} \subseteq \mathbb{F}$ *be a connected open set, let* $\mathcal{U} \subseteq \mathsf{M}$ *be an open set, and let* $\Phi\colon \mathrm{I} \times \mathsf{E}|\mathcal{U} \to \mathsf{E}$ *be a* C^{r}*-map with the following properties:*

(i) the map $\mathrm{e} \mapsto \Phi(\mathrm{t}, \mathrm{e})$ *is a* C^{r}*-diffeomorphism onto its image for every* $\mathrm{t} \in \mathrm{I}$*;*

(ii) $\Phi(0, \mathrm{e}) = \mathrm{e}$ *for every* $\mathrm{e} \in \mathsf{E}|\mathcal{U}$*;*

(iii) $\Phi(\mathrm{s} + \mathrm{t}, \mathrm{e}) = \Phi(\mathrm{s}, \Phi(\mathrm{t}, \mathrm{e}))$ *for every* $\mathrm{s}, \mathrm{t} \in \mathbb{F}$ *such that* $\mathrm{s}, \mathrm{t}, \mathrm{s} + \mathrm{t} \in \mathrm{I}$ *and* $\mathrm{e} \in \mathsf{E}|\mathcal{U}$*;*

(iv) for each $\mathrm{t} \in \mathrm{I}$, *the map* $\mathrm{e} \mapsto \Phi(\mathrm{t}, \mathrm{e})$, *when restricted to the fibre* E_{x}, *is an* $\mathbb{F}$*-isomorphism onto the fibre* E_{y}, *where* $\mathrm{y} = \pi(\Phi(\mathrm{t}, \mathrm{e}))$ *for some* $\mathrm{e} \in \mathsf{E}_{\mathrm{x}}$.

Then the vector field $\mathrm{X} \in \Gamma^{\mathrm{r}}(\mathsf{E}|\mathcal{U})$ *defined by* $\mathrm{X}(\mathrm{e}) = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{\mathrm{t}=0}\Phi(\mathrm{t}, \mathrm{e})$ *is a linear vector field on* $\mathsf{E}|\mathcal{U}$.

Proof By [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.4] (and its easily derived complex analogue) we have $\pi \circ \Phi_t^X(e) = \Phi_t^{\pi X} \circ \pi(e)$ for $(t, e) \in \mathrm{Dom}(X)$. Thus $\Phi_t^X(\mathsf{E}_x) \subseteq \mathsf{E}_{\Phi_t^{\pi X}(x)}$. We claim that $\Phi_t^X(\mathsf{E}_x) = \mathsf{E}_{\Phi_t^{\pi X}(x)}$. Indeed, let $f \in \mathsf{E}_{\Phi_t^{\pi X}(x)}$. Then $\Phi_{-t}^X(f) \in \mathsf{E}_x$ and so $\Phi_t^X(\Phi_{-t}^X(f)) = f$, giving $\Phi_t^X(\mathsf{E}_x) = \mathsf{E}_{\Phi_t^{\pi X}(x)}$, as claimed. Moreover, since $\Phi_t^X|\mathsf{E}_x$ is a local $\mathbb{F}$-diffeomorphism, we have that it is a surjective local $\mathbb{F}$-diffeomorphism, and so a covering map. Since the fibres of E are simply connected, $\Phi^X_t|\mathsf{E}_x$ is a $\mathbb{F}$-diffeomorphism [Lee 2004, Corollary 11.24]. It remains to show that $\Phi^X_t|\mathsf{E}_x$ is an $\mathbb{F}$-isomorphism. To prove this, let $e_x, f_x \in \mathsf{E}_x$ and let $t \mapsto \Phi^X_t(e_x)$ and $t \mapsto \Phi^X_t(f_x)$ be the corresponding integral curves. By our arguments just above, for each t for which the flow is defined, the expression $\Phi^X_t(e_x)+\Phi^X_t(f_x)$ makes sense since the summands are in the same fibre. Moreover, since X is π-projectable and since $\pi(e_x) = \pi(f_x)$,

$$T_{e_x}\pi(X(e_x)) = T_{f_x}\pi(X(f_x)) = \pi X(x).$$

Therefore, since X is a vector bundle mapping over πX,

$$X(\Phi^X_t(e_x)) + X(\Phi^X_t(f_x)) = X(\Phi^X_t(e_x) + \Phi^X_t(f_x)).$$

Thus we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^X_t(e_x) + \Phi^X_t(f_x)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi^X_t(e_x) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi^X_t(f_x) = X(\Phi^X_t(e_x)) + X(\Phi^X_t(f_x))\\ &= X(\Phi^X_t(e_x) + \Phi^X_t(f_x)),\end{aligned}$$

and we also have $\Phi^X_0(e_x) + \Phi^X_0(f_x) = e_x + f_x$. Thus $t \mapsto \Phi^X_t(e_x) + \Phi^X_t(f_x)$ is the integral curve of X through $e_x + f_x$. One similarly shows that $t \mapsto a\Phi^X_t(e_x)$ is the integral curve through ae_x. In particular, this gives the desired linearity of $\Phi^X_t|\mathsf{E}_x$.

For the second assertion of the proposition, let $\Phi\colon I \times \mathsf{E}|\mathcal{U} \to \mathsf{E}$ have the stated properties. By (iv) it follows that there exists a map $\Phi_0\colon I \times \mathcal{U} \to \mathsf{M}$ such that

1. the map $x \mapsto \Phi_0(t, x)$ is a $\mathbb{F}$-diffeomorphism onto its image for every $t \in I$;
2. $\Phi_0(0, x) = x$ for every $x \in \mathcal{U}$;
3. $\Phi_0(s + t, x) = \Phi_0(s, \Phi_0(t, x))$ for every $s, t \in \mathbb{F}$ such that $s, t, s + t \in I$ and $x \in \mathcal{U}$.

We then define $X_0(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi_0(0, x)$, and we note that $\Phi^{X_0}_t(x) = \Phi_0(t, x)$. Since $\pi \circ \Phi^X_t(e) = \Phi^{X_0}_t \circ \pi(e)$, it follows from [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.4] that X_0 is π-related to X. It remains to show that X is a vector bundle mapping over X_0. Let $e_x, f_x \in \mathsf{E}_x$ and compute

$$\begin{aligned}X(e_x + f_x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi^X_t(e_x + f_x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\Phi^X_t(e_x) + \Phi^X_t(f_x))\\ &= X(\Phi^X_0(e_x)) + X(\Phi^X_0(f_x)) = X(e_x) + X(f_x),\end{aligned}$$

using the hypothesised linearity of $\Phi^X_t|\mathsf{E}_x$. ■

The way to read the preceding result is as follows: A vector field on a vector bundle is a linear vector field if and only if its flow is comprised of local $\mathbb{F}$-vector bundle isomorphisms.

Let us give the coordinate form for a linear vector bundle.

4.7.4 Proposition (Coordinate form for linear vector fields) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be an* $\mathbb{F}$*-vector bundle of class* C^{r}. *Let* $(\mathcal{V}, \psi)$ *be a* $\mathbb{F}$*-vector bundle chart for* E *with* $(\mathcal{U}, \phi)$ *the corresponding* $\mathbb{F}$*-chart for* M.

Denote the coordinates in the chart by $(\mathrm{x}^j, \mathrm{v}^a)$ *where* $j \in \{1, \dots, n\}$ *and* $a \in \{1, \dots, m\}$. *If* X *is a linear vector field of class* C^r, *then in coordinates we have*

$$\mathrm{X} = \sum_{j=1}^{n} \mathrm{X}_0^j \frac{\partial}{\partial \mathrm{x}^j} + \sum_{a,b=1}^{m} \mathrm{A}_a^b \mathrm{v}^a \frac{\partial}{\partial \mathrm{v}^b},$$

for C^r*-functions* X_0^j, $j \in \{1, \dots, n\}$, *and* A_a^b, $a, b \in \{1, \dots, m\}$, *on* $\mathcal{U}$.

Proof A general vector field X on E is given in coordinates by

$$X = \sum_{j=1}^{n} X_0^j \frac{\partial}{\partial x^j} + \sum_{a=1}^{m} X_1^a \frac{\partial}{\partial v^a},$$

for functions X_0^j, $j \in \{1, \dots, n\}$, and X_1^a, $a \in \{1, \dots, m\}$, on $\mathcal{V}$. Note that

$$T\pi(X) = \sum_{j=1}^{n} X_0^j \frac{\partial}{\partial x^j}.$$

Since X is π-projectable, the functions $X_0^1, \dots, X_0^n$ do not depend on the fibre coordinates $v^1, \dots, v^m$. Thus these are the components of the projected vector field πX. Since X is a vector bundle mapping over πX, the functions $X_1^1, \dots, X_1^m$ must be linear functions of the fibre coordinates $v^1, \dots, v^m$, and from this the result follows. ■

4.7.3 Connections in a few different ways

The notion of a connection is often useful in making global constructions on manifolds. In this section we provide a few characterisations of connections in vector bundles.

The following is thus our initial definition.

4.7.5 Definition (Connection in a vector bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an $\mathbb{F}$-vector bundle of class C^r. A $\mathbf{C^r}$***-connection*** in E is an assignment to a vector field $X \in \Gamma^r(\mathsf{TM})$ and a section $\xi \in \Gamma^r(\mathsf{E})$ a section $\nabla_X \xi \in \Gamma^r(\mathsf{E})$, and the assignment has the following properties:

(i) the map $(X, \xi) \mapsto \nabla_X \xi$ is $\mathbb{F}$-bilinear;

(ii) $\nabla_{fX} \xi = f \nabla_X \xi$ for every $X \in \Gamma^r(\mathsf{TM})$, $\xi \in \Gamma^r(\mathsf{E})$, and $f \in C^r(\mathsf{M})$;

(iii) $\nabla_X (f \xi) = f \nabla_X \xi + (Xf)\xi$ for every $X \in \Gamma^r(\mathsf{TM})$, $\xi \in \Gamma^r(\mathsf{E})$, and $f \in C^r(\mathsf{M})$.

We call $\nabla_X \xi$ the ***covariant derivative*** of ξ with respect to X. •

Let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart for $\pi\colon \mathsf{E} \to \mathsf{M}$ with $(\mathcal{U}, \phi)$ the associated $\mathbb{F}$-chart for M. Let $(e_1, \dots, e_m)$ be the corresponding basis of sections for $\mathsf{E}|\mathcal{U}$. Then we define C^r-functions $\Gamma_{ja}^b\colon \mathcal{U} \to \mathbb{R}$, $a, b \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, called the ***connection coefficients***, by

$$\nabla_{\frac{\partial}{\partial x^j}} e_a = \sum_{b=1}^{m} \Gamma_{ja}^b e_b.$$

Using the defining properties of connections, one directly verifies that, if $X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}$ and $\xi = \sum_{a=1}^{m} \xi^a e_a$, then

$$\nabla_X \xi = \sum_{j=1}^{m} \sum_{b=1}^{m} \Big(\frac{\partial \xi^b}{\partial x^j} + \sum_{a=1}^{m} \Gamma^b_{ja} \xi^a \Big) X^j e_b.$$

Thus the connection coefficients determine the connection in coordinates.

If ∇ is a C^r-connection on E and if $\xi \in \Gamma^r(\mathsf{E})$, then we define a section $\nabla\xi \in \Gamma^r(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E})$ by $\nabla\xi(X) = \nabla_X \xi$. This definition makes sense by virtue of the second of the defining properties of a connection, which ensures that the map $X \mapsto \nabla_X \xi$ is tensorial.

If $\xi \colon \mathsf{M} \to \mathsf{E}$ is a section on E, we denote by $\xi^*\mathsf{VE}$ the pull-back of the vertical bundle to M by ξ. Note that if $\eta \colon \mathsf{M} \to \xi^*\mathsf{VE}$ is a section of this pull-back bundle, then $\eta(x) \in \mathsf{V}_{\xi(x)}\mathsf{E} \simeq \mathsf{E}_x$. Thus sections of $\xi^*\mathsf{VE}$ are naturally thought of as simply sections. We shall make this identification implicitly below.

Next we make some constructions involving jet bundles. We will formally discuss jet bundles for vector bundles in Section 5.5. Let us provide, associated with a connection ∇ on $\pi \colon \mathsf{E} \to \mathsf{M}$, a section of the bundle $\pi^1_0 \colon \mathsf{J}^1\mathsf{E} \to \mathsf{E}$.

4.7.6 Lemma (Connections and sections of jet bundles) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* ∇ *be a* C^{r}*-connection on a* $\mathbb{F}$*-vector bundle* $\pi \colon \mathsf{E} \to \mathsf{M}$ *of class* C^{r}. *Then there exists a unique* C^{r}*-section* $\mathrm{S}_\nabla \colon \mathsf{E} \to \mathsf{J}^1\mathsf{E}$ *satisfying*

$$\nabla\xi(\mathrm{x}) = \mathrm{j}_1\xi(\mathrm{x}) - \mathrm{S}_\nabla(\xi(\mathrm{x})),$$

for every section $\xi \in \Gamma^{\mathrm{r}}(\mathsf{E})$, *noting that* $\mathrm{j}_1\xi(\mathrm{x}) - \mathrm{S}_\nabla(\xi(\mathrm{x})) \in \mathsf{T}^*_{\mathrm{x}}\mathsf{M} \otimes \mathsf{V}_{\xi(\mathrm{x})}\mathsf{E}$ *and that* $\mathsf{V}_{\xi(\mathrm{x})}\mathsf{E} \simeq \mathsf{E}_{\mathrm{x}}$.

