Chapter 5

Holomorphic and real analytic jet bundles

In this chapter we study quite carefully the structure of jet bundles. Jets can be thought of as a way of adapting the notion of Taylor series to the differential geometric setting. As we shall see, this adaptation requires a little care if one is to properly describe the algebraic structure of these jets. In order to present the theory in appropriate context, our development will include some discussion of the smooth case, along with the holomorphic and real analytic case. We shall see that jet bundles, in the holomorphic and real analytic case, have a nice correspondence with germs that one does not have in the smooth case.

We give a completely self-contained account of jet bundles here. However, further details and some applications can be found in Saunders [1989] in [Kolář, Michor, and Slovák 1993, Chapter IV], at least in the real case. We do not know of a reference that covers the holomorphic case, although the constructions are straightforward adaptations of the real ones.

5.1 Preliminaries to jet bundle constructions

Prior to embarking on our construction of jet bundles, it is helpful to organise a few preliminary constructions. Throughout this section, as usual, we use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. We will talk about objects of class C^r with $r \in \{\infty, \omega\}$ if $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We will, as in the preceding section, denote $\mathsf{TM} = \mathsf{T}^{1,0}\mathsf{M}$ and $\mathsf{T}^*\mathsf{M} = \bigwedge^{1,0}(\mathsf{M})$ if M is a holomorphic manifold. This will greatly facilitate simultaneously treating the real and complex cases of the constructions we make in this section.

5.1.1 Affine spaces and affine bundles

Intuitively, an affine space is a "vector space without an origin." In an affine space, one can add a vector to an element, and one can take the difference of two elements to get a vector. But one cannot add two elements. Precisely, we have the following definition.

5.1.1 Definition (Affine space) Let F be a field and let V be an F-vector space. An ***affine space*** modelled on V is a set A and a map $\phi\colon \mathsf{V} \times \mathsf{A} \to \mathsf{A}$ with the following properties:

(i) for every $x, y \in \mathsf{A}$ there exists an $v \in \mathsf{V}$ such that $y = \phi(v, x)$ (***transitivity***);

(ii) $\phi(v, x) = x$ for every $x \in \mathsf{A}$ implies that $v = 0$ (***faithfulness***);

(iii) $\phi(0, x) = x$, and

(iv) $\phi(u + v, x) = \phi(u, \phi(v, x))$.

The notation $x + v$ if often used for $\phi(v, x)$ and, for $x, y \in \mathsf{A}$, we denote by $y - x \in \mathsf{V}$ the unique vector such that $\phi(y - x, x) = y$. •

An affine space is "almost" a vector space. The following result says that, if one chooses any point in an affine space as an "origin," then the affine space becomes a vector space.

5.1.2 Proposition (Vector spaces from affine spaces) *Let* A *be an affine space modelled on the* F*-vector space* V*. For* $\mathsf{x}_0 \in \mathsf{A}$ *define vector addition on* A *by*

$$\mathsf{x}_1 + \mathsf{x}_2 = \mathsf{x}_0 + ((\mathsf{x}_1 - \mathsf{x}_0) + (\mathsf{x}_2 - \mathsf{x}_0))$$

and scalar multiplication on A *by*

$$\mathsf{a}\,\mathsf{x} = \mathsf{x}_0 + (\mathsf{a}\,(\mathsf{x} - \mathsf{x}_0)).$$

These operations make A *into a* F*-vector space and the map* $\mathsf{x} \mapsto \mathsf{x} - \mathsf{x}_0$ *is an isomorphism of this* F*-vector space with* V*.*

Proof The boring verification of the satisfaction of the vector space axioms we leave to the reader. To verify that the map $x \mapsto x - x_0$ is a vector space isomorphism, compute

$$(x_1 + x_2) - x_0 = (x_0 + ((x_1 - x_0) + (x_2 - x_0))) - x_0 = (x_1 - x_0) + (x_2 - x_0)$$

and

$$a\,x - x_0 = (x_0 + (a\,(x - x_0))) - x_0 = a\,(x - x_0),$$

as desired. ■

Let us denote by $\Phi_{x_0} \colon \mathsf{A}_{x_0} \to \mathsf{V}$ the isomorphism defined in Proposition 5.1.2. Note that we have

$$\Phi_{x_0}(x) = x - x_0, \quad \Phi_{x_0}^{-1}(v) = x_0 + v.$$

We shall use these formulae below.

We have the notion of an affine subspace of an affine space.

5.1.3 Definition (Affine subspace) Let V be a F-vector space and let A be an affine space modelled on V with $\phi \colon \mathsf{V} \times \mathsf{A} \to \mathsf{A}$ the map defining the affine structure. A subset B of A is an ***affine subspace*** if there is a subspace U of V with the property that $\phi|\mathsf{U} \times \mathsf{B}$ takes values in B. •

Let us give a list of alternative characterisations of affine subspaces.

5.1.4 Proposition (Characterisations of affine subspaces) *Let* A *be an affine space modelled on the* F*-vector space* V *and let* $\mathsf{B} \subseteq \mathsf{A}$*. The following statements are equivalent:*

(i) B *is an affine subspace of* A*;*

(ii) there exists a subspace U *of* V *such that, for each* $x_0 \in \mathsf{B}$*,* $\mathsf{B} = \{x_0 + u \mid u \in \mathsf{U}\}$*;*

(iii) if $x_0 \in \mathsf{B}$ *then* $\{y - x_0 \mid y \in \mathsf{B}\} \subseteq \mathsf{V}$ *is a subspace.*

Proof (i) $\Longrightarrow$ (ii) Let $\mathsf{B} \subseteq \mathsf{A}$ be an affine subspace and let $\mathsf{U} \subseteq \mathsf{V}$ be a subspace for which $\phi|\mathsf{U} \times \mathsf{B}$ takes values in B. Let $x_0 \in \mathsf{B}$. For $y \in \mathsf{B}$ there exists a unique $u \in \mathsf{V}$ such that $y = x_0 + u$. Since $\phi|\mathsf{U} \times \mathsf{B}$ takes values in B it follows that $u \in \mathsf{U}$. Therefore,

$$\mathsf{B} \subseteq \{x_0 + u \mid u \in \mathsf{U}\}.$$

Also, if $u \in \mathsf{U}$ then $x_0 + u \in \mathsf{B}$ by definition of an affine subspace, giving

$$\mathsf{B} \supseteq \{x_0 + u \mid u \in \mathsf{U}\},$$

and so giving this part of the result.

(ii) $\Longrightarrow$ (iii) Let $\mathsf{U} \subseteq \mathsf{V}$ be a subspace for which, for each $x_0 \in \mathsf{B}$, $\mathsf{B} = \{x_0 + u \mid u \in \mathsf{U}\}$. Obviously, $\{y - x_0 \mid y \in \mathsf{B}\} = \mathsf{U}$ and so this part of the result follows.

(iii) $\Longrightarrow$ (i) Let $x_0 \in \mathsf{B}$ and denote $\mathsf{U} = \{y - x_0 \mid y \in \mathsf{B}\}$; by hypothesis, U is a subspace. Moreover, for $u \in \mathsf{U}$ and $y \in \mathsf{B}$ we have

$$\phi(u, y) = \phi(u, x_0 + (y - x_0)) = x_0 + (u + y - x_0) \in \mathsf{B},$$

giving the result. ■

We also have notions of maps between affine spaces. To make this definition, it is convenient to denote by A_{x_0} the set A with the F-vector space structure defined as in Proposition 5.1.2 by a choice of $x_0 \in \mathsf{A}$.

5.1.5 Definition (Affine map) If A and B are affine spaces modelled on F-vector spaces V and U, respectively, a map $\phi \colon \mathsf{A} \to \mathsf{B}$ is an ***affine map*** if, for some $x_0 \in \mathsf{A}$, ϕ is a linear map between the vector spaces A_{x_0} and $\mathsf{B}_{\phi(x_0)}$. •

Associated with an affine map is an induced linear map between the corresponding vector spaces.

5.1.6 Proposition (Linear map associated to an affine map) *Let* V *and* U *be* F*-vector spaces, let* A *and* B *be affine spaces modelled on* V *and* U*, respectively, and let* $\phi \colon \mathsf{A} \to \mathsf{B}$ *be an affine map. Let* $x_0 \in \mathsf{A}$ *be such that* $\phi \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{A}_{x_0}; \mathsf{B}_{\phi(x_0)})$*. Then the map* $L(\phi) \colon \mathsf{V} \to \mathsf{U}$ *defined by*

$$L(\phi)(v) = \phi(x_0 + v) - \phi(x_0)$$

is linear. Moreover,

(i) if $x_1, x_2 \in \mathsf{A}$ *are such that* $x_2 = x_1 + v$*, then* $L(\phi)(v) = \phi(x_2) - \phi(x_1)$ *and*

(ii) if $x_0' \in \mathsf{A}$ *then* $\phi(x) = \phi(x_0') + L(\phi)(x - x_0')$ *for every* $x \in \mathsf{V}$*.*

Proof Note that $L(\phi) = \Phi_{\phi(x_0)} \circ \phi \circ \Phi_{x_0}^{-1}$. Linearity of $L(\phi)$ follows since all maps in the composition are linear.

(i) Now let $x_1, x_2 \in \mathsf{A}$ and denote $v = x_2 - x_1$. Write $x_1 = x_0 + v_1$ and $x_2 = x_0 + v_2$ for $v_1, v_2 \in \mathsf{V}$. Then

$$v_2 - v_1 = (x_0 + v_2) - (x_0 + v_1) = x_2 - x_1 = v,$$

and so

$$\begin{aligned}\phi(x_2) - \phi(x_1) &= \phi(x_0 + v_2) - \phi(x_0 + v_1)\\ &= (\phi(x_0) + \phi(x_0 + v_2)) - (\phi(x_0) + \phi(x_0 + v_1))\\ &= (\phi(x_0 + v_2) - \phi(x_0)) - (\phi(x_0 + v_1) - \phi(x_0))\\ &= \Phi_{\phi(x_0)} \circ \phi \circ \Phi_{x_0}^{-1}(v_2) - \Phi_{\phi(x_0)} \circ \phi \circ \Phi_{x_0}^{-1}(v_1)\\ &= L(\phi)(v_2 - v_1) = L(\phi)(v),\end{aligned}$$

as desired.

(ii) By the previous part of the result,

$$L(\phi)(x - x_0') = \phi(x) - \phi(x_0'),$$

from which the result follows by rearrangement. ■

The linear map $L(\phi)$ is called the ***linear part*** of ϕ. The last assertion of the proposition says that an affine map is determined by its linear part and what it does to a single element in its domain.

It is possible to give a few equivalent characterisations of affine maps.

5.1.7 Proposition (Characterisations of affine maps) *Let V and U be F-vector spaces, let A and B be affine spaces modelled on U and V, respectively, and let $\phi\colon \mathsf{A} \to \mathsf{B}$ be a map. Then the following statements are equivalent:*

(i) ϕ is an affine map;

(ii) $\phi \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{A}_{\mathsf{x}_0}; \mathsf{B}_{\phi(\mathsf{x}_0)})$ for every $\mathsf{x}_0 \in \mathsf{A}$;

(iii) $\Phi_{\phi(\mathsf{x}_0)} \circ \phi \circ \Phi_{\mathsf{x}_0}^{-1} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{U})$ for some $\mathsf{x}_0 \in \mathsf{V}$;

(iv) $\Phi_{\phi(\mathsf{x}_0)} \circ \phi \circ \Phi_{\mathsf{x}_0}^{-1} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{U})$ for all $\mathsf{x}_0 \in \mathsf{V}$.

Proof (i) $\Longrightarrow$ (ii) By Proposition 5.1.6 we have

$$\phi(x) = \phi(x_0) + L(\phi)(x - x_0)$$

for every $x, x_0 \in \mathsf{A}$, and from this the result follows.

(ii) $\Longrightarrow$ (iii) This follows immediately from Proposition 5.1.6.

(iii) $\Longrightarrow$ (iv) This also follows immediately from Proposition 5.1.6.

(iv) $\Longrightarrow$ (i) Let $x_0 \in \mathsf{A}$. Define a linear map $L(\phi) = \Phi_{\phi(x_0)} \circ \phi \circ \Phi_{x_0}^{-1}$. Then

$$\phi(x) = \phi(x_0) + L(\phi)(x - x_0).$$

Clearly, then, ϕ is an affine map. ■

Now we make the preceding algebraic constructions geometric.

5.1.8 Definition (Affine bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an $\mathbb{F}$-vector bundle of class C^r. A ***$\mathbf{C^r}$-affine bundle*** over M modelled on E is a fibre bundle $\tau\colon \mathsf{A} \to \mathsf{M}$ of class C^r and a map $\phi\colon \mathsf{E} \times_{\mathsf{M}} \mathsf{A} \to \mathsf{A}$ such that the diagram

define

$$\begin{tikzcd} \mathsf{E} \times_{\mathsf{M}} \mathsf{A} \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}_1"'] & \mathsf{A} \arrow[d, "\tau"] \\ \mathsf{E} \arrow[r, "\pi"'] & \mathsf{M} \end{tikzcd}$$

commutes and such that $a + e \triangleq \phi(e, a)$ makes A_x into an affine space modelled on E_x for each $x \in \mathsf{M}$. Here pr_1 is projection onto the first factor. •

We can define subbundles of affine bundles and maps between affine bundles.

5.1.9 Definition (Affine subbundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let F be an $\mathbb{F}$-vector subbundle of class C^r of the $\mathbb{F}$ vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ of class C^r and let $\tau\colon \mathsf{A} \to \mathsf{M}$ be a C^r-affine bundle modelled on E. A ***$\mathbf{C^r}$-affine subbundle*** of A modelled on F is a C^r-subbundle B of the fibre bundle $\tau\colon \mathsf{A} \to \mathsf{M}$ such that B_x is an affine subspace of A_x associated with F_b for each $x \in \mathsf{M}$. •

5.1.10 Definition (Affine bundle map) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. If $\tau_1\colon \mathsf{A}_1 \to \mathsf{M}_1$ and $\tau_2\colon \mathsf{A}_2 \to \mathsf{M}_2$ are C^r-affine bundles then a ***$\mathbf{C^r}$-affine bundle map*** between these affine bundles is a C^r-map $\Phi\colon \mathsf{A}_1 \to \mathsf{A}_2$ for which there exists a C^r-map $\Phi_0\colon \mathsf{M}_1 \to \mathsf{M}_2$ such that the diagram

$$\begin{tikzcd} \mathsf{A}_1 \arrow[r, "\Phi"] \arrow[d, "\tau_1"'] & \mathsf{A}_2 \arrow[d, "\tau_2"] \\ \mathsf{M}_1 \arrow[r, "\Phi_0"'] & \mathsf{M}_2 \end{tikzcd}$$

commutes and with the property that $\Phi|\tau_1^{-1}(x)\colon \tau_1^{-1}(x) \to \tau_2^{-1}(\Phi_0(x))$ is an affine map. If Φ is a C^r-diffeomorphism we say it is an ***affine bundle isomorphism***. •

5.1.2 Inverse systems

We shall be interested in spaces of infinite jets, and shall consider the algebraic structure on these. To do so, it is useful to have at hand the notion of the inverse limit, and in this section we make this a little precise. In Section GA2.2.3.7 we discuss inverse limits more comprehensively in their proper setting of category theory.

The approach for all of our various sorts of jet bundles is the same, and relies on the following category theory-based construction.

5.1.11 Definition (Inverse system, inverse limit) Let $(I, \succeq)$ be an inverse totally ordered set, i.e., $(I, \preceq)$ is an totally ordered set where $\preceq$ is defined by $i \preceq j$ if $j \succeq i$. Let $\mathcal{C} = (\mathscr{O}, \mathscr{F})$ be a category, i.e., $\mathscr{O}$ is a family of objects and $\mathscr{F}$ is a family of morphisms. An **I-*inverse system*** in $\mathcal{C}$ is a family $\mathscr{S} = (S_i)_{i\in I}$ of objects from $\mathscr{C}$ with a family

$$\mathscr{M} = (\pi_i^j \colon S_j \to S_i |\ j, i \in I,\ j \succeq i)$$

of morphisms from $\mathscr{F}$ satisfying

(i) $\pi_i^j = \pi_i^k \circ \pi_k^j$ for all $i, j, k \in \mathbb{Z}_{>0}$ such that $j \succeq k \succeq i$ and

(ii) $\pi_i^i = \mathrm{id}_{S_i}$ for all $i \in I$.

A ***inverse limit*** of an inverse system $(\mathscr{S}, \mathscr{M})$ is a pair $(S_\infty, (\pi_i^\infty)_{i\in I})$, where $S_\infty \in \mathscr{O}$ is an object and $\pi_i^\infty \colon S_\infty \to S_i$, $i \in I$, are morphisms from $\mathscr{F}$ such that

(iii) the diagram

$$\begin{tikzcd} & S_\infty \arrow[dl, "\pi_j^\infty"'] \arrow[dr, "\pi_i^\infty"] & \\ S_j \arrow[rr, "\pi_i^j"'] & & S_i \end{tikzcd} \tag{5.1}$$

commutes for every $i, j \in I$ such that $j \succeq i$ and

(iv) if $T \in \mathscr{O}$ and if $f_i \colon T \to S_i$, $i \in I$, are morphisms from $\mathscr{F}$ such that the diagram

$$\begin{tikzcd} & T \arrow[dl, "f_j"'] \arrow[dr, "f_i"] & \\ S_j \arrow[rr, "\pi_i^j"'] & & S_i \end{tikzcd} \tag{5.2}$$

commutes for every $i, j \in I$ such that $j \succeq i$, then there exists a unique morphism $g \colon T \to S_\infty$ from $\mathscr{F}$ such that the diagram

$$\begin{tikzcd} T \arrow[d, "g"'] \arrow[dr, "f_i"] & \\ S_\infty \arrow[r, "\pi_i^\infty"'] & S_i \end{tikzcd} \tag{5.3}$$

commutes for every $i \in I$.

We often denote $S_\infty = \mathrm{inv}\lim_I S_i$, suppressing all of the maps involved when they are understood. •

The so-called universal property of the inverse limit expressed by condition (iv) is encapsulated by the diagram

$$\begin{tikzcd}
& T \arrow[d, "g"] \arrow[ddl, "f_j"'] \arrow[ddr, "f_i"] & \\
& S_\infty \arrow[dl, "\pi_j^\infty"] \arrow[dr, "\pi_i^\infty"'] & \\
S_j \arrow[rr, "\pi_i^j"'] & & S_i
\end{tikzcd}$$

which commutes for $i, j \in I$ with $j \geq i$ for a unique morphism g.

We shall certainly not make use of the preceding definition in general categories. Our interest will be restricted to sets, vector spaces, and commutative algebras. In these cases we can describe inverse limits more or less concretely. Let

$$(\mathscr{S} = (S_k)_{k\in\mathbb{Z}_{>0}}, \mathscr{M} = (\pi_l^k \colon S_k \to S_l)_{k,l\in\mathbb{Z}_{>0},\ k\geq l})$$

be an inverse system in the category of vector spaces or commutative algebras over some field F. Define

$$S_\infty = \{\phi \colon I \to \mathring{\bigcup_{i\in I}} S_i \mid \ \phi(i) \in S_i,\ \pi_i^j \circ \phi(j) = \phi(i)\}$$

and define $\pi_i^\infty \colon S_\infty \to S_i$ by $\pi_i^\infty(\phi) = \phi(i)$, $i \in I$. If the category is that of F-vector spaces, we define addition and scalar multiplication in S_∞ component-wise:

$$(\phi + \psi)(i) = \phi(i) + \psi(i), \quad (a\phi)(i) = a(\phi(i)), \qquad i \in I,\ \phi, \psi \in S_\infty,\ a \in \mathsf{F}.$$

If the category is that of F-algebras, then we additionally define the product in S_∞ component-wise:

$$(\phi \cdot \psi)(i) = \phi(i) \cdot \psi(i), \qquad i \in I,\ \phi, \psi \in S_\infty.$$

Let us verify that $(S_\infty, (\pi_i^\infty)_{i\in I})$ is an inverse limit.

5.1.12 Lemma (Inverse limits exist and are unique) *Let* F *be a field and let* $(\mathrm{I}, \geq)$ *be an inverse totally ordered set. Let*

$$(\mathscr{S} = (\mathrm{S_i})_{\mathrm{i}\in\mathrm{I}}, \mathscr{M} = (\pi_{\mathrm{i}}^{\mathrm{j}} \colon \mathrm{S_j} \to \mathrm{S_i})_{\mathrm{i,j}\in\mathrm{I},\ \mathrm{j}\geq\mathrm{i}})$$

be an I*-inverse system in the category of sets,* F*-vector spaces, or* F*-algebras. If* $(\mathrm{S}_\infty, (\pi_{\mathrm{i}}^\infty)_{\mathrm{i}\in\mathrm{I}})$ *is as constructed above, then it is an inverse limit. Moreover, if* $(\mathrm{T}_\infty, (\rho_{\mathrm{i}}^\infty)_{\mathrm{i}\in\mathrm{I}})$ *is an inverse limit, then there exists a unique isomorphism* $\iota \colon \mathrm{T}_\infty \to \mathrm{S}_\infty$ *such that the diagram*

$$\begin{tikzcd}
\mathrm{T}_\infty \arrow[rr, "\iota"] \arrow[dr, "\rho_{\mathrm{i}}^\infty"'] & & \mathrm{S}_\infty \arrow[dl, "\pi_{\mathrm{i}}^\infty"] \\
& \mathrm{S_i} &
\end{tikzcd}$$

commutes for every $\mathrm{i} \in \mathrm{I}$*.*

Proof First of all, note that the direct product of $(S_i)_{i\in I}$ is

$$\prod_{i\in I} S_i = \{\phi\colon I \to \mathring{\bigcup_{i\in I}} S_i \mid \phi(i) \in S_i\}.$$

In the category of F-vector spaces or algebras, the operations are defined component-wise. Let us verify that S_∞ is a subspace or subalgebra in the case we are looking at these categories. For $a \in \mathsf{F}$, $\phi, \psi \in S_\infty$, and $i, j \in I$ with $j \geq i$ we have

$$\pi_i^j(a\phi(j)) = \pi_i^j(a(\phi(j))) = a\pi_i^j(\phi(j)) = a\phi(i),$$
$$\pi_i^j((\phi + \psi)(j)) = \pi_i^j(\phi(j) + \psi(j)) = \pi_i^j(\phi(j)) + \pi_i^j(\psi(j)) = \phi(i) + \psi(i) = (\phi + \psi)(i),$$

and, in the case of commutative algebras,

$$\pi_i^j((\phi \cdot \psi)(j)) = \pi_i^j(\phi(j) \cdot \psi(j)) = (\pi_i^j(\phi(j))) \cdot \pi_i^j(\psi(j)) = \phi(i) \cdot \psi(i) = (\phi \cdot \psi)(i).$$

This shows that S_∞ is an object in the appropriate category. Similarly styled computations show that the maps π_i^∞, $i \in I$, are morphisms in the appropriate category.

Next we show that $(S_\infty, (\pi_i^\infty)_{i\in I})$ is indeed an inverse limit. Firstly,

$$\pi_i^j \circ \pi_j^\infty(\phi) = \pi_i^j(\phi(j)) = \phi(i) = \pi_i^\infty(\phi),$$

giving commutativity of the diagram (5.1). Next let $T \in \mathscr{O}$ and if $f_i\colon T \to S_i$, $i \in I$, are morphisms from $\mathscr{F}$ such that the diagram (5.2) commutes, then define $g\colon T \to S_\infty$ by $g(y)(i) = f_i(y)$. We leave to the reader the elementary exercise of verifying that g is a morphism in the appropriate category. We also immediately have

$$\pi_i^\infty \circ g(y) = f_i(y),$$

giving the commutativity of the diagram (5.3). If $g'\colon T \to S_\infty$ is any other such morphism, then the commutativity of the diagram (5.3) commutes, then it immediately follows that $g'(y)(i) = f_i(y)$, giving $g' = g$.

