Chapter 6

Stein and real analytic manifolds

In Corollary 4.2.11 we saw that there will generally be significant restrictions on the character of holomorphic functions on holomorphic manifolds. In Chapter 3 we saw that there are domains in $\mathbb{C}^n$ (these are holomorphic manifolds, of course) for which it is possible to find holomorphic functions satisfying finitely many algebraic conditions. This raises the question of how one can distinguish between manifolds with few holomorphic functions and those with many holomorphic functions. It is this question we study in this chapter, adapting the methods of Chapter 3. The theory we present in this chapter originated in the work of Stein [1951], building on work of many, including [Behnke and Thullen 1934, Cartan 1931, Cartan and Thullen 1932, Oka 1939], Indeed, it is difficult to ascribe the development of the theory of so-called "Stein manifolds" to one person. All of this aside, a detailed presentation of the theory is given by Grauert and Remmert [2004].

As with the material in the strongly related Chapter 3, the treatment in this chapter is mostly holomorphic. In Section 6.4 we apply holomorphic techniques to prove some important results concerning real analytic manifolds. In Section 6.5 we consider embedding theorems in the cases of interest to us.

6.1 Various forms of convexity for holomorphic manifolds

In Chapter 3 we introduced various three principle forms of convexity, holomorphic convexity, (weak and strong) pseudoconvexity, and Levi pseudoconvexity. These notions can be extended to subsets of holomorphic manifolds, although just what happens in these cases does not always mirror what happens in the case of subsets of $\mathbb{C}^n$. Moreover, the considerations on manifolds are interesting when one considers the subset consisting of the entire manifold, and this is unlike the situation in $\mathbb{C}^n$. In this section we explore these concepts.

6.1.1 Holomorphic convexity on manifolds

The development of the notion of holomorphic convexity on manifolds proceeds in a manner essentially identical with that of holomorphic convexity in $\mathbb{C}^n$. Indeed, much of the development is a mere repetition of the material from Section 3.1.2, just reproduced for convenience to encapsulate the required minor changes of notation.

The starting point is the following definition. We refer the reader to (4.9) for the notation $\|\cdot\|_K$ used in the definition.

6.1.1 Definition (Holomorphically convex hull) If M is a holomorphic manifold and if $K \subseteq \mathsf{M}$, the ***holomorphically convex hull*** of K is the set

$$\mathrm{hconv}_{\mathsf{M}}(K) = \{z \in \mathsf{M} \mid |f(z)| \le \|f\|_K \text{ for all } f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})\}.$$

A set K is called $\mathbf{C^{hol}(M)}$***-convex*** if $\mathrm{hconv}_{\mathsf{M}}(K) = K$. •

Let us give some elementary properties of the holomorphically convex hull.

6.1.2 Proposition (Properties of the holomorphically convex hull) *Let* M *be a holomorphic manifold and let* $\mathrm{K}, \mathrm{L} \subseteq \mathsf{M}$*. Then the following statements hold:*

(i) $\mathrm{K} \subseteq \mathrm{hconv}_{\mathsf{M}}(\mathrm{K})$*;*

(ii) if $\mathrm{K} \subseteq \mathrm{L}$ *then* $\mathrm{hconv}_{\mathsf{M}}(\mathrm{K}) \subseteq \mathrm{hconv}_{\mathsf{M}}(\mathrm{L})$*;*

(iii) $\mathrm{hconv}_{\mathsf{M}}(\mathrm{hconv}_{\mathsf{M}}(\mathrm{K})) = \mathrm{hconv}_{\mathsf{M}}(\mathrm{K})$*;*

(iv) $\mathrm{hconv}_{\mathsf{M}}(\mathrm{K})$ *is a closed subset of* M*.*

Proof (i) This is obvious.

(ii) This too is obvious.

(iii) By parts (i) and (ii) we have

$$\mathrm{hconv}_{\mathsf{M}}(K) \subseteq \mathrm{hconv}_{\mathsf{M}}(\mathrm{hconv}_{\mathsf{M}}(K)).$$

To prove the opposite inclusion, let $z \notin \mathrm{hconv}_{\mathsf{M}}(K)$. Then there exists $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that $|f(z)| > \|f\|_K$. This implies, however, that $|f(z)| > |f(w)|$ for every $w \in \mathrm{hconv}_{\mathsf{M}}(K)$ (by definition of $\mathrm{hconv}_{\mathsf{M}}(K)$) and so $|f(z)| > \|f\|_{\mathrm{hconv}_{\mathsf{M}}(K)}$, showing that $z \notin \mathrm{hconv}_{\mathsf{M}}(\mathrm{hconv}_{\mathsf{M}}(K))$.

(iv) For $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ note that

$$C_f \triangleq \{z \in \mathsf{M} \mid |f(z)| \le \|f\|_K\}$$

is a closed subset of M. Moreover, one can see easily that

$$\mathrm{hconv}_{\mathsf{M}}(K) = \cap_{f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})} C_f,$$

giving closedness of $\mathrm{hconv}_{\mathsf{M}}(K)$ in M. ■

From the previous result, the holomorphically convex hull is always closed. The situation where it is always compact is of interest to us.

6.1.3 Definition (Holomorphically convex manifold) A holomorphic manifold M is ***holomorphically convex*** if $\mathrm{hconv}_{\mathsf{M}}(K)$ is compact for every compact $K \subseteq \mathsf{M}$. •

There are two principal examples of holomorphically convex manifolds.

6.1.4 Examples (Holomorphically convex manifolds)

1. If $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy, it is also a holomorphic manifold, being an open subset of the holomorphic manifold $\mathbb{C}^n$. Moreover, it is holomorphically convex by Theorem 3.5.1.
2. If M is a compact holomorphic manifold and if $K \subseteq \mathsf{M}$ is compact, then $\mathrm{hconv}_{\mathsf{M}}(K)$ is closed by Proposition 6.1.2(iv) and so compact ([Runde 2005, Proposition 3.3.6]). Thus compact holomorphic manifolds are holomorphically convex. •

Let us give some properties of holomorphically convex sets.

6.1.5 Proposition (Basic properties of holomorphically convex manifolds) *For holomorphic manifolds* M *and* N *the following statements hold:*

(i) if M *and* N *are holomorphically convex, then so too is* $\mathsf{M} \times \mathsf{N}$*;*

(ii) if N *is holomorphically convex and if* $\Phi \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathsf{N})$ *is proper, then* M *is holomorphically convex;*

(iii) if M *is holomorphically convex and second countable, there exists a sequence* $(\mathrm{K}_j)_{j\in\mathbb{Z}_{>0}}$ *of compact subsets of* M *with the following properties:*

(a) $\mathrm{hconv}_{\mathsf{M}}(\mathrm{K}_j) = \mathrm{K}_j$*;*

(b) $\mathrm{K}_j \subseteq \mathrm{int}(\mathrm{K}_{j+1})$ *for* $j \in \mathbb{Z}_{>0}$*;*

(c) $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}} \mathrm{K}_j$.

Proof (i) Let $K \subseteq \mathsf{M} \times \mathsf{N}$ be compact and let $L \subseteq \mathsf{M}$ and $M \subseteq \mathsf{N}$ be compact subsets for which $K \subseteq L \times M$. Note that if $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ then, $\hat{f}(z,w) = f(z)$ defines $\hat{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M}\times\mathsf{N})$. If $(z,w) \in \mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(L \times M)$ then

$$|\hat{f}(z,w)| = |f(z)| \le \|f\|_L.$$

Thus $z \in \mathrm{hconv}_{\mathsf{M}}(L)$ and so

$$\mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(L \times M) \subseteq \mathrm{hconv}_{\mathsf{M}}(L) \times \mathsf{N}.$$

Similarly,

$$\mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(L \times M) \subseteq \mathsf{M} \times \mathrm{hconv}_{\mathsf{N}}(M)$$

and so

$$\mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(L \times M) \subseteq \mathrm{hconv}_{\mathsf{M}}(L) \times \mathrm{hconv}_{\mathsf{N}}(M).$$

By hypothesis, the set on the right is compact. Since

$$\mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(K) \subseteq \mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(L \times M),$$

we have that $\mathrm{hconv}_{\mathsf{M}\times\mathsf{N}}(K)$ is a closed subset of a compact set, and so is compact.

(ii) Let $K \subseteq \mathsf{M}$ be compact so that $\Phi(K) \subseteq \mathsf{N}$ is also compact [Runde 2005, Proposition 3.3.6]. We claim that $\mathrm{hconv}_{\mathsf{M}}(K) \subseteq \Phi^{-1}(\mathrm{hconv}_{\mathsf{N}}(\Phi(K)))$. Indeed, let $z \in \mathrm{hconv}_{\mathsf{M}}(K))$ so that $|f(z)| \le \|f\|_K$ for every $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$. Note that $\Phi^*g \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ for every $g \in \mathrm{C}^{\mathrm{hol}}(\mathsf{N})$ by Proposition 1.2.2. Therefore, for every $g \in \mathrm{C}^{\mathrm{hol}}(\mathsf{N})$, $|g \circ \Phi(z)| \le \|g \circ \Phi\|_{\Phi(K)}$, which means that $\Phi(z) \in \mathrm{hconv}_{\mathsf{N}}(\Phi(K))$, as claimed. Now we have that $\mathrm{hconv}_{\mathsf{M}}(K)$ is a closed (by Proposition 6.1.2(iv)) subset of the compact (because Φ is proper) subset $\Phi^{-1}(\mathrm{hconv}_{\mathsf{N}}(\Phi(K)))$. Thus $\mathrm{hconv}_{\mathsf{M}}(K)$ is compact [Runde 2005, Proposition 3.3.6].

(iii) Let $(L_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of M such that $L_j \subseteq \mathrm{int}(L_{j+1})$ and $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}} L_j$ (using [Aliprantis and Border 2006, Lemma 2.76]). We let $K_1 = \mathrm{hconv}_{\mathsf{M}}(L_1)$. Now suppose that we have defined $K_1, \dots, K_m$ with the desired properties. Choose $N_m \geq m$ sufficiently large that $K_m \subseteq L_{N_m}$ and take $K_{m+1} = \mathrm{hconv}_{\mathsf{M}}(L_{N_m})$. One readily verifies, using Proposition 3.1.6, that the sequence $(K_j)_{j\in\mathbb{Z}_{>0}}$ has the asserted properties. ■

In Section 3.1.2, at this point we presented some ideas concerning interpolation and singular functions in holomorphically convex domains in $\mathbb{C}^n$, Theorems 3.1.12 and 3.1.13, respectively. Note that these results do not translate to holomorphically convex manifolds. For example, compact holomorphic manifolds are holomorphically convex, but certainly there will be no general solution to the interpolation problem for holomorphic functions. Also, the notion of a singular function on a holomorphic manifold is problematic as we are working with boundaryless manifolds. Thus these features of holomorphic convexity do not figure into our discussion here.

We can, however, give some properties of holomorphic convexity as it relates to holomorphic differential geometry.

6.1.6 Proposition (Holomorphic convexity of submanifolds) *If M and N are holomorphic manifolds, then the following statements hold:*

(i) if M is holomorphically convex and if $\mathsf{S} \subseteq \mathsf{M}$ is a closed holomorphic submanifold, then S is holomorphically convex;

(ii) if $\mathsf{S}_1, \dots, \mathsf{S}_k \subseteq \mathsf{M}$ are holomorphically convex holomorphic submanifolds for which $\mathsf{S} = \cap_{j=1}^{k} \mathsf{S}_j$ is a holomorphic submanifold, then S is holomorphically convex;

(iii) if $\Phi\colon \mathsf{M} \to \mathsf{N}$ is holomorphic, if $\mathsf{S} \subseteq \mathsf{M}$ and $\mathsf{T} \subseteq \mathsf{N}$ are holomorphically convex submanifolds, and if $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ is a holomorphic submanifold of M, then $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ is holomorphically convex.

Proof (i) By Proposition 6.1.5(ii) it suffices to show that the inclusion map of S in M is proper. For this, it suffices to show that if $K \subseteq \mathsf{M}$ is compact then $K \cap \mathsf{S}$ is compact in S. If $K \cap \mathsf{S} = \emptyset$ this is clear. Otherwise, let $(z_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $K \cap \mathsf{S}$. Since K is compact, there is a convergent subsequence converging to $z \in K$ by the Bolzano–Weierstrass Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 1.5.4]. Since S is closed, $z \in \mathsf{S}$ and so $K \cap \mathsf{S}$ is compact, again by the Bolzano–Weierstrass Theorem.

(ii) Let M^k be the k-fold Cartesian product of M with itself and let $\iota_k\colon \mathsf{M} \to \mathsf{M}^k$ be the inclusion $\iota_k(z) = (z, \dots, z)$. We first claim that

$$\iota_k(\mathsf{S}) = \Big(\prod_{j=1}^{k} \mathsf{S}_j\Big) \cap \iota_k(\mathsf{M}).$$

Indeed, if $z \in \mathsf{S}$ then $z \in \mathsf{S}_j$, $j \in \{1, \dots, k\}$. Thus $\iota_k(z) \in \prod_{j=1}^{k} \mathsf{S}_j$, giving the inclusion "$\subseteq$." Conversely, if $\iota_k(z) \in \prod_{j=1}^{k} \mathsf{S}_j$ then $z \in \mathsf{S}_j$, $j \in \{1, \dots, k\}$, giving the inclusion "$\supseteq$." Let us give the set $\mathsf{T} \triangleq \prod_{j=1}^{k} \mathsf{S}_j$ the holomorphic structure inherited from M^k, noting that this makes sense even if the submanifolds $\mathsf{S}_1, \dots, \mathsf{S}_k$ are not closed (immersed submanifolds have natural differential or holomorphic structures). We claim that $\iota_k(\mathsf{S})$ is a closed submanifold of T. First of all, it is a submanifold since ι_k is an injective immersion and since S is a submanifold by hypothesis. To check that it is closed, let $(z_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in S such that the sequence $(\iota_k(z_j))_{j\in\mathbb{Z}_{>0}}$ converges to $(w_1, \dots, w_k) \in \mathsf{T}$. First of all, since $\iota_k(\mathsf{M})$ is closed in M^k and since $\iota_k(z_j) \in \iota_k(\mathsf{M})$ for every $j \in \mathbb{Z}_{>0}$, we must have $w_1 = \dots = w_k = w$. Thus the sequence $(\iota_k(z_j))_{j\in\mathbb{Z}_{>0}}$ converges to $\iota_k(w) \in \mathsf{T} \cap \iota_k(\mathsf{M}) = \iota_k(\mathsf{S})$. Thus $w \in \mathsf{S}$ and $\iota_k(\mathsf{S})$ is a closed submanifold of T, as claimed. By part (i) and Proposition 6.1.5(i) it follows that $\iota_k(\mathsf{S})$ is holomorphically convex.

Now let $K \subseteq \mathsf{S}$ be compact. We claim that

$$\mathrm{hconv}_{\mathsf{S}}(K) \subseteq \iota_k^{-1}(\mathrm{hconv}_{\mathsf{T}}(\iota_k(K))). \tag{6.1}$$

Indeed, let $z \in \mathrm{hconv}_{\mathsf{S}}(K)$. Thus $|f(z)| \le \|f\|_K$ for every $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S})$. Thus $|f \circ \iota_k(z)| \le \|f \circ \iota_k\|_{\iota_k(\mathsf{S})}$ for every $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S})$. Since $g|\iota_k(\mathsf{S}) \in \mathsf{C}^{\mathrm{hol}}(\mathsf{T})$ for every $g \in \mathsf{C}^{\mathrm{hol}}(\mathsf{T})$, this implies that $|g \circ \iota_k(z)| \le \|g\|_{\iota_k(\mathsf{S})}$ for every $g \in \mathsf{C}^{\mathrm{hol}}(\mathsf{T})$. Therefore, $\iota_k(z) \in \mathrm{hconv}_{\mathsf{T}}(\iota_k(K))$ and so $z \in \iota_k^{-1}(\mathrm{hconv}_{\mathsf{T}}(\iota_k(K)))$, as claimed.

Since T is holomorphically convex by hypothesis, $\mathrm{hconv}_{\mathsf{T}}(\iota_k(K))$ is compact in T. Thus $\mathrm{hconv}_{\mathsf{T}}(\iota_k(K)) \cap \iota_k(\mathsf{M})$ is compact in T, being the intersection of a compact set with a closed set. Since

$$\iota_k^{-1}(\mathrm{hconv}_{\mathsf{T}}(\iota_k(K))) = \iota_k^{-1}(\mathrm{hconv}_{\mathsf{T}}(\iota_k(K)) \cap \iota_k(\mathsf{M}))$$

we have that $\iota_k^{-1}(\mathrm{hconv}_{\mathsf{T}}(\iota_k(K)))$ is compact, being the preimage of a compact set under a map that is a homeomorphism onto its image. Thus, by (6.1), $\mathrm{hconv}_{\mathsf{S}}(K)$ is compact, being a closed subset of a compact set.

(iii) Let $K \subseteq \mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ be compact. We first claim that

$$\mathrm{hconv}_{\mathsf{S}\cap\Phi^{-1}(\mathsf{T})}(K) \subseteq \mathrm{hconv}_{\mathsf{S}}(K) \cap \Phi^{-1}(\mathrm{hconv}_{\mathsf{T}}(\Phi(K))). \tag{6.2}$$

Indeed, let $z \in \mathrm{hconv}_{\mathsf{S}\cap\Phi^{-1}(\mathsf{T})}(K)$ so that $|f(z)| \le \|f\|_K$ for every $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S} \cap \Phi^{-1}(\mathsf{T}))$. Since $g|\mathsf{S} \cap \Phi^{-1}(\mathsf{T}) \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S} \cap \Phi^{-1}(\mathsf{T}))$ for every $g \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S})$, it follows that $|g(z)| \le \|g\|_K$ for every $g \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S})$ and so $z \in \mathrm{hconv}_{\mathsf{S}}(K)$. Also note that $\Phi^*h|\mathsf{S} \cap \Phi^{-1}(\mathsf{T}) \in \mathsf{C}^{\mathrm{hol}}(\mathsf{S} \cap \Phi^{-1}(\mathsf{T}))$ for every $h \in \mathsf{C}^{\mathrm{hol}}(\mathsf{T})$. Thus $|\Phi^*h(z)| \le \|\Phi^*h\|_K$ for every $h \in \mathsf{C}^{\mathrm{hol}}(\mathsf{T})$. Said otherwise, $\Phi(z) \in \mathrm{hconv}_{\mathsf{T}}(\Phi(K))$ and so $z \in \Phi^{-1}(\mathrm{hconv}_{\mathsf{T}}(\Phi(K)))$. Thus (6.2) holds. All one needs to note now is that the right-hand side of (6.2) is compact, being the intersection of a compact set with a closed set, and so $\mathrm{hconv}_{\mathsf{S}\cap\Phi^{-1}(\mathsf{T})}(K)$ is compact, being a closed subset of a compact set. ■

We close this section with a discussion of a class of holomorphically convex sets.

6.1.7 Definition (Holomorphic polyhedron) A subset $P \subseteq \mathsf{M}$ of a holomorphic manifold is a ***holomorphic polyhedron of order* m** if there exist $f_1, \dots, f_m \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M})$ such that P is a union of some number of connected components of the set

$$\{z \in \mathsf{M} \mid \ |f_j(z)| < 1,\ j \in \{1, \dots, m\}\}. \bullet$$

We then have the following result, generalising Example 3.1.8–2.

6.1.8 Proposition (Holomorphic polyhedron are sometimes holomorphically convex) *If* M *is a holomorphically convex holomorphic manifold and if* $\mathrm{P} \subseteq \mathsf{M}$ *is a holomorphic polyhedron, then* P *is holomorphically convex.*

Proof Let $f_1, \dots, f_k \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be such that P is a union of connected components of

$$\mathcal{U} = \{z \in \mathsf{M} \mid \ |f_j(z)| < 1,\ j \in \{1, \dots, k\}\}.$$

Let $K \subseteq P$ be compact and let $r \in [0, 1)$ be such that $|f_j(z)| \leq r$ for each $j \in \{1, \dots, k\}$ and $z \in K$. Thus $|f_j(z)| \leq r$ for each $j \in \{1, \dots, k\}$ and $z \in \mathrm{hconv}_P(K)$ since $f_j|P \in \mathrm{C}^{\mathrm{hol}}(P)$, $j \in \{1, \dots, k\}$. Therefore,

$$\mathrm{hconv}_P(K) \subseteq \{z \in \mathsf{M} \mid \ |f_j(z)| \leq r\}.$$

Thus $\mathrm{hconv}_P(K)$ is a closed subset of the compact (because M is holomorphically convex) set $\mathrm{hconv}_{\mathsf{M}}(K)$, and so is compact [Runde 2005, Proposition 3.3.6]. ■

6.1.2 Plurisubharmonic functions on manifolds

The matter of defining plurisubharmonic functions on manifolds can take various flavours. We shall try to flesh this out by starting with a slightly ungainly definition, and showing that this definition is equivalent, in an appropriate sense, to a nicer characterisation.

6.1.9 Definition (Plurisubharmonic function) A function $u\colon \mathsf{M} \to [-\infty, \infty)$ on a holomorphic manifold M is ***plurisubharmonic*** if, for each $z \in \mathsf{M}$, there exists a $\mathbb{C}$-coordinate chart $(\mathcal{U}, \phi)$ about z such that $u \circ \phi$ is plurisubharmonic in the sense of Definition 3.2.5. By $\mathrm{Psh}(\mathsf{M})$ we denote the set of plurisubharmonic functions on M. •

A consequence of Proposition 3.2.19 is that this definition is well-defined in the sense that it will not be the case that a function can be verified to be plurisubharmonic with respect to one choice of charts, but not with respect to another choice of charts.

Let us record the basic properties of plurisubharmonic functions inherited from their definition. All of these properties follow from their corresponding assertions for open subsets of $\mathbb{C}^n$, along with the fact that the property of plurisubharmonicity is a local one.

6.1.10 Proposition (Properties of plurisubharmonic functions on manifolds) *If* M *is a holomorphic manifold, the following statements hold:*

(i) if $(\mathrm{u_j})_{\mathrm{j} \in \mathbb{Z}_{>0}}$ *is a sequence in* $\mathrm{Psh}(\mathsf{M})$ *such that* $\mathrm{u_{j+1}(z) \leq u_j(z)}$ *for each* $\mathrm{j} \in \mathbb{Z}_{>0}$ *and* $\mathrm{z} \in \mathsf{M}$, *then the function* u *on* M *defined by* $\mathrm{u(z) = \lim_{j\to\infty} u_j(z)}$ *is plurisubharmonic;*

(ii) if $(\mathrm{u_a})_{\mathrm{a} \in \mathrm{A}}$ *is a family of functions in* $\mathrm{Psh}(\mathsf{M})$ *then the function* u *on* M *defined by*

$$\mathrm{u(z) = \sup\{u_a(z) \mid \ a \in A\}}$$

is plurisubharmonic if it is upper semicontinuous and everywhere finite;

(iii) if M *is connected and if* u *is plurisubharmonic and has a global maximum in* M, *then* u *is constant;*

(iv) *if* $u_1, \dots, u_k \colon M \to [-\infty, \infty)$ *are plurisubharmonic and if* $F \colon \mathbb{R}^k \to \mathbb{R}$ *is continuous, convex, and nondecreasing in each component, and if we extend* F *to* $\bar{F} \colon ([-\infty, \infty))^k \to [-\infty, \infty)$ *as in Theorem 3.2.2(vii), then the function*

$$z \mapsto F(u_1(z), \dots, u_k(z))$$

is plurisubharmonic.

We can immediately see one of the ways in which the analysis of plurisubharmonic functions differs on holomorphic manifolds as compared with open subsets of $\mathbb{C}^n$.

6.1.11 Proposition (Plurisubharmonic functions on compact manifolds are locally constant) *If* M *is a compact holomorphic manifold and if* $u \in \mathrm{Psh}(M)$*, then* u *is locally constant.*

Proof Since M is compact, there exists $z_0 \in M$ such that $u(z) \le u(z_0)$ for every $z \in M$ [Aliprantis and Border 2006, Theorem 2.43]. Let

$$\mathcal{C} = \{z \in M \mid u(z) = u(z_0)\},$$

noting that $\mathcal{C}$ is closed since $\mathcal{C} = u^{-1}([u(z_0), \infty))$ and since u is upper semicontinuous. By Proposition 6.1.10(iii) it follows that for any $z \in \mathcal{C}$ there is a neighbourhood $\mathcal{U}$ of z on which u takes the constant value $u(z_0)$. Thus $\mathcal{C}$ is open, and so u is constant on each of its connected components. ■

Note that this is property of being locally constant on compact manifolds is a property shared between holomorphic and plurisubharmonic functions.

For functions with some regularity, one can use the Levi form to give a differential characterisation of plurisubharmonic functions. Thus we need to define the Levi form. It is possible to do so by making the definition in coordinates and noting by Lemma 3.2.11 that this definition makes sense. However, it is interesting to make the definition intrinsically. To do so, we first make the following observation.

6.1.12 Lemma (Reality of complex Hessian) *If* $\phi \in C^2(M)$ *is a* $\mathbb{R}$*-valued function on a holomorphic manifold* M*, then* $i\partial \circ \bar{\partial}\phi$ *is a real form of bidegree* $(1, 1)$*.*

Proof We compute

$$\overline{\partial \circ \bar{\partial}\phi} = \bar{\partial} \circ \overline{\bar{\partial}\phi} = \bar{\partial} \circ \partial\bar{\phi} = -\partial \circ \bar{\partial}\phi,$$

using Proposition 4.6.7, and this shows that $i\partial \circ \bar{\partial}\phi$ is real. The definitions of ∂ and $\bar{\partial}$ ensure that it is also of bidegree $(1, 1)$. ■

By Proposition 4.1.22 the following definition makes sense.

6.1.13 Definition (Levi form) If M is a holomorphic manifold and if $u \in C^2(M)$, the ***Levi form*** associated to u is the Hermitian form $\mathrm{Lev}(u)$ associated to the real bidegree $(1, 1)$ form $\frac{1}{2i}\partial \circ \bar{\partial}u$. •

In a $\mathbb{C}$-chart $(\mathcal{U}, \phi)$ with coordinates $(z^1, \dots, z^n)$ we have

$$\frac{1}{2i}\partial \circ \bar{\partial}u|\mathcal{U} = \frac{1}{2i}\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} dz^j \wedge d\bar{z}^k.$$

Referring to Proposition 4.1.23 we thus have

$$\mathrm{Lev}(u)|\mathcal{U} = \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} \mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k, \tag{6.3}$$

showing that the Levi form agrees with our definition in $\mathbb{C}^n$ from Definition 3.2.7. As with open subsets of $\mathbb{C}^n$, we shall be using plurisubharmonic functions to characterise certain types of manifolds (as we shall see in the next section). It will be useful, therefore, to know that we can impose smoothness assumptions on our plurisubharmonic functions without losing any information. For open subsets of $\mathbb{C}^n$, this is the point of Lemma 3.3.7. For manifolds, the issues are a little more subtle as Lemma 3.3.7 is no longer true in general. For example, since plurisubharmonic functions on compact manifolds are locally constant, one cannot expect on such manifolds to be able to approximate such functions with functions for which the Levi form is strictly positive.

In order to state the appropriate approximation result, we shall need to have at hand a notion of strict plurisubharmonicity. For functions of class C^2, this is not problematic since we can use the natural adaptation of Definition 3.2.16. For plurisubharmonic functions that are merely continuous, the condition that they be strictly plurisubharmonic needs more care, however.

6.1.14 Definition (Strictly plurisubharmonic function) Let M be a holomorphic manifold, let h be a continuous Hermitian metric on M, and let $\epsilon \in \mathrm{C}^0(\mathsf{M}; \mathbb{R}_{\geq 0})$.

(i) A continuous function $u \in \mathrm{C}^0(\mathsf{M})$ is ***(h, ε)-plurisubharmonic*** if, for each $z_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ and a function $\phi \in \mathrm{C}^2(\mathcal{U})$ such that

(a) $u - \phi$ is plurisubharmonic on $\mathcal{U}$ and

(b) the eigenvalues of the linear map $h^\sharp \circ \mathrm{Lev}(\phi)^\flat(z_0)$ exceed $\epsilon(z_0)$.

(ii) A continuous function $u \in \mathrm{C}^0(\mathsf{M})$ is ***strictly plurisubharmonic*** if it is $(h, 0)$-plurisubharmonic.

(iii) The set of strictly plurisubharmonic functions is denoted by $\mathrm{SPsh}(\mathsf{M})$. •

One readily checks that this definition of strictly plurisubharmonic agrees with that that $\mathrm{Lev}(u)(Z_z) > 0$ for all nonzero $Z_z \in \mathsf{T}^{1,0}\mathsf{M}$ in the case that u is of class C^2. Indeed, one need only take $\phi = u$ in the definition in this case. In particular, the definition of the notion of strictly plurisubharmonic is not dependent on the choice of Hermitian metric h. The notion of (h, ϵ)-plurisubharmonic, however, does depend on h when ϵ is nonzero.

With this notion of a strictly plurisubharmonic function, we can prove that continuous strictly plurisubharmonic functions are approximated by smooth strictly plurisubharmonic functions. We state the result here, and also state it with proof as Theorem GA2.7.1.5. The result is originally proved by Richberg [1968], but our proof in Section GA2.7.1.3 follows the strategy of Greene and Wu [1979].

6.1.15 Theorem (Approximation of continuous strictly plurisubharmonic functions) *Let* M *be a paracompact holomorphic manifold, let* $u \in \mathrm{SPsh}(\mathsf{M}) \cap \mathrm{C}^0(\mathsf{M})$*, and let* $\epsilon \in \mathrm{C}^0(\mathsf{M}; \mathbb{R}_{>0})$*. Then there exists* $\hat{u}\colon \mathsf{M} \to \mathbb{R}$ *with the following properties:*

(i) $\hat{u} \in \mathrm{SPsh}(\mathsf{M}) \cap \mathrm{C}^\infty(\mathsf{M})$*;*

(ii) $u(x) \le \hat{u}(x) \le u(x) + \epsilon(x)$ *for* $x \in \mathsf{M}$*.*

Moreover, if h *is a continuous Hermitian metric on* M *and if* u *additionally has the property that* $\mathrm{Lev}(u) - h \ge 0$*, then* $\hat{u}$ *can be chosen so that*

(iii) $\mathrm{Lev}(\hat{u}) - (1 - \epsilon)h \ge 0$*.*

Let us close this section by defining the plurisubharmonic convex hull. We shall explore its relationship with the holomorphically convex hull in Proposition 6.3.10.

6.1.16 Definition (Plurisubharmonic convex hull) If M is a holomorphic manifold and if $K \subseteq \mathsf{M}$, the ***plurisubharmonic convex hull*** of K is the set

$$\mathrm{pconv}_{\mathsf{M}}(K) = \{z \in \mathsf{M} \mid\ u(z) \le \sup_K u \text{ for all } u \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^0(\mathsf{M})\}.$$

A set K is called ***Psh(M)-convex*** if $\mathrm{pconv}_{\mathsf{M}}(K) = K$. •

6.1.3 Pseudoconvexity on manifolds

Having at hand the notion of plurisubharmonic functions on manifolds, the various notions of pseudoconvexity carry over directly from the versions in $\mathbb{C}^n$. We begin by giving the natural adaptation of the notion of an exhaustion function.

6.1.17 Definition (Exhaustion function) A function $u\colon \mathsf{M} \to [-\infty, \infty)$ is an ***exhaustion function*** if the sublevel set $u^{-1}((-\infty, \alpha))$ is a relatively compact subset of M for every $\alpha \in \mathbb{R}$. •

With this, we have the following notions of pseudoconvexity.

6.1.18 Definition (Weak and strong pseudoconvexity) A holomorphic manifold M is ***weakly pseudoconvex*** (resp. ***strongly pseudoconvex***) if there exists a smooth plurisubharmonic (resp. strictly plurisubharmonic) exhaustion function on M. •

As with holomorphic convexity, there are two basic classes of pseudoconvex manifolds.

6.1.19 Examples (Pseudoconvex manifolds)

1. If $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy, it is strongly pseudoconvex by Theorem 3.3.8.
2. If M is a compact holomorphic manifold then it is weakly pseudoconvex. Indeed, by Proposition 6.1.11, plurisubharmonic functions on M are locally constant and the Levi form of all such functions is zero. Note that this also shows that M is not strongly pseudoconvex. •

Let us consider some elementary properties of pseudoconvex manifolds.

6.1.20 Proposition (Basic properties of pseudoconvex manifolds) *For holomorphic manifolds* M *and* N *the following statements hold:*

(i) *if* M *and* N *are weakly (resp. strongly) pseudoconvex, then so too is* $\mathsf{M}\times\mathsf{N}$*;*

(ii) *if* M *is weakly pseudoconvex and if there exists a smooth, strictly plurisubharmonic, nonnegative-valued function on* M*, then* M *is strongly pseudoconvex.*

Proof (i) Let $u \in \mathrm{Psh}(\Omega)\cap \mathrm{C}^\infty(\mathsf{M})$ and $v \in \mathrm{Psh}(\Delta)\cap \mathrm{C}^\infty(\mathsf{N})$ be exhaustion functions. Let $\hat{u}\colon \mathsf{M}\times\mathsf{N}\to\mathbb{R}$ and $\hat{v}\colon \mathsf{M}\times\mathsf{N}\to\mathbb{R}$ be defined by

$$\hat{u}(z,w) = u(z),\quad \hat{v}(z,w) = v(w).$$

We claim that both $\hat{u}$ and $\hat{v}$ are plurisubharmonic. To see this for $\hat{u}$, let $\mathrm{pr}_1\colon \mathsf{M}\times\mathsf{N}\to\mathsf{M}$ be the projection onto the first factor, which is a holomorphic map. By Proposition 3.2.19 and the fact that plurisubharmonicity is a local property, it follows that $\hat{u} = u\circ\mathrm{pr}_1$ is plurisubharmonic. Similarly, $\hat{v}$ is plurisubharmonic. Now define

$$\sigma(z,w) = \max\{u(z), v(w)\} = \max\{\hat{u}(z,w), \hat{v}(z,w)\}.$$

The function σ is continuous (obvious) and plurisubharmonic (by Proposition 6.1.10(ii)). Since

$$\sigma^{-1}((-\infty,\alpha)) \subseteq u^{-1}((-\infty,\alpha))\times v^{-1}((-\infty,\alpha)),$$

it follows that σ is also an exhaustion function. Note that if u and v are strictly plurisubharmonic, then so too is σ.

(ii) Let $u \in \mathrm{Psh}(\mathsf{M})\cap\mathrm{C}^\infty(\mathsf{M})$ be an exhaustion function and let v be a smooth strictly plurisubharmonic $\mathbb{R}_{\geq 0}$-valued function. We claim that $u+v$ is a strictly plurisubharmonic exhaustion function. Clearly $u+v$ is strictly plurisubharmonic since $\mathrm{Lev}(u+v) = \mathrm{Lev}(u) + \mathrm{Lev}(v)$. To see that $u+v$ is an exhaustion function, let $\alpha\in\mathbb{R}$ and note that

$$(u+v)^{-1}((-\infty,\alpha)) = \{z\in\mathsf{M} \mid u(z)+v(z)<\alpha\} \subseteq \{z\in\mathsf{M} \mid u(z)<\alpha\},$$

giving relative compactness of $(u+v)^{-1}((-\infty,\alpha))$ since $u^{-1}((-\infty,\alpha))$ is relatively compact. ■

Let us now give some results indicating how pseudoconvexity interacts with submanifolds.

6.1.21 Proposition (Pseudoconvexity of submanifolds) *If* M *and* N *are holomorphic manifolds, then the following statements hold:*

(i) *if* M *is weakly (resp. strongly) pseudoconvex and if* $\mathsf{S}\subseteq\mathsf{M}$ *is a closed holomorphic submanifold, then* S *is weakly (resp. strongly) pseudoconvex;*

(ii) *if* $\mathsf{S}_1,\dots,\mathsf{S}_k\subseteq\mathsf{M}$ *are weakly (resp. strongly) pseudoconvex holomorphic submanifolds for which* $\mathsf{S} = \cap_{j=1}^k \mathsf{S}_j$ *is a holomorphic submanifold, then* S *is weakly (resp. strongly) pseudoconvex;*

(iii) *if* $\Phi\colon \mathsf{M}\to\mathsf{N}$ *is holomorphic, if* $\mathsf{S}\subseteq\mathsf{M}$ *and* $\mathsf{T}\subseteq\mathsf{N}$ *are weakly (resp. strongly) pseudoconvex submanifolds, and if* $\mathsf{S}\cap\Phi^{-1}(\mathsf{T})$ *is a holomorphic submanifold of* M*, then* $\mathsf{S}\cap\Phi^{-1}(\mathsf{T})$ *is weakly (resp. strongly) pseudoconvex;*

(iv) *if* M *is weakly (resp. strongly) pseudoconvex and if* $u \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^\infty(\mathsf{M})$*, then* $u^{-1}([-\infty, \alpha))$ *is weakly (resp. strongly) pseudoconvex for every* $\alpha \in \mathbb{R}$.

Proof (i) Let u be a smooth, plurisubharmonic exhaustion function. Then $u|\mathsf{S}$ is smooth since S is closed. By Lemma 3.2.11 it follows that $u|\mathsf{S}$ is plurisubharmonic (and is strictly plurisubharmonic if u is). To show that $u|\mathsf{S}$ is an exhaustion function, let $\alpha \in \mathbb{R}$ and note that

$$(u|\mathsf{S})^{-1}((-\infty, \alpha)) = \{z \in \mathsf{S} \mid u(z) < \alpha\} \subseteq \{z \in \mathsf{M} \mid u(z) < \alpha\}$$

so that $(u|\mathsf{S})^{-1}((-\infty, \alpha))$ is relatively compact in M. Since S is closed, $\mathrm{cl}((u|\mathsf{S})^{-1}((-\infty, \alpha)) \subseteq \mathsf{S}$ giving relative compactness of $(u|\mathsf{S})^{-1}((-\infty, \alpha))$ in S.

(ii) Here we make use of a lemma.

1 **Lemma** *Let* $\theta \in \mathrm{C}^\infty(\mathbb{R})$ *be such that* $\mathrm{supp}(\theta) \subseteq [-1, 1]$ *and such that*

$$\int_{\mathbb{R}} \theta(x)\, dx = 1, \qquad \int_{\mathbb{R}} x\theta(x)\, dx = 0.$$

For $\mathbf{c} \in \mathbb{R}^k_{>0}$ *define* $M_{\mathbf{c}}\colon \mathbb{R}^k \to \mathbb{R}$ *by*

$$M_{\mathbf{c}}(\mathbf{x}) = \int_{\mathbb{R}^k} \max\{x_1 + y_1, \dots, x_k + y_k\} \prod_{j=1}^{k} \theta\Big(\frac{y_j}{c_j}\Big)\, d\lambda(\mathbf{y}).$$

Then the following statements hold:

(i) $M_{\mathbf{c}}$ *is smooth, convex, and nondecreasing in all variables;*

(ii) $\max\{x_1, \dots, x_k\} \le M_{\mathbf{c}}(\mathbf{x}) \le \max\{x_1 + c_1, \dots, x_k + c_k\}$ *for all* $\mathbf{x} \in \mathbb{R}^k$;

(iii) *if* $x_j + c_j \le \max\{x_1 - c_1, \dots, \hat{x}_j - \hat{c}_j, x_k - c_k\}$ *for some* $j \in \{1, \dots, k\}$*, then* $M_{\mathbf{c}}(\mathbf{x}) = M_{\hat{\mathbf{c}}_j}(\hat{\mathbf{x}}_j)$*, where* $\hat{\mathbf{x}}_j = (x_1, \dots, \hat{x}_j, \dots, x_k)$ *and* $\hat{\mathbf{c}}_j = (c_1, \dots, \hat{c}_j, \dots, c_k)$*, and where* $\hat{}$ *means the term is omitted from the list;*

(iv) $M_{\mathbf{c}}(\mathbf{x} + (a, \dots, a)) = M_{\mathbf{c}}(\mathbf{x}) + a$ *for all* $a \in \mathbb{R}$;

(v) *if* $u_1, \dots, u_k \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^2(\mathsf{M})$ *satisfy* $\mathrm{Lev}(u_j)(Z_z) - h(Z_z, Z_z) \ge 0$ *for every* $Z_z \in \mathsf{T}^{1,0}\mathsf{M}$*, then the function*

$$z \mapsto M_{\mathbf{c}}(u_1(z), \dots, u_k(z)) \triangleq u(z)$$

is plurisubharmonic and has the property that $\mathrm{Lev}(u)(Z_z) - h(Z_z, Z_z) \ge 0$ *for every* $Z_z \in \mathsf{T}^{1,0}\mathsf{M}$.

Proof (i) We have

$$M_{\boldsymbol{c}}(\boldsymbol{x}) = \int_{\mathbb{R}^k} \max\{y_1, \dots, y_k\} \prod_{j=1}^{k} \theta\Big(\frac{y_j - x_j}{c_j}\Big)\, d\lambda(\boldsymbol{y}).$$

using a change of variable, and the smoothness of $M_{\boldsymbol{c}}$ follows by standard theorems on dependence of integrals on parameters. Convexity and the nondecreasing property of $M_{\boldsymbol{c}}$ follow from the same properties for the function

$$\boldsymbol{x} \mapsto \max\{x_1, \dots, x_k\}.$$

(ii) Note that the support of the function $y \mapsto \theta(\frac{y}{c_j})$ is contained in $[-c_j, c_j]$. Thus

$$M_{\boldsymbol{c}}(\boldsymbol{x}) = \int_{\overline{\mathsf{D}}^k(\boldsymbol{c},\boldsymbol{0})} \max\{x_1 + y_1, \dots, x_k + y_k\} \prod_{j=1}^{k} \theta\Big(\frac{y_j}{c_j}\Big)\, \mathrm{d}\lambda(\boldsymbol{y})$$

For $\boldsymbol{y} \in \overline{\mathsf{D}}^k(\boldsymbol{c}, 0)$ we have

$$\max\{x_1, \dots, x_k\} \le \max\{x_1 + y_1, \dots, x_k + y_k\} \le \max\{x_1 + c_1, \dots, x_k + c_k\}.$$

This, along with the fact that

$$\int_{\overline{\mathsf{D}}^k(\boldsymbol{c},\boldsymbol{0})} \prod_{j=1}^{k} \theta\Big(\frac{y_j}{c_j}\Big)\, \mathrm{d}\lambda(\boldsymbol{y}) = 1$$

gives this part of the result.

(iii) If

$$x_j + c_j \le \max\{x_1 - c_1, \dots, \hat{x}_j - \hat{c}_j, x_k - c_k\}$$

then, for $\boldsymbol{y} \in \overline{\mathsf{D}}^k(\boldsymbol{c}, \boldsymbol{0})$,

$$x_j + y_j \le x_j + c_j \le x_l - c_l \le x_l + y_l, \qquad l \in \{1, \dots, \hat{j}, \dots, k\},$$

and so

$$\max\{x_1 + y_1, \dots, x_k + y_k\} = \max\{x_1 + y_1, \dots, \hat{x}_j + \hat{y}_j, \dots, x_k + y_k\}.$$

This means that the integrand in the definition of $M_{\boldsymbol{c}}$ is independent of y_j except through the dependence of $\theta(y_j)$, and the result follows since $\int_{\mathbb{R}} \theta(x)\, \mathrm{d}x = 1$.

(iv) This follows since

$$\max\{x_1 + a, \dots, x_k + a\} = \max\{x_1, \dots, x_k\} + a$$

for all $\boldsymbol{x} \in \mathbb{R}^k$ and $a \in \mathbb{R}$.