Proof The existence of the section S_∇ is simply a matter of defining $S_\nabla(e_x) = j_1\xi(x) - \nabla\xi(x)$, where ξ is any section satisfying $\xi(x) = e_x$. We should verify that this definition is well-defined. Let $f \in \mathrm{C}^r(\mathsf{M})$ be such that $\mathrm{d}f(x) = 0$ (here, d stands for "d" if $\mathbb{F} = \mathbb{R}$ and "$\mathrm{d}_{\mathbb{C}}$" if $\mathbb{F} = \mathbb{C}$. Then, for any section ξ,

$$\nabla f\xi(x) = f(x)\nabla\xi(x) + \mathrm{d}f(x) \otimes \xi(x) = \mathrm{d}f(x) \otimes \xi(x). \tag{4.36}$$

Similarly,

$$j_1(f\xi)(x) = f(x) j_1\xi(x) + \mathrm{d}f(x) \otimes \xi(x) = \mathrm{d}f(x) \otimes \xi(x). \tag{4.37}$$

Thus, if ζ is a section vanishing at x, since $\Gamma^r(\mathsf{E})$ is a locally free module (cf. the detailed discussion in), in a neighbourhood of x we can write

what?

$$\zeta = f_1\xi_1 + \cdots + f_n\xi_n$$

for linearly independent (over $\mathrm{C}^r(\mathsf{M})$) sections $\xi_1, \dots, \xi_n$ and for functions $f_1, \dots, f_n$ which vanish at x. By (4.36) and (4.37) we then have $j_1\zeta(x) = \nabla\zeta(x)$. Now let ξ' be a section such that $\xi'(x) = \xi(x) = e_x$ and compute

$$(j_1\xi(x) - \nabla\xi(x)) - (j_1\xi'(x) - \nabla\xi'(x)) = j_1(\xi - \xi')(x) - \nabla(\xi - \xi')(x).$$

Note that the section $\zeta = \xi - \xi'$ vanishes at x, and so our computations above give $j_1\zeta(x) = \nabla\zeta(x)$, giving well-definedness of S_∇.

It is clear that S_∇ is uniquely determined from ∇ by the condition in the statement of the result. ■

We now use the map S_∇ as the basis for our further constructions. Let $e_1 \in \mathsf{J}^1\mathsf{E}$, let $e = \pi^1_0(e_1)$, and let $x = \pi(e)$. Let $\xi\colon \mathsf{M} \to \mathsf{E}$ be a section satisfying $j_1\xi(x) = v_1$. Define $L_{e_1} \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{T}_x\mathsf{M}; \mathsf{T}_e\mathsf{E})$ by $L_{e_1}(v_x) = T_x\xi(v_x)$.

4.7.7 Lemma (Properties of $\mathbf{L_{e_1}}$) *Let* $e_1 \in \mathsf{J}^1\mathsf{E}$*, let* $e = \pi^1_0(e_1)$*, and let* $x = \pi(e)$*. The following statements hold:*

(i) L_{e_1} *is a well-defined linear injection;*

(ii) $T_e\pi \circ L_{e_1} = \mathrm{id}_{\mathsf{T}_x\mathsf{M}}$*;*

(iii) $\mathrm{image}(L_{e_1})$ *is a complement to* $\mathsf{V}_e\mathsf{E}$ *in* $\mathsf{T}_e\mathsf{E}$*.*

Moreover, if $e \in \mathsf{E}$ *and* $x = \pi(e)$*, and if* $L\colon \mathsf{T}_x\mathsf{M} \to \mathsf{T}_e\mathsf{E}$ *is a linear map satisfying* $T_e\pi \circ L = \mathrm{id}_{\mathsf{T}_x\mathsf{M}}$*, then there exists a unique* $e_1 \in (\pi^1_0)^{-1}(e)$ *such that* $L = L_{e_1}$*.*

Proof (i) Suppose that $\xi_1, \xi_2\colon \mathsf{M} \to \mathsf{E}$ satisfy $j_1\xi_1(x) = j_1\xi_2(x) = e_1$. This means that, for any curve $\gamma\colon I \to \mathsf{M}$ of class C^r satisfying $0 \in \mathrm{int}(I)$ (remember that I is an open subset of $\mathbb{F}$) and $\gamma(0) = x$ it holds that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi_1 \circ \gamma)(s) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi_2 \circ \gamma)(s)$. This immediately gives $T_x\xi_1(v_x) = T_x\xi_2(v_x)$ for every $v_x \in \mathsf{T}_x\mathsf{M}$. Thus the definition of L_{e_1} is independent of the choice of local section ξ. Linearity of L_{e_1} is now obvious. To see that L_{e_1} is injective note that $\pi \circ \xi = \mathrm{id}_\mathsf{M}$ and so $T_{\xi(x)}\pi \circ T_x\xi = \mathrm{id}_{\mathsf{T}_x\mathsf{M}}$. Thus L_{e_1} possesses a left-inverse and so is injective.

(ii) This was proved as part of the proof of the previous assertion.

(iii) Suppose that $u_e \in \mathrm{image}(L_{e_1}) \cap \mathsf{V}_e\mathsf{E}$. Let $u_e \in \mathrm{image}(L_{e_1})$ write $u_e = L_{e_1}(v_x)$ for $v_x \in \mathsf{T}_x\mathsf{M}$. Then let $\gamma\colon I \to \mathsf{M}$ be a curve of class C^r satisfying $0 \in \mathrm{int}(I)$ and $\gamma'(0) = v_x$. If $\xi\colon \mathsf{M} \to \mathsf{E}$ is a section satisfying $j_1\xi(x) = e_1$, then this means that $u_e = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi \circ \gamma)(s)$. Since u_e is vertical we have $T_e\pi(u_e) = 0$. This in turn means that

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\pi \circ \xi \circ \gamma)(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\gamma(s) = v_x.$$

This means, therefore, that $u_e = 0$. This gives $\mathrm{image}(L_{e_1}) \cap \mathsf{V}_e\mathsf{E} = \{0\}$. This part of the result then follows from a dimension count.

For the last assertion, let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart with $(\mathcal{U}, \phi)$ be the corresponding $\mathbb{F}$-chart for M. Denote the coordinates for E by (x^j, v^a), $j \in \{1, \dots, n\}$, $a \in \{1, \dots, m\}$. Suppose that $\phi(x) = \mathbf{0}$ and that $\psi(e) = (\mathbf{0}, \boldsymbol{v})$. An arbitrary linear map between $\mathsf{T}_x\mathsf{X}$ and $\mathsf{T}_e\mathsf{E}$ will have the coordinate representation

$$L = \sum_{j,k=1}^n A^j_k \mathrm{d}x^k \otimes \frac{\partial}{\partial x^j} + \sum_{j=1}^n \sum_{a=1}^m B^a_j \mathrm{d}x^j \otimes \frac{\partial}{\partial v^a}.$$

The condition that $T_e\pi \circ L = \mathrm{id}_{\mathsf{T}_x\mathsf{M}}$ is readily seen to imply that $A^j_k = \delta^j_k$, $j, k \in \{1, \dots, n\}$. Now, if we define a local section ξ on $\mathcal{U}$ with local representative

$$\boldsymbol{\xi}(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{B}\boldsymbol{x})$$

where $\boldsymbol{B}$ is the $m \times n$ matrix with components B^a_j, $j \in \{1, \dots, n\}$, $a \in \{1, \dots, m\}$, then it is immediate that if we take $e_1 = j_1\xi(x)$ we have $L = L_{e_1}$. This gives the existence assertion. For uniqueness suppose that $L_{e_1} = L_{e_2}$ and let ξ_1 and ξ_2 be local sections such that $L_{e_1} = T_x\xi_1$ and $L_{e_2} = T_x\xi_2$. This means that $j_1\xi_1(x) = j_1\xi_2(x)$ and so $e_1 = e_2$, as desired. ■

Now, associated with S_∇, we define an endomorphism P^H_∇ of TE by

$$P^H_\nabla(u_e) = L_{S_\nabla(e)} \circ T_e\pi(u_e), \qquad u_e \in \mathsf{T}_e\mathsf{E}.$$

The following assertions are more or less obvious.

4.7.8 Lemma (Properties of the horizontal projection) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* ∇ *be a* C^r*-connection on the* $\mathbb{F}$*-vector bundle* $\pi \colon \mathsf{E} \to \mathsf{M}$ *of class* C^r *and let* $\mathrm{S}_\nabla \colon \mathsf{E} \to \mathsf{J}^1\mathsf{E}$ *be the corresponding* C^r*-section. The endomorphism* $\mathrm{P}^\mathrm{H}_\nabla \in \Gamma^\mathrm{r}(\mathsf{T}^*\mathsf{E} \otimes \mathsf{TE})$ *has the following properties:*

(i) $\ker(\mathrm{P}^\mathrm{H}_\nabla) = \mathsf{VE}$;

(ii) $\mathsf{TE} = \ker(\mathrm{P}^\mathrm{H}_\nabla) \oplus \mathrm{image}(\mathrm{P}^\mathrm{H}_\nabla)$.

Proof (i) It is clear that $\mathsf{VE} \subseteq \ker(P^H_\nabla)$. For the opposite inclusion, suppose that $P^H_\nabla(u_e) = 0$. Since $L_{S_\nabla(e)}$ is injective this implies that $T_e\pi(u_e) = 0$ whence u_e is vertical.

(ii) Since $T_e\pi$ is surjective it follows that $\mathrm{image}(P^H_\nabla(e)) = \mathrm{image}(L_{S_\nabla(e)})$ which is complementary to $\mathsf{V}_e\mathsf{E}$ by Lemma 4.7.7. ■

The endomorphism P^H_∇ is called the ***horizontal projection*** associated with the connection S_∇ and the endomorphism $P^V_\nabla = \mathrm{id}_{\mathsf{TE}} - P^H_\nabla$ is called the ***vertical projection***. It is easy to see that P^V_∇ is a projection onto VE and P^H_∇ is a projection onto a subbundle, denoted by HE, that is complimentary to VE. For $e_x \in \mathsf{E}_x$, note that $\mathsf{H}_{e_x}\mathsf{E}$ is naturally isomorphic to $\mathsf{T}_x\mathsf{M}$ and $\mathsf{V}_{e_x}\mathsf{E}$ is isomorphic to E_x. For $\mathsf{H}_{e_x}\mathsf{E}$, we note that the restriction of $T_{e_x}\pi$ to $\mathsf{H}_{e_x}\mathsf{E}$ is the desired isomorphism with $\mathsf{T}_x\mathsf{M}$. For $\mathsf{V}_{e_x}\mathsf{E}$, we note that this subspace is the tangent space to the fibre E_x, and so is naturally isomorphic to E_x without having to say more.

We will have need of the coordinate expressions for the horizontal and vertical projections. Let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-vector bundle chart for E with $(\mathcal{U}, \phi)$ the corresponding $\mathbb{F}$-chart for M, and denote the corresponding coordinates for E by (x^j, v^a), $j \in \{1, \dots, n\}$, $a \in \{1, \dots, m\}$. Parsing the above constructions, one can show that

$$P^H_\nabla = \sum_{j,k=1}^n \delta^j_k \mathrm{d}x^k \otimes \frac{\partial}{\partial x^j} + \sum_{j=1}^n \sum_{a,b=1}^m \Gamma^a_{jb} v^b \mathrm{d}x^j \otimes \frac{\partial}{\partial v^a},$$

$$P^V_\nabla = \sum_{a,b=1}^m \delta^a_b \mathrm{d}v^b \otimes \frac{\partial}{\partial v^a} - \sum_{j=1}^n \sum_{a,b=1}^m \Gamma^a_{jb} v^b \mathrm{d}x^j \otimes \frac{\partial}{\partial v^a}.$$

4.7.4 Tensor products of connections

Connections on a vector bundle lead to extensions to many of the standard algebraic constructions. Some of these are straightforward. For example, if ∇^1 and ∇^2 are connections on $\mathbb{F}$-vector bundles $\pi_1 \colon \mathsf{E}_2 \to |manM$ and $\pi_2 \colon \mathsf{E}_2 \to \mathsf{M}$, then there is a naturally induced connection $\nabla^1 \oplus \nabla^2$ on $\mathsf{E}_1 \oplus \mathsf{E}_2$ defined by

$$(\nabla^1 \oplus \nabla^2)_X(\xi_1 \oplus \xi_2) = (\nabla^1_X \xi_1) \oplus (\nabla^2_X \xi_2).$$

In this section we focus on connections in tensor products of vector bundles which themselves have connections.