Finally, we prove the last assertion of the lemma, supposing that $(T_\infty, (\rho_i^\infty)_{i\in I})$ is an inverse limit. By the second of the properties of inverse limits, let $\iota\colon T_\infty \to S_\infty$ be the unique morphism for which $\pi_i^\infty \circ \iota = \rho_i^\infty$ for each $i \in I$. All that remains to show is that ι is an isomorphism. Since $(T_\infty, (\rho_i^\infty)_{i\in I})$ and $(S_\infty, (\pi_i^\infty)_{i\in I})$ are inverse limits, by the second of the defining properties of inverse limits, there exists unique morphisms $\iota\colon T_\infty \to S_\infty$ and $\iota'\colon S_\infty \to T_\infty$ such that $\pi_i^\infty \circ \iota = \rho_i^\infty$ and $\rho_i^\infty \circ \iota' = \pi_i^\infty$ for each $i \in I$. Thus

$$\pi_i^\infty \circ \iota \circ \iota' = \rho_i^\infty \circ \iota' = \pi_i^\infty.$$

Thus we have

$$\pi_i^j \circ \pi_j^\infty \circ \iota \circ \iota' = \pi_i^j \circ \pi_j^\infty = \pi_i^\infty.$$

Thus if we replace "T" with "S_∞" and "f_i" with "$\pi_i^\infty \circ \iota \circ \iota'$" in the diagram (5.2), the diagram commutes. Since $(S_\infty, (\pi_i^\infty)_{i\in I})$ is an inverse limit, there exists a unique morphism $g\colon S_\infty \to S_\infty$ such that the corresponding diagram (5.3) commutes. However, the identity morphism make this diagram commutative, and so we must have $g = \mathrm{id}_{S_\infty}$. This means that the diagram

$$\begin{tikzcd} S_\infty \arrow[r, "\mathrm{id}_{S_\infty}"] \arrow[dr, "\pi_i^\infty \circ \iota \circ \iota'"'] & S_\infty \arrow[d, "\pi_i^\infty"] \\ & S_i \end{tikzcd}$$

commutes for every $i \in I$. Thus the morphism $\iota \circ \iota'$ has the property that the diagram

$$\begin{tikzcd} S_\infty \arrow[r, "\iota \circ \iota'"] \arrow[dr, "\pi_i^\infty"'] & S_\infty \arrow[d, "\pi_i^\infty"] \\ & S_i \end{tikzcd}$$

commutes for every $i \in I$. However, the identity morphism in S_∞ is the unique morphism which makes the preceding diagram commute since $(S_\infty, (\pi_i^\infty)_{i\in I})$ is an inverse limit. Thus $\iota \circ \iota' = \mathrm{id}_{S_\infty}$. By reversing arrows, one similarly shows that $\iota' \circ \iota = \mathrm{id}_{T_\infty}$, and this completes the proof. ∎

5.1.3 Symmetric tensors and derivatives

The kth $\mathbb{F}$-derivative $\boldsymbol{D}^k\boldsymbol{f}(\boldsymbol{x}_0)$ of a map $\boldsymbol{f}\colon \mathcal{U} \to \mathbb{F}^m$ from an open subset $\mathcal{U} \subseteq \mathbb{F}^n$ is an element of the set $\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m)$ of symmetric multilinear maps from $(\mathbb{F}^n)^k$ to $\mathbb{F}^m$. This set is naturally isomorphic (essentially by definition) to $\mathrm{TS}^k((\mathbb{F}^n)^*) \otimes \mathbb{F}^m$. Given the results of Section F.2.4, this $\mathbb{F}$-vector space is isomorphic to $\mathrm{S}^k((\mathbb{F}^n)^*) \otimes \mathbb{F}^m$. Let us discuss making this observation geometric.

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let us consider manifolds M and N of class C^r and let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$. Given our discussion of the previous paragraph, one might be tempted to say that the kth derivative of Φ at $x_0 \in \mathsf{M}$ is to be regarded as an element of $\mathrm{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{\Phi(x_0)}\mathsf{N}$. This temptation leads one into trouble, since the kth partial derivative of the local representative of Φ at x_0 is *not* generally the local representative of an element of $\mathrm{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{\Phi(x_0)}\mathsf{N}$. It is when $k = 1$, of course, and the reader is invited to see that the coordinate transformation rules are not satisfied in the case when $k \geq 2$. What is true, however, is the following.

5.1.13 Proposition (kth derivatives of maps whose first k − 1 derivatives vanish) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M and N be manifolds of class C^r, let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$, let $(\mathcal{U}, \phi)$ be a $\mathbb{F}$-chart for M about x_0, let $(\mathcal{V}, \psi)$ be a $\mathbb{F}$-chart for N about $\phi(x_0)$, and let $k \in \mathbb{Z}_{>0}$. Suppose that Φ is such that the jth-derivative, $j \in \{1, \dots, k-1\}$ of the local representative $\Phi_{\phi\psi}$ vanishes at $\phi(x_0)$. Then the following statements hold:*

(i) the jth derivative, $j \in \{1, \dots, k-1\}$ of the local representative $\Phi_{\phi'\psi'}$ vanishes at $\phi'(x_0)$ for any $\mathbb{F}$-charts $(\mathcal{U}', \phi')$ about x_0 and $(\mathcal{V}', \psi')$ about $\phi(x_0)$;

(ii) the local representative of kth derivative of Φ at x_0 is the coordinate representative of an element of $\mathrm{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{\phi(x_0)}\mathsf{N}$.

Proof (i) This follows directly from Lemma A.1.1.

(ii) Let $(x^1, \dots, x^n)$ and $(\tilde{x}^1, \dots, \tilde{x}^n)$ be coordinates for M about x_0 and let $(y^1, \dots, y^m)$ and $(\tilde{y}^1, \dots, \tilde{y}^m)$ be coordinates for N about $\phi(x_0)$. Let $(\Phi^1, \dots, \Phi^m)$ and $(\tilde{\Phi}^1, \dots, \tilde{\Phi}^m)$ be the corresponding components of the local representatives of Φ. An inspection of Lemma A.1.1 shows that

$$\frac{\partial^k \tilde{\Phi}^a}{\partial \tilde{x}^{j_1} \cdots \partial \tilde{x}^{j_k}} = \sum_{l_1,\dots,l_k=1}^{n} \sum_{b=1}^{m} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_k}}{\partial \tilde{x}^{j_k}} \frac{\partial \Phi^b}{\partial x^{j_1} \cdots \partial x^{j_k}} \frac{\partial \tilde{y}^a}{\partial y^b} + \Psi,$$

where Ψ is a linear combination of the first $k-1$ derivatives of the components $\Phi^1, \dots, \Phi^m$. Thus, when evaluated at $\phi(x_0)$, these terms are zero, and from this this part of the result follows. ■

Let $\Phi \colon \mathsf{M} \to \mathsf{N}$ be of class C^r. The point of the preceding discussion is that the kth derivative of a function cannot really be talked about intrinsically unless the first $k-1$ derivatives vanish. Thus, one is led to the understanding that the object of interest is *all* derivatives from the 0th to the kth. What kind of space describes these derivatives? An obvious guess, based on Proposition 5.1.13, is that the totality of all derivatives 0 through k at x_0 should take values in

$$\bigoplus_{j=0}^{k} \mathsf{S}^j(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{\Phi(x_0)}\mathsf{N}.$$

More generally, we might guess that the Taylor series of Φ at x_0 takes values in the direct product

$$\prod_{j \in \mathbb{Z}_{\geq 0}} \mathsf{S}^j(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{\Phi(x_0)}\mathsf{N}$$

(cf. Proposition 4.2.6). This obvious guess is, in fact, incorrect. The reason is that the derivatives do not change coordinates in the right way, and the reader is encouraged to explore this by considering again the case of how derivatives from 0 to 2 transform under changes of coordinate. Thus the pressing question now is, "What is the structure of the set of derivatives of maps between manifolds?" This is the subject of this chapter, and in fact we turn to this right now.

5.2 Jet bundles of $\mathbb{F}$-valued maps

We shall consider three settings for the study of jet bundles: (1) jets of functions; (2) jets of general maps between manifolds; (3) jets of sections of vector bundles. These three settings are not mutually exclusive; for example the first and third obviously are subsumed by the second. However, as we shall see, each setting has distinct structure and the structure of the second two settings is described by understanding the first. Thus we start by looking at jets of functions. Throughout this section, and indeed this chapter, we let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ and adopt the convention that $r \in \{\infty, \omega\}$ when $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ when $\mathbb{F} = \mathbb{C}$. We also use the same symbol d to stand for the real or complex differential.

5.2.1 Definitions

We begin with the definition.

5.2.1 Definition (Functions agreeing to order k) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, let $x_0 \in \mathsf{M}$, let $\mathcal{U}$ and $\mathcal{V}$ be neighbourhoods of x_0, let $f \in \mathrm{C}^r(\mathcal{U})$ and $g \in \mathrm{C}^r(\mathcal{V})$, and let $k \in \mathbb{Z}_{\geq 0}$. The pairs $(f, \mathcal{U})$ and $(g, \mathcal{V})$ ***agree to order*** $\mathbf{k}$ at x_0 if, for every C^r-curve $\gamma\colon I \to \mathsf{M}$ for which $0 \in \mathrm{int}(I)$ and $\gamma(0) = x_0$,

$$(f \circ \gamma)^{(j)}(0) = (g \circ \gamma)^{(j)}(0),$$

$j \in \{0, 1, \dots, k\}$. •

5.2.2 Remark (The rôle of functions defined on neighbourhoods) In the preceding definition we defined agreement of functions defined only on a neighbourhood of a point. This is inessential in the smooth case, since a smooth function defined in a neighbourhood of x_0 can be extended, using the Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8], to a globally defined function agreeing to order k with the locally defined function. In the real analytic and holomorphic case, this is no longer generally true. For example, on a compact holomorphic manifold, a non-constant locally defined function cannot be extended to one that is globally defined and agrees to any order greater than zero. Other examples of this appear at the end of Section 5.6.3. Most standard treatments of jet bundles are developed in the smooth setting, and so work with globally defined functions. •

Let us verify that this definition has the desired characterisation in coordinates.

5.2.3 Proposition (Agreement to order k in coordinates) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *be a manifold of class* C^{r}*, let* $\mathrm{x}_0 \in \mathsf{M}$*, let* $\mathcal{U}$ *and* $\mathcal{V}$ *be neighbourhoods of* x_0*, let* $\mathrm{f} \in \mathrm{C}^{\mathrm{r}}(\mathcal{U})$ *and* $\mathrm{g} \in \mathrm{C}^{\mathrm{r}}(\mathcal{V})$*, and let* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$*. Then the following statements are equivalent:*

(i) $(\mathrm{f}, \mathcal{U})$ *and* $(\mathrm{g}, \mathcal{V})$ *agree to order* k *at* x_0*;*

(ii) for any $\mathbb{F}$*-chart* $(\mathcal{W}, \phi)$ *about* x_0 *with* $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ *and with coordinates* $(\mathrm{x}_1, \dots, \mathrm{x}_n)$*, it holds that*

$$\frac{\partial^{\mathrm{m}}(\mathrm{f} \circ \phi^{-1})}{\partial \mathrm{x}_{\mathrm{j}_1} \cdots \partial \mathrm{x}_{\mathrm{j}_{\mathrm{m}}}}(\phi(\mathrm{x}_0)) = \frac{\partial^{\mathrm{m}}(\mathrm{g} \circ \phi^{-1})}{\partial \mathrm{x}_{\mathrm{j}_1} \cdots \partial \mathrm{x}_{\mathrm{j}_{\mathrm{m}}}}(\phi(\mathrm{x}_0))$$

for $\mathrm{j}_1, \dots, \mathrm{j}_{\mathrm{m}} \in \{1, \dots, \mathrm{n}\}$ *and* $\mathrm{m} \in \{0, 1, \dots, \mathrm{k}\}$.

Proof (i) $\Longrightarrow$ (ii) Let $(\mathcal{W}, \phi)$ be a $\mathbb{F}$-chart about x_0 with coordinates $(x_1, \dots, x_n)$. Let us suppose, without loss of generality that $\phi(x_0) = \mathbf{0}$. Let $\gamma\colon I \to \mathsf{M}$ be a C^r-curve (recalling that I is an open subset of $\mathbb{F}$) such that $\gamma(0) = x_0$. Since $(f, \mathcal{U})$ and $(g, \mathcal{V})$ agree to order k we have $f \circ \gamma(0) = g \circ \gamma(0)$ giving $f(x_0) = g(x_0)$, and giving the desired statement for $m = 0$. Now let $m \in \{1, \dots, k\}$ and let $j_1, \dots, j_m \in \{1, \dots, m\}$. Let $l \in \{1, \dots, n\}$ and let i_l be the number of occurrences of l in the list $j_1, \dots, j_m$. Let $\epsilon \in \mathbb{R}_{>0}$ be sufficiently small that

$$\Big(\frac{t^{i_1}}{i_1!}, \dots, \frac{t^{i_n}}{i_n!}\Big) \in \phi(\mathcal{W})$$

for $t \in \mathsf{D}^1(0,\epsilon)$. Define $\gamma\colon \mathsf{D}^1(0,\epsilon) \to \mathsf{M}$ by

$$\phi \circ \gamma(t) = \Big(\frac{t^{i_1}}{i_1!}, \dots, \frac{t^{i_n}}{i_n!}\Big),$$

and note that, by symmetry of partial derivatives [Abraham, Marsden, and Ratiu 1988, Proposition 2.4.14], we have

$$\frac{\partial^m(\phi\circ\gamma)}{\partial x_{j_1}\cdots\partial x_{j_m}}(0) = \frac{\partial^m(\phi\circ\gamma)}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}(0) = 1,$$

and all other derivatives of $\phi\circ\gamma$ are zero at $t = 0$. It follows immediately from Lemma A.1.1 that

$$(f\circ\gamma)^{(m)}(0) = (f\circ\phi^{-1}\circ\phi\circ\gamma)^{(m)}(0) = \frac{\partial^m(f\circ\phi^{-1})}{\partial x_{j_1}\cdots\partial x_{j_m}}(\phi(x_0)),$$

and similarly for g. This gives the desired assertion immediately.

(ii) $\implies$ (i) Under the stated hypotheses we obviously have $f(x_0) = g(x_0)$ and so $f\circ\gamma(0) = g\circ\gamma(0)$ for any curve γ for which $\gamma(0) = x_0$. Let $m \in \{1,\dots,k\}$ and let $\gamma\colon I \to \mathsf{M}$ be a curve such that $0 \in \operatorname{int}(I)$ and $\gamma(0) = x_0$. Then

$$(f\circ\gamma)^{(m)}(0) = (f\circ\phi^{-1}\circ\phi\circ\gamma)^{(m)}(0)$$

and similarly for g. By Lemma A.1.1 it follows that the mth derivative of the composition $(f\circ\phi^{-1})\circ(\phi\circ\gamma)$ involves the derivatives 0 to m at $\phi(x_0)$ and 0, respectively, of the two components of the composition. The same statement holds for g, of course. By hypothesis, the derivatives 0 to m of $f\circ\phi^{-1}$ and $g\circ\phi^{-1}$ agree $\phi(x_0)$, and so it follows that

$$(f\circ\gamma)^{(m)}(0) = (g\circ\gamma)^{(m)}(0),$$

as desired. ■

Let M be a manifold of class C^r, let $x_0 \in \mathsf{M}$, let $\mathcal{U}$ and $\mathcal{V}$ be neighbourhoods of x_0, let $f \in \mathrm{C}^r(\mathcal{U})$ and $g \in \mathrm{C}^r(\mathcal{V})$, and let $k \in \mathbb{Z}_{\geq 0}$. Let us write $(f,\mathcal{U}) \sim_{k,x_0} (g,\mathcal{V})$ if $(f,\mathcal{U})$ and $(g,\mathcal{V})$ agree to order k at x_0. The relation $\sim_{k,x_0}$ is obviously an equivalence relation in the set of pairs $(f,\mathcal{U})$ with $\mathcal{U}$ a neighbourhood of x_0 and $f \in \mathrm{C}^r(\mathcal{U})$. For convenience, let us abbreviate by $\mathscr{F}^r(x_0)$ the set of such pairs.

5.2.4 Definition (Jets of functions) Let $r \in \{\infty,\omega,\mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty,\omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, let $x_0 \in \mathsf{M}$, and let $k \in \mathbb{Z}_{\geq 0}$.

(i) A ***k-jet of functions*** at x_0 is an equivalence class under the equivalence relation $\sim_{k,x_0}$.

(ii) The equivalence class of $(f,\mathcal{U}) \in \mathscr{F}^r(x_0)$ is denoted by $j_k f(x_0)$.

(iii) We denote

$$\mathsf{J}^k_{(x_0,s_0)}(\mathsf{M};\mathbb{F}) = \{j_k f(x_0) \mid (f,\mathcal{U}) \in \mathscr{F}^r(x_0),\ f(x_0) = s_0\}$$

and $\mathsf{J}^k_{x_0}(\mathsf{M};\mathbb{F}) = \cup_{s_0\in\mathbb{F}} \mathsf{J}^k_{(x_0,s_0)}(\mathsf{M};\mathbb{F})$.

(iv) We denote $\mathsf{J}^k(\mathsf{M};\mathbb{F}) = \cup_{(x,s)\in\mathsf{M}\times\mathbb{F}}\mathsf{J}^k_{(x,s)}(\mathsf{M};\mathbb{F})$ which we call the ***bundle of* k*-jets*** of functions. By convention, $\mathsf{J}^0(\mathsf{M};\mathbb{F}) = \mathsf{M}\times\mathbb{F}$.

(v) For $k,l\in\mathbb{Z}_{\geq 0}$ with $k\geq l$ we denote by $\rho^k_l\colon \mathsf{J}^k(\mathsf{M};\mathbb{F})\to\mathsf{J}^l(\mathsf{M};\mathbb{F})$ the projection defined by $\rho^k_l(j_kf(x)) = j_lf(x)$. We abbreviate ρ^k_0 by ρ_k.

(vi) We abbreviate $\mathsf{T}^{*k}_{x_0}\mathsf{M} = \mathsf{J}^k_{(x_0,0)}(\mathsf{M};\mathbb{F})$ and $\mathsf{T}^{*k}\mathsf{M} = \cup_{x\in\mathsf{M}}\mathsf{T}^{*k}_x\mathsf{M}$. •

5.2.2 Geometric structure

Let us now understand the structure of the jet bundles $\mathsf{J}^k(\mathsf{M};\mathbb{F})$. We begin by verifying that the jet bundles are manifolds.

5.2.5 Lemma (Differentiable structure of jet bundles of functions) *Let* $\mathrm{r}\in\{\infty,\omega,\mathrm{hol}\}$ *and let* $\mathbb{F}=\mathbb{R}$ *if* $\mathrm{r}\in\{\infty,\omega\}$ *and let* $\mathbb{F}=\mathbb{C}$ *if* $\mathrm{r}=\mathrm{hol}$. *Let* M *be a manifold of class* C^{r}, *let* $\mathrm{k}\in\mathbb{Z}_{\geq 0}$, *and let* $(\mathcal{U},\phi)$ *be a* $\mathbb{F}$*-chart for* M. *Define* $(\mathrm{j_k}\mathcal{U},\mathrm{j_k}\phi)$ *by*

$$\mathrm{j_k}\mathcal{U} = \{\mathrm{j_kf(x)} \mid (\mathrm{f},\mathcal{W})\in\mathscr{F}^{\mathrm{r}}(\mathrm{x}),\ \mathrm{x}\in\mathcal{U}\}$$

and

$$\begin{aligned}\mathrm{j_k}\phi\colon \mathrm{j_k}\mathcal{U}&\to \phi(\mathcal{U})\times\mathbb{F}\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F})\times\cdots\times\mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F})\\ \mathrm{j_kf(x)}&\mapsto (\phi(\mathrm{x}),\mathrm{f(x)},\mathbf{D}(\mathrm{f}\circ\phi^{-1})(\phi(\mathrm{x})),\dots,\mathbf{D}^{\mathrm{k}}(\mathrm{f}\circ\phi^{-1})(\phi(\mathrm{x}))).\end{aligned}$$

Then $(\mathrm{j_k}\mathcal{U},\mathrm{j_k}\phi)$ *is an* $\mathbb{F}$*-chart for* $\mathsf{J}^{\mathrm{k}}(\mathsf{M};\mathbb{F})$. *Moreover, if* $((\mathcal{U}_{\mathrm{a}},\phi_{\mathrm{a}}))_{\mathrm{a}\in\mathrm{A}}$ *is an atlas for* M, *then* $((\mathrm{j_k}\mathcal{U}_{\mathrm{a}},\mathrm{j_k}\phi_{\mathrm{a}}))_{\mathrm{a}\in\mathrm{A}}$ *is an atlas for* $\mathsf{J}^{\mathrm{k}}(\mathsf{M};\mathbb{F})$.

Proof For the first assertion of the lemma, we must show that $j_k\phi$ is a bijection from $j_k\mathcal{U}$ onto an open subset of

$$\mathbb{F}^n\times\mathbb{F}\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})\times\cdots\times\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}).$$

First note that

$$j_k\phi(j_k\mathcal{U}) = \phi(\mathcal{U})\times\mathbb{F}\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})\times\cdots\times\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}).$$

This can be shown by, for each

$$(\boldsymbol{x},s,\boldsymbol{A}_1,\dots,\boldsymbol{A}_k)\in\phi(\mathcal{U})\times\mathbb{F}\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})\times\cdots\times\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}),$$

explicitly constructing a polynomial function in coordinates such that $f(\boldsymbol{x}) = s$ and $\boldsymbol{D}^jf(\boldsymbol{x}) = \boldsymbol{A}_j$, $j\in\{1,\dots,k\}$. This was done, for example, as part of our proof of Theorem 1.1.4. Next we show that $j_k\phi$ is a bijection. Suppose that $j_k\phi(j_kf(x)) = j_k\phi(j_kg(y))$ for $j_kf(x), j_kg(y)\in j_k\mathcal{U}$. Thus, by definition of $j_k\phi$, the first k derivatives of f and g (including the zeroth) agree. By Proposition 5.2.3 it follows that $j_kf(x) = j_kg(y)$. Thus $(j_k\mathcal{U}, j_k\phi)$ is an $\mathbb{F}$-chart.

To verify that an atlas for M induces an atlas for $\mathsf{J}^k(\mathsf{M};\mathbb{F})$, we must verify that the overlap maps are $\mathbb{F}$-diffeomorphisms. Thus let $(\mathcal{U}_a,\phi_a)$ and $(\mathcal{U}_b,\phi_b)$ be $\mathbb{F}$-charts for M such that $\mathcal{U}_a\cap\mathcal{U}_b\neq\emptyset$. Note that, obviously, $j_k\mathcal{U}_a\cap j_k\mathcal{U}_b\neq\emptyset$. Let us suppose that $\mathcal{U}_a=\mathcal{U}_b=\mathcal{U}$, for simplicity and without loss of generality. For a C^r-map $\psi\colon\mathcal{N}\to\mathbb{F}$ with domain an open subset $\mathcal{N}\subseteq\mathbb{F}^n$, let us abbreviate

$$j_k\psi\colon\mathcal{N}\to\mathcal{N}\times\mathbb{F}\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})\times\cdots\times\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})$$

as the map

$$j_k\psi(\boldsymbol{x}) = (\boldsymbol{x}, \psi(\boldsymbol{x}), \boldsymbol{D}\psi(\boldsymbol{x}), \dots, \boldsymbol{D}^k\psi(\boldsymbol{x})).$$

Let $x \in \mathcal{U}$. With the notation above we have

$$j_k\phi_b(j_kf(x)) = j_k(f \circ \phi_b^{-1})(\phi_b(x)) = j_k(f \circ \phi_a^{-1} \circ \phi_a \circ \phi_b^{-1})(\phi_b(x)). \tag{5.4}$$

Now we use a lemma.

1 **Sublemma** *Let $\psi\colon \mathcal{N} \to \mathcal{N}'$ be a C^{r}-diffeomorphism of open subsets $\mathcal{N}$ and $\mathcal{N}'$ of $\mathbb{F}^{\mathrm{n}}$ and define a map*

$$\mathrm{J_k}\psi\colon \mathcal{N} \times \mathbb{F} \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}) \times \dots \times \mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}) \to \mathcal{N}' \times \mathbb{F} \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F};\mathbb{F}^{\mathrm{n}}) \times \dots \times \mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F})$$

by asking that

$$\begin{aligned}&\mathrm{J_k}\psi(\mathbf{x}, \mathrm{g}(\mathbf{x}), \mathbf{D}\mathrm{g}(\mathbf{x}), \dots, \mathbf{D}^{\mathrm{k}}\mathrm{g}(\mathbf{x}))\\ &\qquad = (\psi(\mathbf{x}), (\mathrm{g} \circ \psi^{-1})(\psi(\mathbf{x})), \mathbf{D}(\mathrm{g} \circ \psi^{-1})(\psi(\mathbf{x})), \dots, \mathbf{D}^{\mathrm{k}}(\mathrm{g} \circ \psi^{-1})(\psi(\mathbf{x}))),\end{aligned}$$

for any C^{r}-function $\mathrm{g}\colon \mathcal{N} \to \mathbb{F}$. Then $\mathrm{J_k}\psi$ is a C^{r}-diffeomorphism.

Proof First of all, note that the map $J_k\psi$ is well-defined by Lemma A.1.1.

