Chapter 1

Sheaf theory

The theory of sheaves has come to play a central rôle in the theories of several complex variables and holomorphic differential geometry. The theory is also essential to real analytic geometry. The theory of sheaves provides a framework for solving "local to global" problems of the sort that are normally solved using partitions of unity in the smooth case. In this chapter we provide a fairly comprehensive overview of sheaf theory. The presentation in this chapter is thorough but basic. When one delves deeply into sheaf theory, a categorical approach is significantly more efficient than the direct approach we undertake here. However, for many first-timers to the world of sheaves—particularly those coming to sheaves from the differential geometric rather than the algebraic world—the categorical setting for sheaf theory is an impediment to understanding the point of the theory. In Chapter 4 we discuss the cohomology of sheaves and use category theory to do so. We use this opportunity to review the more categorical approach to sheaf theory, as this provides a very nice nontrivial application of category theory.

There are many references available for the theory of sheaves. A classical reference is that of Godement [1958], where the subject is developed from the point of view of algebraic topology. An updated treatment along the same lines is that of Bredon [1997]. The theory is developed quite concisely in the book of Tennison [1976] and in Chapter 5 of [Warner 1983]. A comprehensive review of applications of sheaf theory in differential geometry is given in [Kashiwara and Schapira 1990]. A quite down to earth development of differential geometry with the language of sheaves playing an integral rôle is given by Ramanan [2005]. Regardless of one's route to their understanding of the theory of sheaves, it is a subject that will consume some time in order to develop a useful understanding.

1.1 The basics of sheaf theory

In this section we review those parts of the theory that will be useful for us. Our interest in sheaves arises primarily in the context of holomorphic and real analytic functions and sections of real analytic vector bundles. However, in order to provide some colour for the particular setting in which we are interested, we give a treatment with greater generality. The treatment, however, is far from comprehensive, and we refer to the references at the beginning of the chapter for more details.

One of the places we do engage in some degree of generality is the class of functions and sections for which we consider sheaves. While our applications of sheaf theory will focus on the holomorphic and real analytic cases, we will also treat the cases of general differentiability. Specifically, we consider sheaves of functions and sections of class C^r for $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$. The manifolds on which we consider a certain class of differentiability will, of course, vary with the degree of differentiability. To encode this, we shall use the language, " let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required." By this we mean that $r' = \infty$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, that $r' = \omega$ if $r = \omega$, and $r' = \mathrm{hol}$ if $r = \mathrm{hol}$. Also, we shall implicitly or explicitly let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$.

We shall deal with three classes of sheaves in this book: sheaves of sets, sheaves of rings, and sheaves of modules. We shall on occasion separate the presentation according to these three classes. This will serve to clarify that many of the constructions have their basis in sheaves of sets, and the application to sheaves of rings or modules is a matter of invoking the algebraic structure on the constructions on sets. This manner of presentation has the benefit of being unambiguous—and sometimes this is useful—but is also pointlessly repetitive. You lose where you win, sometimes.

1.1.1 Presheaves

The basic ingredient in the theory of sheaves is a presheaf. We shall need various sorts of presheaves, and will define these separately. This is admittedly a little laboured, and is certainly a place where a categorical presentation of the subject is more efficient. But we elect not to follow this abstract approach.

Presheaves of sets

Since nothing is made more complicated by doing so at this point, we give our general definition of presheaf in terms of topological spaces.

1.1.1 Definition (Presheaf of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space. A ***presheaf of sets*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $\mathscr{F}(\mathcal{U})$ and to each $\mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a mapping $r_{\mathcal{U},\mathcal{V}} \colon \mathscr{F}(\mathcal{U}) \to \mathscr{F}(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r_{\mathcal{U},\mathcal{W}} = r_{\mathcal{V},\mathcal{W}} \circ r_{\mathcal{U},\mathcal{V}}$.

We shall frequently use a single symbol, like $\mathscr{F}$, to refer to a presheaf, with the understanding that $\mathscr{F} = (\mathscr{F}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$, and that the restriction maps are understood. •

Let us introduce the common terminology for presheaves.

1.1.2 Definition (Local section, global section) Let $\mathscr{F}$ be a presheaf of sets over a topological space $(\mathcal{S}, \mathscr{O})$. An element $s \in \mathscr{F}(\mathcal{U})$ is called a ***section of $\mathscr{F}$ over $\mathcal{U}$*** and an element of $\mathscr{F}(\mathcal{S})$ is called a ***global section***. •

Presheaves can be restricted to open sets.

1.1.3 Definition (Restriction of a presheaf) Let $\mathscr{F}$ be a presheaf of sets over a topological space $(\mathcal{S}, \mathscr{O})$. If $\mathcal{U} \in \mathscr{O}$ then we denote by $\mathscr{F}|\mathcal{U}$ the ***restriction*** of $\mathscr{F}$ to $\mathcal{U}$, which is the presheaf over $\mathcal{U}$ whose sections over $\mathcal{V} \subseteq \mathcal{U}$ are simply $\mathscr{F}(\mathcal{V})$. •

Let us look at the principal examples we shall use in this book.

1.1.4 Examples (Presheaves of sets)

1. Let $\mathcal{S} = \{\mathrm{pt}\}$ be a one point set. A presheaf of sets over $\mathcal{S}$ is then defined by $F(x_0) = X$ and $\mathscr{F}(\emptyset) = \{\mathrm{pt}\}$ where X is a set.
   (We shall see in Lemma 1.1.12 that it is natural to take sections over the empty set to be singletons, even though this is not required by the definition of a presheaf.)
2. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $x_0 \in \mathcal{S}$. Let X be a set. We define a presheaf of sets $\mathscr{S}_{x_0,X}$ by
   $$\mathscr{S}_{x_0,X}(\mathcal{U}) = \begin{cases} X, & x_0 \in \mathcal{U}, \\ \{\mathrm{pt}\}, & x_0 \notin \mathcal{U}. \end{cases}$$
   The restriction maps are prescribed as the natural maps that can be defined. To be clear, if $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, then, if $x_0 \in \mathcal{U}$), we define
   $$r_{\mathcal{U},\mathcal{V}}(x) = \begin{cases} X, & x \in \mathcal{V}, \\ \{\mathrm{pt}\}, & x_0 \notin \mathcal{V} \end{cases}$$
   and, if $x_0 \notin \mathcal{U}$, we define $r_{\mathcal{U},\mathcal{V}}(\mathrm{pt}) = \mathrm{pt}$. This is called a ***skyscraper presheaf***.
3. If X is a set, a ***constant presheaf*** of sets $\mathscr{F}_X$ on a topological space $(\mathcal{S}, \mathscr{O})$ is defined by $\mathscr{F}_X(\mathcal{U}) = X$ for every $\mathcal{U} \in \mathscr{O}$. The restriction maps are taken to be $r_{\mathcal{U},\mathcal{V}} = \mathrm{id}_X$ for every $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$. •

Presheaves of rings

Now we adapt the preceding constructions to rings rather than sets. Let us make an assumption on the rings we shall use is sheaf theory (and almost everywhere else).

1.1.5 Assumption (Assumption about rings) "Ring" means "commutative ring with unit." •

We can now go ahead and make our definition of presheaves of rings.

1.1.6 Definition (Presheaf of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space. A ***presheaf of rings*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $\mathscr{R}(\mathcal{U})$ and to each $\mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a ring homomorphism $r_{\mathcal{U},\mathcal{V}} \colon \mathscr{R}(\mathcal{U}) \to \mathscr{R}(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r_{\mathcal{U},\mathcal{W}} = r_{\mathcal{V},\mathcal{W}} \circ r_{\mathcal{U},\mathcal{V}}$.

We shall frequently use a single symbol, like $\mathscr{R}$, to refer to a presheaf of rings, with the understanding that $\mathscr{R} = (\mathscr{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, and that the restriction maps are understood. •

The notions of a ***local section*** and a ***global section*** of a presheaf of rings, and of the ***restriction*** of a presheaf of rings is exactly as in the case of a presheaf of sets; see Definitions 1.1.2 and 1.1.3.

Let us give some examples of presheaves of rings.

1.1.7 Examples (Presheaves of rings)

1. If $\mathcal{S} = \{\mathrm{pt}\}$ is a one point set, we can define presheaves of rings by taking a ring R and defining $F(x_0) = \mathsf{R}$ and $\mathscr{F}(\emptyset) = \{0\}$.
2. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $x_0 \in \mathcal{S}$. We let R be a ring and take define $\mathscr{S}_{x_0,\mathsf{R}}$ by
$$\mathscr{S}_{x_0,\mathsf{R}}(\mathcal{U}) = \begin{cases} \mathsf{R}, & x_0 \in \mathcal{U}, \\ \{0\}, & x_0 \notin \mathcal{U}. \end{cases}$$
This is a ***skyscraper presheaf*** of rings. The restriction maps are as in Example 1.1.4–2.
3. In Example 1.1.4–3, if the set X has a ring structure, then we have a constant presheaf of rings. The next few examples give some specific instances of this.
4. Let us denote by $\mathbb{Z}_{\mathcal{S}}$ the constant presheaf over a topological space $(\mathcal{S}, \mathscr{O})$ assigning the ring $\mathbb{Z}$ to every open set.
5. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and denote by $\mathbb{F}_{\mathcal{S}}$ the constant presheaf over a topological space $(\mathcal{S}, \mathscr{O})$ assigning the ring $\mathbb{F}$ to every open set.
6. Let $\mathcal{W} \subseteq \mathbb{R}^n$ be an open subset and let $\mathscr{L}^1_{\mathcal{W}} = (\mathrm{L}^1(\mathcal{U}; \mathbb{R}))_{\mathcal{U}\subseteq\mathcal{W}\text{ open}}$ be the presheaf assigning to an open subset $\mathcal{U} \subseteq \mathcal{W}$ the set of integrable $\mathbb{R}$-valued functions on $\mathcal{U}$. The restriction maps are just restriction of functions in the usual sense.
7. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We let M be a manifold of class $\mathrm{C}^{r'}$. The presheaf of functions on M of class C^r assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ the ring $\mathrm{C}^r(\mathcal{U})$. The restriction map $r_{\mathcal{U},\mathcal{V}}$ for open sets $\mathcal{V}, \mathcal{U} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$ is simply the restriction of functions on $\mathcal{U}$ to $\mathcal{V}$. These maps clearly satisfy the conditions for a presheaf of rings. This presheaf we denote by $\mathscr{C}^r_{\mathsf{M}}$. •

The value of a presheaf is that it allows us to systematically deal with objects that are not globally defined, but are only locally defined. We have seen in various places, most explicitly at the end of Section GA1.4.2.3, that there is value in doing this, especially in the holomorphic and real analytic cases.

An obvious question that suggests itself at this early point is what properties the restrictions maps might have. Are they injective? surjective? These are actually crucial questions in the theory of sheaves, so let us take a look at this even at this early stage.