(v) Plurisubharmonicity of u follows from Proposition 6.1.10(iv) and part (i). Also let $z_0 \in \mathsf{M}$ and let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart for M about z_0 such that $\phi(z_0) = \boldsymbol{0}$. Let $\boldsymbol{h}\colon \mathcal{U} \to \mathbb{C}^{n\times n}$ be the matrix of components of the Hermitian metric h in these coordinates (which we think of as itself defining a Hermitian inner product $\boldsymbol{h}(z)$ on $\mathbb{C}^n$ for each $z \in \mathcal{U}$) and, for $j \in \{1, \dots, k\}$ and $\epsilon \in \mathbb{R}_{>0}$, define a function $\hat{u}_{j,\epsilon}$ on $\mathcal{U}$ by

$$\hat{u}_{j,\epsilon}(z) = u_j(z) - \boldsymbol{h}(z)(\phi(z), \phi(z)) + \epsilon\|\phi(z)\|^2.$$

Then, using the first part of the proof of this part of the lemma, we have that

$$z \mapsto M_{\boldsymbol{c}}(u_{1,\epsilon}(z), \dots, u_{k,\epsilon}(z))$$

is plurisubharmonic. By part (iv) we have

$$M_{\boldsymbol{c}}(u_{1,\epsilon}(z), \dots, u_{k,\epsilon}(z)) = M_{\boldsymbol{c}}(u_1(z), \dots, u_k(z)) - \boldsymbol{h}(z)(\phi(z), \phi(z)) + \epsilon\|\phi(z)\|^2.$$

Since the function on the left is plurisubharmonic, we can apply Proposition 6.1.10(i) in the limit as $\epsilon \to 0$ to deduce that the function

$$z \mapsto M_c(u_1(z), \dots, u_k(z)) - \boldsymbol{h}(z)(\phi(z), \phi(z))$$

is plurisubharmonic. It follows from Proposition 3.2.12 and (6.3) that its Levi form is nonnegative evaluated on any vector field $Z \in \Gamma^\infty(\mathsf{T}^{1,0}\mathcal{U})$. But this Levi form is precisely $\mathrm{Lev}(u)(Z) - h(Z, Z)$, which gives the result. ▼

Now let u_j be a smooth plurisubharmonic exhaustion function on S_j, $j \in \{1, \dots, k\}$ so that, as in the previous part of the proof, $u_j|\mathsf{S}$ is also a smooth plurisubharmonic exhaustion function (and is strictly plurisubharmonic if u_j is) for each $j \in \{1, \dots, k\}$. Then, using the construction of the lemma above, the function

$$u\colon z \mapsto M_{(1,\dots,1)}(u_1(z), \dots, u_k(z))$$

is smooth and plurisubharmonic on S. Moreover, if each of the functions $u_1, \dots, u_k$ is strictly plurisubharmonic, then so too is u. These fact follow from part (v) of the lemma above. It remains to show that u is an exhaustion function. Let $\alpha \in \mathbb{R}$ and, using part (ii) from the previously cited lemma, compute

$$\begin{aligned} u^{-1}((-\infty, \alpha)) &= \{z \in \mathsf{S} \mid\ M_{(1,\dots,1)}(u_1(z), \dots, u_k(z)) < \alpha\} \\ &\subseteq \{z \in \mathsf{S} \mid\ \max\{u_1(z), \dots, u_j(z)\} < \alpha\} \\ &\subseteq\ \cap_{j=1}^{j} \{z \in \mathsf{S} \mid\ u_j(z) < \alpha\}. \end{aligned}$$

Since the sets $u_j^{-1}((-\infty, \alpha))$, $j \in \{1, \dots, k\}$, are relatively compact and since S is a closed submanifold, it follows that $u^{-1}((-\infty, \alpha))$ is also relatively compact, as desired.

(iii) Let u and v be smooth plurisubharmonic exhaustion functions on S and T, respectively. We claim that $u + \Phi^* v$ is a smooth plurisubharmonic exhaustion function on $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$. Under the existing hypothesis that $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ is a holomorphic submanifold, it follows that $u + \Phi^* v$ is smooth on $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ (this is standard) and plurisubharmonic (by Lemma 3.2.11 and locality of plurisubharmonicity). Moreover, it u and v are strictly plurisubharmonic, then so too is $u + \Phi^* v$ (again by Lemma 3.2.11).

It remains to show that $u + \Phi^* v$ is an exhaustion function. Let $\alpha \in \mathbb{R}$. Let $K_1 \subseteq \mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ be a compact set such that $u(z) > \alpha$ for all $z \in (\mathsf{S} \cap \Phi^{-1}(\mathsf{T})) \setminus K_1$, this being possible by Lemma 3.3.2 and the fact that u is an exhaustion function. Let $K_2 \subseteq \mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ be such that $v \circ \Phi(z) > \alpha$ for all $z \in (\mathsf{S} \cap \Phi^{-1}(\mathsf{T})) \setminus K_2$. This is possible because Φ maps compact sets onto compact sets and is surjective onto T, and since v is an exhaustion function. Let $K = K_1 \cup K_2$. Then $u(z) + \Phi^* v(z) > \alpha$ for all $z \in (\mathsf{S} \cap \Phi^{-1}(\mathsf{T})) \setminus K$. Another application of Lemma 3.3.2 allows us to conclude that since $u + \Phi^* v$ is an exhaustion function on $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$.

(iv) Let v be a plurisubharmonic (resp. strictly plurisubharmonic) exhaustion function. Without loss of generality, suppose that α is such that $u^{-1}((-\infty, \alpha)) \neq \emptyset$. In this case we claim that the function

$$\sigma(z) = v(z) + \frac{1}{\alpha - u(z)}$$

is a smooth plurisubharmonic (resp. strictly plurisubharmonic) exhaustion function on $u^{-1}((-\infty, \alpha))$. The smoothness of σ is clear. If $(\mathcal{U}, \phi)$ is a $\mathbb{C}$-chart for M with coordinates $(z^1, \dots, z^n)$, then we compute

$$\begin{aligned}\frac{\partial^2(\sigma\circ\phi^{-1})}{\partial z^j\partial\bar{z}^k}(z) &= \frac{\partial^2(v\circ\phi^{-1})}{\partial z^j\partial\bar{z}^k}(z)\\ &\quad + \frac{2}{(\alpha - u\circ\phi^{-1}(z))^3}\frac{\partial(u\circ\phi^{-1})}{\partial z^j}(z)\frac{\partial(u\circ\phi^{-1})}{\partial\bar{z}^k}(z) + \frac{1}{(\alpha - u\circ\phi^{-1}(z))^2}\frac{\partial^2(u\circ\phi^{-1})}{\partial z^j\partial\bar{z}^k}(z),\end{aligned}$$

from which we deduce that $\mathrm{Lev}(\sigma)(Z_z) > 0$ for nonzero $Z_z \in \mathsf{T}^{1,0}\mathsf{M}$. Thus σ is strictly plurisubharmonic.

It remains to show that σ is an exhaustion function. Let $\beta \in \mathbb{R}$. Let $K' \subseteq \mathsf{M}$ be such that $v(z) > \beta$ for all $z \in \mathsf{M} \setminus K'$, this being possible by Lemma 3.3.2 and the fact that v is an exhaustion function. Let $C \subseteq u^{-1}((-\infty, \alpha))$ be such that $\frac{1}{\alpha - u(z)} > \beta$ for all $z \in C$. Let $K = K' \cap C$ so that K is compact and $\sigma(z) > \beta$ for all $z \in (u^{-1}((-\infty, \alpha))) \setminus K$ since $\frac{1}{\alpha - u(z)} > 0$ for all $z \in u^{-1}((-\infty, \alpha))$. By Lemma 3.3.2 we conclude that σ is an exhaustion function. ■

6.1.4 Connections between holomorphic convexity and pseudoconvexity

We close this section by considering connections between the two forms of convexity we have introduced.

6.1.22 Theorem (Holomorphically convex manifolds are weakly pseudoconvex) *If* M *is a holomorphically convex holomorphic manifold, then it is weakly pseudoconvex.*

Proof By Proposition 6.1.5(iii) let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of M such that

1. $\mathrm{hconv}_{\mathsf{M}}(K_j) = K_j$,
2. $K_j \subseteq \mathrm{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$, and
3. $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}} K_j$.

Denote $L_j = K_{j+2} \setminus \mathrm{int}(K_{j+1})$, $j \in \mathbb{Z}_{>0}$. For $w \in L_j$ let $f_{w,j} \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M})$ be such that $|f_{w,j}(w)| > \|f_{w,j}\|_{K_j}$, this being possible since $w \notin K_j$ and since $\mathrm{hconv}(K_j) = K_j$. By rescaling we can suppose that $\|f_{w,j}\|_{K_j} < 1$ and $|f_{w,j}(w)| > 1$. Let $\mathcal{U}_{w,j}$ be a neighbourhood of w such that $|f_{w,j}(z)| > 1$ for all $z \in \mathcal{U}_{w,j}$. Since L_j is compact, let $w_1, \dots, w_{k_j}$ be such that $L_j \subseteq \cup_{l=1}^{k_j} \mathcal{U}_{w_l,j}$. Then, for all $z \in L_j$ we have

$$\max\{|f_{w_l,j}(z)| \mid l \in \{1, \dots, k_j\}\} > 1$$

and for all $z \in K_j$ we have

$$|f_{w_l,j}(z)| < 1, \qquad l \in \{1, \dots, k_j\}.$$

Now choose $m_j \in \mathbb{Z}_{>0}$ sufficiently large that

$$\sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j} \geq j, \qquad z \in L_j$$

and

$$\sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j} \leq \frac{1}{2^j}, \qquad z \in K_j.$$

Now let

$$u(z) = \sum_{j=1}^{\infty} \sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j}, \qquad z \in \mathsf{M}.$$

Clearly this series converges uniformly on compact subsets of M and so converges to a continuous function. We moreover claim that this function is real analytic and is a plurisubharmonic exhaustion function.

To see that u is real analytic, we use the following lemma.

1 Lemma *If $(\mathrm{f_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ is a sequence of holomorphic functions on a holomorphic manifold M for which the series $\sum_{\mathrm{j}=1}^{\infty}|\mathrm{f_j}|^2$ converges uniformly on compact sets, then the limit function is real analytic and plurisubharmonic.*

Proof Let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart for M for which $\phi(\mathcal{U})$ is a ball in $\mathbb{C}^n$ with centre $\mathbf{0}$. Denote $\mathcal{B} = \phi(\mathcal{U})$. Define $F_j \colon \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ by

$$F_j(\boldsymbol{z}, \boldsymbol{w}) = f_j \circ \phi^{-1}(\boldsymbol{z})\overline{f_j \circ \phi^{-1}(\boldsymbol{w})}.$$

We compute

$$\frac{\partial F_j}{\partial \bar{z}^j}(\boldsymbol{z}, \boldsymbol{w}) = \frac{\partial (f_j \circ \phi^{-1})}{\partial \bar{z}^j}(\boldsymbol{z})\overline{f_j \circ \phi^{-1}(\boldsymbol{w})} = 0$$

and

$$\frac{\partial F_j}{\partial \bar{w}^j}(\boldsymbol{z}, \boldsymbol{w}) = f_j \circ \phi^{-1}(\boldsymbol{z})\overline{\frac{\partial (f_j \circ \phi^{-1})}{\partial \bar{w}^j}(\boldsymbol{w})} = 0$$

for every $j \in \{1, \dots, n\}$ and $(\boldsymbol{z}, \boldsymbol{w}) \in \mathcal{B} \times \mathcal{B}$. Thus F_j is holomorphic on $\mathcal{B} \times \mathcal{B}$. We claim that $\sum_{j=1}^{\infty} F_j$ converges uniformly on compact subsets of $\mathcal{B} \times \mathcal{B}$. Indeed, let $K \subseteq \mathcal{B} \times \mathcal{B}$ be compact and let $\epsilon \in \mathbb{R}_{>0}$. Let $K' \subseteq \mathsf{M}$ be a compact set sufficiently large that $\phi^{-1}(K) \subseteq K'$ and let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\sum_{j=k+1}^{l} |f_j(z)|^2 < \epsilon$$

for $z \in K'$, this being possible since uniformly convergent sequences of continuous functions on compact sets are Cauchy in the ∞-norm. Then, using the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big|\sum_{j=k+1}^{l} F_j(\boldsymbol{z}, \boldsymbol{w})\Big| &\leq \sum_{j=k+1}^{l} |f_j \circ \phi^{-1}(\boldsymbol{z})||\overline{f_j \circ \phi^{-1}(\boldsymbol{w})}| \\ &\leq \Big(\sum_{j=k+1}^{l} |f_j \circ \phi^{-1}(\boldsymbol{z})|^2\Big)^{1/2}\Big(\sum_{j=k+1}^{l} |\overline{f_j \circ \phi^{-1}(\boldsymbol{w})}|^2\Big)^{1/2} \leq \epsilon\end{aligned}$$

for $k, l \geq N$ with $l > k$. By completeness of the space of continuous functions on a compact set in the ∞-norm ([Hewitt and Stromberg 1975, Theorem 7.9]), we conclude that we have the desired uniform convergence on compact sets. Therefore, by , $F \triangleq \sum_{j=1}^{\infty} F_j$ converges to a holomorphic function on $\mathcal{B} \times \mathcal{B}$. Therefore, the restriction of F to the closed submanifold what

$$D = \{(z, w) \in \mathcal{B} \times \mathcal{B} \mid z = w\}$$

is real analytic. But we have

$$\sum_{j=1}^{\infty} |f_j \circ \phi^{-1}(z)|^2 = F(z, z),$$

showing that the local representative of $\sum_{j=1}^{\infty} |f_j|^2$ is real analytic, as desired.

To verify that $f \triangleq \sum_{j=1}^{\infty} |f_j|^2$ is plurisubharmonic we directly compute

$$\mathrm{Lev}(f)|\mathcal{U} = \sum_{j=1}^{\infty} \sum_{r,s=1}^{n} \frac{\partial f_j}{\partial z^r} \frac{\partial f_j}{\partial \bar{z}^s} \mathrm{d}z^r \otimes \mathrm{d}\bar{z}^s$$

for any $\mathbb{C}$-chart $(\mathcal{U}, \phi)$. From this we see that

$$\mathrm{Lev}(f)(Z_z) = \sum_{j=1}^{\infty} |Z_z f_j|^2 \geq 0,$$

giving plurisubharmonicity. ▼

Finally, we show that u constructed before the lemma is an exhaustion function. Let $\alpha \in \mathbb{R}$ and let $N \in \mathbb{Z}_{>0}$ be the least integer such that $N \geq \alpha$. Then, for every $j \geq N$, $u(z) \geq j \geq \alpha$ for every $z \in L_j$. Therefore,

$$u^{-1}((-\infty, \alpha)) \subseteq \cap_{j \geq N}(\mathsf{M} \setminus L_j) = \mathsf{M} \setminus \cup_{j \geq N} L_j \subseteq K_{N+2}.$$

Thus $u^{-1}((-\infty, \alpha))$ is relatively compact, as desired. ■

Note that the theorem does not say that a holomorphically convex manifold is strongly pseudoconvex; indeed this is false since compact holomorphic manifolds are holomorphically convex (by Example 6.1.4–2), but not strongly pseudoconvex (by Proposition 6.1.11). We shall see that the property of a holomorphic manifold being strongly pseudoconvex is equivalent to its being a Stein manifold (Section 6.3.2).

Moreover, the converse of Theorem 6.1.22 does not hold.

6.1.23 Example (A weakly pseudoconvex manifold that is not holomorphically convex) Define $f_1, f_2 \in \mathrm{C}^{\mathrm{hol}}(\mathbb{C}^2; \mathbb{C}^2)$ by

$$f_1(z_1, z_2) = (z_1 + 1, \mathrm{e}^{\mathrm{i}\theta_1} z_2), \quad f_2(z_1, z_2) = (z_1 + \mathrm{i}, \mathrm{e}^{\mathrm{i}\theta_2} z_2)$$

for some $\theta_1, \theta_2 \in \mathbb{R}$. Note that each of these maps is a diffeomorphism of the holomorphic manifold $\mathbb{C}^2$. Let Γ be the group of holomorphic diffeomorphisms generated by f_1

and f_2, and let $\mathsf{M} = \mathbb{C}^2/\Gamma$ be the orbit space and let $\pi\colon \mathbb{C}^2 \to \mathsf{M}$ be the canonical projection. For $(z_1, z_2) \in \mathbb{C}^2$, $\pi^{-1}(\pi(z_1, z_2))$ is a countable union of discrete points possessing disjoint neighbourhoods that are mapped diffeomorphically to one another by elements of Γ. Thus each of these neighbourhoods is mapped by π onto a neighbourhood of $\pi(z_1, z_2)$, and in this way M acquires a holomorphic atlas.

We claim that M is weakly pseudoconvex. Indeed, note that the function $\hat{u}\colon \mathbb{C}^2 \to \mathbb{R}$ defined by $\hat{u}(z_1, z_2) = |z_2|^2$ is plurisubharmonic and Γ-invariant. Thus there exists a unique function $u\colon \mathsf{M} \to \mathbb{R}$ such that $\hat{u} = \pi^* u$. By Lemma 3.2.11 we have

$$\mathrm{Lev}(\hat{u})(Z_z) = \mathrm{Lev}(u)(T_z\pi(Z_z)).$$

Since $T_z\pi$ is surjective, we conclude that u is also plurisubharmonic. We can also show that u is an exhaustion function. Indeed, let $\alpha \in \mathbb{R}$ and note that

$$\begin{aligned} u^{-1}((-\infty, \alpha)) &= \{\pi(z_1, z_2) \in \mathsf{M} \mid\ u \circ \pi(z_1, z_2) \in (-\infty, \alpha)\} \\ &= \pi(\hat{u}^{-1}((-\infty, \alpha))). \end{aligned}$$

Continuity of π and relative compactness of $\hat{u}^{-1}((-\infty, \alpha))$ ensure that $u^{-1}((-\infty, \alpha))$ is also relatively compact. Thus we conclude that M is indeed weakly pseudoconvex.

Let us now consider whether M is holomorphically convex. Let us first consider the character of holomorphic functions on M. Such holomorphic functions are in one-to-one correspondence with Γ-invariant holomorphic functions on $\mathbb{C}^2$. Let us suppose that $\hat{f} \in \mathrm{C}^{\mathrm{hol}}(\mathbb{C}^2)$ is Γ-invariant and consider the function $z_1 \mapsto \hat{f}(z_1, z_2)$ for $z_2 \in \mathbb{C}$ fixed. The image of this function is evidently contained in

$$\hat{f}(\{z_1 \in \mathbb{C} \mid\ \ \mathrm{Re}(z_1), \mathrm{Im}(z_1) \in [0, 1]\} \times \overline{\mathsf{D}}^1(0, |z_2|)).$$

As this latter set is bounded, the function $z_1 \mapsto \hat{f}(z_1, z_2)$ is constant for each $z_2 \in \mathbb{C}$, i.e., $\hat{f}$ is independent of z_1. From this we conclude that holomorphic functions on M are in one-to-one correspondence with functions $\hat{f}$ on $\mathbb{C}$ for which

$$\hat{f}(\mathrm{e}^{\mathrm{i}\theta_1} z_2) = \hat{f}(\mathrm{e}^{\mathrm{i}\theta_2} z_2) = \hat{f}(z_2)$$

for every $z_2 \in \mathbb{C}$. If either θ_1 or θ_2 are irrational, this implies that $\hat{f}$ is constant on circles with centre at 0. By the polar coordinate form of the Cauchy–Riemann equations,

$$\frac{\partial\, \mathrm{Re}(f)}{\partial r} = \frac{1}{r}\frac{\partial\, \mathrm{Im}(f)}{\partial \theta}, \quad \frac{\partial\, \mathrm{Im}(f)}{\partial r} = -\frac{1}{r}\frac{\partial\, \mathrm{Re}(f)}{\partial \theta},$$

it follows that $\hat{f}$ is constant if θ_1 or θ_2 is irrational. Moreover, if θ_1 and θ_2 are rational and if their least common denominator is $k \in \mathbb{Z}_{>0}$, then any convergent power series of the form $\sum_{j=0}^{\infty} a_j z_2^{jk}$ is easily verified to be a Γ-invariant holomorphic function.

We claim that M is holomorphically convex if and only if θ_1 and θ_2 are rational. First of all if one of θ_1 and θ_2 are irrational, the only holomorphic functions on M are constant functions. Therefore, for any compact $K \subseteq \mathsf{M}$, $\mathrm{hconv}_{\mathsf{M}}(K) = \mathsf{M}$ and, since M is not compact (the function $\pi(z_1, z_2) \mapsto |z_2|^2$ is an unbounded continuous function on M), it follows that M is not holomorphically convex. Conversely, suppose that θ_1 and θ_2 are rational with least common denominator $k \in \mathbb{Z}_{>0}$. Let $K \subseteq \mathsf{M}$ be compact and suppose that $\mathrm{hconv}_{\mathsf{M}}(K)$ is not compact. Then $\hat{L} = \pi^{-1}(\mathrm{hconv}_{\mathsf{M}}(K))$ is a closed set projecting to $\mathrm{hconv}_{\mathsf{M}}(K)$. Since $\pi(\hat{L})$ is not compact, it must be unbounded. Let us consider the Γ-invariant holomorphic function $\hat{f}(z_1, z_2) = z_2^k$ on $\mathbb{C}^2$. Since $\hat{L}$ is unbounded, $\hat{f}$ is unbounded on $\hat{L}$. Thus the induced function $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ is unbounded on $\mathrm{hconv}_{\mathsf{M}}(K)$. This, however, contradicts the fact that $|f(\pi(z_1, z_2))| \leq \|f\|_K$ for $\pi(z_1, z_2) \in \mathrm{hconv}_{\mathsf{M}}(K)$. Therefore, $\mathrm{hconv}_{\mathsf{M}}(K)$ is compact and so M is holomorphically convex.

In any event, by taking θ_1 and θ_2 irrational, we have an example of a holomorphic manifold that is weakly pseudoconvex but not holomorphically convex. •

6.2 Hörmander's solution to the $\bar{\partial}$-problem on manifolds

In this section we present the solution of Hörmander [1965] to the $\bar{\partial}$-problem. Hörmander makes great use of the solution of the $\bar{\partial}$-problem and the techniques associated with the solution to investigate many problems of interest in complex analysis and complex differential geometry. We shall use the existence theorems for the $\bar{\partial}$-problem that we prove in this section to solve the Levi problem on manifolds in Section 6.3.2.

6.2.1 Formulation of $\bar{\partial}$-problem and its solution method

Let M be a holomorphic manifold and let $u \in \Gamma^{\infty}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and $f \in \Gamma^{\infty}(\wedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. The $\bar{\partial}$-problem on manifolds asks when the condition that $\bar{\partial} f = 0$ is sufficient for the existence of solutions to the equation $\bar{\partial} u = f$ (the condition is clearly necessary). This is an example of an overdetermined system of partial differential equations, and so the question is whether the obvious integrability condition $\bar{\partial} f = 0$ is sufficient for the existence of solutions. In the real analytic category, such problems can be studied locally using formal methods built around the Cartan–Kähler theory [e.g., Goldschmidt 1967]. In the smooth case and when one wants global solutions, these local methods do not give the required existence theorems, and other methods must be employed. The solution of the $\bar{\partial}$-problem by Hörmander [1965] uses Hilbert space techniques. Specifically, the problem is first studied when the data u and f are differential forms with coefficients in appropriately defined L^2-spaces. In this section we give the machinery required to formulate the problem in this way.

We let M be a second countable holomorphic manifold. To define the appropriate L^2-spaces requires a Hermitian inner product on the tangent spaces of M. We thus let h be a smooth Hermitian metric on M, the existence of which is proved in Theorem 4.9.2.

Locally in a $\mathbb{C}$-chart $(\mathcal{U}, \phi)$ we write

$$h|\mathcal{U} = \sum_{j,k=1}^{n} h_{jk} \mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k$$

for smooth functions $h_{jk} \in \mathrm{C}^\infty(\mathcal{U}; \mathbb{C})$ satisfying $h_{kj} = \bar{h}_{jk}$, $j, k \in \{1, \dots, n\}$, cf. Proposition 4.1.23. Let $h^{jk} \in \mathrm{C}^\infty(\mathcal{U}; \mathbb{C})$, $j, k \in \{1, \dots, n\}$, be defined by

$$\sum_{k=1}^{n} h^{jk} h_{kl} = \delta^j_l, \qquad j, l \in \{1, \dots, n\}.$$

We can then define an inner product on the fibres of $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ by

$$\langle \alpha, \beta \rangle |\mathcal{U} = \sum_{j,k=1}^{n} h^{jk} \alpha_j \bar{\beta}_k,$$

if

$$\alpha|\mathcal{U} = \sum_{j=1}^{n} \alpha_j \mathrm{d}z^j, \quad \beta|\mathcal{U} = \sum_{j=1}^{n} \beta_j \mathrm{d}z^j.$$

It will be convenient to use local bases other than the coordinates bases, bases that are orthonormal with respect to h. Thus, in a chart $(\mathcal{U}, \phi)$ we let $(\omega^1, \dots, \omega^n)$ be a basis of $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ that is orthonormal with respect to the inner product induced by h; this can be done via the usual Gram–Schmidt procedure. The inner product can be easily extended to differential forms of general bidegree. Thus we let $\alpha, \beta \in \Gamma^\infty(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and write

$$\alpha = \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' \alpha_{I,J} \omega^I \wedge \bar{\omega}^J, \quad \beta = \sum_{I \in \boldsymbol{n}^r} \sum_{J \in \boldsymbol{n}^s} \beta_{I,J} \omega^I \wedge \bar{\omega}^J,$$

where, as usual, $\sum'$ denotes sum over multi-indices that are increasing and where

$$\omega^I = \omega^{i_1} \wedge \dots \wedge \omega^{i_r}, \quad \bar{\omega}^J = \bar{\omega}^{j_1} \wedge \dots \wedge \bar{\omega}^{j_s}.$$

We then define

$$\langle \alpha, \beta \rangle = \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' \alpha_{I,J} \bar{\beta}_{I,J}.$$

It is a straightforward verification, using the change of basis formula, that this definition of the inner product is independent of the orthonormal basis. In the usual manner, we denote $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$.

We now define a sequence of functions that will be useful in our constructions.

6.2.1 Lemma (A sequence of cutoff functions) *Let* M *be a second countable holomorphic manifold. There exists a sequence* $(\chi_j)_{j\in\mathbb{Z}}$ *of smooth functions with the following properties:*

(i) χ_j *has compact support for each* $j \in \mathbb{Z}_{>0}$*;*

(ii) $\chi_j(z) \in [0,1]$ *for all* $z \in \mathsf{M}$*;*

(iii) for any compact set $K \subseteq \mathsf{M}$*, there exists* $N \in \mathbb{Z}_{>0}$ *such that* $\chi_j(z) = 1$ *for all* $z \in K$ *and* $j \geq N$*.*

Moreover, given such a sequence $(\chi_j)_{j\in\mathbb{Z}_{>0}}$*, there exists a Hermitian metric* h *on* M *such that* $\|\bar{\partial}\chi_j(z)\| \leq 1$ *for all* $j \in \mathbb{Z}_{>0}$ *and* $z \in \mathsf{M}$*.*

Proof By [Aliprantis and Border 2006, Lemma 2.76] we let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of M such that $K_j \subseteq \operatorname{int}(K_j)$, $j \in \mathbb{Z}_{>0}$, and such that $\mathsf{M} = \cup_{j=1}^{\infty} K_j$. Using the discussion in [Abraham, Marsden, and Ratiu 1988, §5.5], let $\chi_j \in \mathrm{C}^\infty(\mathsf{M})$ be such that $\chi_j(z) \in [0,1]$, $\chi_j(z) = 1$ for $z \in K_j$ and $\chi_j(z) = 0$ for $z \in \mathsf{M} \setminus \operatorname{int}(K_{j+1})$. This sequence clearly has the desired properties.

Now let h' be an arbitrary Hermitian metric. For each $j \in \mathbb{Z}_{>0}$, the one-form $\bar{\partial}\chi_j$ is nonzero on at most $\operatorname{int}(K_{j+1}) \setminus K_j$. Thus there exists $m \in \mathrm{C}^\infty(\mathsf{M};\mathbb{R}_{>0})$ such that

$$\|\bar{\partial}\chi_j(z)\|' \leq m(z), \qquad z \in \mathsf{M},\ j \in \mathbb{Z}_{>0},$$

where $\|\cdot\|'$ denotes the norm induced by h'. By taking $h = m^2 h'$ we see that h has the desired property. ■

6.2.2 Notation (Standing constructions) For the remainder of this section we assume that a sequence $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ of smooth functions and a Hermitian metric h have been chosen satisfying the condition of the lemma. Along with this construction comes the compact exhaustion $(K_j)_{j\in\mathbb{Z}_{>0}}$ used to define the sequence $(\chi_j)_{j\in\mathbb{Z}_{>0}}$. We let μ be the volume form induced by the Hermitian metric. Precisely, we define $\mu = \frac{1}{n!}\omega^n$, where ω is the two-form of bidegree $(1,1)$ given by $\omega(X,Y) = h(J(X),Y)$. The integral with respect to μ we denote by $\int_{\mathsf{M}} \mathrm{d}\mu(z)$. •

Now we introduce the function spaces we use. By $\mathrm{L}^2_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ we denote the set of maps $\alpha\colon \mathsf{M} \to \bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ with the property that, if we write α in a $\mathbb{C}$-chart $(\mathcal{U},\phi)$ as

$$\alpha|\mathcal{U} = \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \alpha_{I,J}\mathrm{d}\boldsymbol{z}^I \wedge \mathrm{d}\bar{\boldsymbol{z}}^J,$$

the coefficients $\alpha_{I,J}$ are measurable and with the property that

$$\int_K \|\alpha(z)\|^2 \,\mathrm{d}\mu(z) < \infty$$

for every compact $K \subseteq \mathsf{M}$. Next suppose that $\varphi \in \mathrm{C}^\infty(\mathsf{M})$ and let $\mathrm{L}^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ be the set of maps $\alpha\colon \mathsf{M} \to \bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ with the property that, if we write α in a $\mathbb{C}$-chart $(\mathcal{U},\phi)$ as

$$\alpha|\mathcal{U} = \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \alpha_{I,J}\mathrm{d}\boldsymbol{z}^I \wedge \mathrm{d}\bar{\boldsymbol{z}}^J,$$

the coefficients $\alpha_{I,J}$ are measurable and with the property that

$$\|\alpha\|_\varphi \triangleq \Big(\int_{\mathsf{M}} \|\alpha(z)\|^2 e^{-\varphi(z)} d\mu(z)\Big)^{1/2} < \infty.$$

we also denote

$$\langle\alpha,\beta\rangle_\varphi = \int_{\mathsf{M}} \langle\alpha(z),\beta(z)\rangle e^{-\varphi(z)} d\mu(z).$$

The choice of the weight φ will come up as a technical device rather near the end of the proof.

We wish to define the operator $\bar{\partial}$ on $L^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. To do this we denote by $\mathscr{D}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ (borrowing notation from the theory of distributions) the set of smooth forms of bidegree (r,s) with compact support.

6.2.3 Lemma (Density of smooth compactly supported forms) *The subspace $\mathscr{D}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ is dense in $L^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Moreover,*

$$\bar{\partial}\colon \mathscr{D}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \mathscr{D}(\textstyle\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$$

is a closable linear operator in the L^2_φ-topology.

Proof Let us abbreviate

$$L^2_\varphi(\mathsf{M};r,s) = L^2_\varphi(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})),$$
$$\mathscr{D}(\mathsf{M};r,s) = \mathscr{D}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})).$$

Let $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ be the sequence of smooth functions from Notation 6.2.2. We claim that if $\alpha \in L^2_\varphi(\mathsf{M};r,s)$ then the sequence $(\chi_j\alpha)_{j\in\mathbb{Z}_{>0}}$ converges to α in the L^2_φ-norm. Indeed, the sequence $(y_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}_{\geq 0}$ defined by

$$y_j = \int_{\mathsf{M}\setminus K_{j+1}} \|\chi_j(z)\alpha(z)\|^2 e^{-\varphi(z)}\, d\mu(z)$$

is a decreasing sequence. Moreover, by the Dominated Convergence Theorem, the sequence $(y_j)_{j\in\mathbb{Z}_{>0}}$ converges to 0. Thus, for $\epsilon \in \mathbb{R}_{>0}$, let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $y_j < \epsilon^2$ for $j \geq N$. We then have

$$\|\alpha - \chi_j\alpha\|^2_\varphi = y_j < \epsilon^2,$$

giving the desired convergence.

Now we approximate compactly supported forms in $L^2_\varphi(\mathsf{M};r,s)$ by elements of $\mathscr{D}(\mathsf{M};r,s)$. Let $K \subseteq \mathsf{M}$ be compact and let $\alpha \in L^2_\varphi(\mathsf{M};r,s)$ have support in K. For $z \in K$ let $(\mathcal{U}_z,\phi_z)$ be a $\mathbb{C}$-chart for M about z. By the usual regularisation procedure (e.g., using convolution [e.g., Kecs 1982]) in the chart codomain, there is a neighbourhood $\mathcal{V}_z \subseteq \mathcal{U}_z$ and sequence $(\beta_{z,j})_{j\in\mathbb{Z}_{>0}}$ of smooth forms of bidegree (r,s) with compact support in $\mathcal{U}_z$ such that

$$\lim_{j\to\infty} \int_{\mathsf{M}} \|\beta_{z,j}(\zeta) - \chi_{\mathcal{V}_z}(\zeta)\alpha(\zeta)\|^2 e^{-\varphi(\zeta)}\, d\mu(\zeta) = 0,$$

with $\chi_{\mathcal{V}_z}$ being the characteristic function of $\mathcal{V}_z$. By compactness let $z_1, \dots, z_k \in K$ be such that $K \subseteq \cup_{l=1}^k \mathcal{V}_{z_l}$. Let $(\rho_1, \dots, \rho_k, \rho_{k+1})$ be a partition of unity subordinate to the open cover $(\mathcal{V}_{z_1}, \dots, \mathcal{V}_{z_k}, \mathsf{M} \setminus K)$ [Abraham, Marsden, and Ratiu 1988, Theorem 5.5.12]. Let $\epsilon \in \mathbb{R}_{>0}$ and choose $N \in \mathbb{Z}_{>0}$ sufficiently large that

$$\int_{\mathsf{M}} \|\beta_{z_l,j}(\zeta) - \chi_{\mathcal{V}_{z_l}}(\zeta)\alpha(\zeta)\|^2 \mathrm{e}^{-\varphi(\zeta)}\,\mathrm{d}\mu(\zeta) < \frac{\epsilon^2}{k}, \qquad j \geq N,\ l \in \{1, \dots, N\}.$$

This implies that

$$\int_{\mathsf{M}} \|\rho_l(\zeta)\beta_{z_l,j}(\zeta) - \rho_l(\zeta)\chi_{\mathcal{V}_{z_l}}(\zeta)\alpha(\zeta)\|^2 \mathrm{e}^{-\varphi(\zeta)}\,\mathrm{d}\mu(\zeta) < \frac{\epsilon^2}{k}, \qquad j \geq N,\ l \in \{1, \dots, N\},$$

since $\rho_l(\zeta) \in [0,1]$, $l \in \{1, \dots, k\}$, $z \in \mathsf{M}$. Let

$$\beta_j(\zeta) = \sum_{l=1}^{k} \rho_l(\zeta)\beta_{z_l,j}(\zeta), \qquad j \in \mathbb{Z}_{>0},\ \zeta \in \mathsf{M},$$

noting that this sum makes sense for every $\zeta \in \mathsf{M}$. Also note that

$$\sum_{l=1}^{k} \rho_l(\zeta)\chi_{\mathcal{V}_{z_l}}(\zeta)\alpha(\zeta) = \sum_{l=1}^{k} \rho_l(\zeta)\alpha(\zeta) = \alpha(\zeta), \qquad \zeta \in \mathsf{M}.$$

Thus

$$\int_{\mathsf{M}} \|\beta_j(\zeta) - \alpha(\zeta)\|^2 \mathrm{e}^{-\varphi(\zeta)}\,\mathrm{d}\mu(\zeta) < \epsilon^2, \qquad j \geq N.$$

Since β_j has compact support for each $j \in \mathbb{Z}_{>0}$, we get the desired approximation of forms in $\mathrm{L}^2_\varphi(\mathsf{M}; r, s)$ by forms in $\mathscr{D}(\mathsf{M}; r, s)$.

Next let $\alpha \in \mathrm{L}^2_\varphi(\mathsf{M}; r, s)$ be arbitrary and let $\epsilon \in \mathbb{R}_{>0}$. From the first paragraph of the proof, let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\|\alpha - \chi_j\alpha\|_\varphi < \frac{\epsilon}{2}$ for $j \geq N$. From the second paragraph of the proof let $\beta \in \mathscr{D}(\mathsf{M}; r, s)$ be such that $\|\chi_j\alpha - \beta\|_\varphi < \frac{\epsilon}{2}$. The triangle inequality then gives $\|\alpha - \beta\| < \epsilon$ which proves the first part of the lemma.

For the second part of the proof we will show that $\bar{\partial}$ restricted to $\mathscr{D}(\mathsf{M}; r, s)$ is closable in $\mathrm{L}^2_\varphi(\mathsf{M}; r, s)$. Thus we will show that for any sequence $(\beta_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathsf{M}; r, s)$ converging to 0 in $\mathrm{L}^2_\varphi(\mathsf{M}; r, s)$ and for which $(\bar{\partial}\beta_j)_{j \in \mathbb{Z}_{>0}}$ converges to $\alpha \in \mathrm{L}^2_\varphi(\mathsf{M}; r, s+1)$ in the L^2_φ-norm, it holds that $\alpha = 0$, cf. Proposition E.1.4. As above, since $\alpha \in \mathrm{L}^2_\varphi(\mathsf{M}; r, s)$, for $\epsilon \in \mathbb{R}_{>0}$ there exists a compact set K such that $\|\chi_{\mathsf{M} \setminus K}\alpha\|_\varphi < \epsilon$. Now let $\mathcal{U}_1, \dots, \mathcal{U}_k$ be coordinate neighbourhoods whose images under the corresponding $\mathbb{C}$-chart maps $\phi_1, \dots, \phi_k$ are balls centred at $\mathbf{0} \in \mathbb{C}^n$. Let $(\rho_1, \dots, \rho_{k+1})$ be a smooth partition of unity subordinate to the open cover $(\mathcal{U}_1, \dots, \mathcal{U}_k, \mathsf{M} \setminus K)$. Since the functions ρ_l, $l \in \{1, \dots, k\}$, are bounded, we easily deduce that each of the sequences $(\rho_l\beta_j)_{j \in \mathbb{Z}_{>0}}$, $l \in \{1, \dots, k\}$, converges to zero in $\mathrm{L}^2_\varphi(\mathcal{U}_l; r, s)$. Similarly, $(\rho_l\bar{\partial}\beta_j)_{j \in \mathbb{Z}_{>0}}$ converges to $\rho_l\alpha$ in $\mathrm{L}^2_\varphi(\mathcal{U}_l; r, s+1)$ for $l \in \{1, \dots, k\}$. If we think of $\bar{\partial}$ as being defined on $\mathscr{D}'(\mathcal{U}_l; r, s)$ (the latter being distributions on $\mathcal{U}_l$ acting on test functions that are smooth compactly supported (r, s)-forms), the fact that $(\rho_l\bar{\partial}\beta_j)_{j \in \mathbb{Z}_{>0}}$ converges to $\rho_l\alpha$ in $\mathrm{L}^2_\varphi(\mathcal{U}_l; r, s)$ means that $(\rho_l\bar{\partial}\beta_j)_{j \in \mathbb{Z}_{>0}}$ converges to $\rho_l\alpha$ in $\mathscr{D}'(\mathcal{U}_l; r, s+1)$ [Renardy and Rogers 1993, Remark 6.15]. Note that

$$\rho_l\bar{\partial}\beta_j = \bar{\partial}(\rho_l\beta_j) - \bar{\partial}\rho_l \wedge \beta_j.$$

Since differentiation of distributions and multiplication by smooth functions are continuous operations on distributions, it follows that $(\bar{\partial}(\rho_l\beta_j))_{j\in\mathbb{Z}_{>0}}$ converges to 0 in $\mathscr{D}'(\mathcal{U}_l; r, s+1)$. Since multiplication by smooth functions is a continuous operation on distributions, so is wedge product. Thus $(\bar{\partial}\rho_l \wedge \beta_j)_{j\in\mathbb{Z}_{>0}}$ converges to 0 in $\mathscr{D}'(\mathcal{U}_l; r, s+1)$. Thus $(\rho_l\bar{\partial}\beta_j)_{j\in\mathbb{Z}_{>0}}$ converges to 0 in $\mathscr{D}'(\mathcal{U}; \mathbb{R})$. Since the L^p-spaces are injectively included is the corresponding distribution spaces [Renardy and Rogers 1993, Lemma 6.16], this means that $\rho_l\alpha$ is 0 in $\mathrm{L}^2_\varphi(\mathcal{U}_l; r, s+1)$. Therefore, for any $\epsilon \in \mathbb{R}_{>0}$,

$$\|\alpha\|_\varphi \leq \sum_{l=1}^{k+1} \|\rho_l\alpha\|_\varphi < \epsilon,$$

since the functions ρ_l, $l \in \{1, \dots, k+1\}$, take values in $[0, 1]$. Thus $\bar{\partial}$ is closable in $\mathrm{L}^2_\varphi(\mathsf{M}; r, s)$ as claimed. ■

From the lemma, we can extend $\bar{\partial}$ from the subspace of compactly supported forms to unique closed linear maps

$$T\colon \mathrm{L}^2_\varphi(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \mathrm{L}^2_\varphi(\textstyle\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \tag{6.4}$$

and

$$S\colon \mathrm{L}^2_\varphi(\textstyle\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \to \mathrm{L}^2_\varphi(\textstyle\bigwedge^{r,s+2}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \tag{6.5}$$

with dense domains containing $\mathscr{D}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. The $\bar{\partial}$-problem in L^2 then becomes to show that $\mathrm{image}(T) = \ker(S)$, noting that the inclusion $\mathrm{image}(T) \subseteq \ker(S)$ is obvious. We do this by first establishing a few general Hilbert space results.

6.2.2 General Hilbert space inequalities

In this section we translate the solution of the $\bar{\partial}$-problem in L^2 into the establishment of an estimate, namely that of the following lemma.

6.2.4 Lemma (A general Hilbert space estimate) *Let* H *and* G *be* $\mathbb{C}$*-Hilbert spaces, let* $\mathrm{T}\colon \mathsf{H} \to \mathsf{G}$ *be a closed linear map defined on a dense subset* $\mathrm{dom}(\mathrm{T})$ *of* H*, and let* $\mathsf{F} \subseteq \mathsf{G}$ *be a closed subspace for which* $\mathrm{image}(\mathrm{T}) \subseteq \mathsf{F}$*. Then* $\mathrm{image}(\mathrm{T}) = \mathsf{F}$ *if and only if there exists* $\mathrm{C} \in \mathbb{R}_{>0}$ *such that* $\|\mathrm{f}\|_{\mathsf{G}} \leq \mathrm{C}\|\mathrm{T}^*(\mathrm{f})\|_{\mathsf{H}}$ *for all* $\mathrm{f} \in \mathsf{F} \cap \mathrm{dom}(\mathrm{T}^*)$*.*

Moreover, if this equality is satisfied, then, for each $\mathrm{g} \in \mathsf{F}$ *and* $\mathrm{u} \in \mathsf{H}$ *such that* $\mathrm{T}(\mathrm{u}) = \mathrm{g}$*, we have* $\|\mathrm{u}\|_{\mathsf{H}} \leq \|\mathrm{g}\|_{\mathsf{G}}$*.*

Proof First suppose that there exists $C \in \mathbb{R}_{>0}$ such that $\|f\|_{\mathsf{G}} \leq C\|T^*(f)\|_{\mathsf{H}}$ for all $f \in \mathsf{F} \cap \mathrm{dom}(T^*)$. Let $g \in \mathsf{F}$ and let $f \in \mathrm{dom}(T^*)$. Let us write $f = f_1 + f_2$ for $f_1 \in \mathsf{F}$ and $f_2 \in \mathsf{F}^\perp$ (the orthogonal complement of F). We then have

$$|\langle g, f\rangle_{\mathsf{G}}| = |\langle g, f_1\rangle_{\mathsf{G}}| \leq \|g\|_{\mathsf{G}}\|f_1\|_{\mathsf{G}} \leq \|g\|_{\mathsf{G}}\|f\|_{\mathsf{G}} \leq C\|g\|_{\mathsf{G}}\|T^*(f)\|_{\mathsf{H}},$$

using the Cauchy–Schwartz inequality and the hypotheses of the lemma. This shows that the linear map from $\mathrm{image}(T^*)$ to $\mathbb{C}$ defined by $T^*(f) \mapsto \langle g, f\rangle_{\mathsf{G}}$ on $\mathrm{image}(T^*)$ is bounded by $C\|g\|_{\mathsf{G}}$, and moreover is $\mathbb{C}$-antilinear. Let $P\colon \mathsf{H} \to \mathrm{image}(T^*)$ be the orthogonal projection and consider the map from H to $\mathbb{C}$ defined by $v \mapsto \langle g, f\rangle_{\mathsf{G}}$, where $f \in \mathrm{dom}(T^*)$ is such that

$P(v) = T^*(f)$. By the Riesz Representation Theorem, let $u \in \mathsf{H}$ be such that $\langle u, v\rangle_{\mathsf{H}} = \langle g, f\rangle_{\mathsf{G}}$ for every $v \in \mathsf{H}$ and where $P(v) = T^*(f)$. We then have $\langle u, T^*(f)\rangle_{\mathsf{H}} = \langle g, f\rangle_{\mathsf{G}}$ for every $f \in \mathrm{dom}(T^*)$. Thus $\langle T(u), f\rangle_{\mathsf{G}} = \langle g, f\rangle_{\mathsf{G}}$ for every $f \in \mathrm{dom}(T^*)$ since $T^{**} = T$ [Reed and Simon 1980, Theorem VIII.1]. Thus $T(u) - g \in \mathrm{dom}(T^*)^\perp$ and so $T(u) - g = 0$ again by [Reed and Simon 1980, Theorem VIII.1].