Next we prove that $J_k\psi$ is of class C^r. We prove this by induction on k. For $k = 1$ we have

$$J_1\psi(\boldsymbol{x}, y, \boldsymbol{\alpha}) = (\psi(\boldsymbol{x}), y, (\boldsymbol{D}\psi^{-1}(\boldsymbol{x}))^*(\boldsymbol{\alpha})),$$

and the lemma follows in this case from the Inverse Function Theorem. Now suppose that the lemma holds for $k \in \{1, \dots, m\}$. Then, for $g \in \mathrm{C}^r(\mathcal{N})$, the first m derivatives of $g \circ \psi^{-1}$ are C^r-functions of the first m derivatives of g by the induction hypothesis. Then, by Lemma A.1.1,

$$\boldsymbol{D}^{m+1}(g \circ \psi^{-1})(\psi(\boldsymbol{x})) = \boldsymbol{D}^{m+1}g(\boldsymbol{x}) \cdot (\boldsymbol{D}\psi^{-1}(\psi(\boldsymbol{x})), \dots, \boldsymbol{D}\psi^{-1}(\psi(\boldsymbol{x}))) + G(\boldsymbol{x}, \boldsymbol{D}g(\boldsymbol{x}), \dots, \boldsymbol{D}^m g(\boldsymbol{x}),$$

where the function G is a C^r-function of $\boldsymbol{x}$ and the first m derivatives of g. Thus the first $m+1$ derivatives of $g \circ \psi^{-1}$ are C^r-functions of the first $m+1$ derivatives of g, and this gives $J_k\psi$ as being of class C^r.

Now we prove that $J_k\psi$ is invertible. To see this, one needs only to note that the map

$$(\boldsymbol{y}, x, \boldsymbol{D}h(\boldsymbol{y}), \dots, \boldsymbol{D}^k h(\boldsymbol{y})) \mapsto (\psi^{-1}(\boldsymbol{y}), x, \boldsymbol{D}(h \circ \psi)(\psi^{-1}(\boldsymbol{y})), \dots, \boldsymbol{D}^k(h \circ \psi)(\psi^{-1}(\boldsymbol{y}))),$$

for $h \in \mathrm{C}^r(\mathcal{N}')$, is the inverse of $J_k\psi$. Moreover, this inverse is of class C^r by the same argument as in the preceding paragraph. ▼

By (5.4) we have

$$j_k\phi_b(j_kf(x)) = J_k(\phi_b \circ \phi_a^{-1})(j_k(f \circ \phi_a^{-1})(\phi_a(x))).$$

Since $\phi_a \circ \phi_b^{-1}$ is an $\mathbb{F}$-diffeomorphism, the overlap condition for jet bundle charts holds. ■

An $\mathbb{F}$-chart for $\mathsf{J}^k(\mathsf{M};\mathbb{F})$ as in the lemma is called a ***natural chart***. The lemma gives the following result which further refines the differentiable structure of the jet bundles $\mathsf{J}^k(\mathsf{M};\mathbb{F})$.

5.2.6 Theorem (Fibre and vector bundle structure for jet bundles of functions) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r and let $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$. Then*

(i) $\rho^k_l \colon J^k(M;\mathbb{F}) \to J^l(M;\mathbb{F})$ is a locally trivial fibre bundle.

Moreover, if $\mathrm{pr}_1 \colon M \times \mathbb{F} \to M$ denotes the projection onto the first factor, then

(ii) $\mathrm{pr}_1 \circ \rho_k \colon J^k(M;\mathbb{F}) \to M$ is a vector bundle and

(iii) $(\mathrm{pr}_1 \circ \rho_k)|T^{*k}M \colon T^{*k}M \to M$ is a vector bundle.

Proof (i) This follows since the local representative of ρ^k_l is

$$(\boldsymbol{x}, s, \boldsymbol{A}_1, \dots, \boldsymbol{A}_k) \mapsto (\boldsymbol{x}, s, \boldsymbol{A}_1, \dots, \boldsymbol{A}_l),$$

which shows that ρ^k_l is a surjective submersion and that $\rho^k_l \colon J^k(M;\mathbb{F}) \to J^l(M;\mathbb{F})$ is locally trivial with respect to the natural coordinate charts.

Let $(\mathcal{U}_a, \phi_a)$ and $(\mathcal{U}_b, \phi_b)$ be $\mathbb{F}$-charts for M with $(j_k\mathcal{U}_a, j_k\phi_a)$ and $(j_k\mathcal{U}_b, j_k\phi_b)$ the associated natural charts for $J^k(M;\mathbb{F})$. Parts (ii) and (iii) follow since, by Lemma A.1.1 and our computations from Lemma 5.2.5, the overlap map relating $j_k\phi_b(j_kf(x))$ to $j_k\phi_a(j_kf(x))$ are linear in the coordinate components of the derivatives of f. ■

Note that the vector bundle operations in

$$\mathrm{pr}_1 \circ \rho_k \colon J^k(M;\mathbb{F}) \to M \ \text{ and } \ (\mathrm{pr}_1 \circ \rho_k)|T^{*k}M \colon T^{*k}M \to M$$

are both defined by

$$j_kf(x) + j_kg(x) = j_k(f+g)(x), \quad a\,j_kf(x) = j_k(af)(x).$$

5.2.3 Algebraic structure

Now that the jet bundles are manifolds, let us examine their algebraic structure. A key to doing this is the following result.

5.2.7 Lemma (Products of vanishing functions) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, let $x_0 \in M$, let $\mathcal{U}$ be a neighbourhood of x_0, let $f_1, \dots, f_k \in C^r(\mathcal{U})$ be functions vanishing at x_0, and define $f = f_1 \cdots f_k$. Then the first $k-1$ derivatives of f vanish at x_0 and the kth derivative of f at x_0 (which makes sense by Proposition 5.1.13) is*

$$df_1(x_0) \odot \cdots \odot df_k(x_0) \in S^k(T^*_{x_0}M).$$

Moreover,

$$S^k(T^*_{x_0}M) = \mathrm{span}_{\mathbb{F}}(df_1(x_0) \odot \cdots \odot df_k(x_0) \mid \ f_1, \dots, f_k \in C^r(M),\ f_1(x_0) = \cdots = f_k(x_0) = 0).$$

Proof Let us first prove that the first $k-1$ derivatives of f vanish at x_0. To do so, we use the general form of the Leibniz Rule from Lemma A.2.2. We apply this result to the case where $n_1, \dots, n_k = m = 1$, where $L(x_1, \dots, x_k) = x_1 \cdots x_k$, and where $f_1, \dots, f_j$ are (abusing notation) the local representatives. One easily sees that the jth derivatives of f vanish at x_0 since each term in the sum in Lemma A.2.2 will involve a term with the zeroth derivative of one of the functions $f_1, \dots, f_k$. Since these functions vanish at x_0, the derivatives must also vanish.

Now we prove that the derivative has the stated form. By Corollary F.2.13,

$$\mathrm{d}f_1(x_0) \odot \cdots \odot \mathrm{d}f_k(x_0) = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{d}f_{\sigma(1)}(x_0) \otimes \cdots \otimes \mathrm{d}f_{\sigma(k)}(x_0).$$

Therefore, in an $\mathbb{F}$-chart $(\mathcal{U}, \phi)$ with coordinates $(x^1, \dots, x^n)$, we can use Lemma A.2.2 and the fact that the functions $f_1, \dots, f_k$ vanish to x_0 to compute

$$\begin{aligned}
\boldsymbol{D}^k(f \circ \phi^{-1})(\phi(x_0)) &= \sum_{j_1,\dots,j_k=1}^{n} \frac{\partial^k (f_1 \circ \phi^{-1} \cdots f_k \circ \phi^{-1})}{\partial x^{j_1} \cdots \partial x^{j_k}}(\phi(x_0)) \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_k} \\
&= \sum_{j_1,\dots,j_k=1}^{n} \sum_{\sigma \in \mathfrak{S}_k} \frac{\partial (f_1 \circ \phi^{-1})}{\partial x^{j_{\sigma(1)}}}(\phi(x_0)) \cdots \frac{\partial (f_k \circ \phi^{-1})}{\partial x^{j_{\sigma(k)}}}(\phi(x_0)) \mathrm{d}x^{j_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{j_{\sigma(k)}} \\
&= \sum_{j_1,\dots,j_k=1}^{n} \sum_{\sigma \in \mathfrak{S}_k} \frac{\partial (f_{\sigma(1)} \circ \phi^{-1})}{\partial x^{j_1}}(\phi(x_0)) \cdots \frac{\partial (f_{\sigma(k)} \circ \phi^{-1})}{\partial x^{j_k}}(\phi(x_0)) \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_k} \\
&= \mathrm{d}(f_1 \circ \phi^{-1})(\phi(x_0)) \odot \cdots \odot \mathrm{d}(f_k \circ \phi^{-1})(\phi(x_0)),
\end{aligned}$$

as desired.

For the final assertion of the lemma, let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-chart about x_0 such that $\phi(x_0) = \mathbf{0}$. For $j \in \{1, \dots, n\}$ let $f_j \in \mathrm{C}^r(\mathcal{U})$ have the property that $f_j \circ \phi^{-1}(\boldsymbol{x}) = x^j$ for $\boldsymbol{x} \in \mathcal{U}$. Since

$$\mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) = \mathrm{span}_{\mathbb{F}}(\mathrm{d}f_1(x_0) \odot \cdots \odot \mathrm{d}f_k(x_0)),$$

the result follows. ■

Let $k \in \mathbb{Z}_{>0}$ and let $x_0 \in \mathsf{M}$, We define a map

$$\epsilon_{k,x_0} \colon \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \to \mathsf{J}^k_{(x_0,0)}(\mathsf{M}; \mathbb{F})$$

by

$$\epsilon_k(\mathrm{d}f_1(x_0) \odot \cdots \odot \mathrm{d}f_k(x_0)) = j_k(f_1 \cdots f_k)(x_0),$$

where $f_1, \dots, f_k$ all vanish at x_0. That this gives a well-defined map on all of $\mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M})$ follows from Lemma 5.2.7. The following lemma is then important.

5.2.8 Lemma (Structure of jets of functions vanishing to order k − 1) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^r *and let* $\mathrm{x}_0 \in \mathsf{M}$. *Then the following sequence of* $\mathbb{F}$*-vector spaces is exact:*

$$0 \longrightarrow \mathsf{S}^\mathrm{k}(\mathsf{T}^*_{\mathrm{x}_0}\mathsf{M}) \xrightarrow{\epsilon_{\mathrm{k},\mathrm{x}_0}} \mathsf{J}^\mathrm{k}_{(\mathrm{x}_0,0)}(\mathsf{M}; \mathbb{F}) \xrightarrow{\rho^\mathrm{k}_{\mathrm{k}-1}} \mathsf{J}^{\mathrm{k}-1}_{(\mathrm{x}_0,0)}(\mathsf{M}; \mathbb{F}) \longrightarrow 0$$

Proof First let us show that ϵ_{k,x_0} is injective. Let $A \in \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M})$ be such that $\epsilon_{k,x_0}(A) = 0$. Let $(f, \mathcal{U}) \in \mathscr{F}^r(x_0)$ be such that the first $k-1$ derivatives of f at x_0 vanish and such that $j_k f(x_0) = \epsilon_{k,x_0}(A)$. Thus $j_k f(x_0) = 0$ and so the first k derivatives of f vanish at x_0 and so $A = 0$, giving the desired injectivity.

By construction, the image of ϵ_{k,x_0} consists of k-jets of functions whose first $k-1$ derivatives vanish. Thus $\operatorname{image}(\epsilon_{k,x_0}) \subseteq \ker(\rho^k_{k-1})$. Let n be the dimension of the connected component of M containing x_0. By Corollary F.2.10 it follows that $\dim_{\mathbb{F}}(\operatorname{image}(\epsilon_{k,x_0})) = \binom{n+k-1}{n-1}$. By Corollary F.2.10 and Lemma 5.2.5,

$$\dim_{\mathbb{F}}(\mathsf{J}^k(\mathsf{M};\mathbb{F})) = n + 1 + \sum_{j=1}^{k} \binom{n+k-1}{n-1}.$$

Thus $\dim_{\mathbb{F}}(\mathsf{J}^k(\mathsf{M};\mathbb{F})) - \dim_{\mathbb{F}}(\mathsf{J}^{k-1}(\mathsf{M};\mathbb{F})) = \binom{n+k-1}{n-1}$, showing that $\operatorname{image}(\epsilon_{k,x_0}) = \ker(\rho^k_{k-1})$ by dimension counting and since ϵ_{k,x_0} is injective. Since it is clear that ρ^k_{k-1} is surjective, the lemma follows. ■

Now let us make the preceding pointwise construction global. Note that $\mathsf{S}^k(\mathsf{T}^*\mathsf{M})$ is an $\mathbb{F}$-vector bundle over M. Let us denote by $\sigma_k\colon \mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \to \mathsf{M}$ the canonical projection. Note that we then have the pull-back vector bundle $\rho^*_{k-1}\sigma_k\colon \rho^*_{k-1}\mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \to \mathsf{J}^{k-1}(\mathsf{M};\mathbb{F})$. Our constructions above all took place, not in $\mathsf{J}^k(\mathsf{M};\mathbb{F})$, but in $\mathsf{T}^{*k}\mathsf{M}$ since we were considering functions which vanish at the point under consideration. Let us make a mild abuse of notation and denote $\rho^k_{k-1} = \rho^{k-1}_k|\mathsf{T}^{*k}\mathsf{M}$ and $\rho_k = (\mathrm{pr}_1 \circ \rho_k)|\mathsf{T}^{*k}\mathsf{M}$. We then have a mapping $\epsilon_k\colon \rho^*_{k-1}\mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \to \mathsf{T}^{*k}\mathsf{M}$ obtained by extending the above pointwise construction. Since the local representative of ϵ_k in natural coordinates is

$$((\boldsymbol{x}, 0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}), \boldsymbol{A}_k) \mapsto (\boldsymbol{x}, 0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}, \boldsymbol{A}_k),$$

it follows that ϵ_k is a vector bundle mapping of class C^r.

5.2.9 Theorem (Affine bundle structure for jet bundles of functions) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r} *and let* $\mathrm{k} \in \mathbb{Z}_{>0}$. *Then the sequence of* $\mathbb{F}$*-vector bundles*

$$0 \longrightarrow \rho^*_{\mathrm{k}-1}\mathsf{S}^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}) \xrightarrow{\epsilon_{\mathrm{k}}} \mathsf{T}^{*\mathrm{k}}\mathsf{M} \xrightarrow{\rho^{\mathrm{k}}_{\mathrm{k}-1}} \mathsf{T}^{*\mathrm{k}-1}\mathsf{M} \longrightarrow 0$$

is exact, and, as a consequence, $\rho^{\mathrm{k}}_{\mathrm{k}-1}\colon \mathsf{T}^{*\mathrm{k}}\mathsf{M} \to \mathsf{T}^{*\mathrm{k}-1}\mathsf{M}$ *is an affine bundle of class* C^{r} *modelled on the pull-back vector bundle* $\rho^*_{\mathrm{k}-1}\mathsf{S}^{\mathrm{k}}(\mathsf{T}^*\mathsf{M})$.

Proof The exactness of the sequence follows from Lemma 5.2.8. For the final statement we prove the following general fact.

1 Lemma *Let* U, V, *and* W *be vector bundles over* M. *If*

$$0 \longrightarrow \mathsf{U} \xrightarrow{\Phi} \mathsf{V} \xrightarrow{\Psi} \mathsf{W} \longrightarrow 0$$

is an exact sequence of vector bundles over id_{M}, *then* $\ker(\Psi)$ *is an affine bundle modelled on* $\operatorname{image}(\Phi)$, *where, by definition,* $\mathrm{v} + \mathrm{u} = \mathrm{v} + \Phi(\mathrm{u})$, *i.e., the affine structure of* V *is addition restricted to* $\mathsf{V} \times \Phi(\mathsf{U})$.

Proof The only possibly nontrivial facts to verify are that the affine structure is faithful and transitive. Let us prove transitivity. Let $v_1, v_2 \in \ker(\Psi)_x$. Thus $v_2 - v_1 \in \ker(\Psi)_x$ and so, by exactness of the sequence, there exists $u \in \mathsf{U}_x$ such that $v_2 - v_1 = u$, i.e., $v_2 = v_1 + u$. This gives transitivity. Now we prove faithfulness. Let $v \in \mathsf{V}_x$ and let $u_1, u_2 \in \mathsf{U}_x$ be such that $v + \Phi(u_1) = v + \Phi(u_2)$. Thus $\Phi(u_1) = \Phi(u_2)$, and injectivity of Φ gives $u_1 = u_2$. ▼

The theorem follows immediately from the lemma. ■

The preceding constructions can be generalised when considering general projections $\rho^k_l\colon \mathsf{J}^k(\mathsf{M};\mathbb{F}) \to \mathsf{J}^l(\mathsf{M};\mathbb{F})$. Let us outline how this works since it gives some context to our constructions above. First we need some notation. Let $k, l \in \mathbb{Z}_{>0}$ with $k > l$ and let $x_0 \in \mathsf{M}$. Denote by $\mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F})$ the subset of $\mathsf{J}^k_{(x_0,0)}(\mathsf{M};\mathbb{F})$ consisting of k-jets of functions whose first l derivatives vanish at x_0. Note that $\mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F})$ is a subspace of $\mathsf{T}^{*k}_{x_0}\mathsf{M}$. This can be seen by noting that, in natural coordinates, elements of $\mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F})$ are represented as

$$(\mathsf{x}, 0, \mathbf{0}, \dots, \mathbf{0}, \boldsymbol{A}_{l+1}, \dots, \boldsymbol{A}_k),$$

and that this form is preserved by the overlap maps. Define $\epsilon^k_{l,x_0}\colon \mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F}) \to \mathsf{J}^k_{(x_0,0)}(\mathsf{M};\mathbb{F})$ to be the inclusion. Then one shows that the sequence

$$0 \longrightarrow \mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F}) \xrightarrow{\epsilon^k_{l,x_0}} \mathsf{J}^k_{(x_0,0)}(\mathsf{M};\mathbb{F}) \xrightarrow{\rho^k_l} \mathsf{J}^l_{(x_0,0)}(\mathsf{M};\mathbb{F}) \longrightarrow 0$$

of vector bundles is exact. What is interesting about the case when $l = k-1$ is that the structure of $\mathsf{Z}^k_{l,x_0}(\mathsf{M};\mathbb{F})$ can be understood in terms of the symmetric algebra of $\mathsf{T}^*_{x_0}\mathsf{M}$.

We shall see the preceding constructions concerning the affine structure of jets mirrored in the structure of our other jet bundles below. The point is that the natural structure of sets of derivatives is not that of a vector bundle, but rather an affine bundle.

There is additional structure of $\mathsf{T}^{*k}\mathsf{M}$ that is of interest. As we noted in Lemma 5.2.5, $\mathsf{T}^{*k}_{x_0}\mathsf{M}$ has the structure of a $\mathbb{F}$-vector space, and the vector space operations are

$$j_k f(x_0) + j_k g(x_0) = j_k(f+g)(x_0), \quad a\, j_k f(x_0) = j_k(af)(x_0).$$

It is also true that $\mathsf{T}^{*k}_{x_0}\mathsf{M}$ possesses a product defined by

$$(j_k f(x_0)) \cdot (j_k g(x_0)) = j_k(fg)(x_0).$$

That this product makes sense in $\mathsf{T}^{*k}_{x_0}\mathsf{M}$ follows from the Leibniz Rule, Lemma A.2.2. Thus we have the following result.

5.2.10 Theorem (Algebra structure of jets of functions) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r}, *let* $\mathrm{x}_0 \in \mathsf{M}$, *and let* $\mathrm{k} \in \mathbb{Z}_{>0}$. *Then* $\mathsf{T}^{*\mathrm{k}}_{\mathrm{x}_0}\mathsf{M}$ *has the structure of am* $\mathbb{F}$*-algebra with the algebra operations being those inherited from the pointwise algebra operations on functions.*

Moreover, let $(\mathcal{U},\phi)$ *be an* $\mathbb{F}$*-chart for* M *about* x_0 *such that* $\phi(x_0) = \mathbf{0}$ *and let* $\chi^1_\phi,\dots,\chi^n_\phi\colon\mathcal{U}\to\mathbb{F}$ *be the coordinate functions for the chart. Then the coordinate functions generate the algebra* $\mathsf{T}^{*k}_{x_0}\mathsf{M}$*, i.e.,*

$$\mathsf{T}^{*k}_{x_0}\mathsf{M} = \operatorname{span}\Big(j_m((\chi^1_\phi)^{p_1}\cdots(\chi^n_\phi)^{p_n})(x_0) \;\Big|\; m\in\{1,\dots,k\},\ p_1+\dots+p_n = m\Big).$$

Proof Only the final assertion remains to be proved. Note that, if $p_1+\dots+p_n = m$, the coordinate representation of $j_m((\chi^1_\phi)^{p_1}\cdots(\chi^n_\phi)^{p_n})(x_0)$ is

$$\mathrm{d}x_1^{p_1}\odot\dots\odot\mathrm{d}x_n^{p_n}, \tag{5.5}$$

following the constructions at the end of Section 1.1.2. As we saw in the proof of Corollary F.2.10, the set

$$\left\{\mathrm{d}x_1^{p_1}\odot\dots\odot\mathrm{d}x_n^{p_n} \;\middle|\; p_1+\dots+p_n = m\right\}$$

is a basis for $\mathsf{S}^m(\mathsf{T}^*_x\mathsf{M})$. Note that we have the following exact sequence,

$$\mathsf{T}^{*k}_{x_0}\mathsf{M} \longrightarrow \mathsf{T}^{*k-1}_{x_0}\mathsf{M} \longrightarrow \cdots \longrightarrow \mathsf{T}^{*2}_{x_0}\mathsf{M} \longrightarrow \mathsf{T}^{*}_{x_0}\mathsf{M} \longrightarrow 0$$

the horizontal arrows being the canonical projections. By Lemma 5.2.8, it follows that

$$\dim_{\mathbb{F}}(\mathsf{T}^{*k}_{x_0}\mathsf{M}) = \sum_{m=1}^{k}\dim_{\mathbb{F}}(\mathsf{S}^m(\mathsf{T}^*_x\mathsf{M})).$$

From Lemma 1.1.1 and Corollary F.2.10 and since $\mathsf{T}^{*k}_{x_0}\mathsf{M}$ contains all vectors of the form (5.5), the theorem follows. ■

5.2.4 Infinite jets

In this section we define precisely the notion of an infinite jet of an $\mathbb{F}$-valued function, and consider the algebraic structure of the set of such infinite jets, using our constructions from Section 5.1.2. It is also possible to consider topological and differentiable structures on infinite jets, but we will not make use of this structure here. We shall, however, need to understand the topological structure for infinite jet bundles in Section 7.5, and in Section 7.5.1 we shall take the requisite measures to describe this topology. We also refer to [Saunders 1989, Chapter 7].

Let $r\in\{\infty,\omega,\mathrm{hol}\}$ and let $\mathbb{F}=\mathbb{R}$ if $r\in\{\infty,\omega\}$ and let $\mathbb{F}=\mathbb{C}$ if $r=\mathrm{hol}$. We let M be a manifold of class C^r and let $x_0\in\mathsf{M}$. Note that

$$((\mathsf{T}^{*k}_{x_0}\mathsf{M})_{k\in\mathbb{Z}_{>0}}, (\rho^k_l)_{k,l\in\mathbb{Z}_{>0},\ k\geq l})$$

is a inverse system of $\mathbb{F}$-algebras. By Lemma 5.1.12, we can define the $\mathbb{F}$-algebra

$$\mathsf{T}^{*\infty}_{x_0}\mathsf{M} = \operatorname*{inv\,lim}_{k\to\infty}\mathsf{T}^{*k}_{x_0}\mathsf{M}$$

with the associated $\mathbb{F}$-algebra homomorphisms $\rho^\infty_k\colon\mathsf{T}^{*\infty}_{x_0}\mathsf{M}\to\mathsf{T}^{*k}_{x_0}\mathsf{M}$. If $(f,\mathcal{U})\in\mathscr{F}^r(x_0)$ satisfies $f(x_0)=0$ then we define $j_\infty f(x_0)\in\mathsf{T}^{*\infty}_{x_0}\mathsf{M}$ by asking that $j_\infty f(x_0)(k) = j_k f(x_0)$ for each $k\in\mathbb{Z}_{\geq0}$.

5.3 Jet bundles of curves

In this section we define jet bundles associated with curves. We shall not make much direct use of this construction, but it will provide a useful way of thinking about jets between manifolds in Lemma 5.4.1 below. Also, it serves to provide a more complete picture of the general theory of jets which is useful in and of itself. As in all of our discussions about jets, we let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ and adopt the convention that $r \in \{\infty, \omega\}$ when $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ when $\mathbb{F} = \mathbb{C}$. We also remind the reader that we are talking about curves in the general sense of having a domain I as an open subset of $\mathbb{F}$.