The main result is the following.

4.7.9 Proposition (Feasibility of the tensor product of two connections) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$. *Let* $\pi_1 \colon \mathsf{E}_1 \to \mathsf{M}$ *and* $\pi_2 \colon \mathsf{E}_2 \to \mathsf{M}$ *be* $\mathbb{F}$ *vector bundles of class* C^r *and let* ∇^1 *and* ∇^2 *be* C^r*-connections in* E_1 *and* E_2*, respectively. Then there exists a unique* C^r*-connection* $\nabla^1 \otimes \nabla^2$ *on* $\mathsf{E}_1 \otimes \mathsf{E}_2$ *satisfying*

$$(\nabla^1 \otimes \nabla^2)_X(\xi_1 \otimes \xi_2) = (\nabla^1_X \xi_1) \otimes \xi_2 + \xi_1 \otimes (\nabla_X \xi_2) \tag{4.38}$$

for $\xi_a \in \Gamma^r(\mathsf{E}_a)$, $a \in \{1, 2\}$*, and* $X \in \Gamma^r(\mathsf{TM})$*.*

Proof The existence assertion comes from the condition (4.38), since any section of $\mathsf{E}_1 \otimes \mathsf{E}_2$ is a ■

4.7.5 Affine connections and the Levi-Civita connection

The following result further characterises pseudo-Riemannian affine connections.

4.7.10 Proposition *Let* (M, g) *be a pseudo-Riemannian manifold and let* ∇ *be an affine connection on* M*. The following are equivalent:*

(i) ∇ *is a pseudo-Riemannian affine connection;*

(ii) $\tau_c(t_1, t_2)$ *is an isometry for every curve* $c \colon [a, b] \to M$ *of class* C^2 *and* $a < t_1 < t_2 < b$*.*

Proof Suppose that ∇ is a pseudo-Riemannian affine connection. Let $c \colon [a, b] \to M$ be a curve of class C^2 with $a < t_1 < t_2 < b$. Let $u, v \in \mathsf{T}_{c(t_1)}M$ and let X and Y be the vector fields along c obtained by parallel translating u and v along c, respectively. We then have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(X(t), Y(t)) = (\nabla_{c'(t)} g)(X(t), Y(t)) + g(\nabla_{c'(t)} X(t), Y(t)) + g(X(t), \nabla_{c'(t)} Y(t)) = 0$$

since X and Y are parallel along c and since ∇ is pseudo-Riemannian. This shows in particular that $g_{c(t_1)}(u, v) = g(\tau_c(t_1, t_2) \cdot u, \tau_c(t_1, t_2) \cdot v)$. In other words, $\tau_c(t_1, t_2)$ is an isometry.

Now suppose that $\tau_c(t_1, t_2)$ is an isometry for every curve $c \colon [a, b] \to M$ of class C^2 curve and $a < t_1 < t_2 < b$. Let $x \in M$ and let $u, v, w \in \mathsf{T}_x M$. Let $c \colon (-\epsilon, \epsilon) \to M$ be a curve of class C^2 such that $c(0) = x$ and $c'(0) = w$. Let X and Y be the vector fields along c obtained by parallel translation of u and v, respectively. We then have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(X(t), Y(t)) = (\nabla_{c'(t)} g)(X(t), Y(t))$$

where we have used the fact that parallel translation consists of isometries and that X and Y are parallel along c. Evaluating this expression at $t = 0$ gives $(\nabla_w g)(u, v) = 0$. As x and $u, v, w \in \mathsf{T}_x M$ are arbitrary, the result follows. ■

The following important theorem asserts that pseudo-Riemannian metrics exist.

4.7.11 Theorem (The Fundamental Theorem of Riemannian Geometry) *If* $(\mathsf{M}, \mathbb{g})$ *is a pseudo-Riemannian manifold then there exists a unique affine connection* $\overset{\mathbb{G}}{\nabla}$ *on* M *such that*

(i) $\overset{\mathbb{G}}{\nabla}$ *is a pseudo-Riemannian affine connection, and*

(ii) $\overset{\mathbb{G}}{\nabla}$ *is torsion-free.*

The affine connection $\overset{\mathbb{G}}{\nabla}$ *is called the* ***Levi-Civita affine connection*** *associated with* $(\mathsf{M}, \mathbb{g})$.

Proof First we establish existence. We define $\overset{\mathbb{G}}{\nabla}$ by asking that for vector fields X, Y, and Z on M we have

$$g(\overset{\mathbb{G}}{\nabla}_X Y, Z) = \frac{1}{2}\Big(\mathscr{L}_X(g(Y,Z)) + \mathscr{L}_Y(g(Z,X)) - \mathscr{L}_Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\Big). \quad (4.39)$$

To ensure that this definition make sense one must verify that it is linear with respect to multiplication by functions in the second argument. Thus, for $f \in \mathrm{C}^\infty(M)$ we must verify that

$$g(\overset{\mathbb{G}}{\nabla}_X Y, fZ) = f g(\overset{\mathbb{G}}{\nabla}_X Y, Z)$$

using the relation (4.39). This is straightforwardly done using the derivation properties of the Lie bracket. We must also ensure that (4.39) does define an affine connection. This consists of checking that $\overset{\mathbb{G}}{\nabla}$ satisfies the defining properties of an affine connection. The property is obvious. To check let $f \in \mathrm{C}^\infty(M)$ and then verify that

what
what

$$g(\overset{\mathbb{G}}{\nabla}_{fX} Y, Z) = g(f\overset{\mathbb{G}}{\nabla}_X Y, Z)$$

which one does easily by using the derivation properties of the Lie derivative. In a similar manner one may show that

$$g(\overset{\mathbb{G}}{\nabla}_X fY, Z) = g(f\overset{\mathbb{G}}{\nabla}_X Y + (\mathscr{L}_X f)Y, Z).$$

One also needs to verify that $\overset{\mathbb{G}}{\nabla}$ so defined is torsion-free. It is straightforward to see that indeed

$$g(\overset{\mathbb{G}}{\nabla}_X Y - \overset{\mathbb{G}}{\nabla}_Y X, Z) = g([X,Y],Z)$$

which means that $\overset{\mathbb{G}}{\nabla}$ is torsion-free. Finally, one needs to show that $\overset{\mathbb{G}}{\nabla}$ is a pseudo-Riemannian affine connection. To verify this one can use (4.39) to show that

$$\mathscr{L}_X(g(Y,Z)) = g(\overset{\mathbb{G}}{\nabla}_X Y, Z) + g(Y, \overset{\mathbb{G}}{\nabla}_X Z).$$

From the derivation properties of $\overset{\mathbb{G}}{\nabla}$ one then sees that this implies that $\overset{\mathbb{G}}{\nabla}_X(Y,Z) = 0$ for all vector fields X, Y, and Z. That is, $\overset{\mathbb{G}}{\nabla}$ is pseudo-Riemannian. This establishes the existence of $\overset{\mathbb{G}}{\nabla}$.

Now we show that conditions (i) and (ii) uniquely define $\overset{\mathrm{G}}{\nabla}$. So suppose that ∇ is an affine connection satisfying these conditions. Then, for vector fields X, Y, and Z one has

$$\begin{aligned}\mathscr{L}_X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\\ \mathscr{L}_Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X)\\ \mathscr{L}_Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y).\end{aligned}$$

Now add the first two equations add subtract from this the third to get

$$\mathscr{L}_X(g(Y,Z)) + \mathscr{L}_Y(g(Z,X)) - \mathscr{L}_Z(g(X,Y)) = \\ g(\nabla_X Y, Z) + g(\nabla_Y X, Z) + g([X,Z],Y) + g([Y,Z],X).$$

In arriving at this formula we have used the relation $\nabla_X Y - \nabla_Y X = [X,Y]$ as ∇ is torsion-free. Now we use the same relation to note that $\nabla_Y X = \nabla_X Y - [X,Y]$ to obtain

$$2g(\nabla_X Y, Z) = \mathscr{L}_X(g(Y,Z)) + \mathscr{L}_Y(g(Z,X)) - \mathscr{L}_Z(g(X,Y)) + \\ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).$$

Thus ∇ satisfies (4.39). ■

The formula (4.39) is called the ***Koszul formula***.

From the proof we may obtain an expression for the Christoffel symbols of the Levi-Civita affine connection:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\Big(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\Big)$$

for the Christoffel symbols.

4.8 Correspondence between holomorphic manifolds and complex structures

In Proposition 4.5.8 we saw that any holomorphic manifold possesses a natural almost complex structure. In this section we shall explore the converse relationship, holomorphic structures on manifolds with almost complex structures.

4.8.1 The Nijenhuis tensor

In this section we recall a construction of Nijenhuis [1951] on the structure of (1, 1)-tensor fields.

4.8.1 Definition (Nijenhuis tensor) Let M be a smooth manifold. The ***Nijenhuis tensor*** of a (1, 1)-tensor field A on M is the (1, 2) tensor field given by

$$N_A(X,Y) = [AX,AY] + A^2[X,Y] - A[AX,Y] - A[X,AY].$$ •

Of course, we should verify that this is indeed a tensor.

4.8.2 Lemma (The Nijenhuis tensor is well-defined) *The mapping* $(X, Y) \mapsto N_A(X, Y)$ *is* $C^\infty(M)$*-bilinear.*

Proof We let $f, g \in C^\infty(M)$ and $X, Y \in \Gamma^\infty(TM)$ and compute

$$\begin{aligned} N_A(fX, gY) &= [fAX, gAY] + A^2([fX, gY]) - A([fAX, gY]) - A([fX, gAY]) \\ &= fg[AX, AY] + f(\mathscr{L}_{AX}g)AY - g(\mathscr{L}_{AY}f)AX + \\ &\quad fgA^2[X, Y] + f(\mathscr{L}_X g)A^2(Y) - g(\mathscr{L}_Y f)A^2(X) - \\ &\quad fgA([AX, Y]) - f(\mathscr{L}_{AX}g)AY + g(\mathscr{L}_Y f)A^2(X) - \\ &\quad fgA([X, AY]) - f(\mathscr{L}_X g)A^2(Y) + g(\mathscr{L}_{AY}f)AX \\ &= fgN_A(X, Y), \end{aligned}$$

as desired. ■

Let us give the local representation of the Nijenhuis tensor. Let $(\mathcal{U}, \phi)$ be a chart for M with coordinates $(x^1, \dots, x^n)$. A straightforward computation gives the local representative of N_A as

$$N_A|\mathcal{U} = \sum_{i,j,k,l=1}^{n} \Big(A^l_i \frac{\partial A^k_j}{\partial x^l} - A^l_j \frac{\partial A^k_i}{\partial x^l} + A^k_l \frac{\partial A^l_i}{\partial x^j} - A^k_l \frac{\partial A^l_j}{\partial x^i}\Big) \frac{\partial}{\partial x^k} \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{4.40}$$

The definition of the Nijenhuis tensor does not make immediately apparent its usefulness, so let us describe a typical generic problem that comes up in the study of $(1, 1)$-tensor fields, and how it relates to the Nijenhuis tensor.

The study of deformations of geometric structures was instigated by Cartan [1955]. For our purposes, let us say that a $(1, 1)$-tensor field A on a smooth manifold M is ***0-deformable*** if, for each $x \in M$, there is a chart $(\mathcal{U}, \phi)$ around x with $\phi(x) = \mathbf{0}$ and such that, for each $\boldsymbol{x} \in \phi(\mathcal{U})$, there exists $\boldsymbol{P}(\boldsymbol{x}) \in GL(n; \mathbb{R})$ such that

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) \circ \boldsymbol{A}(\mathbf{0}) \circ \boldsymbol{P}^{-1}(\boldsymbol{x}).$$

We say that A is ***integrable*** if, for every $x \in M$ there is a chart $(\mathcal{U}, \phi)$ around x for which the matrix function of the components of A, $\boldsymbol{x} \mapsto \boldsymbol{A}(\boldsymbol{x})$, is constant. Clearly, if A is integrable then it is 0-deformable. The converse is not generally true. For certain classes of 0-deformable $(1, 1)$-tensor fields, one can show that integrability is implied by the vanishing of the Nijenhuis tensor N_A. This is often related to the Jordan form of $\boldsymbol{A}(\mathbf{0})$ (see [Kobayashi 1962] and [Lehmann-Lejeune 1966]). However, it is *not* true that a 0-deformable $(1, 1)$-tensor field is always integrable if its Nijenhuis tensor vanishes (see [Kobayashi 1962]).

4.8.2 Equivalent characterisations of complex structures

In this section we consider various conditions equivalent to an almost complex structure being a complex structure. The equivalence of many of the conditions we give is proved in a more or less straightforward manner, but one implication is proved using the difficult results of Section 6.2.