For the converse, suppose that $\mathrm{image}(T) = \mathsf{F}$. We will first show that the set

$$B = \{f \in \mathsf{F} \cap \mathrm{dom}(T^*) \mid \ \|T^*(f)\|_{\mathsf{H}} \leq 1\}$$

is bounded. For this it is sufficient to show that the function $f \mapsto \langle f, g\rangle$ on B is bounded for every $g \in \mathsf{G}$. Write $g = g_1 + g_2$ for $g_1 \in \mathsf{F}$ and $g_2 \in \mathsf{F}^\perp$. Let $u \in \mathsf{H}$ be such that $T(u) = g_1$. Then, for $f \in B$,

$$|\langle f, g\rangle_{\mathsf{G}}| = |\langle f, g_1\rangle_{\mathsf{G}}| = |\langle T^*(f), u\rangle_{\mathsf{H}}| \leq \|u\|_{\mathsf{H}}$$

since $T^{**} = T$ [Reed and Simon 1980, Theorem VIII.1]. Thus B is indeed bounded. Then there exists $C \in \mathbb{R}_{>0}$ such that $B \subseteq \overline{\mathsf{B}}(C, 0)$. Now let $f \in \mathsf{F} \cap \mathrm{dom}(T^*)$ so that $\frac{f}{\|T^*(f)\|_{\mathsf{H}}} \in B \subseteq \overline{\mathsf{B}}(C, 0)$ and so $\|f\|_{\mathsf{G}} \leq C\|T^*(f)\|_{\mathsf{H}}$, as desired.

For the final assertion, note that

$$T(u) = g \quad \iff \quad \langle u, T^*(f)\rangle_{\mathsf{H}} = \langle g, f\rangle_{\mathsf{G}}, \qquad f \in \mathrm{dom}(T^*).$$

We also showed that $|\langle u, T^*(f)\rangle_{\mathsf{H}}| \leq C\|g\|_{\mathsf{G}}\|T^*(f)\|_{\mathsf{H}}$. By the Hahn–Banach Theorem [Kreyszig 1978, Theorem 4.2-1] applied to the map

$$\mathrm{image}(T^*) \ni T^*(f) \mapsto \langle u, T^*(f)\rangle_{\mathsf{H}_2} \in \mathbb{C},$$

we extend this map to all of H by requiring it to be zero on $\mathrm{image}(T^*)^\perp$, and note that the resulting map is bounded by $C\|g\|_{\mathsf{G}}$. This resulting map, however, is simply the element of H^* defined by u under the Riesz Representation Theorem. Thus $\|u\|_{\mathsf{H}} \leq C\|g\|_{\mathsf{G}}$. ■

Our solution of the $\bar\partial$-problem in L^2 will consist of showing that the estimate of the lemma holds for $T = \bar\partial$ in strongly pseudoconvex manifolds. To do so, we shall use the operator S defined by (6.5) and establish instead the estimate

$$\|f\|_\varphi^2 \leq C^2(\|T^*(f)\|_\varphi^2 + \|S(f)\|_\varphi^2),$$

which gives the estimate of Lemma 6.2.4 when the subspace F of the lemma is taken to be $\ker(S)$.

Some of the important consequences of the solution of the $\bar\partial$-problem will rely on another general Hilbert space estimate that we now give.

6.2.5 Lemma (Another general Hilbert space estimate) *Let* H *and* G *be* $\mathbb{C}$*-Hilbert spaces, let* $\mathrm{T}\colon \mathsf{H} \to \mathsf{G}$ *be a closed linear map defined on a dense subset* $\mathrm{dom}(\mathrm{T})$ *of* H*, and let* $\mathsf{F} \subseteq \mathsf{G}$ *be a closed subspace for which* $\mathrm{image}(\mathrm{T}) \subseteq \mathsf{F}$*. Suppose that there exists* $\mathrm{C} \in \mathbb{R}_{>0}$ *such that* $\|\mathrm{f}\|_{\mathsf{G}} \leq \mathrm{C}\|\mathrm{T}^*(\mathrm{f})\|_{\mathsf{H}}$ *for all* $\mathrm{f} \in \mathsf{F} \cap \mathrm{dom}(\mathrm{T}^*)$*. Then, for* $\mathrm{v} \in \ker(\mathrm{T})^\perp$*, there exists* $\mathrm{f} \in \mathrm{dom}(\mathrm{T}^*)$ *such that* $\mathrm{v} = \mathrm{T}^*(\mathrm{f})$ *and such that* $\|\mathrm{f}\|_{\mathsf{G}} \leq \mathrm{C}\|\mathrm{v}\|_{\mathsf{H}}$*.*

Proof We have

$$v \in \ker(T)^{\perp} = (\mathrm{image}(T^*)^{\perp})^{\perp} = \mathrm{cl}(\mathrm{image}(T^*))$$

using the fact that the orthogonal complement of the orthogonal complement of a subspace is the closure of the subspace, cf. [Kreyszig 1978, Lemma3.3-6]. Also, using Theorem E.1.8,

$$\mathsf{F} \supseteq \mathrm{image}(T) \implies \mathsf{F}^{\perp} \subseteq \mathrm{image}(T)^{\perp} = \ker(T^*).$$

Thus, if we write $g \in \mathrm{dom}(T^*)$ as $g = g_1 + g_2$ for $g_1 \in \mathsf{F}$ and $g_2 \in \mathsf{F}^{\perp}$, we have $T^*(f) = T^*(f_1)$, i.e., $\mathrm{image}(T^*) = T^*(\mathsf{F} \cap \mathrm{dom}(T^*))$. The condition $\|f\|_{\mathsf{G}} \leq C\|T^*(f)\|_{\mathsf{H}}$, $f \in \mathsf{F} \cap \mathrm{dom}(T^*)$, implies that $T^*(\mathsf{F} \cap \mathrm{dom}(T^*))$ is closed [Kato 1980, Theorem IV.5.2]. Thus, if $v \in \ker(T)^{\perp} = \mathrm{cl}(\mathrm{image}(T^*))$ there exists $f \in \mathsf{F} \cap \mathrm{dom}(T^*)$ such that $v = T^*(f)$. The lemma now follows directly from the hypotheses. ■

These general results involve the adjoint operator T^*, and so we need to say a few things about this operator.

6.2.3 Estimates for $\bar{\partial}$ and its adjoint

The results of Section 6.2.2 illustrate that one needs to understand the adjoint of T. In this section we undertake to understand those facets of this adjoint that will be important, as well as some similar elementary facets of the operator S.

First we give a local coordinate expression for S and T^*. In order to do so, it is convenient to exploit notation associated with an h-orthonormal local basis $\omega^1, \dots, \omega^n$ for $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$. In particular, for a $\mathbb{C}$-chart $(\mathcal{U}, \phi)$ and for $f \in \mathrm{C}^{\infty}(\mathcal{U}; \mathbb{C})$, we define $\frac{\partial f}{\partial \omega^j}$ and $\frac{\partial f}{\partial \bar{\omega}^j}$, $j \in \{1, \dots, n\}$, by

$$\mathrm{d}_{\mathbb{C}} f = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial \omega^j} \omega^j + \frac{\partial f}{\partial \bar{\omega}^j} \bar{\omega}^j \Big).$$

The exact expressions for $\frac{\partial f}{\partial \omega^j}$ and $\frac{\partial f}{\partial \bar{\omega}^j}$ are easily determined by the change of basis formula. To wit, if $\omega^j = \sum_{k=1}^{n} \omega^j_k \mathrm{d}z^k$, then

$$\frac{\partial f}{\partial \omega^j} = \sum_{k=1}^{n} v^k_j \frac{\partial f}{\partial z^k}, \quad \frac{\partial f}{\partial \bar{\omega}^j} = \sum_{k=1}^{n} \bar{v}^k_j \frac{\partial f}{\partial \bar{z}^k}, \qquad j \in \{1, \dots, n\},$$

where $\sum_{l=1}^{n} \omega^j_l v^l_k = \delta^j_k$. Note that if $u \in \Gamma^{\infty}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ is locally given by

$$u = \sum_{I \in \boldsymbol{n}^s}{}' \sum_{J \in \boldsymbol{n}^r}{}' u_{I,J} \omega^I \wedge \bar{\omega}^J,$$

then we can locally write

$$\bar{\partial} u | \mathcal{U} = (-1)^r \sum_{I \in \boldsymbol{n}^s}{}' \sum_{J \in \boldsymbol{n}^r}{}' \sum_{j=1}^{n} \frac{\partial u_{I,J}}{\partial \bar{\omega}^j} \omega^I \wedge \bar{\omega}^j \wedge \bar{\omega}^J + \sum_{I \in \boldsymbol{n}^s}{}' \sum_{J \in \boldsymbol{n}^r}{}' \alpha_{I,J} u_{I,J},$$

where $\alpha_{I,J} \in \Gamma^\infty(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. For $f \in \Gamma^\infty(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, let us abbreviate

$$A(f) = (-1)^r \sum_{I\in\boldsymbol{n}^s}{}' \sum_{J\in\boldsymbol{n}^r}{}' \sum_{j=1}^{n} \frac{\partial u_{I,J}}{\partial\bar{\omega}^j}\omega^I \wedge \bar{\omega}^j \wedge \bar{\omega}^J. \tag{6.6}$$

The following lemma then gives an estimate for the operator S.

6.2.6 Lemma (An estimate for S(f)) *Let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart for* M *and let* $\mathrm{K} \subseteq \mathcal{U}$ *be a compact set. Then there exists* $\mathrm{C}_1 \in \mathbb{R}_{>0}$ *such that, for any* $\mathrm{f} \in \mathscr{D}(\bigwedge^{\mathrm{r,s+1}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *with support contained in* K, $\|\mathrm{S(f)} - \mathrm{A(f)}\| \le \mathrm{C}_1\|\mathrm{f}\|$. *Moreover,*

$$\|\mathrm{A(f)}(\mathbf{z})\|^2 = \sum_{\mathrm{I}\in\mathbf{n}^{\mathrm{r}}}{}' \sum_{\mathrm{J}\in\mathbf{n}^{\mathrm{s+1}}}{}' \sum_{\mathrm{j}=1}^{\mathrm{n}} \Big|\frac{\partial \mathrm{f}_{\mathrm{I,J}}}{\partial\bar{\omega}^{\mathrm{j}}}(\mathbf{z})\Big|^2 - \sum_{\mathrm{I}\in\mathbf{n}^{\mathrm{r}}}{}' \sum_{\mathrm{J}\in\mathbf{n}^{\mathrm{s}}}{}' \sum_{\mathrm{j,k}=1}^{\mathrm{n}} \frac{\partial \mathrm{f}_{\mathrm{I,jJ}}}{\partial\bar{\omega}^{\mathrm{k}}}(\mathbf{z})\overline{\frac{\partial \mathrm{f}_{\mathrm{I,kJ}}}{\partial\bar{\omega}^{\mathrm{j}}}(\mathbf{z})}.$$

Proof For $j,k \in \{1,\dots,n\}$ and $J,K \in \boldsymbol{n}^s$, define

$$\epsilon^{jJ}_{kK} = \begin{cases} \operatorname{sign}\left(\begin{smallmatrix} jJ \\ kK \end{smallmatrix}\right), & j \notin J,\ k \notin K,\ \{j\}\cup J = \{k\}\cup K, \\ 0, & \text{otherwise.} \end{cases}$$

One then directly verifies that

$$\|A(f)(z)\|^2 = \sum_{I\in\boldsymbol{n}^r}{}' \sum_{J,K\in\boldsymbol{n}^{s+1}}{}' \sum_{j,k=1}^{n} \frac{\partial f_{I,J}}{\partial\bar{\omega}^j}(z)\overline{\frac{\partial f_{I,K}}{\partial\bar{\omega}^k}(z)}\epsilon^{jJ}_{kK}.$$

Let us examine this expression. First consider terms in the sum with $j = k$. For the corresponding term to be nonzero, we must have $k = j$ and $j \notin J$. Thus the terms for which $j = k$ sum to

$$\sum_{I\in\boldsymbol{n}^r}{}' \sum_{J\in\boldsymbol{n}^{s+1}}{}' \sum_{j\notin J} \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^j}(z)\Big|^2. \tag{6.7}$$

Now we consider terms in the sum for which $j \neq k$. In order for ϵ^{jJ}_{kK} to be nonzero, we must have $j \in K$, $k \in J$, and $J \setminus \{k\} = K \setminus \{j\} \triangleq L$. One readily verifies that $\epsilon^{jJ}_{lL}\epsilon^{lL}_{kK} = \epsilon^{jJ}_{kK}$ for every j, k, l, J, K, and L. Thus, for the particular j, k, l, J, K, and L given the assumptions we are making, we have

$$\epsilon^{jJ}_{kK} = \epsilon^{jJ}_{jkL}\epsilon^{jkL}_{kjL}\epsilon^{kjL}_{kK} = -\epsilon^{jJ}_{jkL}\epsilon^{kjL}_{kK} = -\epsilon^{J}_{kL}\epsilon^{jL}_{K}.$$

This gives the terms corresponding to $j \neq k$ to be

$$-\sum_{I\in\boldsymbol{n}^r}{}' \sum_{L\in\boldsymbol{n}^s}{}' \sum_{\substack{j,k=1\\ j\neq k}}^{n} \frac{\partial f_{I,kL}}{\partial\bar{\omega}^j}(z)\overline{\frac{\partial f_{I,jL}}{\partial\bar{\omega}^k}(z)}. \tag{6.8}$$

Now note that by taking $j = k$ in the inner sum of the preceding expression we get

$$\sum_{I\in\boldsymbol{n}^r}{}' \sum_{L\in\boldsymbol{n}^s}{}' \Big|\frac{\partial f_{I,jL}}{\partial\bar{\omega}^j}(z)\Big|^2 = \sum_{I\in\boldsymbol{n}^r}{}' \sum_{\substack{J\in\boldsymbol{n}^{s+1}\\ j\in J}}{}' \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^j}(z)\Big|^2. \tag{6.9}$$

Combining (6.7) and (6.8), and using the equality (6.9), we arrive at the formula of the lemma. ■

Now we arrive at a similar estimate for T^*. We do so by also providing a local form for T^*. To do this, we use the notation

$$\delta_j(g) = \mathrm{e}^{\varphi} \frac{\partial(\mathrm{e}^{-\varphi} g)}{\partial \omega^j}, \qquad j \in \{1, \dots, n\},$$

where g is, for the sake of generality, a distribution defined on a $\mathbb{C}$-chart domain $\mathcal{U}$. With this notation in hand we define

$$B(f) = (-1)^{r-1} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' \sum_{j=1}^{n} \delta_j(f_{I,jJ}) \omega^I \wedge \bar{\omega}^J, \tag{6.10}$$

for $f \in \mathrm{dom}(T^*)$ with support in the chart domain $\mathcal{U}$ given by

$$f = \sum_{I \in \boldsymbol{n}^r} \sum_{J \in \boldsymbol{n}^{s+1}} f_{I,J} \omega^I \wedge \bar{\omega}^J.$$

With this notation we have the following lemma.

6.2.7 Lemma (A local expression and an estimate for T*) *If* $\mathrm{f} \in \mathrm{dom}(\mathrm{T}^*)$ *has support in a* $\mathbb{C}$*-chart domain* $\mathcal{U}$ *for* M, *then* $\mathrm{T}^*(\mathrm{f}) = \mathrm{B}(\mathrm{f}) + \beta(\mathrm{f})$, *where* β *is a differential operator of order zero, i.e.,* $\beta(\mathrm{f})$ *is an expression linear in* f *and involving no derivatives of* f. *Moreover* β *is independent of* φ.

Consequently, if $\mathrm{K} \subseteq \mathcal{U}$ *is compact then there exists* $\mathrm{C}_2 \in \mathbb{R}_{>0}$ *such that* $\|\mathrm{T}^*(\mathrm{f}) - \mathrm{B}(\mathrm{f})\| \leq \mathrm{C}_2 \|\mathrm{f}\|$ *for* $\mathrm{f} \in \mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *having compact support contained in* K.

Proof Let

$$u = \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' u_{I,J} \omega^I \wedge \bar{\omega}^J \in \mathscr{D}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$$

and

$$f = \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^{s+1}}{}' f_{I,J} \omega^I \wedge \bar{\omega}^J \in \mathrm{L}^2_{\varphi}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U})).$$

If $f \in \mathrm{dom}(T^*)$ then we have

$$\begin{aligned}
\int_{\mathcal{U}} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' (T^*(f))_{I,J}(z) \overline{u_{I,J}(z)} \mathrm{e}^{-\varphi(z)} \, \mathrm{d}\mu(z) &= \langle T^*(f), u \rangle_{\varphi} = \langle f, T(u) \rangle_{\varphi} \\
&= (-1)^r \int_{\mathcal{U}} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' \sum_{j=1}^{n} f_{I,jJ}(z) \overline{\frac{\partial u_{I,J}}{\partial \bar{\omega}^j}}(z) \mathrm{e}^{-\varphi(z)} \, \mathrm{d}\mu(z) \\
&\quad + \int_{\mathcal{U}} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^{s+1}}{}' \langle f(z), \alpha_{I,J}(z) \rangle \overline{u_{I,J}(z)} \mathrm{e}^{-\varphi(z)} \, \mathrm{d}\mu(z) \\
&= (-1)^{r-1} \int_{\mathcal{U}} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^s}{}' \sum_{j=1}^{n} \frac{\partial(\mathrm{e}^{-\varphi} f_{I,jJ})}{\partial \omega^j}(z) \overline{u_{I,J}(z)} \, \mathrm{d}\mu(z) \\
&\quad + \int_{\mathcal{U}} \sum_{I \in \boldsymbol{n}^r}{}' \sum_{J \in \boldsymbol{n}^{s+1}}{}' \langle f(z), \alpha_{I,J}(z) \rangle \overline{u_{I,J}(z)} \mathrm{e}^{-\varphi(z)} \, \mathrm{d}\mu(z),
\end{aligned}$$

using the distributional derivative. Here the terms $\alpha_{I,J}$ depend on u, but not on the derivatives of u. Matching coefficients gives

$$(T^*(f))_{I,J} = (-1)^{r-1}\sum_{j=1}^{n} \mathrm{e}^{\varphi}\frac{\partial(\mathrm{e}^{-\varphi}f_{I,jJ})}{\partial\omega^j} + \beta(f),$$

as asserted in the first part of the lemma.

The second assertion of the lemma follows immediately from the first. ■

We will also need a technical lemma regarding $\mathrm{dom}(T^*)$.

6.2.8 Lemma *If* $\chi \in \mathscr{D}(\mathsf{M};\mathbb{R})$ *and if* $\mathrm{f} \in \mathrm{dom}(\mathrm{T}^*)$, *then* $\chi\mathrm{f} \in \mathrm{dom}(\mathrm{T}^*)$.

Proof For $u \in \dim(T)$ we compute

$$\begin{aligned}\langle\chi f, T(u)\rangle_{\varphi} = \langle f, \chi T(u)\rangle_{\varphi} &= \langle f, T(\chi u)\rangle_{\varphi} + \langle f, \chi T(u) - T(\chi u)\rangle_{\varphi}\\ &= \langle\chi T^*(f), u\rangle_{\varphi} - \langle f, \bar{\partial}\chi \wedge u)\rangle_{\varphi}.\end{aligned}$$

This last expression is continuous in u and so there exists $u' \in \mathrm{L}^2(\mathsf{M}; r, s)$ such that $\langle u', u\rangle = \langle\chi f, T(u)\rangle$ for every $u \in \mathrm{dom}(T)$. This is the definition for χf to be in $\mathrm{dom}(T^*)$ and, moreover, gives $T^*(\chi f) = v$. ■

6.2.4 Reduction to smooth compactly supported forms

In this section we show that it is possible to verify the estimate of Lemma 6.2.4 for the operator T by establishing it for smooth compactly supported forms. We do this by proving a density result that will be an essential part of establishing our main estimate.

6.2.9 Lemma (Density of smooth compactly supported forms in $\mathrm{dom}(\mathrm{T}^*) \cap \mathrm{dom}(\mathrm{S})$) *The subspace* $\mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *is dense in* $\mathrm{dom}(\mathrm{T}^*) \cap \mathrm{dom}(\mathrm{S})$ *with respect to the norm*

$$\mathrm{f} \mapsto \|\mathrm{f}\|_{\varphi} + \|\mathrm{T}^*(\mathrm{f})\|_{\varphi} + \|\mathrm{S}(\mathrm{f})\|_{\varphi}.$$

Proof We shall abbreviate

$$\begin{aligned}\mathrm{L}^2_{\varphi}(\mathsf{M}; r, s) &= \mathrm{L}^2_{\varphi}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})),\\ \mathscr{D}(\mathsf{M}; r, s) &= \mathscr{D}(\textstyle\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})).\end{aligned}$$

Using Proposition 4.6.7(v), the definition of S, and the standing assumptions of Notation 6.2.2, we have

$$\|S(\chi_j f)(z) - \chi_j(z)S(f)(z)\| = \|\bar{\partial}\chi_j(z) \wedge f(z)\| \le \|f(z)\|, \qquad f \in \mathrm{dom}(S),\ j \in \mathbb{Z}_{>0},\ z \in \mathsf{M}. \quad (6.11)$$

Now we perform a similar estimate for T^*, but this will require a little effort. Using the computation from the proof of Lemma 6.2.8, the definition of the adjoint T^*, Proposition 4.6.7(v), and the standing assumptions of Notation 6.2.2, we have

$$\langle T^*(\chi_j f) - \chi_j T^*(f), u\rangle_{\varphi} = -\langle f, \bar{\partial}\chi_j \wedge u\rangle_{\varphi}, \qquad f \in \mathrm{dom}(T^*),\ u \in \mathscr{D}(\mathsf{M}; r, s),\ j \in \mathbb{Z}_{>0}.$$

Using the explicit expression for $\langle\cdot,\cdot\rangle_\varphi$ this gives

$$|\langle T^*(\chi_j f) - \chi_j T^*(f), u\rangle|_\varphi \leq \int_{\mathsf{M}} \|f(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z). \tag{6.12}$$

Let us abbreviate $v(z) = T^*(\chi_j f)(z) - \chi_j(z)T^*(f)(z)$. We claim that $\|v(z)\| \leq \|f(z)\|$ for almost every $z \in \mathsf{M}$. Suppose otherwise so that

$$A = \{z \in \mathsf{M} \mid \ \|v(z)\| > \|f(z)\|\}$$

has positive measure. Let $\mathcal{U}$ be a relatively compact open set such that $\mathcal{U} \cap A$ has positive measure. Note that $\mathrm{L}^2_\varphi(\mathcal{U}; r, s) \subseteq \mathrm{L}^1_\varphi(\mathcal{U}; r; s)$ since $\mathcal{U}$ has finite measure. Thus

$$C \triangleq \max\Big\{\int_{\mathcal{U}} \|v(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z), \int_{\mathcal{U}} \|f(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big\} < \infty.$$

Let $K \subseteq \mathcal{U} \cap A$ be compact and such that $\mu(K) > 0$. Let $\mathcal{U}_1, \dots, \mathcal{U}_k \subseteq \mathcal{U}$ cover K, be such that $\mathrm{cl}(\mathcal{U}_j) \subseteq \mathcal{U}$, $j \in \mathbb{Z}_{>0}$, and be such that

$$\mu(\cup_{j=1}^k \mathcal{U}_j) - \mu(K) < \frac{\epsilon}{2C}.$$

Define $u \in \mathscr{D}(\mathcal{U}; r, s)$ as follows. We require that $\|u(z)\| \in [0,1]$ for all $z \in \mathcal{U}$. For $z \in K$ we require that $\|u(z)\| = 1$ and that $\bar{u}$ be collinear with $v(z)$. And, for $z \in \mathcal{U} \setminus \cup_{j=1}^k \mathrm{cl}(\mathcal{U}_j)$, we require that $u(z) = 0$. For u so defined and for $z \in K$,

$$\langle v(z), u(z)\rangle = \|v(z)\|\,\|u(z)\| = \|v(z)\| > \|f(z)\|\,\|u(z)\|.$$

Thus

$$\int_K \langle v(z), u(z)\rangle \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) > \int_K \|f(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z).$$

We also have

$$\begin{aligned}\Big|\int_{\mathcal{U}\setminus K} \langle v(z), u(z)\rangle \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big| &\leq \int_{\mathcal{U}\setminus K} |\langle v(z), u(z)\rangle| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\\ &\leq \int_{\mathcal{U}\setminus K} \|v(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) < \frac{\epsilon}{2}\end{aligned}$$

and

$$\int_{\mathcal{U}\setminus K} \|f(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) < \frac{\epsilon}{2}.$$

Thus

$$\begin{aligned}&\int_{\mathcal{U}\setminus K} \|f(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) - \int_{\mathcal{U}\setminus K} \langle v(z), u(z)\rangle \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\\ &\qquad \leq \Big|\int_{\mathcal{U}\setminus K} \|f(z)\|\,\|u(z)\| \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) - \int_{\mathcal{U}\setminus K} \langle v(z), u(z)\rangle \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big| < \epsilon.\end{aligned}$$

Now take $\epsilon \in \mathbb{R}_{>0}$ sufficiently small that

$$\epsilon < \int_K \langle v(z), u(z)\rangle e^{-\varphi(z)}\, d\mu(z) - \int_K \|f(z)\|\,\|u(z)\| e^{-\varphi(z)}\, d\mu(z),$$

this being possible since the expression on the right does not depend on our choice of cover $\mathcal{U}_1, \dots, \mathcal{U}_k$ for K. Now, extending u to be defined on M by requiring that it be zero off $\mathcal{U}$, we have

$$\begin{aligned}\int_{\mathsf{M}\setminus K} \|f(z)\|\,\|u(z)\| e^{-\varphi(z)}\, d\mu(z) - \int_{\mathsf{M}\setminus K} \langle v(z), u(z)\rangle e^{-\varphi(z)}\, d\mu(z)\\ < \int_K \langle v(z), u(z)\rangle e^{-\varphi(z)}\, d\mu(z) - \int_K \|f(z)\|\,\|u(z)\| e^{-\varphi(z)}\, d\mu(z).\end{aligned}$$

Upon rearrangement this contradicts (6.12) and so we indeed have

$$\|T^*(\chi_j f)(z) - \chi_j(z) T^*(f)(z)\| \le \|f(z)\|, \qquad j \in \mathbb{Z}_{>0},\ f \in \operatorname{dom}(T), \tag{6.13}$$

for almost every $z \in \mathsf{M}$.

Now let $f \in \operatorname{dom}(T^*) \cap \operatorname{dom}(S)$. The bounds (6.11) and (6.13) allow us to use the Dominated Convergence Theorem to assert that

$$\lim_{j\to\infty} \|S(\chi_j f) - \chi_j S(f)\|_\varphi = 0, \quad \lim_{j\to\infty} \|T^*(\chi_j f) - \chi_j T^*(f)\|_\varphi = 0.$$

Thus the sequence $(\chi_j f)_{j\in\mathbb{Z}_{>0}}$ converges to f in the norm from the statement of the lemma. It, therefore, suffices to show that $\mathscr{D}(\mathsf{M}; r, s+1)$ is dense in the subspace of $\operatorname{dom}(T^*)\cap\operatorname{dom}(S)$ consisting of compactly supported forms.

We do this first for forms with compact support in a $\mathbb{C}$-chart domain $\mathcal{U}$. Thus we let $f \in \operatorname{dom}(T^*) \cap \operatorname{dom}(S)$ have compact support in $\mathcal{U}$. We let $\rho \in C^\infty(\mathbb{R}^n)$ be a nonnegative-valued function with support in $\overline{\mathsf{B}}_n(1, \mathbf{0})$ and define $\rho_\epsilon(\boldsymbol{x}) = \epsilon^{-n}\rho(\epsilon^{-1}\boldsymbol{x})$. As we showed in the proof of Proposition E.2.16,

$$\lim_{\epsilon\to 0} \|S(\rho_\epsilon * f) - \rho_\epsilon * S(f)\|_\varphi = 0.$$

Therefore,

$$\lim_{\epsilon\to 0} S(\rho_\epsilon * f) = \lim_{\epsilon\to 0} \rho_\epsilon * S(f) = S(f) \tag{6.14}$$

in the appropriate L^2_φ-spaces, using the usual convergence arguments for convolutions [Kecs 1982]. Taking into account Lemma 6.2.7, we can also apply the constructions from the proof of Proposition E.2.16 for T^*. Specifically, we have

$$\lim_{\epsilon\to 0} \|T^*(\rho_\epsilon * f) - \rho_\epsilon * (T^*(f))\|_\varphi = 0.$$

Thus

$$\lim_{\epsilon\to 0} T^*(\rho_\epsilon * f) = \lim_{\epsilon\to 0} \rho_\epsilon * T^*(f) = T^*(f), \tag{6.15}$$

the limits being taken in L^2_φ-spaces. Combining equations (6.14) and (6.15) with the usual limit $\lim_{\epsilon\to 0} \rho_\epsilon * f = f$ (again in L^2_φ), we have the density of $\mathscr{D}(\mathcal{U}; r, s+1)$ in $\operatorname{dom}(T^*)\cap\operatorname{dom}(S)$ in the norm of the statement of the lemma.

Now, to complete the proof of the lemma, we can use the partition of unity argument from the proof of Lemma 6.2.3 to show that $\mathscr{D}(\mathsf{M}; r, s+1)$ is dense in the subspace of $L^2_\varphi(\mathsf{M}; r, s+1)$ consisting of compactly supported forms. ■

6.2.5 A useful estimate

In this section we prove two technical lemmata—one local and a similar global lemma—that will allow us to easily establish the estimate of Lemma 6.2.4 for the $\bar{\partial}$-problem.

In the course of the construction, we will require the local representation of the Levi form in our orthonormal basis $(\omega^1,\dots,\omega^n)$ in a $\mathbb{C}$-chart $(\mathcal{U},\phi)$. To do so, let us introduce local structure constants, i.e., smooth functions c^j_{kl}, $j,k,l\in\{1,\dots,n\}$, on $\mathcal{U}$ given by

$$\bar{\partial}\omega^j=\sum_{k,l=1}^n c^j_{kl}\omega^k\wedge\bar{\omega}^l,\qquad j\in\{1,\dots,n\}.$$

For $g\in C^\infty(\mathcal{U};\mathbb{C})$ let us also denote

$$g_{jk}=\frac{\partial^2 g}{\partial\omega^j\partial\bar{\omega}^k}+\sum_{l=1}^n\bar{c}^l_{jk}\frac{\partial g}{\partial\bar{\omega}^l}.$$

With this notation, we have the following result.

6.2.10 Lemma (A local representation for the Levi form) *If* $\mathrm{g}\in C^2(\mathsf{M})$ *then, in a* $\mathbb{C}$*-chart* $(\mathcal{U},\phi)$ *for* M *and in an orthonormal basis* $(\omega^1,\dots,\omega^{\mathrm{n}})$ *for* $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$*, we have*

$$\mathrm{Lev}(\mathrm{g})=\sum_{\mathrm{j,k=1}}^{\mathrm{n}}\mathrm{g_{jk}}\omega^{\mathrm{j}}\wedge\bar{\omega}^{\mathrm{k}}.$$

Proof This is a direct computation using $\mathrm{Lev}(g)=\partial\circ\bar{\partial}g$ and the definition of g_{jk}. ■

Let us denote by $\lambda\in C^0(\mathsf{M})$ the smallest eigenvalue of the endomorphism $h^\sharp\circ\mathrm{Lev}(\varphi)^\flat$ of $\mathsf{T}^{\mathbb{C}}\mathsf{M}$. The local lemma is the following.

6.2.11 Lemma (A local estimate) *If* $(\mathcal{U},\phi)$ *is a* $\mathbb{C}$*-chart for* M*, if* $(\omega^1,\dots,\omega^{\mathrm{n}})$ *is an orthonormal basis for* $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathcal{U})$*, if* $\varphi\in C^\infty(\mathsf{M})$*, and if* λ *is the smallest eigenvalue function as defined above, and if* $\mathrm{K}\subseteq\mathcal{U}$ *is compact, then there exists* $\mathrm{C}_3\in\mathbb{R}_{>0}$ *such that*

$$\int_{\mathcal{U}}\sum_{\mathrm{I}\in\mathbf{n}^{\mathrm{r}}}{}'\sum_{\mathrm{J}\in\mathbf{n}^{\mathrm{s+1}}}{}'|\mathrm{f_{I,J}}(\mathbf{z})|^2\lambda(\mathbf{z})\mathrm{e}^{-\varphi(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z})+\frac{1}{2}\int_{\mathcal{U}}\sum_{\mathrm{I}\in\mathbf{n}^{\mathrm{r}}}{}'\sum_{\mathrm{J}\in\mathbf{n}^{\mathrm{s+1}}}{}'\sum_{\mathrm{j=1}}^{\mathrm{n}}\left|\frac{\partial\mathrm{f_{I,J}}}{\partial\bar{\omega}^{\mathrm{j}}}(\mathbf{z})\right|^2\mathrm{e}^{-\varphi(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z})$$
$$\leq 2(\|\mathrm{T}^*(\mathrm{f})\|_\varphi^2+\|\mathrm{S}(\mathrm{f})\|_\varphi^2+\mathrm{C}_3\|\mathrm{f}\|_\varphi^2)$$

for every $\mathrm{f}\in\mathscr{D}(\bigwedge^{\mathrm{r,s+1}}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ *with support contained in* K*.*

Proof From Lemmata 6.2.6 and 6.2.7 there exists $C_1,C_2\in\mathbb{R}_{>0}$ such that

$$\|S(f)-A(f)\|_\varphi\leq C_1\|f\|_\varphi,\quad\|T^*(f)-B(f)\|_\varphi\leq C_2\|f\|_\varphi$$

for every f satisfying the conditions of the lemma, and where A and B are as given by (6.6) and (6.10), respectively. Thus

$$\|A(f)\|_\varphi\leq\|A(f)-S(f)\|_\varphi+\|S(f)\|_\varphi\leq\|S(f)\|_\varphi+C_1\|f\|_\varphi.$$

Squaring both sides of this inequality and using the fact that $2(a^2+b^2) \geq (a^2+b^2+2ab)$ for $a, b \in \mathbb{R}_{\geq 0}$, we get

$$\|A(f)\|_\varphi^2 \leq 2(\|S(f)\|_\varphi^2 + C_1^2\|f\|_\varphi^2).$$

A similar computation for $\|B(f)\|_\varphi$ applies, and adding the resulting expression to that just derived gives

$$\|A(f)\|_\varphi^2 + \|B(f)\|_\varphi^2 \leq 2(\|S(f)\|_\varphi^2 + \|T^*(f)\|_\varphi^2) + C_3'\|f\|_\varphi^2 \tag{6.16}$$

for some $C_3' \in \mathbb{R}_{>0}$, and where $f \in \mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ has support in K. Combining the conclusions of Lemmata 6.2.6 and 6.2.7 we have

$$\begin{aligned}\|A(f)\|_\varphi^2 + \|B(f)\|_\varphi^2 = &\int_{\mathcal{U}} \sum_{I\in \boldsymbol{n}^r}{}' \sum_{J\in \boldsymbol{n}^{s+1}}{}' \sum_{j=1}^n \left|\frac{\partial f_{I,J}}{\partial \bar{\omega}^j}(z)\right|^2 \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\\ &+ \int_{\mathcal{U}} \sum_{I\in \boldsymbol{n}^r}{}' \sum_{J\in \boldsymbol{n}^{s}}{}' \sum_{j,k=1}^n \left(\delta_j(f_{I,jJ})(z)\overline{\delta_k(f_{I,kJ})(z)} - \frac{\partial f_{I,jJ}}{\partial \bar{\omega}^k}(z)\overline{\frac{\partial f_{I,kJ}}{\partial \bar{\omega}^j}(z)}\right)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z).\end{aligned} \tag{6.17}$$

To manipulate the right-hand side into something we want, we shall swap the operators δ_j for the operators $\frac{\partial}{\partial \bar{\omega}^j}$. To do so, let us first indicate the appropriate integration by parts formula.

1 Sublemma *If* $\mathrm{g}, \mathrm{h} \in \mathscr{D}(\mathcal{U};\mathbb{C})$ *then*

$$\int_{\mathcal{U}} \mathrm{g}(\mathbf{z})\overline{\frac{\partial \mathrm{h}}{\partial \mathrm{z}^j}(\mathbf{z})}\,\mathrm{d}\lambda(\mathbf{z}) = -\int_{\mathcal{U}} \frac{\partial \mathrm{g}}{\partial \bar{\mathrm{z}}^j}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\lambda(\mathbf{z})$$

and

$$\int_{\mathcal{U}} \mathrm{g}(\mathbf{z})\overline{\frac{\partial \mathrm{h}}{\partial \bar{\mathrm{z}}^j}(\mathbf{z})}\,\mathrm{d}\lambda(\mathbf{z}) = -\int_{\mathcal{U}} \frac{\partial \mathrm{g}}{\partial \mathrm{z}^j}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\lambda(\mathbf{z})$$

for $\mathrm{j} \in \{1,\dots,\mathrm{n}\}$.

Proof This is a direct computation using the definitions of $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$, and the usual integration by parts formula. ▼

The following closely related lemma is one we shall implicitly make use of.

2 Sublemma *If* $\mathrm{g}, \mathrm{h} \in \mathscr{D}(\mathcal{U};\mathbb{C})$ *then*

$$\int_{\mathcal{U}} \mathrm{g}(\mathbf{z})\overline{\frac{\partial \mathrm{h}}{\partial \omega^j}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) = -\int_{\mathcal{U}} \frac{\partial \mathrm{g}}{\partial \bar{\omega}^j}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) + \int_{\mathcal{U}} \beta_j(\mathbf{z})\mathrm{g}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z})$$

and

$$\int_{\mathcal{U}} \mathrm{g}(\mathbf{z})\overline{\frac{\partial \mathrm{h}}{\partial \bar{\omega}^j}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) = -\int_{\mathcal{U}} \frac{\partial \mathrm{g}}{\partial \omega^j}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) + \int_{\mathcal{U}} \bar{\beta}_j(\mathbf{z})\mathrm{g}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z})$$

for $\mathrm{j} \in \{1,\dots,\mathrm{n}\}$, *where* $\beta_j \in \mathrm{C}^\infty(\mathcal{U};\mathbb{C})$ *are independent of* g *and* h.

Proof The proof is a direct computation using the preceding lemma and the fact that $\frac{\partial}{\partial \omega^j}$ and $\frac{\partial}{\partial \bar{\omega}^j}$ are linear combinations of $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}$ and $\frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}$, respectively, with coefficients in $\mathrm{C}^\infty(\mathcal{U};\mathbb{C})$. ▼

Now we have the following lemma relating the operators $\frac{\partial}{\partial \bar{\omega}^j}$ and δ_j.

3 Sublemma *If* $\mathrm{g}, \mathrm{h} \in \mathscr{D}(\mathcal{U};\mathbb{C})$ *then*

$$\int_{\mathcal{U}} \frac{\partial \mathrm{g}}{\partial \bar{\omega}^{\mathrm{j}}}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\mathrm{e}^{-\varphi(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) = -\int_{\mathcal{U}} \mathrm{g}(\mathbf{z})\overline{\delta_{\mathrm{j}}(\mathrm{h})(\mathbf{z})}\mathrm{e}^{-\varphi(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}) + \int_{\mathcal{U}} \alpha_{\mathrm{j}}(\mathbf{z})\mathrm{g}(\mathbf{z})\overline{\mathrm{h}(\mathbf{z})}\mathrm{e}^{-\varphi(\mathbf{z})}\,\mathrm{d}\mu(\mathbf{z}),$$

for some $\alpha_{\mathrm{j}} \in \mathrm{C}^{\infty}(\mathcal{U};\mathbb{C})$ *that is independent of* g *and* h.

Proof This is a direct computation using the definition of δ_j, integration by parts, and the fact that g and h have compact support. ▼

Now, by moving both differentiations to the left in the first summand in the second integral of (6.17), we see that we need to understand the commutator $\delta_j \circ \frac{\partial}{\partial \bar{\omega}^k} - \frac{\partial}{\partial \bar{\omega}^k} \circ \delta_j$. The following lemma gives us this.

4 Sublemma *For* $\mathrm{g} \in \mathrm{C}^{\infty}(\mathcal{U};\mathbb{C})$ *we have*

$$\delta_{\mathrm{j}} \circ \frac{\partial}{\partial \bar{\omega}^{\mathrm{k}}}(\mathrm{g}) - \frac{\partial}{\partial \bar{\omega}^{\mathrm{k}}} \circ \delta_{\mathrm{j}}(\mathrm{g}) = \mathrm{g}\varphi_{\mathrm{jk}} + \sum_{\mathrm{l}=1}^{\mathrm{n}} \mathrm{c}_{\mathrm{kj}}^{\mathrm{l}}\delta_{\mathrm{l}}(\mathrm{g}) - \sum_{\mathrm{l}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{jk}}^{\mathrm{l}}\frac{\partial \mathrm{g}}{\partial \bar{\omega}^{\mathrm{l}}}.$$

Proof This is another direct computation using the definitions. ▼

We now put together the preceding two sublemmata to deduce that

$$\int_{\mathcal{U}} \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k=1}^{n} \left(\delta_j(f_{I,jJ})(z)\overline{\delta_k(f_{I,kJ})(z)} - \frac{\partial f_{I,jJ}}{\partial\bar{\omega}^k}(z)\overline{\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^j}(z)}\right)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)$$

$$= \int_{\mathcal{U}} \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k=1}^{n} \varphi_{jk}(z) f_{I,jJ}(z)\overline{f_{I,kJ}(z)}\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \tag{6.18}$$

$$+ \int_{\mathcal{U}} \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k,l=1}^{n} f_{I,jJ}(z)c_{kj}^l(z)\overline{\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^l}(z)}\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \tag{6.19}$$

$$- \int_{\mathcal{U}} \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k,l=1}^{n} f_{I,jJ}(z)\bar{c}_{kj}^l(z)\overline{\delta_l(f_{I,kJ})(z)}\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \tag{6.20}$$

$$+ \int_{\mathcal{U}} \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k=1}^{n} \left(f_{I,jJ}(z)\alpha_k(z)\overline{\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^j}(z)} - f_{I,jJ}(z)\bar{\alpha}_j(z)\overline{\delta_k(f_{I,kJ})(z)}\right)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z). \tag{6.21}$$

We next estimate the terms in lines (6.18)–(6.21).

Since the basis $(\omega^1,\dots,\omega^n)$ is orthonormal, φ_{jk}, $j,k \in \{1,\dots,n\}$, are the components of both the Hermitian form $\mathrm{Lev}(\varphi)$ and the Hermitian endomorphism $h^{\sharp} \circ \mathrm{Lev}(\varphi)^{\flat}$. It follows, therefore, that

$$\sum_{j,k=1}^{n} \varphi_{jk}(z) f_{I,jJ}(z)\overline{f_{I,kJ}(z)} \geq \lambda(z)\sum_{j=1}^{n} f_{I,jJ}(z)\overline{f_{I,jJ}(z)}. \tag{6.22}$$

Therefore,

$$\sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j,k=1}^{n} \varphi_{jk}(z) f_{I,jJ}(z)\overline{f_{I,kJ}(z)} \geq \lambda(z) \sideset{}{'}\sum_{I\in\boldsymbol{n}^r} \sideset{}{'}\sum_{J\in\boldsymbol{n}^s} \sum_{j=1}^{n} f_{I,jJ}(z)\overline{f_{I,jJ}(z)} = \lambda(z)\|f(z)\|^2,$$

giving an estimate for line (6.18).