5.3.1 Definitions

We begin with the following definition, using the terminology that a curve at $x_0 \in \mathsf{M}$ is a curve $\gamma\colon I \to \mathsf{M}$ where $I \subseteq \mathbb{F}$ is an open subset for which $0 \in I$ and $\gamma(0) = x_0$. We also recall from the preceding section that $\mathscr{F}^r(x_0)$ is the set of pairs $(f, \mathcal{U})$ where $\mathcal{U}$ is a neighbourhood of x_0 and $f \in \mathrm{C}^r(\mathcal{U})$.

5.3.1 Definition (Curves agreeing to order k) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, let $I, J \subseteq \mathbb{F}$ be open sets for which $0 \in I$ and $0 \in J$, let $x_0 \in \mathsf{M}$, let $\gamma\colon I \to \mathsf{M}$ and let $\sigma\colon J \to \mathsf{M}$ be curves at x_0 of class C^r, and let $k \in \mathbb{Z}_{\geq 0}$. The curves γ and σ ***agree to order* k** at x_0 if, for every $(f, \mathcal{U}) \in \mathscr{F}^r(x_0)$,

$$(f \circ \gamma)^{(j)}(0) = (f \circ \sigma)^{(j)}(0),$$

$j \in \{0, 1, \dots, k\}$. •

As with functions agreeing to order k, one can readily verify that curves agreeing to order k have the expected characterisation in coordinates.

5.3.2 Proposition (Agreement to order k in coordinates) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *be a manifold of class* C^{r}*, let* $\mathrm{I}, \mathrm{J} \subseteq \mathbb{F}$ *be open sets for which* $0 \in \mathrm{I}$ *and* $0 \in \mathrm{J}$*, let* $\mathrm{x}_0 \in \mathsf{M}$*, let* $\gamma\colon \mathrm{I} \to \mathsf{M}$ *and let* $\sigma\colon \mathrm{J} \to \mathsf{M}$ *be curves at* x_0 *of class* C^{r}*, and let* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$*. Then the following two statements are equivalent:*

(i) *γ and σ agree to order k at x_0;*

(ii) *for any $\mathbb{F}$-chart $(\mathcal{U}, \phi)$ about x_0, it holds that*

$$\frac{\mathrm{d}^{\mathrm{m}}(\phi \circ \gamma)}{\mathrm{d}t^{\mathrm{m}}}(0) = \frac{\mathrm{d}^{\mathrm{m}}(\phi \circ \sigma)}{\mathrm{d}t^{\mathrm{m}}}(0), \qquad \mathrm{m} \in \{0, 1, \dots, \mathrm{k}\}.$$

Proof (i) $\Longrightarrow$ (ii) Let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-chart about x_0 and let $\chi^j_\phi\colon \mathcal{U} \to \mathbb{F}$, $j \in \{1, \dots, n\}$, be the coordinate functions for the chart. Let $\epsilon \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{B}}^n(\epsilon, \phi(x_0)) \subseteq \phi(\mathcal{U})$. Then, if γ and σ agree to order k at x_0,

$$(\chi^j_\phi \circ \gamma)^{(l)}(0) = (\chi^j_\phi \circ \sigma)^{(l)}(0), \qquad l \in \{0, 1, \dots, k\}.$$

By Lemma A.1.1,

$$(\chi^j_\phi \circ \gamma)^{(m)}(0) = ((\chi^j_\phi \circ \phi^{-1}) \circ (\phi \circ \gamma))^{(m)}(0) = \frac{\mathrm{d}^m(\phi \circ \gamma)}{\mathrm{d}t^m}(0), \qquad j \in \{1, \dots, n\},\ m \in \{0, 1, \dots, k\},$$

and similarly for σ. From this, this part of the result follows.

(ii) $\implies$ (i) Let $(\mathcal{U}, \phi)$ be a $\mathbb{F}$-chart about x_0 and let $(f, \mathcal{V}) \in \mathscr{F}^r(x_0)$. We have

$$(f \circ \gamma)^{(m)}(0) = ((f \circ \phi^{-1}) \circ (\phi \circ \gamma))^{(m)}(0), \qquad m \in \{0, 1, \dots, k\},$$

and similarly for σ. Using this formula, Lemma A.1.1, and the hypotheses of this part of the proposition, it follows that

$$(f \circ \gamma)^{(m)}(0) = (f \circ \sigma)^{(m)}(0), \qquad m \in \{0, 1, \dots, k\},$$

giving the result. ■

For $x_0 \in \mathsf{M}$ we define an equivalence relation $\sim_{k,x_0}$ in the set of C^r-curves at x_0 by asking that $\gamma \sim_{k,x_0} \sigma$ if γ and σ agree to order k at x_0. We can now define jets for curves. For convenience, denote by $\mathscr{C}^r(x_0)$ the set of pair (γ, I) where $I \subseteq \mathbb{F}$ is open with $0 \in I$ and where $\gamma \colon I \to \mathsf{M}$ is of class C^r and satisfies $\gamma(0) = x_0$.

5.3.3 Definition (Jets of curves) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, let $x_0 \in \mathsf{M}$, and let $k \in \mathbb{Z}_{\geq 0}$.

(i) A ***k-jet of curves*** at x_0 is an equivalence class in $\mathscr{C}^r(x_0)$ under the equivalence relation $\sim_{k,x_0}$.

(ii) The equivalence class of $(\gamma, I) \in \mathscr{C}^r(x_0)$ is denoted by $j_k\gamma(0)$.

(iii) We denote

$$\mathsf{T}^k_{x_0}\mathsf{M} = \{j_k\gamma(0) \mid (\gamma, I) \in \mathscr{C}^r(x_0)\}.$$

(iv) We denote $\mathsf{T}^k\mathsf{M} = \cup_{x \in \mathsf{M}} \mathsf{T}^k_x\mathsf{M}$ which we call the ***bundle of k-jets*** of curves. By convention, $\mathsf{T}^0\mathsf{M} = \mathsf{M}$.

(v) For $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$ we denote by $\rho^k_l \colon \mathsf{T}^k\mathsf{M} \to \mathsf{T}^l\mathsf{M}$ the projection defined by $\rho^k_l(j_k\gamma(0)) = j_l\gamma(0)$. We abbreviate ρ^k_0 by ρ_k. •

Note that, by definition, $\mathsf{T}^1\mathsf{M} = \mathsf{TM}$. Thus $\mathsf{T}^1\mathsf{M}$ is to be thought of as the space of "velocities" on M. Correspondingly, the sets $\mathsf{T}^k\mathsf{M}$, $k > 1$, are to be thought of as spaces of higher-order derivatives, accelerations, etc.

5.3.2 Geometric structure

As with the bundle of jets of $\mathbb{F}$-valued functions, one can give some structure to the bundles $\mathsf{T}^k\mathsf{M}$. We begin by describing the natural differentiable structure of these sets. We make an abuse of notation regarding charts, using the same notation as in Lemma 5.2.5. This ought not cause any confusion since the context should make clear the meaning of the symbols.

5.3.4 Lemma (Differentiable structure of jet bundles of curves) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *be a manifold of class* C^{r}*, let* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$*, and let* $(\mathcal{U}, \phi)$ *be an* $\mathbb{F}$*-chart for* M*. Define* $(\mathrm{j_k}\mathcal{U}, \mathrm{j_k}\phi)$ *by*

$$\mathrm{j_k}\mathcal{U} = \{\mathrm{j_k}\gamma(0) \mid \gamma \text{ a curve at } \mathrm{x},\ \mathrm{x} \in \mathcal{U}\}$$

and

$$\mathrm{j_k}\phi\colon \mathrm{j_k}\mathcal{U} \to \phi(\mathcal{U}) \times \underbrace{\mathbb{F}^{\mathrm{n}} \times \cdots \times \mathbb{F}^{\mathrm{n}}}_{\mathrm{k} \text{ times}}$$

$$\mathrm{j_k}\gamma(0) \mapsto ((\phi \circ \gamma)(0), (\phi \circ \gamma)'(0), \dots, (\phi \circ \gamma)^{(\mathrm{k})}(0)).$$

Then $(\mathrm{j_k}\mathcal{U}, \mathrm{j_k}\phi)$ *is an* $\mathbb{F}$*-chart for* $\mathsf{T}^{\mathrm{k}}\mathsf{M}$*. Moreover, if* $((\mathcal{U}_{\mathrm{a}}, \phi_{\mathrm{a}}))_{\mathrm{a} \in \mathrm{A}}$ *is an atlas for* M*, then* $((\mathrm{j_k}\mathcal{U}_{\mathrm{a}}, \mathrm{j_k}\phi_{\mathrm{a}}))_{\mathrm{a} \in \mathrm{A}}$ *is an atlas for* $\mathsf{T}^{\mathrm{k}}\mathsf{M}$*.*

Proof We first show that $j_k\phi$ is a bijection from $j_k\mathcal{U}$ onto the open set $\phi(\mathcal{U}) \times (\mathbb{F}^n)^k \subseteq (\mathbb{F}^n)^{k+1}$. First we show that $j_k\phi$ is surjective. Let $(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k) \in \phi(\mathcal{U}) \times (\mathbb{F}^n)^k$. Let γ be a curve whose local representative is

$$t \mapsto \boldsymbol{x} + t\boldsymbol{v}_1 + \cdots + \frac{t^k}{k!}\boldsymbol{v}_k$$

for t sufficiently small. Note that $j_k\phi(j_k\gamma(0)) = (\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$, giving the desired surjectivity. To show that $j_k\phi$ is injective, suppose that $j_k\phi(j_k\gamma(0)) = j_k\phi(j_k\sigma(0))$ for C^r-curves γ and σ at $x \in \mathcal{U}$. This implies that the first k Taylor coefficients at 0 of $\phi \circ \gamma$ and $\phi \circ \sigma$ in coordinates agree. By Lemma A.1.1 this implies that the first k Taylor coefficients at 0 of $f \circ \gamma$ and $f \circ \sigma$ agree, noting that

$$f \circ \gamma = (f \circ \phi^{-1}) \circ (\phi \circ \gamma),$$

and similarly for σ. Thus $j_k\gamma(0) = j_k\sigma(0)$, giving bijectivity of $j_k\phi$.

For the final assertion of the lemma, we must show that the overlap condition is satisfied. Thus let $(\mathcal{U}_a, \phi_a)$ and $(\mathcal{U}_b, \phi_b)$ be $\mathbb{F}$-charts for which $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$. It is clear that $j_k\mathcal{U}_a \cap j_k\mathcal{U}_b \neq \emptyset$. We suppose without loss of generality that $\mathcal{U} = \mathcal{U}_a = \mathcal{U}_b$. For an open set $I \subseteq \mathbb{F}$ for which $0 \in I$, for an open set $\mathcal{N} \subseteq \mathbb{F}^n$, and for a C^r-curve $\boldsymbol{\gamma}\colon I \to \mathcal{N}$, we denote

$$j_k\boldsymbol{\gamma}(0) = (\boldsymbol{\gamma}(0), \boldsymbol{\gamma}'(0), \dots, \boldsymbol{\gamma}^{(k)}(0)).$$

Note that

$$j_k\phi_b(j_k\gamma(0)) = j_k(\phi_b \circ \gamma)(0) = j_k(\phi_a \circ \phi_a^{-1} \circ \phi_b \circ \gamma)(0). \tag{5.6}$$

With this in mind, we give a lemma.

1 Sublemma *Let* $\psi\colon \mathcal{N} \to \mathcal{N}'$ *be a* C^{r}*-diffeomorphism of open subsets* $\mathcal{N}$ *and* $\mathcal{N}'$ *of* $\mathbb{F}^{\mathrm{n}}$ *and define a map*

$$\mathrm{J_k}\psi\colon \mathcal{N} \times (\mathbb{F}^{\mathrm{n}})^{\mathrm{k}} \to \mathcal{N}' \times (\mathbb{F}^{\mathrm{n}})^{\mathrm{k}}$$

by asking that

$$\mathrm{J_k}\psi(\boldsymbol{\gamma}(0), \boldsymbol{\gamma}'(0), \dots, \boldsymbol{\gamma}^{(\mathrm{k})}(0)) = (\psi \circ \boldsymbol{\gamma}(0), (\psi \circ \boldsymbol{\gamma})'(0), (\psi \circ \boldsymbol{\gamma})^{(\mathrm{k})}(0))$$

for any C^{r}*-curve* $\boldsymbol{\gamma}\colon \mathrm{I} \to \mathcal{N}$ *for which* $0 \in \mathrm{I}$*. Then* $\mathrm{J_k}\psi$ *is a diffeomorphism.*

Proof First of all, note that the map $J_k\psi$ is well-defined by Lemma A.1.1.

Next we prove that $J_k\psi$ is of class C^r. We prove this by induction on k. For $k = 1$ we have

$$J_1\psi(\boldsymbol{x}, \boldsymbol{v}) = (\psi(\boldsymbol{x}), \boldsymbol{D}\psi(\boldsymbol{x}) \cdot \boldsymbol{v}),$$

and the lemma follows immediately in this case. Now suppose that the lemma holds for $k \in \{1, \dots, m\}$. Then, for $\boldsymbol{\gamma}\colon I \to \mathcal{N}$ with $0 \in I$, the first m derivatives of $\psi \circ \boldsymbol{\gamma}$ are C^r-functions of the first m derivatives of $\boldsymbol{\gamma}$ by the induction hypothesis. Then, by Lemma A.1.1,

$$\boldsymbol{D}^{m+1}(\psi \circ \boldsymbol{\gamma})(0) = \boldsymbol{D}\psi(\boldsymbol{\gamma}(0)) \cdot \boldsymbol{\gamma}^{(m+1)}(0) + G(\boldsymbol{\gamma}(0), \boldsymbol{\gamma}'(0), \dots, \boldsymbol{\gamma}^{(m)}(0)),$$

where the function G is a C^r-function of the first m derivatives of $\boldsymbol{\gamma}$. Thus the first $m+1$ derivatives of $\psi \circ \boldsymbol{\gamma}$ are C^r-functions of the first $m+1$ derivatives of $\boldsymbol{\gamma}$, and this gives that $J_k\psi$ is of class C^r.

Now we prove that $J_k\psi$ is invertible. To see this, one needs only to note that the map

$$(\boldsymbol{\sigma}(0), \boldsymbol{\sigma}'(0), \dots, \boldsymbol{\sigma}^{(k)}(0)) \mapsto (\psi \circ \boldsymbol{\sigma}(0), (\psi \circ \boldsymbol{\sigma})'(0), (\psi \circ \boldsymbol{\sigma})^{(k)}(0))$$

for a C^r-curve $\boldsymbol{\sigma}\colon I \to \mathcal{N}'$ for which $0 \in I$, is the inverse of $J_k\psi$. Moreover, this inverse is of class C^r by the same argument as in the preceding paragraph. ▼

By (5.6) we have

$$j_k\phi_b(j_k\gamma(0)) = J_k(\phi_b \circ \phi_a^{-1})(j_k(\phi_a \circ \gamma)(0)).$$

Since $\phi_a \circ \phi_b^{-1}$ is a diffeomorphism, the overlap condition for jet bundle charts holds. ■

An $\mathbb{F}$-chart for $\mathsf{T}^k\mathsf{M}$ as in the lemma is called a ***natural chart***. The lemma gives the following result which further refines the differentiable structure of the jet bundles $\mathsf{T}^k\mathsf{M}$.

5.3.5 Theorem (Fibre bundle structure for jet bundles of curves) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r} *and let* $\mathrm{k}, \mathrm{l} \in \mathbb{Z}_{\geq 0}$ *with* $\mathrm{k} \geq \mathrm{l}$. *Then* $\rho^{\mathrm{k}}_{\mathrm{l}}\colon \mathsf{T}^{\mathrm{k}}\mathsf{M} \to \mathsf{T}^{\mathrm{l}}\mathsf{M}$ *is a locally trivial fibre bundle.*

Proof This follows since the local representative of ρ^k_l is

$$(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k) \mapsto (\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_l),$$

which shows that ρ^k_l is a surjective submersion and that $\rho^k_l\colon \mathsf{T}^k\mathsf{M} \to \mathsf{T}^l\mathsf{M}$ is locally trivial with respect to the natural coordinate charts. ■

5.3.3 Algebraic structure

Note that $\mathsf{T}^k\mathsf{M}$ does not have an obvious vector bundle structure like $\mathsf{T}^{*k}\mathsf{M}$, except in the case that $k = 1$, in which case $\mathsf{T}^1\mathsf{M}$ is simply the tangent bundle. (We shall see in Section 5.4.3 that $\mathsf{T}^k_x\mathsf{M}$ *does* possess algebraic structure.) However, $\mathsf{T}^k\mathsf{M}$ does have an affine structure. First let us describe an injection of $\mathsf{T}_x\mathsf{M}$ into $\mathsf{T}^k_x\mathsf{M}$. Let $k \in \mathbb{Z}_{>0}$ and let $I \subseteq \mathbb{F}$ be an open set for which $0 \in I$. Denote

$$I_k = \{t \in \mathbb{F} \mid t^k \in I\}$$

and let $\tau_k\colon I_k \to I$ be defined by $\tau_k(t) = \frac{t^k}{k!}$. We let $x \in \mathsf{M}$ and define $\epsilon_{k,x}\colon \mathsf{T}_x\mathsf{M} \to \mathsf{T}^k_x\mathsf{M}$ by

$$\epsilon_{k,x}(v) = j_k(\gamma \circ \tau_k)(0),$$

where $\gamma\colon I \to \mathsf{M}$ is a C^r-curve for which $\gamma'(0) = v$. Now, to define the affine structure, let $j_k\gamma(0) \in \mathsf{T}^k_x\mathsf{M}$ and let $v \in \mathsf{T}_x\mathsf{M}$. Suppose that γ has domain $I \subseteq \mathbb{F}$, let $J \subseteq \mathbb{F}$ be an open set for which $0 \in J$, and let $\sigma\colon J \times I \to \mathsf{M}$ be a C^r-map for which

1. $\sigma(0, t) = \gamma(t)$ for every $t \in I$ and
2. $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma(s, 0) = v$.

Then define $\hat{I} = I \cap J$ and $\hat{\gamma}\colon \hat{I} \to \mathsf{M}$ by $\hat{\gamma}(t) = \sigma(\tau_k(t), t)$. We then define

$$j_k\gamma(0) + v = j_k\hat{\gamma}(0).$$

The notation is intended to suggest that $j_k\hat{\gamma}(0)$ is the affine addition of v to $j_k\gamma(0)$. The proof of the following result contains all of the ingredients to ensure that this makes sense.

5.3.6 Theorem (Affine bundle structure for jet bundles of curves) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r}, *let* $\mathrm{k} \geq 0$, *let* $\mathrm{x} \in \mathsf{M}$, *and let* $\mathrm{v} \in \mathsf{T}_{\mathrm{x}}\mathsf{M}$. *Then the diagram*

$$\begin{tikzcd} \mathsf{T}^{\mathrm{k}}_{\mathrm{x}}\mathsf{M} \arrow[rr, "+\mathrm{v}"] \arrow[dr, "\rho^{\mathrm{k}}_{\mathrm{k}-1}"'] & & \mathsf{T}^{\mathrm{k}}_{\mathrm{x}}\mathsf{M} \arrow[dl, "\rho^{\mathrm{k}}_{\mathrm{k}-1}"] \\ & \mathsf{T}^{\mathrm{k}-1}_{\mathrm{x}}\mathsf{M} & \end{tikzcd}$$

commutes and the map $(\mathrm{v}, \mathrm{j_k}\gamma(0)) \mapsto \mathrm{j_k}\gamma(0) + \mathrm{v}$ *makes* $(\rho^{\mathrm{k}}_{\mathrm{k}-1})^{-1}(\mathrm{j_{k-1}}\gamma(0))$ *an affine space modelled on* $\mathsf{T}_{\mathrm{x}}\mathsf{M}$.

Proof Let us first derive the local representative in natural coordinates for $j_k\gamma(0) + v$. Let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-chart about x and let

$$(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k) \in \phi(\mathcal{U}) \times (\mathbb{F}^n)^k, \quad \boldsymbol{v} \in \mathbb{F}^n$$

be the coordinate representations for $j_k\gamma(0)$ and v, respectively. The local representative for σ has the form $(s, t) \mapsto \boldsymbol{\sigma}(s, t)$ for a C^r-map $\boldsymbol{\sigma}\colon J' \times I' \to \phi(\mathcal{U})$ satisfying

1. $\boldsymbol{\sigma}(0, t) = \boldsymbol{\gamma}(t)$ and
2. $\boldsymbol{D}_1\boldsymbol{\sigma}(0, 0) = \boldsymbol{v}$,

where $\boldsymbol{\gamma}$ is the local representative of γ. Thus the local representative $\hat{\boldsymbol{\gamma}}$ of $\hat{\gamma}$ is given by $\hat{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\tau_k(t), t)$. To facilitate taking derivatives of $\hat{\boldsymbol{\gamma}}$, let us denote $\boldsymbol{\iota}_k(t) = (\tau_k(t), t)$ so that $\hat{\boldsymbol{\gamma}} = \boldsymbol{\sigma} \circ \boldsymbol{\iota}_k$. Note that

1. $\boldsymbol{D}\boldsymbol{\iota}_k(0, 0) = (0, 1)$,
2. $\boldsymbol{D}^m\boldsymbol{\iota}_k(0, 0) = (0, 0)$ for $m \in \{2, \dots, k-1\}$, and
3. $\boldsymbol{D}^k\boldsymbol{\iota}_k(0, 0) = (1, 0)$.

Then, by Lemma A.1.1,

$$\boldsymbol{D}^m\hat{\boldsymbol{\gamma}}(0) = \boldsymbol{D}^m\boldsymbol{\sigma}\circ\boldsymbol{\iota}_k(0) = \boldsymbol{D}_2^m\boldsymbol{\sigma}(0) = \boldsymbol{v}_m, \qquad m \in \{1,\dots,k-1\}$$

and

$$\boldsymbol{D}^k\hat{\boldsymbol{\gamma}}(0) = \boldsymbol{D}^k\boldsymbol{\sigma}\circ\boldsymbol{\iota}_k(0) = \boldsymbol{D}_2^k\boldsymbol{\sigma}(0,0) + \boldsymbol{D}_1\boldsymbol{\sigma}(0,0) = \boldsymbol{v}_k + \boldsymbol{v}.$$

Thus the local representative of $j_k\gamma(0) + v$ is

$$(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k + \boldsymbol{v}).$$

This shows that $j_k\gamma(0) + v$ does not depend on the choice of σ. Moreover, the assertions of the theorem follow directly from the form of this local representative. ■

5.4 Jet bundles of maps between manifolds

In this section we consider jets of mappings between manifolds. We do this by making use of notion of jets of functions and curves as considered in the preceding sections. As we have done all along in this chapter, we let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \{\infty, \omega, \mathrm{hol}\}$ and adopt the convention that $r \in \{\infty, \omega\}$ when $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ when $\mathbb{F} = \mathbb{C}$. We also use the same symbol d to stand for the real or complex differential. Also, when we say "curve" we mean a map of class C^r from I where $I \subseteq \mathbb{F}$ is open.

5.4.1 Definitions

The key to our construction is the following lemma.

5.4.1 Lemma (Characterisations of kth-order agreement for maps) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *and* N *be manifolds of class* C^{r}*, let* $\mathrm{x}_0 \in \mathsf{M}$ *and let* $\mathrm{y}_0 \in \mathsf{N}$*, let* $\mathcal{U}$ *be a neighbourhood of* x_0 *and let* $\mathcal{V}$ *be a neighbourhood of* y_0*, and let* $\Phi, \Psi \in \mathrm{C}^{\mathrm{r}}(\mathcal{U}; \mathcal{V})$ *be such that* $\Phi(\mathrm{x}_0) = \Psi(\mathrm{x}_0) = \mathrm{y}_0$*. Let* $(\mathcal{U}', \phi)$ *be an* $\mathbb{F}$*-chart for* M *about* x_0 *and let* $(\mathcal{V}', \psi)$ *be an* $\mathbb{F}$*-chart for* N *about* y_0 *such that* $\Phi(\mathcal{U}'), \Psi(\mathcal{U}') \subseteq \mathcal{V}'$*. Then, for* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$*, the following statements are equivalent:*

(i) $\mathbf{D}^{\mathrm{m}}\Phi_{\phi\psi}(\Phi(\mathrm{x}_0)) = \mathbf{D}^{\mathrm{m}}\Psi_{\phi\psi}(\Psi(\mathrm{x}_0))$, $\mathrm{m} \in \{0, 1, \dots, \mathrm{k}\}$;

(ii) $\mathrm{j}_{\mathrm{k}}(\Phi\circ\gamma)(0) = \mathrm{j}_{\mathrm{k}}(\Psi\circ\gamma)(0)$ *for every* $(\gamma, \mathrm{I}) \in \mathscr{C}^{\mathrm{r}}(\mathrm{x}_0)$*;*

(iii) $\mathrm{j}_{\mathrm{k}}(\mathrm{f}\circ\Phi)(\mathrm{x}_0) = \mathrm{j}_{\mathrm{k}}(\mathrm{f}\circ\Psi)(\mathrm{x}_0)$ *for every* $(\mathrm{f}, \mathcal{W}) \in \mathscr{F}^{\mathrm{r}}(\mathrm{y}_0)$*.*

Proof (i) $\Longrightarrow$ (ii) By Lemma A.1.1, the local representative for the mth derivative of $\Phi\circ\gamma$ involves the local representatives for the first m derivatives of Φ and γ, and similarly for $\Psi\circ\gamma$. Thus, if the local representatives for the first k derivatives of Φ and Ψ agree, then so too do the local representative for the first k derivatives of $\Phi\circ\gamma$ and $\Psi\circ\gamma$. This part of the lemma then follows from Proposition 5.3.2.