To state our main result in this section, we first note that if J is an almost complex structure on a smooth manifold M, the tangent bundle constructions of $T^{\mathbb{C}}M$, $T^{1,0}M$, and $T^{0,1}M$ can be made fibrewise from the constructions of Section 4.1.1. In like manner, the differential form constructions of $\bigwedge^{k,l}(T^{*\mathbb{C}}M)$ can be made fibrewise following Section 4.1.4. We still have the decomposition

$$\bigwedge^{m}(T^{*\mathbb{C}}M) = \bigoplus_{\substack{k,l\\ k+l=m}} \bigwedge^{k,l}(T^{*\mathbb{C}}M),$$

as this is purely an algebraic construction. That is, the algebraic structures associated with vector spaces with linear complex structures all can be adapted to manifolds with an almost complex structure. What fails are the differential constructions involving the Lie bracket and the exterior derivative. Specifically, Lemmata 4.5.21 and 4.6.5 do not generally hold for manifolds with an almost complex structure. However, we can still define the operators ∂ and $\bar{\partial}$, thanks to the following lemma.

4.8.3 Lemma (The complex exterior derivative on manifolds with an almost complex structure) *If* M *is a smooth manifold with an almost complex structure* J, *then*

$$\begin{aligned} &d_{\mathbb{C}}(\Gamma^\infty(\textstyle\bigwedge^{k,l}(T^{*\mathbb{C}}M))) \\ &\quad\subseteq \Gamma^\infty(\textstyle\bigwedge^{k-1,l+2}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\textstyle\bigwedge^{k,l+1}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\textstyle\bigwedge^{k+1,l}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\textstyle\bigwedge^{k+2,l-1}(T^{*\mathbb{C}}M)). \end{aligned}$$

Proof Since $\bigwedge^1(T^{*\mathbb{C}}M) = \bigwedge^{1,0}(T^{*\mathbb{C}}M) \oplus \bigwedge^{0,1}(T^{*\mathbb{C}}M)$ the result holds when $k = l = 0$. Since

$$\bigwedge^2(T^{*\mathbb{C}}M) = \bigwedge^{2,0}(T^{*\mathbb{C}}M) \oplus \bigwedge^{1,0}(T^{*\mathbb{C}}M) \oplus \bigwedge^{0,1}(T^{*\mathbb{C}}M) \oplus \bigwedge^{0,2}(T^{*\mathbb{C}}M),$$

the result also holds for $(k,l) = (1,0)$ and $(k,l) = (0,1)$. Since $\Gamma^\infty(\bigwedge^{k,l}(T^{*\mathbb{C}}M))$ is locally generated by $\Gamma^\infty(\bigwedge^{0,0}(T^{*\mathbb{C}}M))$, $\Gamma^\infty(\bigwedge^{1,0}(T^{*\mathbb{C}}M))$, and $\Gamma^\infty(\bigwedge^{0,1}(T^{*\mathbb{C}}M))$, and since the result is local, the result follows from the cases proved and Proposition 4.6.4(v). ■

Thus we can define ∂ and $\bar{\partial}$ applied to $\Gamma^\infty(\bigwedge^{k,l}(T^{*\mathbb{C}}M))$ as the $(k+1,l)$- and $(k,l+1)$-components of $d_{\mathbb{C}}$.

With this development, we have the following summary of equivalent characterisations of complex structures.

4.8.4 Theorem (Equivalent characterisations of complex structures) *If* M *is a smooth manifold with an almost complex structure* J, *the following statements are equivalent:*

(i) $[Z_1, Z_2]_{\mathbb{C}} \in \Gamma^\infty(T^{1,0}M)$ *for every* $Z_1, Z_2 \in \Gamma^\infty(T^{1,0}M)$;

(ii) $[Z_1, Z_2]_{\mathbb{C}} \in \Gamma^\infty(T^{0,1}M)$ *for every* $Z_1, Z_2 \in \Gamma^\infty(T^{0,1}M)$;

(iii) $d_{\mathbb{C}}(\Gamma^\infty(\bigwedge^{1,0}(T^{*\mathbb{C}}M))) \subseteq \Gamma^\infty(\bigwedge^{2,0}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\bigwedge^{1,1}(T^{*\mathbb{C}}M))$;

(iv) $d_{\mathbb{C}}(\Gamma^\infty(\bigwedge^{0,1}(T^{*\mathbb{C}}M))) \subseteq \Gamma^\infty(\bigwedge^{1,1}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\bigwedge^{0,2}(T^{*\mathbb{C}}M))$;

(v) $d_{\mathbb{C}}(\Gamma^\infty(\bigwedge^{k,l}(T^{*\mathbb{C}}M))) \subseteq \Gamma^\infty(\bigwedge^{k,l+1}(T^{*\mathbb{C}}M)) \oplus \Gamma^\infty(\bigwedge^{k+1,l}(T^{*\mathbb{C}}M))$ *for* $k, l \in \mathbb{Z}_{>0}$;

(vi) $d_{\mathbb{C}} = \partial + \bar{\partial}$;

(vii) $N_J = 0$;

(viii) J *is a complex structure.*

Proof (i) $\iff$ (ii) By Proposition 4.1.5, a section of $\mathsf{T}^{1,0}\mathsf{T}^{\mathbb{C}}\mathsf{M}$ has the form

$$Z = 1 \otimes X - \mathrm{i} \otimes J(X)$$

and a section of $\mathsf{T}^{0,1}\mathsf{T}^{\mathbb{C}}\mathsf{M}$ has the form

$$Z = 1 \otimes X + \mathrm{i} \otimes J(X)$$

for a smooth (real) vector field X on M. As in the proof of Lemma 4.5.21, the condition that the Lie bracket $[Z_1, Z_2]_{\mathbb{C}}$ of two such $\mathsf{T}^{1,0}\mathsf{M}$-vector fields be a $\mathsf{T}^{1,0}\mathsf{M}$-valued vector field is that

$$J_{\mathbb{C}}([Z_1, Z_2]_{\mathbb{C}}) - \mathrm{i}[Z_1, Z_2]_{\mathbb{C}} = 0.$$

The corresponding condition that the Lie bracket $[Z_1, Z_2]_{\mathbb{C}}$ of two $\mathsf{T}^{0,1}\mathsf{M}$-vector fields be a $\mathsf{T}^{0,1}\mathsf{M}$-valued vector field is that

$$J_{\mathbb{C}}([Z_1, Z_2]_{\mathbb{C}}) + \mathrm{i}[Z_1, Z_2]_{\mathbb{C}} = 0.$$

As each of these conditions is equivalent to the conjugate of the other, this part of the proof follows.

(i) $\iff$ (iv) For $\alpha \in \Gamma^\infty(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and $Z_1, Z_2 \in \Gamma^\infty(\mathsf{T}^{1,0}\mathsf{M})$ we have, using (4.25),

$$\mathrm{d}_{\mathbb{C}}(Z_1, Z_2) = \mathscr{L}_{Z_1}(\alpha(Z_2)) - \mathscr{L}_{Z_2}(\alpha(Z_1)) - \alpha([Z_1, Z_2]) = -\alpha([Z_1, Z_2]).$$

Note that (i) is equivalent to the right-hand side vanishing for every α, Z_1, and Z_2, while (iv) is equivalent to the left-hand side vanishing for every α, Z_1, and Z_2.

(iii) $\iff$ (iv) Using Proposition 4.1.15(iii) and Proposition 4.6.4(iv), these conditions are easily seen to be conjugate to one another.

(iii,iv) $\implies$ (v) As in our proof of Lemma 4.8.3, this follows since $\Gamma^\infty(\bigwedge^{k,l}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ is locally generated by $\Gamma^\infty(\bigwedge^{0,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, $\Gamma^\infty(\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, and $\Gamma^\infty(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$.

(v) $\implies$ (iii,iv) This is clear.

(v) $\iff$ (vi) This follows by definition of ∂ and $\bar{\partial}$ and Lemma 4.8.3.

(i) $\iff$ (vii) Consider sections

$$Z_a = 1 \otimes X_a - \mathrm{i} \otimes J(X_a), \qquad a \in \{1, 2\},$$

of $\mathsf{T}^{1,0}\mathsf{M}$, where X_1 and X_2 are smooth vector fields on M. The computation from the proof of Lemma 4.5.21 of the formula

$$J_{\mathbb{C}}([Z_1, Z_2]_{\mathbb{C}}) - \mathrm{i}[Z_1, Z_2]_{\mathbb{C}} = -1 \otimes J \circ N_J(X_1, X_2)) + \mathrm{i} \otimes N_J(X_1, X_2)$$

is valid for almost complex structures. Thus $N_J = 0$ if and only if

$$[Z_1, Z_2]_{\mathbb{C}} \in \Gamma^\infty((\ker(J_{\mathbb{C}} - \mathrm{i}\ \mathrm{id}_{\mathsf{T}^{\mathbb{C}}\mathsf{M}}))) = \Gamma^\infty(\mathsf{T}^{1,0}\mathsf{M}).$$

(viii) $\implies$ (vii) This follows from (4.40) since the components of J in a $\mathbb{C}$-chart are constant.

(vi) $\implies$ (viii) In the above we have shown that the first seven conditions in the theorem are equivalent and that they are implied by the eighth. The remaining implication is difficult, and the proof we give relies on the constructions and results of Section 6.2. We will assume without mention these constructions and results, acknowledging that this presupposition will only be realised after a great deal of effort from the reader.

Let us first do some preliminary constructions to justify our use of the results of Section 6.2 for almost complex structures. As in Section 6.2.1 we let h be a Hermitian metric on M that is compatible with the almost complex structure J. Also as in Section 6.2.1 we use an orthonormal local basis $(\omega^1, \dots, \omega^n)$ for $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$, so that we define $\frac{\partial f}{\partial \omega^j}$ and $\frac{\partial f}{\partial \bar{\omega}^j}$, $j \in \{1, \dots, n\}$, by

$$\mathrm{d}_{\mathbb{C}} f = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial \omega^j} \omega^j + \frac{\partial f}{\partial \bar{\omega}^j} \bar{\omega}^j \Big).$$

Next note that the condition $\mathrm{d}_{\mathbb{C}} = \partial + \bar{\partial}$ implies that

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0.$$

If one now goes through the constructions and arguments of Section 6.2, one sees that these properties of ∂ and $\bar{\partial}$ are the only ones used. Thus all existence and regularity results for the partial differential equation $\bar{\partial} u = f$ from that section can be applied to the case when ∂ and $\bar{\partial}$ are defined for an almost complex structure for which condition (vi) holds.

Next, the result is local so we assume that M is a neighbourhood of $\mathbf{0} \in \mathbb{C}^n$. We denote Euclidean coordinates for M by $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ and denote $\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \in \mathbb{C}^n$. By a linear change of coordinates, we can without loss of generality suppose that

$$\Big(\frac{\partial}{\partial x^1}(\mathbf{0}), \dots, \frac{\partial}{\partial x^n}(\mathbf{0}), \frac{\partial}{\partial y^1}(\mathbf{0}), \dots, \frac{\partial}{\partial y^n}(\mathbf{0}) \Big)$$

is a $J(\mathbf{0})$-adapted basis for $\mathsf{T}_{\mathbf{0}}\mathsf{M}$. Let $\psi \in \mathsf{C}^\infty(\mathsf{M})$ be given by $\psi(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$. We readily verify that $\frac{1}{2\mathrm{i}} \partial \circ \bar{\partial} \psi(0)$ is positive-definite since $J(0)$ is the canonical complex structure on $\mathbb{C}^n$. We can assume, possibly after shrinking M, that ϕ is "strictly plurisubharmonic" on M, i.e., strictly plurisubharmonic after the natural adaptation of those words to almost complex manifolds. Let us take $r \in \mathbb{R}_{>0}$ such that $\mathsf{B}_{2n}(r, \mathbf{0}) \subseteq \mathsf{M}$ and, moreover, redefine M to be this ball. Then

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto \frac{1}{r^2 - \psi(\boldsymbol{x}, \boldsymbol{y})}$$

is a plurisubharmonic exhaustion function on M, cf. the proof of Proposition 6.1.21(iv). As in the proof of Theorem 6.2.13 we can thus choose φ to be a convex increasing function of the right-hand side of the preceding equation such that, if $f \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfies $\bar{\partial} f = 0$, then there exists $u \in \mathsf{L}^2_{\mathrm{loc}}(\mathsf{M}; \mathbb{C})$ satisfying $\bar{\partial} u = f$ and $\|u\|_\varphi \le \|f\|_\varphi$.