Now let us consider line (6.19). First we use a lemma.

5 Sublemma *There exists* $c_n \in \mathbb{R}_{>0}$ *such that, for any* $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$,

$$\sum_{j,k=1}^{n} |w_j||z_k| \leq c_n \Big(\sum_{j=1}^{n} |w_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{n} |z_j|^2\Big)^{1/2}.$$

Proof Our argument is somewhat indirect. We first define a norm on the set $\mathbb{C}^{n\times n}$ matrices by

$$\|\boldsymbol{A}\|_1 = \sum_{j,k=1}^{n} |A_{jk}|.$$

Note for $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{C}^n$ we have

$$\sum_{j,k=1}^{n} = |w_j||z_k| = \|\boldsymbol{w}\boldsymbol{z}^T\|_2.$$

Now denote by $\|\cdot\|_{2,2}$ the norm on $\mathbb{C}^{n\times n}$ induced by the Euclidean norm on $\mathbb{C}_{<0}$¡ thinking of $\mathbb{C}^{n\times n}$ as being the set of endomorphisms of $\mathbb{C}^n$, That is,

$$\|\boldsymbol{A}\|_{2,2} = \inf\{\|\boldsymbol{A}\boldsymbol{\zeta}\| \mid \|\zeta\| = 1\}.$$

Then we have

$$\|\boldsymbol{w}\boldsymbol{z}^T\boldsymbol{\zeta}\| = |\langle \boldsymbol{z}, \boldsymbol{\zeta}\rangle|\|\boldsymbol{w}\| \leq \|\boldsymbol{w}\|\|\boldsymbol{z}\|\|\boldsymbol{\zeta}\|,$$

from which we conclude that $\|\boldsymbol{w}\boldsymbol{z}^T\|_{2,2} \leq \|\boldsymbol{w}\|\,\|\boldsymbol{z}\|$. The lemma follows by noting that equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_{2,2}$ implies the existence of the constant c_n in the statement. ▼

Note also that there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{j=1}^{n} \sum_{I\in\boldsymbol{n}^r}{}' \sum_{J\in\boldsymbol{n}^s}{}' |\alpha_{I,jJ}|^2 \leq N \sum_{I\in\boldsymbol{n}^r}{}' \sum_{J\in\boldsymbol{n}^{s+1}}{}' |\alpha_{I,J}|^2,$$

since the sum on the left contains each term in the sum on the right a finite number (bounded above by some $N \in \mathbb{Z}_{>0}$) of times.

Let

$$M_1 = \sup\{|c^l_{kj}(\boldsymbol{z})| \mid \boldsymbol{z} \in K,\ j,k,l \in \{1,\dots,n\}\}.$$

Putting all of the above together we compute

$$\begin{aligned}
&\Big|\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \sum_{j,k,l=1}^{n} f_{I,jJ}(z)c^l_{kj}(z)\overline{\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^l}(z)}\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big| \\
&\leq M_1\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \sum_{j,k,l=1}^{n} |f_{I,jJ}(z)|\Big|\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^l}(z)\Big|\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \\
&\leq M_1\sum_{j,k,l=1}^{n}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' |f_{I,jJ}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \Big|\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^l}(z)\Big|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2} \\
&\leq c_nM_1\sum_{l=1}^{n}\Big(\int_{\mathcal{U}}\sum_{j=1}^{n} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' |f_{I,jJ}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}\Big(\sum_{k=1}^{n} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \int_{\mathcal{U}}\Big|\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^l}(z)\Big|^2\Big)^{1/2} \\
&\leq Nc_nM_1\sum_{l=1}^{n}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\Big)^{1/2}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\Big|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}.
\end{aligned}$$

By combining Sublemma 3 with computations like those above, the term in line (6.20) is estimated by

$$\begin{aligned}
&\Big|\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \sum_{j,k,l=1}^{n} f_{I,jJ}(z)\bar{c}^l_{kj}(z)\overline{\delta_l(f_{I,kJ})(z)}\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big| \\
&\leq Nc_nM_2\sum_{l=1}^{n}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\Big|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2} \\
&\qquad + M_3\int_{\mathcal{U}}\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)
\end{aligned}$$

for suitable constants $M_2, M_3 \in \mathbb{R}_{>0}$.

We can estimate the terms in the line (6.21) in a similar way, and combining the estimates for lines (6.19)–(6.21) we obtain that the sum of these lines is bounded above in magnitude by

$$\begin{aligned}
&D_1\sum_{l=1}^{n}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\Big|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2} \\
&\qquad + D_2\int_{\mathcal{U}}\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)
\end{aligned}$$

for some $D_1, D_2 \in \mathbb{R}_{>0}$. We now apply the equality $ab \leq \frac{1}{2}(a^2 + b^2)$, valid for $a, b \in \mathbb{R}_{\geq 0}$, for

$$\begin{aligned}
a &= \Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' |f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}, \\
b &= D_1\Big(\int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^{s+1}}}' \Big|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\Big|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\Big)^{1/2}.
\end{aligned}$$

In this case the sum of the lines (6.19)–(6.21) is bounded above in magnitude by

$$\frac{1}{2}\int_{\mathcal{U}}\sum_{l=1}^{n}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s+1}}{}'\left|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\right|^2 \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) + D\int_{\mathcal{U}}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s+1}}{}'|f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)$$

for some $D\in\mathbb{R}_{>0}$.

Assembling all of the estimates for lines (6.18)–(6.21) we finally obtain

$$\begin{aligned}\int_{\mathcal{U}}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s}}{}'\sum_{j,k=1}^{n}\left(\delta_j(f_{I,jJ})(z)\overline{\delta_k(f_{I,kJ})(z)} - \frac{\partial f_{I,jJ}}{\partial\bar{\omega}^k}(z)\overline{\frac{\partial f_{I,kJ}}{\partial\bar{\omega}^j}(z)}\right)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\\ \ge \int_{\mathcal{U}}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s+1}}{}'|f_{I,J}(z)|^2\lambda(z)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) - \frac{1}{2}\int_{\mathcal{U}}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s+1}}{}'|f_{I,J}(z)|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z)\\ - D\int_{\mathcal{U}}\sum_{I\in\boldsymbol{n}^r}{}'\sum_{J\in\boldsymbol{n}^{s+1}}{}'\left|\frac{\partial f_{I,J}}{\partial\bar{\omega}^l}(z)\right|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z).\end{aligned}$$

Combining the preceding estimate with the equality (6.17) and the inequality (6.16) gives the lemma. ■

We now give a global version of the preceding lemma.

6.2.12 Lemma (A global estimate) *If* $\varphi\in\mathrm{C}^\infty(\mathsf{M})$ *and if* λ *is the smallest eigenvalue function as above, there exists* $\mathrm{C}\in\mathrm{C}^0(\mathsf{M})$ *such that*

$$\int_{\mathsf{M}}(\lambda(\mathrm{z})-\mathrm{C}(\mathrm{z}))\|\mathrm{f}(\mathrm{z})\|\mathrm{e}^{-\varphi(\mathrm{z})}\,\mathrm{d}\mu(\mathrm{z}) \le 4(\|\mathrm{T}^*(\mathrm{f})\|_\varphi^2 + \|\mathrm{S}(\mathrm{f})\|_\varphi^2)$$

for every $\mathrm{f}\in\mathscr{D}(\bigwedge^{\mathrm{r,s+1}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$.

Proof We cover M by a locally finite open cover $(\mathcal{U}_a)_{a\in A}$ whose open sets are $\mathbb{C}$-chart domains. For each $a\in A$ let $\rho_a\in\mathscr{D}(\mathcal{U}_a;[0,1])$ be such that

$$\sum_{a\in A}\rho_a(z)^2 = 1, \qquad z\in\mathsf{M}.$$

(One can take the functions ρ_a, $a\in A$, to be the square roots of the functions defining a partition of unity subordinate to $(\mathcal{U}_a)_{a\in A}$.) For $a\in A$, note that $K_a=\operatorname{supp}(\rho_a)$ is a compact subset of $\mathcal{U}_a$. Then, by Lemma 6.2.11, there exists $C_a\in\mathbb{R}_{>0}$ such that, for any $f\in\mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$,

$$\int_{\mathsf{M}}\|\rho_a(z)f(z)\|^2\lambda(z)\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \le 2(\|T^*(\rho_a f)\|_\varphi^2 + \|S(\rho_a f)\|_\varphi^2) + C_a\int_{\mathsf{M}}\|\rho_a(z)f(z)\|^2\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z).$$

Now note that

$$\|S(\rho_a f)\|_\varphi \le \|\rho_a S(f)\|_\varphi + \|\rho_a S(f) - S(\rho_a f)\|_\varphi.$$

We square both sides and using the fact that $2(a^2+b^2)\ge(a^2+b^2+2ab)$ for $a,b\in\mathbb{R}_{\ge0}$, we get

$$\|S(\rho_a f)\|_\varphi^2 \le 2(\|\rho_a S(f)\|_\varphi^2 + \|\rho_a S(f)-S(\rho_a f)\|_\varphi^2) \le 2\|\rho_a S(f)\|_\varphi^2 + 2\|f\|_\varphi^2,$$

using (6.11). Similarly, using (6.12), we have

$$\|T^*(\rho_a f)\|_\varphi^2 \le 2\|\rho_a T^*(f)\|_\varphi^2 + 2\|f\|_\varphi^2.$$

Thus, for each $a \in A$ we have the estimate

$$\int_{\mathsf{M}} \|\rho_a(z) f(z)\|^2 \lambda(z) \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \le 4(\|\rho_a T^*(f)\|_\varphi^2 + \|\rho_a S(f)\|_\varphi^2) + (4 + C_a)\int_{\mathsf{M}} \|f(z)\|^2 \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z).$$

By summing this inequality over A, it only remains to show that there exists a continuous function C such that

$$C(z) \ge \sum_{a\in A_z} (4 + C_a),$$

where $A_z = \{a \in A \mid z \in \mathcal{U}_a\}$. As M is assumed to be second countable, let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a compact exhaustion of M. For $j \in \mathbb{Z}_{>0}$ let

$$M_j = \max\Big\{ \sum_{a\in A_z} (4 + C_a) \;\Big|\; z \in K_j \Big\}.$$

By local finiteness of the open cover $(\mathcal{U}_a)_{a\in A}$, M_j is finite for every $j \in \mathbb{Z}_{>0}$. Define $g_1 \in \mathsf{C}^0(\mathsf{M}; [0,1])$ so that $g_1(z) = 1$ for $z \in K_1$ and $g_1(z) = 0$ for $z \in \mathsf{M} \setminus \mathrm{int}(K_2)$. For $j \ge 2$ let $g_j \in \mathsf{C}^0(\mathsf{M}; [0,1])$ be such that $g_j(z) = 1$ for $z \in K_j$ and $g_j(z) = 0$ for $z \in (\mathsf{M} \setminus \mathrm{int}(K_{j+1})) \cup (\mathsf{M} \setminus \mathrm{int}(K_{j-1}))$. Then the function

$$C(z) = \sum_{j=1}^{\infty} |M_j| g_j(z)$$

has the desired property. ■

6.2.6 Existence theorems for the $\bar\partial$-problem

Now we use our estimates from the preceding section to give existence theorems for the $\bar\partial$-problem. We do this first for data in $\mathsf{L}^2_{\mathrm{loc}}$.

6.2.13 Theorem (Existence of solutions to the $\bar\partial$-problem in L^2) *If M is a second countable strongly pseudoconvex holomorphic manifold then there exists $\varphi \in \mathsf{C}^\infty(\mathsf{M})$ such that, given $\mathsf{f} \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfying $\bar\partial\mathsf{f} = 0$, there exists $\mathsf{u} \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfying $\bar\partial\mathsf{u} = \mathsf{f}$ and $\|\mathsf{u}\|_\varphi \le \|\mathsf{f}\|_\varphi$.*

Proof Let ψ be a strictly pseudoconvex exhaustion function on M and let $\lambda \in \mathsf{C}^0(\mathsf{M})$ be the smallest eigenvalue function for $h^\sharp \circ \mathrm{Lev}(\psi)^\flat$. We let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be smooth, convex, and strictly increasing so that $\varphi \triangleq \sigma \circ \psi$ is strictly plurisubharmonic. It is plurisubharmonic by Proposition 6.1.10(iv) and strictly plurisubharmonic since, in a $\mathbb{C}$-chart,

$$\frac{\partial^2\varphi}{\partial z^k \partial\bar z^j}(z) = \sigma''(\psi(z))\frac{\partial\psi}{\partial z^k}(z)\frac{\partial\psi}{\partial\bar z^j}(z) + \sigma'(\psi(z))\frac{\partial^2\psi}{\partial z^k\partial\bar z^j}(z),$$

from which we deduce that $\mathrm{Lev}(\varphi)$ is positive definite. From this last formula observe that the inequality (6.22) implies that

$$\sum_{I\in\boldsymbol{n}^r}{}' \sum_{J\in\boldsymbol{n}^s}{}' \sum_{j,k=1}^{n} \varphi_{jk}(z) f_{I,jJ}(z)\overline{f_{I,kJ}(z)} \ge \sigma'(\psi(z))\lambda(z)\|f(z)\|^2$$

for $f \in \mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Therefore, by Lemma 6.2.12, there exists a continuous function C such that

$$\int_{\mathsf{M}} (\sigma'(\psi(z))\lambda(z) - C(z))\|f(z)\|\mathrm{e}^{-\varphi(z)}\,\mathrm{d}\mu(z) \leq 4(\|T^*(f)\|_\varphi^2 + \|S(f)\|_\varphi^2)$$

for every $f \in \mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$. We claim that it is possible to choose σ so that

$$\sigma'(\psi(z))\lambda(z) - C(z) \geq 4, \qquad z \in \mathsf{M}. \tag{6.23}$$

Indeed, for $x \in \mathbb{R}$ denote

$$K_x = \{z \in \mathsf{M} \mid \psi(z) \leq x\},$$

noting that this set is compact since ψ is an exhaustion function. With this notation, (6.23) is equivalent to the requirement that

$$\sigma'(x) \geq \gamma(x) \triangleq \sup\left\{\frac{4 + C(z)}{\lambda(z)} \;\middle|\; z \in K_x\right\}$$

for every $x \in \mathbb{R}$. Note that γ is increasing and bounded on all compact subsets of $\mathbb{R}$. It is then a straightforward exercise (e.g., first use a piecewise linear function and then smooth it) to find a smooth increasing function σ' that exceeds γ. One can without loss of generality also ensure that σ' is positive. Integrating gives us the desired smooth, convex, and strictly increasing σ so that, if $\varphi = \sigma \circ \psi$, then

$$\|f\|_\varphi^2 \leq \|T^*(f)\|_\varphi^2 + \|S(f)\|_\varphi^2.$$

for every $f \in \mathscr{D}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. By Lemma 6.2.9 we conclude that this inequality holds for every $f \in \operatorname{dom}(T^*) \cap \ker(S)$. From Lemma 6.2.4 we conclude that for every $f \in \ker(S)$ there exists $u \in \mathsf{L}^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ such that $T(u) = f$ and such that $\|u\|_\varphi \leq \|f\|_\varphi$. This establishes the required existence result in $\mathsf{L}^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$.

Now let $f \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. By taking σ sufficiently large, for $\varphi = \sigma \circ \psi$ we have $f \in \mathsf{L}^2_\varphi(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. By our arguments above there is a solution $u \in \mathsf{L}^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ to $T(u) = f$. Since $\mathsf{L}^2_\varphi(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})) \subseteq \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, the theorem follows. ■

Now we investigate the regularity of solutions to the $\bar{\partial}$-problem when the problem data is regular. We shall refer here to the Sobolev space constructions of Section E.2.1. We consider a holomorphic manifold M. By $\mathsf{H}^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ we denote the sections of $\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$ for which the components of the local representatives in any relatively compact $\mathbb{C}$-chart $(\mathcal{U}, \phi)$ are in $\mathsf{H}^q(\phi(\mathcal{U}); \mathbb{C})$. It is easy to show that this definition makes sense since the notion of being locally in L^2 is independent of changes of coordinate. This notation is used in the following theorem.

6.2.14 Theorem (Existence of solutions to the $\bar{\partial}$-problem in H^q) *If M is a second countable strongly pseudoconvex holomorphic manifold and if $\mathrm{q} \in \mathbb{Z}_{>0}$, then, given $\mathrm{f} \in \mathsf{H}^{\mathrm{q}}_{\mathrm{loc}}(\bigwedge^{\mathrm{r,s+1}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfying $\bar{\partial}\mathrm{f} = 0$, there exists $\mathrm{u} \in \mathsf{H}^{\mathrm{q+1}}_{\mathrm{loc}}(\bigwedge^{\mathrm{r,s}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ satisfying $\bar{\partial}\mathrm{u} = \mathrm{f}$ and $\|\mathrm{u}\|_\varphi \leq \|\mathrm{f}\|_\varphi$.*

Moreover, if $\mathrm{s} = 0$ then $\mathrm{u} \in \mathsf{H}^{\mathrm{q+1}}_{\mathrm{loc}}(\bigwedge^{\mathrm{r,s}}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ for every u satisfying $\bar{\partial}\mathrm{u} = \mathrm{f}$.

Proof Throughout the proof, we make use of regularisations by convolution, and refer to the constructions of Lemma GA2.7.1.4 for details. Here, we shall simply denote by $(f_j)_{j\in\mathbb{Z}_{>0}}$ the sequence of compactly supported smooth approximations of a function or form f.

We first prove a technical lemma.

1 **Lemma** *Let* M *be a holomorphic manifold, let* $(\mathcal{U},\phi)$ *be a* $\mathbb{C}$*-chart for* M*, and let* $(\omega^1,\dots,\omega^n)$ *be a basis for* $\bigwedge^{1,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})$*. If* $g \in L^2(\mathcal{U};\mathbb{C})$ *has compact support and if* $\frac{\partial g}{\partial \bar{\omega}^j} \in L^2(\mathcal{U};\mathbb{C})$ *for each* $j \in \{1,\dots,n\}$*, then* $g \in H^1(\mathcal{U};\mathbb{C})$*.*

Proof First let us suppose that g is smooth. In this case, two applications of Sublemma 2 from the proof of Lemma 6.2.11, using the fact that g has compact support, gives

$$\begin{aligned}\int_{\mathcal{U}}\left|\frac{\partial g}{\partial \omega^j}(z)\right|^2 \mathrm{d}\mu(z) &= \int_{\mathcal{U}} \frac{\partial g}{\partial \omega^j}(z)\overline{\frac{\partial g}{\partial \omega^j}}(z)\,\mathrm{d}\mu(z)\\ &= -\int_{\mathcal{U}} \frac{\partial^2 g}{\partial\bar{\omega}^j\partial\omega^j}(z)\overline{g(z)}\,\mathrm{d}\mu(z) + \int_{\mathcal{U}} \beta_j(z)\frac{\partial g}{\partial \omega^j}(z)\overline{g(z)}\,\mathrm{d}\mu(z)\\ &= \int_{\mathcal{U}}\left|\frac{\partial g}{\partial \bar{\omega}^j}(z)\right|^2 \mathrm{d}\mu(z) - \int_{\mathcal{U}} \bar{\beta}_j(z)\frac{\partial g}{\partial \bar{\omega}^j}(z)\overline{g(z)}\,\mathrm{d}\mu(z)\\ &\quad - \int_{\mathcal{U}} g(z)\overline{\frac{\partial(\bar{\beta}^j g)}{\partial\bar{\omega}^j}}(z)\,\mathrm{d}\mu(z) + \int_{\mathcal{U}} \bar{\beta}_j(z)g(z)\overline{\bar{\beta}_j(z)g(z)}\,\mathrm{d}\mu(z).\end{aligned}$$

Now let $(g_k)_{k\in\mathbb{Z}_{>0}}$ be a sequence of smooth compactly supported approximations to g, cf. the proof of Lemma GA2.7.1.4. By hypothesis and since differentiation, even of distributions, commutes with convolution, the sequence $(\frac{\partial g_k}{\partial\bar{\omega}^j})_{k\in\mathbb{Z}_{>0}}$ converges in L^2. Therefore, by using our computation above, for $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\int_{\mathcal{U}}\left|\frac{\partial g_k}{\partial \omega^j}(z) - \frac{\partial g_l}{\partial \omega^j}(z)\right|^2 \mathrm{d}\mu(z) < \epsilon$$

for $k,l \geq N$. Thus the sequence $(\frac{\partial g_l}{\partial\bar{\omega}^j})_{k\in\mathbb{Z}_{>0}}$ also converges in L^2. Therefore, the sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ has a limit, namely g, that is an element of $H^1(\mathcal{U};\mathbb{C})$. ▼

Now we prove the theorem in the case $s = 0$. Thus we let $f \in H^q_{\mathrm{loc}}(\bigwedge^{r,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and note that Theorem 6.2.13 shows that there exists $u \in L^2_{\mathrm{loc}}(\bigwedge^{r,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ such that $T(u) = f$. We will prove by induction that $u \in H^k_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ for every $k \in \{0,1,\dots,q+1\}$. We certainly have $u \in H^0_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Suppose that $u \in H^k_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ for $k \in \{0,1,\dots,q\}$. For $z \in \mathsf{M}$ let $(\mathcal{U}_z,\phi_z)$ be a $\mathbb{C}$-chart about z. Let $\mathcal{V}_z$ be a relatively compact open set such that $z \in \mathcal{V}_z$ and such that $\mathrm{cl}(\mathcal{V}_z) \subseteq \mathcal{U}_z$. Let $\chi_z \in \mathscr{D}(\mathcal{U}_z;\mathbb{C})$ be such that $\chi_z(w) = 1$ for w in a neighbourhood of $\mathcal{V}_z$. Note that if

$$u|\mathcal{U}_z = {\sum_{I\in\boldsymbol{n}^r}}' u_I \mathrm{d}z^I,$$

we have $\frac{\partial u_I}{\partial \bar{z}^j} \in H^k(\mathcal{U}_z;\mathbb{C})$ for each increasing multi-index $I \in \boldsymbol{n}^r$ and $j \in \{1,\dots,n\}$. Therefore, by Leibniz's Rule,

$$\bar{\partial}(\chi_z u) \in H^k(\mathcal{U}_z;\mathbb{C}). \tag{6.24}$$

Now let $I \in \mathbb{Z}_{\geq 0}^n$ satisfy $|I| = k$. By (6.24) it follows that $\bar{\partial}(\boldsymbol{D}^I(\chi_z u)) \in \mathrm{L}^2(\bigwedge^{r,1}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$. By Lemma 1 above, $\chi_z u \in \mathrm{H}^{k+1}(\bigwedge^{r,1}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$. Since χ_z is equal to the constant function 1 on $\mathcal{V}_z$, $u|\mathcal{V}_z \in \mathrm{H}^{k+1}(\bigwedge^{r,1}(\mathsf{T}^{*\mathbb{C}}\mathcal{V}_z))$. Thus we have a covering of M by coordinate charts $\mathcal{V}_z$, $z \in \mathsf{M}$, for which the components of u are in $\mathrm{H}^{k+1}(\mathcal{V}_z;\mathbb{C})$ for each $z \in \mathsf{M}$. Thus $u \in \mathrm{H}^{k+1}_{\mathrm{loc}}(\bigwedge^{r,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, and by induction $u \in \mathrm{H}^{q+1}_{\mathrm{loc}}(\bigwedge^{r,0}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, as desired. Note that this also proves the second assertion of the theorem.

Now let $s \geq 2$ and let $f \in \mathrm{H}^{q}_{\mathrm{loc}}(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. We will prove by induction on k that $u \in \mathrm{H}^{k}_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ for every $k \in \{0,1,\dots,q+1\}$. By Theorem 6.2.13 this holds for $k = 0$. So suppose that this holds for $k \in \{0,1,\dots,q\}$. For $z \in \mathsf{M}$ let $(\mathcal{U}_z,\phi_z)$ be a $\mathbb{C}$-chart about z. Let $\mathcal{V}_z$ be a relatively compact open set such that $z \in \mathcal{V}_z$ and such that $\mathrm{cl}(\mathcal{V}_z) \subseteq \mathcal{U}_z$. Let $\chi_z \in \mathscr{D}(\mathcal{U}_z;\mathbb{C})$ be such that $\chi_z(w) = 1$ for w in a neighbourhood of $\mathcal{V}_z$. Let $\alpha \in \mathscr{D}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$, let D be a partial differential operator of order k, and compute

$$\langle \bar{\partial} D u_j, \alpha\rangle = \langle u_j, D^* \bar{\partial}^* \alpha\rangle = \langle u, \bar{\partial}^* D^* \alpha_j\rangle = \langle f, D^* \alpha_j\rangle = \langle D f_j, \alpha\rangle,$$

using change of variables in the convolution formula, and using the fact that convolution commutes with differentiation with respect to coordinates. From the above computation we deduce that $\bar{\partial} D u_j = D f_j$, equality being as distributions with support in some compact subset of $\mathcal{U}_z$. Now, according to the proof of Lemma 6.2.4, we can choose $u \in \mathrm{cl}(\mathrm{image}(\bar{\partial}^*))$ such that $\bar{\partial} u = f$. Thus, by Theorem E.1.8, $\langle u, \mathrm{e}^{-\varphi}\bar{\partial}\beta\rangle = 0$ for every $\beta \in \mathscr{D}(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$ for $\varphi \in \mathrm{C}^\infty(\mathsf{M})$ chosen as in the proof of Theorem 6.2.13. This implies, therefore, that $\langle u, \mathrm{e}^{-\varphi}\bar{\partial}(\mathrm{e}^{\varphi}\beta)\rangle = 0$ for every $\beta \in \mathscr{D}(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$. Thus we have

$$\langle u, \bar{\partial}\beta\rangle = -\langle u, \bar{\partial}\varphi \wedge \beta\rangle$$

for every $\beta \in \mathscr{D}(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}_z))$. Let us write

$$-\langle u, \bar{\partial}\varphi \wedge \beta\rangle = \langle \beta, \gamma_\varphi\rangle$$

for an appropriate $\gamma_\varphi \in \mathrm{L}^2(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, this being possible since the expression on the left is linear in β. Note that, in fact, $\gamma_\varphi \in \mathrm{H}^{k}_{\mathrm{loc}}(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ by the induction hypothesis. As above, let D be a linear constant coefficient differential operator of order k and compute

$$\langle \bar{\partial}^* D u_j, \beta\rangle = \langle u_j, D^* \bar{\partial}\beta\rangle = \langle u, \bar{\partial} D^* \beta_j\rangle = \langle \gamma_\varphi, D^* \beta_j\rangle = \langle D(\gamma_\varphi), \beta_j\rangle = \langle (D(\gamma_\varphi))_j, \beta\rangle,$$

using the fact that $\gamma_\varphi \in \mathrm{H}^{k}_{\mathrm{loc}}(\bigwedge^{r,s-2}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. As with the computation above, in the above sequence of calculations we used a change of variable in the convolution and the commutativity of convolution with differentiation. In any event, we conclude that $\bar{\partial}^* D u_j = (D(\gamma_\varphi))_j$, equality being of distributions with support in compact subsets of $\mathcal{U}_z$. We also conclude from the arguments above that the sequences $(\bar{\partial} D u_j)_{j\in\mathbb{Z}_{>0}}$ and $(\bar{\partial}^* D u_j)_{j\in\mathbb{Z}_{>0}}$ converge to $D(f)$ and $D(\gamma_\varphi)$ in the sense of converging in L^2 on compact sets.

Next note that

$$\bar{\partial}(\chi_z D u_j) = \chi_z \bar{\partial} D u_j + \bar{\partial}\chi_z \wedge D u_j$$

and, using the local form for $\bar{\partial}^*$ from Lemma 6.2.7,

$$\bar{\partial}^*(\chi_z D u_j) = \chi_z \bar{\partial}^* D u_j + B(\partial\chi_z, D u_j),$$

where B is a zeroth-order differential operator, the exact form of which is of no particular consequence. Now, by the induction hypothesis and our arguments above, the sequences $(\bar{\partial}(\chi_z Du_j))_{j\in\mathbb{Z}_{>0}}$ and $(\bar{\partial}^*(\chi_z Du_j))_{j\in\mathbb{Z}_{>0}}$ converge in the L^2-topology on $\mathrm{cl}(\mathcal{V}_x)$.

Next let $m \in \{1,\dots,n\}$, let $I \in \boldsymbol{n}^r$ and $J \in \boldsymbol{n}^s$, and note from Lemma 6.2.11 that we have

$$\int_{\mathcal{U}_z}\Big|\frac{\partial(\chi_z D(u_{I,J})_j)}{\partial\bar{\omega}^m}(\boldsymbol{z})\Big|^2\,\mathrm{d}\mu(\boldsymbol{z}) \le 4(\|\bar{\partial}^*(\chi_z Du_j)\|^2 + \|\bar{\partial}(\chi_z Du_j)\|^2 + C_3\|\chi_z Du_j\|^2).$$

Since the terms on the right-hand side correspond to converging sequences, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\int_{\mathcal{U}_z}\Big|\frac{\partial(\chi_z D(u_{I,J})_j)}{\partial\bar{\omega}^m} - \frac{\partial(\chi_z D(u_{I,J})_k)}{\partial\bar{\omega}^m}(\boldsymbol{z})\Big|^2\,\mathrm{d}\mu(\boldsymbol{z}) < \epsilon$$

for $j,k \ge N$. Thus $\big(\frac{\partial(\chi_z D(u_{I,J})_j)}{\partial\bar{\omega}^m}\big)_{j\in\mathbb{Z}_{>0}}$ converges in L^2. Referring to the proof of Lemma 1, we see that this implies that the sequence $\big(\frac{\partial(\chi_z D(u_{I,J})_j)}{\partial\omega^m}\big)_{j\in\mathbb{Z}_{>0}}$ also converges in L^2. Therefore, arguing as in the proof of Lemma 1, $\chi_z D(u_{I,J}) \in H^1(\mathcal{U}_z;\mathbb{C})$ and so $u_{I,J} \in H^{k+1}(\mathcal{V}_z;\mathbb{C})$, giving the result. ■

We can now state the main result of this section.

6.2.15 Theorem (Existence of solutions to the $\bar{\partial}$-problem in C^∞) *If* M *is a second countable strongly pseudoconvex holomorphic manifold then, given* $f \in \Gamma^\infty(\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$, *there exists* $u \in \Gamma^\infty(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ *satisfying* $\bar{\partial}u = f$.

Proof By Theorem 6.2.14 we have that there exists $u \in \cap_{q\in\mathbb{Z}_{>0}}H^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ such that $\bar{\partial}u = f$, with equality being in the sense of distributions. It remains to show that if $g \in \cap_{q\in\mathbb{Z}_{>0}}H^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ then g is almost everywhere equal to an element of $\Gamma^\infty(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Thus let $g \in \cap_{q\in\mathbb{Z}_{>0}}H^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ and let $z_0 \in \mathsf{M}$. Let $\phi \in \mathscr{D}(\mathsf{M};\mathbb{R})$ have the property that $\phi(z) = 1$ for z in a neighbourhood of z_0. Since $g \in \cap_{q\in\mathbb{Z}_{>0}}H^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$ we easily conclude that $\phi g \in \cap_{q\in\mathbb{Z}_{>0}}H^q_{\mathrm{loc}}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}))$. Let us suppose that $\mathrm{supp}(\phi) \subseteq \mathcal{U}$ where $\mathcal{U}$ is the domain of a holomorphic chart for M. To simplify notation, let us identify objects with their local representatives. Let us also suppose for simplicity that the image of $\mathcal{U}$ under the chart map is a polydisk. Let $\rho \in C^\infty(\mathbb{C}^n)$ have the following properties:

1. $\rho(\boldsymbol{z}) \ge 0$;
2. $\mathrm{supp}(\rho) = \mathsf{B}^n(1,\boldsymbol{0})$;
3. $\rho(\boldsymbol{z}_1) = \rho(\boldsymbol{z}_2)$ whenever $\|\boldsymbol{z}_1\| = \|\boldsymbol{z}_2\|$;
4. $\int_{\mathbb{C}^n}\rho(\boldsymbol{z})\,\mathrm{d}\lambda(\boldsymbol{z}) = 1$.

For $j \in \mathbb{Z}_{>0}$ define

$$(\phi g)_j(\boldsymbol{z}) = \int_{\mathbb{C}^n}(\phi g)(\boldsymbol{\zeta})\rho(j(\boldsymbol{z}-\boldsymbol{\zeta}))j^{2n}\,\mathrm{d}\lambda(\boldsymbol{\zeta}),$$

noting that $(\phi g)_j$ is a smooth differential form and has compact support contained in $\mathcal{U}$ for j sufficiently large.

We claim that, for any linear partial differential operator D of degree $k \in \mathbb{Z}_{\geq 0}$ with constant coefficients, $D((\phi g)_j)$ is a Cauchy sequence with respect to uniform convergence. To see this, consider the linear partial differential operator D_0 defined by

$$D_0\psi = \frac{\partial^{2n}\psi}{\partial x^1 \cdots \partial x^n \partial y^1 \cdots \partial y^n}$$

where $z^j = x^j + \mathrm{i}y^j$ are the coordinates in the chart. For $\psi \in \mathrm{L}^2(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ let us denote

$$\|\psi\|_2^2 = \int_{\mathcal{U}} {\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' |\psi_{I,J}(\boldsymbol{z})|^2 \,\mathrm{d}\lambda(\boldsymbol{z}).$$

ref Since convolution commutes with partial differentiation of distributions, since the inclusion of $\mathrm{L}^2(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ in $\mathscr{D}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$ is continuous, and since $\phi g \in \mathrm{H}^{2n+k}(\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathcal{U}))$, we have

$$\lim_{j\to\infty} \|D_0 D((\phi g)_j) - D_0 D(\phi g)\|_2 = 0.$$

Let $h \in \mathrm{C}^\infty(\mathbb{C}^n)$ have compact support in $\mathcal{U}$, let $\boldsymbol{z} \in \mathcal{U}$, and let $\boldsymbol{w} \in \mathbb{C}^n \setminus \mathcal{U}$. We then estimate

$$\begin{aligned} |h(\boldsymbol{z})| &= \Big| \int_{\mathrm{Re}(w^1)}^{\mathrm{Re}(z^1)} \cdots \int_{\mathrm{Re}(w^n)}^{\mathrm{Re}(z^n)} \int_{\mathrm{Im}(w^1)}^{\mathrm{Im}(z^n)} \cdots \int_{\mathrm{Im}(w^n)}^{\mathrm{Im}(z^n)} D_0 h(\boldsymbol{\zeta}) \,\mathrm{d}\lambda(\boldsymbol{\zeta}) \Big| \\ &\leq \int_{\mathcal{U}} |D_0 h(\boldsymbol{\zeta})| \,\mathrm{d}\lambda(\boldsymbol{\zeta}) \leq \lambda(\mathcal{U})^{1/2} \|D_0 h\|_2, \end{aligned}$$

using the Cauchy–Schwartz inequality in the last step. This shows that

$${\sum_{I\in\boldsymbol{n}^r}}' {\sum_{J\in\boldsymbol{n}^s}}' \sup\{|D((\phi g_{I.J})_j(\boldsymbol{z}))| \mid \boldsymbol{z} \in \mathcal{U}\} \leq \lambda(\mathcal{U})^{1/2} \|D_0 D((\phi g)_j)\|_2.$$

It follows, therefore, that the coefficients $(D((\phi g_{I.J})_j)_{j\in\mathbb{Z}_{>0}}$ form a Cauchy sequence for uniform convergence, as was claimed at the beginning of this paragraph.

As the conclusions of the preceding paragraph hold for any linear partial differential operator with constant coefficients, it follows that all partial derivatives of the sequence $((\phi g)_j)_{j\in\mathbb{Z}_{>0}}$ converge uniformly on $\mathcal{U}$, and so converge to some smooth limit that we denote by h. Moreover, we have

$$\|h - \phi g\|_2 = \lim_{j\to\infty} \|(\phi g)_j - \phi g\|_2 = 0,$$

the first inequality holding since uniform convergence implies L^2-convergence. From this we conclude that ϕg is almost everywhere equal to h. Thus ϕg is almost everywhere equal to a smooth differential form. Since ϕ has value 1 in a neighbourhood of z_0 we conclude that g is almost everywhere equal to a smooth differential form in a neighbourhood of z_0, and from this the theorem follows. ■

6.3 Stein manifolds

As we have seen, there are two quite different sorts of holomorphically convex holomorphic manifolds: (1) domains of holomorphy in $\mathbb{C}^n$ and (2) compact holomorphic manifolds. The character of holomorphic functions on these two sorts of holomorphically convex manifolds are quite different. For example, in a domain of holomorphy we can solve "reasonable" interpolation problems (Theorem 3.5.3), whereas on compact holomorphic manifolds all holomorphic functions are locally constant (Corollary 4.2.11). What we seek is a class of holomorphic manifolds that resemble domains of holomorphy in that they have a plentiful supply of holomorphic functions.

6.3.1 The ingredients for Stein manifolds

There are a number of equivalent definitions of what is meant by a Stein manifold. In this section we introduce the concepts used to define what is meant by a Stein manifold, and the relationships between them. Some of the relationships are trivial, and others much less so, relying on the solution to the Levi problem we give in Section 6.3.2.

6.3.1 Definition (Holomorphically separable, holomorphically spreadable, global coordinate functions) Let M be a holomorphic manifold.

(i) M is ***holomorphically spreadable*** if, for every $z \in \mathsf{M}$, there exists $k \in \mathbb{Z}_{>0}$, a map $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^k)$, and a neighbourhood $\mathcal{U}$ of z such that

$$f^{-1}(f(z)) \cap \mathcal{U} = \{z\}.$$

(ii) M is ***holomorphically separable*** if, for every distinct $z_1, z_2 \in \mathsf{M}$, there exists $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M})$ such that $f(z_1) \neq f(z_2)$.

(iii) M is ***locally holomorphically separable*** if, for each $z \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of z such that, for every $w \in \mathcal{U}$, there exists $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M})$ satisfying $f(w) \neq f(z)$.

(iv) M ***possesses global coordinate functions*** if, for every $z \in \mathsf{M}$, there exists $f \in \mathsf{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^n)$ and a neighbourhood $\mathcal{U}$ of z such that $(\mathcal{U}, f|\mathcal{U})$ is a $\mathbb{C}$-chart. •

Let us prove some relationships between these various concepts.

6.3.2 Lemma (Connections between holomorphically spreadable, holomorphically separable, and global coordinate functions) *For a holomorphic manifold M, the following statements hold:*

(i) if M is holomorphically spreadable then it is locally holomorphically separable;

(ii) if M is holomorphically separable then it is locally holomorphically separable;

(iii) if M is holomorphically separable then it is holomorphically spreadable;

(iv) if M possesses global coordinate functions then it is holomorphically spreadable;

(v) *if* M *is second countable, Hausdorff, and holomorphically convex, then the following statements hold:*

(a) *if* M *is locally holomorphically separable then it is holomorphically separable;*

(b) *if* M *is holomorphically spreadable then it is holomorphically separable;*

(c) *if* M *is holomorphically separable then it possess global coordinate functions.*

Proof (i) Let $z \in \mathsf{M}$, and let $\mathcal{U}$ be a neighbourhood of z and $\boldsymbol{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M}; \mathbb{C}^k)$ be such that $\boldsymbol{f}^{-1}(\boldsymbol{f}(z)) \cap \mathcal{U} = \{z\}$. Define $g \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ by

$$g(w) = \sum_{j=1}^{k} (f_j(w) - f_j(z))^2,$$

and note that $g(z) = 0$ and $g(w) \in \mathbb{R}_{>0}$ for $w \in \mathcal{U}$. From this we conclude that M is locally holomorphically separable.

(ii) This is obvious.

Make references to analytic sets chapter

(iii) Let $z_0 \in \mathsf{M}$ and let $z_1 \neq z_0$. Let $f_1 \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ satisfy $f_1(z_0) \neq f_1(z_1)$. By shifting and scaling the values of f_1, we may suppose that $f_1(z_0) = 0$ and $f_1(z_1) = 1$. If z_0 lies in a component of dimension 0 of the analytic set $f_1^{-1}(0)$ then $f_1^{-1}(f_1(z_0)) \cap \mathcal{U} = \{z_0\}$ for some sufficiently small neighbourhood $\mathcal{U}$ of z_0. Otherwise, let $\mathsf{S}_1, \dots, \mathsf{S}_k$ be the irreducible components of $f_1^{-1}(0)$. For each $j \in \{1, \dots, k\}$ choose $w_j \in \mathsf{S}_j$ and let $g_j \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be such that $g_j(z_0) = 0$ and $g_j(w_j) = 1$. Let

$$f_2(z) = 1 - \prod_{j=1}^{k} (1 - g_j(z))$$

so that $f_2(z_0) = 0$ and $f_2(w_j) = 1$ for each $j \in \{1, \dots, k\}$. Note that f_2 then does not vanish identically on the irreducible components of $f_1^{-1}(0)$. If z_0 lies in a component of dimension 0 of the analytic set $f_1^{-1}(0) \cap f_2^{-1}(0)$ then $\boldsymbol{f}^{-1}(\boldsymbol{f}(z_0)) = \{z_0\}$ with $\boldsymbol{f}(z) = (f_1(z), f_2(z))$. Otherwise, we continue on this way, next on the irreducible components of $\boldsymbol{f}^{-1}(\boldsymbol{0})$. As the dimension of the irreducible components decreases at each step, this procedure will terminate with the existence of $\boldsymbol{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M}; \mathbb{C}^k)$ for which $\boldsymbol{f}^{-1}(\boldsymbol{f}(z_0)) = \{z_0\}$.

(iv) Let $z \in \mathsf{M}$ and let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart for which $\phi(z) = \boldsymbol{0}$, supposing since M possesses global coordinate functions, that $\phi = \boldsymbol{f}|\mathcal{U}$ for $\boldsymbol{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M}; \mathbb{C}^n)$. Then we have $\boldsymbol{f}^{-1}(\boldsymbol{f}(z)) \cap \mathcal{U} = \{z\}$.

(v a) By Theorem 6.3.3, M is strongly pseudoconvex. By Theorem 6.3.5, M is holomorphically separable.

(v b) By part (i) and Theorem 6.3.3, M is strongly pseudoconvex. By Theorem 6.3.5, M is holomorphically separable.

(v c) By part (ii) and Theorem 6.3.4, M is strongly pseudoconvex. By Theorem 6.3.5, M possesses global coordinate functions. ■

By Theorems 3.1.10 and 3.1.12 we deduce that domains of holomorphy are holomorphically separable. By Corollary 4.2.11 we have that compact holomorphic manifolds are not holomorphically separable.

6.3.2 The Levi problem on manifolds

In this section we shall present the solution to the Levi problem on manifolds using the solution of the $\bar{\partial}$-problem that we presented at length in Section 6.2. As we saw in Section 3.4, the Levi problem for open subsets of $\mathbb{C}^n$ is the assertion that weakly pseudoconvex (or equivalently strongly pseudoconvex) open sets are domains of holomorphy. This problem was solved by [Oka 1953]. The Levi problem in manifolds is slightly different in that one must begin with the hypothesis of strong pseudoconvexity and the conclusion is that the manifold is a Stein manifold. This version of the Levi problem was solved by Grauert [1958] using methods from sheaf theory. Here we give the solution of the Levi problem using partial differential equation methods. Our approach follows the presentation of Hörmander [1973].

The Levi problem gives the equivalence of strong pseudoconvexity with holomorphic convexity in the presence of other conditions. We begin with some of the easier implications, beginning with the assumption of local holomorphic separability.