(ii) $\Longrightarrow$ (iii) Let $(f, \mathcal{V}) \in \mathscr{F}^r(y_0)$. By hypothesis, the curves $\Phi\circ\gamma$ and $\Psi\circ\gamma$ agree to order k at y_0 for every $(\gamma, I) \in \mathscr{R}^r(x_0)$. Thus, by definition,

$$(f\circ(\Phi\circ\gamma))^{(j)}(0) = (f\circ(\Psi\circ\gamma))^{(j)}(0), \qquad j \in \{0, 1, \dots, k\},$$

for every $(\gamma, I) \in \mathscr{C}^r(x_0)$. Again by definition, this means that $f \circ \Phi$ and $f \circ \Psi$ agree to order k at y_0. But this means, by a final application of the definitions, that $j_k(f \circ \Phi)(x_0) = j_k(f \circ \Psi)(x_0)$.

(iii) $\Longrightarrow$ (i) Without loss of generality, suppose that the chart $(\mathcal{V}', \psi)$ has the property that $\psi(\Phi(x_0)) = \mathbf{0} \in \mathbb{F}^m$. Also suppose that ϕ takes values in $\mathbb{F}^n$. Let coordinates in the chart $(\mathcal{U}', \phi)$ be denoted by $(x_1, \dots, x_n)$ and coordinates in the chart $(\mathcal{V}', \psi)$ be denoted by $(y_1, \dots, y_m)$. Let $j \in \{1, \dots, k\}$ and let $j_1, \dots, j_n \in \mathbb{Z}_{\geq 0}$ be such that $j_1 + \dots + j_n = j$. Let $J = (j_1, \dots, j_n)$ and let $l \in \{1, \dots, m\}$. Now define $f_{l,j} \colon \psi(\mathcal{V}) \to \mathbb{F}$ by $f_{l,j}(\boldsymbol{y}) = y_l^j$. By Lemma A.1.1,

$$\frac{\partial^j (f_{l,j} \circ \Phi \circ \phi^{-1})}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(\phi(x_0)) = \frac{\partial^j (f_{l,j} \circ \psi^{-1})}{y_l^j}(\mathbf{0}) \frac{\partial^j (\psi \circ \Phi \circ \phi^{-1})^l}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(\phi(x_0)) = j! \frac{\partial^j (\psi \circ \Phi \circ \phi^{-1})^l}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(\phi(x_0)).$$

A similar expression holds for Ψ. Since $j_k(f_{l,j} \circ \Phi)(x_0) = j_k(f_{l,j} \circ \Psi)(x_0)$, we may conclude from Proposition 5.2.3 that

$$\frac{\partial^j (\psi \circ \Phi \circ \phi^{-1})^l}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(\phi(x_0)) = \frac{\partial^j (\psi \circ \Psi \circ \phi^{-1})^l}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(\phi(x_0)).$$

Since this holds for every $l \in \{1, \dots, m\}$ and for every multi-index J for which $|J| \leq k$, this implication of the lemma follows. ■

Let $x_0 \in \mathsf{M}$, $y_0 \in \mathsf{N}$, and $k \in \mathbb{Z}_{\geq 0}$. For convenience, let us denote by $\mathscr{M}^r(x_0, y_0)$ the set of triples $(\Phi, \mathcal{U}, \mathcal{V})$ such that $\mathcal{U}$ is a neighbourhood of x_0, $\mathcal{V}$ is a neighbourhood of y_0, and $\Phi \in \mathrm{C}^r(\mathcal{U}; \mathcal{V})$ satisfies $\Phi(x_0) = y_0$. Based on the above lemma, we can have the following equivalent characterisations of an equivalence relation $\sim_{k,x_0,y_0}$ in $\mathscr{M}^r(x_0, y_0)$:

1. $\Phi \sim_{k,x_0,y_0} \Psi$ if $\boldsymbol{D}^m \Phi_{\phi\psi}(\Phi(x_0)) = \boldsymbol{D}^m \Psi_{\phi\psi}(\Psi(x_0))$, $m \in \{0, 1, \dots, k\}$, for all $\mathbb{F}$-charts $(\mathcal{U}, \phi)$ for M about x_0 and $\mathbb{F}$-charts $(\mathcal{V}, \psi)$ for N about y_0 such that $\Phi(\mathcal{U}), \Psi(\mathcal{U}) \subseteq \mathcal{V}$;
2. $\Phi \sim_{k,x_0,y_0} \Psi$ if $j_k(\Phi \circ \gamma)(0) = j_k(\Psi \circ \gamma)(0)$ for every $(\gamma, I) \in \mathscr{C}^r(x_0)$;
3. $\Phi \sim_{k,x_0,y_0} \Psi$ if $j_k(f \circ \Phi)(x_0) = j_k(f \circ \Psi)(x_0)$ for every $(f, \mathcal{W}) \in \mathscr{F}^r(y_0)$.

This allows us to make the following definition.

5.4.2 Definition (Jets of maps between manifolds) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M and N be manifolds of class C^r, let $x_0 \in \mathsf{M}$, let $y_0 \in \mathsf{N}$, and let $k \in \mathbb{Z}_{\geq 0}$.

(i) A **k-*jet of maps*** at (x_0, y_0) is an equivalence class in $\mathscr{M}^r(x_0, y_0)$ under the equivalence relation $\sim_{k,x_0,y_0}$.

(ii) The equivalence class of $(\Phi, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^r(x_0, y_0)$ is denoted by $j_k\Phi(x_0)$.

(iii) We denote
$$\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M}; \mathsf{N}) = \{ j_k\Phi(x_0) \mid (\Phi, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^r(x_0, y_0) \}.$$

(iv) We denote $\mathsf{J}^k(\mathsf{M}; \mathsf{N}) = \cup_{(x,y) \in \mathsf{M} \times \mathsf{N}} \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M}; \mathsf{N})$. By convention, $\mathsf{J}^0(\mathsf{M}; \mathsf{N}) = \mathsf{M} \times \mathsf{N}$.

(v) For $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$ we denote by $\rho^k_l \colon \mathsf{J}^k(\mathsf{M}; \mathsf{N}) \to \mathsf{J}^l(\mathsf{M}; \mathsf{N})$ the projection defined by $\rho^k_l(j_k\Phi(x_0)) = j_l(\Phi(x_0))$. We abbreviate ρ^k_0 by ρ_k. •

5.4.2 Geometric structure

Let us describe the differentiable structure for jet bundles of maps between manifolds.

5.4.3 Lemma (Differentiable structure of jet bundles of maps between manifolds) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *and* N *be manifolds of class* C^{r}*, let* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$*, and let* $(\mathcal{U}, \phi)$ *and* $(\mathcal{V}, \psi)$ *be* $\mathbb{F}$*-charts for* M *and* N*. Define* $(\mathrm{j}_{\mathrm{k}}(\mathcal{U} \times \mathcal{V}), \mathrm{j}_{\mathrm{k}}(\phi \times \psi))$ *by*

$$\mathrm{j}_{\mathrm{k}}(\mathcal{U} \times \mathcal{V}) = \{\mathrm{j}_{\mathrm{k}}\Phi(\mathrm{x}) \mid \ \mathrm{x} \in \mathcal{U},\ (\Phi, \mathcal{U}', \mathcal{V}') \in \mathscr{M}^{\mathrm{r}}(\mathrm{x}_0, \mathrm{y}_0)\}$$

and

$$\begin{aligned}\mathrm{j}_{\mathrm{k}}(\phi \times \psi)\colon\ \mathrm{j}_{\mathrm{k}}(\mathcal{U} \times \mathcal{V}) &\to \phi(\mathcal{U}) \times \psi(\mathcal{V}) \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}}; \mathbb{F}^{\mathrm{m}}) \times \cdots \times \mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}}; \mathbb{F}^{\mathrm{m}})\\ \mathrm{j}_{\mathrm{k}}\Phi(\mathrm{x}) &\mapsto \phi(\mathrm{x}), \psi \circ \Phi(\mathrm{x}), \mathbf{D}\Phi_{\phi\psi}(\phi(\mathrm{x})), \dots, \mathbf{D}^{\mathrm{k}}\Phi_{\phi\psi}(\phi(\mathrm{x})).\end{aligned}$$

Then $(\mathrm{j}_{\mathrm{k}}(\mathcal{U} \times \mathcal{V}), \mathrm{j}_{\mathrm{k}}(\phi \times \psi))$ *is an* $\mathbb{F}$*-chart for* $\mathsf{J}^{\mathrm{k}}(\mathsf{M}; \mathsf{N})$*. Moreover, if* $((\mathcal{U}_{\mathrm{a}}, \phi_{\mathrm{a}}))_{\mathrm{a} \in \mathrm{A}}$ *and* $((\mathcal{V}_{\mathrm{b}}, \psi_{\mathrm{b}}))_{\mathrm{b} \in \mathrm{B}}$ *are atlases for* M *and* N*, respectively, then* $((\mathrm{j}_{\mathrm{k}}(\mathcal{U}_{\mathrm{a}} \times \mathcal{V}_{\mathrm{b}}), \mathrm{j}_{\mathrm{k}}(\phi_{\mathrm{a}} \times \psi_{\mathrm{b}})))_{(\mathrm{a},\mathrm{b}) \in \mathrm{A} \times \mathrm{B}}$ *is an atlas for* $\mathsf{J}^{\mathrm{k}}(\mathsf{M}; \mathsf{N})$*.*

Proof Let us first show that $j_k(\phi \times \psi)$ is a bijection from $j_k(\mathcal{U} \times \mathcal{V})$ to

$$\phi(\mathcal{U}) \times \psi(\mathcal{V}) \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m).$$

To see that $j_k(\phi \times \psi)$ is surjective, let

$$(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_k) \in \phi(\mathcal{U}) \times \psi(\mathcal{V}) \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m)$$

and define $\boldsymbol{\Phi}\colon \mathbb{F}^n \to \mathbb{F}^m$ so that $\boldsymbol{\Phi}(\boldsymbol{x}_0) = \boldsymbol{y}_0$ and $\boldsymbol{D}^j\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{A}_j$, $j \in \{1, \dots, k\}$. This construction was done as part of the proof of Theorem 1.1.4. Let $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{B}}^n(r, \boldsymbol{x}_0) \subseteq \phi(\mathcal{U})$ and such that $\boldsymbol{\Phi}(\overline{\mathsf{B}}^n(r, \boldsymbol{x}_0)) \subseteq \psi(\mathcal{V})$. Then the function $\Phi(x) = \boldsymbol{\Phi}(\phi(x))$ defined on $\phi^{-1}(\overline{\mathsf{B}}^n(r, \boldsymbol{x}_0))$ has the property that

$$j_k(\phi \times \psi)(j_k\Phi(x)) = (\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_k).$$

Next we show that $j_k(\phi \times \psi)$ is injective. Suppose that

$$j_k(\phi \times \psi)(j_k\Phi(x)) = j_k(\phi \times \psi)(j_k\Psi(x)).$$

This implies that $\Phi(x) = \Psi(x)$ and that

$$\boldsymbol{D}^j\boldsymbol{\Phi}_{\phi\psi}(\phi(x)) = \boldsymbol{D}^j\boldsymbol{\Psi}_{\phi\psi}(\phi(x)), \qquad j \in \{1, \dots, k\}.$$

By Lemma 5.4.1 it follows that $j_k\Phi(x) = j_k\Psi(x)$, and so $(j_k(\mathcal{U} \times \mathcal{V}), j_k(\phi \times \psi))$ is an $\mathbb{F}$-chart.

Next we verify that the charts satisfy the overlap condition. Let $(\mathcal{U}_a \times \mathcal{V}_b, \phi_a \times \psi_b)$ and $(\mathcal{U}_\alpha \times \mathcal{U}_\beta, \phi_\alpha \times \psi_\beta)$ be product charts for $\mathsf{M} \times \mathsf{N}$ such that their intersection is nonempty. Note that

$$(\mathcal{U}_a \times \mathcal{V}_b) \cap (\mathcal{U}_\alpha \times \mathcal{U}_\beta) = (\mathcal{U}_a \cap \mathcal{U}_\alpha) \times (\mathcal{V}_b \cap \mathcal{V}_\beta).$$

Thus we may assume, without loss of generality, that $\mathcal{U}_a = \mathcal{U}_\alpha = \mathcal{U}$ and that $\mathcal{V}_b = \mathcal{V}_\beta = \mathcal{V}$. If $\boldsymbol{\Psi}\colon \mathcal{N} \to \mathcal{M}$ is a C^r-map between open subsets $\mathcal{N} \subseteq \mathbb{F}^n$ and $\mathcal{M} \subseteq \mathbb{F}^m$, define

$$\begin{aligned} j_k\boldsymbol{\Psi}\colon\ \mathcal{N} &\to\ \mathcal{N} \times \mathcal{M} \times \mathrm{L}_{\mathrm{sym}}(1;\mathbb{F}^n)\mathbb{F}^m \times \cdots \times \mathrm{L}_{\mathrm{sym}}(k;\mathbb{F}^n)\mathbb{F}^m \\ \boldsymbol{x} &\mapsto\ (\boldsymbol{x}, \boldsymbol{\Psi}(x), \boldsymbol{D\Psi}(\boldsymbol{x}), \dots, \boldsymbol{D}^k\boldsymbol{\Psi}(\boldsymbol{x})). \end{aligned}$$

With this notation we have

$$j_k(\phi_\alpha \times \psi_\beta)(j_k\Phi(x)) = j_k(\psi_\beta \circ \Phi \circ \phi_\alpha^{-1})(\phi_\alpha(x)) = j_k(\psi_\beta \circ \psi_b^{-1} \circ \psi_b \circ \Phi \circ \phi_a^{-1} \circ \phi_a \circ \phi_\alpha^{-1})(\phi_\alpha(x)). \tag{5.7}$$

We now use a lemma.

1 **Sublemma** *Let $\phi\colon \mathcal{N} \to \mathcal{N}'$ be a C^{r}-diffeomorphism of open subsets $\mathcal{N}, \mathcal{N}' \subseteq \mathbb{F}^{\mathrm{n}}$, let $\psi\colon \mathcal{M} \to \mathcal{M}'$ be a C^{r}-diffeomorphism of open subsets $\mathcal{M}, \mathcal{M}' \subseteq \mathbb{F}^{\mathrm{m}}$, and define a map*

$$\begin{aligned} \mathrm{J}_{\mathrm{k}}(\phi \times \psi)\colon \mathcal{N} \times \mathcal{M} \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}^{\mathrm{m}}) \times \cdots \times \mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}^{\mathrm{m}}) \\ \to \mathcal{N}' \times \mathcal{M}' \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}^{\mathrm{m}}) \times \cdots \times \mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathbb{F}^{\mathrm{n}};\mathbb{F}^{\mathrm{m}}) \end{aligned}$$

by asking that

$$\begin{aligned} &\mathrm{J}_{\mathrm{k}}(\phi \times \psi)(\mathbf{x}, \boldsymbol{\Psi}(\mathbf{x}), \mathbf{D}\boldsymbol{\Psi}(\mathbf{x}), \mathbf{D}^{\mathrm{k}}\boldsymbol{\Psi}(\mathbf{x})) \\ &\qquad = (\phi(\mathbf{x}), (\psi \circ \boldsymbol{\Psi} \circ \phi^{-1})(\phi(\mathbf{x})), \mathbf{D}(\psi \circ \boldsymbol{\Psi} \circ \phi^{-1})(\phi(\mathbf{x})), \dots, \mathbf{D}^{\mathrm{k}}(\psi \circ \boldsymbol{\Psi} \circ \phi^{-1})(\phi(\mathbf{x}))) \end{aligned}$$

for any C^{r}-map $\boldsymbol{\Psi}\colon \mathcal{N} \to \mathcal{M}$. Then $\mathrm{J}_{\mathrm{k}}(\phi \times \psi)$ is a C^{r}-diffeomorphism.

Proof By Lemma A.1.1, $J_k(\phi \times \psi)$ is well-defined.

To prove that the map is of class C^r, we use induction on k. For $k = 1$ we have

$$J_1(\phi \times \psi)(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}) = (\phi(\boldsymbol{x}), \psi(\boldsymbol{y}), \boldsymbol{D}\psi(\boldsymbol{y}) \circ \boldsymbol{A} \circ \boldsymbol{D}\phi^{-1}(\boldsymbol{x})).$$

By the Inverse Function Theorem, it follows that $J_1(\phi \times \psi)$ is of class C^r. Now suppose that $J_k(\phi \times \psi)$ is of class C^r for $k \in \{1, \dots, m\}$. By Lemma A.1.1,

$$\begin{aligned} &\boldsymbol{D}^{m+1}(\psi \circ \boldsymbol{\Psi} \circ \phi^{-1})(\phi(\boldsymbol{x}))(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{m+1}) \\ &\qquad = \boldsymbol{D}\psi(\boldsymbol{y}) \circ \boldsymbol{D}^{m+1}\boldsymbol{\Psi}(\boldsymbol{x}) \cdot (\boldsymbol{D}\phi^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}_1, \dots, \boldsymbol{D}\phi^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}_{m+1}) \\ &\qquad\qquad + \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{D\Psi}(\boldsymbol{x}), \dots, \boldsymbol{D}^m\boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{v}_1, \dots, \boldsymbol{v}_{m+1}), \end{aligned}$$

where $\boldsymbol{G}$ is a C^r-function of its arguments. By the induction hypothesis, $J_{m+1}(\phi \times \psi)$ is of class C^r.

To verify that $J_k(\phi \times \psi)$ is a bijection, we note that its inverse is defined by the map satisfying

$$\begin{aligned} &(\boldsymbol{x}', \boldsymbol{\Phi}(\boldsymbol{x}'), \boldsymbol{D\Phi}(\boldsymbol{x}'), \boldsymbol{D}^k\boldsymbol{\Phi}(\boldsymbol{x}')) \\ &\quad = (\phi^{-1}(\boldsymbol{x}'), (\psi^{-1} \circ \boldsymbol{\Phi} \circ \phi)(\phi^{-1}(\boldsymbol{x}')), \boldsymbol{D}(\psi^{-1} \circ \boldsymbol{\Phi} \circ \phi)(\phi^{-1}(\boldsymbol{x}')), \dots, \boldsymbol{D}^k(\psi^{-1} \circ \boldsymbol{\Phi} \circ \phi)(\phi^{-1}(\boldsymbol{x}'))) \end{aligned}$$

for a C^r-map $\boldsymbol{\Phi}\colon \mathcal{N}' \to \mathcal{M}'$. Moreover, this inverse is of class C^r by the same arguments used to prove that $J_k(\phi \times \psi)$ is of class C^r. ▼

Rephrasing (5.7), we have

$$j_k(\phi_\alpha \times \psi_\beta)(j_k\Phi(x)) = J_k((\phi_\alpha \circ \phi_a^{-1}) \times (\psi_\beta \circ \psi_b^{-1}))(j_k(\psi_b \circ \Phi \circ \phi_a^{-1})(\phi_a(x))).$$

By the lemma, this implies that the overlap map is a diffeomorphism. ■

As with the analogous charts constructed for jet bundles of curves and functions, the charts from the preceding lemma are called ***natural charts*** for $\mathsf{J}^k(\mathsf{M};\mathsf{N})$. Also as in the previous cases, these charts allow us to give fibre bundle structure to jet bundles.

5.4.4 Theorem (Fibre bundle structure for jet bundles of maps) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$. *Let* M *and* N *be manifolds of class* C^r *and let* $k, l \in \mathbb{Z}_{\geq 0}$ *with* $k \geq l$. *Then*

(i) $\rho^k_l \colon \mathsf{J}^k(\mathsf{M};\mathsf{N}) \to \mathsf{J}^l(\mathsf{M};\mathsf{N})$ *is a locally trivial fibre bundle.*

Moreover, if $\mathrm{pr}_1 \colon \mathsf{M} \times \mathsf{N} \to \mathsf{M}$ *and* $\mathrm{pr}_2 \colon \mathsf{M} \times \mathsf{N} \to \mathsf{N}$ *are the projections, then*

(ii) $\mathrm{pr}_1 \circ \rho_k \colon \mathsf{J}^k(\mathsf{M};\mathsf{N}) \to \mathsf{M}$ *and* $\mathrm{pr}_1 \circ \rho_k \colon \mathsf{J}^k(\mathsf{M};\mathsf{N}) \to \mathsf{M}$ *are locally trivial fibre bundles.*

Proof The local representative of ρ^k_l in natural charts is

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_k) \mapsto (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_l).$$

From this the first assertion follows immediately. The second assertion also follows easily from this formula for the local representative. ■

The following result characterises jets of compositions of maps.

5.4.5 Proposition (Jets commute with composition) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$. *Let* $k \in \mathbb{Z}_{\geq 0}$. *Let* M, N, *and* P *be manifolds of class* C^r, *let* $x_0 \in \mathsf{M}$, $y_0 \in \mathsf{N}$, *and* $z_0 \in \mathsf{P}$, *and let* $\mathcal{U}$ *be a neighbourhood of* x_0, $\mathcal{V}$ *be a neighbourhood of* y_0, *and* $\mathcal{W}$ *be a neighbourhood of* z_0. *If* $(\Phi_1, \mathcal{U}, \mathcal{V}), (\Phi_2, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^r(x_0, y_0)$ *and* $(\Psi_1, \mathcal{V}, \mathcal{W}), (\Psi_2, \mathcal{V}, \mathcal{W}) \in \mathscr{M}^r(y_0, z_0)$ *satisfy* $j_k\Phi_1(x_0) = j_k\Phi_2(x_0)$ *and* $j_k\Psi_1(y_0) = j_k\Psi_2(y_0)$, *then* $j_k(\Psi_1 \circ \Phi_1)(x_0) = j_k(\Psi_2 \circ \Phi_2)(x_0)$.

Proof Let $(\gamma, I) \in \mathscr{C}^r(x_0)$. Since $j_k\Phi_1(x_0) = j_k\Phi_2(x_0)$ it follows from Lemma 5.4.1 that

$$j_k(\Phi_1 \circ \gamma)(0) = j_k(\Phi_2 \circ \gamma)(0). \tag{5.8}$$

We next claim that

$$j_k(\Psi_1 \circ \Phi_1 \circ \gamma)(0) = j_k(\Psi_1 \circ \Phi_2 \circ \gamma)(0). \tag{5.9}$$

Indeed, let $(f, \mathcal{W}') \in \mathscr{F}^r(z_0)$. Since $f \circ \Psi_1$ is of class C^r on some neighbourhood of y_0, it follows from Definition 5.3.1 and (5.8) that

$$(f \circ \Psi_1 \circ \Phi_1 \circ \gamma)^{(j)}(0) = (f \circ \Psi_1 \circ \Phi_2 \circ \gamma)^{(j)}(0), \qquad j \in \{1, \dots, k\}.$$

Since this holds for every $(f, \mathcal{W}') \in \mathscr{F}^r(z_0)$, it follows by definition that (5.9) holds. Similarly,

$$j_k(\Psi_2 \circ \Phi_1 \circ \gamma)(0) = j_k(\Psi_2 \circ \Phi_2 \circ \gamma)(0). \tag{5.10}$$

Since $j_k\Psi_1(y_0) = j_k\Psi_2(y_0)$ it follows from Lemma 5.4.1 that

$$j_k(\Psi_1 \circ \Phi_1 \circ \gamma)(0) = j_k(\Psi_2 \circ \Phi_1 \circ \gamma)(0).$$

From the preceding equation and (5.10) we have

$$j_k(\Psi_1 \circ \Phi_1 \circ \gamma)(0) = j_k(\Psi_2 \circ \Phi_2 \circ \gamma)(0),$$

giving the result by Lemma 5.4.1 since the preceding equality holds for every curve γ. ■

The point of the result is that the composition of jets can be defined to be the jet of composition. That is, if $(\Phi, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^r(x_0, y_0)$ and $(\Psi, \mathcal{V}, \mathcal{W}) \in \mathscr{M}^r(y_0, z_0)$, then

$$j_k\Psi(y_0) \circ j_k\Phi(x_0) = j_k(\Psi \circ \Phi)(x_0). \tag{5.11}$$

5.4.3 Algebraic structure

Now let us examine the algebraic structure of jet bundles of maps. This algebraic structure will come in two parts. The first characterisation relates jets of maps to jets of functions.