1.1.8 Examples (Properties of restriction maps)

1. Let us show that restriction maps are generally not surjective. This happens very often and in rather simple ways, and to illustrate we take the presheaf $\mathscr{C}^0_{\mathbb{F}}$ of continuous functions on $\mathbb{F}$. Let us take $\mathcal{U} = \mathsf{D}^1(2,0)$ and $\mathcal{V} = \bar{\mathsf{D}}^1(2,0)$. Let us consider $f \in \mathrm{C}^0(\mathcal{V})$ defined by $\frac{1}{1-|x|^2}$. It is clear that f is not in the image of $r_{\mathcal{U},\mathcal{V}}$.
2. Let us consider a way in which restriction maps may fail to be injective. Here, as in the first of our examples, we take the presheaf $\mathscr{C}^0_{\mathbb{F}}$ of continuous functions on $\mathbb{F}$, and we let $\mathcal{U} = \mathsf{D}^1(2,0)$ and $\mathcal{V} = \mathsf{D}^1(1,0)$. Let $f,g \in \mathrm{C}^0(\mathcal{U})$ have the property that $r_{\mathcal{U},\mathcal{V}}(f) = r_{\mathcal{U},\mathcal{V}}(g)$. This obviously does not imply that $f = g$ since there are many continuous functions on $\mathcal{U}$ agreeing on $\mathcal{V}$.
3. Next we consider another variant on the theme of injectivity of restriction maps. Let us first consider the presheaf $\mathscr{C}^r_{\mathbb{F}}$, $r \in \{\omega, \mathrm{hol}\}$ of analytic or holomorphic functions on $\mathbb{F}$. Let $\mathcal{U}$ be a connected open set and let $\mathcal{V} \subseteq \mathcal{U}$. Let $f,g \in \mathrm{C}^r(\mathcal{U})$ and suppose that $r_{\mathcal{U},\mathcal{V}}(f) = r_{\mathcal{U},\mathcal{V}}(g)$. Then, by Theorem GA1.1.1.18, we must have $f = g$ and so $r_{\mathcal{U},\mathcal{V}}$ is injective in this case.
4. We work with the same presheaf as the preceding example, and now relax the condition that $\mathcal{U}$ is connected. Let $\mathcal{V} \subseteq \mathcal{U}$ be a subset of a connected component of $\mathcal{U}$. In this case, the requirement that, for $f,g \in \mathrm{C}^r(\mathcal{U})$, we have $r_{\mathcal{U},\mathcal{V}}(f) = r_{\mathcal{U},\mathcal{V}}(g)$ only requires that f and g agree on the connected component of $\mathcal{U}$ containing $\mathcal{V}$. The specification of f and g on the other connected components of $\mathcal{U}$ is arbitrary, and so $r_{\mathcal{U},\mathcal{V}}$ is not injective.
5. Another example of where the restriction map is interesting is specific to holomorphic functions. We consider the presheaf $\mathrm{C}^{\mathrm{hol}}(\mathbb{C}^n)$ with $n \geq 2$. We let $\mathcal{U} = \mathbb{C}^n$ and $\mathcal{V} = \mathbb{C}^n \setminus \{\mathbf{0}\}$. In this case, as we saw in Example GA1.3.1.8–**??**, the restriction map $r_{\mathcal{U},\mathcal{V}}$ is a bijection since every holomorphic function on $\mathcal{V}$ is extended uniquely to a holomorphic function on $\mathcal{U}$. •

Presheaves of modules

We now consider the third setting for presheaves, that when a module structure is present.

1.1.9 Definition (Presheaf of modules) Let $(\mathcal{S},\mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$ with restriction maps denote by $r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}$. A ***presheaf of $\mathscr{R}$-modules*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $\mathscr{E}(\mathcal{U})$ and to each $\mathcal{V},\mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a mapping $r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}} \colon \mathscr{E}(\mathcal{U}) \to \mathscr{E}(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W},\mathcal{V},\mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r^{\mathscr{E}}_{\mathcal{U},\mathcal{W}} = r^{\mathscr{E}}_{\mathcal{V},\mathcal{W}} \circ r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}$;

(iii) $r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}$ is a morphism of Abelian groups with respect to addition in modules $\mathscr{E}(\mathcal{U})$ and $\mathscr{E}(\mathcal{V})$;

(iv) the diagram

$$\begin{tikzcd} \mathscr{R}(\mathcal{U}) \times \mathscr{E}(\mathcal{U}) \arrow[r] \arrow[d] & \mathscr{E}(\mathcal{U}) \arrow[d] \\ \mathscr{R}(\mathcal{V}) \times \mathscr{E}(\mathcal{V}) \arrow[r] & \mathscr{E}(\mathcal{V}) \end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication and the vertical arrows are the restriction maps.

We shall frequently use a single symbol, like $\mathscr{E}$, to refer to a presheaf of $\mathscr{R}$-modules, with the understanding that $\mathscr{E} = (\mathscr{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, and that the restriction maps are understood. •

Note that if $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$ then $\mathscr{E}(\mathcal{V})$ is actually an $\mathscr{R}(\mathcal{U})$-module with multiplication defined by $f s = r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}(f)s$. This being the case, the restriction map from $\mathscr{E}(\mathcal{U})$ to $\mathscr{E}(\mathcal{V})$ for an $\mathscr{R}$-module $\mathscr{E}$ is defined so that it is a homomorphism of $\mathscr{R}(\mathcal{U})$-modules.

1.1.10 Examples (Presheaves of modules)

1. If $\mathcal{S} = \{\mathrm{pt}\}$ is a one point set and if A is an R-module, then we can define a sheaf of modules by $F(x_0) = \mathsf{A}$ and $\mathscr{F}(\emptyset) = \{0\}$.
2. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $x_0 \in \mathcal{S}$. We let R be a ring and let A be a R-module, and take define $\mathscr{S}_{x_0,\mathsf{A}}$ by

$$\mathscr{S}_{x_0,\mathsf{A}}(\mathcal{U}) = \begin{cases} \mathsf{A}, & x_0 \in \mathcal{U}, \\ \{0\}, & x_0 \notin \mathcal{U}. \end{cases}$$

   This is a ***skyscraper presheaf*** of modules. The restriction maps are as in Example 1.1.4–2.
3. Referring to Example 1.1.7–4, an $\mathbb{Z}_{\mathcal{S}}$-module is a presheaf of Abelian groups, in the sense that to every $\mathcal{U} \in \mathscr{O}$ we assign an $\mathbb{Z}$-module, i.e., an Abelian group.
4. Referring to Example 1.1.7–5, an $\mathbb{F}_{\mathcal{S}}$-module is a presheaf of $\mathbb{F}$-modules, in the sense that to every $\mathcal{U} \in \mathscr{O}$ we assign an $\mathbb{F}$-module, i.e., an $\mathbb{F}$-vector space.
5. In Example 1.1.7–7 we introduced the presheaves $\mathscr{C}^r_{\mathsf{M}}$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$ of functions on manifolds of class $r' \in \{\infty, \omega, \mathrm{hol}\}$, for appropriate r'. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $\mathrm{C}^{r'}$. The presheaf of sections of E of class C^r assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ the $\mathrm{C}^r(\mathcal{U})$-module $\Gamma^r(\mathsf{E}|\mathcal{U})$. The restriction map $r_{\mathcal{U},\mathcal{V}}$ for open sets $\mathcal{V}, \mathcal{U} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$ is again just the restriction of sections on $\mathcal{U}$ to $\mathcal{V}$. These maps satisfy the conditions for a presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. This presheaf we denote by $\mathscr{G}^r_{\mathsf{E}}$.
6. Generalising the preceding example a little, a ***presheaf of $\mathscr{C}^{\mathbf{r}}_{\mathbf{M}}$-modules*** is a presheaf $\mathscr{E}$ such that $\mathscr{E}(\mathcal{U})$ is a $\mathrm{C}^r(\mathcal{U})$-module and such that the restriction maps satisfy the

natural algebraic conditions

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}}(s+t) &= r_{\mathcal{U},\mathcal{V}}(s) + r_{\mathcal{U},\mathcal{V}}(t), \qquad && s,t \in \mathscr{E}(\mathcal{U}), \\ r_{\mathcal{U},\mathcal{V}}(f\,s) &= r_{\mathcal{U},\mathcal{V}}(f) r_{\mathcal{U},\mathcal{V}}(s), \qquad && f \in \mathrm{C}^r(\mathcal{U}),\ s \in \mathscr{E}(\mathcal{U}). \end{aligned}$$ •

1.1.2 Sheaves

The notion of a sheaf, which we are about to define, allows us to patch locally defined objects together to produce an object defined on a union of open sets.

Sheaves of sets

The properties intrinsic to sheaves are the following.

1.1.11 Definition (Sheaf of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and suppose that we have a presheaf $\mathscr{F}$ of sets with restriction maps $r_{\mathcal{U},\mathcal{V}}$ for $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfying $\mathcal{V} \subseteq \mathcal{U}$.

(i) The presheaf $\mathscr{F}$ is ***separated*** when, if $\mathcal{U} \in \mathscr{O}$, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if $s, t \in \mathscr{F}(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_a}(s) = r_{\mathcal{U},\mathcal{U}_a}(t)$ for every $a \in A$, then $s = t$;

(ii) The presheaf $\mathscr{F}$ has the ***gluing property*** when, if $\mathcal{U} \in \mathscr{O}$, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if, for each $a \in A$, there exists $s_a \in \mathscr{F}(\mathcal{U}_a)$ with the family $(s_a)_{a\in A}$ satisfying

$$r_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}) = r_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_2})$$

for each $a_1, a_2 \in A$, then there exists $s \in \mathscr{F}(\mathcal{U})$ such that $s_a = r_{\mathcal{U},\mathcal{U}_a}(s)$ for each $a \in A$.

(iii) The presheaf of sets $\mathscr{F}$ is a ***sheaf*** of sets if it is separated and has the gluing property. •

Let us get one boring and mostly unimportant technicality out of the way.

1.1.12 Lemma (Sections over the empty set) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F}$ is a sheaf of sets, then $\mathscr{F}(\emptyset)$ is a one point set.*

Proof Since we can cover $\emptyset$ with the empty cover, the gluing property ensures that $\mathscr{F}(\emptyset) \neq \emptyset$. The separation property ensures that any two sections over $\emptyset$ agree, since any cover of $\emptyset$ is by empty sets. ■

As a consequence of the lemma, if $\mathscr{F}$ is a sheaf of sets then $\mathscr{F}(\emptyset) = \{\mathrm{pt}\}$ is a one point set. We shall assume without mention that all presheaves have this structure.

Let us look at some other examples of presheaves that are sheaves.

1.1.13 Examples (Presheaves of sets that are sheaves)

1. Presheaves described in Example 1.1.4–1 over topological spaces comprised of one point are sheaves.
2. Skyscraper presheaves as described in Example 1.1.4–2 are sheaves. •

Let us also give some examples of presheaves that are not sheaves.