6.3.3 Theorem (Holomorphic convexity plus local holomorphic separability imply strong pseudoconvexity) *Let* M *be a second countable Hausdorff holomorphic manifold. If* M *is holomorphically convex and locally holomorphically separable, then* M *is strongly pseudoconvex.*

Proof Let $z \in \mathsf{M}$ and let $\mathcal{V}$ be a neighbourhood of z such that, for every $w \in \mathcal{V}$, there exists $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ satisfying $f(w) \neq f(z)$. Let $(\mathcal{W}, \phi)$ be a $\mathbb{C}$-chart about z such that $\phi(z) = \mathbf{0}$ and such that $\phi^{-1}(\overline{\mathsf{B}}^n(r, \mathbf{0})) \subseteq \mathcal{V}$. Let $w \in \phi^{-1}(\mathrm{bd}(\overline{\mathsf{B}}^n(r, \mathbf{0})))$ and let $g_w \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be such that $g_w(w) \neq g_w(z)$. Without loss of generality we can suppose that $g_w(z) = 0$ and $g_w(w) > 1$. By continuity of g_w there is a neighbourhood $\mathcal{N}_w$ of w such that $g_w(w') > q$ for all $w' \in \mathcal{N}_w$. By compactness of $\phi^{-1}(\mathrm{bd}(\overline{\mathsf{B}}^n(r, \mathbf{0})))$ there exists $w_1, \dots, w_{k_z} \in \phi^{-1}(\mathrm{bd}(\overline{\mathsf{B}}^n(r, \mathbf{0})))$ such that, if we take $h_z = \sum_{j=1}^{k_z} |g_{w_j}|^2$, then $h_z(z) = 0$ and $h_z(w) > 1$ for $w \in \phi^{-1}(\mathrm{bd}(\overline{\mathsf{B}}^n(r, \mathbf{0})))$. By Lemma 1 from the proof of Theorem 6.1.22, h_z is smooth and plurisubharmonic. Now define

$$f_z(w) = \begin{cases} h_z(w), & w \in \mathsf{M} \setminus \phi^{-1}(\mathsf{B}^n(1, \mathbf{0})), \\ M_{(\epsilon,\epsilon)}(h_z(w), \|\phi(w)\|^2), & w \in \phi^{-1}(\mathsf{B}^n(1, \mathbf{0})), \end{cases}$$

where $M_{(\epsilon,\epsilon)}$ is as defined in Lemma 1 from the proof of Proposition 6.1.21. Note that f_z is plurisubharmonic and strictly plurisubharmonic (by Lemma 1 from the proof of Proposition 6.1.21) in a neighbourhood $\mathcal{N}_z$ of z. Since M is second countable, there exists $(z_j)_{j \in \mathbb{Z}_{>0}}$ such that $(\mathcal{N}_{z_j})_{j \in \mathbb{Z}_{>0}}$ covers M.

We shall construct a strictly plurisubharmonic function v as the limit of a sequence of smooth functions converging in the weak C^∞-topology. We prove the following lemma of general utility.

1 Lemma *If* M *is a smooth paracompact Hausdorff manifold and if* $(\mathrm{f}_j)_{j \in \mathbb{Z}_{>0}}$ *is a sequence in* $\mathrm{C}^\infty(\mathsf{M})$*, then there exists a sequence* $(\epsilon_j)_{j \in \mathbb{Z}_{>0}}$ *in* $\mathbb{R}_{>0}$ *such that the sum* $\sum_{j=1}^\infty \epsilon_j \mathrm{f}_j$ *converges in the weak* C^∞*-topology.*

Proof We equip M with a Riemannian metric $\mathbb{G}$, this by paracompactness of M [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13]. We denote by ∇ the Levi-Civita connection of

$\mathbb{G}$. Let $g \in \mathrm{C}^\infty(\mathsf{M})$. If $K \subseteq \mathsf{M}$ is compact and if $r \in \mathbb{Z}_{\geq 0}$, we define

$$\|g\|_{r,K} = \sup\{\|\nabla^j g(x)\| \mid x \in K,\ j \in \{0, 1, \dots, r\}\},$$

where $\|\cdot\|$ indicates the norm induced on tensors by the norm associated with the Riemannian metric. One readily sees that the family of seminorms $\|\cdot\|_{r,K}$, $r \in \mathbb{Z}_{\geq 0}$, $K \subseteq \mathsf{M}$ compact, defines a locally convex topology agreeing with other definitions of the weak C^∞-topology. Thus, if a sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ satisfies

reference for this

$$\lim_{j\to\infty} \|g - g_j\|_{r,K} = 0, \qquad r \in \mathbb{Z}_{\geq 0},\ K \subseteq \mathsf{M} \text{ compact},$$

then g is infinitely differentiable [Michor 1980, §4.3].

Without loss of generality, we suppose that M is connected, since otherwise the construction can be applied separately to the connected components of M. If M is paracompact, connectedness allows us to conclude that M is second countable [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.11]. Using Lemma 2.76 of [Aliprantis and Border 2006], we let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of $\mathcal{U}$ such that $K_j \subseteq \mathrm{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$ and such that $\cup_{j\in\mathbb{Z}_{>0}} K_j = \mathcal{U}$. Let us define $\alpha_j = \|f_j\|_{j,K_j}$ and take $\epsilon_j \in \mathbb{R}_{>0}$ to satisfy $\epsilon_j < (\alpha_j 2^j)^{-1}$. We define f by

$$f(x) = \sum_{j=1}^\infty \epsilon_j f_j(x),$$

and claim that the sum converges to f in the weak C^∞-topology. Let us define $g_m \in \mathrm{C}^\infty(\mathsf{M})$ by

$$g_m(x) = \sum_{j=1}^m \epsilon_j f_j(x).$$

Let $K \subseteq \mathsf{M}$ be compact, let $r \in \mathbb{Z}_{\geq 0}$, and let $\epsilon \in \mathbb{R}_{>0}$. Take $N \in \mathbb{Z}_{>0}$ sufficiently large that $K \subseteq K_N$ and such that

$$\sum_{m=m_1+1}^{m_2} \frac{1}{2^m} < \epsilon,$$

for $m_1, m_2 \geq N$ with $m_1 < m_2$, this being possible by convergence of $\sum_{j=1}^\infty \frac{1}{2^j}$. Then, for $m_1, m_2 \geq N$,

$$\begin{aligned}
\|g_{m_1} - g_{m_2}\|_{r,K} &= \sup\{\|\nabla^j g_{m_1}(x) - \nabla^j g_{m_2}(x)\| \mid x \in K,\ j \in \{0,1,\dots,r\}\} \\
&= \sup\left\{\left\| \sum_{m=m_1+1}^{m_2} \epsilon_m \nabla^j f_m(x) \right\| \,\middle|\, x \in K,\ j \in \{0,1,\dots,r\}\right\} \\
&\leq \sup\left\{ \sum_{m=m_1+1}^{m_2} \epsilon_m \|\nabla^j f_m(x)\| \,\middle|\, x \in K,\ j \in \{0,1,\dots,r\}\right\} \leq \sum_{m_1+1}^{m_2} \frac{1}{2^m} < \epsilon.
\end{aligned}$$

Thus, for every $r \in \mathbb{Z}_{\geq 0}$ and $K \subseteq \mathsf{M}$ compact, $(f_m)_{m\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in the norm $\|\cdot\|_{r,K}$. Completeness of the weak C^∞-topology implies that the sequence $(f_m)_{m\in\mathbb{Z}_{>0}}$ converges to a function that is infinitely differentiable. ▼

By the lemma, let $(\epsilon_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}_{>0}$ for which the sum

$$v = \sum_{j=1}^{\infty} \epsilon_j f_{z_j}$$

converges in the weak C^∞-topology. Let $z \in M$. Since v is an infinite sum of smooth plurisubharmonic functions, at least one of which is strictly plurisubharmonic in a neighbourhood of z, it follows from that v is strictly plurisubharmonic in a neighbourhood of z, and so it strictly plurisubharmonic.

Finally, we show that there exists a smooth strictly plurisubharmonic exhaustion function on M. Since M is holomorphically convex, it is weakly pseudoconvex by Theorem 6.1.22. Our assertion follows from Proposition 6.1.20. ■

A related result with stronger hypotheses and stronger conclusions is the following.

6.3.4 Theorem (Holomorphic convexity and global coordinate functions imply strong pseudoconvexity) *Let* M *be a second countable Hausdorff holomorphic manifold, let* $K \subseteq M$ *be compact, and let* $\mathcal{U}$ *be a neighbourhood of* K*. If* M *is holomorphically convex and possesses global coordinate functions, then there exists* $u \in C^\infty(M) \cap \mathrm{SPsh}(M)$ *such that*

(i) u *is an exhaustion function,*

(ii) $u(z) < 0$ *for every* $z \in K$*, and*

(iii) $u(z) > 0$ *for every* $z \in M \setminus \mathcal{U}$*.*

Proof By Proposition 6.1.5(iii) let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of M such that

1. $K_1 = \mathrm{hconv}_M(K)$,
2. $\mathrm{hconv}_M(K_j) = K_j$,
3. $K_j \subseteq \mathrm{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$, and
4. $M = \cup_{j\in\mathbb{Z}_{>0}} K_j$.

For each $j \in \mathbb{Z}_{>0}$ let $\mathcal{U}_j$ be such that $K_j \subseteq \mathcal{U}_j \subseteq K_{j+1}$, this being possible by a simple construction using the fact that $K_j \subseteq \mathrm{int}(K_{j+1})$ and that K_j is compact. We may suppose, moreover, that $\mathcal{U}_1 \subseteq \mathcal{U}$. Denote $L_j = K_{j+2} \setminus \mathcal{U}_j$, $j \in \mathbb{Z}_{>0}$. For $w \in L_j$ let $f_{w,j} \in C^{\mathrm{hol}}(M)$ be such that $|f_{w,j}(w)| > \|f_{w,j}\|_{K_j}$, this being possible since $w \notin K_j$ and since $\mathrm{hconv}_M(K_j) = K_j$. By rescaling we can suppose that $\|f_{w,j}\|_{K_j} < 1$ and $|f_{w,j}(w)| > 1$. Also let $g_{w,1}, \dots, g_{w,n} \in C^{\mathrm{hol}}(M)$ be such that the map

$$w' \mapsto (g_{w,1}(w'), \dots, g_{w,n}(w'))$$

is a $\mathbb{C}$-chart on a neighbourhood of w for which w is mapped to $\mathbf{0}$. By rescaling if necessary, we can suppose that $\|g_{w,l}\|_{K_j} < 1$ for $l \in \{1, \dots, n\}$. Let $\mathcal{U}_{w,j}$ be a neighbourhood of w such that $|f_{w,j}(z)| > 1$ for all $z \in \mathcal{U}_{w,j}$. Since L_j is compact, let $w_1, \dots, w_{k_j}$ be such that $L_j \subseteq \cup_{l=1}^{k_j} \mathcal{U}_{w_l,j}$. Then, for all $z \in L_j$ we have

$$\max(\{|f_{w_l,j}(z)| \mid l \in \{1, \dots, k_j\}\} \cup \{|g_{w_l,r}(z)| \mid l \in \{1, \dots, k_j\},\ r \in \{1, \dots, n\}\}) > 1$$

and for all $z \in K_j$ we have

$$|f_{w_l,j}(z)| < 1, \qquad l \in \{1, \dots, k_j\},$$

and

$$|g_{w_l,r}(z)| < 1, \qquad l \in \{1, \dots, k_j\},\ r \in \{1, \dots, n\}.$$

We combine the two families

$$(f_{w_1,j}, \dots, f_{w_{k_j},j}), \quad (g_{w_1,1}, \dots, g_{w_1,n}, \dots, g_{w_{k_j},1}, \dots, g_{w_{k_j},n})$$

into a single family that, with an abuse of notation, we denote by

$$(f_{w_1,j}, \dots, f_{w_{k_j},j}).$$

Now choose $m_j \in \mathbb{Z}_{>0}$ sufficiently large that

$$\sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j} \geq j, \qquad z \in L_j$$

and

$$\sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j} \leq \frac{1}{2^j}, \qquad z \in K_j.$$

Now let

$$u(z) = \sum_{j=1}^{\infty} \sum_{l=1}^{k_j} |f_{w_l,j}(z)|^{2m_j} - 1, \qquad z \in \mathsf{M}.$$

By Lemma 1 from the proof of Theorem 6.1.22, u is real analytic and plurisubharmonic. Following the proof of that lemma, we showed that u is also an exhaustion function. Moreover, by construction, $u(z) < 0$ for $z \in K$ and $u(z) > 0$ for $z \in \mathsf{M} \setminus \mathcal{U}$.

It remains to show that u is strictly plurisubharmonic. Suppose that $\mathrm{Lev}(u)(Z_z, Z_z) = 0$ for some $Z_z \in \mathsf{TM}$. As we saw in the proof of Lemma 1 from the proof of Theorem 6.1.22,

$$\mathrm{Lev}(u)(Z_z) = \sum_{j=1}^{\infty} \sum_{l=1}^{k_j} |Z_z f_{w_l,j}|^2.$$

Thus we have $Z_z f_{w_l,j} = 0$ for every $l \in \{1, \dots, k_j\}$, $j \in \mathbb{Z}_{>0}$. Since the list of functions $f_{w_l,j}$, $l \in \{1, \dots, k_j\}$, $j \in \mathbb{Z}_{>0}$, contains functions that form a $\mathbb{C}$-chart about z, it follows that $Z_z = 0_z$, giving strictly plurisubharmonicity of u as desired. ■

Now we can prove a converse to results such as the preceding two. While the preceding results are more or less easy to prove, the proof we give of the following result depends on the results of Section 6.2.

6.3.5 Theorem (The Levi problem on manifolds) *If* M *is a second countable, strongly pseudoconvex holomorphic manifold, then* M *is holomorphically convex, holomorphically separable, and possesses global coordinate functions.*

Proof We first prove a lemma.

1 Lemma *If* M *is a holomorphic manifold, if* u *is a smooth strictly plurisubharmonic exhaustion function on* M*, and if* $z_0 \in \mathsf{M}$*, there exists a neighbourhood* $\mathcal{U}_0$ *of* z_0 *and* $f_0 \in C^{\mathrm{hol}}(\mathcal{U}_0)$ *such that* $f_0(z_0) = 0$ *and*

$$\mathrm{Re}(f_0(z)) < u(z) - u(z_0)$$

for all $z \in \mathcal{U}_0 \setminus \{z_0\}$.

Proof Let $(\mathcal{U}, \phi)$ be a $\mathbb{C}$-chart for M about z_0 such that $\phi(z_0) = \mathbf{0}$. By Lemma 3.2.10 we can then write the Taylor series for the local representative u_ϕ as

$$u_\phi(z) = u_\phi(\mathbf{0}) + \mathrm{Re}(P(z)) + \mathrm{Lev}(u_\phi)(\mathbf{0}; z) + O(\|z\|^3)$$

for a polynomial P of degree at most two and satisfying $P(\mathbf{0}) = 0$. Since $\mathrm{Lev}(u_\phi)$ is positive-definite,

$$\mathrm{Re}(P(z)) < u_\phi(z) - u_\phi(\mathbf{0})$$

for z in some neighbourhood of $\mathbf{0}$. Taking f_0 to have local representative P gives the result. ▼

Let u be a strictly plurisubharmonic exhaustion function on M. Let $z_0, z_1 \in \mathsf{M}$ be distinct and choose a neighbourhood $\mathcal{U}_0$ of z_0 and $f_0 \in C^{\mathrm{hol}}(\mathcal{U}_0)$ as in the lemma. We suppose, without loss of generality, that $u(z_1) \leq u(z_0)$. We may also suppose that $z_1 \notin \mathcal{U}_0$, cf. the proof of the lemma. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be neighbourhoods of z_0 satisfying

$$\mathrm{cl}(\mathcal{U}_1) \subseteq \mathcal{U}_2 \subseteq \mathrm{cl}(\mathcal{U}_2) \subseteq \mathcal{U}_0.$$

By the Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8], let $\psi \in C^\infty(\mathsf{M})$ be such that $\mathrm{supp}(\psi) \subseteq \mathcal{U}_2$ and $\psi(z) = 1$ for z in some neighbourhood of $\mathcal{U}_1$. Let $a > u(z_0)$ and let

$$z \in \mathrm{supp}(\bar{\partial}\psi) \cap \{w \in \mathsf{M} \mid u(w) < a\}.$$

By choosing a sufficiently small, continuity of u ensures that such a z exists. Then, since $z \in \mathrm{supp}(\bar{\partial}\psi) \subseteq \mathrm{cl}(\mathcal{U}_2) \setminus \mathcal{U}_1 \subseteq \mathcal{U}_0 \setminus \{z_0\}$, $\mathrm{Re}(f_0(z)) < u(z) - u(z_0)$. Therefore, $u(z_0) > a$,

$$\mathrm{Re}(f_0(z)) < a - a = 0.$$

By compactness of $\mathrm{supp}(\bar{\partial}\psi)$, there exists $\epsilon \in \mathbb{R}_{>0}$ such that

$$\mathrm{Re}(f_0(z)) < -\epsilon, \qquad z \in \mathrm{supp}(\bar{\partial}\psi) \cap \{w \in \mathsf{M} \mid u(w) < a\}. \tag{6.25}$$

Define

$$\mathsf{M}_a = \{z \in \mathsf{M} \mid u(z) < a\}.$$

By Proposition 6.1.21(iv) and Theorem 6.2.13 there exists $\varphi_a \in C^\infty(\mathsf{M}_a)$ such that, if $g \in \mathsf{L}^2_{\mathrm{loc}}(\bigwedge^{0,1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M}_a))$ satisfies $\bar{\partial}g = 0$, then there exists $v \in \mathsf{L}^2_{\mathrm{loc}}(\mathsf{M}_a, \varphi_a)$ for which $\bar{\partial}v = g$ and $\|v\|_{\varphi_a} \leq \|g\|_{\varphi_a}$. Moreover, by Theorem 6.2.14 and the argument of the proof of Theorem 6.2.15, if g is of class C^∞, then so too will be v.

Now, with f_0 and ψ as above, and for $f \in C^{\mathrm{hol}}(\mathcal{U}_0)$ and $t \in \mathbb{R}_{>0}$, define

$$g_t = f e^{t f_0} \bar{\partial}\psi.$$

Note that $g_t = \bar{\partial}(f e^{t f_0} \psi)$ and so $\bar{\partial} g_t = 0$ by Proposition 4.6.4. By (6.25) we have

$$\|g_t\|_{\varphi_a} \leq C_1 e^{-\epsilon t}$$

for some $C_1 \in \mathbb{R}_{>0}$. Thus there exists v_t such that $\bar{\partial} v_t = g_t$ and

$$\|v_t\|_{\varphi_a} \leq C_2 e^{-\epsilon t}$$

for some $C_2 \in \mathbb{R}_{>0}$. Since $g_t(z) = 0$ for $z \in \mathsf{M}_a \setminus \operatorname{supp}(\bar{\partial}\psi)$, v_t is holomorphic in $\mathsf{M}_a \setminus \operatorname{supp}(\bar{\partial}\psi)$. Finally, define

$$f_t = f e^{t f_0} \psi - v_t, \tag{6.26}$$

and note that f_t is also holomorphic in $\mathsf{M}_a \setminus \operatorname{supp}(\bar{\partial}\psi)$.

By applying Lemma 1 from the proof of Theorem GA2.7.1.7, along with the discussion following that lemma, we have

$$\lim_{t\to\infty} f_t(z_1) = -\lim_{t\to\infty} v_t(z_1) = 0$$

(noting that $z_1 \in \mathsf{M}_a$ since we are assuming that $u(z_1) \leq u(z_0)$) and

$$\lim_{t\to\infty} f_t(z_0) = f(z_0) - \lim_{t\to\infty} v_t(z_0) = f(z_0).$$

Now take f so that $f(z_0) = 1$ so that $f_t(z_0) \neq f_t(z_1)$ for t sufficiently large. By Theorem GA2.7.1.7 it follows that there exists $g \in C^{\mathrm{hol}}(\mathsf{M})$, approximating f_t on $\mathrm{cl}(\mathsf{M}_{\varphi(z_0)})$ closely enough that $g(z_0) \neq g(z_1)$. Thus M is holomorphically separable.

For $a \in \mathbb{R}$ let us denote

$$\overline{\mathsf{M}}_a = \{z \in \mathsf{M} \mid \; u(z) \leq a\} = \mathrm{cl}(\mathsf{M}_a).$$

We claim that $\operatorname{hconv}_{\mathsf{M}}(\overline{\mathsf{M}}_a) = \overline{\mathsf{M}}_a$ for every $a \in \mathbb{R}$. To see this, let $z_0 \notin \overline{\mathsf{M}}_a$ and let $a' \in (a, u(z_0))$. By our constructions above, and using z_0 as used in these constructions, we have a family f_t, $r \in \mathbb{R}_{>0}$, of smooth functions on M for which

$$\lim_{t\to\infty} f_t(z_0) = 1$$

and

$$\lim_{t\to\infty} \int_{\mathsf{M}_{a'}} |f_t(z)|^2 \, \mathrm{d}\mu(z) = 0,$$

(using the Dominated Convergence Theorem). The arguments from the proof of Theorem GA2.7.1.7 show that f_t converges as $t \to \infty$ uniformly to zero on compact sets. In particular, then f_t converges uniformly to zero on $\overline{\mathsf{M}}_a$. Choose t sufficiently large that $\|f_t\|_{\overline{\mathsf{M}}_a} < \frac{1}{2}$ and $f_t(z_0) > \frac{1}{2}$. Using Theorem GA2.7.1.7 we can approximate f_t uniformly on $\overline{\mathsf{M}}_{\alpha'}$ by functions holomorphic on M. Thus there exists $g \in C^{\mathrm{hol}}(\mathsf{M})$ for which $|g(z_0)| > \|g\|_{\overline{\mathsf{M}}_a}$. Thus $z_0 \notin \operatorname{hconv}_{\mathsf{M}}(\overline{\mathsf{M}}_a)$, as desired.

Next we show that our conclusion of the preceding paragraph implies that M is holomorphically convex. Let $K \subseteq \mathsf{M}$ be convex. Since u is an exhaustion function, there exists $a \in \mathbb{R}$ such that $K \subseteq \overline{\mathsf{M}}_a$. By Proposition 6.1.2(ii), and the previous paragraph, $\operatorname{hconv}_{\mathsf{M}}(K) \subseteq \overline{\mathsf{M}}_a$. Since $\overline{\mathsf{M}}_a$ is compact (u being an exhaustion function) and since $\operatorname{hconv}_{\mathsf{M}}(K)$

is closed by Proposition 6.1.2(iv), it follows that $\mathrm{hconv}_{\mathsf{M}}(K)$ is compact by [Runde 2005, Proposition 3.3.6]. Thus M is holomorphically convex.

Finally, we show that M possesses global coordinate functions. We again make reference to the constructions of the first part of the proof, i.e., that portion of the proof following the lemma. In particular, we start with $z_0 \in \mathsf{M}$. By shrinking $\mathcal{U}_0$ if necessary, we choose $f^1, \dots, f^n \in \mathrm{C}^{\mathrm{hol}}(\mathcal{U}_0)$ such that $z \mapsto (f^1(z), \dots, f^n(z))$ forms a coordinate system in a neighbourhood of z_0 mapping z_0 to $\mathbf{0}$. Then define smooth functions $f_t^1, \dots, f_t^n$ on M as in (6.26), i.e., replace "f" in (6.26) with "f^j", $j \in \{1, \dots, n\}$. Note that

$$\partial f_t^j(z_0) = \partial f^f(z_0) - \partial g_t(z_0), \qquad j \in \{1, \dots, n\},$$

and so, by our arguments above,

$$\lim_{t\to\infty} \partial f_t^j(z_0) = \partial f^j(z_0), \qquad j \in \{1, \dots, n\}. \tag{6.27}$$

Let J_t be the holomorphic function defined in a neighbourhood of z_0 by computing the Jacobian determinant with respect to the coordinates $z \mapsto (f^1(z), \dots, f^n(z))$ of the $\mathbb{C}^n$-valued function $z \mapsto (f_t^1(z), \dots, f_t^n(z))$. By (6.27) it follows that $\lim_{t\to\infty} J_t(z_0) = 1$. Thus, for t sufficiently large, we have that $z \mapsto (f_t^1(z), \dots, f_t^n(z))$ are globally defined coordinates for M about z_0. ■

6.3.3 Stein manifolds and their basic properties

There are many ways one can characterise Stein manifolds, but they all amount share being equivalent to the fact that a manifold is a Stein manifold if and only if it is strongly pseudoconvex. The results of Section 6.3.2, along with Lemma 6.3.2, lead to many equivalent such characterisations, all of which appear in the literature in various places. We formulate our definition of a Stein manifold to capture all of these equivalent characterisations, just to summarise what they are so that we can use the characterisation that is most convenient for our various purposes.

6.3.6 Definition (Equivalent definitions of "Stein manifold") A second countable holomorphic manifold M is a ***Stein manifold*** if it satisfies one of the following equivalent definitions:

(i) M is holomorphically convex and holomorphically spreadable;

(ii) M is holomorphically convex and locally holomorphically separable;

(iii) M is holomorphically convex and holomorphically separable;

(iv) M is holomorphically convex and possesses global coordinate functions;

(v) M is strongly pseudoconvex. •

As we shall see as we go along, Stein manifolds possess many deep and ultimately useful properties. In this section we shall present some of the simpler properties of Stein manifolds, most of which follow directly from their defining properties.

Let us consider some examples of Stein manifolds.

6.3.7 Examples (Stein manifolds and not Stein manifolds)

1. If $\mathcal{U} \subseteq \mathbb{C}^n$ is an open set, then the coordinate functions are holomorphic functions on $\mathcal{U}$ that provide a global $\mathbb{C}$-chart for $\mathcal{U}$. Thus an open subset of $\mathbb{C}^n$ is a Stein manifold if and only if it is holomorphically convex. Thus, by Theorems 3.1.10 and 3.1.13, the open subsets of $\mathbb{C}^n$ that are Stein manifolds are the domains of holomorphy.
2. The only compact holomorphic manifolds that are Stein manifolds are those of zero dimension. Indeed, holomorphic functions on a connected compact manifold are constant by Corollary 4.2.11, and this precludes a compact holomorphic manifold from being locally holomorphically separable, even though it is holomorphically convex.
3. The holomorphic manifold from Example 6.1.23 is not Stein, although it is weakly pseudoconvex. In that example, the manifold is not holomorphically convex. •

The following result is a consequence of Propositions 6.1.20 and 6.1.21.

6.3.8 Proposition (Basic properties of Stein manifolds) *If* M *and* N *are Stein manifolds, then the following statements hold:*

(i) $\mathsf{M} \times \mathsf{N}$ *is a Stein manifold;*

(ii) if M *is a Stein manifold and if* $\mathsf{S} \subseteq \mathsf{M}$ *is a closed holomorphic submanifold, then* S *is a Stein manifold;*

(iii) if $\mathsf{S}_1, \dots, \mathsf{S}_k \subseteq \mathsf{M}$ *are Stein submanifolds for which* $\mathsf{S} = \cap_{j=1}^{k} \mathsf{S}_j$ *is a holomorphic submanifold, then* S *is a Stein manifold;*

(iv) if $\Phi\colon \mathsf{M} \to \mathsf{N}$ *is holomorphic, if* $\mathsf{S} \subseteq \mathsf{M}$ *and* $\mathsf{T} \subseteq \mathsf{N}$ *are Stein submanifolds, and if* $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ *is a holomorphic submanifold of* M*, then* $\mathsf{S} \cap \Phi^{-1}(\mathsf{T})$ *is a Stein manifold;*

(v) if M *is a second countable Stein manifold and if* $u \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^\infty(\mathsf{M})$*, then* $u^{-1}([-\infty, \alpha))$ *is a Stein manifold for every* $\alpha \in \mathbb{R}$*.*

The following result gives a criterion for a holomorphically convex manifold to be Stein. We refer to Chapter GA2.6 for a discussion of analytic sets.

6.3.9 Proposition (Analytic subsets of Stein manifolds) *If* M *is holomorphically convex and if there are no compact analytic subsets of positive dimension, then* M *is Stein.*

Proof Let $z_0 \in \mathsf{M}$ so $K = \mathrm{hconv}_{\mathsf{M}}\{z_0\}$ is compact. Let $\mathcal{U}$ be a relatively compact neighbourhood of K. For all $z \in \mathrm{bd}(\mathcal{U})$ there exists $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ with $|f(z)| > |f(z_0)|$. Since $\mathrm{bd}(\mathcal{U})$ is compact there exists $f_1, \dots, f_k \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that, for $z \in \mathrm{bd}(\mathcal{D})$, there exists $j \in \{1, \dots, k\}$ such that $|f_j(z)| > |f_j(z_0)|$. Let

$$\mathsf{S} = \{z \in \mathsf{M} \mid f_j(z) = f_j(z_0)\}.$$

Then $\mathsf{S} \cap \mathcal{D}$ is a closed subset of M and hence is a closed analytic subset of M contained in $\mathrm{cl}(\mathcal{D})$. Thus $\mathsf{Z} \cap \mathcal{D}$ is finite and so M is holomorphically spreadable. ■

For Stein manifolds, one can also relate the holomorphically convex hull with the plurisubharmonic convex hull.

6.3.10 Proposition (Convex hulls for Stein manifolds) *If* M *is a Stein manifold and if* $\mathrm{K} \subseteq \mathrm{M}$ *is convex, then* $\mathrm{hconv}_{\mathsf{M}}(\mathrm{K}) = \mathrm{pconv}_{\mathsf{M}}(\mathrm{K})$.

Proof Let $z \notin \mathrm{hconv}_{\mathsf{M}}(K)$ so that there exists $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ for which $|f|(z) > |||f|||_K$. This implies that $|f|^2(z) > |||f|^2||_K$, and since $|f|^2$ is plurisubharmonic (cf. the proof of Lemma 1 from the proof of Theorem 6.1.22) it follows that $z \notin \mathrm{pconv}_{\mathsf{M}}(K)$. This proves that $\mathrm{pconv}_{\mathsf{M}}(K) \subseteq \mathrm{hconv}_{\mathsf{M}}(K)$.

For the opposite inclusion, suppose that $z \notin \mathrm{pconv}_{\mathsf{M}}(K)$. Thus there exists $u \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^0(\mathsf{M})$ such that $u(z) > \sup_K u$. We can then choose $\epsilon \in \mathbb{R}_{>0}$ sufficiently small that

$$u(z) + \epsilon u_0(z) > \sup_K (u + \epsilon u_0),$$

where u_0 is a smooth strictly plurisubharmonic exhaustion function (which exists since M is Stein). Now we can use Theorem GA2.7.1.5 to infer the existence of $v \in \mathrm{Psh}(\mathsf{M}) \cap \mathrm{C}^\infty(\mathsf{M})$ such that $v(z) > \sup_K v$. Let $c \in (\sup_K v, v(z))$. As we proved during the course of the proof of Theorem 6.3.5, $v^{-1}((-\infty, c])$ is holomorphically convex. Since $c < \sup_K v$ we have $K \subseteq v^{-1}((-\infty, c])$. Therefore,

$$\mathrm{hconv}_{\mathsf{M}}(K) \subseteq \mathrm{hconv}_{\mathsf{M}}(v^{-1}((-\infty, c])) = v^{-1}((-\infty, c]).$$

Thus $z \notin \mathrm{hconv}_{\mathsf{M}}(K)$ since $v(z) > c$. ■

6.4 Real analytic manifolds

In this section we apply techniques from the theory of Stein manifold to say useful things about real analytic manifolds. As we shall see, real analytic manifolds should not be thought of as being analogous to general holomorphic manifolds, but rather only to holomorphic manifolds that are Stein. Thus real analytic manifolds do not share the sorts of restrictions on their real analytic functions that one can expect for general holomorphic manifolds.

6.4.1 Totally real submanifolds of holomorphic manifolds

In this section we consider a generalisation of the notion of the fact that we have a natural embedding

$$(x^1, \dots, x^n) \mapsto (x^1 + \mathrm{i}0, \dots, x^n + \mathrm{i}0)$$

of $\mathbb{R}^n$ as the "real part" of $\mathbb{C}^n$. To generalise this idea to a holomorphic manifold is no longer natural. For example, were one to attempt to formulate the definition of a real submanifold as being one whose image under a $\mathbb{C}$-chart lies in $\mathbb{R}^n \subseteq \mathbb{C}^n$, this notion fails to be chart independent. The following definition captures the notion we are after, using the complex structure on a holomorphic manifold as per Definition 4.5.7 and the notion of a totally real subspace from Section 4.1.7.

6.4.1 Definition (Totally real submanifold) Let M be a holomorphic manifold with complex structure J. A smooth real submanifold S of M, i.e., S is a submanifold of the smooth manifold M, is ***totally real*** if $\mathsf{T}_z\mathsf{S}$ is a totally real subspace of $\mathsf{T}_x\mathsf{M}$ for every $z \in \mathsf{S}$. •

Totally real submanifolds are distinguished by the fact that they admit particular coordinate charts.

6.4.2 Lemma ($\mathbb{C}$-charts near totally real submanifolds) *Let* M *be a holomorphic manifold with complex structure* J *and let* S *be a smooth totally real submanifold. Then, for* $\mathrm{z}_0 \in \mathsf{S}$*, there exists a* $\mathbb{C}$*-chart* $(\mathcal{U}, \phi)$ *for* M *about* z *with the following properties:*

(i) ϕ *takes values in* $\mathbb{C}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^k \times \mathbb{R}^{2n-k}$*, and we denote coordinates by* $\phi(\mathrm{z}) = \mathbf{z}(\mathrm{z}) = \mathbf{x}(\mathrm{z}) + \mathrm{i}\mathbf{y}(\mathrm{z}) \in \mathbb{C}^n$*,* $\phi(\mathrm{z}) = (\mathbf{x}(\mathrm{z}), \mathbf{y}(\mathrm{z})) \in \mathbb{R}^n \times \mathbb{R}^n$*, or* $\phi(\mathrm{z}) = (\boldsymbol{\xi}(\mathrm{z}), \boldsymbol{\eta}(\mathrm{z})) \in \mathbb{R}^k \times \mathbb{R}^{2n-k}$*;*

(ii) $\phi(\mathrm{z}_0) = \mathbf{0}$*;*

(iii) $\mathsf{T}_{\mathbf{0}}(\phi(\mathcal{U} \cap \mathsf{S})) = \mathsf{T}_{\mathbf{0}}\mathbb{R}^k \subseteq \mathsf{T}_{\mathbf{0}}(\mathbb{R}^k \times \mathbb{R}^{2n-k})$.

If, moreover, S *is additionally a real analytic submanifold, then* $(\mathcal{U}, \phi)$ *may be additionally chosen so that*

(iv) $\phi(\mathcal{U}|\mathsf{S}) \subseteq \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \mid \boldsymbol{\xi} = \mathbf{0}\}$.

Proof Let $(\mathcal{U}, \psi)$ be any $\mathbb{C}$-chart about z_0 such that $\psi(z_0) = \mathbf{0}$. Since $\psi(\mathcal{V} \cap \mathsf{S})$ is a totally real submanifold of $\psi(\mathcal{V}) \subseteq \mathbb{C}^n$, $\mathsf{T}_{\mathbf{0}}(\psi(\mathcal{V} \cap \mathsf{S}))$ is a totally real subspace of $\mathsf{T}_{\mathbf{0}}\mathbb{C}^n \simeq \mathbb{C}^n$. Let $(v_1, \dots, v_k)$ be a (real) basis for $\mathsf{T}_{\mathbf{0}}(\psi(\mathcal{V} \cap \mathsf{S}))$. By Lemma 4.1.27 there exists $v_{k+1}, \dots, v_n \in \mathbb{C}^n$ such that

$$(v_1, \dots, v_n, \boldsymbol{J}(v_1), \dots, \boldsymbol{J}(v_n))$$

is $\boldsymbol{J}$-adapted, where $\boldsymbol{J}$ denotes the standard linear complex structure on $\mathbb{C}^n$. Thus, by Proposition 4.1.6, the map $\boldsymbol{L}\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\boldsymbol{L}\Big(\sum_{j=1}^n a^j v_j + \sum_{j=1}^n b^j \boldsymbol{J}(v_j)\Big) = (a^1 + \mathrm{i}b^1, \dots, a^n + \mathrm{i}b^n)$$

is $\mathbb{C}$-linear. Then the map $\phi\colon \mathcal{U} \to \mathbb{C}^n$ defined by $\phi(z) = \boldsymbol{L} \circ \psi(z)$ has the properties (i)–(iii), as can be verified directly.

For the final assertion of the lemma it is convenient to rename the coordinates $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^k \times \mathbb{R}^{2n-k}$ as $(z_1, z_2) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ as follows:

$$\begin{aligned} z_1^j &= \xi^j + \mathrm{i}\eta^{k+j}, \qquad & j \in \{1, \dots, k\}, \\ z_2^j &= \eta^j + \mathrm{i}\eta^{n+j}, \qquad & j \in \{1, \dots, n-k\}. \end{aligned}$$

We also denote $x_1 = \mathrm{Re}(z_1)$, $y_1 = \mathrm{Im}(z_1)$, $x_2 = \mathrm{Re}(z_2)$, and $y_1 = \mathrm{Im}(z_2)$. Then, from the first part of the proof and by shrinking $\mathcal{U}$ if necessary, we have real analytic maps $\chi_1\colon \phi(\mathcal{U}) \cap \mathbb{R}^k \to \mathbb{R}^k$ and $\chi_2\colon \phi(\mathcal{U}) \cap \mathbb{R}^k \to \mathbb{C}^{n-k}$ with the property that

$$\phi(\mathcal{U} \cap \mathsf{S}) = \{(x_1 + \mathrm{i}y_1, z_2) \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid y_1 = \chi_1(x_1),\ z_2 = \chi_2(x_1),\ x_1 \in \phi(\mathcal{U}) \cap \mathbb{R}^k\}.$$

Moreover, χ_1 and χ_2 and their first derivatives vanish at $\mathbf{0} \in \mathbb{R}^k$. By Lemma 1 from the proof of Proposition 6.4.3 below, there exists a neighbourhood $\overline{\mathcal{U}}$ of $\mathbf{0} \in \mathbb{C}^k$ and holomorphic maps $\overline{\chi}_1\colon \overline{\mathcal{U}} \to \mathbb{C}^k$ and $\overline{\chi}_2\colon \overline{\mathcal{U}} \to \mathbb{C}^{n-k}$ that agree with χ_1 and χ_2, respectively, on $\overline{\mathcal{U}} \cap \mathbb{R}^k$. Define $\boldsymbol{\Phi}\colon \overline{\mathcal{U}} \times \overline{\mathcal{U}} \to \mathbb{C}^k$ by

$$\boldsymbol{\Phi}(z_1, \zeta_1) = z_1 - \zeta_1 - \mathrm{i}\overline{\chi}_1(\zeta_1).$$

By the holomorphic Implicit Function Theorem, Theorem 1.2.7, we can write ζ_1 as a holomorphic function of z_1 and such that $\mathbf{\Phi}(z_1, \zeta_1(z_1)) = \mathbf{0}$. We have

$$\mathrm{Re}(\zeta_1(z_1)) = \mathrm{Re}(z_1) + \mathrm{Im}(\overline{\chi}_1(\mathrm{Re}(\zeta_1(z_1)))) = \mathrm{Re}(z_1)$$

since $\overline{\chi}_1$ is real for real arguments. Therefore, we have that $\mathrm{Im}(\zeta_1(z_1)) = \mathbf{0}$ if and only if

$$\mathbf{0} = \mathrm{Im}(z_1) - \mathrm{Re}(\overline{\chi}_1(\mathrm{Re}(\zeta_1(z_1))) = y_1 - \chi_1(\mathrm{Re}(\zeta_1(z_1))) = y_1 - \chi_1(\mathrm{Re}(z_1)),$$

Let us also define

$$\zeta_2(z_1, z_2) = z_2 - \overline{\chi}_2(\zeta_1(z_1)).$$

If $\mathrm{Im}(\zeta(z_1)) = \mathbf{0}$ then

$$\zeta_2(z_1, z_2) = z_2 - \overline{\chi}_2(\mathrm{Re}(\zeta_1(z_1))) = \chi_1(\mathrm{Re}(z_1)).$$

so that $(z_1, z_2) \in \phi(\mathcal{U} \cap \mathsf{S}) \cap \overline{\mathcal{U}}$ if and only if $\mathrm{Im}(\zeta_1(z_1)) = \mathbf{0}$ and $\zeta_2(z_1, z_2) = \mathbf{0}$. Thus the coordinates (ζ_1, ζ_2) achieve the desired result. ■

The lemma says, roughly, that totally real submanifolds are linearly approximated by real subspaces of $\mathbb{C}^n$ (this is unsurprising) and, if the submanifold is real analytic, this approximation is not only linear, but local.

It is also possible to extend real analytic maps from real analytic totally real submanifolds.

6.4.3 Proposition (Extension of real analytic maps) *Let M and N be holomorphic manifolds, let $\mathsf{S} \subseteq \mathsf{M}$ be a totally real real analytic submanifold for which $\dim_{\mathbb{R}}(\mathsf{S}) = \dim_{\mathbb{C}}(\mathsf{M})$, and let $\Phi \in \mathsf{C}^{\omega}(\mathsf{S}; \mathsf{N})$ (thinking of N also as a real analytic manifold). Then there exists a neighbourhood $\mathcal{U}$ of S in M and $\overline{\Phi} \in \mathsf{C}^{\mathrm{hol}}(\mathcal{U}; \mathsf{N})$ such that $\overline{\Phi}|\mathsf{S} = \Phi$. Moreover, if $\mathcal{U}$ and $\mathcal{U}'$ are two neighbourhoods of S in M and if $\overline{\Phi} \in \mathsf{C}^{\mathrm{hol}}(\mathcal{U}; \mathsf{N})$ and $\overline{\Phi}' \in \mathsf{C}^{\mathrm{hol}}(\mathcal{U}'; \mathsf{N})$ satisfy $\overline{\Phi}|\mathsf{S} = \overline{\Phi}'|\mathsf{S} = \Phi$, then $\overline{\Phi}$ and $\overline{\Phi}'$ agree on any connected component of $\mathcal{U} \cap \mathcal{U}'$ that intersects S.*

Moreover, if $\dim_{\mathbb{C}}(\mathsf{M}) = \dim_{\mathbb{C}}(\mathsf{N})$ and if Φ is a real analytic diffeomorphism onto a totally real submanifold of N, then the neighbourhood $\mathcal{U}$ and the extension $\overline{\Phi}$ can be chosen so that $\overline{\Phi}$ is a holomorphic diffeomorphism onto its image.

Proof Let us first do this for Euclidean space, using the following (usual) identification of $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$:

$$(z_1, \dots, z_n) \mapsto ((\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_n)), (\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_n))). \tag{6.28}$$

With this convention, we have the following result.

1 Lemma *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and let $\mathbf{\Phi} \in \mathsf{C}^{\omega}(\mathcal{U}; \mathbb{C}^m)$. Then there exists a neighbourhood $\overline{\mathcal{U}} \subseteq \mathbb{C}^n$ of $\mathcal{U} \times \{\mathbf{0}\}$ and $\overline{\mathbf{\Phi}} \in \mathsf{C}^{\mathrm{hol}}(\overline{\mathcal{U}}; \mathbb{C}^m)$ such that $\overline{\mathbf{\Phi}}|\mathcal{U} \times \{\mathbf{0}\} = \mathbf{\Phi}$. Moreover, for any other neighbourhood $\overline{\mathcal{U}}'$ of $\mathcal{U} \times \{\mathbf{0}\}$ in $\mathbb{C}^n$ and for any other mapping $\overline{\mathbf{\Phi}}' \in \mathsf{C}^{\mathrm{hol}}(\overline{\mathcal{U}}'; \mathbb{C}^m)$ for which $\overline{\mathbf{\Phi}}'|\mathcal{U} \times \{\mathbf{0}\} = \mathbf{\Phi}$, $\overline{\mathbf{\Phi}}$ and $\overline{\mathbf{\Phi}}'$ agree on any connected component of $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$ intersecting $\mathcal{U}$.*

Moreover, if $\mathrm{m} = \mathrm{n}$ and if $\mathbf{\Phi}$ is a real analytic diffeomorphism onto its image in $\mathbb{R}^{\mathrm{m}} \times \{\mathbf{0}\}$, then $\overline{\mathcal{U}}$ and the extension $\overline{\mathbf{\Phi}}$ can be chosen so that $\overline{\mathbf{\Phi}}$ is a holomorphic diffeomorphism onto its image.