5.4.6 Theorem (Maps between spaces of jets of functions) *Let $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $\mathrm{r} \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $\mathrm{r} = \mathrm{hol}$. Let M and N be manifolds of class C^{r}, let $\mathrm{x}_0 \in \mathsf{M}$ and let $\mathrm{y}_0 \in \mathsf{N}$, and let $(\Phi, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^{\mathrm{r}}(\mathrm{x}_0, \mathrm{y}_0)$. Then the map $\mathrm{j}_{\mathrm{k}}\Phi(\mathrm{x}_0)\colon \mathsf{T}^{*\mathrm{k}}_{\mathrm{y}_0}\mathsf{N} \to \mathsf{T}^{*\mathrm{k}}_{\mathrm{x}_0}\mathsf{M}$ defined by $\mathrm{j}_{\mathrm{k}}\Phi(\mathrm{x}_0)(\mathrm{j}_{\mathrm{k}}\mathrm{f}(\mathrm{y}_0)) = \mathrm{j}_{\mathrm{k}}(\Phi^*\mathrm{f})(\mathrm{x}_0)$ is a well-defined homomorphism of $\mathbb{F}$-algebras.*

Moreover, if $\Psi\colon \mathsf{T}^{*\mathrm{k}}_{\mathrm{y}_0}\mathsf{N} \to \mathsf{T}^{*\mathrm{k}}_{\mathrm{x}_0}\mathsf{M}$ is a homomorphism of $\mathbb{F}$-algebras, then there exists $(\Phi, \mathcal{U}, \mathcal{V}) \in \mathscr{M}^{\mathrm{r}}(\mathrm{x}_0, \mathrm{y}_0)$ such that $\Psi = \mathrm{j}_{\mathrm{k}}\Phi(\mathrm{x}_0)$.

Proof First let us verify that $j_k\Phi(x_0)$ is well-defined. Let $(f, \mathcal{V}), (g, \mathcal{V}) \in \mathscr{F}^r(y_0)$ be such that $j_kf(y_0) = j_kg(y_0)$. Note that $\Phi^*f = f \circ \Phi$ and so $\Phi^*f(x_0) = 0$. By the higher-order Chain Rule, Lemma A.1.1, the coordinate expression for $j_k(\Phi^*f)(x_0)$ depends on the first k-derivatives of f at y_0 and Φ at x_0. By Proposition 5.2.3 it follows that $j_k(\Phi^*f)(x_0) = j_k(\Phi^*g)(x_0)$ and so $j_k\Phi(x_0)$ is well-defined, as claimed.

Let us now verify that $j_k\Phi(x_0)$ is a homomorphism of $\mathbb{F}$-algebras. Using elementary properties of pull-back [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.3] we have

$$\begin{aligned} j_k\Phi(x_0)(j_kf(x_0) + j_kg(x_0)) &= j_k\Phi(x_0)(j_k(f+g)(x_0)) = j_k(\Phi^*(f+g))(x_0) \\ &= j_k(\Phi^*f)(x_0) + j_k(\Phi^*g)(x_0) = j_k\Phi(x_0)(j_kf(y_0)) + j_k\Phi(x_0)(j_kg(y_0)). \end{aligned}$$

correct? In like manner we compute

$$j_k\Phi(x_0)(j_k(fg)(y_0)) = j_k\Phi(x_0)(j_kf(y_0))\, j_k\Phi(x_0)(j_kg(y_0)),$$

and from this we deduce that $j_k\Phi(x_0)$ is an algebra homomorphism as desired.

Now we verify the last assertion of the theorem. Let $(\mathcal{U}, \phi)$ be an $\mathbb{F}$-chart for M about x_0 and let $(\mathcal{V}, \psi)$ be an $\mathbb{F}$-chart for N about y_0. Suppose, without loss of generality, that $\phi(x_0) = \mathbf{0}$ and that $\psi(y_0) = \mathbf{0}$. Suppose that n is the dimension of the connected component of M containing x_0 and that m is the dimension of the connected component of N containing y_0. By Lemma 5.2.5, the charts $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ establish $\mathbb{F}$-algebra isomorphisms

$$\begin{aligned} \hat{\phi}_k&\colon \mathsf{T}^{*k}_{x_0}\mathsf{M} \to \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}), \\ \hat{\psi}_k&\colon \mathsf{T}^{*k}_{y_0}\mathsf{N} \to \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^m; \mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^m; \mathbb{F}). \end{aligned}$$

Let

$$\hat{\Psi}\colon \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^m;\mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^m;\mathbb{F}) \to \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})$$

be given by $\hat{\Psi} = \hat{\phi}_k \circ \Psi \circ \hat{\psi}_k^{-1}$. Note that $\hat{\Psi}$ is an $\mathbb{F}$-algebra homomorphism. If $(\boldsymbol{e}^1, \dots, \boldsymbol{e}^n)$ denotes the standard dual basis for $\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}) \simeq (\mathbb{F}^n)^*$ and if $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^m)$ denotes the standard dual basis for $\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^m;\mathbb{F}) \simeq (\mathbb{F}^m)^*$, then write

$$\hat{\Psi}(\boldsymbol{f}^a, \boldsymbol{0}, \dots, \boldsymbol{0}) = \Big(\sum_{j=1}^n \Psi^a_j \boldsymbol{e}^j, \boldsymbol{0}, \dots, \boldsymbol{0}\Big), \qquad a \in \{1, \dots, m\},$$

noting that the right-hand side will have the given form by the high-order Leibniz Rule, Lemma A.2.2, since the argument on the left corresponds to a function whose derivatives of order higher than one vanish, i.e., a linear function. Now define $\hat{\Phi}\colon \mathbb{F}^n \to \mathbb{F}^m$ by

$$\hat{\Phi}(x^1, \dots, x^n) = \Big(\sum_{j=1}^n \Psi^1_j x^j, \dots, \sum_{j=1}^n \Psi^m_j x^j\Big),$$

and note that

$$\hat{\Psi}(\boldsymbol{f}^a, \boldsymbol{0}, \dots, \boldsymbol{0}) = j_k\hat{\Phi}(\boldsymbol{0})(\boldsymbol{f}^a, \boldsymbol{0}, \dots, \boldsymbol{0}), \qquad a \in \{1, \dots, m\}.$$

Since the elements

$$\begin{aligned}(\boldsymbol{e}^j, \boldsymbol{0}, \dots, \boldsymbol{0}) &\in \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}), \qquad j \in \{1, \dots, n\},\\ (\boldsymbol{f}^a, \boldsymbol{0}, \dots, \boldsymbol{0}) &\in \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^m;\mathbb{F}) \times \cdots \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^m;\mathbb{F}), \qquad a \in \{1, \dots, m\},\end{aligned}$$

generate the $\mathbb{F}$-algebras by Theorem 5.2.10, it follows that $\hat{\Psi} = j_k\hat{\Phi}(\boldsymbol{0})$. ■

Next let us examine the affine bundle structure of $\mathsf{J}^k(\mathsf{M};\mathsf{N})$. To do this we use our interpretation from the preceding theorem of elements of $\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ as algebra homomorphisms. We first establish a mapping from $\epsilon_{k,(x_0,y_0)}\colon \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{y_0}\mathsf{N} \to \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. We shall first simply state the constructions, keeping in mind that these constructions will be made sense of in the ensuing proofs. Let $(f_1, \mathcal{U}), \dots, (f_k, \mathcal{U}) \in \mathscr{F}^r(x_0)$, let $(g, \mathcal{V}) \in \mathscr{F}^r(x_0)$, and let Y be a vector field defined on $\mathcal{V}$. Let us abbreviate $f = f_1 \cdots f_k$. We then define a homomorphism $H_{f,Y}\colon \mathsf{T}^{*k}_{y_0}\mathsf{N} \to \mathsf{T}^{*k}_{x_0}\mathsf{M}$ by

$$H_{f,Y}(j_k g(y_0)) = j_k(\mathscr{L}_Y g(y_0) f_1 \cdots f_k)(x_0).$$

Let us be clear what the preceding expression means. The function whose k-jet is being taken at x_0 is the function

$$x \mapsto \mathscr{L}_Y g(y_0) f_1 \cdots f_k(x),$$

with $\mathscr{L}_Y g(y_0)$ being thought of as a scalar.

Now one defines the proposed map $\epsilon_{k,(x_0,y_0)}$ by asking that

$$\epsilon_{k,(x_0,y_0)}(\mathrm{d}f_1(x_0) \odot \cdots \odot \mathrm{d}f_k(x_0)) \otimes Y(y_0) = H_{f,Y},$$

understanding that we identify homomorphisms of the $\mathbb{F}$-algebras $\mathsf{T}^{*k}_{y_0}\mathsf{N}$ and $\mathsf{T}^{*x_0}_k\mathsf{M}$ with $\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. The following lemma gives some meaning to the preceding constructions.

5.4.7 Lemma (k-jets whose first k − 1 derivatives are zero) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$*. Let* M *and* N *be manifolds of class* C^r*, let* $(x_0, y_0) \in \mathsf{M} \times \mathsf{N}$*, and let* $k \in \mathbb{Z}_{>0}$*. Let* $\Phi_{y_0} \in \mathrm{C}^\infty(\mathsf{M}; \mathsf{N})$ *be defined by* $\mathrm{Phi}_{y_0}(x) = y_0$*. Then the map* $\epsilon_{k,(x_0,y_0)} \colon \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{TN} \to \mathsf{J}^k(\mathsf{M}; \mathsf{N})$ *is a well-defined bijection onto* $(\rho^k_{k-1})^{-1}(j_{k-1}\Phi_{y_0}(x_0))$*.*

Proof We adopt the notation preceding the statement of the lemma. We first claim that $H_{f,Y}$ is, in fact, a homomorphism of $\mathbb{F}$-algebras. Clearly $H_{f,Y}$ is $\mathbb{F}$-linear. Also,

$$\begin{aligned} H_{f,Y}(j_k g(y_0) j_k h(y_0)) &= H_{f,Y}(j_k(gh)(x_0)) = \mathscr{L}_Y(gh)(y_0) j_k(f_1 \cdots f_k)(x_0) \\ &= (\mathscr{L}_Y g(y_0) h(y_0) + g(y_0) \mathscr{L}_Y h(y_0)) j_k(f_1 \cdots f_k)(x_0) \\ &= H_{f,Y}(j_k g(y_0)) h(y_0) + g(y_0) H_{f,Y}(j_k h(y_0)), \end{aligned}$$

as desired.

We next claim that $H_{f,Y} \in (\rho^k_{k-1})^{-1}(j_{k-1}\Phi_{y_0}(x_0))$. By Lemma 5.2.7 it follows that $j_{k-1}(H_{f,Y}) = 0$. Since $j_{k-1}\Phi_{y_0}(x_0) = 0$ we have $j_{k-1}(H_{f,Y}) = j_{k-1}\Phi_{y_0}(x_0)$, as desired.

We next note that $H_{f,Y}$ is actually a map from $\mathsf{T}^{*1}_{y_0}\mathsf{N} \simeq \mathsf{T}^*_{y_0}\mathsf{N}$ to $\mathsf{T}^{*k}_{x_0}\mathsf{M}$, and the domain is regarded as $\mathsf{T}^{*k}_{y_0}\mathsf{N}$ by composition on the right with the projection ρ^k_1. That is, we have

$$H_{f,Y}(j_k g(y_0)) = j_k((\mathrm{d}g(y_0) \cdot Y(y_0)) f_1 \cdots f_k)(x_0).$$

With the above discussion in hand, let us make a construction. Let $\alpha_1, \dots, \alpha_k \in \mathsf{T}^*_{x_0}\mathsf{M}$ and let $u \in \mathsf{T}_{y_0}\mathsf{N}$. Let $(f_j, \mathcal{U}) \in \mathscr{F}^r(x_0)$ be such that $f_j(x_0) = 0$ and $\mathrm{d}f_j(x_0) = \alpha_j$, $j \in \{1, \dots, k\}$, and let Y be a vector field defined on a neighbourhood $\mathcal{V}$ of y_0 such that $Y(y_0) = u$. Abbreviate $\alpha = (\alpha_1, \dots, \alpha_k)$. Then define an element $H_{\alpha,u}$ of $\mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{y_0}\mathsf{N} \simeq \mathrm{Hom}_{\mathbb{F}}(\mathsf{T}^*_{y_0}\mathsf{N}; \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}))$ by

$$H_{\alpha,u} = \alpha_1 \odot \cdots \odot \alpha_k \otimes u.$$

The constructions of the two preceding paragraphs and Lemma 5.2.7 show that

$$H_{f,Y} = \epsilon_{k,x_0} \circ H_{\alpha,u} \circ \rho^k_1,$$

where f and Y are related to α and u as above, where $\epsilon_{k,x_0} \colon \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \to \mathsf{T}^{*k}_{x_0}\mathsf{M}$ is as defined preceding Lemma 5.2.8, and where $\rho^k_1 \colon \mathsf{T}^{*k}_{y_0}\mathsf{N} \to \mathsf{T}^*_{y_0}\mathsf{N}$ is the projection. By Lemma 5.2.7 and since $H_{f,Y}$ depends only on the value of Y at y_0, it follows that this expression is independent of the choices made for f and Y. This all shows that $\epsilon_{k,(x_0,y_0)}$ is a well-defined map from $\mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{y_0}\mathsf{N}$ to $(\rho^k_{k-1})^{-1}(j_k\Phi_{y_0}(x_0))$. Moreover, noting that $(\rho^k_{k-1})^{-1}(j_k\Phi_{y_0}(x_0))$ is an $\mathbb{F}$-vector space (cf. the proof of Proposition 5.1.13), the map constructed is a linear map.

Thus it remains to show that this map is a bijection. Suppose that $\epsilon_{k,(x_0,y_0)}(A) = 0$. By [Hungerford 1980, Theorem IV.5.11] let us write

$$A = A_1 \otimes u_1 + \cdots + A_m \otimes u_m$$

for $A_1, \dots, A_m \in \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M})$ and for a basis $(u_1, \dots, u_m)$ for $\mathsf{T}_{y_0}\mathsf{N}$. Thus, by definition of $\epsilon_{k,(x_0,y_0)}$, we have

$$(\mathrm{d}g(y_0) \cdot u_1) A_1 + \cdots + (\mathrm{d}g(y_0) \cdot u_m) A_m = 0$$

for every $(g, \mathcal{V}) \in \mathscr{F}^r(y_0)$. Taking g such that $\mathrm{d}g(y_0)$ is the ath basis vector dual to $(u_1, \dots, u_m)$ gives $A_a = 0$. Since this can be done for every $a \in \{1, \dots, m\}$, we have $A = 0$. Thus $\epsilon_{k,(x_0,y_0)}$ is injective. Surjectivity of $\epsilon_{k,(x_0,y_0)}$ follows from Lemma 5.2.7. ■

With the preceding constructions, we can state the following result.

5.4.8 Theorem (Affine structure of jets of maps) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *and* N *be manifolds of class* $\mathrm{C^r}$ *and let* $\mathrm{k} \in \mathbb{Z}_{>0}$. *Then* $\rho^{\mathrm{k}}_{\mathrm{k}-1}\colon \mathsf{J}^{\mathrm{k}}(\mathsf{M};\mathsf{N}) \to \mathsf{J}^{\mathrm{k}-1}(\mathsf{M};\mathsf{N})$ *is an affine bundle modelled on the pull-back of* $\mathsf{S}^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TN}$ *(as a vector bundle with base* $\mathsf{M} \times \mathsf{N}$*) to* $\mathsf{J}^{\mathrm{k}-1}(\mathsf{M};\mathsf{N})$.

Proof Let $(x_0, y_0) \in \mathsf{M} \times \mathsf{N}$. Let $\Psi \in \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ and let $A \in \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{T}_{y_0}\mathsf{N}$. We regard Ψ and $\epsilon_{k,(x_0,y_0)}(A)$ as homomorphisms from $\mathsf{T}^{*k}_{y_0}\mathsf{N}$ to $\mathsf{T}^{*k}_{x_0}\mathsf{M}$. We have $\rho_k(\Psi(j_kg(y_0))) = 0$ for every $(g, \mathcal{V}) \in \mathscr{F}^r(y_0)$ since $\mathsf{T}^{*0}_{x_0}\mathsf{M}$ is the equivalence class of functions taking the value zero. By Lemma 5.4.7 it follows that $\rho^k_{k-1}(\epsilon_{k,(x_0,y_0)}(A))(j_kg(y_0)) = 0$ for every $(g, \mathcal{V}) \in \mathscr{F}^r(y_0)$. Thus $\Psi(j_kg(y_0)) = j_kg_1(x_0)$ for $(g_1, \mathcal{U}) \in \mathscr{F}^r(x_0)$ vanishing at x_0 and $\epsilon_{k,(x_0,y_0)}(A)(j_kg(y_0)) = j_kg_2(x_0)$ for $(g_2, \mathcal{U}) \in \mathscr{F}^r(x_0)$ vanishing to order $k-1$ at x_0. It follows from Lemma A.2.2 that $j_k(g_1h_2)(x_0) = 0$ and $j_k(g_2h_2)(x_0) = 0$ for every $g, h \in \mathrm{C}^r(\mathsf{N})$. Therefore,

$$\begin{aligned}\Psi(j_kg(y_0)j_kh(y_0)) + \epsilon_{k,(x_0,y_0)}(A)(j_kg(y_0)j_kh(y_0)) &= \Psi(j_kg(y_0))\Psi(j_kh(y_0))\\ &= (\Psi(j_kg(y_0)) + \epsilon_{k,(x_0,y_0)}(A)(j_kg(y_0))) \cdot (\Psi(j_kh(y_0)) + \epsilon_{k,(x_0,y_0)}(A)(j_kh(y_0))).\end{aligned}$$

This shows that the homomorphisms Ψ and $\epsilon_{k,(x_0,y_0)}(A)$ can be added.

Moreover, in natural local coordinates for $\mathsf{J}^k(\mathsf{M};\mathsf{N})$ and $\mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TN}$, the local representatives of Ψ, A, and $\Psi + \epsilon_{k,(x_0,y_0)}(A)$ are

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}, \boldsymbol{A}_k), \quad (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{B}), \quad (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}, \boldsymbol{A}_k + \boldsymbol{B}),$$

respectively, which gives the desired affine bundle structure. ■

Let us make a few cautionary remarks about what structure is not possessed by jets bundles of maps between manifolds.

1. Unlike the fibres $\mathsf{T}^{*k}_{x_0}\mathsf{M}$, the fibres $\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ are not generally $\mathbb{F}$-vector spaces since there is not generally a notion of addition in the codomain N.
2. As a consequence, one *cannot* generally write an exact sequence

$$0 \longrightarrow \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{TN} \xrightarrow{\epsilon_{k,(x_0,y_0)}} \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N}) \xrightarrow{\rho^k_{k-1}} \mathsf{J}^{k-1}_{(x_0,y_0)}(\mathsf{M};\mathsf{N}) \longrightarrow 0$$

since the entries in the sequence are not all $\mathbb{F}$-vector spaces.

5.4.4 Infinite jets

Now we shall consider the structure of the infinite jets for maps between manifolds. Thus we let M and N be C^r-manifolds and let $x_0 \in \mathsf{M}$ and $y_0 \in \mathsf{N}$. Note that

$$((\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N}))_{k\in\mathbb{Z}_{>0}}, (\rho^k_l)_{k,l\in\mathbb{Z}_{>0},\, k\geq l})$$

is an inverse system of sets. We also have the projections $\rho^\infty_k\colon \mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N}) \to \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. By Lemma 5.1.12 we have the set

$$\mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N}) = \operatorname*{inv\,lim}_{k\to\infty} \mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N}).$$

Let us consider now the algebraic structure of the set $\mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$.

5.4.9 Proposition (Elements of $J^\infty_{(x_0,y_0)}(M;N)$ as algebra homomorphisms) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$*. Let* M *and* N *be manifolds of class* C^r *and let* $x_0 \in M$ *and* $y_0 \in N$*. If* $\Phi \in J^\infty_{(x_0,y_0)}(M;N)$ *then the map* $A_\Phi\colon T^{*\infty}_{y_0}N \to T^{*\infty}_{x_0}M$ *defined by*

$$A_\Phi(\phi)(k) = \Phi(k)(\phi(k))$$

is a homomorphism of $\mathbb{F}$*-algebras for which the diagram*

$$\begin{tikzcd}
& T^{*\infty}_{y_0}N \arrow[dl, "\Phi\circ\rho^\infty_k"'] \arrow[d, "A_\Phi"] \arrow[dr, "\Phi\circ\rho^\infty_l"] & \\
T^{*k}_{x_0}M \arrow[rr, "\rho^k_l"'] & T^{*\infty}_{x_0}M \arrow[l, "\rho^\infty_k"] \arrow[r, "\rho^\infty_l"'] & T^{*l}_{x_0}M
\end{tikzcd} \tag{5.12}$$

commutes for every $k, l \in \mathbb{Z}_{>0}$ *with* $k \geq l$*.*

Moreover, if $A\colon T^{*\infty}_{y_0}N \to T^{*\infty}_{x_0}M$ *is a homomorphism of* $\mathbb{F}$*-algebras for which the diagram*

$$\begin{tikzcd}
& T^{*\infty}_{y_0}N \arrow[dl, "A\circ\rho^\infty_k"'] \arrow[dr, "A\circ\rho^\infty_l"] & \\
T^{*k}_{y_0}M \arrow[rr, "\rho^k_l"'] & & T^{*l}_{x_0}M
\end{tikzcd} \tag{5.13}$$

commutes for every $k, l \in \mathbb{Z}_{>0}$ *with* $k \geq l$*, then there exists* $\Phi \in J^\infty_{(x_0,y_0)}(M;N)$ *such that* $A = A_\Phi$*.*

Proof Using the fact that, for each $k \in \mathbb{Z}_{>0}$, $\Phi(k)$ is a homomorphism of the $\mathbb{F}$-algebras $T^{*k}_{y_0}N$ and $T^{*k}_{x_0}M$, one easily verifies that A_Φ is a homomorphism of $\mathbb{F}$-algebras. The commutativity of the diagram (5.12) follows by direct verification.

For the second assertion of the proposition, since $T^{*\infty}_{y_0}N$ is an $\mathbb{F}$-algebra, since the maps $A\circ\rho^\infty_k$, $k \in \mathbb{Z}_{>0}$, are $\mathbb{F}$-algebra homomorphisms, and since $T^{*\infty}_{x_0}M$ is an inverse limit, by the definition of inverse limit there exists a unique $\mathbb{F}$-algebra homomorphism $B\colon T^{*\infty}_{y_0}N \to T^{*\infty}_{x_0}M$ for which the diagram

$$\begin{tikzcd}
& T^{*\infty}_{y_0}N \arrow[dl, "A\circ\rho^\infty_k"'] \arrow[d, "B"] \arrow[dr, "A\circ\rho^\infty_l"] & \\
T^{*k}_{x_0}M \arrow[rr, "\rho^k_l"'] & T^{*\infty}_{x_0}M \arrow[l, "\rho^\infty_k"] \arrow[r, "\rho^\infty_l"'] & T^{*l}_{x_0}M
\end{tikzcd}$$

commutes for every $k, l \in \mathbb{Z}_{>0}$ such that $k \geq l$. Let us verify that $B = A$. For $\phi \in T^{*\infty}_{y_0}N$ we have

$$B(\phi)(k) = B\circ\rho^\infty_k(\phi) = A\circ\rho^\infty_k(\phi) = A(\phi)(k),$$

using commutativity of the preceding diagram. Next we must show that $A = A_\Phi$ for some $\Phi \in \mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. However, this follows by defining $\Phi\colon \mathbb{Z}_{>0} \to \mathring{\cup}_{k\in\mathbb{Z}_{>0}} \mathsf{T}^{*k}_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ by

$$\Phi(k)(j_k f(y_0)) = A \circ \rho^\infty_k(\phi),$$

where $\phi \in \mathsf{T}^{*\infty}_{y_0}\mathsf{N}$ is such that $\phi(k) = j_k f(y_0)$. By the commutativity of the diagram (5.13) it follows that this definition of Φ ensures that $\Phi(k)$ is a map from $\mathsf{T}^{*k}_{y_0}\mathsf{N}$ to $\mathsf{T}^{*k}_{x_0}\mathsf{M}$. Moreover, using the fact that A and ρ^∞_k are homomorphisms of $\mathbb{F}$-algebras, one easily verifies that $\Phi(k)$ is a homomorphism of $\mathbb{F}$-algebras. By Theorem 5.4.6 it follows that Φ so defined is an element of $\mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. ■

Thus $\mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ is to be thought of as the set of homomorphisms of the $\mathbb{F}$-algebras $\mathsf{J}^\infty_{y_0}\mathsf{N}$ and $\mathsf{J}^\infty_{x_0}\mathsf{M}$, exactly as is the case for finite jets in Theorem 5.4.6. If $\Phi \in \mathscr{M}^r(x_0, y_0)$ then we define $j_\infty\Phi(x_0) \in \mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ by $j_\infty\Phi(x_0)(k) = j_k\Phi(x_0)$ for each $k \in \mathbb{Z}_{\geq 0}$.