1.1.14 Examples (Presheaves of sets that are not sheaves)

1. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. As in Example 1.1.4–3, $\mathscr{F}_X$ denotes the constant presheaf defined by $\mathscr{F}_X(\mathcal{U}) = X$. It is clear that $\mathscr{F}_X$ satisfies the separation condition. We claim that $\mathscr{F}_X$ does not generally satisfy the gluing condition. Indeed, let $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}$ be disjoint and take $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. Let $s_1 \in \mathscr{F}_X(\mathcal{U}_1)$ and $s_2 \in \mathscr{F}(\mathcal{U}_2)$. If $s_1 \neq s_2$ then there is no $s \in \mathscr{F}_X(\mathcal{U})$ for which $r_{\mathcal{U},\mathcal{U}_1}(s) = s_1$ and $r_{\mathcal{U},\mathcal{U}_2}(s) = s_2$.
2. An example of a presheaf that is not separated is a little less relevant, but we give it for the sake of completeness. Let $\mathcal{S} = \{0, 1\}$ have the discrete topology and define a presheaf $\mathscr{F}$ by requiring that $\mathscr{F}(\emptyset) = \emptyset$ and that $\mathscr{F}(\mathcal{U}) = \mathbb{R}^{\mathcal{U}}$ (i.e., the set of maps from $\mathcal{U}$ into $\mathbb{R}$). The restriction maps are defined by asking that $r_{\mathcal{U},\mathcal{V}}(s) = \zeta_{\mathcal{V}}$ whenever $\mathcal{V}$ is a proper subset of $\mathcal{U}$, where $\zeta_{\mathcal{V}} \colon \mathcal{V} \to \mathbb{R}$ is defined by $\zeta_{\mathcal{V}}(x) = 0$. Now let $s, t \in F(\{0, 1\})$ be defined by
$$s(0) = s(1) = 1, \quad t(0) = t(1) = -1.$$
Note that $(\{0\}, \{1\})$ is an open cover for $\{0, 1\}$ and
$$r_{\{0,1\},\{0\}}(s) = r_{\{0,1\},\{0\}}(t), \quad r_{\{0,1\},\{1\}}(s) = r_{\{0,1\},\{1\}}(t).$$
But it does not hold that $s = t$. •

The gluing condition is the one that will fail most often in practice, and a reason for this is the following result, characterising a large class of presheaves that are separated.

1.1.15 Proposition (Presheaves of mappings are separated) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if X is a set, and if $\mathscr{F}$ is a presheaf over $\mathcal{S}$ such that*

(i) each element $\mathrm{f} \in \mathscr{F}(\mathcal{U})$ is a mapping from $\mathcal{U}$ to X and

(ii) if $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ are such that $\mathcal{V} \subseteq \mathcal{U}$, then the restriction map $\mathrm{r}_{\mathcal{U},\mathcal{V}}$ is given by
$$\mathrm{r}_{\mathcal{U},\mathcal{V}}(\mathrm{f})(\mathrm{x}) = \mathrm{f}(\mathrm{x}), \qquad \mathrm{x} \in \mathcal{V},$$

then $\mathscr{F}$ is separated.

Proof Suppose that $\mathcal{U} \in \mathscr{O}$, that $(\mathcal{U}_a)_{a \in A}$ is an open cover of $\mathcal{U}$, and that $f, g \in \mathscr{F}(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_a}(f) = r_{\mathcal{U},\mathcal{U}_a}(g)$ for every $a \in A$. For $x \in \mathcal{U}$ let $a \in A$ be such that $x \in \mathcal{U}_a$. It follows immediately from the definition of the restriction maps that $f(x) = g(x)$. ■

In practice, one often wishes to patch together locally defined objects and have these be a sheaf. The following result shows how this can be done, the statement referring ahead to Section 1.1.5 for the notion of morphisms of sheaves.

1.1.16 Proposition (Building a sheaf of sets from local constructions) *Let $(\mathcal{S},\mathscr{O})$ be a topological space and let $(\mathcal{U}_{\mathrm{a}})_{\mathrm{a}\in\mathrm{A}}$ be an open cover for $\mathcal{S}$. Suppose that, for each $\mathrm{a}\in\mathrm{A}$, $\mathscr{F}_{\mathrm{a}}$ is a sheaf of sets over $\mathcal{U}_{\mathrm{a}}$ and denote the restriction maps for $\mathscr{F}_{\mathrm{a}}$ by $\mathrm{r}^{\mathrm{a}}_{\mathcal{U},\mathcal{V}}$ for $\mathcal{U},\mathcal{V}\subseteq\mathcal{U}_{\mathrm{a}}$ open with $\mathcal{V}\subseteq\mathcal{U}$. If, for $\mathrm{a}_1,\mathrm{a}_2\in\mathrm{A}$ satisfying $\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2}\neq\emptyset$, we have a sheaf isomorphism*

$$\phi_{\mathrm{a}_1\mathrm{a}_2}\colon \mathscr{F}_{\mathrm{a}_1}(\mathcal{U}_{\mathrm{a}_1})|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2}\to\mathscr{F}_{\mathrm{a}_2}(\mathcal{U}_{\mathrm{a}_2})|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2},$$

then there exists a sheaf $\mathscr{F}$ over $\mathcal{S}$, unique up to isomorphism, and isomorphisms $\phi_{\mathrm{a}}\colon\mathscr{F}|\mathcal{U}_{\mathrm{a}}\to\mathscr{F}_{\mathrm{a}}$, $\mathrm{a}\in\mathrm{A}$, such that the diagram

$$\begin{tikzcd}
\mathscr{F}|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2} \arrow[r, "\phi_{\mathrm{a}_1}"] \arrow[d, equal] & \mathscr{F}_{\mathrm{a}_1}|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2} \arrow[d, "\phi_{\mathrm{a}_1\mathrm{a}_2}"] \\
\mathscr{F}|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2} \arrow[r, "\phi_{\mathrm{a}_2}"'] & \mathscr{F}_{\mathrm{a}_1}|\mathcal{U}_{\mathrm{a}_1}\cap\mathcal{U}_{\mathrm{a}_2}
\end{tikzcd} \tag{1.1}$$

commutes for every $\mathrm{a}_1,\mathrm{a}_2\in\mathrm{A}$.

Proof For $\mathcal{U}\in\mathscr{O}$ we define

$$\mathscr{F}(\mathcal{U})=\Big\{(s_a)_{a\in A}\,\Big|\; s_a\in\mathscr{F}_a(\mathcal{U}\cap\mathcal{U}_a),\ a\in A,\\ \phi_{a_1a_2}(r^{a_1}_{\mathcal{U}\cap\mathcal{U}_{a_1},\mathcal{U}\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}))=r^{a_2}_{\mathcal{U}\cap\mathcal{U}_{a_2},\mathcal{U}\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_2}),\ a_1,a_2\in A\Big\}.$$

For $\mathcal{U},\mathcal{V}\in\mathscr{O}$ satisfying $\mathcal{V}\subseteq\mathcal{U}$, we define $r_{\mathcal{U},\mathcal{V}}\colon\mathscr{F}(\mathcal{U})\to\mathscr{F}(\mathcal{V})$ by

$$r_{\mathcal{U},\mathcal{V}}((s_a)_{a\in A})=(r^a_{\mathcal{U}\cap\mathcal{U}_a,\mathcal{V}\cap\mathcal{U}_a}(s_a))_{a\in A}.$$

We will verify that $\mathscr{F}$ is a sheaf over $\mathcal{S}$.

Let $\mathcal{W}\in\mathscr{O}$ and let $(\mathcal{W}_i)_{i\in I}$ be an open cover for $\mathcal{W}$. Let $s,t\in\mathscr{F}(\mathcal{W})$ satisfy $r_{\mathcal{W},\mathcal{W}_i}(s)=r_{\mathcal{W},\mathcal{W}_i}(t)$ for each $i\in I$. We write $s=(s_a)_{a\in A}$ and $t=(t_a)_{a\in A}$ and note that we have

$$r^a_{\mathcal{W}\cap\mathcal{U}_a,\mathcal{W}_i\cap\mathcal{U}_a}(s_a)=r^a_{\mathcal{W}\cap\mathcal{U}_a,\mathcal{W}_i\cap\mathcal{U}_a}(t_a),\qquad a\in A,\ i\in I.$$

Since $\mathscr{F}_a$ is separated, $s_a=t_a$ for each $a\in A$ and so $s=t$.

Let $\mathcal{W}\in\mathscr{O}$ and let $(\mathcal{W}_i)_{i\in I}$ be an open cover for $\mathcal{W}$. For each $i\in I$ let $s_i\in\mathscr{F}(\mathcal{W}_i)$ and suppose that $r_{\mathcal{W}_i,\mathcal{W}_i\cap\mathcal{W}_j}(s_i)=r_{\mathcal{W}_j,\mathcal{W}_i\cap\mathcal{W}_j}(s_j)$ for each $i,j\in I$. We write $s_i=(s_{i,a})_{a\in A}$, $i\in I$, and note that

$$r^a_{\mathcal{W}_i\cap\mathcal{U}_a,\mathcal{W}_i\cap\mathcal{W}_j\cap\mathcal{U}_a}(s_{i,a})=r^a_{\mathcal{W}_j\cap\mathcal{U}_a,\mathcal{W}_i\cap\mathcal{W}_j\cap\mathcal{U}_a}(s_{j,a}),\qquad i,j\in I,\ a\in A.$$

Since $\mathscr{F}_a$ satisfies the gluing property, there exists $s_a\in\mathscr{F}_a(\mathcal{W}\cap\mathcal{U}_a)$ such that

$$r^a_{\mathcal{W}\cap\mathcal{U}_a,\mathcal{W}_i\cap\mathcal{U}_a}(s_a)=s_{i,a},\qquad i\in I,\ a\in A.$$

Let us define $s=(s_a)_{a\in A}$. We have

$$\phi_{a_1a_2}(r^{a_1}_{\mathcal{W}_i\cap\mathcal{U}_{a_1},\mathcal{W}_i\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{i,a_1}))=r^{a_2}_{\mathcal{W}_i\cap\mathcal{U}_{a_2},\mathcal{W}_i\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{i,a_2})),\qquad i\in A,\ a_1,a_2\in A.$$

Therefore,

$$r^{a_2}_{\mathcal{W}_i\cap\mathcal{U}_{a_2},\mathcal{W}_i\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(\phi_{a_1a_2}(s_{i,a_1}))=r^{a_2}_{\mathcal{W}_i\cap\mathcal{U}_{a_2},\mathcal{W}_i\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{i,a_2})),\qquad i\in A,\ a_1,a_2\in A.$$

Since $\mathscr{F}_{a_2}$ is a sheaf we conclude that $\phi_{a_1,a_2}(s_{i,a_1}) = s_{i,a_2}$ for every $i \in I$ and $a_1, a_2 \in A$. Thus

$$r^{a_2}_{\mathcal{W}\cap\mathcal{U}_{a_2},\mathcal{W}_i\cap\mathcal{U}_{a_2}}(\phi_{a_1a_2}(s_{a_1})) = r^{a_2}_{\mathcal{W}\cap\mathcal{U}_{a_2},\mathcal{W}_i\cap\mathcal{U}_{a_2}}(s_{a_2})$$

and so we conclude that $\phi_{a_1a_2}(s_{a_1}) = s_{a_2}$ for $a_1, a_2 \in A$. Finally, from this we conclude that

$$\phi_{a_1a_2}(r^{a_1}_{\mathcal{W}\cap\mathcal{U}_{a_1},\mathcal{W}\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1})) = r^{a_2}_{\mathcal{W}\cap\mathcal{U}_{a_2},\mathcal{W}\cap\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_2}), \qquad a_1, a_2 \in A,$$

and so s as constructed is an element of $\mathscr{F}(\mathcal{W})$. In the preceding computation, we have repeatedly used the fact that $\phi_{a_2a_2}$ commutes with restrictions.