Now, keeping in mind that M has been assumed to be a ball of radius r about the origin in $\mathbb{C}^n$, let $\zeta^1, \dots, \zeta^n$ be linear $\mathbb{C}$-valued functions on M for which $\mathrm{d}_{\mathbb{C}} \zeta^j(\mathbf{0}) = \omega^j(\mathbf{0})$, $j \in \{1, \dots, n\}$. For $\epsilon \in (0, 1)$, by $\delta_\epsilon \colon \mathbb{C}^n \to \mathbb{C}^n$ denote the mapping $\delta_\epsilon(\boldsymbol{x}, \boldsymbol{y}) = (\epsilon \boldsymbol{x}, \epsilon \boldsymbol{y})$. We then have the almost complex structure $J_\epsilon = \delta_\epsilon^* J$ on M, and $\delta_\epsilon^* \omega^1, \dots, \delta_\epsilon^* \omega^n$ generate the subbundle of forms of type $(1, 0)$ for this almost complex structure. Let ∂_ϵ and $\bar{\partial}_\epsilon$ denote the differential operators associated with the almost complex structures J_ϵ. Since

$$\frac{1}{2\mathrm{i}}\partial_\epsilon \circ \bar{\partial}_\epsilon \psi(0) = \frac{1}{2\mathrm{i}\epsilon^2}\partial_\epsilon \circ \bar{\partial}_\epsilon \psi(0),$$

we can sift through the arguments to see that the function φ above has the property that, for every $\epsilon \in (0,1)$, if $f \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfies $\bar{\partial}_\epsilon f = 0$, then there exists $u \in \mathsf{L}^2_{\mathrm{loc}}(\mathsf{M};\mathbb{C})$ satisfying $\bar{\partial}_\epsilon u = f$ and $\|u\|_\varphi \le \|f\|_\varphi$.

Let us prove a simple lemma that we shall call upon a couple of times in the remainder of the proof.

1 **Lemma** *The form* $\mathrm{d}_{\mathbb{C}}\zeta^j - \epsilon^{-1}\delta_\epsilon^*\omega^j$ *and all of its derivatives converge uniformly to zero as* $\epsilon \to 0$.

Proof Let $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,2n\}$, and note that

$$\delta_\epsilon^*\omega^j(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{e}_k = \omega^j(T_{(\boldsymbol{x},\boldsymbol{y})}\delta_\epsilon \cdot \boldsymbol{e}_k) = \epsilon\omega^j(\epsilon\boldsymbol{x},\epsilon\boldsymbol{y}) \cdot \boldsymbol{e}_k.$$

From this and the fact that $\mathrm{d}_{\mathbb{C}}\zeta^j(\boldsymbol{0}) = \omega^j(\boldsymbol{0})$ we conclude that $\mathrm{d}_{\mathbb{C}}\zeta^j - \epsilon^{-1}\delta_\epsilon^*\omega^j$ converges uniformly to zero as claimed. For the corresponding statement for the derivatives of this form, note that all derivatives of ζ^j are zero, and, for $I \in \mathbb{Z}^{2n}_{\ge 0}$,

$$\boldsymbol{D}^I(\epsilon^{-1}\delta_\epsilon^*\omega^j)(\boldsymbol{x},\boldsymbol{y}) = \epsilon^{|I|}\boldsymbol{D}^I\omega^j(\epsilon\boldsymbol{x},\epsilon\boldsymbol{y}).$$

From this the lemma follows. ▼

Now we give a few technical lemmata that will be important for us.

2 **Lemma** *Let* $\mathcal{U} \subseteq \mathbb{C}^n$ *be open, let* $f \in \mathsf{L}^2(\mathcal{U};\mathbb{C})$ *have compact support, and suppose that* $\frac{\partial f}{\partial \bar{\omega}^j} \in \mathsf{L}^2(\mathcal{U};\mathbb{C})$, $j \in \{1,\dots,n\}$. *Then* $f \in \mathsf{H}^1(\mathcal{U};\mathbb{C})$ *and, moreover, if* $K \subseteq \mathcal{U}$ *is compact, then there exists* $C \in \mathbb{R}_{>0}$ *such that*

$$\sum_{|I|\le 1}\|\boldsymbol{D}^I f\|_2 \le C\Big(\|f\|_2 + \sum_{j=1}^n\Big\|\frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2^2\Big)$$

for every $f \in \mathsf{L}^2(\mathcal{U};\mathbb{C})$ *for which* $\operatorname{supp}(f) \subseteq K$.

Proof Let δ_j be the formal adjoint, with respect to the integral using the Lebesgue measure on $\mathbb{C}^n$, of $\frac{\partial}{\partial\bar{\omega}^j}$. We first suppose that $f \in \mathscr{D}(\mathcal{U};\mathbb{C})$. Computations like those used for Sublemma 3 from the proof of Lemma 6.2.11 give

$$\Big\|\frac{\partial f}{\partial \omega^j}\Big\|_2^2 = \Big\|\frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2^2 + \int_{\mathcal{U}}\Big(\Big(\delta_j \circ \frac{\partial}{\partial\bar{\omega}^j} - \bar{\delta}_j \circ \frac{\partial}{\partial \omega^j}\Big)f(\boldsymbol{x},\boldsymbol{y})\Big)\bar{f}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\lambda(\boldsymbol{x},\boldsymbol{y}). \tag{4.41}$$

One can verify, cf. Lemmata 6.2.6 and 6.2.7, that the operator

$$\delta_j \circ \frac{\partial}{\partial\bar{\omega}^j} - \bar{\delta}_j \circ \frac{\partial}{\partial \omega^j}$$

is of order (not necessarily homogeneous order) one. Note that any linear partial differential operator of homogeneous order one is a linear combination of the operators $\frac{\partial}{\partial\omega^j}$ and $\frac{\partial}{\partial\bar{\omega}^j}$, $j \in \{1,\dots,n\}$. In particular, if $|I| = 1$ then

$$\boldsymbol{D}^I f = \sum_{j=1}^n c_I^j\frac{\partial f}{\partial\omega^j} + \sum_{j=1}^n d_I^j\frac{\partial}{\partial\bar{\omega}^j}$$

for $c^j_I, d^j_I \in C^\infty(\mathcal{U};\mathbb{C})$, $j \in \{1,\dots,n\}$. Thus one obtains an estimate

$$\|\boldsymbol{D}^I f\|_2^2 \le C_{I,1}\sum_{j=1}^n \Big\|\frac{\partial f}{\partial \omega^j}\Big\|_2^2 + C_{I,2}\sum_{j=1}^n \Big\|\frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2^2.$$

for suitable $C_{I,1}, C_{I,2} \in \mathbb{R}_{>0}$. We also write

$$\delta_j \circ \frac{\partial}{\partial \bar{\omega}^j} - \bar{\delta}_j \circ \frac{\partial}{\partial \omega^j} f = \sum_{|I|\le 1} \alpha_I \boldsymbol{D}^I f$$

for $\alpha_I \in C^\infty(\mathcal{U};\mathbb{C})$. If we apply (4.41) and the Cauchy–Schwarz inequality we then have

$$\sum_{|I|=1}\|\boldsymbol{D}^I f\|_2^2 \le C'\Big(\sum_{j=1}^n \Big\|\frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2^2 + \|f\|_2 \sum_{|I|\le 1}\|\boldsymbol{D}^I f\|_2\Big)$$

for some suitable $C' \in \mathbb{R}_{>0}$. The inequality $(\frac{b}{\sqrt{2}} - a)^2 \ge 0$ gives $\frac{b^2}{2} + a^2 \ge \sqrt{2}ab \ge ab$, and this gives

$$C'\|f\|_2\|\boldsymbol{D}^I f\|_2 \le C'^2\|f\|_2^2 + \frac{1}{2}\|\boldsymbol{D}^I f\|_2^2.$$

Thus we have

$$\sum_{|I|=1}\|\boldsymbol{D}^I f\|_2^2 \le C'\Big\|\frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2^2 + (n+1)C'^2\|f\|_2^2 + \frac{1}{2}\sum_{|I|\le 1}\|\boldsymbol{D}^I f\|_2^2.$$

This gives the inequality of the lemma when $f \in]sD(\mathcal{U};\mathbb{C})$.

In the general case, we let $\rho \in C^\infty(\mathbb{R}^n)$ be a nonnegative-valued function with support in $\overline{\mathsf{B}}_n(1,\boldsymbol{0})$ and define $\rho_\epsilon(\boldsymbol{x}) = \epsilon^{-n}\rho(\epsilon^{-1}\boldsymbol{x})$. As we showed in the proof of Proposition E.2.16,

$$\lim_{\epsilon\to 0}\Big\|\frac{\partial(\rho_\epsilon * f)}{\partial \bar{\omega}^j} - \rho_\epsilon * \frac{\partial f}{\partial \bar{\omega}^j}\Big\|_2 = 0.$$

Using this fact, we can apply the inequality of the lemma to the smooth compactly supported function $\rho_\epsilon * f - \rho_\delta * f$ gives that $\boldsymbol{D}^I(\rho_\epsilon * f)$ converges in L^2, and this proves that $\boldsymbol{D}^I f \in L^2(\mathcal{U};\mathbb{C})$ for $|I| \le 1$. ▼

The following lemma is essential for us.

3 Lemma *Let* $\mathcal{U} \subseteq \mathbb{C}^n$ *and let* $f \in L^2_{\mathrm{loc}}(\mathcal{U};\mathbb{C})$ *satisfy* $\frac{\partial f}{\partial \bar{\omega}^j} \in H^q_{\mathrm{loc}}(\mathcal{U};\mathbb{C})$, $j \in \{1,\dots,n\}$, *for some* $q \in \mathbb{Z}_{>0}$. *Then* $f \in H^{q+1}_{\mathrm{loc}}(\mathcal{U};\mathbb{C})$. *Moreover, if* $K \subseteq \mathcal{U}$ *is compact and if* $\mathcal{V} \subseteq \mathcal{U}$ *is a neighbourhood of* K, *then there exists* $C \in \mathbb{R}_{>0}$ *such that*

$$\sum_{|I|\le q+1}\int_K \|\mathbf{D}^I f(\mathbf{x},\mathbf{y})\|^2 \, d\lambda(\mathbf{x},\mathbf{y}) \le C\Big(\sum_{|I|\le q}\sum_{j=1}^n \int_{\mathcal{V}} \Big\|\mathbf{D}^I\Big(\frac{\partial f}{\partial \bar{\omega}^j}\Big)(\mathbf{x},\mathbf{y})\Big\|^2 \, d\lambda(\mathbf{x},\mathbf{y}) + \int_{\mathcal{V}} |f(\mathbf{x},\mathbf{z})|^2 \, d\lambda(\mathbf{x},\mathbf{y})\Big).$$

Proof This is proved using the previous lemma along the lines of the corresponding part of Theorem 6.2.14. ▼

Taking $q = 2n$ we have

$$\sum_{|I|\le 1}\int_K \|\boldsymbol{D}^I f(\boldsymbol{x},\boldsymbol{y})\|^2\,\mathrm{d}\lambda(\boldsymbol{x},\boldsymbol{y}) \\ \le C\Big(\sum_{|I|\le q}\sum_{j=1}^n \int_{\mathcal{V}} \Big\|\boldsymbol{D}^I\Big(\frac{\partial f}{\partial \bar{\omega}^j}\Big)(\boldsymbol{x},\boldsymbol{y})\Big\|^2\,\mathrm{d}\lambda(\boldsymbol{x},\boldsymbol{y}) + \int_{\mathcal{V}} |f(\boldsymbol{x},\boldsymbol{z})|^2\,\mathrm{d}\lambda(\boldsymbol{x},\boldsymbol{y})\Big). \quad (4.42)$$

Now we can conclude the proof. For each $\epsilon \in (0,1)$ and $j \in \{1,\dots,n\}$, we can find ζ^j_ϵ such that $\bar{\partial}_\epsilon \zeta^j_\epsilon = \bar{\partial}_\epsilon \zeta^j$ and such that $\|\zeta^j_\epsilon\|_\varphi \le \|\bar{\partial}_\epsilon \zeta^j\|_\varphi$. This last inequality, along with the lemma above, allow us to conclude that $\lim_{\epsilon\to 0}\|\zeta^j_\epsilon\|_\varphi = 0$. Applying the projection onto $\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ to the conclusion of Lemma 1, we have

$$\lim_{\epsilon\to 0}\|\boldsymbol{D}^I(\bar{\partial}_\epsilon \zeta^j)\|_2 = 0$$

for any $I \in \mathbb{Z}^n_{\ge 0}$. We can then apply (4.42) with $f = \zeta^j_\epsilon$ to conclude that

$$\lim_{\epsilon\to 0}\|\boldsymbol{D}^I \zeta^j_\epsilon\|_2 = 0$$

for $|I| = 1$. Thus, if we define $z^j_\epsilon = \zeta^j - \zeta^j_\epsilon$, we see that $\mathrm{d}_{\mathbb{C}} z^1_\epsilon(\boldsymbol{0}),\dots,\mathrm{d}_{\mathbb{C}} z^n_\epsilon(\boldsymbol{0})$ are linearly independent for ϵ sufficiently small. Moreover, since $\bar{\partial}_\epsilon z^j_\epsilon = 0$ it follows that, for fixed ϵ,

$$(\boldsymbol{x},\boldsymbol{y}) \mapsto z^j(\boldsymbol{x},\boldsymbol{y}) \triangleq z^j_\epsilon(\epsilon^{-1}\boldsymbol{x},\epsilon^{-1}\boldsymbol{y})$$

satisfies $\bar{\partial} z^j = 0$. Thus the functions $z^1,\dots,z^n$ are holomorphic functions in a neighbourhood of $\boldsymbol{0}$ for which $\partial z^1,\dots,\partial z^n$ are linearly independent. These are, thus, holomorphic coordinates for M. ■

One can see from the proof of the theorem that the equivalence of the first seven conditions is proved more or less easily, and that the difficult part of the proof is that these conditions imply the eighth. This was first proved by Newlander and Nirenberg [1957] and the proof given above is that of Kohn [1963].