5.5 Jet bundles of vector bundles

In this section we study the jet bundles associated with a vector bundle. Thus we let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle over M. Since sections of E are smooth maps between manifolds, we can talk about jets of sections as jets of maps in the usual sense. However, since sections have additional structure, this structure is reflected in the structure of the jets. In this section we explicate this additional structure.

5.5.1 Definitions

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We consider a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ of class C^r. For $x_0 \in \mathsf{M}$ we denote by $\mathscr{S}^r(x_0)$ the set of pairs $(\xi, \mathcal{U})$ where $\mathcal{U}$ is a neighbourhood of x_0 and $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$. For $k \in \mathbb{Z}_{\geq 0}$ we define an equivalence relation $\sim_{k,x_0}$ in $\mathscr{S}^r(x_0)$ by saying that $\xi \sim_{k,x_0} \eta$ if $\xi \simeq_{k,x_0,\xi(x_0)} \eta$ in $\mathscr{M}^r(x_0, \xi(x_0))$. With this notation we can make the following definition.

5.5.1 Definition (Jet bundles associated with a vector bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, let $x_0 \in \mathsf{M}$, and let $k \in \mathbb{Z}_{\geq 0}$.

(i) A **k-*jet of sections*** at x_0 is an equivalence class in $\mathscr{S}^r(x_0)$ under the equivalence relation $\sim_{k,x_0}$.

(ii) The equivalence class of $(\xi, \mathcal{U}) \in \mathscr{S}^r(x_0)$ is denoted by $j_k\xi(x_0)$.

(iii) We denote

$$\mathsf{J}^k_{x_0}\mathsf{E} = \{j_k\xi(x_0) \mid (\xi, \mathcal{U}) \in \mathscr{S}^r(x_0)\}.$$

(iv) We denote $\mathsf{J}^k\mathsf{E} = \cup_{x\in\mathsf{M}}\mathsf{J}^k_x\mathsf{E}$. By convention, $\mathsf{J}^0\mathsf{E} = \mathsf{E}$.

(v) For $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$ we denote by $\rho^k_l\colon \mathsf{J}^k\mathsf{E} \to \mathsf{J}^l\mathsf{E}$ the projection defined by $\rho^k_l(j_k\xi(x_0)) = j_l\xi(x_0)$. We abbreviate ρ^k_0 by ρ_k. •

We can give a coordinate characterisation of the equivalence relation used to characterise jets of sections of vector bundles.

5.5.2 Lemma (Characterisation of sections whose k-jets agree) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$*. Let* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^r*, let* $x_0 \in \mathsf{M}$*, let* $k \in \mathbb{Z}_{\geq 0}$*, and let* $\xi, \eta \in \Gamma^r(\mathsf{E})$*. Let* $(\mathcal{V}, \psi)$ *be an* $\mathbb{F}$*-vector bundle chart for* E *with* $(\mathcal{U}, \phi)$ *the induced* $\mathbb{F}$*-chart for* M*. Denote by*

$$\boldsymbol{\xi}_\psi = \mathrm{pr}_2 \circ \psi \circ \xi \circ \phi^{-1} \colon \phi(\mathcal{U}) \to \mathbb{F}^m, \quad \boldsymbol{\eta}_\psi = \mathrm{pr}_2 \circ \psi \circ \eta \circ \phi^{-1} \colon \phi(\mathcal{U}) \to \mathbb{F}^m$$

the local representatives of ξ *and* η*, respectively, where* $\mathrm{pr}_2 \colon \phi(\mathcal{U}) \times \mathbb{F}^m \to \mathbb{F}^m$ *is the projection onto the second factor and where* $n, m \in \mathbb{Z}_{>0}$ *are such that* ψ *is* $\mathbb{F}^n \times \mathbb{F}^m$*-valued. Then the following statements are equivalent:*

(i) $j_k\xi(x_0) = j_k\eta(x_0)$*;*

(ii) $\mathbf{D}^j\boldsymbol{\xi}_\psi(\phi(\mathbf{x}_0)) = \mathbf{D}^j\boldsymbol{\eta}_\psi(\phi(\mathbf{x}_0))$, $j \in \{1, \dots, k\}$.

Proof This follows from Lemma 5.4.1, noting that the local representatives of ξ and η, as maps from M to E, are

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}_\psi(\boldsymbol{x})), \quad \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\eta}_\psi(\boldsymbol{x})).$$

Since these local representatives agree in the first component, the lemma follows. ■

5.5.2 Geometric structure

Of course we have $\mathsf{J}^k\mathsf{E} \subseteq \mathsf{J}^k(\mathsf{M}; \mathsf{E})$. It happens that $\mathsf{J}^k\mathsf{E}$ is a submanifold. It is also a vector bundle.

5.5.3 Lemma (Vector bundle structure of jet bundles of vector bundles) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$*. If* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *is a vector bundle of class* C^r*, then* $\mathsf{J}^k\mathsf{E}$ *is a* C^r*-submanifold of* $\mathsf{J}^k(\mathsf{M}; \mathsf{E})$*. If* $(\mathcal{V}, \psi)$ *is an* $\mathbb{F}$*-vector bundle chart for* E *with* $(\mathcal{U}, \phi)$ *the associated* $\mathbb{F}$*-chart for* M*, then define* $(j_k\mathcal{V}, j_k\psi)$ *by*

$$j_k\mathcal{V} = \{j_k\xi(x) \mid (\xi, \mathcal{W}) \in \mathscr{S}^r(x_0), \ x \in \mathcal{V}\}$$

and

$$\begin{aligned} j_k\psi \colon \ j_k\mathcal{V} &\to \ \phi(\mathcal{U}) \times \mathbb{F}^m \times \mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m) \times \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F}^m) \\ j_k\xi(x) &\mapsto \ (\phi(x), \boldsymbol{\xi}_\psi(\phi(x)), \mathbf{D}\boldsymbol{\xi}_\psi(\phi(x)), \dots, \mathbf{D}^k\boldsymbol{\xi}_\psi(\phi(x)), \end{aligned}$$

where $n, m \in \mathbb{Z}_{>0}$ *are such that* ψ *is* $\mathbb{F}^n \times \mathbb{F}^m$*-valued and* ϕ *is* $\mathbb{F}^n$*-valued, and* $\boldsymbol{\xi}_\psi$ *is the local representative of* ξ *as in the preceding lemma. Then* $(j_k\mathcal{V}, j_k\psi)$ *is an* $\mathbb{F}$*-vector bundle chart for* $\mathsf{J}^k\mathsf{E}$*. Moreover, if* $((j_k\mathcal{V}_a, j_k\psi_a))_{a \in A}$ *is a vector bundle atlas for* E*, then* $((j_k\mathcal{V}_a, j_k\psi_a))_{a \in A}$ *is a vector bundle atlas for* $\mathsf{J}^k\mathsf{E}$*. Finally, the vector bundle operations in* $\mathsf{J}^k\mathsf{E}$ *are given by*

$$j_k\xi(x) + j_k\eta(x) = j_k(\xi + \eta)(x), \quad a j_k\xi(x) = j_k(a\xi)(x),$$

where $\xi, \eta \in \Gamma^r(\mathsf{E})$ *and* $a \in \mathbb{F}$*.*

Proof We shall prove the first assertion of the lemma along the way to proving the second assertion. Associated to the $\mathbb{F}$-vector bundle chart $(\mathcal{V},\psi)$ for E and the corresponding $\mathbb{F}$-chart $(\mathcal{U},\phi)$ for M is the natural chart $(j_k(\mathcal{U}\times\mathcal{V}), j_k(\phi\times\psi))$ for $\mathsf{J}^k(\mathsf{M};\mathsf{N})$ given in Lemma 5.4.3. If we restrict the domain of this chart to $\mathsf{J}^k\mathsf{E}\cap j_k(\mathcal{U}\times\mathcal{V})$ we get

$$j_k(\phi\times\psi)(j_k\xi(x)) = (\phi(x),(\phi(x),\boldsymbol{\xi}_\psi(\phi(x))),(\boldsymbol{I}_n,\boldsymbol{D}\boldsymbol{\xi}_\psi(\phi(x))),\dots,(\boldsymbol{0},\boldsymbol{D}^k\boldsymbol{\xi}_\psi(\phi(x)))),$$

noting that $\mathrm{L}^j_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^n\oplus\mathbb{F}^m)\simeq\mathrm{L}^j_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^n)\oplus\mathrm{L}^j_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m)$ for $j\in\mathbb{Z}_{\geq 0}$. It is clear that by mere rearranging of the components in the domain of $j_k(\phi\times\psi)$ we arrive at a submanifold chart with domain $j_k\mathcal{V}$, codomain

$$\phi(\mathcal{U})\times\mathbb{F}^m\times\mathrm{L}^1_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m)\times\cdots\times\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m)\times\mathbb{F}^N$$

for an appropriate N, and where the first component of the chart is $j_k\psi$ and the second part of which is $\boldsymbol{0}\in\mathbb{F}^N$. This establishes that $\mathsf{J}^k\mathsf{E}$ is a submanifold of $\mathsf{J}^k(\mathsf{M};\mathsf{N})$ and that $(j_k\mathcal{V}, j_k\psi)$ is an $\mathbb{F}$-chart. Moreover, it also satisfies the hypotheses for an $\mathbb{F}$-vector bundle chart.

Next we verify that the overlap condition for vector bundle charts is satisfied by the charts just constructed. Let $(\mathcal{V}_a,\psi_a)$ and $(\mathcal{V}_b,\psi_b)$ be overlapping $\mathbb{F}$-vector bundle charts. For simplicity and without loss of generality, suppose that $\mathcal{V}_a=\mathcal{V}_b=\mathcal{V}$. Denote by $(\mathcal{U},\phi_a)$ and $(\mathcal{U},\phi_b)$ the corresponding induced $\mathbb{F}$-charts for M. The overlap map for the vector bundle E that has the form

$$\psi_b\circ\psi_a^{-1}(\boldsymbol{x},\boldsymbol{v}) = (\phi_b\circ\phi_a^{-1}(\boldsymbol{x}),\boldsymbol{B}(\boldsymbol{x})\cdot\boldsymbol{v}),$$

where $\boldsymbol{B}\colon\phi_a(\mathcal{U})\to GL(m;\mathbb{F})$ is of class C^r with m the fibre dimension of $\mathsf{E}|\mathcal{U}$. For $\xi\in\Gamma^r(\mathsf{E})$ we write

$$\boldsymbol{\xi}_{\psi_a}(\boldsymbol{x}) = \mathrm{pr}_2\circ\psi_a\circ\xi\circ\phi_a^{-1}(\boldsymbol{x}).$$

Thus we have

$$\boldsymbol{\xi}_{\psi_b}(\boldsymbol{x}) = \mathrm{pr}_2\circ\psi_b\circ\psi_a^{-1}\circ\psi_a\circ\xi\circ\circ\phi_a^{-1}\circ\phi_a\circ\phi_b^{-1}(\boldsymbol{x}) = (\boldsymbol{B}(\phi_a\circ\phi_b^{-1}(\boldsymbol{x}))\cdot\boldsymbol{\xi}_{\psi_a}(\phi_a\circ\phi_b^{-1}(\boldsymbol{x}))).$$

We claim that the jth derivative of $\boldsymbol{\xi}_{\psi_b}$ at $\boldsymbol{x}$ is linear in ξ_{ψ_a} and its derivatives up to order j at $\phi_a\circ\phi_b(\boldsymbol{x})$. This is proved by induction on j. For $j=0$ the assertion is true by virtue of the vector bundle overlap condition for E. Assume the assertion is true for $j=r$. Thus

$$\boldsymbol{D}^r\boldsymbol{\xi}_{\psi_b}(\boldsymbol{x}) = \sum_{j=0}^{r}\boldsymbol{L}_{r,j}(\boldsymbol{D}^j\boldsymbol{\xi}_{\psi_a}(\phi_a\circ\phi_b^{-1}(\boldsymbol{x}))$$

where $\boldsymbol{L}_{r,j}\in\mathrm{L}(\mathrm{L}^j_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m);\mathrm{L}^r_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m))$. Differentiating this expression with respect to $\boldsymbol{x}$ yields an expression that is linear in the first $r+1$ derivatives of ξ_{ψ_a} at $\phi_a\circ\phi_b(\boldsymbol{x})$, as desired. This shows that the overlap map for the charts for $\mathsf{J}^k\mathsf{E}$ are $\mathbb{F}$-linear in the fibre coordinates, and so satisfy the compatibility conditions for $\mathbb{F}$-vector bundle charts.

The final assertion of the lemma follows since the local representation of the vector bundle operations for the vector bundle charts are exactly the local vector bundle operations. ■

5.5.3 Algebraic structure

Next let us examine the affine structure of jet bundles of vector bundles. Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, let $x_0 \in \mathsf{M}$, and let $(\xi, \mathcal{U}) \in \mathscr{S}^r(x_0)$. Let $(f_1, \mathcal{U}), \dots, (f_k, \mathcal{U}) \in \mathscr{F}^r(x_0)$ be such that $f_j(x_0) = 0$, $j \in \{1, \dots, k\}$. Then define $\epsilon_{k,x_0}\colon \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{E}_{x_0} \to \mathsf{J}^k_{x_0}\mathsf{E}$ by

$$\epsilon_{k,x_0}(\boldsymbol{d}f_1(x_0) \odot \cdots \odot \boldsymbol{d}f_k(x_0) \otimes \xi(x_0)) = j_k((f_1 \cdots f_k)\xi)(x_0).$$

The following lemma shows that ϵ_{k,x_0} is well-defined, and gives it some meaning.

5.5.4 Lemma (Structure of jets of sections vanishing to order k − 1) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r} *and let* $\mathrm{x}_0 \in \mathsf{M}$. *Then the following sequence of* $\mathbb{F}$*-vector spaces is exact:*

$$0 \longrightarrow \mathsf{S}^{\mathrm{k}}(\mathsf{T}^*_{\mathrm{x}_0}\mathsf{M}) \otimes \mathsf{E}_{\mathrm{x}_0} \xrightarrow{\epsilon_{\mathrm{k},\mathrm{x}_0}} \mathsf{J}^{\mathrm{k}}_{\mathrm{x}_0}\mathsf{E} \xrightarrow{\rho^{\mathrm{k}}_{\mathrm{k}-1}} \mathsf{J}^{\mathrm{k}-1}_{\mathrm{x}_0}\mathsf{E} \longrightarrow 0$$

Proof Let us first show that ϵ_{k,x_0} is well-defined. That is, suppose that $(f_j, \mathcal{U}), (g_j, \mathcal{U}) \in \mathscr{F}^r(x_0)$ satisfy $f_j(x_0) = g_j(x_0) = 0$ and $\boldsymbol{d}f_j(x_0) = \boldsymbol{d}g_j(x_0)$, $j \in \{1, \dots, k\}$, and suppose that $(\xi, \mathcal{U}), (\eta, \mathcal{U}) \in \mathscr{S}^r(x_0)$ satisfy $\xi(x_0) = \eta(x_0)$. One then shows, using coordinates, Lemma A.2.2, and a moments thought, that

$$j_k((f_1 \cdots f_k)\xi)(x_0) = j_k((g_1 \cdots g_k)\eta)(x_0).$$

This shows that ϵ_{k,x_0} is indeed well-defined.

Next we show that ϵ_{k,x_0} is injective. Suppose that $\epsilon_{k,x_0}(A) = 0$ for $A \in \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{E}_{x_0}$. Let $(u_1, \dots, u_m)$ be a basis for E_{x_0} and, using [Hungerford 1980, Theorem 5.11], write

$$A = A_1 \otimes u_1 + \cdots + A_m \otimes u_m$$

for $A_1, \dots, A_m \in \mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M})$. Since $(A_1 \otimes u_1, \dots, A_m \otimes u_m)$ are linearly independent, it follows that $A = 0$. The exactness of the sequence now follows from the dimension counting arguments from the proof of Lemma 5.2.8. ■

To make the preceding constructions global, we denote by $\sigma_k\colon \mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \to \mathsf{M}$ the vector bundle projection and so have the pull-back vector bundle

$$\rho^*_{k-1}(\sigma_k \otimes \pi_{\mathsf{E}})\colon \rho^*_{k-1}(\mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \to \mathsf{J}^{k-1}\mathsf{E}.$$

We also have the vector bundle mapping

$$\epsilon_k\colon \rho^*_{k-1}(\mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \to \mathsf{J}^k\mathsf{E}$$

over $\mathsf{J}^{k-1}\mathsf{E}$. Then the local representative of ϵ_k is

$$((\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}), (\boldsymbol{x}, \boldsymbol{A}_k)) \mapsto (\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{k-1}, \boldsymbol{A}_k),$$

showing that ϵ_k is a C^r-vector bundle map. Moreover, we have the following theorem.

5.5.5 Theorem (Affine bundle structure for jet bundles of vector bundles) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r} *and let* $\mathrm{k} \in \mathbb{Z}_{>0}$. *Then the sequence of vector bundles*

$$0 \longrightarrow \rho^*_{\mathrm{k}-1}(\mathsf{S}^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \xrightarrow{\epsilon_{\mathrm{k}}} \mathsf{J}^{\mathrm{k}}\mathsf{E} \xrightarrow{\rho^{\mathrm{k}}_{\mathrm{k}-1}} \mathsf{J}^{\mathrm{k}-1}\mathsf{E} \longrightarrow 0$$

is exact, and, as a consequence, $\rho^{\mathrm{k}}_{\mathrm{k}-1} \colon \mathsf{J}^{\mathrm{k}}\mathsf{E} \to \mathsf{J}^{\mathrm{k}-1}\mathsf{E}$ *is an affine bundle modelled on the pull-back vector bundle* $\rho^*_{\mathrm{k}-1}(\mathsf{S}^{\mathrm{k}}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E})$.

Proof This follows immediately from the lemma from the proof of Theorem 5.2.9. ■

As we observed with jets of functions following Theorem 5.2.9, the preceding constructions can be generalised. We briefly describe this generalisation; the verification of the validity of all statements is an elementary exercise. Let $k, l \in \mathbb{Z}_{>0}$ satisfy $k > l$, and let us denote by Z^k_{l,x_0} the k-jets of sections whose l-jets are zero at x_0. Let $\epsilon^k_{l,x_0} \colon Z^k_{l,x_0} \to \mathsf{J}^k_{x_0}\mathsf{E}$ be the inclusion. We then have the following exact sequence of $\mathbb{F}$-vector spaces:

$$0 \longrightarrow Z^k_{l,x_0} \xrightarrow{\epsilon^k_{l,x_0}} \mathsf{J}^k_{x_0}\mathsf{E} \xrightarrow{\rho^k_l} \mathsf{J}^l_{x_0}\mathsf{E} \longrightarrow 0$$

The case where $l = k-1$ is distinguished since one has an isomorphism of Z^k_{k-1,x_0} with $\mathsf{S}^k(\mathsf{T}^*_{x_0}\mathsf{M}) \otimes \mathsf{E}_{x_0}$.

The following result gives an inclusion that will be of use to us.

5.5.6 Lemma (Character of iterated jet bundles) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r} *and let* $\mathrm{k}, \mathrm{l} \in \mathbb{Z}_{>0}$. *Then the mapping*

$$\begin{aligned} \iota_{\mathrm{k,l}} \colon \mathsf{J}^{\mathrm{k+l}}\mathsf{E} &\to \mathsf{J}^{\mathrm{k}}(\mathsf{J}^{\mathrm{l}}\mathsf{E}) \\ j_{\mathrm{k+l}}\xi(\mathrm{x}) &\mapsto j_{\mathrm{k}}(j_{\mathrm{l}}\xi(\mathrm{x})) \end{aligned}$$

is a well-defined monomorphism of vector bundles.

Proof We work locally in a vector bundle chart $(\mathcal{V}, \psi)$ for E with $(\mathcal{U}, \phi)$ the associated chart for M. Let ξ be a local section about x with local representative $\boldsymbol{x} \mapsto \boldsymbol{\xi}(\boldsymbol{x})$. The local representative of $j_l\xi$ is then

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(x), \boldsymbol{D}\boldsymbol{\xi}, \dots, \boldsymbol{D}^l\boldsymbol{\xi}),$$

and so the local representative of $j_k(j_l\xi)$ is

$$\boldsymbol{x} \mapsto ((\boldsymbol{x}, \boldsymbol{\xi}(x), \boldsymbol{D}\boldsymbol{\xi}, \dots, \boldsymbol{D}^l\boldsymbol{\xi}), (\boldsymbol{D}\boldsymbol{\xi}(x), \boldsymbol{D}^2\boldsymbol{\xi}, \dots, \boldsymbol{D}^{l+1}\boldsymbol{\xi}), \dots, (\boldsymbol{D}^{k+1}\boldsymbol{\xi}(x), \boldsymbol{D}^{k+2}\boldsymbol{\xi}, \dots, \boldsymbol{D}^{k+l}\boldsymbol{\xi})).$$

Therefore, the local representative of $\iota_{k,l}$ is given by

$$(\boldsymbol{x}, \xi_0, \xi_1, \dots, \xi_{k+l}) \mapsto ((\boldsymbol{x}, \xi_0, \xi_1, \dots, \xi_{k+l}), (\xi_1, \xi_2, \dots, \xi_{k+1}), \dots, (\xi_{k+1}, \xi_{k+2}, \dots, \xi_{k+l})).$$

This shows that $\iota_{k,l}$ is well-defined, and keeping in mind the local characterisation of the vector bundle structure for the domain and codomain, the map is readily seen to be injective and fibre-linear. ■

5.5.4 Infinite jets

The only remaining jet bundle structure for which we need to understand the structure of infinite jets is the infinite jet bundle of a vector bundle. Thus let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r and let $x_0 \in \mathsf{M}$. Note that

$$((\mathsf{J}^k_{x_0}\mathsf{E})_{k\in\mathbb{Z}_{>0}}, (\rho^k_l)_{k,l\in\mathbb{Z}_{>0},\ k\geq l})$$

is an inverse system of $\mathbb{F}$-vector spaces. Thus we can use Lemma 5.1.12 to define the inverse limit

$$\mathsf{J}^\infty_{x_0}\mathsf{E} = \operatorname{inv\,lim}_{k\to\infty} \mathsf{J}^k_{x_0}\mathsf{E},$$

which is a $\mathbb{F}$-vector space. We also have the projections $\rho^\infty_k\colon \mathsf{J}^\infty_{x_0}\mathsf{E} \to \mathsf{J}^k_{x_0}\mathsf{E}$, $k \in \mathbb{Z}_{>0}$. If $(\xi, \mathcal{U}) \in \mathscr{S}^r(x_0)$ then we define $j_\infty\xi(x_0) \in \mathsf{J}^\infty_{x_0}\mathsf{E}$ by $j_\infty\xi(x_0)(k) = j_k\xi(x_0)$ for each $k \in \mathbb{Z}_{\geq 0}$.

5.6 Jets and germs

In this section we clarify the relationship between jets of functions (resp. maps, sections of vector bundles) with germs of functions (resp. maps, sections of vector bundles). While our interest is mainly in holomorphic and real analytic objects, for appropriate context we also present the smooth case. We also take the opportunity to present the notion of germs in the most general setting we shall need. This general notion of germ is useful for some of the constructions we shall make .

where?

5.6.1 Germs of functions, maps, and sections

We have given definitions of germs in Sections 2.3.1, 4.2.3, and 4.3.3. In this section we expand these existing definitions and also define for the first time the notion of a germ of maps between manifolds. The definitions of germs for various objects all have a similar character, so we make all of our definitions together. We shall mostly use the notion of germs at a point, but we will on occasion use germs for general sets. Thus we give the definitions in this general case, but the reader is welcome to substitute the set "A" with a point x_0 in the following definitions if it is initially helpful.

5.6.1 Definition (Equivalence of locally defined functions, maps, and sections) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M and N be manifolds of class C^r and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. Let $A \subseteq \mathsf{M}$ be a set. Let $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ be neighbourhoods of A, let $f \in \mathrm{C}^r(\mathcal{U})$ and $g \in \mathrm{C}^r(\mathcal{V})$, let $\Phi \in \mathrm{C}^r(\mathcal{U}; \mathsf{N})$ and $\Psi \in \mathrm{C}^r(\mathcal{V}; \mathsf{N})$, and let $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$ and $\eta \in \Gamma^r(\mathsf{E}|\mathcal{V})$.