We must also show the commutativity of the diagram (1.1). To do so, let $a \in A$, let $\mathcal{U} \subseteq \mathcal{U}_a$, let $s_a = (s_{a,b})_{b\in A} \in \mathscr{F}(\mathcal{U})$, let $t_a \in \mathscr{F}_a(\mathcal{U})$ be defined by the requirement that $r_{\mathcal{U},\mathcal{U}\cap\mathcal{U}_b}(s_{a,b}) = r_{\mathcal{U},\mathcal{U}\cap\mathcal{U}_b}(t_a)$, $b \in A$, noting that this makes sense since $\mathscr{F}_a$ is a sheaf. We then define $\phi_a(s_a) = t_a$. It is now a routine computation to verify that, if $s = (s_b)_{b\in A} \in \mathscr{F}(\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2})$ then

$$\phi_{a_1a_2} \circ \phi_{a_1}(s) = \phi_{a_2}(s), \qquad a_1, a_2 \in A.$$

Finally, we must show that $\mathscr{F}$ is uniquely defined up to isomorphism by the requirements in the statement of the proposition. A moment's reflection shows that this will follow from the following assertion.

1 **Lemma** *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $(\mathcal{U}_\mathrm{a})_{\mathrm{a}\in\mathrm{A}}$ be an open cover of $\mathcal{S}$, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of sets over $\mathcal{S}$. Suppose that, for each $\mathrm{a} \in \mathrm{A}$, there exists a morphism of sheaves $\psi_\mathrm{a} \colon \mathscr{F}|\mathcal{U}_\mathrm{a} \to \mathscr{G}|\mathcal{U}_\mathrm{a}$ such that*

$$\psi_\mathrm{a}|(\mathscr{F}|\mathcal{U}_\mathrm{a} \cap \mathcal{U}_\mathrm{b}) = \psi_\mathrm{b}|(\mathscr{F}|\mathcal{U}_\mathrm{a} \cap \mathcal{U}_\mathrm{b}), \qquad \mathrm{a}, \mathrm{b} \in \mathrm{A}.$$

Then there exists a sheaf morphism $\psi \colon \mathscr{F} \to \mathscr{G}$ such that $\psi|(\mathscr{F}|\mathcal{U}_\mathrm{a}) = \psi_\mathrm{a}$ for each $\mathrm{a} \in \mathrm{A}$.

Proof To define ψ, let $\mathcal{U} \in \mathscr{O}$ and let $s \in \mathscr{F}(\mathcal{U})$. Note that $(\mathcal{U} \cap \mathcal{U}_a)_{a\in A}$ is an open cover for $\mathcal{U}$ and that

$$\psi_a(r^{\mathscr{F}}_{\mathcal{U},\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(s)) = \psi_b(r^{\mathscr{F}}_{\mathcal{U},\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(s)), \qquad a, b \in A.$$

Thus

$$r^{\mathscr{G}}_{\mathcal{U}\cap\mathcal{U}_a,\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\psi_a(r^{\mathscr{F}}_{\mathcal{U}\cap\mathcal{U}_a,\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(s))) = r^{\mathscr{G}}_{\mathcal{U}\cap\mathcal{U}_b,\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\psi_b(r^{\mathscr{F}}_{\mathcal{U}\cap\mathcal{U}_b,\mathcal{U}\cap\mathcal{U}_a\cap\mathcal{U}_b}(s))), \qquad a, b \in A.$$

Therefore, since $\mathscr{G}$ satisfies the gluing condition, there exists $t \in \mathscr{G}(\mathcal{U})$ satisfying

$$r^{\mathscr{G}}_{\mathcal{U},\mathcal{U}_a}(\psi_a(r^{\mathscr{F}}_{\mathcal{U},\mathcal{U}_a}(s))) = r^{\mathscr{G}}_{\mathcal{U},\mathcal{U}_a}(t), \qquad a \in A.$$

We define $\psi(s) = t$. One has to verify (1) that ψ is a sheaf morphism, i.e., it commutes with restriction and (2) that ψ satisfies the final condition of the lemma. All of these are now straightforward, perhaps tedious, verifications. ▼

Note that, by applying the lemma to the inverse, if the sheaf morphisms ψ_a, $a \in A$, in the lemma are isomorphisms, then ψ is also an isomorphism. This completes the proof. ■

Sheaves of rings

The constructions from the preceding section can be applied directly to presheaves of rings.

1.1.17 Definition (Sheaf of rings) A presheaf $\mathscr{R}$ of rings over a topological space $(\mathcal{S}, \mathscr{O})$ is a ***sheaf*** of rings if, as a presheaf of sets, it is a sheaf. •

As a consequence of Lemma 1.1.12, if $\mathscr{R}$ is a sheaf of rings, then $\mathscr{R}(\emptyset)$ is the zero ring. We shall assume without mention that all presheaves have this structure.

It is fairly easy to show that the presheaf $\mathscr{C}^r_{\mathsf{M}}$ is a sheaf, and let us record this here.

1.1.18 Proposition (Presheaves of functions are sheaves) *Let* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, *let* $\mathrm{r}' \in \{\infty, \omega, \mathrm{hol}\}$ *be as required, and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r}. *Then the presheaf* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$ *is a sheaf of rings.*

Proof Let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $(\mathcal{U}_a)_{a \in A}$ be an open cover for $\mathcal{U}$. To prove condition (i), if $f, g \in \mathrm{C}^r(\mathcal{U})$ agree on each neighbourhood $\mathcal{U}_a$, $a \in A$, then it follows that $f(x) = g(x)$ for every $x \in \mathcal{U}$ since $(\mathcal{U}_a)_{a \in A}$ covers $\mathcal{U}$. To prove condition (ii) let $f_a \in \mathrm{C}^r(\mathcal{U}_a)$ satisfy

$$r_{\mathcal{U}_{a_1}, \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}}(f_{a_1}) = r_{\mathcal{U}_{a_2}, \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}}(f_{a_2})$$

for each $a_1, a_2 \in A$. Define $f \colon \mathcal{U} \to \mathbb{F}$ by $f(x) = f_a(x)$ if $x \in \mathcal{U}_a$. This gives f as being well-defined by our hypotheses on the family $(f_a)_{a \in A}$. It remains to show that f is of class C^r. This, however, follows since f as defined agrees with f_a on $\mathcal{U}_a$, and f_a is of class C^r for each $a \in A$. ■

Let us give some examples of presheaves of rings that are not sheaves.

1.1.19 Examples (Presheaves of rings that are not sheaves)

1. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and take $\mathsf{M} = \mathbb{R}$. Let us define a presheaf $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ over $\mathbb{R}$ by

$$\mathscr{C}^r_{\mathrm{bdd}}(\mathcal{U}) = \{f \in \mathrm{C}^r(\mathcal{U}) \mid f \text{ is bounded}\}.$$

The restriction maps are, of course, just restriction of functions, and one readily verifies that this defines a presheaf of rings. It is not a sheaf. Indeed, let $(\mathcal{U}_a)_{a \in A}$ be a covering of $\mathbb{R}$ by bounded open sets and define $f_a \in \mathscr{C}^r_{\mathrm{bdd}}(\mathcal{U})$ by $f_a(x) = x$. Then we certainly have $f_a(x) = f_b(x)$ for $x \in \mathcal{U}_a \cap \mathcal{U}_b$. However, it does not hold that there exists $f \in \mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ such that $f(x) = f_a(x)$ for every $x \in \mathcal{U}_a$ and for every $a \in A$, since any such function would necessarily be unbounded. The difficulty in this case is that presheaves are designed to carry local information, and so they do not react well to cases where local information does not carry over to global information, in this case boundedness. Note that the defect in this example comes in the form of the violation of gluing condition (ii) in Definition 1.1.11; condition (i) still holds.

2. We consider the presheaf $\mathscr{L}^1_{\mathcal{W}} = (\mathrm{L}^1(\mathcal{U}; \mathbb{R}))_{\mathcal{U} \subseteq \mathcal{W} \text{ open}}$ of integrable functions on open subsets of an open subset $\mathcal{W} \subseteq \mathbb{R}^n$. This presheaf was considered in Example 1.1.7–6. This presheaf is not a sheaf. For example, let us consider $\mathcal{W} = \mathbb{R}^n$ and take, in the definition of the gluing property, $\mathcal{U} = \mathbb{R}^n$ and any open cover $(\mathcal{U}_a)_{a \in A}$ of $\mathcal{U}$ by balls of radius 1. On $\mathcal{U}_a$ take the local section f_a of $\mathscr{L}^1_{\mathbb{R}^n}$ defined by $f_a(\boldsymbol{x}) = 1$. Then there is no integrable function on $\mathbb{R}^n$ whose restriction to $\mathcal{U}_a$ is f_a for each $a \in A$. While we have done this only in the case that $\mathcal{W} = \mathbb{R}^n$, a little thought shows that $\mathscr{L}^1_{\mathcal{W}}$ is not a sheaf for *any* $\mathcal{W}$. •

As with sheaves of sets, we can patch together sheaves of rings from local constructions.

1.1.20 Proposition (Building a sheaf of rings from local constructions) *Let* $(\mathcal{S}, \mathscr{O})$ *be a topological space and let* $(\mathcal{U}_a)_{a\in A}$ *be an open cover for* $\mathcal{S}$. *Suppose that, for each* $a \in A$, $\mathscr{R}_a$ *is a sheaf of rings over* $\mathcal{U}_a$ *and denote the restriction maps for* $\mathscr{R}_a$ *by* $r^a_{\mathcal{U},\mathcal{V}}$ *for* $\mathcal{U}, \mathcal{V} \subseteq \mathcal{U}_a$ *open with* $\mathcal{V} \subseteq \mathcal{U}$. *If, for* $a_1, a_2 \in A$ *satisfying* $\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \neq \emptyset$, *we have a sheaf isomorphism*

$$\phi_{a_1 a_2} : \mathscr{R}_{a_1}(\mathcal{U}_{a_1})|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \to \mathscr{R}_{a_2}(\mathcal{U}_{a_2})|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2},$$

then there exists a sheaf of rings $\mathscr{R}$ *over* $\mathcal{S}$, *unique up to isomorphism, and isomorphisms* $\phi_a : \mathscr{R}|\mathcal{U}_a \to \mathscr{R}_a$, $a \in A$, *such that the diagram*

$$\begin{tikzcd}
\mathscr{R}|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \arrow[r, "\phi_{a_1}"] \arrow[d, equal] & \mathscr{R}_{a_1}|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \arrow[d, "\phi_{a_1 a_2}"] \\
\mathscr{R}|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \arrow[r, "\phi_{a_2}"'] & \mathscr{R}_{a_1}|\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}
\end{tikzcd}$$

commutes for every $a_1, a_2 \in A$.