Proof Let $x \in \mathcal{U}$ and let $\mathcal{V}_x \subseteq \mathcal{U}$ be a neighbourhood of x in $\mathbb{R}^n$ such that the Taylor series for $\mathbf{\Phi}$ converges on $\mathcal{V}_x$. Thus, for each $y \in \mathcal{V}_x$, we have

$$\sum_{k=0}^{\infty} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} \frac{\partial^{|I|} \Phi^j}{\partial x^I}(x)(y - x)^I, \qquad j \in \{1, \dots, m\}.$$

Then, by Theorem 1.1.17, there exists a neighbourhood $\overline{\mathcal{V}}_x$ of $(x, \mathbf{0})$ in $\mathbb{C}^n$ such that

$$\sum_{k=0}^{\infty} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} \frac{\partial^{|I|} \Phi^j}{\partial x^I}(x_0)(z - (x, \mathbf{0}))^I, \qquad j \in \{1, \dots, m\},$$

converges for every $z \in \overline{\mathcal{V}}_x$. Let $\overline{\mathbf{\Phi}}_x \in \mathrm{C}^{\mathrm{hol}}(\overline{\mathcal{V}}_x; \mathbb{C}^m)$ be the holomorphic function defined by the preceding Taylor series.

We claim that if $\overline{\mathcal{V}}$ is a neighbourhood of $(x, \mathbf{0})$ and if $\overline{\mathbf{\Psi}} \in \mathrm{C}^{\mathrm{hol}}(\overline{\mathcal{V}}; \mathbb{C}^m)$ agrees with $\overline{\mathbf{\Phi}}_x$ when restricted to a neighbourhood of x in $\mathbb{R}^n \times \{\mathbf{0}\}$, then $\overline{\mathbf{\Phi}}_x$ and $\overline{\mathbf{\Psi}}$ agree on the connected component of the intersection of their domains containing $(x, \mathbf{0})$. To see this, we note that the real Taylor series of $\overline{\mathbf{\Phi}}_x$ and $\overline{\mathbf{\Psi}}$, restricted to $\mathbb{R}^n \times \{\mathbf{0}\}$, must agree. Thus $\overline{\mathbf{\Phi}}_x$ and $\overline{\mathbf{\Psi}}$ have the same Taylor series at $(x, \mathbf{0})$, and so must agree on a neighbourhood of $(x, \mathbf{0})$. Our claim follows from Theorem 1.1.18.

Thus we have a family $(\overline{\mathcal{V}}_x)_{x \in \mathcal{U}}$ and $(\overline{\mathbf{\Phi}})_{x \in \mathcal{U}}$ of open sets and holomorphic functions defined on these open sets. Define $\overline{\mathcal{U}}$ to be the connected component of $\cup_{x \in \mathcal{U}} \overline{\mathcal{V}}_x$ containing $\mathcal{U} \times \{\mathbf{0}\}$ and define $\overline{\mathbf{\Phi}} \in \mathrm{C}^{\mathrm{hol}}(\overline{\mathcal{U}}; \mathbb{R})$ by asking that $\overline{\mathbf{\Phi}}(z) = \overline{\mathbf{\Phi}}_x(z)$ where $x \in \mathcal{U}$ is such that $z \in \overline{\mathcal{V}}_x$. By the second paragraph of the proof, $\overline{\mathbf{\Phi}}$ is well-defined on $\overline{\mathcal{U}}$ since $\overline{\mathcal{U}}$ is connected. That $\overline{\mathbf{\Phi}}$ is holomorphic follows by holomorphicity of the functions $\overline{\mathbf{\Phi}}_x$, $x \in \mathcal{U}$. The final assertion of the theorem follows from the Identity Theorem, Theorem 1.1.18.

The final assertion follows by the Inverse Function Theorem, Theorem 1.2.6, along with the fact that the complexification of a $\mathbb{R}$-linear isomorphism is a $\mathbb{C}$-linear isomorphism. ▼

For the general case, let $z \in \mathsf{S}$ and, by Lemma 6.4.2, choose a $\mathbb{C}$-chart $(\mathcal{U}_z, \phi_z)$ for M about z such that

$$\phi_z(\mathcal{U}_z \cap \mathsf{S}) \subseteq \{z \in \mathbb{C}^n \mid \operatorname{Im}(z) = \mathbf{0}\}.$$

Also choose a $\mathbb{C}$-chart $(\mathcal{V}_z, \psi_z)$ for N about $\Phi(z)$ and suppose that $\Phi(\mathcal{U}_z) \subseteq \mathcal{V}_z$. Then the local representative $\Phi_{\phi_z \psi_z} \colon z \mapsto \psi_z \circ \Phi \circ \psi_z^{-1}(z)$ satisfies the hypotheses of the lemma. By the conclusions of the lemma we can extend Φ to $\overline{\Phi}_z \in \mathrm{C}^{\mathrm{hol}}(\overline{\mathcal{U}}_z; \mathsf{N})$ defined on a neighbourhood $\overline{\mathcal{U}}_z$ of $\mathcal{U}_z$. We can argue as in the proof of the lemma that, by doing this for each $z \in \mathsf{S}$, we obtain a neighbourhood $\mathcal{U} = \cup_{z \in \mathsf{S}} \overline{\mathcal{U}}_z$ and a well defined map $\overline{\Phi} \in \mathrm{C}^{\mathrm{hol}}(\mathcal{U}; \mathsf{N})$ given by requiring that $\overline{\Phi}(w) = \overline{\Phi}_z(w)$ if $w \in \overline{\mathcal{U}}_z$. The Identity Theorem for manifolds, Theorem 4.2.5, gives the uniqueness assertion of the theorem.

The final assertion of the theorem follows from the final assertion of the lemma above, along with an application of Lemma 6.4.2 at $z \in \mathsf{S}$ and $\Phi(z) \in \mathsf{N}$. ■

An important property of totally real submanifolds is that they have Stein tubular neighbourhoods. This was first proved by [Grauert 1958] in, more or less, the case of totally real submanifolds of half the (real) dimension of M. The proof in general that we give is that of [Cieliebak and Eliashberg 2012]. To state the theorem and give the proof, we need to recall some facts about Riemannian manifolds.

We suppose that M is equipped with a Hermitian metric $\mathbb{H}$, and that the associated Riemannian metric is denoted by $\mathbb{G}$, cf. Section 4.1.5. We shall additionally suppose that M is connected. For an absolutely continuous curve $\gamma\colon [0,T] \to \mathsf{M}$ we define its ref
length by

$$\ell_{\mathbb{H}}(\gamma) = \int_0^T \mathbb{G}(\gamma'(t),\gamma'(t))^{1/2}\,\mathrm{d}t.$$

It is easily and directly shown that this definition of the length of a curve depends only on the trace of the curve and not its parameterisation. Thus we may as well suppose that $T = 1$ when we speak of the length of a curve. Thus, for $x, y \in \mathsf{M}$, we define the ***distance*** between x and y to be

$$\mathrm{d}_{\mathbb{H}}(x,y) = \inf\{\ell_{\mathbb{H}}(\gamma) \mid \gamma\colon [0,1] \to \mathsf{M} \text{ is absolutely continuous with } \gamma(0) = x,\ \gamma(1) = y\}.$$

One verifies that $\mathrm{d}_{\mathbb{H}}$ is a metric on M. Thus, for a subset $S \subseteq \mathsf{M}$, we can define $\mathrm{dist}_S\colon \mathsf{M} \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{dist}_S(x) = \inf\{\mathrm{d}_{\mathbb{H}}(x,y) \mid y \in S\}$$

as the distance function from S. With all this as backdrop, we state the following ref
theorem.

6.4.4 Theorem (Totally real submanifolds possess Stein neighbourhoods) *Let M be a paracompact holomorphic manifold, let $\mathbb{H}$ be an Hermitian metric on M with associated Riemannian metric $\mathbb{G}$, and let S be a properly embedded, totally real submanifold of M. Then*

(i) $\mathrm{dist}_{\mathsf{S}}^2$ *is strictly plurisubharmonic in a neighbourhood of S and*

(ii) for any neighbourhood $\mathcal{N}$ of S, there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{N}$ of S that is a Stein manifold with a smooth strictly pseudoconvex boundary, i.e., a boundary that is a level set of a strictly plurisubharmonic function.

Proof We suppose that M is connected, and so second countable [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.11]. If this is not so, the construction of the proof can be made on all connected components of M. We let $\mathsf{NS} \subseteq \mathsf{TM}|\mathsf{S}$ be the $\mathbb{G}$-normal bundle of S. For $z \in \mathsf{S}$ and $v_z \in \mathsf{N}_z\mathsf{S}$ define $\tau(v_z) = \gamma_{v_z}(1)$, where γ_{v_z} is the unique geodesic satisfying $\gamma'_{v_z} = v_z$. For $\rho\colon \mathsf{S} \to \mathbb{R}_{\geq 0}$ define

$$\mathsf{NS}_\rho = \{v_z \in \mathsf{NS} \mid \|v_z\|_{\mathbb{G}} < \rho(z)\}.$$

By the Tubular Neighbourhood Theorem [Bröcker and Jänich 1982, Theorem 12.11] there exists a continuous map $\rho\colon \mathsf{S} \to \mathbb{R}_{\geq 0}$ such that τ is a diffeomorphism from NS_ρ onto a neighbourhood of S in M. Moreover, by properties of geodesic normal coordinates [Kobayashi and Nomizu 1963, §IV.3], the distance from $z \in \mathsf{NS}_\rho$ to S is $\|\tau^{-1}(z)\|_{\mathbb{G}}$.

The following lemma gives particular coordinates for $\mathcal{T}_\rho \triangleq \tau(\mathsf{NS}_\rho)$ about every point in S.

1 Lemma *For each $z_0 \in \mathsf{S}$ there exists a chart $(\mathcal{U}, \phi)$ about z_0 with the following properties:*

(i) *ϕ takes values in $\mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^k \times \mathbb{R}^{2n-k}$, and we denote coordinates by $\phi(w) = (\mathbf{x}(w), \mathbf{y}(w)) \in \mathbb{R}^n \times \mathbb{R}^n$ or $\phi(w) = (\boldsymbol{\xi}(w), \boldsymbol{\eta}(w)) \in \mathbb{R}^k \times \mathbb{R}^{2n-k}$;*

(ii) $\mathsf{S} \cap \mathcal{U} = \{w \in \mathcal{U} \mid \boldsymbol{\eta}(w) = \mathbf{0}\}$;

(iii) *for $w \in \mathcal{U}$, $\mathrm{dist}_{\mathsf{S}}(w) = \|\boldsymbol{\eta}(w)\|$;*

(iv) *for each $z \in \mathsf{S} \cap \mathcal{U}$ the function $f_z \in C^\infty(\mathcal{U})$ given by*

$$f_z(w) = 2\|\boldsymbol{\eta}(w)\|^2 - \|\boldsymbol{\xi}(w) - \boldsymbol{\xi}(z)\|^2$$

is strictly plurisubharmonic on $\mathcal{U}$.

Proof Let $(\mathcal{V}, \psi)$ be a chart for S about z_0, and denote coordinates in this chart by $w \mapsto \boldsymbol{\xi}(w)$. We suppose this chart has been chosen so that $(\frac{\partial}{\partial \xi^1}(z_0), \dots, \frac{\partial}{\partial \xi^k}(z_0))$ is a $\mathbb{G}(z_0)$-orthonormal basis for $\mathsf{T}_{z_0}\mathsf{S}$. By Proposition 4.1.26 and the fact that $J(z_0)$ is $\mathbb{G}(z_0)$-orthogonal, it follows that

$$\frac{\partial}{\partial \xi^1}(z_0), \dots, \frac{\partial}{\partial \xi^k}(z_0), J\Big(\frac{\partial}{\partial \xi^1}(z_0)\Big), \dots, \Big(\frac{\partial}{\partial \xi^k}(z_0)\Big)$$

form an orthonormal basis for $\mathsf{T}_{z_0}\mathsf{S} \oplus J(\mathsf{T}_{z_0}\mathsf{S})$.

Now let $X_1, \dots, X_{n-k}$ be orthonormal sections of $\mathsf{N}\mathcal{V}$ that are orthogonal to $J(\mathsf{TS})$. Let $Y_1, \dots, Y_k$ be orthonormal sections of $\mathsf{N}\mathcal{V}$ for which $Y_j(z_0) = J(\frac{\partial}{\partial \xi^j})$, $j \in \{1, \dots, k\}$. Finally, let $Y_{k+1}, \dots, Y_n$ be orthonormal sections of $\mathsf{N}\mathcal{V}$ for which $Y_j(z_0) = J(X_{j-k}(z_0))$ for $j \in \{k+1, \dots, n\}$. Note that with these sections as defined, $J(z_0)$ has the matrix representation

$$\begin{bmatrix} \mathbf{0} & -\mathsf{I}_n \\ \boldsymbol{I}_n & \mathbf{0} \end{bmatrix}$$

in the basis

$$\frac{\partial}{\partial \xi^1}(z_0), \dots, \frac{\partial}{\partial \xi^k}(z_0), X_1(z_0), \dots, X_{n-k}(z_0), Y_1(z_0), \dots, Y_n(z_0)$$

for $\mathsf{T}_{z_0}\mathsf{M}$.

For $z \in \mathcal{V}$ and for $(\eta^1, \dots, \eta^{2n-k}) \in \mathbb{R}^{2n-k}$ sufficiently small, we can define

$$\exp_z(\eta^1, \dots, \eta^{2n-k}) = (\gamma_{Y_1(z)}(\eta_1), \dots, \gamma_{Y_{2n-k}(z)}(\eta_{2n-k})),$$

where $t \mapsto \gamma_v(t)$ denotes the geodesic of $\mathbb{G}$ for which $\gamma'(0) = v$. Moreover, the Tubular Neighbourhood Theorem shows that the map

$$\eta^1 Y_1(z) + \dots + \eta^{2n-k} Y_{2n-k}(z) \mapsto (\boldsymbol{\xi}(z), \boldsymbol{\eta})$$

is a diffeomorphism from a neighbourhood of $Z(\mathsf{NS})$ to a neighbourhood of S. Thus we have a chart $(\mathcal{U}, \phi)$ for M defined in a neighbourhood of z_0. Moreover, the distance from $z \in \mathcal{U}$ to S is exactly $\boldsymbol{\eta}(z)$.

It remains to show that f_z is strictly plurisubharmonic in a neighbourhood of z_0 for each z sufficiently close to z_0. Let $\boldsymbol{J}$ be the complex structure on $\phi(\mathcal{U}) \subseteq \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{C}^n$ induced by the coordinate chart and the complex structure J on M. That is, $\boldsymbol{J} = \phi_*(J|\mathcal{U})$. Note that $\boldsymbol{J}$ agrees with the canonical complex structure at $\phi(z_0)$ by the manner in which we have constructed our coordinates. Next note that the function

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto 2\|\boldsymbol{\eta}\|^2 - \|\boldsymbol{\xi} - \boldsymbol{\xi}(z)\|^2$$

is strictly plurisubharmonic on $\mathbb{C}^n$ for any $z \in \mathcal{V}$, as can be directly verified. By this we mean that it is strictly plurisubharmonic with respect to the standard complex structure. Define a map $\boldsymbol{L}\colon \mathcal{U} \times \mathcal{U} \to \mathbb{C}^{n\times n}$ by asking that $\boldsymbol{L}_{ij}(\boldsymbol{z}, \boldsymbol{\zeta})$ be the Levi form for $\phi_* f_{\phi^{-1}(\zeta)}$ with respect to the complex structure $\boldsymbol{J}(\boldsymbol{z})$ evaluated at the standard basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$. The matrix $\boldsymbol{L}(\phi(z_0), \phi(z_0))$ is Hermitian and positive-definite. Thus there is a neighbourhood of z_0 such that the Levi form of f_z is positive-definite for every z in that neighbourhood, and this concludes the proof. ▼

By virtue of second countability of M, we can thus choose a locally finite covering of S with a countable family $((\mathcal{U}_l, \phi_l))_{l\in\mathbb{Z}_{>0}}$ of submanifold charts as in the lemma, and with closed subsets $A_l \subseteq \mathcal{U}_l$, $l \in \mathbb{Z}_{>0}$, for which $\mathsf{S} = \cup_{l\in\mathbb{Z}_{>0}} A_l$. Borrowing the notation from the lemma, we denote

$$\phi_l(w) = (\boldsymbol{x}_l(w), \boldsymbol{y}_l(w)) = (\boldsymbol{\xi}_l(w), \boldsymbol{\eta}_l(w)).$$

For each $l \in \mathbb{Z}_{>0}$ and each $z \in \mathcal{S} \cap A_l$, define $f_{l,z} \in \mathrm{C}^\infty(\mathcal{U}_l)$ by

$$f_{l,z}(w) = 2\|\boldsymbol{\eta}_l(w)\|^2 - \|\boldsymbol{\xi}_l(w) - \boldsymbol{\xi}_l(z)\|^2.$$

By the lemma the neighbourhood $\mathcal{U}_l$ is chosen so that $f_{l.z}$ is strictly plurisubharmonic on $\mathcal{U}_l$ for each $z \in \mathsf{S} \cap \mathcal{U}_l$.

Let $\mathcal{N}$ be a neighbourhood of S. Since $\mathrm{dist}^2_{\mathsf{S}}(w) = \|\tau^{-1}(w)\|^2_{\mathbb{G}}$ the Hessian of $\mathrm{dist}^2_{\mathsf{S}}$ at $z \in \mathsf{S}$ is

$$\mathrm{Hess}\,\mathrm{dist}^2_{\mathsf{S}}(z)(u_z, v_z) = \mathbb{G}(z)(\nu(u_z), \nu(v_z)),$$

where ν is the projection onto $\mathsf{N}_z\mathsf{S}$. From Lemma 3.2.9 we have

$$\mathrm{Lev}(\mathrm{dist}^2_{\mathsf{S}})(v_z) = \mathbb{G}(z)(\nu(v_z), \nu(v_z)) + \mathbb{G}(z)(\nu(J(v_z)), \nu(J(v_z))).$$

Since S is totally real, we conclude that $\mathrm{dist}^2_{\mathsf{S}}$ is strictly plurisubharmonic along S. By Lemma 3.3.3 let ψ be a smooth exhaustion function on S that we extend arbitrarily to a smooth function on the tubular neighbourhood $\mathcal{T}_\rho$ by asking that $\tau^*\psi(v_z) = \psi(z)$ for every $v_z \in \mathsf{N}_z\mathsf{S}_\rho$. That is, if we regard $\mathcal{T}_\rho$ as a bundle over S whose fibre over $z \in \mathsf{S}$ is a disk, the extension of ψ just described prescribes ψ as being constant on these fibres. Now, following the argument given in the proof of Theorem 6.2.13, let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be smooth, convex, and strictly increasing and such that $\psi + \sigma \circ \mathrm{dist}^2_{\mathsf{S}}$ is strictly plurisubharmonic on S, and so in a neighbourhood $\mathcal{U}$ of S. We choose $\mathcal{U}$ to be a neighbourhood of S contained in $\mathcal{N} \cap \mathrm{cl}(\mathcal{T}_\rho)$.

For each $z \in \mathsf{S}$ let $\mathcal{V}_z$ be a neighbourhood of z such that (1) $\mathcal{V}_z \subseteq \mathcal{U}$ and (2) if $w \in \mathcal{V}_z$ and if $l \in \mathbb{Z}_{>0}$ is such that $z \in \mathcal{U}_l$, then $3\|\boldsymbol{\eta}(w)\|^2 < \mathrm{dist}(A_l, \mathrm{bd}(\mathcal{U}_l))^2$, the distance between sets being given as in . This is possible since the covering $(\mathcal{U}_l)_{l\in\mathbb{Z}_{>0}}$ is locally finite. Let ref $\mathcal{V} = \mathrm{cl}(\cup_{z\in\mathsf{S}} \mathcal{V}_z)$. For $z \in \mathsf{S}$ and $l \in \mathbb{Z}_{>0}$ such that $z \in \mathcal{U}_l$, denote

$$C_{l,z} = \{w \in \mathcal{U}_l \cap \mathcal{V} \mid f_{l,z}(w) > 0\}.$$

We claim that, with $\mathcal{V}$ so chosen,

$$\mathrm{bd}_{\mathcal{V}}(C_{l,z}) = \{w \in \mathcal{U}_l \cap \mathcal{V} \mid f_{l,z}(w) = 0\},$$

i.e., the boundary of $C_{l,z}$ in $\mathcal{V}$ does not contain points from $\mathrm{bd}(\mathcal{U}_l)$. Indeed, suppose this is not so, so that there exists $w \in \mathrm{cl}(C_{l,p}) \cap \mathrm{bd}(\mathcal{U}_l)$. Therefore,

$$0 \leq f_{l,z}(w) = 2\|\boldsymbol{\eta}(w)\|^2 - \|\xi(w)\|^2.$$

Since $w \in \mathrm{bd}(\mathcal{U}_l)$ and $z \in A_l$ we have

$$d_{\mathbb{H}}(w, z) \geq \mathrm{dist}(A_l, \mathrm{bd}(\mathcal{U}_l)).$$

By the lemma,

$$d_{\mathbb{H}}(w, z) \geq \mathrm{dist}_{\mathsf{S}}(w) = \|\eta(w)\|.$$

Combining the preceding inequalities, we have

$$\mathrm{dist}(A_l, \mathrm{bd}(\mathcal{U}_l))^2 \leq \|\boldsymbol{\eta}(w)\|^2 + \|\boldsymbol{\xi}(w) - \boldsymbol{\xi}(z)\|^2 \leq 3\|\boldsymbol{\eta}(w)\|^2 < \mathrm{dist}(A_l, \mathrm{bd}(\mathcal{U}_l))^2,$$

which is a contradiction. Thus we conclude that (6.4.4) holds, as desired. For $z \in \mathsf{S}$ fix $l_z \in \mathbb{Z}_{>0}$ such that $z \in A_{l_z}$. Since $\mathcal{V} \setminus \mathsf{S} = \cup_{z \in \mathsf{S}} C_{l_z, z}$, the condition (6.4.4) ensures that there exists a countable set $(z_j)_{j \in \mathbb{Z}_{>0}}$ in S such that $\mathrm{bd}(\mathcal{V}) \subseteq \cup_{j \in \mathbb{Z}_{>0}} C_{l_{z_j}, z_j}$. (To see this, note that $\mathrm{bd}(\mathcal{V})$ is a countable union of compact sets, each of which can be covered by a finite number of the sets $C_{l_z, z}$.)

Now let

$$\phi(t) = \begin{cases} \mathrm{e}^{-1/t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

and define $g_j = \phi \circ f_{l_{z_j}, z_j}$, $j \in \mathbb{Z}_{>0}$. Note that g_j is defined on $\mathcal{U}_{l_j}$, but can be smoothly extended to $\mathcal{V}$. Moreover, by Proposition 6.1.10(iv), g_j is strictly plurisubharmonic on $C_{l_{z_j}, z_j}$ and plurisubharmonic on $\mathcal{V}$. We can then choose suitable $\alpha_j \in \mathbb{R}_{>0}$, $j \in \mathbb{Z}_{>0}$, such that, if we define

$$g(z) = \sum_{j \in \mathbb{Z}_{>0}} \alpha_j g_j(z), \qquad z \in \mathcal{V},$$

then $g^{-1}([0,1)) \subseteq \mathcal{V}$. If $\epsilon \in (0,1)$ is a regular value for g (of which there are densely many by Sard's Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 3.6.5]), then $\mathcal{V}_\epsilon = g^{-1}([0,\epsilon))$ is a neighbourhood of S with a smooth strictly pseudoconvex boundary.

Now let ϵ be a regular value of g and $\kappa\colon [0,\epsilon) \to \mathbb{R}_{\geq 0}$ be defined by $\kappa(s) = \tan(\frac{\pi s}{2\epsilon})$, noting that κ is a strictly convex diffeomorphism. Thus $\kappa \circ g$ is plurisubharmonic by Proposition 6.1.10(iv). Thus $u = \kappa \circ g + \psi + \sigma \circ \mathrm{dist}_{\mathsf{S}}^2$ is strictly plurisubharmonic. Let us verify that it is also an exhaustion function on $\mathcal{V}_\epsilon$. Let $\alpha \in \mathbb{R}$. Since $\psi|\mathsf{S}$ is an exhaustion function, $(\psi|\mathsf{S})^{-1}((-\infty,\alpha])$ is compact. Since S is properly embedded in M, $K_\alpha \triangleq \psi^{-1}((-\infty,\alpha]) \cap \mathsf{S}$ is compact. Note that, by the manner in which ψ is extended from S to $\mathcal{T}_\rho$,

$$\psi^{-1}((-\infty,\alpha]) = \{z \in \mathcal{T}_\rho \mid \pi_{\mathsf{TM}}(\tau^{-1}(z)) \in K_\alpha\}.$$

By the following lemma, $\psi^{-1}((-\infty,\alpha])$ is compact.

2 **Lemma** *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces with $\mathcal{X}$ compact and let $\pi\colon \mathcal{Y} \to \mathcal{X}$ be a continuous map with the following properties:*

(i) π is a surjective open mapping;

(ii) $\pi^{-1}(\mathrm{x})$ is compact in $\mathcal{Y}$ for every $\mathrm{x} \in \mathcal{X}$.

Then $\mathcal{Y}$ is compact.

Proof Let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{Y}$. Let $x \in \mathcal{X}$ and note that $(\mathcal{U}_a \cap \pi^{-1}(x))_{a\in A}$ is an open cover for $\pi^{-1}(x)$. Since $\pi^{-1}(x)$ is compact there exists a finite subset $\{a_1(x), \dots, a_{k_x}(x)\} \subseteq A$ such that $\pi^{-1}(x) = \cup_{j=1}^{k_x} \mathcal{U}_{a_j(x)}$. Now, note that $\mathcal{N}_x = \cap_{j=1}^{k_x} \pi(\mathcal{U}_{a_j(x)})$ is a neighborhood of x since π is open. In turn, this implies that $\mathscr{N} = (\mathcal{N}_x)_{x\in\mathcal{X}}$ is an open cover of the compact set $\mathcal{X}$. Therefore, there exits a finite subset $\{x_1, \dots, x_m\} \subseteq \mathcal{X}$ such that $\mathcal{X} = \cup_{l=1}^m \mathcal{N}_{x_l}$. In summary we have

$$\mathcal{Y} = \cup_{x\in\mathcal{X}} \pi^{-1}(x) = \cup_{x\in\mathcal{X}} \cup_{j=1}^{k_x} \mathcal{U}_{a_j(x)} = \cup_{l=1}^m \cup_{j=1}^{k_{x_l}} \mathcal{U}_{a_j(x_l)},$$

giving a finite subcover of $\mathcal{Y}$ as desired. ▼

Since g and $\mathrm{dist}_{\mathsf{S}}^2$ are $\mathbb{R}_{\geq 0}$-valued we have $u^{-1}((-\infty, \alpha]) \subseteq \psi^{-1}((-\infty, \alpha])$. Compactness of $u^{-1}((-\infty, \alpha])$ now follows since it is a closed subset of the compact set $\psi^{-1}((-\infty, \alpha])$. Thus $\mathcal{V}_\epsilon$ is strongly pseudoconvex, and so Stein. ■

6.4.2 Complexification of real analytic manifolds

Very often in real analytic geometry it is useful to complexify, by which we mean extend real analytic objects to the corresponding holomorphic objects. The first step in doing this, of course, is to complexify real analytic manifolds. It is relatively easy to do this in a dumb direct way, but upon doing so one would like for the complexification to have some nice properties. In this section we indicate what these nice properties might be, and prove that a complexification does indeed exist with these properties. The first substantial work in this direction was done by Whitney and Bruhat [1959] (see also [Cartan 1957]). Further exploration of the idea of complexification was carried out by Kulkarni [1978].

Let us first define what we mean by the complexification of a real analytic manifold.

6.4.5 Definition (Complexification) If M is a real analytic manifold, a ***complexification*** of M is a pair $(\overline{\mathsf{M}}, \iota)$ where $\overline{\mathsf{M}}$ is a holomorphic manifold $\overline{\mathsf{M}}$ and $\iota\colon \mathsf{M} \to \overline{\mathsf{M}}$ is a real analytic embedding for which $\iota(\mathsf{M})$ is a totally real submanifold. A complexification $\overline{\mathsf{M}}$ is ***minimal*** if ι is a homotopy equivalence, i.e., if there exists continuous maps $p\colon \overline{\mathsf{M}} \to \mathsf{M}$, $h\colon [0,1] \times \mathsf{M} \to \overline{\mathsf{M}}$, and $\bar{h}\colon [0,1] \times \overline{\mathsf{M}} \to \mathsf{M}$ such that

(i) $h(0, x) = p \circ \iota(x)$ for all $x \in \mathsf{M}$,

(ii) $h(1, x) = x$ for all $x \in \mathsf{M}$,

(iii) $\bar{h}(0, z) = \iota \circ p(z)$ for all $z \in \overline{\mathsf{M}}$, and

(iv) $\bar{h}(1, z) = z$ for all $z \in \overline{\mathsf{M}}$. •

With these definitions we have the following result.

6.4.6 Theorem (Real analytic manifolds possess minimal complexifications) *If* M *is a Hausdorff, paracompact real analytic manifold then there exists a minimal Hausdorff complexification* $\overline{\mathsf{M}}$ *of* M*.*

Proof For $x \in \mathsf{M}$ let $\mathcal{U}_x$, $\mathcal{V}_x$, and $\mathcal{W}_x$ be neighbourhoods x in M for which $\mathcal{U}_x$ and $\mathcal{V}_x$ are relatively compact, for which

$$\mathrm{cl}(\mathcal{U}_x) \subseteq \mathcal{V}_x, \quad \mathrm{cl}(\mathcal{V}_x) \subseteq \mathcal{W}_x,$$

and for which $\mathcal{W}_x$ is the domain of a real analytic chart map $\phi_x : \mathcal{W}_x \to \mathbb{R}^n$. We denote $\mathcal{A}_x = \phi_x(\mathcal{U}_x)$, $\mathcal{B}_x = \phi_x(\mathcal{V}_x)$, and $\mathcal{C}_x = \phi_x(\mathcal{W}_x)$. Since M is paracompact, by [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.5] we choose a subcover of $(\mathcal{U}_x)_{x\in\mathsf{M}}$ of M with index set I such that $(\mathcal{U}_i)_{i\in I}$, $(\mathcal{V}_i)_{i\in I}$ and $(\mathcal{W}_i)_{i\in I}$ are locally finite open covers. We use the symbols $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, and ϕ_i for the obvious entities for each $i \in I$. Let us also denote

$$\mathcal{A}_{ij} = \phi_i(\mathcal{U}_i \cap \mathcal{U}_j), \quad \mathcal{B}_{ij} = \phi_i(\mathcal{V}_i \cap \mathcal{V}_j), \quad \mathcal{C}_{ij} = \phi_i(\mathcal{W}_i \cap \mathcal{W}_j)$$

for $i, j \in I$. We shall assume whenever necessary and without saying so that $i, j \in I$ are such that desired intersections are nonempty. Thus we have a real analytic diffeomorphism $\phi_{ij} : \mathcal{C}_{ij} \to \mathcal{C}_{ji}$ if $\mathcal{C}_{ij} \neq \emptyset$. Obviously we have

$$\mathcal{A}_{ij} \subseteq \mathcal{B}_{ij} \subseteq \mathcal{C}_{ij}$$

and

$$\phi_{ij}(\mathcal{A}_{ij}) = \mathcal{A}_{ji}, \quad \phi_{ij}(\mathcal{B}_{ij}) = \mathcal{B}_{ji}.$$

By Proposition 6.4.3 we can extend ϕ_{ij} to a holomorphic diffeomorphism $\overline{\phi}_{ij} : \overline{\mathcal{C}}_{ij} \to \overline{\mathcal{C}}_{ji}$ from a neighbourhood $\overline{\mathcal{C}}_{ij}$ of $\mathcal{C}_{ij} \subseteq \mathbb{R}^n$ in $\mathbb{C}^n$. By restricting if necessary, we suppose that $\overline{\phi}_{ij}^{-1} = \overline{\phi}_{ji}$ for each $i, j \in I$. For $i, j \in I$ let $\overline{\mathcal{B}}_{ij}$ be an open subset of $\mathbb{C}^n$ with the following properties for each $i, j \in I$:

1. $\mathrm{cl}(\overline{\mathcal{B}}_{ij}))$ is a compact subset of $\mathcal{C}_{ij}$;
2. $\overline{\mathcal{B}}_{ij} \cap \mathbb{R}^n = \mathcal{B}_{ji}$;
3. $\mathrm{cl}(\overline{\mathcal{B}}_{ij}) \cap \mathbb{R}^n = \mathrm{cl}(\mathcal{B}_{ji})$;
4. $\overline{\phi}_{ij}(\overline{\mathcal{B}}_{ij}) = \overline{\mathcal{B}}_{ji}$.

Note that $\mathrm{cl}(\mathcal{A}_i) \cap \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji}))$ is a compact subset of $\overline{\mathcal{B}}_{ij}$. Thus, for each $i, j \in I$, there exists an open subset $\overline{\mathcal{D}}_{ij}$ of $\mathbb{C}^n$ with the following properties for each $i, j \in I$:

5. $\mathrm{cl}(\overline{\mathcal{D}}_{ij})$ is a compact subset of $\overline{\mathcal{B}}_{ij}$;
6. $\mathrm{cl}(\mathcal{A}_i) \cap \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji})) \subseteq \overline{\mathcal{D}}_{ij}$;
7. $\overline{\phi}_{ij}(\overline{\mathcal{D}}_{ij}) = \overline{\mathcal{D}}_{ji}$.

Since $\overline{\mathcal{D}}_{ij}$ is an open set containing the intersection of the two compacts sets $\mathrm{cl}(\mathcal{A}_i)$ and $\overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji}))$, it follows that the sets

$$\mathrm{cl}(\mathcal{A}_i) - \overline{\mathcal{D}}_{ij}, \quad \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji})) - \overline{\mathcal{D}}_{ij}$$

are disjoint and each compact. Thus there exist disjoint open sub sets $\overline{\mathcal{X}}_i$ and $\overline{\mathcal{Y}}_i$ in $\mathbb{C}^n$ containing these complements, and thus having the property that

$$\mathrm{cl}(\mathcal{A}_i) \subseteq \overline{\mathcal{D}}_{ij} \cup \overline{\mathcal{X}}_{ij}, \quad \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji})) \subseteq \overline{\mathcal{D}}_{ij} \cup \overline{\mathcal{Y}}_{ij}.$$

Let $i \in I$. Let $\overline{\mathcal{X}}_i$ be an open subset of $\mathbb{C}^n$ having the following properties:

8. $\overline{\mathcal{X}}_i \cap \mathbb{R}^n = \mathcal{A}_i$;
9. $\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathbb{R}^n = \mathrm{cl}(\mathcal{A}_i)$;
10. $\overline{\mathcal{X}}_i \subseteq \overline{\mathcal{D}}_{ij} \cup \overline{\mathcal{X}}_{ij}$ for each $j \in I$ (of which there are finitely many) for which $\mathcal{C}_{ij} \neq \emptyset$.

Note that
$$\mathrm{cl}(\overline{\mathcal{X}}_i \cap \overline{\mathcal{B}}_{ij}) \subseteq \mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij})$$
and so
$$\mathrm{cl}(\overline{\phi}_{ij}(\overline{\mathcal{X}}_i \cap \overline{\mathcal{B}}_{ij})) \subseteq \overline{\phi}_{ij}(\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij})).$$
Therefore,
$$\begin{aligned}\mathrm{cl}(\overline{\phi}_{ij}(\overline{\mathcal{X}}i1 \cap \overline{\mathcal{B}}_{ij})) \cap \mathbb{R}^n &\subseteq \overline{\phi}_{ij}(\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij})) \cap \mathbb{R}^n \\ &= \overline{\phi}_{ij}(\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij}) \cap \mathbb{R}^n) \\ &= \phi_{ij}(\mathrm{cl}(\mathcal{A}_i) \cap \mathrm{cl}(\mathcal{B}_{ij})), \end{aligned} \tag{6.29}$$
by virtue of the properties of 2 and 3 of $\overline{\mathcal{B}}_{ij}$ and 8 and 9 of $\overline{\mathcal{X}}_i$.

For $x \in \mathcal{B}_i$, let $\overline{\mathcal{B}}_{i,x}$ be an open subset of $\mathbb{C}^n$ satisfying the following properties:

11. for each $j \in I$ (of which there are finitely many) for which $x \in \mathcal{B}_{ij}$, $\overline{\mathcal{B}}_{i,x} \subseteq \overline{\mathcal{B}}_{ij}$;
12. for each $j \in I$ (of which there are finitely many) for which $x \in \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji}))$, $\overline{\mathcal{B}}_{i,x} \subseteq \overline{\mathcal{D}}_{ij} \cup \overline{\mathcal{Y}}_{ij}$;
13. since, by (6.29), if $\phi_{ij}(x) \notin \mathrm{cl}(\mathcal{A}_j)$ then we have $x \notin \overline{\phi}_{ij}(\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij}))$, we require that, for each $j \in I$ (of which there are finitely many) for which $\phi_{ij}(x) \notin \mathrm{cl}(\mathcal{A}_j)$, $\overline{\mathcal{B}}_{i,x} \cap \overline{\phi}_{ij}(\mathrm{cl}(\overline{\mathcal{X}}_i) \cap \mathrm{cl}(\overline{\mathcal{B}}_{ij})) = \emptyset$;
14. for each ordered pair $(j,k) \in J \times J$ (of which there are finitely many) for which $x \in \mathcal{B}_{ij} \cap \mathcal{B}_{ik}$, we have
$$\overline{\mathcal{B}}_{i,x} \subseteq \overline{\phi}_{ji}(\overline{\mathcal{B}}_{ji} \cap \overline{\mathcal{B}}_{jk}) \cap \overline{\phi}_{ki}(\overline{\mathcal{B}}_{ki} \cap \overline{\mathcal{B}}_{kj})$$
and $\overline{\phi}_{ij}(z) = \overline{\phi}_{kj} \circ \overline{\phi}_{ik}(z)$ for every $z \in \overline{\mathcal{B}}_{i,x}$.

Now denote $\overline{\mathcal{B}}_i = \cup_{x \in \mathcal{B}_i} \overline{\mathcal{B}}_{i,x}$ and let $\overline{\mathcal{A}}_i$ be a neighbourhood of $\mathcal{A}_i$ in $\mathbb{C}^n$ contained in $\overline{\mathcal{X}}_i \cap \overline{\mathcal{B}}_i$ and for which $\mathrm{cl}(\overline{\mathcal{A}}_i) \subseteq \overline{\mathcal{B}}_i$. By properties 8 and 9 of $\overline{\mathcal{X}}_i$ we have
$$\overline{\mathcal{A}}_i \cap \mathbb{R}^n = \mathcal{A}_i, \quad \mathrm{cl}(\overline{\mathcal{A}}_i) \cap \mathbb{R}^n = \mathrm{cl}(\mathcal{A}_i).$$
Let us also define
$$\overline{\mathcal{A}}_{ij} = \overline{\mathcal{A}}_i \cap \overline{\psi}_{ji}(\overline{\mathcal{A}}_i \cap \overline{\mathcal{B}}_{ji}), \quad \overline{\mathcal{A}}_{ijk} = \overline{\mathcal{A}}_{ij} \cap \overline{\mathcal{A}}_{ik}.$$

Note that $\overline{\phi}_{ij}|\overline{\mathcal{A}}_{ij}$ is a holomorphic diffeomorphism onto $\overline{\mathcal{A}}_{ji}$. Also let $z \in \overline{\mathcal{A}}_{ijk}$ so $z \in \overline{\mathcal{A}}_{ij}$ and, therefore, $z \in \overline{\phi}_{ji}(\overline{\mathcal{A}}_j \cap \overline{\mathcal{B}}_{ji})$. Similarly, $z \in \overline{\phi}_{ki}(\overline{\mathcal{A}}_k \cap \overline{\mathcal{B}}_{ki})$. From this we conclude that
$$z \in \overline{\phi}_{ji}(\overline{\mathcal{X}}_j \cap \overline{\mathcal{B}}_{ji}) \cap \overline{\phi}_{ki}(\overline{\mathcal{X}}_k \cap \overline{\mathcal{B}}_{ki}).$$
Therefore, if $z \in \overline{\mathcal{B}}_{i,x}$ for some $x \in \mathcal{B}_i$ (as must be the case), then $\overline{\mathcal{B}}_{i,x}$ must intersect both $\overline{\phi}_{ji}(\overline{\mathcal{X}}_j \cap \overline{\mathcal{B}}_{ji})$ and $\overline{\phi}_{ki}(\overline{\mathcal{X}}_k \cap \overline{\mathcal{B}}_{ki})$. By property 13 of the sets $\overline{\mathcal{B}}_{i,x}$ we conclude that $\phi_{ij}(x) \in \mathrm{cl}(\mathcal{A}_j)$

and $\phi_{ik}(\boldsymbol{x}) \in \mathrm{cl}(\mathcal{A}_k)$. From this we conclude that $\boldsymbol{x} \in \mathcal{B}_{ij} \cap \mathcal{B}_{ik}$. By property 11 of the sets $\overline{\mathcal{B}}_{i,\boldsymbol{x}}$ we conclude that $\boldsymbol{z} \in \overline{\mathcal{B}}_{ij} \cap \overline{\mathcal{B}}_{ik}$. Thus we have

$$\overline{\phi}_{ik}(\boldsymbol{z}) = \overline{\phi}_{jk} \circ \overline{\phi}_{ij}(\boldsymbol{z}) \in \overline{\mathcal{A}}_{kj}.$$

Since $\boldsymbol{z} \in \overline{\mathcal{A}}_{ik}$ we also have $\overline{\phi}_{ik}(\boldsymbol{z}) \in \overline{\mathcal{A}}_k$. Therefore, we also have $\overline{\phi}_{kj} \circ \overline{\phi}_{ik}(\boldsymbol{z}) = \overline{\phi}_{ij}(\boldsymbol{z})$. From this we conclude that

$$\overline{\phi}_{ij}(\boldsymbol{z}) \in \overline{\mathcal{A}}_j, \quad \overline{\phi}_{ij}(\boldsymbol{z}) \in \overline{\phi}_{ij}(\overline{\mathcal{A}}_i \cap \overline{\mathcal{B}}_{ij}), \quad \overline{\phi}_{ij}(\boldsymbol{z}) \in \overline{\phi}_{kj}(\overline{\mathcal{A}}_k \cap \overline{\mathcal{B}}_{kj}).$$

From the preceding three formulae we conclude that $\overline{\phi}_{ij}(\boldsymbol{z}) \in \overline{\mathcal{A}}_{jik}$. Thus $\overline{\phi}_{ij}(\overline{\mathcal{A}}_{ijk}) \subseteq \overline{\mathcal{A}}_{jik}$. In like manner we conclude that $\overline{\phi}_{ji}(\mathcal{A}_{jik}) \subseteq \overline{\mathcal{A}}_{ijk}$. Therefore, $\overline{\phi}_{ij}$ is a holomorphic diffeomorphism from $\overline{\mathcal{A}}_{ijk}$ onto $\overline{\mathcal{A}}_{jik}$.