4.9 Why is there not a chapter on Kähler geometry?

An important notion in holomorphic differential geometry is that of a Kähler manifold. We shall not have much to say about Kähler manifolds in this book, and in this section we say a few words about what we are not talking about.

4.9.1 Definitions and main structural results

Let us introduce the main player.

4.9.1 Definition (Hermitian metric) A ***Hermitian metric*** on a holomorphic manifold M is a section $h \in \Gamma^\infty(\mathbb{C}_\mathsf{M} \otimes \mathsf{S}^2(\mathsf{T}^*\mathsf{M}))$ for which $h(z)$ is a Hermitian inner product on $\mathsf{T}_z\mathsf{M}$. •

Let us note that Hermitian metrics exist.

4.9.2 Theorem (Holomorphic manifolds possess Hermitian metrics) *If* M *is a paracompact holomorphic manifold, then there exists a Hermitian metric on* M*.*

Proof Let $((\mathcal{U}_a, \phi))_{a\in A}$ be an atlas of holomorphic charts. Since M is paracompact, for each $a \in A$ let $\rho_a \in \mathrm{C}^\infty(\mathsf{M})$ be such that $\mathrm{image}(\rho_a) \subseteq [0,1]$ and $\mathrm{supp}(\rho_a) \subseteq \mathcal{U}_a$, $a \in A$, and such that $(\mathrm{supp}(\rho_a))_{a\in A}$ is locally finite and $\sum_{a\in A} \rho_a = 1$ [Abraham, Marsden, and Ratiu 1988, Theorem 5.5.12]. For $a \in A$ define $h_a \in \Gamma^\infty(\mathbb{C}_\mathsf{M} \otimes \mathsf{S}^2(\mathsf{T}^*\mathsf{M}))$ by

$$h_a(z) = \begin{cases} \phi_a^*(\mathbb{H}|\phi_a(\mathcal{U}_a))(z), & z \in \mathcal{U}_a, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbb{H}$ is the canonical Hermitian metric on $\mathbb{C}^n$. Then define $h = \sum_{a\in A} h_a$, this sum making sense since it is locally finite. Moreover, since the set of Hermitian inner products on a vector space is convex (as can easily be verified), and since $h(z)$ is a convex combination of Hermitian inner products, it follows that $h(z)$ is a Hermitian inner product. ■

Now, given a Hermitian metric h on a holomorphic manifold M, we follow the constructions of Section 4.1.5 and write $h = g - \mathrm{i}\omega$ for a Riemannian metric g and an exterior two-form ω of bidegree $(1,1)$. Additional requirements are specified to ensure that h is a Kähler metric.

4.9.3 Definition (Kähler metric) A Hermitian metric $h = g - \mathrm{i}\omega$ on a holomorphic manifold M is a ***Kähler metric*** if $\mathrm{d}\omega = 0$. •

The requirement that ω be closed has a few equivalent characterisations.

4.9.4 Theorem (Characterisations of Kähler metrics) *For a Hermitian metric* $\mathrm{h} = \mathrm{g} - \mathrm{i}\omega$ *on a holomorphic manifold* M*, the following statements are equivalent:*

(i) $\mathrm{d}\omega = 0$*;*

(ii) $\nabla \mathrm{J} = 0$*;*

(iii) $\nabla\omega = 0$*.*

In the above, J *is the canonical almost complex structure on* M *and* ∇ *is the Levi-Civita connection associated to* g*.*

Proof We first assemble a few technical lemmata.

1 Lemma *If* $\mathrm{h} = \mathrm{g} - \mathrm{i}\omega$ *is a Hermitian metric on a holomorphic manifold* M *then*

$$\mathrm{g}(\nabla_\mathrm{X}\mathrm{Y}, \mathrm{Z}) = \frac{1}{2}\mathrm{d}\omega(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) - \frac{1}{2}\mathrm{d}\omega(\mathrm{X}, \mathrm{J}\mathrm{Y}, \mathrm{J}\mathrm{Z})$$

for every $\mathrm{X}, \mathrm{Y}, \mathrm{Z} \in \Gamma^\infty(\mathsf{T}\mathsf{M})$*, where* ∇ *is the Levi-Civita connection associated with* g*.*

Proof By the formula of [Palais 1954] we have

$$\mathrm{d}\omega(X_0, X_1, X_2) = \mathscr{L}_{X_0}(\omega(X_1, X_2)) - \mathscr{L}_{X_1}(\omega(X_0, X_2)) + \mathscr{L}_{X_2}(\omega(X_0, X_1)) - \omega([X_0, X_1], X_2) + \omega([X_0, X_2], X_1) - \omega([X_1, X_2], X_0).$$

Using Proposition 4.1.18 we may rewrite this as

$$\mathrm{d}\omega(X_0, X_1, X_2) = \mathscr{L}_{X_0}(g(JX_1, X_2)) - \mathscr{L}_{X_1}(g(JX_0, X_2)) + \mathscr{L}_{X_2}(g(JX_0, X_1)) - g(J[X_0, X_1], X_2) + g(J[X_0, X_2], X_1) - g(J[X_1, X_2], X_0) = 0. \tag{4.43}$$

We shall also make use of the identity

$$\mathrm{d}\omega(X_0, JX_1, JX_2) = \mathscr{L}_{X_0}(g(JX_1, X_2)) - \mathscr{L}_{JX_1}(g(X_0, X_2)) + \mathscr{L}_{JX_2}(g(X_0, X_1)) - g([X_0, JX_1], X_2) + g([X_0, JX_2]), X_1) - g(J[JX_1, JX_2], X_0). \tag{4.44}$$

Since ∇ is the Levi-Civita connection for g we have

$$\mathscr{L}_{X_0}(g(X_1, X_2)) = g(\nabla_{X_0} X_1, X_2) + g(X_1, \nabla_{X_0} X_2).$$

By cyclically permuting this identity and using the fact that ∇ is torsion free, one may easily arrive at the formula

$$g(\nabla_{X_0} X_1, X_2) = \frac{1}{2}(\mathscr{L}_{X_0}(g(X_1, X_2)) + \mathscr{L}_{X_1}(g(X_0, X_2)) - \mathscr{L}_{X_2}(g(X_0, X_1)) + g([X_0, X_1], X_2) + g([X_2, X_0], X_1) - g([X_1, X_2], X_0)). \tag{4.45}$$

Now for $X, Y, Z \in \Gamma^\infty(\mathsf{TM})$ we compute

$$\begin{aligned}
g((\nabla_X J)Y, Z) &= g(\nabla_X(JY), Z) - g(J(\nabla_X Y), Z) \\
&= g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ) \\
&= \frac{1}{2}(\mathscr{L}_X(g(JY, Z)) + \mathscr{L}_{JY}(g(X, Z)) - \mathscr{L}_Z(g(X, JY)) \\
&\quad + g([X, JY], Z) + g([Z, X], JY) - g([JY, Z], X) \\
&\quad + \mathscr{L}_X(g(Y, JZ)) + \mathscr{L}_Y(g(X, JZ)) - \mathscr{L}_{JZ}(g(X, Y)) \\
&\quad + g([X, Y], JZ) + g([JZ, X], Y) - g([Y, JZ], X)) \\
&= \frac{1}{2}(\mathscr{L}_Y(g(X, JZ)) - \mathscr{L}_Z(g(X, JY)) + \mathscr{L}_X(g(JY, Z)) \\
&\quad - \mathrm{d}\omega(X, JY, JZ) - g([X, JY], Z) + g([X, JZ], Y) + g([JY, JZ], JX) \\
&\quad + g([X, JY], Z) + g([Z, X], JY) - g([JY, Z], X) \\
&\quad + g([X, Y], JZ) + g([JZ, X], Y) - g([Y, JZ], X)) \\
&= \frac{1}{2}(\mathscr{L}_Y(g(X, JZ)) - \mathscr{L}_Z(g(X, JY)) + \mathscr{L}_X(g(JY, Z)) \\
&\quad - \mathrm{d}\omega(X, JY, JZ) + g([JY, JZ], JX) + g(J[X, Z], Y) - g(J[JY, Z], JX) \\
&\quad - g(J[X, Y], Z) - g(J[Y, JZ], JX) - g([Y, Z], JX) - g(J[Y, Z], X) \\
&= \frac{1}{2}(\mathrm{d}\omega(X, Y, Z) - \mathrm{d}\omega(X, JY, JZ) + g(N_J(Y, Z), JX)).
\end{aligned}$$

Here we have made use of the relations (4.43), (4.44), and (4.45). We have also repeatedly used Proposition 4.1.18. The result follows since $N_J = 0$. ▼

2 Lemma *An almost complex structure* J *on a manifold* M *is a complex structure if and only if there exists a torsion-free affine connection* ∇ *on* M *such that* $\nabla J = 0$.

Proof Let J be an almost complex structure on M. We may first suppose that M possesses *some* torsion free affine connection $\widetilde{\nabla}$ (see [Lang 1995]). Given such an affine connection, let us define a $(1,2)$ tensor field Q on M by

$$Q(X,Y) = \frac{1}{4}(\widetilde{\nabla}_{JY}J)X + \frac{1}{4}J((\widetilde{\nabla}_Y J)X) + \frac{1}{2}J((\widetilde{\nabla}_X J)Y).$$

As Q is a $(1,2)$ tensor field,

$$\nabla_X Y = \widetilde{\nabla}_X Y - Q(X,Y)$$

defines another affine connection on M. We first claim that $\nabla J = 0$. We note that $0 = \widetilde{\nabla}_X J^2 = (\widetilde{\nabla}_X J)J + J(\widetilde{\nabla}_X J)$ from which we arrive at

$$J((\widetilde{\nabla}_X J)JY) = (\widetilde{\nabla}_X J)Y. \tag{4.46}$$

Now for $X, Y \in \Gamma^\infty(\mathsf{TM})$ we compute

$$\begin{aligned}
(\nabla_X J)Y &= \nabla_X(JY) - J(\nabla_X Y) \\
&= \widetilde{\nabla}_X(JY) - J(\widetilde{\nabla}_X Y) - Q(X,JY) + JQ(X,Y) \\
&= (\widetilde{\nabla}_X J)Y + \frac{1}{4}(\widetilde{\nabla}_Y J)X - \frac{1}{4}J((\widetilde{\nabla}_{JY}J)X) - \frac{1}{2}J((\widetilde{\nabla}_X J)JY) \\
&\quad + \frac{1}{4}J((\widetilde{\nabla}_{JY}J)X) - \frac{1}{4}(\widetilde{\nabla}_Y J)X - \frac{1}{2}(\widetilde{\nabla}_X J)Y \\
&= \frac{1}{2}(\widetilde{\nabla}_X J)Y - \frac{1}{2}J((\widetilde{\nabla}_X J)JY).
\end{aligned}$$

Substitution of (4.46) shows that $\nabla J = 0$. Next we show that $N_J = 4T$ where T is the torsion tensor of ∇. We have

$$\begin{aligned}
N_J(X,Y) &= [JX,JY] - [X,Y] - J([JX,Y]) - J([X,JY]) \\
&= \nabla_{JX}JY - \nabla_{JY}JX - \nabla_X Y + \nabla_Y X \\
&\quad - J(\nabla_{JX}Y - \nabla_Y JX) - J(\nabla_X JY - \nabla_{JY}X) \\
&= (\nabla_{JX}J)Y + J((\nabla_Y J)X) - (\nabla_{JY}J)X - J((\nabla_X J)Y) \\
&= 4(Q(X,Y) - Q(Y,X)) = 4T(X,Y).
\end{aligned}$$

We have repeatedly used here the derivation properties of ∇.

To summarise the above, given an almost complex structure on M we have constructed an affine connection ∇ such that $\nabla J = 0$ and $N_J = 4T$. Now suppose that J is a complex structure. Then by Theorem 4.8.4 $N_J = 0$ and so $T = 0$ and ∇ as constructed above is a torsion free affine connection on M with $\nabla J = 0$. Now suppose that there exists a torsion free affine connection ∇ such that $\nabla J = 0$. Following through the above computations with $\widetilde{\nabla} = \nabla$ and $Q = 0$ shows that $N_J = 0$ since ∇ is torsion free. Thus J is a complex structure by Theorem 4.8.4. ▼

3 Lemma *If* ∇ *is a torsion-free affine connection on* M *then* $\mathrm{d}\alpha = (-1)^k(k+1)\,\mathrm{Alt}(\nabla\alpha)$ *for* $\alpha \in \Gamma^\infty(\bigwedge^k(\mathsf{T}^*\mathsf{M}))$.