Proof We can construct $\mathscr{R}$ as a sheaf of sets as in Proposition 1.1.16. To verify that it is, appropriately, a sheaf of rings follows by defining the algebraic operations in the obvious way. For example, if $\mathscr{R}_a$, $a \in A$, are sheaves of rings, then we can define addition and multiplication in $\mathscr{R}(\mathcal{U})$ by

$$(r_a)_{a\in A} + (s_a)_{a\in A} = (r_a + s_a)_{a\in A}, \qquad \big((r_a)_{a\in A}\big) \cdot \big((s_a)_{a\in A}\big) = (r_a \cdot s_a)_{a\in A},$$

respectively. One easily verifies that these operations are well-defined, and that the restriction morphisms for $\mathscr{R}$ are ring homomorphisms. One also needs to verify that the morphism ψ from Lemma 1 from the proof of Proposition 1.1.16 is a morphism of sheaves of rings. ■

Sheaves of modules

Now we turn to constructions with modules.

1.1.21 Definition (Sheaf of modules) Let $\mathscr{R}$ be a sheaf of rings over a topological space $(\mathcal{S}, \mathscr{O})$. A presheaf $\mathscr{E}$ of $\mathscr{R}$-modules over a topological space $(\mathcal{S}, \mathscr{O})$ is a ***sheaf*** of $\mathscr{R}$-modules if, as a presheaf of sets, it is a sheaf. •

As a consequence of Lemma 1.1.12, if $\mathscr{E}$ is a sheaf of $\mathscr{R}$-modules, then $\mathscr{E}(\emptyset)$ is the zero ring. We shall assume without mention that all presheaves have this structure.

It is fairly easy to show that the presheaf $\mathscr{G}^r_{\mathsf{E}}$ is a sheaf, and let us record this here.

1.1.22 Proposition (Presheaves of sections are sheaves) *Let* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$*, let* $\mathrm{r}' \in \{\infty, \omega, \mathrm{hol}\}$ *be as required, and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* M *be a manifold of class* C^{r} *and let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r}*. Then* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ *is a sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules.*

Proof This follows, *mutatis mutandis*, as does the proof for Proposition 1.1.18. ■

As with sets and rings, one can patch together modules from local constructions.

1.1.23 Proposition (Building a sheaf of modules from local constructions) *Let* $(\mathcal{S}, \mathscr{O})$ *be a topological space and let* $(\mathcal{U}_{\mathrm{a}})_{\mathrm{a}\in A}$ *be an open cover for* $\mathcal{S}$*. Suppose that, for each* $\mathrm{a} \in A$*,* $\mathscr{R}_{\mathrm{a}}$ *is a sheaf of rings over* $\mathcal{U}_{\mathrm{a}}$ *and* $\mathscr{E}_{\mathrm{a}}$ *is a sheaf of* $\mathscr{R}_{\mathrm{a}}$*-modules, and denote the restriction maps for* $\mathscr{E}_{\mathrm{a}}$ *by* $\mathrm{r}^{\mathrm{a}}_{\mathcal{U},\mathcal{V}}$ *for* $\mathcal{U}, \mathcal{V} \subseteq \mathcal{U}_{\mathrm{a}}$ *open with* $\mathcal{V} \subseteq \mathcal{U}$*. If, for* $\mathrm{a}_1, \mathrm{a}_2 \in A$ *satisfying* $\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2} \neq \emptyset$*, we have a sheaf isomorphism*

$$\phi_{\mathrm{a}_1\mathrm{a}_2}\colon \mathscr{E}_{\mathrm{a}_1}(\mathcal{U}_{\mathrm{a}_1})|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2} \to \mathscr{E}_{\mathrm{a}_2}(\mathcal{U}_{\mathrm{a}_2})|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2},$$

then there exists a sheaf of $\mathscr{R}$*-modules (here* $\mathscr{R}$ *is the sheaf of rings from Proposition 1.1.20)* $\mathscr{E}$ *over* $\mathcal{S}$*, unique up to isomorphism, and isomorphisms* $\phi_{\mathrm{a}}\colon \mathscr{E}|\mathcal{U}_{\mathrm{a}} \to \mathscr{E}_{\mathrm{a}}$*,* $\mathrm{a} \in A$*, such that the diagram*

$$\begin{tikzcd}
\mathscr{E}|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2} \arrow[r, "\phi_{\mathrm{a}_1}"] \arrow[d, equal] & \mathscr{E}_{\mathrm{a}_1}|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2} \arrow[d, "\phi_{\mathrm{a}_1\mathrm{a}_2}"] \\
\mathscr{E}|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2} \arrow[r, "\phi_{\mathrm{a}_2}"'] & \mathscr{E}_{\mathrm{a}_1}|\mathcal{U}_{\mathrm{a}_1} \cap \mathcal{U}_{\mathrm{a}_2}
\end{tikzcd}$$

commutes for every $\mathrm{a}_1, \mathrm{a}_2 \in A$*.*

Proof As with Proposition 1.1.20, this follows from Proposition 1.1.16, along with some bookkeeping which we leave to the reader. ■

1.1.3 The étalé space of a presheaf

The examples of presheaves we are most interested in, the presheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$, arise naturally as sections of some geometric object. However, there is nothing built into our definition of a presheaf that entails that it arises in this way. In this section we associate to a presheaf a space which realises sections of a presheaf as sections of some object, albeit a sort of peculiar one.

The étalé space of a presheaf of sets

In Section GA1.5.6.1 we saw the notions of germs of C^r-functions and germs of C^r-sections of a vector bundle. We begin our constructions of this section by understanding the germ construction for general presheaves. For the purposes of this discussion, we work with a presheaf $\mathscr{F}$ of sets over a topological space $(\mathcal{S}, \mathscr{O})$. We let $x \in \mathcal{S}$ let $\mathscr{O}_x$ be the collection of open subsets of $\mathcal{S}$ containing x. This is a directed set using inclusion since, given $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}_x$, we have $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathscr{O}_x$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \subseteq \mathcal{U}_1$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \subseteq \mathcal{U}_2$. What we want is the direct limit in $(\mathscr{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}_x}$. This we define using the equivalence relation where, for $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}_x$, $s_1 \in \mathscr{F}(\mathcal{U}_1)$ and $s_2 \in \mathscr{F}(\mathcal{U}_2)$ are ***equivalent*** if there exists $\mathcal{V} \in \mathscr{O}_x$ such that $\mathcal{V} \subseteq \mathcal{U}_1$, $\mathcal{V} \subseteq \mathcal{U}_2$ and $r_{\mathcal{U}_1,\mathcal{V}}(s_1) = r_{\mathcal{U}_2,\mathcal{V}}(s_2)$. The equivalence class of a section $s \in \mathscr{F}(\mathcal{U})$ we denote by $r_{\mathcal{U},x}(s)$, or simply by $[s]_x$ if we are able to forget about the neighbourhood on which s is defined.

The preceding constructions allow us to make the following definition.

1.1.24 Definition (Stalk of a sheaf of sets, germ of a section) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$. For $x \in \mathcal{S}$, the ***stalk*** of $\mathscr{F}$ at x is the set of equivalence classes under the equivalence relation defined above, and is denoted by $\mathscr{F}_x$. The equivalence class $r_{\mathcal{U},x}(s)$ of a section $s \in \mathscr{F}(\mathcal{U})$ is called the ***germ*** of s at x. •

With stalks at hand, we can make another useful construction associated with a presheaf.

1.1.25 Definition (Étalé space of a presheaf of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf of sets. The ***étalé space*** of $\mathscr{F}$ is the disjoint union of the stalks of $\mathscr{F}$:

$$\mathrm{Et}(\mathscr{F}) = \mathop{\mathring{\bigcup}}_{x \in \mathcal{S}} \mathscr{F}_x.$$

The ***étalé topology*** on $\mathrm{Et}(\mathscr{F})$ is that topology whose basis consists of subsets of the form

$$\mathcal{B}(\mathcal{U}, s) = \{r_{\mathcal{U},x}(s) \mid x \in \mathcal{U}\}, \qquad \mathcal{U} \in \mathscr{O},\ s \in \mathscr{F}(\mathcal{U}).$$

By $\pi_{\mathscr{F}} \colon \mathrm{Et}(\mathscr{F}) \to \mathcal{S}$ we denote the canonical projection $\pi_{\mathscr{F}}(r_{\mathcal{U},x}(s)) = x$ which we call the ***étalé projection***. •

Let us give some properties of étalé spaces, including the verification that the proposed basis we give for the étalé topology is actually a basis.

1.1.26 Proposition (Properties of the étalé topology) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space with $\mathscr{F}$ a presheaf of sets over $\mathcal{S}$. The étalé topology on $\mathrm{Et}(\mathscr{F})$ has the following properties:*

(i) the sets $\mathcal{B}(\mathcal{U}, \mathrm{s})$, $\mathcal{U} \in \mathscr{O}$, $\mathrm{s} \in \mathscr{F}(\mathcal{U})$, form a basis for a topology;

(ii) the projection $\pi_{\mathscr{F}}$ is a local homeomorphism, i.e., about every $[\mathrm{s}]_{\mathrm{x}} \in \mathrm{Et}(\mathscr{F})$ there exists a neighbourhood $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{F})$ such that $\pi_{\mathscr{F}}$ is a homeomorphism onto its image.

Proof (i) According to [Willard 1970, Theorem 5.3] this means that we must show that for sets $\mathcal{B}(\mathcal{U}_1, s_1)$ and $\mathcal{B}(\mathcal{U}_2, s_2)$ and for $[s]_x \in \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$, there exists $\mathcal{B}(\mathcal{V}, t) \subseteq \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$ such that $[s]_x \in \mathcal{B}(\mathcal{V}, t)$. We let $\mathcal{V} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ be a neighbourhood of x such that $s(y) = s_1(y) = s_2(y)$ for each $y \in \mathcal{V}$, this being possible since $[s]_x \in \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$. We then clearly have $\mathcal{B}(\mathcal{V}, t) \subseteq \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$ as desired.