Now consider the disjoint union $\mathring{\cup}_{i\in I}\overline{\mathcal{A}}_i$; thus an element of this set is a pair $(i, \boldsymbol{z})$ where $\boldsymbol{z} \in \overline{\mathcal{A}}_i$. On $\mathring{\cup}_{i\in I}\overline{\mathcal{A}}_i$ we use the disjoint union topology, i.e., the topology generated by the sets $\{i\} \times \mathcal{O}$ for $i \in I$ and for $\mathcal{O} \subseteq \overline{\mathcal{A}}_i$ open. We define an equivalence relation on this set by $(i, \boldsymbol{z}) \sim (j, \boldsymbol{w})$ if $\boldsymbol{z} \in \overline{\mathcal{A}}_{ij}$, $\boldsymbol{w} \in \overline{\mathcal{A}}_{ji}$, and $\boldsymbol{w} = \overline{\phi}_{ij}(\boldsymbol{z})$. Our constructions above, particularly that $\overline{\phi}_{ij}$ is a holomorphic diffeomorphism from $\overline{\mathcal{A}}_{ij}$ onto $\overline{\mathcal{A}}_{ji}$ and from $\overline{\mathcal{A}}_{ijk}$ onto $\overline{\mathcal{A}}_{jik}$, ensure that $\sim$ is indeed an equivalence relation. (We adopt the natural convention that $\overline{\mathcal{A}}_{ii} = \overline{\mathcal{A}}_i$ and $\overline{\phi}_{ii} = \mathrm{id}_{\overline{\mathcal{A}}_i}$.) Let us denote by $\overline{\mathsf{M}}$ the quotient of $\mathring{\cup}_{i\in I}\overline{\mathcal{A}}_i$ by the equivalence relation, and we equip $\overline{\mathsf{M}}$ with the quotient topology. By π we denote the projection. We claim that $\overline{\mathsf{M}}$ has a holomorphic structure and an inclusion of M whose image is real analytic and totally real. First we specify the holomorphic structure by prescribing a holomorphic atlas. For $i \in I$ let $\overline{\mathcal{U}}_i = \pi(\{i\} \times \overline{\mathcal{A}}_i)$ and $\overline{\phi}_i \colon \overline{\mathcal{U}}_i \to \mathbb{C}^n$ by $\overline{\phi}_i(z) = (i, z)$, where $\pi(i, z) = z$. Since $\overline{\phi}_i$ is a bijection onto $\overline{\mathcal{A}}_i$, $(\overline{\mathcal{U}}_i, \overline{\phi}_i)$ is a $\mathbb{C}$-chart. For two overlapping charts $(\overline{\mathcal{U}}_i, \overline{\phi}_i)$ and $(\overline{\mathcal{U}}_j, \overline{\phi}_j)$ we have $\overline{\phi}_j \circ \overline{\phi}_i^{-1}(\boldsymbol{z}) = \overline{\phi}_{ij}(\boldsymbol{z})$ for $\boldsymbol{z} \in \overline{\phi}_i(\overline{\mathcal{U}}_i \cap \overline{\mathcal{U}}_j)$. Thus $((\overline{\mathcal{U}}_i, \overline{\phi}_i))_{i\in I}$ is a holomorphic atlas. Now we need to embed M in $\overline{\mathsf{M}}$ as a totally real submanifold. If $x \in \mathsf{M}$ then $x \in \mathcal{U}_i$ for some $i \in I$. We then define $\iota(x) = \pi \circ \phi_i(x)$. To see that ι is well-defined, suppose that $x \in \mathcal{U}_j$. We then have $\phi_i(x) = \phi_{ij} \circ \phi_j(x)$ so that $(i, \phi_i(x)) \sim (j, \phi_j(x))$, giving well-definedness. Since the local representative of ι with respect to the chart $(\mathcal{U}_i, \phi)$ for M and the chart $(\overline{\mathcal{U}}_i, \overline{\phi}_i)$ for $\overline{\mathsf{M}}$ is the canonical embedding of $\mathcal{A}_i \subseteq \mathbb{R}^n$ in $\overline{\mathsf{A}}_i \subseteq \mathbb{C}^n$, it follows that ι is real analytic and that its image it totally real.

Next we show that the topology on $\overline{\mathsf{M}}$ is Hausdorff. To do so, we first show that $\mathrm{cl}(\overline{\mathcal{A}}_{ij}) \subseteq \overline{\mathcal{B}}_{ij}$. To do so, we suppose without loss of generality that $\mathcal{C}_{ij} \neq \emptyset$. Let $\boldsymbol{z} \in \overline{\mathcal{A}}_{ij}$. Noting that $\overline{\mathcal{A}}_{ij} \subseteq \overline{\mathcal{A}}_i \subseteq \overline{\mathcal{B}}_i$, it follows that there exists $\boldsymbol{x} \in \mathcal{B}_i$ such that $\boldsymbol{z} \in \overline{\mathcal{B}}_{i,\boldsymbol{x}}$. We claim that $\boldsymbol{x} \in \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji}))$. Indeed, suppose otherwise so $\phi_{ij}(\boldsymbol{x}) \notin \mathrm{cl}(\mathcal{A}_j)$. By property 13 of the sets $\overline{\mathcal{B}}_{i,\boldsymbol{x}}$ we have $\boldsymbol{z} \notin \overline{\phi}_{ji}(\overline{\mathcal{X}}_j \cap \overline{\mathcal{B}}_{ji})$. This implies that $\boldsymbol{z} \notin \overline{\phi}_{ji}(\overline{\mathcal{A}}_j \cap \overline{\mathcal{B}}_{ji})$, contradicting the fact that $\boldsymbol{z} \in \overline{\mathcal{A}}_{ij}$ by the definition of $\overline{\mathcal{A}}_{ij}$. We, therefore, have $\boldsymbol{x} \in \overline{\phi}_{ji}(\mathrm{cl}(\mathcal{A}_j) \cap \mathrm{cl}(\mathcal{B}_{ji}))$. By property 12 of the sets $\overline{\mathcal{B}}_{i,\boldsymbol{x}}$ we have $\boldsymbol{z} \in \overline{\mathcal{D}}_{ij} \cup \overline{\mathcal{Y}}_{ij}$. We also have $\boldsymbol{z} \in \overline{\mathcal{A}}_i \subseteq \overline{\mathcal{X}}_i \subseteq \overline{\mathcal{X}}_{ij} \cup \overline{\mathcal{D}}_{ij}$. Since $\overline{\mathcal{X}}_{ij}$ and $\overline{\mathcal{Y}}_{ij}$ are prescribed as being disjoint, we must have $\boldsymbol{z} \in \overline{\mathcal{D}}_{ij}$. Since $\overline{\mathcal{D}}_{ij}$ is prescribed as being relatively compact in $\overline{\mathcal{B}}_{ij}$, it follows that $\mathrm{cl}(\overline{\mathcal{A}}_{ij}) \subseteq \overline{\mathcal{B}}_{ij}$, as desired.

To show that $\overline{\mathsf{M}}$ is Hausdorff, let $z, w \in \overline{\mathsf{M}}$ be distinct and suppose that $\pi(i, \boldsymbol{z}) = z$ and $\pi(j, \boldsymbol{w}) = w$. We claim that there exists a neighbourhood $\mathcal{S}$ of $\boldsymbol{z}$ in $\overline{\mathcal{A}}_i$ and a neighbourhood $\mathcal{T}$ of $\boldsymbol{w}$ in $\overline{\mathcal{A}}_j$ such that the set

$$\{((i, \boldsymbol{z}'), (j, \boldsymbol{w}')) \in (\{i\} \times \mathcal{S}) \times (\{j\} \times \mathcal{T}) \mid (i, \boldsymbol{z}') \sim (j, \boldsymbol{w}')\}$$

is empty (this obviously suffices to show that z and w have disjoint neighbourhoods). Suppose otherwise. This implies that there exists sequences $(\boldsymbol{z}_l)_{l\in\mathbb{Z}_{>0}}$ in $\overline{\mathcal{A}}_{ij}$ and $(\boldsymbol{w}_l)_{l\in\mathbb{Z}_{>0}}$ in $\overline{\mathcal{A}}_{ji}$ converging to $\boldsymbol{z}$ and $\boldsymbol{w}$, respectively, and such that $\boldsymbol{w}_l = \overline{\phi}_{ij}(\boldsymbol{z}_l)$, $l \in \mathbb{Z}_{>0}$. By the preceding paragraph, we conclude that $\boldsymbol{z} \in \overline{\mathcal{B}}_{ij}$ and $\boldsymbol{w} \in \overline{\mathcal{B}}_{ji}$. This contradicts the fact that $(i, \boldsymbol{z}) \nsim (j, \boldsymbol{w})$.

Finally, we show that there is a minimal complexification. This, however, follows from the Tubular Neighbourhood Theorem in the following manner. We suppose that $\overline{\mathsf{M}}$ possesses a Riemannian metric $\mathbb{G}$ as a smooth real manifold. By that theorem we have a function $\rho\colon \mathsf{M} \to \mathbb{R}_{>0}$ and a diffeomorphism $\tau\colon \mathsf{NM}_\rho \to \overline{\mathsf{M}}$ from a neighbourhood of the zero section of the normal bundle to M in $\overline{\mathsf{M}}$. By shrinking, we suppose that $\overline{\mathsf{M}} = \operatorname{image}(\tau)$. We then define the smooth map $p\colon \overline{\mathsf{M}} \to \mathsf{M}$ by $p(z) = \nu \circ \tau^{-1}(z)x$, where $\nu\colon \mathsf{NM} \to \mathsf{M}$ is the vector bundle projection. We define smooth maps $h\colon [0,1]\times\mathsf{M} \to \overline{\mathsf{M}}$ and $\overline{h}\colon [0,1]\times\overline{\mathsf{M}} \to \mathsf{M}$ by $h(s, x) = x$ and $\overline{h}(s, z) = \tau(s\tau^{-1}(z))$. These maps clearly satisfy the conditions in the definition of minimality. ■

If we combine the preceding theorem with Theorem 6.4.4 we have the following result.

6.4.7 Corollary (Real analytic manifolds possess Stein complexifications) *If* M *is a Hausdorff, paracompact real analytic manifold, then it possesses a minimal Stein complexification.*

6.5 Embedding theorems

Embedding theorems for manifolds have to do with the question, "When can a manifold be embedded as a submanifold of Euclidean space?" The abstract definition of a manifold by no means makes it clear that such an object can, indeed, be regarded as a submanifold of Euclidean space. Embedding theorems, when they exist, are often quite useful. For example, an embeddable manifold will always possess a Riemannian or Hermitian metric, simply by restricting the standard Euclidean inner product or Hermitian inner product, respectively. A host of similar type existence theorems follow from embeddability, for example Theorem GA2.5.3.3 and its Corollary GA2.5.3.4.

The basic embedding theorem for smooth manifolds in differential geometry is that of Whitney [1936], which says that a second countable smooth Hausdorff manifold of dimension n can be embedded as a smooth submanifold of $\mathbb{R}^N$ for some sufficiently large $2n+1$. The embedding theorem of Whitney is comparatively easy to prove, using partitions of unity, and we give a proof below. Another important smooth embedding theorem is that of Nash [1956], which says that if a second countable smooth Hausdorff manifold possesses a smooth Riemannian metric, its embedding into $\mathbb{R}^N$ (generally we will have $N > 2n+1$) can be chosen such that the Riemannian metric on the embedded submanifold agrees with that induced by the standard Euclidean metric. Nash's proof is a technical one, and involved proving a powerful Inverse Function Theorem in infinite dimensions.

Thus, in the smooth case, one has at one's disposal exactly the sort of embedding theorems on would like. However, it is not the case that embedding theorems hold universally; for instance, Example 4.2.13–3 shows that not all holomorphic manifolds can be embedded in complex Euclidean space. Thus the question of embeddability does not always have a positive answer. In this section we overview the embedding theorems that will be of interest to us.

6.5.1 Proper mappings

The embedding theorems we give will all assert the our manifold is *properly* embedded in some Euclidean space. In this section we quickly overview the notion of being proper, and why it is important.

6.5.1 Definition (Proper map) For topological spaces $\mathcal{S}$ and $\mathcal{T}$, a map $\Phi\colon \mathcal{S} \to \mathcal{T}$ is ***proper*** if $\Phi^{-1}(K)$ is compact for every compact subset $K \subseteq \mathcal{T}$. •

To gain some insight into this definition, let us consider some examples.

6.5.2 Examples (Proper and not proper maps)

1. If $\mathcal{S}$ is a compact space, if $\mathcal{T}$ is a Hausdorff space, and if $\Phi\colon \mathcal{S} \to \mathcal{T}$ is continuous, then Φ is proper, since a compact set $K \subseteq \mathcal{T}$ is closed ([Runde 2005, Proposition 3.3.6]), so $\Phi^{-1}(K)$ is closed and so compact ([Runde 2005, Proposition 3.3.6]).
2. The map $P\colon \mathbb{R} \to \mathbb{R}$ is a polynomial function of positive degree, then it is proper.
3. The map $\sin\colon \mathbb{R} \to \mathbb{R}$ is not proper.
4. We claim that continuous exhaustion functions are proper. Indeed, if $u\colon \mathcal{S} \to \mathbb{R}$ is such a function and of $K \subseteq \mathbb{R}$, then $K \subseteq (-\infty, \alpha]$ for some α. Therefore, $u^{-1}(K)$ is a closed subset of the compact set $u^{-1}((-\infty, \alpha])$ and so compact [Runde 2005, Proposition 3.3.6].

   Note, however, that not all continuous proper functions are exhaustion functions. For example, an odd polynomial function $P\colon \mathbb{R} \to \mathbb{R}$ is proper but not an exhaustion function. Roughly speaking, proper functions have to "blow up" at "infinity" to plus or minus infinity, whereas exhaustion functions "blow up" at "infinity" to plus infinity. •

The examples, particularly the last two, suggest that a proper map "must go to infinity when evaluated at points going to infinity." This is imprecise, of course,[1] but

[1] It can be made more precise by use of one-point compactifications. A little more specifically and assuming that $\mathcal{S}$ and $\mathcal{T}$ are locally compact but not compact, if $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ are the one-point compactifications of $\mathcal{S}$ and $\mathcal{T}$, respectively, then a continuous map $\Phi\colon \mathcal{S} \to \mathcal{T}$ is proper if and only if it extends to a continuous map $\hat{\Phi}\colon \hat{\mathcal{S}} \to \hat{\mathcal{T}}$ having the property that $\Phi(\infty) = \infty$.

nonetheless it does capture the spirit of why properness is important for our purposes. To flesh this out further, let us introduce the following definition.

6.5.3 Definition (Properly embedded submanifold) A submanifold S of a smooth manifold M is ***properly embedded*** if the inclusion map $\iota_{\mathsf{S}}\colon \mathsf{S} \to \mathsf{M}$ is proper. •

Some examples illustrate the point of this definition.

6.5.4 Examples (Properly and not properly embedded submanifolds)

1. Let $\mathsf{M} = \mathbb{R}^2$ and consider the submanifold

$$\mathsf{S} = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

One readily verifies that S is properly embedded. Intuitively, points at infinity in S are also at infinity in M.

2. Let $\mathsf{M} = \mathbb{R}^2$ and consider the submanifold

$$\mathsf{S} = \{(x, y) \in \mathbb{R}^2 \mid y = 0,\ x \in (-1, 1)\}.$$

Note that the compact subset $K = [-1, 1]$, intersected with S, is not a compact subset of S, and this precludes S from being properly embedded. Intuitively, points at infinity in S, i.e., points tending to the endpoints of S, are not at infinity in M. •

The following result encapsulates the important properties of proper maps.

6.5.5 Proposition (Characterisations of proper maps) *If M and N are smooth manifolds and if $\Phi\colon \mathsf{M} \to \mathsf{N}$ is a continuous injection, then the following statements are equivalent:*

(i) *Φ is proper;*

(ii) *$\Phi(\mathsf{M})$ is closed and $\Phi\colon \mathsf{M} \to \Phi(\mathsf{M})$ is a homeomorphism provided that $\Phi(\mathsf{M})$ is equipped with the subspace topology inherited from N;*

(iii) *$\Phi(\mathrm{A})$ is a closed subset of N for every closed subset $\mathrm{A} \subseteq \mathsf{S}$.*

Proof (i) $\Longrightarrow$ (iii) Here we use a lemma.

1 Lemma *Let M and N be smooth manifolds and let $\Phi \in \mathrm{C}^0(\mathsf{M}; \mathsf{N})$. Then Φ is proper if and only if it holds that, for every sequence $(\mathrm{x_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ for which the sequence $(\Phi(\mathrm{x_j}))_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges to $\mathrm{y} \in \mathsf{N}$, then there is a convergent subsequence $(\mathrm{x_{j_k}})_{\mathrm{k}\in\mathbb{Z}_{>0}}$ for which $(\Phi(\mathrm{x_{j_k}}))_{\mathrm{k}\in\mathbb{Z}_{>0}}$ also converges to y.*

Proof Suppose that Φ is proper. Since the sequence $(\Phi(x_j))_{j\in\mathbb{Z}_{>0}}$ converges, there exists a compact set $K \subseteq \mathsf{N}$ such that $\Phi(x_j) \in K$ for every $j \in \mathbb{Z}_{>0}$. Then $\Phi^{-1}(K)$ is compact since Φ is proper, and x_j belongs to the compact set $\Phi^{-1}(x_j)$ for each $j \in \mathbb{Z}_{>0}$. By the Bolzano–Weierstrass Theorem, there exists a convergent subsequence $(x_{j_k})_{j\in\mathbb{Z}_{>0}}$, and clearly $\lim_{k\to\infty} \Phi(x_{j_k}) = y$.

To prove the converse, let $K \subseteq \mathsf{N}$ be compact and let $(x_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\Phi^{-1}(K)$. Since K is compact, by the Bolzano–Weierstrass Theorem there exists a subsequence $(x_{j_k})_{k\in\mathbb{Z}_{>0}}$ such that $(\Phi(x_{j_k}))_{k\in\mathbb{Z}_{>0}}$ converges to $y \in K$. Then, by hypothesis, there exists a subsubsequence $(x_{j_{k_l}})_{l\in\mathbb{Z}_{>0}}$ in $\Phi^{-1}(K)$ converging to $x \in \mathsf{M}$. Since Φ is continuous, $\Phi^{-1}(K)$ is closed $x \in \Phi^{-1}(K)$. By the Bolzano–Weierstrass Theorem we conclude that $\Phi^{-1}(K)$ is compact and so Φ is proper. ▼

Now let $A \subseteq \mathsf{M}$ be closed and let $(x_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in A for which $(\Phi(x_j))_{j\in\mathbb{Z}_{>0}}$ converges to $y \in \mathsf{N}$. By the lemma and since A is closed, there exists a subsequence $(x_{j_k})_{j\in\mathbb{Z}_{>0}}$ converging to $x \in A$. Since Φ is continuous, $\Phi(x) = y$ and this shows that $\Phi(A)$ is closed.

(ii) $\implies$ (i) Let $K \subseteq \mathsf{N}$ be compact so that $K \cap \Phi(\mathsf{M})$ is a compact subset of the closed set $\Phi(\mathsf{M})$. By [Runde 2005, Proposition 3.3.6] we have that $K \cap \Phi(\mathsf{M})$ is compact. Since Φ is a homeomorphism with the subspace topology on $\Phi(\mathsf{M})$, we conclude that $\Phi^{-1}(K) = \Phi^{-1}(K \cap \Phi(\mathsf{M}))$ is compact, as desired.

(iii) $\implies$ (ii) Let $\Phi^{-1}\colon \Phi(\mathsf{M}) \to \mathsf{M}$ be the inverse of Φ restricted to its image. Since Φ is obviously continuous, it suffices to show that Φ^{-1} is continuous. For this, we must show that $\Phi(\mathcal{U})$ is an open subset of $\Phi(\mathsf{M})$ in the subspace topology. By hypothesis, $\Phi(\mathsf{M} \setminus \mathcal{U})$ is a closed subset of N, and so a closed subset of $\Phi(\mathsf{M})$ in the subspace topology. Injectivity of Φ ensures that $\Phi(\mathcal{U}) = \Phi(\mathsf{M}) \setminus (\Phi(\mathsf{M} \setminus \mathcal{U}))$, and from this we conclude that $\Phi(\mathcal{U})$ is indeed open. ■

6.5.2 The Whitney embedding theorem for smooth manifolds

In this section we prove the grandfather of the embedding theorems, and perhaps the easiest of them to prove.

6.5.6 Theorem (The Whitney Embedding Theorem) *If M is a smooth, paracompact, Hausdorff and connected manifold of dimension n, then there exists a proper smooth embedding of M in $\mathbb{R}^{2\mathrm{n}+1}$.*

Proof The proof will be by a series of lemmata. The first is a weak version of Sard's Theorem, adequate for our needs. For the general version and an elementary proof, we refer to [Aubin 2001].

1 **Lemma** *If $\boldsymbol{\Phi}\colon \mathcal{U} \to \mathbb{R}^{\mathrm{n}}$ is a map of class C^1 from an open subset $\mathcal{U} \subseteq \mathbb{R}^{\mathrm{n}}$, then $\boldsymbol{\Phi}(\mathrm{Z})$ has measure zero for every subset $\mathrm{Z} \subseteq \mathcal{U}$ of measure zero.*

Proof Let $B \subseteq \mathcal{U}$ be a closed ball. Since B is compact, $\boldsymbol{\Phi}|B$ is Lipschitz, and so there exists $C \in \mathbb{R}_{>0}$ such that

$$\|\boldsymbol{\Phi}(\boldsymbol{x}) - \boldsymbol{\Phi}(\boldsymbol{y})\| < C\|\boldsymbol{x} - \boldsymbol{y}\|$$

for every $\boldsymbol{x}, \boldsymbol{y} \in B$. Since $Z \cap B$ has measure zero, given $\delta \in \mathbb{R}_{>0}$ we can cover $Z \cap B$ by balls $(B_j)_{j\in\mathbb{Z}_{>0}}$ whose volumes sum to less than δ. For each $j \in \mathbb{Z}_{>0}$, $\boldsymbol{\Phi}(B_j)$ is contained in a ball whose radius is less than C times that of B_j. Thus $\boldsymbol{\Phi}(B_j)$ is contained in a ball whose volume is less than C^n times that of B_j. In other words, $\boldsymbol{\Phi}(Z\cap B)$ is covered by balls $(B'_j)_{j\in\mathbb{Z}_{>0}}$ whose volumes sum to less than $C^n\delta$. Since δ can be chosen as small as desired, $\boldsymbol{\Phi}(Z \cap B)$ has measure zero. Since Z can be covered by countable many closed balls, $\boldsymbol{\Phi}(Z \cap B)$ is a countable union of sets of measure zero, and so of measure zero. ▼

The next lemma is similar in spirit.

2 **Lemma** *If M is a second countable smooth manifold of dimension n and if $\Phi \in \mathrm{C}^1(\mathsf{M};\mathbb{R}^{\mathrm{m}})$ for $\mathrm{m} > \mathrm{n}$, then $\Phi(\mathsf{M})$ has measure zero.*

Proof First we prove the result in the case that $\mathsf{M} = \mathcal{U}$ is an open subset of $\mathbb{R}^n$. Let $p\colon \mathbb{R}^m \to \mathbb{R}^n$ be the projection onto the first n components and define $\hat{\Phi} \in \mathrm{C}^1(\mathcal{U} \times \mathbb{R}^{m-n};\mathbb{R}^m)$

by $\hat{\Phi}(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{x})$. By the preceding lemma, $\hat{\Phi}(\mathcal{U} \times \{\boldsymbol{0}\})$ has measure zero. Since $\Phi(\mathcal{U}) = \hat{\Phi}(\mathcal{U} \times \{\boldsymbol{0}\})$, the lemma follows in this case.

If M is second countable, it can be covered by countably many coordinate charts. By the first part of the proof, the image of each coordinate chart under Φ has measure zero. Thus image(Φ) is a countable union of sets of measure zero, and so has measure zero. ▼

For the next lemma, let $\mathbb{R}^{n\times m}$ be the set of $\mathbb{R}$-matrices with n rows and columns, and for $k \leq \min\{m, n\}$ let $\mathbb{R}_k^{n\times m}$ be the matrices of rank k.

3 Lemma *The subset* $\mathrm{\mathbb{R}_k^{n\times m}}$ *is a finite union of submanifolds of dimension* $\mathrm{k(n + m - k)}$*.*

Proof A simple computation shows that there are

$$N(n, m, k) = \binom{n}{k}\binom{m}{k}$$

minors of size $k\times k$. Thus we have functions $\Psi_j\colon \mathbb{R}^{n\times m} \to \mathbb{R}$, $j \in \{1, \dots, N(n, m, k)\}$, returning the values of these $k \times k$-minors. Let us denote

$$S_j = \{\boldsymbol{A} \in \mathbb{R}^{n\times m} \mid \Psi_j(\boldsymbol{A}) \neq 0\}.$$

Let us suppose that Ψ_1 is the principal $k \times k$-minor.

If $k = \min\{n, m\}$ then $\mathbb{R}_k^{n\times m}$ is the set of matrices of full rank, and so none of the functions Ψ_j vanishes on $\mathbb{R}_k^{n\times m}$, i.e., $\mathbb{R}_k^{m\times n} = \cup_{j=1}^{N(n,m,k)} S_j$. By continuity, the sets S_j are open, $j \in \{1, \dots, N(n, m, k)\}$. Therefore, $\mathbb{R}_k^{n\times m}$ is also open, and so a submanifold of dimension nm.

In case $k < \min\{n, m\}$ then if $\boldsymbol{A} \in \mathbb{R}_k^{n\times m}$ it must be the case that $\Psi_j(\boldsymbol{A}) \neq 0$ for some $j \in \{1, \dots, N(n, m, k)\}$. Let us suppose consider the set S_1, as all other sets S_j, $j \in \{2, \dots, N(n, m, k)\}$, are mapped to S_1 by reordering row and column indices. In this case, for $a \in \{1, \dots, n - k\}$, $b \in \{1, \dots, m - k\}$, and $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, define $d_{ab}(\boldsymbol{A})$ to be the determinant of the matrix

$$\begin{bmatrix} A_1^1 & \cdots & A_k^1 & A_{k+b}^1 \\ \vdots & \ddots & \vdots & \vdots \\ A_1^k & \cdots & A_k^a & A_{k+b}^k \\ A_1^{k+a} & \cdots & A_k^{k+a} & A_{k+b}^{k+a} \end{bmatrix}.$$

Then define

$$\begin{aligned} \Phi\colon \mathbb{R}^{n\times m} &\to \mathbb{R}^{(n-k)(m-k)} \\ \boldsymbol{A} &\mapsto (d_{11}(\boldsymbol{A}), \dots, d_{1k}(\boldsymbol{A}), \dots, d_{n-k,1}(\boldsymbol{A}), \dots, d_{n-k,m-k}(\boldsymbol{A})), \end{aligned}$$

and note that $S_1 \cap \mathbb{R}_k^{n\times m}$ is given by $\Phi^{-1}(1, 0, \dots, 0)$. We then compute, for $\boldsymbol{A} \in S_1 \cap \mathbb{R}_k^{n\times m}$,

$$\frac{\partial d_{ab}}{\partial A_j^i}(\boldsymbol{A}) = \begin{cases} \Psi_1(\boldsymbol{A}), & i = k + a,\ j = k + b, \\ 0, & i \in \{k + 1, \dots, n\},\ j \in \{k + 1, \dots, m\},\ i \neq k + a,\ j \neq k + b. \end{cases}$$

We conclude that points on $S_1 \cap \mathbb{R}_k^{n\times m}$ are regular points for Φ, and so $S_1 \cap \mathbb{R}_k^{n\times m}$ is a submanifold of dimension $nm - (n - k)(m - k) = k(n + m - k)$. As mentioned above, by reindexing, we also conclude that $S_j \cap \mathbb{R}_k^{n\times m}$ is a submanifold of dimension $nm-(n-k)(m-k) = k(n+m-k)$ for each $j \in \{1,\dots,N(n,m,k)\}$. Finally, since $\mathbb{R}_k^{n\times m} \subseteq \cup_{j=1}^{N(n,m,k)} S_j$, we have

$$\mathbb{R}_k^{n\times m} = \cup_{j=1}^{N(n,m,k)} S_j \cap \mathbb{R}_k^{n\times m},$$

and so $\mathbb{R}_k^{n\times m}$ is indeed of the stated form. ▼

Now we can give our first interesting construction, showing that certain maps can be well approximated by maps all of whose values are regular values.

4 **Lemma** *If $\mathbf{\Phi}\colon \mathcal{U} \to \mathbb{R}^{\mathrm{m}}$ is a mapping of class C^2 from an open subset $\mathcal{U} \subseteq \mathbb{R}^{\mathrm{n}}$ with $\mathrm{m} \geq 2\mathrm{n}$, then, for $\epsilon \in \mathbb{R}_{>0}$, there exists $\mathbf{A} \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$ such that*

(i) $|\mathrm{A}^{\mathrm{i}}_{\mathrm{j}}| < \epsilon$ *for* $\mathrm{i} \in \{1,\dots,\mathrm{m}\}$, $\mathrm{j} \in \{1,\dots,\mathrm{n}\}$, *and*

(ii) the mapping $\mathbf{x} \mapsto \mathbf{\Phi}(\mathbf{x}) + \mathbf{A}\mathbf{x}$ is an immersion.

Proof Let us define $\mathbf{\Psi}\colon \mathcal{U} \times \mathbb{R}_k^{m\times n} \to \mathbb{R}^{m\times n}$ by

$$\mathbf{\Psi}(\boldsymbol{x}, \boldsymbol{B}) = \boldsymbol{D}\mathbf{\Phi}(\boldsymbol{x}) + \boldsymbol{B}.$$

If $k < n$ we have

$$n + k(n +_m -k) \leq n + (n-1)(m+1) \leq nm - 1.$$

Thus, by the first lemma and the third lemma, $\mathbf{\Psi}(\mathcal{U} \times \mathbb{R}_k^{m\times n})$ has measure zero, and so consequently does $\mathbf{\Psi}(\mathcal{U} \times \cup_{k=1}^{n-1} \mathbb{R}_k^{m\times n})$. It follows, therefore, that there are matrices $\boldsymbol{A}$ in the complement to $\cup_{k=1}^{n-1} \mathbb{R}_k^{m\times n}$, i.e., matrices of maximal rank, that are as small as we like, e.g., their components are as small as we like, and which satisfy the condition that $\boldsymbol{D}\mathbf{\Phi}(\boldsymbol{x}) + \boldsymbol{A}$ has maximal rank for every $\boldsymbol{x} \in \mathcal{U}$, as desired. ▼

Next we give a local approximation result.

5 **Lemma** *Let* M *be a smooth manifold of dimension* n *and let* $\Phi\colon \mathsf{M} \to \mathbb{R}^{\mathrm{m}}$ *be of class* C^1 *with* $\mathrm{m} > \mathrm{n}$. *Let $(\mathcal{U}, \phi)$ be a chart, let $\mathrm{K} \subseteq \mathcal{U}$ be compact, and suppose that, for all $\mathrm{x} \in \mathrm{K}$, $\mathrm{T_x}\Phi$ has maximal rank n. Then there exists $\delta \in \mathbb{R}_{>0}$ such that, if $\Psi \in \mathrm{C}^1(\mathsf{M}; \mathbb{R}^{\mathrm{m}})$ satisfies $\|\mathbf{D}(\Psi \circ \phi^{-1})(\mathbf{x})\| < \delta$ for $\mathbf{x} \in \phi(\mathrm{K})$, then $\mathrm{T_x}(\Phi + \Psi)$ has maximal rank n for all $\mathrm{x} \in \mathrm{K}$.*

Proof Let $d\colon K \to \mathbb{R}$ be defined by asking that $d(x)$ be the maximum absolute value of all $n \times n$ minors of $\boldsymbol{D}(\Psi \circ \phi^{-1})(\phi(x))$. Note that d is continuous and $d|K$ is positive. Thus, by virtue of compactness of K, there exists $\epsilon \in \mathbb{R}_{>0}$ such that $d(x) \geq \epsilon$ for all $x \in K$. Continuity of the determinant ensures that there exists $\delta \in \mathbb{R}_{>0}$ such that, if $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ satisfies $\|\boldsymbol{A}\| < \delta$, then $\boldsymbol{D}(\Psi \circ \phi^{-1})(\phi(x)) + \boldsymbol{A}$ has rank n for all $x \in K$. From this the result immediately follows. ▼

Now we use the preceding lemmata to arrive at a global approximation result.

6 **Lemma** *Let* M *be a smooth, connected, paracompact, Hausdorff manifold, let* $\Phi\colon \mathsf{M} \to \mathbb{R}^{\mathrm{m}}$ *be a mapping of class* C^2 *with* $\mathrm{m} \geq 2\mathrm{n}$, *and let* $\delta \in \mathrm{C}^0(\mathsf{M}; \mathbb{R}_{>0})$. *Then there exists a* C^1 *immersion* $\Psi\colon \mathsf{M} \to \mathbb{R}^{\mathrm{m}}$ *satisfying* $\|\Psi(\mathrm{x}) - \Phi(\mathrm{x})\| < \delta(\mathrm{x})$ *for every* $\mathrm{x} \in \mathsf{M}$.

Proof By [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.5] and since M is second countable as it is paracompact and connected [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.11] we let $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$, $(\mathcal{V}_j)_{j\in\mathbb{Z}_{>0}}$, and $(\mathcal{W}_j)_{j\in\mathbb{Z}_{>0}}$ be locally finite open covers by relatively compact sets such that $(\mathcal{W}_j, \phi_j)$ is a coordinate chart, and such that

$$\mathrm{cl}(\mathcal{U}_j) \subseteq \mathcal{V}_j, \quad \mathrm{cl}(\mathcal{V}_j) \subseteq \mathcal{W}_j$$

for each $j \in \mathbb{Z}_{>0}$. For each $j \in \mathbb{Z}_{>0}$ let ρ_j be a smooth function taking values in $[0,1]$ and such that $\rho_j(x) = 1$ for $x \in \mathrm{cl}(\mathcal{U}_j)$ and $\rho_j(x) = 0$ for $x \in \mathsf{M} \setminus \mathrm{cl}(\mathcal{V}_j)$ [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8]. We inductively define a sequence of maps $(\Psi_k)\mathsf{M}\mathbb{R}^m$ such that $\Psi_k - \Phi$ is smooth and has support contained in $K_k = \cup_{j=1}^k \mathrm{cl}(\mathcal{V}_j)$ for each $k \in \mathbb{Z}_{>0}$. For $k = 1$, let $\boldsymbol{A}_1 \in \mathbb{R}^{m\times n}$ and define

$$\Psi_1(x) = \Phi(x) + \rho_1(x)\boldsymbol{A}_1(\phi_1(x)).$$

By the fourth lemma above, we can choose $\boldsymbol{A}_1$ such that (1) $T_x\Psi_1$ is injective and (2) $\|\Psi_1(x) - \Phi(x)\| < \frac{\delta(x)}{2}$ for $x \in \mathrm{cl}(\mathcal{U}_1)$. Now suppose that we have defined $\Psi_1, \dots, \Psi_k$ such that, for each $j \in \{1, \dots, k\}$, $\Psi_j - \Phi$ is smooth, has support in K_j, and satisfies $\|\Psi_j(x) - \Phi(x)\| < \frac{\delta(x)}{2^j}$ for $x \in K_j$. For $\boldsymbol{A}_{k+1} \in \mathbb{R}^{m\times n}$ denote

$$\Psi_{k+1}(x) = \Psi_k(x) + \rho_{k+1}(x)\boldsymbol{A}_{k+1}(\phi_{k+1}(x)).$$

Moreover, choose $\boldsymbol{A}_{k+1}$ such that (1) $T_x\Psi_{k+1}$ is injective for $x \in \mathcal{W}_{k+1}$ (by the fourth lemma) and (2) $\|\Psi_{k+1}(x) - \Psi_k(x)\| < \frac{\delta(x)}{2^{k+1}}$ for $x \in K_k \cap \mathrm{cl}(\mathcal{W}_{k+1})$ (by the fifth lemma). Now take $\Psi(x) = \lim_{k\to\infty} \Psi_k(x)$, this limit existing and Ψ being of class C^2 since, for each $x \in \mathsf{M}$, there exists $N \in \mathbb{Z}_{>0}$ such that $\Psi_k(y) = \Psi_N(y)$ for y in some neighbourhood of x. Moreover, easily verify that Ψ is an immersion and satisfies $\|\Psi(x) - \Phi(x)\| < \delta(x)$ for each $x \in \mathsf{M}$. ▼

Next we modify the immersion of the preceding lemma so that it is injective.

7 Lemma *Let* M *be a smooth, connected, paracompact, Hausdorff manifold, let* $\Phi\colon \mathsf{M} \to \mathbb{R}^{\mathrm{m}}$ *be a mapping of class* C^2 *with* $\mathrm{m} \geq 2\mathrm{n} + 1$*, and let* $\delta \in \mathrm{C}^0(\mathsf{M}; \mathbb{R}_{>0})$*. Then there exists a* C^1 *injective immersion* $\Psi\colon \mathsf{M} \to \mathbb{R}^{\mathrm{m}}$ *satisfying* $\|\Psi(\mathrm{x}) - \Phi(\mathrm{x})\| < \delta(\mathrm{x})$ *for every* $\mathrm{x} \in \mathsf{M}$*.*

Proof By the preceding lemma, let $\Phi\colon \mathsf{M} \to \mathbb{R}^m$ be a C^2 immersion such that $\|\Upsilon(x) - \Phi(x)\| < \frac{\delta(x)}{2}$ for every $x \in \mathsf{M}$. By Theorem 1.2.9 (or the smooth version of this, which follows from the smooth Inverse Function Theorem exactly as Theorem 1.2.9 follows from the real analytic Inverse Function Theorem), for each $x \in \mathsf{M}$ there exists a relatively compact neighbourhood $\mathcal{U}_x$ about x such that $\Upsilon|\mathcal{U}_x$ is injective. By paracompactness (and so second countability, since M is connected) of M let $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ be a locally finite open cover of M by relatively compact open sets such that $\Upsilon|\mathcal{U}_j$ is injective. For each $j \in \mathbb{Z}_{>0}$ let ρ_j be a smooth function on M taking values in $[0,1]$ and satisfying $\rho_j(x) = 0$ for $x \in \mathsf{M} \setminus \mathcal{U}_j$, and suppose that $\sum_{j=1}^\infty \rho_j(x) = 1$ for every $x \in \mathsf{M}$, i.e., $(\rho_j)_{j\in\mathbb{Z}_{>0}}$ is a partition of unity. For any sequence $(\boldsymbol{y}_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}^{2n+1}$ satisfying $\|\boldsymbol{y}_j\| < \frac{\delta(x)}{2^{j+2}}$ for $x \in \mathcal{U}_j$, for $k \in \mathbb{Z}_{>0}$ define

$$\Psi_k(x) = \Upsilon(x) + \sum_{j-1}^{k-1} \rho_j(x)\boldsymbol{y}_j.$$

By the fifth lemma above the sequence $(\boldsymbol{y}_j)_{j\in\mathbb{Z}_{>0}}$ can be chosen so that Ψ_k is an immersion for each $k \in \mathbb{Z}_{>0}$.

For each $k \in \mathbb{Z}_{>0}$ let

$$\mathcal{O}_k = \{(x,y) \in \mathsf{M}\times\mathsf{M} \mid \rho_k(x) \neq \rho_k(y)\}$$

and define $\Gamma_k\colon \mathcal{O}_k \to \mathbb{R}^{2n+1}$ by

$$\Gamma_k(x,y) = \frac{\Psi_k(x) - \Psi_k(y)}{\rho_k(x) - \rho_k(y)}.$$

Noting that Γ_k is of class C^2, by the second lemma above we conclude that image(Γ_k) has measure zero. Thus we can successively amend our section of the sequence $(\boldsymbol{y}_j)_{j\in\mathbb{Z}_{>0}}$ in such a way that we additionally have $\boldsymbol{y}_k \notin \mathrm{image}(\Gamma_k)$. Upon doing so, we define $\Psi(x) = \lim_{k\to\infty}\Psi_k(x)$, noting that Ψ is of class C^2, is an immersion, and satisfies $\|\Psi(x)-\Phi(x)\| < \delta(x)$.

It remains to show that Ψ is injective. Suppose that $\Psi(x) = \Psi(y)$. Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\rho_j(x) = \rho_j(y)$ and $\Psi_j(x) = \Psi_j(y) = \Psi(x) = \Psi(y)$ for every $j \geq N$. Since $\boldsymbol{y}_{N-1} \notin \mathcal{O}_{N-1}$ we have $\rho_{N-1}(x) = \rho_{N-1}(y)$. Consequently, the equality $\Psi_N(x) = \Psi_N(y)$ forces us to conclude that $\Psi_{N-1}(x) = \Psi_{N-1}(y)$. We can proceed inductively backwards to conclude that $\rho_j(x) = \rho_j(y)$ and $\Psi_j(x) = \Psi_j(y)$ for every $j \in \mathbb{Z}_{>0}$. Consequently, $\Upsilon(x) = \Upsilon(y)$. Since $\rho_j(x) \neq 0$ for at least one $j \in \mathbb{Z}_{>0}$, it follows that $x, y \in \mathcal{U}_j$ for some $j \in \mathbb{Z}_{>0}$. However, this is in contradiction with the fact that $\Upsilon|\mathcal{U}_j$ is injective. ▼

Now we can conclude the proof of the theorem. It remains to show that an injective immersion can be chosen that is an embedding. By Proposition 6.5.5 this means that we must be able to choose a *proper* injective immersion. Let δ be the constant function on M taking the value 1 and let u be a smooth exhaustion function on M; such a function exists by Lemma 3.3.3. Now define $\Phi \in \mathrm{C}^\infty(\mathsf{M};\mathbb{R}^{2n+1})$ by

$$\Phi(x) = (u(x), 0, \dots, 0).$$

By the previous lemma let Ψ be an injective immersion such that $\|\Psi(x) - \Phi(x)\| < 1$ for every $x \in \mathsf{M}$. We claim that Ψ is proper. Let $K \subseteq \mathbb{R}^{2n+1}$ be compact. If $x \in \Psi^{-1}(K)$ then

$$\|\Psi(x)\| \leq \sup\{\|\boldsymbol{x}\| \mid \boldsymbol{x} \in K\} \triangleq C_K,$$

and so $\|\Phi(x)\| \leq C_K + 1$. Since $\|\Phi(x)\| = |u(x)|$ we have $\Psi^{-1}(K) \subseteq u^{-1}((-\infty, C_K+1])$. Since u is an exhaustion function we conclude that $\Psi^{-1}(K)$ is compact, being a closed subset of the compact set $u^{-1}((-\infty, C_K+1])$, [Runde 2005, Proposition 3.3.6]. ■

A much harder theorem, also due to Whitney [1944], asserts that an n-dimensional paracompact Hausdorff manifold can be smoothly embedded in $\mathbb{R}^{2n}$. For our purposes, the preceding theorem suffices. For example, it allows us to immediately conclude the following.

6.5.7 Corollary (Existence of smooth Riemannian metrics) *If* M *is a smooth, paracompact, Hausdorff manifold, then there exists a smooth Riemannian metric on* M.

Proof It suffices to give a smooth Riemannian metric on each connected component of M. By the Whitney Embedding Theorem, one can embed each such connected component into $\mathbb{R}^N$ for suitable N, and the restriction of the Euclidean inner product to the embedded component defines a smooth Riemannian metric on that component. ■

6.5.3 The Remmert embedding theorem for Stein manifolds

In this section we give the basic embedding theorem for holomorphic manifolds due to Remmert [1955], with additional contributions by Bishop [1961] and Narasimhan [1961]. The proof we give follows the treatment of Hörmander [1973]. For holomorphic manifolds, note that a general embedding theorem is not possible, for example, by virtue of the fact that there are no compact positive-dimensional holomorphic submanifolds of $\mathbb{C}^N$ (Example 4.2.13–3). Thus we need restrictions on holomorphic manifolds in order for them to admit a holomorphic embedding into $\mathbb{C}^n$; it turns out that the condition is that the manifold be a Stein manifold. Certainly a holomorphic submanifold of $\mathbb{C}^N$ is Stein by Proposition 6.3.8. The difficult thing to prove is the converse, and this is what we now do.

6.5.8 Theorem (The Remmert Embedding Theorem) *If* M *is a holomorphic, paracompact, Hausdorff and connected manifold of dimension* n*, then there exists a proper holomorphic embedding of* M *in* $\mathbb{C}^{2\mathrm{n}+1}$*.*

Proof The first part of the proof bears a strong resemblance in structure to the proof of the Whitney Embedding Theorem. First we show that compact subsets of M can be immersed in sufficiently large copies of Euclidean space.