Proof We proceed by induction on k. For $k = 0$ the result follows since $\nabla_X = \mathscr{L}_X$ on $C^\infty(\mathsf{M})$. Now assume the lemma true for $k = l - 1$ and let $\alpha \in \Gamma^\infty(\bigwedge^l(\mathsf{T}^*\mathsf{M}))$. For $X_0, X_1, \dots, X_l \in \Gamma^\infty(\mathsf{TM})$ we compute

$$\begin{aligned}
\mathrm{d}\alpha(X_0, X_1, \dots, X_l) &= (X_0 \lrcorner\, \mathrm{d}\alpha)(X_1, \dots, X_l) \\
&= (\mathscr{L}_{X_0}\alpha)(X_1, \dots, X_l) - (\mathrm{d}(X_0 \lrcorner\, \alpha))(X_1, \dots, X_l) \\
&= \mathscr{L}_{X_0}(\alpha(X_1, \dots, X_l)) - \sum_{i=1}^{l} \alpha(X_1, \dots, [X_0, X_i], \dots, X_l) \\
&\quad - (-1)^{l-1} l(\mathrm{Alt}(\nabla(X_0 \lrcorner\, \alpha)))(X_1, \dots, X_l) \\
&= \mathscr{L}_{X_0}(\alpha(X_1, \dots, X_l)) - \sum_{i=1}^{l} \alpha(X_1, \dots, \nabla_{X_0}X_i, \dots, X_l) \\
&\quad + \sum_{i=1}^{l} \alpha(X_1, \dots, \nabla_{X_i}X_0, \dots, X_l) + (-1)^l l(\mathrm{Alt}(\nabla(X_0 \lrcorner\, \alpha)))(X_1, \dots, X_l).
\end{aligned} \tag{4.47}$$

Here we have used Cartan's formula, the derivation property of $\mathscr{L}_{X_0}$, the induction hypothesis, and the fact that ∇ is torsion free so that $[X, Y] = \nabla_X Y - \nabla_Y X$. Observe that the first two terms in (4.47) combine to give

$$\mathscr{L}_{X_0}(\alpha(X_1, \dots, X_l)) - \sum_{i=1}^{l} \alpha(X_1, \dots, \nabla_{X_0}X_i, \dots, X_l) = (\nabla_{X_0}\alpha)(X_1, \dots, X_l). \tag{4.48}$$

Let us simplify the last term in (4.47). We compute

$$(X_0 \lrcorner\, \alpha)(X_1, \dots, X_{l-1}) = \alpha(X_0, X_1, \dots, X_{l-1}).$$

If we covariantly differentiate both sides with respect to $X_l \in \Gamma^\infty(\mathsf{TM})$ we get

$$\begin{aligned}
(\nabla_{X_l}(X_0 \lrcorner\, \alpha))(X_1, \dots, X_{l-1}) + \sum_{i=1}^{l-1} (X_0 \lrcorner\, \alpha)(X_1, \dots, \nabla_{X_l}X_i, \dots, X_{l-1}) \\
= (\nabla_{X_l}\alpha)(X_0, X_1, \dots, X_{l-1}) + \alpha(\nabla_{X_l}X_0, X_1, \dots, X_{l-1}) \\
+ \sum_{i=1}^{l-1} \alpha(X_0, X_1, \dots, \nabla_{X_l}X_i, \dots, X_{l-1}).
\end{aligned}$$

This then gives

$$(\nabla(X_0 \lrcorner\, \alpha))(X_1, \dots, X_l) = (\nabla\alpha)(X_0, X_1, \dots, X_l) + \alpha(\nabla_{X_l}X_0, X_1, \dots, X_{l-1}).$$

Therefore,

$$
\begin{aligned}
(-1)^l l(\mathrm{Alt}(\nabla(X_0 \lrcorner \alpha)))(X_1,\dots,X_l) \\
&= \frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\mathrm{sign}\sigma)(\nabla\alpha)(X_0,X_{\sigma(1)},\dots,X_{\sigma(l)}) \\
&\quad + \frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\mathrm{sign}\sigma)\alpha(\nabla_{X_{\sigma(l)}}X_0,X_{\sigma(1)},\dots,X_{\sigma(l-1)}). \qquad (4.49)
\end{aligned}
$$

Substituting (4.48) and (4.49) into (4.47) gives

$$
\begin{aligned}
\mathrm{d}\alpha(X_0,\dots,X_l) &= \nabla\alpha(X_1,\dots,X_l,X_0) + \sum_{i=1}^{l}\alpha(X_1,\dots,\nabla_{X_i}X_0,\dots,X_l) \\
&\quad + \frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\mathrm{sign}\sigma)(\nabla\alpha)(X_0,X_{\sigma(1)},\dots,X_{\sigma(l)}) \\
&\quad + \frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\mathrm{sign}\sigma)\alpha(\nabla_{X_{\sigma(l)}}X_0,X_{\sigma(1)},\dots,X_{\sigma(l-1)}). \qquad (4.50)
\end{aligned}
$$

One may readily verify that since α is alternating in all arguments that

$$
\frac{1}{(l-1)!}\sum_{\substack{\sigma\in S_l\\ \sigma(l)=i}}(\mathrm{sign}\sigma)\alpha(\nabla_{X_i}X_0,X_{\sigma(1)},\dots,X_{\sigma(l-1)}) = (-1)^{i+l}\alpha(\nabla_{X_i}X_0,X_1,\dots,\hat{X}_i,\dots,X_l)
$$

where ˆ means the term is omitted. Therefore, we compute

$$
\begin{aligned}
\frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\mathrm{sign}\sigma)\alpha(\nabla_{X_{\sigma(l)}}X_0,X_{\sigma(1)},\dots,X_{\sigma(l-1)}) \\
&= \frac{(-1)^l}{(l-1)!}\sum_{i=1}^{l}\sum_{\substack{\sigma\in S_l\\ \sigma(l)=i}}(\mathrm{sign}\sigma)\alpha(\nabla_{X_i}X_0,X_{\sigma(1)},\dots,X_{\sigma(l-1)}) \\
&= \sum_{i=1}^{l}(-1)^i\alpha(\nabla_{X_1}X_0,X_1,\dots,\hat{X}_i,\dots,X_l) \\
&= -\sum_{i=1}^{l}\alpha(X_1,\dots,\nabla_{X_i}X_0,\dots,X_l).
\end{aligned}
$$

Thus the second and fourth terms in (4.50) cancel. Since $\nabla\alpha$ is alternating in the first l

entries we may see that

$$\begin{aligned}(-1)^l(l+1)\operatorname{Alt}(\nabla\alpha)(X_0,\dots,X_l) &= \frac{(-1)^l}{l!}\sum_{\sigma\in S_{l+1}}(\operatorname{sign}\sigma)(\nabla\alpha)(X_{\sigma(0)},\dots,X_{\sigma(l)})\\ &= \frac{(-1)^l}{l!}\sum_{i=0}^{l}\sum_{\substack{\sigma\in S_{l+1}\\ \sigma(l)=i}}(\operatorname{sign}\sigma)(\nabla\alpha)(X_{\sigma(0)},\dots,X_{\sigma(l-1)},X_i)\\ &= \sum_{i=0}^{l}(-1)^i(\nabla\alpha)(X_0,\dots,\hat{X}_i,\dots,X_{l-1},X_i). \end{aligned} \tag{4.51}$$

Similarly we compute

$$\frac{(-1)^l}{(l-1)!}\sum_{\sigma\in S_l}(\operatorname{sign}\sigma)(\nabla\alpha)(X_0,X_{\sigma(1)},\dots,X_{\sigma(l)}) = \sum_{i=1}^{l}(-1)^i(\nabla\alpha)(X_0,X_1,\dots,\hat{X}_i,\dots,X_{l-1},X_i). \tag{4.52}$$

Combining (4.51) and (4.52) we see that the first and third terms of (4.50) are exactly

$$(-1)^l(l+1)(\operatorname{Alt}(\nabla\alpha))(X_0,\dots,X_l)$$

which is the lemma. ▼

It is now straightforward to prove the theorem.

(i) $\Longrightarrow$ (ii) Suppose that h is a Kähler metric so that $\mathrm{d}\omega = 0$. By Lemma 1 this implies that $\nabla J = 0$.

(ii) $\Longrightarrow$ (iii) Since $\omega^\flat = -g\circ J$, the fact that $\nabla J = 0$ and $\nabla g = 0$ implies that $\nabla\omega = 0$.

(iii) $\Longrightarrow$ (i) By Lemma 3, if $\nabla\omega = 0$ we have $\mathrm{d}\omega = 0$. ■

The preceding establishes what we mean by a Kähler manifold: it is a Hermitian manifold with an additional integrability condition. Cases where this integrability condition is satisfied are interesting and are extensively studied in algebraic geometry [Griffiths and Harris 1978]. However, the condition is quite non-generic, and is seldom met in situations having to do with applications. Thus it is not one that we will pursue in our treatment that we intend to be broadly applicable.

4.9.2 A simple example of a Kähler manifold

We consider $\mathbb{CP}^1$, which we showed in Example 4.2.2–4 is C^ω-diffeomorphic to $\mathbb{S}^2$. In this section we examine the structure of $\mathbb{CP}^1$ by working explicitly with $\mathbb{S}^2$, and understand its holomorphic geometry by relating it to the geometry of $\mathbb{R}^3$.

We denote by $\mathbb{G}$ the Euclidean inner product on $\mathbb{R}^3$ which we also regard as a Riemannian metric on $\mathbb{R}^3$ by the identification of the tangent spaces of the manifold $\mathbb{R}^3$

with $\mathbb{R}^3$ in the usual way; that is, $\mathsf{T}_x\mathbb{R}^3 \simeq \mathbb{R}^3$. We define a two-form ω and a Riemannian metric g on $\mathbb{S}^2$ by

$$\omega_x(\boldsymbol{u},\boldsymbol{v}) = \mathbb{G}(\boldsymbol{x},\boldsymbol{u}\times\boldsymbol{v}), \qquad g_x(\boldsymbol{u},\boldsymbol{v}) = \mathbb{G}(\boldsymbol{u},\boldsymbol{v}),$$

where $\times$ is the usual vector cross-product and where we think of $\mathsf{T}_x\mathbb{S}^2$ as a subspace of $\mathbb{R}^3$ for $\boldsymbol{x}\in\mathbb{S}^2$. Using the fact that $\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to $\boldsymbol{x}$ for every $\boldsymbol{u},\boldsymbol{v}\in\mathsf{T}_x\mathbb{S}^2$ we conclude that ω is nondegenerate. Clearly $\mathrm{d}\omega = 0$ and so ω is a symplectic form. Let us define a $(1,1)$-tensor field J on $\mathbb{S}^2$ by $J = \omega^\sharp\circ g^\flat$. First we obtain an explicit formula for J.

4.9.5 Lemma *$J_{\mathbf{x}}(\mathbf{v}) = -\mathbf{x}\times\mathbf{v}$ for $\mathbf{x}\in\mathbb{S}^2$ and $\mathbf{v}\in\mathsf{T}_{\mathbf{x}}\mathbb{S}^2$.*

Proof We identify $\mathsf{T}^*_x\mathbb{S}^2$ with $\mathsf{T}_x\mathbb{S}^2$ using g, i.e., we write the pairing of $\mathsf{T}^*_x\mathbb{S}^2$ with $\mathsf{T}_x\mathbb{S}^2$ using the Euclidean inner product. With this identification, we may easily verify that

$$\omega^\flat_x(\boldsymbol{v}) = \boldsymbol{x}\times\boldsymbol{v}, \qquad g^\flat_x(\boldsymbol{v}) = \boldsymbol{v}$$

by checking that

$$\mathbb{G}(\omega^\flat_x(\boldsymbol{v}),\boldsymbol{u}) = \omega_x(\boldsymbol{v},\boldsymbol{u}), \qquad \mathbb{G}(g^\flat_x(\boldsymbol{v}),\boldsymbol{u}) = g_x(\boldsymbol{v},\boldsymbol{u})$$

for every $\boldsymbol{u}\in\mathsf{T}_x\mathbb{S}^2$.