(ii) By definition of the étalé topology, $\pi_{\mathscr{F}} | \mathcal{B}(\mathcal{U}, s)$ is a homeomorphism onto $\mathcal{U}$ (its inverse is s), and this suffices to show that $\pi_{\mathscr{F}}$ is a local homeomorphism. ■

The way in which one should think of the étalé topology is depicted in Figure 1.1. The point is that open sets in the étalé topology can be thought of as the "graphs" of local sections. In Figure 1.2 we illustrate how one might think about the possibilities regarding restriction maps as pointed out in Example 1.1.8.

A good example to illustrate the étalé topology is the constant sheaf.

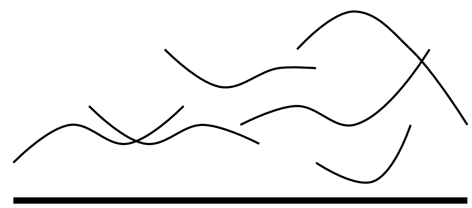

Figure 1.1 How to think of open sets in the étalé topology

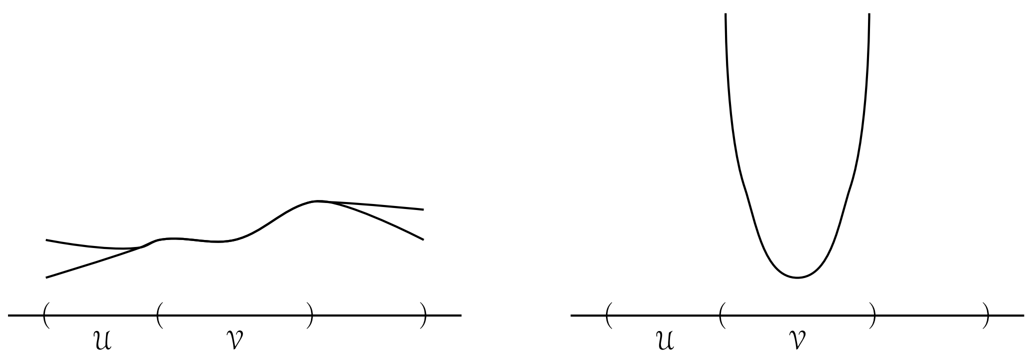

Figure 1.2 A depiction of the lack of injectivity (left) and surjectivity (right) of the restriction map $r_{\mathcal{U},\mathcal{V}}$ for étalé spaces

1.1.27 Example (The étalé space of a constant sheaf) We let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. By $\mathscr{F}_X$ we denote the constant presheaf defined by $\mathscr{F}_X(\mathcal{U}) = X$. Note that the stalk $\mathscr{F}_{X,x}$ is simply X. Thus $\mathrm{Et}(\mathscr{F}_X) = \cup_{x\in\mathcal{S}}(x, X)$ which we identity with $\mathcal{S} \times X$ in the natural way. Under this identification of $\mathrm{Et}(\mathscr{F}_X)$ with $\mathcal{S} \times X$, the étalé projection $\pi\colon \mathcal{S} \times X \to \mathcal{S}$ is identified with projection onto the first factor. Thus a section is, first of all, a map $\sigma\colon \mathcal{S} \to X$. It must also satisfy the criterion of continuity, and so we must understand the étalé topology on $\mathcal{S} \times X$. Let $\mathcal{U} \in \mathscr{O}$ and let $s \in \mathscr{F}_X(\mathcal{U}) = X$. The associated basis set for the étalé topology is then

$$\mathcal{B}(\mathcal{U}, s) = \{(x, s) \mid x \in \mathcal{U}\}.$$

These are precisely the open sets for $\mathcal{S} \times X$ if we equip X with the discrete topology. Thus $\mathrm{Et}(\mathscr{F}_X)$ is identified with the product topological space $\mathcal{S} \times X$ where X has the discrete topology. •

The étalé space of a presheaf of rings

Let us now consider étalé spaces of rings. Presheaves of rings being presheaves of sets, we can define stalks of sheaves of rings and germs of local sections of presheaves of rings. With this, we can make the following definition.

1.1.28 Definition (Étalé space of a presheaf of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings. The ***étalé space*** of $\mathscr{R}$ is the disjoint union of the stalks of

$\mathscr{R}$:

$$\mathrm{Et}(\mathscr{R}) = \mathring{\bigcup_{x\in\mathcal{S}}} \mathscr{R}_x,$$

which we equip with the étalé topology of Definition 1.1.25. We define ring operations on the set $\mathscr{R}_x$ of germs by

$$r_{\mathcal{U},x}(f) + r_{\mathcal{V},x}(g) = r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(f) + r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(g),$$
$$(r_{\mathcal{U},x}(f)) \cdot (r_{\mathcal{V},x}(g)) = (r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(f)) \cdot (r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(g)),$$

where $f \in \mathscr{R}(\mathcal{U})$, $g \in \mathscr{R}(\mathcal{V})$ for neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of x. We denote by $0_x \in \mathscr{R}_x$ and $1_x \in \mathscr{R}_x$ the germs of the sections $\zeta, \mu \in \mathscr{R}(\mathcal{U})$ over some neighbourhood $\mathcal{U}$ of x given by $\zeta = 0$ and $\mu = 1$. •

One readily verifies, just as we did for germs of functions, mappings, and sections of vector bundles, that these ring operations is well-defined and satisfy the ring axioms.

Of course, the basic properties of étalé spaces of sets apply to étalé spaces of rings.

1.1.29 Proposition (Properties of the étalé topology (ring version)) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space with $\mathscr{R}$ a presheaf of rings over $\mathcal{S}$. The étalé topology on $\mathrm{Et}(\mathscr{R})$ has the following properties:*

(i) the sets $\mathcal{B}(\mathcal{U}, \mathrm{f})$, $\mathcal{U} \in \mathscr{O}$, $\mathrm{f} \in \mathscr{R}(\mathcal{U})$, form a basis for a topology;

(ii) the projection $\pi_{\mathscr{R}}$ is a local homeomorphism, i.e., about every $[\mathrm{f}]_{\mathrm{x}} \in \mathrm{Et}(\mathscr{R})$ there exists a neighbourhood $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{R})$ such that $\pi_{\mathscr{R}}$ is a homeomorphism onto its image.

Proof This follows from Proposition 1.1.26. ■

Let us look a little closely at the particular étalé space of rings that will be of most concern for us. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class $\mathrm{C}^{r'}$. It is rather apparent that the stalks of $\mathrm{Et}(\mathscr{C}^r_{\mathsf{M}})$ are exactly the sets $\mathscr{C}^r_{x,\mathsf{M}}$ of germs of functions.

Let us examine some of the properties of these eťalé spaces.

1.1.30 Lemma (The étalé topology for sheaves of smooth functions) *The étalé topology on $\mathrm{Et}(\mathscr{C}^{\mathrm{r}}_{\mathsf{M}})$ is not Hausdorff when $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*

what?

Proof Let $\mathcal{U} \subseteq \mathsf{M}$ be an open set and as in , let $f \in \mathrm{C}^\infty(\mathsf{M})$ be such that $f(x) \in \mathbb{R}_{>0}$ for $x \in \mathcal{U}$ and $f(x) = 0$ for $x \in \mathsf{M} \setminus \mathcal{U}$. Let $g \in \mathrm{C}^\infty(\mathsf{M})$ be the zero function. Now let $x \in \mathrm{bd}(\mathcal{U})$. We claim that any neighbourhoods of $[f]_x$ and $[g]_x$ in $\mathrm{Et}(\mathscr{C}^r_{\mathsf{M}})$ intersect. To see this, let $\mathcal{O}_f$ and $\mathcal{O}_g$ be neighbourhoods in the étalé topology of $[f]_x$ and $[g]_x$. Since any sufficiently small neighbourhood of $[f]_x$ and $[g]_x$ is homeomorphic to a neighbourhood of x under the étalé projection, let us suppose without loss of generality that $\mathcal{O}_f$ and $\mathcal{O}_g$ are both homeomorphic to a neighbourhood $\mathcal{V}$ of x under the projection. For $y \in \mathcal{V} \cap (\mathsf{M} \setminus \mathrm{cl}(\mathcal{U}))$, $[f]_y = [g]_y$. Since $\mathcal{O}_f$ and $\mathcal{O}_g$ are uniquely determined by the germs of f and g in $\mathcal{V}$, respectively, it follows that $[f]_y = [g]_y \in \mathcal{O}_f \cap \mathcal{O}_g$, giving the desired conclusion. ■

1.1.31 Lemma (The étalé topology for sheaves of analytic functions) *If* M *is Hausdorff, then the étalé topology on* $\mathrm{Et}(\mathscr{C}^{\mathrm{r}}_{\mathsf{M}})$ *is Hausdorff when* $\mathrm{r} \in \{\omega, \mathrm{hol}\}$.

Proof Let $[f]_x$ and $[g]_y$ be distinct. If $x \neq y$ then there are disjoint neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of x and y and then $\mathcal{B}(\mathcal{U}, f)$ and $\mathcal{B}(\mathcal{V}, g)$ are disjoint neighbourhoods of $[f]_x$ and $[g]_y$, respectively, since the étalé projection is a homeomorphism from the neighbourhoods in M to the neighbourhoods in $\mathrm{Et}(\mathscr{C}^r_{\mathsf{M}})$. If $x = y$ let $[f]_x$ and $[g]_x$ be distinct and suppose that every neighbourhood of $[f]_x$ and $[g]_x$ in the étalé topology intersect. This implies, in particular, that for every connected neighbourhood $\mathcal{U}$ of x the basic neighbourhoods $\mathcal{B}(\mathcal{U}, f)$ and $\mathcal{B}(\mathcal{U}, g)$ intersect. This implies by Lemma 1.1.40 below the existence of an open subset $\mathcal{V}$ of $\mathcal{U}$ such that f and g agree on $\mathcal{V}$. This, however, contradicts the identity principle, Theorem GA1.4.2.5. Thus the étalé topology is indeed Hausdorff in the holomorphic or real analytic case. ■

Readers who are annoyed by the notation $\mathrm{Et}(\mathscr{C}^r_{\mathsf{M}})$ and $\mathrm{Et}(\mathscr{G}^r_{\mathsf{E}})$ will be pleased to know that we will stop using this notation eventually.

The étalé space of a presheaf of modules

Let us now consider étalé spaces of modules. Presheaves of modules being presheaves of sets, we can define stalks of sheaves of modules and germs of local sections of presheaves of modules. With this, we can make the following definition.

1.1.32 Definition (Étalé space of a presheaf of modules) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules. The ***étalé space*** of $\mathscr{E}$ is the disjoint union of the stalks of $\mathscr{E}$:

$$\mathrm{Et}(\mathscr{E}) = \mathop{\mathring{\bigcup}}_{x \in \mathcal{S}} \mathscr{E}_x,$$

which we equip with the étalé topology of Definition 1.1.25. We define an $\mathscr{R}_x$-module structure on the set $\mathscr{E}_x$ of germs by

$$r_{\mathcal{U},x}(s) + r_{\mathcal{V},x}(t) = r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(s) + r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(t),$$
$$(r_{\mathcal{W},x}(f)) \cdot (r_{\mathcal{V},x}(s)) = (r_{\mathcal{W}\cap\mathcal{V},x} \circ r_{\mathcal{W},\mathcal{W}\cap\mathcal{V}}(f)) \cdot (r_{\mathcal{W}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{W}\cap\mathcal{V}}(s)),$$

where $s \in \mathscr{E}(\mathcal{U})$, $t \in \mathscr{E}(\mathcal{V})$, and $f \in \mathscr{R}(\mathcal{W})$. •

One readily verifies, just as we did for germs of sections of vector bundles, that the module operations are well-defined and satisfy the module axioms.