1 Lemma *Let* M *be a holomorphically separable holomorphic manifold possessing global coordinate functions and let* $\mathrm{K} \subseteq \mathsf{M}$ *be compact. Then there exists* $\mathrm{N} \in \mathbb{Z}_{>0}$ *and a holomorphic map* $\Phi\colon \mathsf{M} \to \mathbb{C}^{\mathrm{N}}$ *such that* $\mathrm{T}^{\mathbb{C}}_{\mathrm{z}}\Phi$ *is injective for each* $\mathrm{z} \in \mathrm{K}$*.*

Proof For $z \in K$ let $\zeta^1, \dots, \zeta^n \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be holomorphic functions such that there exists a neighbourhood $\mathcal{U}_z$ of z for which

$$\phi_z\colon w \mapsto (\zeta^1_z(w), \dots, \zeta^n_z(w))$$

gives a $\mathbb{C}$-chart $(\mathcal{U}_z, \phi_z)$. Since K is compact, we can cover it by finitely many such charts denoted $(\mathcal{U}_j, \phi_j)$, $j \in \{1, \dots, k\}$. Let $\Psi\colon \mathsf{M} \to \mathbb{C}^{nk}$ be the map

$$\Psi(z) = (\phi_1(z), \dots, \phi_k(z)).$$

If $z \in K$ then $z \in \mathcal{U}_j$ for some $j \in \{1, \dots, k\}$. Then $T^{\mathbb{C}}_z\phi_j$ is injective, and so $T^{\mathbb{C}}_z\Psi$ is injective. Let $\mathcal{W} = \cup_{j=1}^k \mathcal{U}_j \times \mathcal{U}_j \subseteq K \times K$ be a neighbourhood of the diagonal. We claim that $\Psi \times \Psi|\mathcal{W}$ is injective. Indeed, if $(z_1, z_2) \in \mathcal{W}$ that $z_1, z_2 \in \mathcal{U}_j$ for for some $j \in \{1, \dots, k\}$. Then we immediately have $z_1 = z_2$ because of injectivity of ϕ_j. Now let $(z, w) \in K \times K \setminus \mathcal{W}$. Since M is holomorphically separable, there exists $f_{(z,w)} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that $f_{(z,w)}(z) \neq f_{(z,w)}(w)$. There thus exists a neighbourhood $\mathcal{W}_{(z,w)}$ of (z, w) such that $f_{(z,w)}(z') \neq f_{(z,w)}(w')$ for each $(z', w') \in \mathcal{W}_{(z,w)}$. By compactness of $K \times K$ we can then find finitely many holomorphic functions, say $f_{nk+1}, \dots, f_m$, such that, if $f_j(z_1) = f_j(z_2)$ for all $j \in \{nk+1, \dots, m\}$, then $z_1 = z_2$, provided that $(z_1, z_2) \in K \times K \setminus \mathcal{W}$. If we define $\Phi\colon \mathsf{M} \to \mathbb{C}^m$ by

$$\Phi(z) = (\phi_1(z), \dots, \phi_k(z), f_{nk+1}(z), \dots, f_m(z)),$$

we see that Φ has the desired properties stated in the lemma. ▼

The next lemma says that, up to a point, we can reduce the number m from the preceding lemma. We suppose, as usual, that $\dim_{\mathbb{C}}(\mathsf{M}) = n$.

2 Lemma *Let* M *be a holomorphic manifold, let* $K \subseteq M$ *be compact, and let* $\Phi \in C^{hol}(M;\mathbb{C}^{m+1})$, $m \geq 2n$, *be an immersion on* K. *Then, for* $\epsilon \in \mathbb{R}_{>0}$, *there exists* $\mathbf{v} \in \mathbb{C}^m$ *such that* $\|\mathbf{v}\| < \epsilon$ *and such that the map*

$$z \mapsto (\Phi^1(z) - v^1\Phi^{m+1}(z), \dots, \Phi^m(z) - v^m\Phi^{m+1}(z)) \in \mathbb{C}^m$$

is an immersion on K. *Moreover, the set of* $\mathbf{v} \in \mathbb{C}^m$ *for which this is possible has a complement of measure zero.*

Proof First let us suppose that $K \subseteq \mathcal{U}$ for a $\mathbb{C}$-chart $(\mathcal{U}, \phi)$. To keep notation simple, let us identify the local representative of Φ with Φ.

Let us define

$$\begin{aligned} \boldsymbol{b}\colon K \times \mathbb{C}^n &\to \mathbb{C}^{m+1} \\ (z, \boldsymbol{\lambda}) &\mapsto \Big(\sum_{j=1}^n \lambda^k \frac{\partial \Phi^1}{\partial z^k}(z), \dots, \sum_{j=1}^n \lambda^k \frac{\partial \Phi^{m+1}}{\partial z^k}(z)\Big). \end{aligned}$$

We claim that if $(\boldsymbol{v}, 1) \notin \operatorname{image}(\boldsymbol{b})$ then $\boldsymbol{v}$ satisfies the conclusions of the lemma. Indeed, suppose that $\boldsymbol{v} \in \mathbb{C}^m$ does not satisfy the conclusions of the lemma. This means that there exists $\boldsymbol{\lambda} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that

$$\sum_{k=1}^n \lambda^k \Big(\frac{\partial \Phi^1}{\partial z^k}(z) - v^1 \frac{\partial \Phi^{m+1}}{\partial z^k}(z), \dots, \frac{\partial \Phi^m}{\partial z^k}(z) - v^m \frac{\partial \Phi^{m+1}}{\partial z^k}(z)\Big) = 0$$

for some $\boldsymbol{z} \in K$. Equivalently, if we take $v^{m+1} = 1$, there exists $\boldsymbol{\lambda} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that

$$\sum_{k=1}^n \lambda^k \frac{\partial \Phi^j}{\partial z^k}(z) = v^j \sum_{k=1}^n \lambda^k \frac{\partial \Phi^{m+1}}{\partial z^k}(z), \qquad j \in \{1, \dots, m+1\},$$

for some $\boldsymbol{x} \in K$. First note, since $T_z\Phi$ is injective, this implies that

$$\sum_{k=1}^n \lambda^k \frac{\partial \Phi^{m+1}}{\partial z^k}(z) \neq 0,$$

since otherwise $\boldsymbol{\lambda}$ is a nonzero element of $\ker(T_z\Phi)$. Then, by scaling $\boldsymbol{\lambda}$, we can suppose that

$$\sum_{k=1}^n \lambda^k \frac{\partial \Phi^{m+1}}{\partial z^k}(z) = 1.$$

It then immediately follows that $\boldsymbol{b}(z, \boldsymbol{\lambda}) = (\boldsymbol{v}, 1)$, and we infer that if $(\boldsymbol{v}, 1) \notin \operatorname{image}(\boldsymbol{b})$ then $\boldsymbol{a}$ satisfies the conclusions of the lemma. Now we note that, by Lemma 2 from the proof of Theorem 6.5.6, $\operatorname{image}(\boldsymbol{b})$ has measure zero in $\mathbb{C}^{m+1}$. Therefore, we also have that

$$\{\boldsymbol{v} \in \mathbb{C}^m \mid (\boldsymbol{v}, 1) \in \operatorname{image}(\boldsymbol{b})\}$$

has measure zero. Thus we can find $\boldsymbol{v} \in \mathbb{C}^m$ as in the statement of the lemma, and moreover can choose $\boldsymbol{v}$ as close to $\mathbf{0}$ as we like.

Finally, if $K \subseteq M$ is not necessarily contained in a $\mathbb{C}$-chart, we can cover K with finitely many compact sets, each of which is contained in a $\mathbb{C}$-chart. The set of $\boldsymbol{v} \in \mathbb{C}^m$ such that the conclusions of the lemma hold has a complement that is the union of finitely many sets of measure zero, and so has measure zero. We can thus still take $\boldsymbol{v}$ as small as desired, and satisfying the conclusions of the lemma. ▼

3 Lemma *Let* M *be a holomorphic manifold, let* $K \subseteq M$ *be compact, and let* $\Phi \in C^{hol}(M; \mathbb{C}^{m+1})$, $m \geq 2n + 1$, *be an injective immersion on* K. *Then, for* $\epsilon \in \mathbb{R}_{>0}$, *there exists* $\mathbf{v} \in \mathbb{C}^m$ *such that* $\|\mathbf{v}\| < \epsilon$ *and such that the map*

$$z \mapsto (\Phi^1(z) - v^1\Phi^{m+1}(z), \dots, \Phi^m(z) - v^m\Phi^{m+1}(z)) \in \mathbb{C}^m$$

is an injective immersion on K. *Moreover, the set of* $\mathbf{v} \in \mathbb{C}^m$ *for which this is possible has a complement of measure zero.*

Proof Let us define

$$\begin{aligned} \boldsymbol{b} \colon K \times K \times \mathbb{C} &\to \mathbb{C}^{m+1} \\ (z_1, z_2, \alpha) &\mapsto \alpha(\Phi^1(z_1) - \Phi(z_2), \dots, \Phi^{m+1}(z_1) - \Phi^{m+1}(z_2)). \end{aligned}$$

We claim that if $(\boldsymbol{b}, 1) \notin \operatorname{image}(\boldsymbol{b})$ then the conclusions of the lemma hold. Indeed, suppose that the conclusions of the lemma do not hold. Then there exists $z_1, z_2 \in K$ such that

$$\Phi^j(z_1) - v^j\Phi^{m+1}(z_1) = \Phi^j(z_2) - v^j\Phi^{m+1}(z_2), \qquad j \in \{1, \dots, m+1\},$$

which we rewrite as

$$\Phi^j(z_1) - \Phi^j(z_2) = v^j(\Phi^{m+1}(z_1) - \Phi^{m+1}(z_2)), \qquad j \in \{1, \dots, m+1\},$$

taking $v^{m+1} = 1$. Since Φ is injective on K, we must have $\Phi^{m+1}(z_1) - \Phi^{m+1}(z_2) \neq 0$, since otherwise we arrive at a contradiction. Consequently, there exists $\alpha \in \mathbb{C}$ such that $\alpha(\Phi^{m+1}(z_1) - \Phi^{m+1}(z_2)) = 1$, and so

$$\alpha(\Phi^j(z_1) - \Phi^j(z_2)) = v^j, \qquad j \in \{1, \dots, m+1\}.$$

Thus $(\boldsymbol{a}, 1) \in \operatorname{image}(\boldsymbol{b})$, showing that, for the conclusions of the lemma to hold, it suffices to choose $\boldsymbol{v}$ so that $(\boldsymbol{v}, 1) \notin \operatorname{image}(\boldsymbol{b})$. By Lemma 2 from the proof of Theorem 6.5.6 we know that $\operatorname{image}(\boldsymbol{b})$ has measure zero. Combining this with the preceding lemma we see that we can choose $\boldsymbol{v} \in \mathbb{C}^m$ from a set with a complement of measure zero that satisfies the conclusions of the lemma. ▼

We may now prove a preliminary result concerning the existence of immersions and injective immersions of a holomorphic manifold in complex Euclidean space. To state the result, we recall two things. First, the set $C^{hol}(M; \mathbb{R}^m)$ is a Fréchet space and so a complete metrisable space . Second, a subset of a complete metric space is of ***first category*** if it is contained in a countable union of closed sets with nonempty interior. A subset of first category has no interior and a dense complement, i.e., can be thought of as being "small." ref

4 Lemma *If* M *is a second countable, holomorphically separable holomorphic manifold that admits global coordinate functions, then*

(i) the set of holomorphic immersions of M *in* $\mathbb{C}^m$ *has a complement of first category in* $C^{hol}(M; \mathbb{C}^m)$ *if* $m \geq 2n$ *and*

(ii) the set of holomorphic injective immersions of M *in* $\mathbb{C}^m$ *has a complement of first category in* $C^{hol}(M; \mathbb{C}^m)$ *if* $m \geq 2n + 1$.

Proof We prove the first assertion. Since M is second countable, there exists a family $(K_j)_{j\in\mathbb{Z}_{>0}}$ of compact sets such that $K_j \subseteq \mathrm{int}(K_{j+1})$, $j \in \mathbb{Z}_{>0}$, and $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}} K_j$ [Aliprantis and Border 2006, Lemma 2.76]. It thus suffices to show that if $K \subseteq \mathsf{M}$ is compact then the set of mappings $\Phi \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^m)$ for which $T_z\Phi$ is not injective for some $z \in K$ has first category. Let us denote this set of mappings by M_K. We first claim that M_K is closed. Suppose that $(\Phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in M_K converging to $\Phi \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^m)$. Let $z_j \in K$ be such that $T_{z_j}\Phi_j$ is not injective. By the Bolzano—Weierstrass Theorem let $(z_{j_k})_{k\in\mathbb{Z}_{>0}}$ be a subsequence converging to $z \in K$. We claim that $T_z\Phi$ is not injective. Indeed, suppose it is injective. By continuity there exists a neighbourhood $\mathcal{U}$ of z such that $T_w\Phi$ is injective for every $w \in \mathcal{U}$. ref Since convergence in $\mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^m)$ implies uniform convergence of derivatives , there also exists a neighbourhood $\mathcal{V}$ of z and $N \in \mathbb{Z}_{>0}$ such that $T_w\Phi_j$ is injective for $w \in \mathcal{V}$ and $j \geq N$. But this contradicts the fact that there exists $N' \in \mathbb{Z}_{>0}$ such that $z_k \in \mathcal{V}$ for $k \geq N'$ and $T_{z_k}\Phi_k$ is not injective. Let us next show that M_K has empty interior. According to Lemma 1 let $k \in \mathbb{Z}_{>0}$ and $\Psi \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^k)$ be such that Ψ is an injective immersion on K. Let $\Phi \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^m)$ and denote by (Φ,Ψ) the holomorphic mapping from M to $\mathbb{C}^{m+k}$ obtained by adjoining the components of Ψ to those of Φ. Now apply Lemma 2 r times to arrive at $a_r^j \in \mathbb{C}$, $j \in \{1,\dots,m\}$, $r \in \{1,\dots,m\}$, such that, if we define

$$\Phi'^j = \Phi^j + \sum_{r=1}^{k} a_r^j \Psi^r,$$

we have that $T_z\Phi'$ is injective for $z \in K$. Thus $\Phi' \notin M_K$. Since the coefficients a_r^j, $j \in \{1,\dots,m\}$, $r \in \{1,\dots,m\}$, can be chosen arbitrarily small, we can moreover conclude that any neighbourhood of Φ contains elements from the complement of M_K, and so M_K must have empty interior. Thus M_K is of first category, as desired.

The second assertion of the lemma follows from Lemma 3 as the first part follows from Lemma 2. ▼

Our first result is that compact sets can be contained in holomorphic polyhedra (see Definition 6.1.7).

5 Lemma *If* M *is a Stein manifold, if* $\mathrm{K} \subseteq \mathsf{M}$ *is a compact holomorphically convex set, and if* $\mathcal{U}$ *is a neighbourhood of* K *in* M*, then there exists a relatively compact holomorphic polyhedron* P *such that* $\mathrm{K} \subseteq \mathrm{P}$ *and* $\mathrm{cl}(\mathrm{P}) \subseteq \mathcal{U}$*.*

Proof It suffices to take $\mathcal{U}$ to be relatively compact. Let $z \in \mathrm{bd}(\mathcal{U})$ and let $f_z \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ have the property that $|f_z(z)| > \|f_z\|_K$. By rescaling we have $|f_z(z)| > 1$ and $\|f\|_K < 1$. Then there exists a relatively compact neighbourhood $\mathcal{V}_z$ of z such that $|f(w)| > 1$ for $w \in \mathrm{cl}(\mathcal{V}_z) \cap \mathrm{bd}(\mathcal{U})$. By compactness of $\mathrm{bd}(\mathcal{U})$ we choose $z_1,\dots,z_m \in \mathrm{bd}(\mathcal{U})$ such that $\mathrm{bd}(\mathcal{U}) \subseteq \cup_{j=1}^m \mathcal{U}_{z_j}$. Let $f_j = f_{z_j}$, $j \in \{1,\dots,m\}$. Note that the set

$$\{z \in \mathsf{M} \mid |f_j(z)| < 1,\ j \in \{1,\dots,m\}\} \tag{6.30}$$

contains K and its closure does not intersect $\mathrm{bd}(\mathcal{U})$. Thus we can take P to be the union of the connected components of the set (6.30) that intersect $\mathcal{U}$. ▼

Now let us reduce, up to a point, the number of holomorphic polyhedra needed to cover a compact set.

6 Lemma *Let* M *be a Stein manifold and let* $\mathrm{K} \subseteq \mathsf{M}$ *be compact. If* P *is a holomorphic polyhedron of order* $\mathrm{m}+1$, $\mathrm{m} \geq 2\mathrm{n}$, *that covers* P, *then there exists a holomorphic polyhedron* P' *of order* m *such that* $\mathrm{K} \subseteq \mathrm{P}' \subseteq \mathrm{P}$.

Proof Let $f_1, \dots, f_{m+1} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be such that P is a union of some connected components of

$$\{z \in \mathsf{M} \mid |f_j(z)| < 1,\ j \in \{1, \dots, m+1\}\}.$$

Let $\alpha_0 \in (0,1)$ be such that $\|f_j\|_K < \alpha_0$ for each $j \in \{1, \dots, m+1\}$, this since the functions $|f_1|, \dots, |f_{m+1}|$ achieve their maximum on K by compactness. Let α_1, α_2, and α_3 satisfy

$$\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < 1.$$

Let us define a compact set

$$K_2 = \{z \in \mathrm{cl}(P) \mid |f_{m+1}(z)| \geq \alpha_2\}.$$

Now choose $f'_1, \dots, f'_{m+1} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ to satisfy the following conditions:

1. $f'_{m+1} = f_{m+1}$;
2. by Lemma 4 choose $f'_1, \dots, f'_m$ so that the map

$$z \mapsto \Big(\frac{f'_1(z)}{f_{m+1}(z)}, \dots, \frac{f'_m(z)}{f_{m+1}(z)}\Big)$$

   is an immersion on K_2;
3. again by Lemma 4 and noting that choosing $\frac{f'_j}{f_{m+1}}$ close enough to $\frac{f_j}{f_{m+1}}$, $j \in \{1, \dots, m\}$, implies that f'_j can be made as close as desired to f_j on K, $\|f'_j\|_K < \alpha_0$, $j \in \{1, \dots, m\}$;
4. arguing as in the preceding point, f'_j can be made sufficiently close to f_j, $j \in \{1, \dots, m\}$, such that $\mathrm{cl}(\mathcal{U}) \subseteq P$, where

$$\mathcal{U} = \{z \in P \mid |f'_j(z)| < \alpha_3,\ j \in \{1, \dots, m+1\}\}.$$

(All statements regarding "closeness" above are made with reference to the Fréchet topology on $\mathrm{C}^{\mathrm{hol}}(\mathsf{M}; \mathbb{C}^m)$.) ref

Next, for $k \in \mathbb{Z}_{>0}$, let

$$\mathcal{V}_k = \{z \in \mathsf{M} \mid |f'_j(z)^k - f_{m+1}(z)^k| < \alpha_1^k,\ j \in \{1, \dots, m\}\}.$$

Let us also denote by P'_k the holomorphic polyhedron that is the union of the connected components of $\mathcal{V}_k$ which have nonempty intersection with K. Noting that $\lim_{k\to\infty}(\frac{\alpha_0}{\alpha_1})^k = 0$, let k be sufficiently large that $2\alpha_0^k < \alpha_1^k$. Moreover,

$$|f'_j(z)^k - f_{m+1}(z)^k| \leq |f'_j(z) - f_{m+1}(z)|^k \leq 2\alpha_0^k, \qquad j \in \{1, \dots, m\},$$

for every $z \in K$ and $k \in \mathbb{Z}_{>0}$. Thus

$$|f'_j(z)^k - f_{m+1}(z)^k| < \alpha_1^k, \qquad j \in \{1, \dots, m\},$$

for $z \in \mathsf{M}$ and for large k, and from this we conclude, therefore, that $K \subseteq \mathcal{V}_k$ for k sufficiently large. Obviously we also have $K \subseteq P'_k$ for sufficiently large k.

To conclude the proof we must show that $P'_k \subseteq P$ for sufficiently large k. We shall do this by proving that $P'_k \subseteq \mathcal{U}$ for k sufficiently large. Suppose that there exists no $N \in \mathbb{Z}_{>0}$ such that $P'_k \subseteq \mathcal{U}$ for all $k \geq N$. Noting that, for every $k \in \mathbb{Z}_{>0}$, every connected component of P'_k intersects $\mathcal{U}$ since it intersects K, this means there for each $N \in \mathbb{Z}_{>0}$ there exists $k \geq N$ such that $P'_k \cap \mathrm{bd}(\mathcal{U}) \neq \emptyset$. Thus we have an increasing sequence $(k_r)_{r\in\mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ and a sequence $(z_r)_{r\in\mathbb{Z}_{>0}}$ such that $z_r \in P_{k_r} \cap \mathrm{bd}(\mathcal{U})$. We claim that $f_{m+1}(z_r) \geq \alpha_2$ for r sufficiently large. Suppose otherwise so that there is an increasing sequence $(r_l)_{l\in\mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ such that $f_{m+1}(z_{r_l}) < \alpha_2$ for each $l \in \mathbb{Z}_{>0}$. Then we have

$$|f'_j(z_{r_l})|^{k_{r_l}} \leq |f'_j(z_{r_l}) - f_{m+1}(z_{r_l})|^{k_{r_l}} + |f_{m+1}(z_{r_l})|^{k_{r_l}} < \alpha_1^{k_{r_l}} + \alpha_2^{k_{r_l}}.$$

We have $\alpha_1^k + \alpha_2^k < \alpha_3^k$ for large k, and so we conclude that $|f'_j(z_{r_l})|^{k_{r_l}} < \alpha_3^{k_{r_l}}$ for large l. This contradicts the fact that $z_{r_l} \in \mathrm{bd}(\mathcal{U})$. Thus we conclude that $f_{m+1}(z_r) \geq \alpha_2$ for r sufficiently large. That is to say, for large r,

$$z_r \in L \triangleq \{z \in \mathrm{bd}(\mathcal{U}) \mid f_{m+1}(z) \geq \alpha_2\}.$$

Let us now arrive at a few estimates. We denote $F_j = \frac{f'_j}{f_{m+1}}$, $j \in \{1, \dots, m\}$, understanding that we will be evaluating F_j only at points where f_{m+1} is nonzero. For some of the estimates, we shall utilise a $\mathbb{C}$-chart $(\mathcal{W}, \phi)$ for M about $z \in L$. For convenience in these cases, we identity $\mathcal{W}$ and $\phi(\mathcal{W})$, and so we work in $\mathbb{C}^n$. In particular, we will let $\boldsymbol{\zeta} \in \mathbb{C}^n$ satisfy $\|\zeta\| = \frac{1}{k^2}$.

1. Let $j \in \{1, \dots, m\}$ and $\boldsymbol{z} \in L$. Using Taylor expansions,

$$|f_{m+1}(\boldsymbol{z} + \boldsymbol{\zeta})|^k = |f_{m+1}(\boldsymbol{z}) + \boldsymbol{D}f_{m+1}(\boldsymbol{z}) \cdot \boldsymbol{\zeta} + O(k^{-4})|^k.$$

   Thus, for k sufficiently large, $|f_{m+1}(\boldsymbol{z} + \boldsymbol{\zeta})|^k \geq \frac{\alpha_2^k}{2}$. Note that this estimate holds uniformly in $\boldsymbol{z} \in L$ since the terms in $O(k^{-4})$ involve derivatives of f_{m+1} and so are uniformly $O(k^{-4})$ on L. Thus there exists $N_1 \in \mathbb{Z}_{>0}$ such that

$$|f_{m+1}(\boldsymbol{z} + \boldsymbol{\zeta})|^k \geq \frac{\alpha_2^k}{2}, \qquad k \geq N_1,\ \boldsymbol{z} \in L,\ \|\zeta\| = \frac{1}{k^2}.$$

2. Let $j \in \{1, \dots, m\}$. A Taylor expansion gives

$$\frac{F_j(\boldsymbol{z} + \boldsymbol{\zeta})}{F_j(\boldsymbol{z})} = 1 + F_j(\boldsymbol{z})^{-1}\boldsymbol{D}F_j(\boldsymbol{z}) \cdot \boldsymbol{\zeta} + O(k^{-4}).$$

   Since F_j is an immersion on L, there exists $C \in \mathbb{R}_{>0}$ such that

$$\max\{|F_j(\boldsymbol{z})^{-1}\boldsymbol{D}F_j(\boldsymbol{z}) \cdot \zeta| \mid j \in \{1, \dots, m\}\} \geq C\|\zeta\|.$$

   Thus we have

$$\max\left\{\left|\frac{F_j(\boldsymbol{z} + \boldsymbol{\zeta})^k}{F_j(\boldsymbol{z})^k} - 1\right| \;\middle|\; j \in \{1, \dots, m\}\right\} \geq kC\|\zeta\| + O(k^{-2}) = \frac{C}{k} + O(k^{-2}),$$

and so

$$\max\left\{\left|\frac{F_j(z+\zeta)^k}{F_j(z)^k}-1\right| \;\middle|\; j\in\{1,\dots,m\}\right\} \geq \frac{C}{2k}$$

for k sufficiently large. More precisely, since the term $O(k^{-2})$ involves derivatives of F_j, it is $O(k^{-2})$ uniformly, and so there exists $N_2 \in \mathbb{Z}_{>0}$ such that

$$\max\left\{\left|\frac{F_j(z+\zeta)^k}{F_j(z)^k}-1\right| \;\middle|\; j\in\{1,\dots,m\}\right\} \geq \frac{C}{2k}, \qquad k \geq N_2,\ z\in L,\ \|\zeta\| = \frac{1}{k^2}.$$

3. Let $j \in \{1,\dots,m\}$. If $z \in L$ then $f_j'(z) = \alpha_3$ and $f_{m+1}(z) \in [\alpha_2,\alpha_3)$ and so $F_j(z) \geq 1$.
4. Let $j \in \{1,\dots,m\}$. If $z \in L\cap\mathcal{V}_k$ then we have, by definition of L and $\mathcal{V}_k$, $|F_j(z)^k-1| \leq (\frac{\alpha_1}{\alpha_2})^k$. Note that $\lim_{k\to\infty} k(\frac{\alpha_1}{\alpha_2})^k = 0$, and so for k sufficiently large we have $(\frac{\alpha_1}{\alpha_2})^k < \frac{C}{4k}$. Therefore, for k sufficiently large, we have $|F_j(z)^k - 1| \leq \frac{C}{4k}$. By compactness of L this bound holds uniformly, and so there exists $N_3 \in \mathbb{Z}_{>0}$ such that

$$|F_j(z)^k - 1| \leq \frac{C}{4k}, \qquad k \geq N_3,\ z \in L\cap\mathcal{V}_k.$$

5. Let $j \in \{1,\dots,m\}$. We shall write

$$|f_j'(z+\zeta)^k - f_{m+1}(z+\zeta)^k| = |F_j(z+\zeta)^k - 1||f_{m+1}(z+\zeta)|^k$$

and

$$F_j(z+\zeta)^k - 1 = F_j(z)^k\left(\frac{F_j(z+\zeta)^k}{F_j(z)^k}-1\right) + F_j(z)^k - 1.$$

Using the preceding two expressions and combining the above estimates gives $N' \in \mathbb{Z}_{>0}$ such that

$$|f_j'(z+\zeta)^k - f_{m+1}(z+\zeta)^k| \geq \frac{C\alpha_2^k}{8k}, \qquad k \geq N',\ z\in L,\ \|\zeta\| = \frac{1}{k^2}.$$

6. Now we note that $\lim_{k\to\infty} k(\frac{\alpha_1}{\alpha_2})^k = 0$ and so $k(\frac{\alpha_1}{\alpha_2})^k < \frac{C}{8}$ for k sufficiently large. Thus we finally arrive at the estimate: There exists $N \in \mathbb{Z}_{>0}$ such that

$$|f_j'(z+\zeta)^k - f_{m+1}(z+\zeta)^k| \geq \alpha_1^k, \qquad k \geq N',\ z\in L\cap\mathcal{V}_k,\ \|\zeta\| = \frac{1}{k^2}.$$

This contradicts the definition of $\mathcal{V}_k$ and so we conclude that, for k sufficiently large, $P_k' \cap \mathcal{U} = \emptyset$ and so P_k' must be contained in $\mathcal{U}$. ▼

By Lemma 4 let $\Psi\colon \mathsf{M} \to \mathbb{C}^{2n+1}$ be a holomorphic injective immersion. By Proposition 6.1.5(iii) and since M is second countable being paracompact and connected [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.5], let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of holomorphically convex compact sets satisfying $K_j \subseteq \operatorname{int}(K_{j_1})$ and $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}} K_j$. By the preceding lemma, there exist holomorphic polyhedra P_j, $j \in \mathbb{Z}_{>0}$, such that $K_j \subseteq P_j \subseteq K_{j+1}$. Let

$$M_j = \sup\{\Psi^j(z) \mid z \in K_j,\ j\in\{1,\dots,2n+1\}\}.$$

We now prove another lemma.

7 Lemma *There exist* $f_1, \dots, f_{2n} \in C^{hol}(M)$ *such that*

$$\sup\{|f_j(z)| \mid z \in \mathrm{bd}(P_k),\ j \in \{1, \dots, 2n\}\} > k + M_{k+1}$$

for $k \in \mathbb{Z}_{>0}$.

Proof For $k \geq 2$ let $h'_{k,1}, \dots, h'_{k,2n} \in C^{hol}(M)$ be such that P_k is a union of connected components of the set

$$\{z \in M \mid |h_{k,j}(z)| < 1\}.$$

We thus have

$$\sup\{|h_{k,j}(z)| \mid z \in \mathrm{cl}(P_{k-1}),\ j \in \{1, \dots, 2n\}\} < 1,\ \ \sup\{|h_{k,j}(z)| \mid z \in \mathrm{bd}(P_k),\ j \in \{1, \dots, 2n\}\} = 1.$$

Now let $a_2 \in (1, \infty)$ be sufficiently small and let $m_2 \in \mathbb{Z}_{>0}$ be sufficiently large that, if we define $f_{2,j} = (a_2 h_{2,j})^{m_2}$, $j \in \{1, \dots, 2n\}$, then we have

$$\sup\{|f_{2,j}(z)| \mid z \in P_1,\ j \in \{1, \dots, 2n\}\} \leq 2^{-k}$$

and

$$\begin{aligned}\sup\{|f_{2,j}(z)| \mid z \in \mathrm{bd}(P_2),\ j \in \{1, \dots, 2n\}\}& \\ \geq M_{k+1} + k + 1 + \sup\{|f_{1,j(z)}| \mid z \in \mathrm{bd}(P_2),\ j \in \{1, \dots, 2n\}\}&.\end{aligned}$$

Then, inductively in $k \geq 3$, define take $a_k \in (1, \infty)$ sufficiently small and $m_k \in \mathbb{Z}_{>0}$ sufficiently large that

$$\sup\{|f_{k,j}(z)| \mid z \in P_{k-1},\ j \in \{1, \dots, 2n\}\} \leq 2^{-k}$$

and

$$\begin{aligned}\sup\{|f_{2,j}(z)| \mid z \in \mathrm{bd}(P_k),\ j \in \{1, \dots, 2n\}\}& \\ \geq M_{k+1} + k + 1 + \sup\left\{\sum_{l=1}^{k-1} |f_{1,j(z)}| \;\middle|\; z \in \mathrm{bd}(P_k),\ j \in \{1, \dots, 2n\}\right\}&.\end{aligned}$$

It is then clear that, on any compact set $K \subseteq M$, the sum

$$f_j(z) = \sum_{k=1}^{\infty} f_{k,j}(z)$$

ref converges uniformly, and so converges to a holomorphic function . It is easy to check that the functions $f_1, \dots, f_{2n}$ have the desired properties. ▼

The next lemma gives us the function we need to append to $f_1, \dots, f_{2n}$ to get the desired proper embedding.

8 Lemma *There exists* $\mathrm{f} \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ *such that*

$$\sup\{|\mathrm{f}(\mathrm{z})| \mid \mathrm{z} \in \mathrm{P}_{\mathrm{k}+1} \setminus \mathrm{P}_{\mathrm{k}}\} \geq \mathrm{k} + \mathrm{M}_{\mathrm{k}+1}.$$

Proof For $k \in \mathbb{Z}_{>0}$ let us denote

$$\begin{aligned} S_k &= \{z \in P_{k+1} \setminus P_k \mid \max\{|f_j(z)| \mid j \in \{1, \dots, 2n\}\} \leq k + M_{k+1}\}, \\ T_k &= \{z \in P_k \mid \max\{|f_j(z)| \mid j \in \{1, \dots, 2n\}\} \leq k + M_{k+1}\}. \end{aligned}$$

By the previous lemma,

$$\sup\{|f_j(z)| \mid z \in \mathrm{bd}(P_k),\ j \in \{1, \dots, 2n\}\} > k + M_{k+1}$$

and

$$\sup\{|f_j(z)| \mid z \in \mathrm{bd}(P_{k+1}),\ j \in \{1, \dots, 2n\}\} > k + 1 + M_{k+2} > k + M_{k+1}.$$

From these facts we conclude that S_k is a compact subset of $P_{k+1} \setminus P_k$ and that T_k is a compact subset of P_k. Moreover, $S_k \cap T_k = \emptyset$. By Proposition 6.1.2(i) we have that $\mathrm{hconv}_{\mathsf{M}}(S_k \cup T_k) \subseteq K_{k+2}$. In fact, by Proposition 6.1.8, $\mathrm{hconv}_{\mathsf{M}}(S_k \cup T_k) \subseteq P_{k+1}$. Now, by Theorem GA2.7.1.7, there exists $h_1 \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that

$$\sup\{|h_1(z)| \mid z \in T_1\} < \tfrac{1}{2}$$

and

$$\sup\{|h_1(z)| \mid z \in S_1\} \geq k + 1 + M_{k+1}.$$

We then iteratively define $h_k \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that

$$\sup\{|h_k(z)| \mid z \in T_1\} < 2^{-k} \tag{6.31}$$

and

$$\sup\{|h_k(z)| \mid z \in S_1\} \geq k + 1 + M_{k+1} + \sup\left\{\left|\sum_{j=1}^{k-1} h_j(z)\right| \;\middle|\; z \in S_k\right\}.$$

We claim that $\mathsf{M} = \cup_{k \in \mathbb{Z}_{>0}} T_k$. Indeed, if $z \in \mathsf{M}$ then there exists $N \in \mathbb{Z}_{>0}$ such that $z \in P_k$ for all $k \geq N$, by the manner in which the holomorphic polyhedra P_k, $k \in \mathbb{Z}_{>0}$, are defined. Also, since P_N is relatively compact, the functions $f_1, \dots, f_{2n}$ are bounded on P_N. Therefore, there exists k sufficiently large that

$$\max\{|f_j(z)| \mid j \in \{1, \dots, 2n\}\} \leq k + M_{k+1},$$

i.e., such that $z \in T_k$. It follows, therefore, that every compact subset of M is contained in T_k for some k and so the definition

$$f(z) = \sum_{k=1}^{\infty} h_k(z)$$

defines a holomorphic function on M by virtue of (6.31) and . Moreover, we easily verify ref
that

$$|f(z)| \geq k + M_{k+1}, \qquad z \in S_k,$$

and since $S_k \subseteq P_{k+1} \setminus P_k$, the lemma follows. ▼

To complete the proof of the theorem we will show that $\Phi' \in C^{\mathrm{hol}}(\mathsf{M};\mathbb{C}^{2n+1})$ defined by

$$\Phi'(z) = (f_1(z), \dots, f_{2n}(z), f(z)),$$

with $f_1, \dots, f_{2n}$ as in Lemma 7 and with f as in Lemma 8 gives the desired proper embedding after a minor modification. From the preceding lemma and the definition of M_k, $k \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}\sup\{|\Phi'^j(z)| \mid z \in P_{k+1} \setminus P_k,\ j \in \{1, \dots, 2n+1\}\}\\ \geq k + \sup\{|\Psi^j(z)| \mid z \in P_{k+1} \setminus P_k,\ j \in \{1, \dots, 2n+1\}\}.\end{aligned}$$

Therefore, noting that $\cup_{j=k}^{\infty} P_{j+1} \setminus P_j = \mathsf{M} \setminus P_k$, we have

$$\begin{aligned}\sup\{|\Phi'^j(z)| \mid z \in \mathsf{M} \setminus P_k,\ j \in \{1, \dots, 2n+1\}\}\\ \geq k + \sup\{|\Psi^j(z)| \mid z \in \mathsf{M} \setminus P_k,\ j \in \{1, \dots, 2n+1\}\}.\end{aligned}$$

Relative compactness of P_k then implies that

$$\{z \in \mathsf{M} \mid \max\{\Phi'^j(z) \mid j \in \{1, \dots, 2n+1\}\} \leq k + \max\{\Psi^j(z) \mid j \in \{1, \dots, 2n+1\}\}\}$$

is relatively compact. We then successively apply Lemma 3 to the map $z \mapsto (\Phi(z), \Psi(z))$ from M to $\mathbb{C}^{4n+2}$ and the compact sets K_k, $k \in \mathbb{Z}_{>0}$, to define

$$\Phi^j = \Phi'^j + \sum_{k=1}^{2n+1} v_k^j \Psi^k, \qquad j \in \{1, \dots, 2n+1\},$$

with the constants v_k^j, $j, k \in \{1, \dots, 2n+1\}$ chosen so that Φ is an injective immersion and such that

$$\sum_{k=1}^{2n+1} |v_k^j| < 1, \qquad j \in \{1, \dots, 2n+1\}.$$

We then have

$$\begin{aligned}&\{z \in \mathsf{M} \mid |\Phi^j(z)| \leq k,\ j \in \{1, \dots, 2n+1\}\}\\ &\quad\subseteq \{z \in \mathsf{M} \mid \max\{|\Phi'^j(z)| \mid j \in \{1, \dots, 2n+1\}\} \leq k + \max\{|\Phi'^j(z)| \mid j \in \{1, \dots, 2n+1\}\}\},\end{aligned}$$

which implies that Φ is proper since any compact subset of $\mathbb{C}^{2n+1}$ lies within a set of the form

$$\{\boldsymbol{z} \mid \max\{|z^j| \leq k \mid j \in \{1, \dots, 2n+1\}\}\}$$

for some $k \in \mathbb{Z}_{>0}$. ■

We have the following corollary that follows from the preceding theorem just as Corollary 6.5.7 follows from Theorem 6.5.6.

6.5.9 Corollary (Existence of holomorphic Hermitian metrics) *If* M *is a paracompact, Hausdorff Stein manifold, then there exists a holomorphic Hermitian metric on* M*.*

6.5.4 The Grauert–Morrey embedding theorem for real analytic manifolds

Our final embedding theorem is that for real analytic manifolds. This is an instance of where real analytic theory differs from holomorphic theory. For holomorphic manifolds, we have a restriction, namely that they be Stein, in order for them to be embedded in Euclidean space. For real analytic manifolds, there are no such restrictions.

6.5.10 Theorem (Embedding of real analytic manifolds) *If* M *is a second countable real analytic manifold of dimension* n*, then there exists an embedding of* M *into* $\mathbb{R}^{4\mathrm{n}+2}$*.*

Proof By Corollary 6.4.7 let $\overline{\mathsf{M}}$ be a Stein manifold of which M is a proper real analytic totally real submanifold. By Theorem 6.5.8 there is a proper embedding of $\overline{\mathsf{M}}$ in $\mathbb{C}^{2n+1} \simeq \mathbb{R}^{4n+2}$. Restricting this map to M gives the desired embedding since holomorphic maps are real analytic. ■

We have the following corollary that follows from the preceding theorem just as Corollary 6.5.7 follows from Theorem 6.5.6.

6.5.11 Corollary (Existence of real analytic Riemannian metrics) *If* M *is a real analytic, paracompact, Hausdorff manifold, then there exists a real analytic Riemannian metric on* M*.*

Bibliography

Abraham, R., Marsden, J. E., and Ratiu, T. S. [1988] *Manifolds, Tensor Analysis, and Applications*, second edition, number 75 in Applied Mathematical Sciences, Springer-Verlag, ISBN 0-387-96790-7.

Aliprantis, C. D. and Border, K. C. [2006] *Infinite-Dimensional Analysis: A Hitchhiker's Guide*, third edition, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 3-540-29586-0.

Aubin, T. [2001] *A Course in Differential Geometry*, number 27 in Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, ISBN 0-8218-2709-X.

Behnke, H. and Thullen, P. [1934] *Theorie der Funktionen mehrerer komplexen Veränderlichen*, number 3 in Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Heidelberg.

Bishop, E. [1961] *Mappings of partially analytic spaces*, American Journal of Mathematics, **83**, 209–242.

Bröcker, T. and Jänich, K. [1982] *Introduction to Differential Topology*, Cambridge University Press, New York/Port Chester/Melbourne/Sydney, ISBN 0-521-28470-8, translation of 1973 German edition by C. B. and M. J. Thomas.

Cartan, H. [1931] *Sur les domaines d'existence des fonctions de plusieurs variables complexes*, Bulletin de la Société Mathématique de France, **59**, 46–69.

— [1957] *Variétés analytiques réelles et variétés analytiques complexes*, Bulletin de la Société Mathématique de France, **85**, 77–99.

Cartan, H. and Thullen, P. [1932] *Zur theorie der singularitäten der funktionen mehrerer komplexen veränderlichen*, Mathematische Annalen, **106**(1), 617–647.

Cieliebak, K. and Eliashberg, Y. [2012] *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*, number 59 in American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, ISBN 0-8218-8533-2.

Goldschmidt, H. L. [1967] *Existence theorems for analytic linear partial differential equations*, Annals of Mathematics. Second Series, **86**(2), 246–270.

Grauert, H. [1958] *On Levi's problem and the imbedding of real-analytic manifolds*, Annals of Mathematics. Second Series, **68**, 460–472.

Grauert, H. and Remmert, R. [1977] *Theorie der Steinschen Räume*, number 227 in Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 3-540-08231-*.

— [2004] *Theory of Stein Spacess*, Classics in Mathematics, Springer-Verlag, New York/-Heidelberg/Berlin, ISBN 3-540-00373-8, reprint of 1979 translation by Alan Huckleberry of [Grauert and Remmert 1977].

Greene, R. E. and Wu, H. [1979] *C^∞-approximations of convex, subharmonic, and plurisubharmonic functions*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série, **12**(1), 47–84.

Hewitt, E. and Stromberg, K. [1975] *Real and Abstract Analysis*, number 25 in Graduate Texts in Mathematics, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-90138-8.

Hörmander, L. [1965] *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Mathematica, **113**, 89–152.

— [1973] *An Introduction to Complex Analysis in Several Variables*, second edition, North-Holland, Amsterdam/New York, ISBN 0-444-10523-9.

Kato, T. [1980] *Perturbation Theory for Linear Operators*, second edition, number 132 in Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York/-Heidelberg/Berlin.

Kecs, W. [1982] *The Convolution Product and Some Applications*, number 2 in Mathematics and its Applications, D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/-Tokyo, ISBN 90-2771-409-6.

Kobayashi, S. and Nomizu, K. [1963] *Foundations of Differential Geometry,* Volume I, number 15 in Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, New York, ISBN 0-470-49647-9.

Kreyszig, E. [1978] *Introductory Functional Analysis With Applications*, John Wiley and Sons, New York, ISBN 0-471-50731-8.

Kulkarni, R. S. [1978] *On complexifications of differentiable manifolds*, Inventiones Mathematicae, **44**(1), 49–64.

Michor, P. W. [1980] *Manifolds of Differentiable Mappings*, number 3 in Shiva Mathematics Series, Shiva Publishing Limited, Orpington, UK, ISBN 0-906812-03-8.

Narasimhan, R. [1961] *Imbedding of holomorphically complete complex spaces*, American Journal of Mathematics, **82**, 917–934.

Nash, J. [1956] *The imbedding problem for Riemannian manifolds*, Annals of Mathematics. Second Series, **63**, 20–63.

Oka, K. [1939] *Sur les fonctions de plusieurs variables. III. Deuxième problème de Cousin*, Journal of Science of the Hiroshima University, **9**, 7–19.

— [1953] *Sur les fonctions analytiques de plusieurs variables. IX. domaines finis sans point critique intérieur*, Japanese Journal of Mathematics, **23**, 97–155.

Reed, M. and Simon, B. [1980] *Methods of Mathematical Physics. I: Functional Analysis*, Academic Press, New York, ISBN 0-12-585050-6.

Remmert, R. [1955] *Theorie der Modifikationen. I. Stetige und eigentliche Modifikationen komplexer Räume*, Mathematische Annalen, **129**, 274–296.

Renardy, M. and Rogers, R. C. [1993] *An Introduction to Partial Differential Equations*, number 13 in Texts in Applied Mathematics, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-97952-2.

Richberg, R. [1968] *Stetige streng pseudokonvexe Funktionen*, Mathematische Annalen, **175**, 257–286.

Runde, V. [2005] *A Taste of Topology*, Universitext, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-25790-X.

Stein, K. [1951] *Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Mathematische Annalen, **123**, 201–222.

Whitney, H. [1936] *Differentiable manifolds*, Annals of Mathematics. Second Series, **37**(3), 645–680.

— [1944] *The self-intersections of a smooth n-manifold in* 2*n-space*, Annals of Mathematics. Second Series, **45**, 220–246.

Whitney, H. and Bruhat, F. [1959] *Quelques propriétés fondamentales des ensembles analytiques-réels*, Commentarii Mathematici Helvetici, **33**, 132–160.