We claim that

$$\omega^\sharp_x(\boldsymbol{v}) = -\boldsymbol{x}\times\boldsymbol{v}.$$

Indeed, if $\omega^\sharp$ is so defined, we have

$$\omega^\sharp_x\circ\omega^\flat_x(\boldsymbol{v}) = \omega^\sharp_x(\boldsymbol{x}\times\boldsymbol{v}) = -\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{v}).$$

We now invoke the vector product identity

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{4.53}$$

to show that $\omega^\sharp_x\circ\omega^\flat_x(\boldsymbol{v}) = \boldsymbol{v}$ thus showing that our proposed formula for $\omega^\sharp_x$ is correct. Now we compute

$$J_x(\boldsymbol{v}) = \omega^\sharp_x\circ g^\flat_x(\boldsymbol{v}) = \omega^\sharp_x(\boldsymbol{v}) = -\boldsymbol{x}\times\boldsymbol{v},$$

as claimed. ■

Note that J is an almost complex structure. Indeed,

$$J_x\circ J_x(\boldsymbol{v}) = -J_x(\boldsymbol{x}\times\boldsymbol{v}) = -\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{v}) = -\boldsymbol{v},$$

using Example 4.53 and the fact that $\|\boldsymbol{x}\| = 1$. We claim that J is the complex structure on $\mathbb{S}^2$ if we think of $\mathbb{S}^2$ as a holomorphic manifold as in Example 4.2.2–4. We do this by showing that the local representative of J is the canonical complex structure in local coordinates. To do this, recall that we cover $\mathbb{S}^2$ with charts $(\mathcal{U}_+,\phi_+)$ and $(\mathcal{U}_-,\phi_-)$ given by

$$\mathcal{U}_+ = \mathbb{S}^2\setminus\{(0,0,1)\}, \quad \mathcal{U}_- = \mathbb{S}^2\setminus\{(0,0,-1)\}$$

and

$$\phi_+(x_1, x_2, x_3) = \frac{x_1}{1 - x_3} + \mathrm{i}\frac{x_2}{1 - x_3}, \quad \phi_-(x_1, x_2, x_3) = \frac{x_1}{1 + x_3} - \mathrm{i}\frac{x_2}{1 + x_3};$$

note that we have changed the notation from Example 4.2.2–3 to ensure the overlap map is holomorphic. Let us denote coordinates for the charts $(\mathcal{U}_+, \phi_+)$ and $(\mathcal{U}_-, \phi_-)$ by z_+ and z_-, respectively. One then determines that, for $(x_1, x_2, x_3) \in \mathcal{U}_+$, we have

$$(x_1, x_2, x_3) = \Big(\frac{2\,\mathrm{Re}(z_+)}{|z_+|^2 + 1}, \frac{2\,\mathrm{Im}(z_+)}{|z_+|^2 + 1}, \frac{|z_+|^2 - 1}{|z_+|^2 + 1}\Big)$$

if $z_+ = \phi_+(x_1, x_2, x_3)$ and, for $(x_1, x_2, x_3) \in \mathcal{U}_-$, we have

$$(x_1, x_2, x_3) = \Big(\frac{2\,\mathrm{Re}(z_-)}{|z_-|^2 + 1}, -\frac{2\,\mathrm{Im}(z_-)}{|z_-|^2 + 1}, \frac{1 - |z_-|^2}{|z_-|^2 + 1}\Big)$$

if $z_- = \phi_-(x_1, x_2, x_3)$.

Let us recall a few elementary facts about one-dimensional $\mathbb{C}$-vector spaces.

4.9.6 Lemma *Let* V *and* V' *be two-dimensional* $\mathbb{R}$*-vector spaces with linear complex structures* J' *and* J' *and compatible inner products* g *and* g'*, let* ω *and* ω' *be the corresponding fundamental forms, and let* $(\mathrm{e}_1, \mathrm{e}_2)$ *and let* $(\mathrm{e}'_1, \mathrm{e}'_2)$ *be* J*-adapted orthonormal bases for* V *and* V'*, respectively. For a nonzero linear map* $\mathrm{A}\colon \mathsf{V} \to \mathsf{V}'$*, the following statements are equivalent:*

(i) $\mathrm{A} \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{V}; \mathsf{V}')$*;*

(ii) the matrix representative of A *with respect to the bases* $(\mathrm{e}_1, \mathrm{e}_2)$ *and* $(\mathrm{e}'_1, \mathrm{e}'_2)$ *has the form*

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ -\mathrm{b} & \mathrm{a} \end{bmatrix}$$

for some $\mathrm{a}, \mathrm{b} \in \mathbb{R}$*;*

(iii) the following conditions hold:

(a) $\mathrm{g}'(\mathrm{A}(\mathrm{e}_1), \mathrm{A}(\mathrm{e}_2)) = 0$*;*

(b) $\mathrm{g}'(\mathrm{A}(\mathrm{e}_1), \mathrm{A}(\mathrm{e}_1)) = \mathrm{g}'(\mathrm{A}(\mathrm{e}_2), \mathrm{A}(\mathrm{e}_2))$*;*

(c) $\mathrm{A}^*\omega' = \alpha\omega$ *for* $\alpha \in \mathbb{R}_{>0}$*.*

Proof (i) $\Longrightarrow$ (ii) Since (e_1, e_2) are J-adapted, $e_2 = J(e_1)$, and similarly $e'_2 = J'(e'_1)$. We thus have $A(e_1) = ae'_1 - be'_2$ for some $a, b \in \mathbb{R}$. We also have

$$A(e_2) = A \circ J(e_1) = J' \circ A(e_1) = aJ'(e'_1) - bJ'(e'_2) = ae'_2 + be'_1,$$

from which the desired form of the matrix representative follows.

(ii) $\Longrightarrow$ (iii) The given matrix representative implies that

$$A(e_1) = ae'_1 - be'_2, \quad A(e_2) = be'_1 + ae'_2.$$

Direct computation, using the fact that (e'_1, e'_2) are g'-orthogonal, shows that the first two assertions hold. Using the fact that $\omega(u, v) = g(J(u), v)$ and similarly for ω', we can also show that $\omega'(A(u), A(v)) = (a^2 + b^2)\omega(u, v)$, giving the final desired conclusion.

(iii) $\implies$ (ii) Let us write $A(e_1) = ae'_1 - be'_2$ for $a, b \in \mathbb{R}$. Since $A(e_2)$ is orthogonal to $A(e_1)$ we must have $A(e_2) = \lambda(be'_1 + ae'_2)$ for some $\lambda \in \mathbb{R}$. The condition that $A(e_1)$ and $A(e_2)$ have the same length implies that $\lambda = \pm 1$. We now directly compute

$$\omega'(A(e_1), A(e_2)) = \lambda(a^2 + b^2),$$

which gives $\lambda = 1$ upon noting that, by hypothesis, we have

$$\lambda(a^2 + b^2) = \omega'(A(e_1), A(e_2)) = \alpha\omega(e_1, e_2) = \alpha > 0.$$

(ii) $\implies$ (i) Using the given form of the matrix representative of A, we compute

$$J' \circ A(e_1) = J'(ae'_1 - be'_2) = ae'_2 + be'_1$$

and

$$J' \circ A(e_2) = J'(be'_1 + ae'_2) = be'_2 - ae'_1.$$

We also compute

$$A \circ J(e_1) = A(e_2) = be'_1 + ae'_2$$

and

$$A \circ J(e_2) = -A(e_1) = -ae'_1 + be'_2,$$

and we indeed see that $J' \circ A = A \circ J$. ■

We now wish to verify the following formula, which shows that the almost complex structure J is indeed the complex structure on $\mathbb{S}^2$ induced by the holomorphic atlas $((\mathcal{U}_+, \phi_+), (\mathcal{U}_-, \phi_-))$.

4.9.7 Lemma *The formulae*

$$\mathrm{T}_{\mathbf{x}}\phi_+(\mathrm{J}_{\mathbf{x}}(\mathbf{v})) = \mathrm{i}\mathrm{T}_{\mathbf{x}}\phi_+(\mathbf{v}), \qquad \mathbf{x} \in \mathcal{U}_+,\ \mathbf{v} \in \mathsf{T}_{\mathbf{x}}\mathbb{S}^2,$$

and

$$\mathrm{T}_{\mathbf{x}}\phi_-(\mathrm{J}_{\mathbf{x}}(\mathbf{v})) = \mathrm{i}\mathrm{T}_{\mathbf{x}}\phi_-(\mathbf{v}), \qquad \mathbf{x} \in \mathcal{U}_-,\ \mathbf{v} \in \mathsf{T}_{\mathbf{x}}\mathbb{S}^2,$$

hold.

Proof We consider the mappings

$$\begin{aligned}\psi_+ \colon \mathbb{C} &\to \mathbb{R}^3\\ z_+ &\mapsto \Big(\frac{2\,\mathrm{Re}(z_+)}{|z_+|^2+1}, \frac{2\,\mathrm{Im}(z_+)}{|z_+|^2+1}, \frac{|z_+|^2-1}{|z_+|^2+1}\Big)\end{aligned}$$

and

$$\begin{aligned}\psi_- \colon \mathbb{C} &\to \mathbb{R}^3\\ z_- &\mapsto \Big(\frac{2\,\mathrm{Re}(z_-)}{|z_-|^2+1}, -\frac{2\,\mathrm{Im}(z_-)}{|z_-|^2+1}, \frac{1-|z_-|^2}{|z_-|^2+1}\Big),\end{aligned}$$

i.e., the inverses of ϕ_+ and ϕ_-, respectively. Note that ψ_+ and ψ_- are diffeomorphisms onto $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. In particular, $T_{z_+}\psi_+$ and $T_{z_-}\psi_-$ are isomorphisms onto $\mathsf{T}_{\psi_+(z_+)}\psi_+$ and $\mathsf{T}_{\psi_-(z_-)}\psi_i$, respectively. Note that the inner product on each of these latter tangent spaces are the restrictions of the Euclidean inner product $\mathbb{G}$. Let us write $z_+ = x_+ + \mathrm{i}y_+$ and $z_- = x_- + \mathrm{i}y_-$ and denote by $\frac{\partial\psi_+}{\partial x_+}$ and $\frac{\partial\psi_+}{\partial y_+}$ the vectors of partial derivatives of ψ_+ and similarly $\frac{\partial\psi_-}{\partial x_-}$ and $\frac{\partial\psi_-}{\partial y_-}$. We can then compute

$$\left\|\frac{\partial\psi_+}{\partial x_+}(z_+)\right\|^2 = \left\|\frac{\partial\psi_+}{\partial y_+}(z_+)\right\|^2 = \frac{4}{(1+|z_+|^2)^2}, \quad \mathbb{G}\Big(\frac{\partial\psi_+}{\partial x_+}(z_+), \frac{\partial\psi_+}{\partial y_+}(z_+)\Big) = 0$$

and

$$\left\|\frac{\partial\psi_-}{\partial x_-}(z_-)\right\|^2 = \left\|\frac{\partial\psi_-}{\partial y_-}(z_-)\right\|^2 = \frac{4}{(1+|z_-|^2)^2}, \quad \mathbb{G}\Big(\frac{\partial\psi_-}{\partial x_-}(z_-), \frac{\partial\psi_-}{\partial y_-}(z_-)\Big) = 0.$$

The proof will now follow from the implication (iii) $\Longrightarrow$ (i) from Lemma 4.9.6 if we can show that $\psi_+^*\omega$ and $\psi_-^*\omega$ are positive multiples of the standard volume form of $\mathbb{C} \simeq \mathbb{R}^2$. Since $\psi_+^*\omega$ and $\psi_-^*\omega$ are each equal multiples of the standard volume form, since the multiple is nowhere zero by virtue of the ψ_+ and ψ_- being diffeomorphisms, and since $\mathbb{C}$ is connected, it suffices to show that $\psi_+^*\omega$ and $\psi_-^*\omega$ are a positive multiples of the standard volume form at a single point in $\mathbb{C}$. Thus let consider the point $(0,0,-1) \in \mathcal{U}_+$ for which $\phi_+(0,0,-1) = 0$. Let (e_1, e_2) be the standard basis for $\mathsf{T}_0\mathbb{C} \simeq \mathbb{R}^2$. A direct computation gives $T_0\psi_+(e_1) = 2\boldsymbol{e}_1$ and $T_0\psi_+(e_2) = 2\boldsymbol{e}_2$, where by slight abuse we let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis for $\mathsf{T}_{(0,0,-1)}\mathbb{R}^3$ so that $(\boldsymbol{e}_2, \boldsymbol{e}_2)$ are a basis for $\mathsf{T}_{(0,0,-1)}\mathbb{S}^2$. From this we directly compute

$$\begin{aligned}\omega_{(0,0,-1)}(T_0\psi_+(e_1), T_0\psi_+(e_2)) &= g_{(0,0,-1)}(J_{(0,0,-1)} \circ T_0\psi_+(e_1), T_0\psi_+(e_2)) \\ &= g_{(0,0,-1)}(2\boldsymbol{e}_2, 2\boldsymbol{e}_2) = 4,\end{aligned}$$

using the fact that

$$J_{(0,0,-1)}(\boldsymbol{e}_1) = -(0,0,-1) \times (1,0,0) = \boldsymbol{e}_2.$$

Since the standard volume form acting on the pair (e_1, e_2) is equal to 1, it follows that $\psi_+^*\omega(0)$ is 4 times the standard volume form. An entirely similar argument applied to the point $(0,0,1)$ gives the same conclusion for $\psi_-^*\omega$, and so the lemma follows. ■

Note that $\mathbb{S}^2$ is automatically a Kähler manifold since, if we define the Hermitian metric $h = g + \mathrm{i}\omega$ on $\mathbb{S}^2$, we have $\mathrm{d}\omega = 0$. According to Theorem 4.9.4, it follows that J and ω are parallel with respect to the Levi-Civita connected associated with the round Riemannian metric on $\mathbb{S}^2$.

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