Of course, the basic properties of étalé spaces of sets apply to étalé spaces of rings.

1.1.33 Proposition (Properties of the étalé topology (module version)) *Let* $(\mathcal{S}, \mathscr{O})$ *be a topological space with* $\mathscr{R}$ *a presheaf of rings over* $\mathcal{S}$ *and* $\mathscr{E}$ *a presheaf of* $\mathscr{R}$*-modules. The étalé topology on* $\mathrm{Et}(\mathscr{E})$ *has the following properties:*

(i) *the sets* $\mathcal{B}(\mathcal{U}, \mathrm{s})$, $\mathcal{U} \in \mathscr{O}$, $\mathrm{s} \in \mathscr{E}(\mathcal{U})$, *form a basis for a topology;*

(ii) *the projection* $\pi_{\mathscr{E}}$ *is a local homeomorphism, i.e., about every* $[\mathrm{s}]_{\mathrm{x}} \in \mathrm{Et}(\mathscr{E})$ *there exists a neighbourhood* $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{E})$ *such that* $\pi_{\mathscr{E}}$ *is a homeomorphism onto its image.*

Proof This follows from Proposition 1.1.26. ■

For sheaves of rings or modules the notion of stalk makes it possible to define the notion of the support of a local section.

1.1.34 Definition (Support of a local section) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules over $\mathcal{S}$. The ***support*** of a local section $s \in \mathscr{E}(\mathcal{U})$ is

$$\operatorname{supp}(s) = \{x \in \mathcal{U} \mid [s]_x \neq 0_x\}. \qquad \bullet$$

Note that the support of a local section $s \in \mathscr{E}(\mathcal{U})$ is necessarily closed since if $[s]_x = 0_x$ then $[s]_y = 0_y$ for y in some neighbourhood of x.

Let us examine closely the structure of the étalé spaces of sheaves of sections of a vector bundle. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class $\mathrm{C}^{r'}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $\mathrm{C}^{r'}$. It is rather apparent that the stalks of $\mathrm{Et}(\mathscr{G}^r_{\mathsf{E}})$ are exactly the sets and $\mathscr{G}^r_{x,\mathsf{E}}$ of germs of functions and sections, respectively.

Let us examine some of the properties of these eťalé spaces.

1.1.35 Lemma (The étalé topology for sheaves of smooth sections) *The étalé topology on both* $\mathrm{Et}(\mathscr{C}^{\mathrm{r}}_{\mathsf{M}})$ *and* $\mathrm{Et}(\mathscr{G}^{\mathrm{r}}_{\mathsf{E}})$ *is not Hausdorff when* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Proof This follows, *mutatis mutandis*, as the proof of Lemma 1.1.30. ■

1.1.36 Lemma (The étalé topology for sheaves of analytic sections) *If* M *is Hausdorff, then the étalé topology both* $\mathrm{Et}(\mathscr{G}^{\mathrm{r}}_{\mathsf{E}})$ *is Hausdorff when* $\mathrm{r} \in \{\omega, \mathrm{hol}\}$.

Proof This follows, *mutatis mutandis*, as the proof of Lemma 1.1.31. ■

1.1.4 Étalé spaces

Let us now talk about étalé spaces in general. As with presheaves and sheaves, we will give a few definitions associated with the various structures we shall use. We begin with sets.

Étalé spaces of sets

The basic flavour of étalé space is that of sets, corresponding to the following definition.

1.1.37 Definition (Étalé space of sets) If $(\mathcal{S}, \mathscr{O})$ is a topological space, an ***étalé space of sets*** over $\mathcal{S}$ is a topological space $\mathscr{S}$ with a surjective map $\pi\colon \mathscr{S} \to \mathcal{S}$, called the ***étalé projection***, such that π is a local homeomorphism. The the ***stalk*** at x is $\mathscr{S}_x = \pi^{-1}(x)$. •

Like presheaves, étalé spaces have restrictions, but these can be defined for arbitrary subsets, not just open subsets.

1.1.38 Definition (Restriction of étalé space) If $\pi\colon \mathscr{S} \to \mathcal{S}$ is an étalé space over a topological space $(\mathcal{S}, \mathscr{O})$ and if $A \subseteq \mathcal{S}$, the ***restriction*** of $\mathscr{S}$ to A is $\mathscr{S}|A = \pi^{-1}(A)$, which we regard as an étalé space over A. •

Similarly, sections of étalé spaces can be defined over arbitrary subsets.

1.1.39 Definition (Sections of étalé space) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\pi\colon \mathscr{S} \to \mathcal{S}$ be an étalé space of sets over $\mathcal{S}$. A ***section*** of $\mathscr{S}$ over $A \subseteq \mathcal{S}$ is a continuous map $\sigma\colon A \to \mathscr{S}$ (with the subspace topology for A) for which $\pi \circ \sigma(x) = x$ for every $x \in A$. The set of sections of $\mathscr{S}$ over A is denoted by $\Gamma(A; \mathscr{S})$. •

Most often one is interested in sections of étalé spaces over open sets, and we shall see why such sections are particularly important as we go along.

The following properties of sections are used often when proving statements about étalé spaces.

1.1.40 Lemma (Properties of sections of étalé spaces) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\pi\colon \mathscr{S} \to \mathcal{S}$ be an étalé space of sets over $\mathcal{S}$, and let $\mathrm{x} \in \mathcal{S}$:*

(i) if $\alpha \in \mathscr{S}_{\mathrm{x}}$ then there exists a neighbourhood $\mathcal{U}$ of x and a section σ of $\mathscr{S}$ over $\mathcal{U}$ such that $\sigma(\mathrm{x}) = \alpha$;

(ii) if σ and τ are sections of $\mathscr{S}$ over neighbourhoods $\mathcal{U}$ and $\mathcal{V}$, respectively, of x for which $\sigma(\mathrm{x}) = \tau(\mathrm{x})$, then there exists a neighbourhood $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ of x such that $\sigma|\mathcal{W} = \tau|\mathcal{W}$.

Proof (i) Let $\mathcal{O}$ be a neighbourhood of α in $\mathscr{S}$, and suppose, without loss of generality, that $\pi|\mathcal{O}$ is a homeomorphism onto its image. The inverse $\sigma\colon \pi(\mathcal{O}) \to \mathcal{O} \subseteq \mathscr{S}$ is continuous, and so it a section.

(ii) Let $\alpha = \sigma(x) = \tau(x)$ and let $\mathcal{O} \subseteq \mathscr{S}$ be a neighbourhood of α such that $\pi|\mathcal{O}$ is a homeomorphism onto its image. Let $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$ be such that $\sigma(\mathcal{U}'), \tau(\mathcal{V}') \subseteq \mathcal{O}$, this by continuity of the sections. Let $\mathcal{W} = \mathcal{U}' \cap \mathcal{V}'$. Note that $\sigma|\mathcal{W}$ and $\tau|\mathcal{W}$ are continuous bijections onto their image and that they are further homeomorphisms onto their image, with the continuous inverse being furnished by π. Thus σ and τ are both inverse for π in the same neighbourhood of α, and so are, therefore, equal. ■

Most of our examples of étalé spaces will come from Proposition 1.1.42 below. Let us give another example for fun.

1.1.41 Example (Étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. We define $\mathscr{S}_X = \mathcal{S} \times X$ and we equip this set with the product topology inherited by using the discrete topology on X. One readily verifies that the projection $\pi\colon \mathcal{S} \times X \to \mathcal{S}$ given by projection onto the first factor then makes $\mathscr{S}_X$ into an étalé space. One also verifies that sections of $\mathscr{S}_X$ over $\mathcal{U} \in \mathscr{O}$ are regarded as locally constant maps from $\mathcal{U}$ to X. This étalé space we call the ***constant étalé space***. Note that, by our constructions of Example 1.1.27, if $\mathscr{F}_X$ is a constant presheaf, its étalé space $\mathrm{Et}(\mathscr{F}_X)$ is a constant étalé space, and is isomorphic to the constant étalé space $\mathscr{S}_X$. •

We should verify that the étalé space of a presheaf is an étalé space in the general sense.

1.1.42 Proposition (Étalé spaces of presheaves of sets are étalé spaces of sets) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F}$ is a presheaf of sets over $\mathcal{S}$, then $\pi_{\mathscr{F}} \colon \mathrm{Et}(\mathscr{F}) \to \mathcal{S}$ is an étalé space of sets and $\mathrm{Et}(\mathscr{F})_{\mathrm{x}} = \mathscr{F}_{\mathrm{x}}$.*

Proof By Proposition 1.1.26 the étalé projection is a local homeomorphism. As it is clearly surjective, it follows that $\mathrm{Et}(\mathscr{F})$ is an étalé space. The final assertion of the proposition is just the definition. ■

1.1.43 Notation (Stalks) We shall write either $\mathscr{F}_x$ or $\mathrm{Et}(\mathscr{F})_x$ for the stalk, depending on what is most appropriate. •

Thus, associated to every presheaf is an étalé space. Moreover, associated to every étalé space is a natural presheaf.

1.1.44 Definition (The presheaf of sections of an étalé space of sets) For a topological space $(\mathcal{S}, \mathscr{O})$ and an étalé space $\mathscr{S}$ of sets, the ***presheaf of sections*** $\mathscr{S}$ is the presheaf $\mathrm{Ps}(\mathscr{S})$ of sets which assigns to $\mathcal{U} \in \mathscr{O}$ the set $\Gamma(\mathcal{U}; \mathscr{S})$ of sections of $\mathscr{S}$ over $\mathcal{U}$ and for which the restriction map for $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ is given by $r_{\mathcal{U},\mathcal{V}}(\sigma) = \sigma|\mathcal{V}$. •

It is readily seen that $\mathrm{Ps}(\mathscr{S})$ is indeed a presheaf. Moreover, it is a sheaf.