(i) The pairs $(f, \mathcal{U})$ and $(g, \mathcal{V})$ are ***equivalent*** if there exists a neighbourhood $\mathcal{W}$ of A such that $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ and $f|\mathcal{W} = g|\mathcal{W}$.

(ii) The pairs $(\Phi, \mathcal{U})$ and $(\Psi, \mathcal{V})$ are ***equivalent*** if there exists a neighbourhood $\mathcal{W}$ of A such that $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ and $\Phi|\mathcal{W} = \Psi|\mathcal{W}$.

(iii) The pairs $(\xi, \mathcal{U})$ and $(\eta, \mathcal{V})$ are ***equivalent*** if there exists a neighbourhood $\mathcal{W}$ of A such that $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ and $\xi|\mathcal{W} = \eta|\mathcal{W}$. •

One readily verifies that the above three notions of equivalence define an equivalence relation on the set of stated pairs. Let us verify this in the case of functions. First of all, it is clear that $(f,\mathcal{U})$ is equivalent to itself by taking $\mathcal{W}=\mathcal{U}$. Symmetry of the relation is also clear from the definition. To verify transitivity of the relation, suppose that $(f,\mathcal{U})$ and $(g,\mathcal{V})$ are equivalent and that $(g,\mathcal{V})$ and $(h,\mathcal{W})$ are equivalent. Taking $\mathcal{W}'=\mathcal{U}\cap\mathcal{V}\cap\mathcal{W}$, we see that $\mathcal{W}'$ is a neighbourhood of x_0 for which $f|\mathcal{W}'=h|\mathcal{W}'$. Thus $(f,\mathcal{U})$ is equivalent to $(h,\mathcal{W})$. The equivalence classes under these equivalence relations are what we are interested in.

5.6.2 Definition (Germs of functions, maps, and sections) Let $r\in\{\infty,\omega,\mathrm{hol}\}$ and let $\mathbb{F}=\mathbb{R}$ if $r\in\{\infty,\omega\}$ and let $\mathbb{F}=\mathbb{C}$ if $r=\mathrm{hol}$. Let M and N be manifolds of class C^r and let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ be a vector bundle of class C^r. Let $A\subseteq\mathsf{M}$, let $x_0\in\mathsf{M}$, and let $y_0\in\mathsf{N}$. Consider the equivalence relations from Definition 5.6.1.

(i) The set of ***C^r-germs of functions*** on M at A is the set of equivalence classes under the equivalence relation Definition 5.6.1(i) and is denoted by $\mathscr{C}^r_{A,\mathsf{M}}$. We abbreviate $\mathscr{C}^r_{x_0,\mathsf{M}}=\mathscr{C}^r_{\{x_0\},\mathsf{M}}$.

(ii) The set of ***C^r-germs of maps*** from M to N at A is the set of equivalence classes under the equivalence relation Definition 5.6.1(ii) and is denoted by $\mathscr{C}^r_A(\mathsf{M};\mathsf{N})$. We abbreviate $\mathscr{C}^r_{x_0}(\mathsf{M};\mathsf{N})=\mathscr{C}^r_{\{x_0\}}(\mathsf{M};\mathsf{M})$. The subset of $\mathscr{C}^r_{x_0}(\mathsf{M};\mathsf{N})$ consisting of equivalence classes for pairs $(\Phi,\mathcal{U})$ satisfying $f(x_0)=y_0$ is denoted by $\mathscr{C}^r_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$.

(iii) The set of ***C^r-germs of sections*** of E at A is the set of equivalence classes under the equivalence relation Definition 5.6.1(iii) and is denoted by $\mathscr{G}^r_{A,\mathsf{E}}$. We abbreviate $\mathscr{G}^r_{x_0,\mathsf{M}}=\mathscr{G}^r_{\{x_0\},\mathsf{M}}$. •

Let us denote the equivalence class of $(f,\mathcal{U})$ (resp. $(\Phi,\mathcal{U})$, $(\xi,\mathcal{U})$) by $[(f,\mathcal{U})]_A$ (resp. $[(\Phi,\mathcal{U})]_A$, $[(\xi,\mathcal{U})]_A$). We use the expected abbreviations $[(f,\mathcal{U})]_{x_0}=[(f,\mathcal{U})]_{\{x_0\}}$, $[(\Phi,\mathcal{U})]_{x_0}=[(\Phi,\mathcal{U})]_{\{x_0\}}$, and $[(\xi,\mathcal{U})]_{x_0}=[(\xi,\mathcal{U})]_{\{x_0\}}$. If $[(\Phi,\mathcal{U})]_{x_0}\in\mathscr{C}^r_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ then we will write the equivalence class as $[(\Phi,\mathcal{U})]_{(x_0,y_0)}$ when we wish to emphasise the rôle of y_0.

Let us provide some algebraic structure for germs of functions and sections.

5.6.3 Proposition (Germs of functions are a ring and germs of sections are a module) *Let* $\mathrm{r}\in\{\infty,\omega,\mathrm{hol}\}$ *and let* $\mathbb{F}=\mathbb{R}$ *if* $\mathrm{r}\in\{\infty,\omega\}$ *and let* $\mathbb{F}=\mathbb{C}$ *if* $\mathrm{r}=\mathrm{hol}$. *Let* M *be a manifolds of class* C^{r} *and let* $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ *be a vector bundle of class* C^{r}. *Let* $\mathrm{A}\subseteq\mathsf{M}$. *Then* $\mathscr{C}^{\mathrm{r}}_{\mathrm{A},\mathsf{M}}$ *is an* $\mathbb{F}$*-algebra with the operations*

$$\begin{aligned}
\mathrm{a}[(\mathrm{f},\mathcal{U})]_{\mathrm{A}} &= [(\mathrm{af},\mathcal{U})]_{\mathrm{A}},\\
[(\mathrm{f},\mathcal{U})]_{\mathrm{A}}+[(\mathrm{g},\mathcal{V})]_{\mathrm{A}} &= [((\mathrm{f}+\mathrm{g})|\mathcal{U}\cap\mathcal{V},\mathcal{U}\cap\mathcal{V})]_{\mathrm{A}},\\
[(\mathrm{f},\mathcal{U})]_{\mathrm{A}}\cdot[(\mathrm{g},\mathcal{V})]_{\mathrm{A}} &= [((\mathrm{f}\cdot\mathrm{g})|\mathcal{U}\cap\mathcal{V},\mathcal{U}\cap\mathcal{V})]_{\mathrm{A}},
\end{aligned}$$

and $\mathscr{G}^{\mathrm{r}}_{\mathrm{A},\mathsf{E}}$ *is a module over the ring* $\mathscr{C}^{\mathrm{r}}_{\mathrm{A},\mathsf{M}}$ *(forgetting the* $\mathbb{F}$*-vector space structure) with the operations*

$$\begin{aligned}
[(\xi,\mathcal{U})]_{\mathrm{A}}+[(\eta,\mathcal{V})]_{\mathrm{A}} &= [((\xi+\eta)|\mathcal{U}\cap\mathcal{V},\mathcal{U}\cap\mathcal{V})]_{\mathrm{A}},\\
[(\mathrm{f},\mathcal{U})]_{\mathrm{A}}\cdot[(\xi,\mathcal{V})]_{\mathrm{A}} &= [((\mathrm{f}\cdot\xi)|\mathcal{U}\cap\mathcal{V},\mathcal{U}\cap\mathcal{V})]_{\mathrm{A}}.
\end{aligned}$$

Proof This is quite routine. One must first verify that the operations are independent of representatives of equivalence classes. Let us verify this in the case of the operation of addition in $\mathscr{C}^r_{A,\mathsf{M}}$, noting that the argument is the same for all other operations. Suppose that $[(f,\mathcal{U}')]_A$ and $[(f',\mathcal{U}')]_A$ are equivalent and that $[(g,\mathcal{V})]_A$ and $[(g',\mathcal{V}')]_A$ are equivalent. Let $\mathcal{W} = \mathcal{U}\cap\mathcal{U}'\cap\mathcal{V}\cap\mathcal{V}'$. Then $\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}$ and $\mathcal{W}\subseteq\mathcal{U}'\cap\mathcal{V}'$ and

$$(f+g)|\mathcal{W} = (f'+g')|\mathcal{W}.$$

Thus $((f+g)|\mathcal{U}\cap\mathcal{V},\mathcal{U}\cap\mathcal{V})$ and $((f'+g')|\mathcal{U}'\cap\mathcal{V}',\mathcal{U}'\cap\mathcal{V}')$ are equivalent.

To complete the proof, one must show that the operations satisfy the ring and module axioms. This is an elementary verification which we leave to the reader. ■

It is also possible to define algebraic structure on $\mathscr{C}^r_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. This rather mirrors what we have already done with jets of mappings. First of all, we note that $\mathscr{C}^r_{(x_0,0)}(\mathsf{M};\mathbb{F})$ is a subalgebra of $\mathscr{C}^r_{x_0,\mathsf{M}} = \mathscr{C}^r_{x_0}(\mathsf{M};\mathbb{F})$. For $[(\Phi,\mathcal{U})]_{(x_0,y_0)}$ we define a mapping from $\mathscr{C}^r_{(y_0,0)}(\mathsf{N};\mathbb{F})$ to $\mathscr{C}^r_{(x_0,0)}(\mathsf{M};\mathbb{F})$ by

$$[(f,\mathcal{V})]_{(y_0,0)} \mapsto [(f\circ\Phi,\mathcal{U})]_{(x_0,0)},$$

where $\mathcal{V}$ is a neighbourhood of y_0 such that $\Phi(\mathcal{U})\subseteq\mathcal{V}$. We leave to the reader the task of verifying that this mapping is well-defined, and is moreover a homomorphism of $\mathbb{F}$-algebras.

We shall often abbreviate $[(f,\mathcal{U})]_A$ (resp. $[(\xi,\mathcal{U})]_A$, $[(\Phi,\mathcal{U})]_A$) with $[f]_A$ (resp. $[\xi]_A$, $[\Phi]_A$) when the neighbourhood is not relevant.

5.6.2 Infinite jets and smooth germs

One can certainly imagine that there are relationships between the set of germs at x_0 and the set of jets at x_0. However, this relationship is not as easy to characterise as one might imagine. For example, the situation differs in a significant way between the smooth and the analytic case.

We begin by considering the smooth case. We first need to define a map from germs into jets.

5.6.4 Proposition (From germs of smooth maps to jets of maps) *Let* M *and* N *be smooth manifolds of positive dimension and let* $\mathrm{x}_0\in\mathsf{M}$ *and* $\mathrm{y}_0\in\mathsf{N}$*. If* $[(\Phi,\mathcal{W})]_{(\mathrm{x}_0,\mathrm{y}_0)}\in\mathscr{C}^\infty_{(\mathrm{x}_0,\mathrm{y}_0)}(\mathsf{M};\mathsf{N})$ *then there exists* $\Psi\in\mathrm{C}^\infty(\mathsf{M};\mathsf{N})$ *such that* $[(\Phi,\mathcal{W})]_{(\mathrm{x}_0,\mathrm{y}_0)} = [(\Psi,\mathsf{M})]_{(\mathrm{x}_0,\mathrm{y}_0)}$*. Moreover, the map*

$$\mathscr{J}_{(\mathrm{x}_0,\mathrm{y}_0)}\colon [(\Phi,\mathcal{W})]_{(\mathrm{x}_0,\mathrm{y}_0)} \mapsto \mathrm{j}_\infty\Psi(\mathrm{x}_0)$$

from $\mathscr{C}^\infty_{(\mathrm{x}_0,\mathrm{y}_0)}(\mathsf{M};\mathsf{N})$ *to* $\mathsf{J}^\infty_{(\mathrm{x}_0,\mathrm{y}_0)}(\mathsf{M};\mathsf{N})$ *is well-defined, surjective, and not injective.*

Proof Let $(\mathcal{U},\phi)$ be a chart for M about x_0 and let $(\mathcal{V},\psi)$ be a chart for N about x_0. By shrinking $\mathcal{U}$ if necessary, we assume that $\mathcal{U}\subseteq\mathcal{W}$ and that $\Phi(\mathcal{U})\subseteq\mathcal{V}$. For simplicity, assume that $\phi(x_0)=\mathbf{0}$ and $\psi(y_0)=\mathbf{0}$. Let $r\in\mathbb{R}_{>0}$ be such that $\overline{\mathsf{B}}^n(r,\mathbf{0})\subseteq\phi(\mathcal{U})$. By the smooth Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8], let $\mathbf{\Phi}_0\colon\mathcal{U}\to\mathbb{R}$

be such that $\mathbf{\Phi}_0|\overline{\mathsf{B}}^n(\frac{r}{2},\mathbf{0}) = \psi\circ\Phi\circ\phi^{-1}$ and such that $\mathbf{\Phi}_0(\boldsymbol{x}) = \mathbf{0}$ for $\boldsymbol{x} \in \phi(\mathcal{U}) \setminus \mathsf{B}^n(r,\boldsymbol{x}_0)$. If $\Psi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$ is given by

$$\Psi(x) = \begin{cases} \mathbf{\Phi}_0(\phi(x)), & x \in \mathcal{U}\cap\phi^{-1}(\overline{\mathsf{B}}^n(r,\boldsymbol{x}_0)), \\ y_0, & \text{otherwise}, \end{cases}$$

then $[(\Phi,\mathcal{W})]_{(x_0,y_0)} = [(\Psi,\mathsf{M})]_{(x_0,y_0)}$ since Φ and Ψ obviously agree on the neighbourhood $\phi^{-1}(\mathsf{B}^n(\frac{r}{2},\mathbf{0}))$ of x_0.

To see that the map $\mathscr{J}_{(x_0,y_0)}$ is well-defined, note that, if $\Psi' \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$ is any mapping such that $[(\Phi,\mathcal{W})]_{(x_0,y_0)} = [(\Psi',\mathsf{M})]_{(x_0,y_0)}$ and if $(\Phi',\mathcal{W}') \in [(\Phi,\mathcal{W})]_{(x_0,y_0)}$, then $j_\infty\Psi'$ depends only on the values of Ψ' in any neighbourhood of x_0. Since any neighbourhood of x_0 contains a neighbourhood on which Φ' and Ψ' agree, the well-definedness of $\mathscr{J}_{(x_0,y_0)}$ follows.

To prove surjectivity of $\mathscr{J}_{(x_0,y_0)}$, let $\Theta \in \mathsf{J}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$. Let $(\mathcal{U},\phi)$ and $(\mathcal{V},\psi)$ be charts for M about x_0 and for N about y_0, respectively. For each $k \in \mathbb{Z}_{\geq 0}$ define $\boldsymbol{A}_k = \boldsymbol{D}^k(\psi\circ\Phi\circ\phi^{-1})(\phi(x_0)) \in \mathsf{L}^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ where $\Phi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$ is such that $\rho^\infty_k(\Theta) = j_k\Phi(x_0)$. By Borel's Theorem, Theorem 1.1.4, let $\mathbf{\Psi} \in \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ be such that $\boldsymbol{D}^k\mathbf{\Phi}(\phi(x_0)) = \boldsymbol{A}_k$ for each $k \in \mathbb{Z}_{\geq 0}$. As in the first part of the proof, let $\Phi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$ be such that the local representative of Φ agrees with $\mathbf{\Psi}$ in a neighbourhood of $\phi(x_0)$. It is then clear that $j_k\Phi(x_0) = \rho^\infty_k(\Theta)$ for each $k \in \mathbb{Z}_{\geq 0}$, and so $j_\infty\Phi(x_0) = \Theta$.

Now we show that $\mathscr{J}_{(x_0,y_0)}$ is not injective. Since $\mathscr{J}_{(x_0,y_0)}$ depends only on the local value of maps, we can assume that $\mathsf{M} = \mathcal{U}$ is a neighbourhood of $\mathbf{0} \in \mathbb{R}^n$ and that $\mathsf{N} = \mathbb{R}^m$. Then $\mathscr{J}_{(x_0,y_0)}$ is not injective if and only if there exist smooth maps $\Phi,\Phi'\colon \mathcal{U} \to \mathbb{R}^m$ whose derivatives of all orders agree and which differ on any neighbourhood of $\mathbf{0}$. Equivalently, $\mathscr{J}_{(x_0,y_0)}$ is not injective if and only if there exists a smooth map $\Phi\colon \mathcal{U} \to \mathbb{R}^m$ whose germ at $\mathbf{0}$ is not the germ of the zero map and whose derivatives of all orders vanish. It is easy to furnish such a map, however. Indeed, define $h\colon \mathcal{U} \to \mathbb{R}$ by

$$h(\boldsymbol{x}) = \exp\Big(\frac{1}{1-\|\boldsymbol{x}\|^2}\Big)$$

and define $\Phi(\boldsymbol{x}) = (h(\boldsymbol{x}),\dots,h(\boldsymbol{x}))$. It is clear that the germ of Φ at $\mathbf{0}$ is not that of the zero function. One can also determine that the derivatives of all orders of Φ vanish at $\mathbf{0}$, cf. [Abraham, Marsden, and Ratiu 1988, Page 82] and Example 1.1.5. ■

A similar construction is possible for section of vector bundles.

5.6.5 Proposition (From germs of smooth sections to jets of sections) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle for which M and the fibres of E have positive dimension and let $\mathsf{x}_0 \in \mathsf{M}$. If $[(\xi,\mathcal{W})]_{\mathsf{x}_0} \in \mathscr{G}^\infty_{(\mathsf{x}_0,\mathsf{y}_0),\mathsf{E}}$ then there exists $\eta \in \Gamma^\infty(\mathsf{E})$ such that $[(\xi,\mathcal{W})]_{\mathsf{x}_0} = [(\eta,\mathsf{M})]_{\mathsf{x}_0}$. Moreover, the map*

$$\mathscr{J}_{\mathsf{x}_0}\colon [(\xi,\mathcal{W})]_{\mathsf{x}_0} \mapsto \mathsf{j}_\infty\xi(\mathsf{x}_0)$$

from $\mathscr{G}^\infty_{\mathsf{x}_0,\mathsf{E}}$ to $\mathsf{J}^\infty_{\mathsf{x}_0}\mathsf{E}$ is well-defined, surjective, and not injective.

Proof The proof here follows closely that of Proposition 5.6.4. We leave to the reader the exercise of making the appropriate adaptations. ■

Note that, in the smooth case, there is nothing gained, in some sense, by using germs of objects rather than objects defined globally. Indeed, the first of the assertions of Propositions 5.6.4 and 5.6.5 is exactly that the information contained in a germ is obtained from a globally defined object. As we shall see in the next section, this is not true in the analytic case.

5.6.3 Infinite jets and holomorphic and real analytic germs

Now we consider the ways in which jets are related to germs of holomorphic or real analytic functions, mappings, and sections of vector bundles. We let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \{\omega, \mathrm{hol}\}$ and adopt the convention that $r = \omega$ when $\mathbb{F} = \mathbb{R}$ and $r = \mathrm{hol}$ when $\mathbb{F} = \mathbb{C}$. Let M and N be manifolds of class C^r and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. Let $x_0 \in \mathsf{M}$ and $y_0 \in \mathsf{N}$. Note that we have inclusions

$$\mathscr{C}^r_{x_0,\mathsf{M}} \subseteq \mathscr{C}^\infty_{x_0}(\mathsf{M};\mathbb{F}), \quad \mathscr{C}^r_{(x_0,y_0)}(\mathsf{M};\mathsf{N}) \subseteq \mathscr{C}^\infty_{(x_0,y_0)}(\mathsf{M};\mathsf{N}), \quad \mathscr{G}^r_{x_0,\mathsf{E}} \subseteq \mathscr{G}^\infty_{x_0,\mathsf{E}}.$$

Therefore, the maps $\mathscr{J}_{(x_0,y_0)}$ and $\mathscr{J}_{x_0}$ from Propositions 5.6.4 and 5.6.5, respectively, restrict to $\mathscr{C}^r_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ and $\mathscr{G}^r_{x_0,\mathsf{E}}$. Let us denote the restrictions by $\hat{\mathscr{J}}_{(x_0,y_0)}$ and $\hat{\mathscr{J}}_{x_0}$, respectively.

Let us characterise the properties of these restrictions.

5.6.6 Proposition (From germs of holomorphic or analytic maps to jets of maps) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. Let M and N be manifolds of class C^r of positive dimension and let $\mathrm{x}_0 \in \mathsf{M}$ and $\mathrm{y}_0 \in \mathsf{N}$. Then the map*

$$\hat{\mathscr{J}}_{(\mathrm{x}_0,\mathrm{y}_0)} \colon [(\Phi, \mathcal{W})]_{(\mathrm{x}_0,\mathrm{y}_0)} \mapsto \mathrm{j}_\infty\Psi(\mathrm{x}_0)$$

from $\mathscr{C}^\mathrm{r}_{(\mathrm{x}_0,\mathrm{y}_0)}(\mathsf{M};\mathsf{N})$ to $\mathsf{J}^\infty_{(\mathrm{x}_0,\mathrm{y}_0)}(\mathsf{M};\mathsf{N})$ is well-defined, injective, and not surjective.

Proof The well-definedness of $\hat{\mathscr{J}}_{(x_0,y_0)}$ follows from the well-definedness of $\mathscr{J}_{(x_0,y_0)}$. Since the definition of $\hat{\mathscr{J}}_{(x_0,y_0)}$ is characterised by local information, we can assume that $\mathsf{M} = \mathcal{U}$ is a neighbourhood of $\mathbf{0} \in \mathbb{F}^n$ and that $\mathsf{N} = \mathbb{F}^m$. Suppose that $\Phi, \Psi \colon \mathcal{U}' \to \mathbb{F}^m$ are analytic maps defined on a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathbf{0}$ and that $j_\infty\Phi(\mathbf{0}) = j_\infty\Psi(\mathbf{0})$. This means that the derivatives of all orders for Φ and Ψ agree at $\mathbf{0}$. By Theorem 1.1.17 it follows that the Taylor series of Φ and Ψ converge. As we showed in Theorem 1.1.17, Φ and Ψ are equal to their Taylor series in a neighbourhood of $\mathbf{0}$. In particular, there exists a neighbourhood of $\mathbf{0}$ on which Φ and Ψ agree, and so $j_\infty\Phi(\mathbf{0}) = j_\infty\Psi(\mathbf{0})$. Thus we have the desired injectivity of $\hat{\mathscr{J}}_{(x_0,y_0)}$. It is clear that $\hat{\mathscr{J}}_{(x_0,y_0)}$ is not surjective since the derivatives of analytic mappings satisfy the derivative conditions prescribed by Theorem 1.1.17. ■

An analogous result holds for analytic sections of vector bundles.

5.6.7 Proposition (From germs of analytic sections to jets of sections) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathrm{r} = \omega$ if $\mathbb{F} = \mathbb{R}$ and $\mathrm{r} = \mathrm{hol}$ if $\mathbb{F} = \mathbb{C}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r for which M and the fibres of E have positive dimension and let $\mathrm{x}_0 \in \mathsf{M}$. Then the map*

$$\hat{\mathscr{J}}_{\mathrm{x}_0} \colon [(\xi, \mathcal{W})]_{\mathrm{x}_0} \mapsto \mathrm{j}_\infty\eta(\mathrm{x}_0)$$

from $\mathscr{G}^\mathrm{r}_{\mathrm{x}_0,\mathsf{E}}$ to $\mathsf{J}^\infty_{\mathrm{x}_0}\mathsf{E}$ is well-defined, injective, and not surjective.

Proof The proof here follows closely that of Proposition 5.6.6. We leave to the reader the exercise of making the appropriate adaptations. ■

The preceding results clearly point out distinctions between the correspondence between germs and sections in the smooth and holomorphic and real analytic cases. Other distinctions are illustrated at the end of Sections 4.2.3 and 4.3.3.

Finally we show that there can be no neighbourhood small enough that jets of holomorphic or real analytic mappings defined on this neighbourhood recover all jets of analytic functions.

5.6.8 Example (There is no fixed neighbourhood to which all jets can be extended) We again take $\mathsf{M} = \mathsf{N} = \mathbb{F}$ and $x_0 = 0$. Let $\epsilon \in \mathbb{R}_{>0}$ and define $f_\epsilon \colon \mathsf{M} \to \mathsf{N}$ by $f_\epsilon(x) = \frac{\epsilon^2}{\epsilon^2 + x^2}$. Note that $f_\epsilon \in \mathsf{C}^r(\mathbb{F})$ but the Taylor series for f_ϵ converges only on $\mathsf{D}^1(0, \epsilon)$. This implies that, for any neighbourhood $\mathcal{U}$ of 0, there exists an element of $\mathsf{J}_0^\infty(\mathbb{F}; \mathbb{F})$ defining a power series that does not converge on $\mathcal{U}$. •

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