1.1.45 Proposition ($\mathrm{Ps}(\mathscr{S})$ is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{S}$ is an étalé space of sets over $\mathcal{S}$, then the presheaf $\mathrm{Ps}(\mathscr{S})$ is a sheaf of sets.*

Proof By Proposition 1.1.15 it follows that $\mathrm{Ps}(\mathscr{S})$ is separated. Let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a \in A}$ be an open cover for $\mathcal{U}$. Suppose that for each $a \in A$ there exists $\sigma_a \in \Gamma(\mathcal{U}_a; \mathscr{S})$ such that $\sigma_{a_1}(x) = \sigma_{a_2}(x)$ for every $x \in \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}$. Then, for $x \in \mathcal{U}$, define $\sigma(x) = \sigma_a(x)$ where $a \in A$ is such that $x \in \mathcal{U}_a$. This is clearly well-defined. We need only show that σ is continuous. But this follows since σ_a is continuous, and σ agrees with σ_a in a neighbourhood of x. ■

Étalé spaces of rings

We next discuss étalé spaces of rings. To do so, we shall require that the ring operations be appropriately continuous, which requires a suitable topology which we now describe. Given étalé spaces $\pi \colon \mathscr{S} \to \mathcal{S}$ and $\tau \colon \mathscr{T} \to \mathcal{S}$ over $(\mathcal{S}, \mathscr{O})$, let us define

$$\mathscr{S} \times_{\mathcal{S}} \mathscr{T} = \{(\alpha, \beta) \in \mathscr{S} \times \mathscr{T} \mid \pi(\alpha) = \tau(\beta)\}.$$

This space is given the relative topology from $\mathscr{S} \times \mathscr{T}$.

1.1.46 Definition (Étalé space of rings) If $(\mathcal{S}, \mathscr{O})$ is a topological space, an ***étalé space of rings*** over $\mathcal{S}$ is a topological space $\mathscr{A}$ with a surjective map $\pi \colon \mathscr{A} \to \mathcal{S}$ such that

(i) $\mathscr{A}$ is an étalé space of sets,

(ii) the stalk $\mathscr{A}_x = \pi^{-1}(x)$ is a ring for each $x \in \mathcal{S}$,

(iii) the ring operations are continuous, i.e., the maps

$$\mathscr{A} \times_{\mathcal{S}} \mathscr{A} \ni (f, g) \mapsto f + g \in \mathscr{A}, \quad \mathscr{A} \times_{\mathcal{S}} \mathscr{A} \ni (f, g) \mapsto f \cdot g \in \mathscr{A}$$

are continuous. •

The essential features of étalé spaces of sets carry over to étalé spaces of rings. In particular, one can define the restriction of an étalé space of rings over $\mathcal{S}$ to any subset $A \subseteq \mathcal{S}$ just as in Definition 1.1.38, and the set of sections of an étalé space of rings over a subset A as in Definition 1.1.39. Sections of étalé spaces of rings have the properties enumerated in Lemma 1.1.40.

Let us give some simple examples of étalé spaces of rings.

1.1.47 Examples (Some constant étalé spaces of rings)

1. Note that the étalé space $\mathrm{Et}(\mathbb{Z}_{\mathcal{S}})$ is an étalé space of rings.
2. Similarly, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the étalé space $\mathrm{Et}(\mathbb{F}_{\mathcal{S}})$ is an étalé space of rings. •

Étalé spaces of presheaves of rings have the expected property of being étalé spaces of rings.

1.1.48 Proposition (Étalé spaces of presheaves of rings are étalé spaces of rings) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, then $\pi_{\mathscr{R}}\colon \mathrm{Et}(\mathscr{R}) \to \mathcal{S}$ is an étalé space of rings and $\mathrm{Et}(\mathscr{R})_{\mathrm{x}} = \mathscr{R}_{\mathrm{x}}$.*

Proof Except for the continuity of the ring operations, the result follows from Proposition 1.1.42. Let us show that the ring operations on $\mathrm{Et}(\mathscr{R})$ are continuous. Let $[f]_x + [g]_x \in \mathrm{Et}(\mathscr{R})$ and let $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{R})$ be a neighbourhood of $[f]_x + [g]_x$. Without loss of generality, suppose that $f, g, f + g \in \mathscr{R}(\mathcal{U})$ for some neighbourhood $\mathcal{U}$ of x. By shrinking $\mathcal{U}$ if necessary, by definition of the basic neighbourhoods for $\mathrm{Et}(\mathscr{R})$, we can suppose that $\mathcal{B}(\mathcal{U}, f + g) \subseteq \mathcal{O}$. Then we have

$$\mathrm{Et}(\mathscr{R}) \times_{\mathcal{S}} \mathrm{Et}(\mathscr{R}) \supseteq \mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g) \ni ([f]_y, [g]_y) \mapsto [f + g]_y \in \mathcal{B}(\mathcal{U}, f + g) \subseteq \mathcal{O},$$

where, of course,

$$\mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g) = \{([f]_y, [g]_z) \in \mathcal{B}(\mathcal{U}, f) \times \mathcal{B}(\mathcal{U}, g) \mid\ y = z\}.$$

This gives continuity of addition since $\mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g)$ is open in $\mathrm{Et}(\mathscr{R}) \times_{\mathcal{S}} \mathrm{Et}(\mathscr{R})$. A similarly styled argument shows that multiplication is continuous. ■

As with stalks of presheaves of sets, we might write $\mathscr{R}_x$ or $\mathrm{Et}(\mathscr{R})_x$ for the stalk of a presheaf $\mathscr{R}$ of rings.

Étalé spaces of rings give rise to natural presheaves of rings.

1.1.49 Definition (The presheaf of sections of an étalé space of rings) For a topological space $(\mathcal{S}, \mathscr{O})$ and an étalé space $\mathscr{A}$ of rings, the ***presheaf of sections*** $\mathscr{A}$ is the presheaf $\mathrm{Ps}(\mathscr{A})$ of rings which assigns to $\mathcal{U} \in \mathscr{O}$ the set $\Gamma(\mathcal{U}; \mathscr{A})$ of sections of $\mathscr{A}$ over $\mathcal{U}$ and for which the restriction map for $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ is given by $r_{\mathcal{U},\mathcal{V}}(f) = f|\mathcal{V}$. The ring operations are

$$(f + g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x) \cdot g(x) \qquad f, g \in \Gamma(\mathcal{U}; \mathscr{A}),\ x \in \mathcal{U}. \qquad \bullet$$

The presheaf $\mathrm{Ps}(\mathscr{A})$ is a sheaf.

1.1.50 Proposition (Ps($\mathscr{A}$) is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, then the presheaf $\mathrm{Ps}(\mathscr{A})$ is a sheaf of rings.*

Proof This follows from Proposition 1.1.45. ■

Étalé spaces of modules

The definition and main results for étalé spaces of modules are now clear, so let us get to it.

1.1.51 Definition (Étalé space of modules) If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, an ***étalé space of $\mathscr{A}$-modules*** over $\mathcal{S}$ is a topological space $\mathscr{E}$ with a surjective map $\pi\colon \mathscr{E} \to \mathcal{S}$ such that

(i) $\mathscr{E}$ is an étalé space of sets,

(ii) the stalk $\mathscr{E}_x = \pi^{-1}(x)$ is an $\mathscr{A}_x$-module for each $x \in \mathcal{S}$,

(iii) the module operations are continuous, i.e., the maps

$$\mathscr{E} \times_{\mathcal{S}} \mathscr{E} \ni (\sigma, \tau) \mapsto \sigma + \tau \in \mathscr{E}, \quad \mathscr{A} \times_{\mathcal{S}} \mathscr{E} \ni (f, \sigma) \mapsto f \cdot \sigma \in \mathscr{E}$$

are continuous. •

The essential features of étalé spaces of sets carry over to étalé spaces of modules. In particular, one can define the restriction of an étalé space of modules over $\mathcal{S}$ to any subset $A \subseteq \mathcal{S}$ just as in Definition 1.1.38, and the set of sections of an étalé space of modules over a subset A as in Definition 1.1.39. Sections of étalé spaces of modules have the properties enumerated in Lemma 1.1.40.

We have a few simple, but useful, examples of étalé spaces of modules.

1.1.52 Examples (Étalé spaces of modules over constant étalé spaces of rings)

1. An étalé space of modules over the constant étalé space $\mathrm{Et}(\mathbb{Z}_{\mathcal{S}})$ is an étalé space of Abelian groups.
2. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, an étalé space of modules over the constant étalé space $\mathbb{F}_{\mathcal{S}}$ is an étalé space of $\mathbb{F}$-vector spaces. •

Étalé spaces of presheaves of modules have the expected property of being étalé spaces of modules.

1.1.53 Proposition (Étalé spaces of presheaves of modules are étalé spaces of modules) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, and if $\mathscr{E}$, then $\pi_{\mathscr{E}}\colon \mathrm{Et}(\mathscr{E}) \to \mathcal{S}$ is an étalé space of $\mathrm{Et}(\mathscr{R})$-modules and $\mathrm{Et}(\mathscr{E})_{\mathrm{x}} = \mathscr{E}_{\mathrm{x}}$.*

Proof This follows in a manner entirely similar to the proof of Proposition 1.1.48. ■

Étalé spaces of modules give rise to natural presheaves of modules.

1.1.54 Definition (The presheaf of sections of an étalé space of modules) For a topological space $(\mathcal{S}, \mathscr{O})$, an étalé space $\mathscr{A}$ of rings, and an étalé space $\mathscr{E}$ of $\mathscr{A}$-modules, the ***presheaf of sections*** $\mathscr{E}$ is the presheaf $\mathrm{Ps}(\mathscr{E})$ of $\mathrm{Ps}(\mathscr{A})$-modules which assigns to $\mathcal{U} \in \mathscr{O}$ the set $\Gamma(\mathcal{U}; \mathscr{E})$ of sections of $\mathscr{E}$ over $\mathcal{U}$ and for which the restriction map for $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ is given by $r_{\mathcal{U},\mathcal{V}}(\sigma) = \sigma|\mathcal{V}$. The module operations are

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x), \ (f \cdot \sigma)(x) = f(x) \cdot \sigma(x) \qquad \sigma, \tau \in \Gamma(\mathcal{U}; \mathscr{E}), \ f \in \Gamma(\mathcal{U}; \mathscr{A}), \ x \in \mathcal{U}. \quad \bullet$$

The presheaf $\mathrm{Ps}(\mathscr{E})$ is a sheaf.

1.1.55 Proposition ($\mathrm{Ps}(\mathscr{E})$ is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, and if $\mathscr{E}$ is an étalé space of $\mathscr{A}$-modules, then the presheaf $\mathrm{Ps}(\mathscr{E})$ is a sheaf of $\mathrm{Ps}(\mathscr{A})$-modules.*

Proof This follows from Proposition 1.1.45. ■

1.1.5 Morphisms of presheaves and étalé spaces

We now study mappings between sheaves and étalé spaces. We break the discussion down into the various cases of sheaves.

Morphisms of presheaves and étalé spaces of sets

We begin by defining morphisms for presheaves and étalé spaces of sets.

1.1.56 Definition (Morphism of presheaves of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be presheaves of sets over $\mathcal{S}$.

(i) A ***morphism*** of the presheaves $\mathscr{F}$ and $\mathscr{G}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a mapping $\Phi_{\mathcal{U}} \colon \mathscr{F}(\mathcal{U}) \to \mathscr{G}(\mathcal{U})$ such that the diagram

$$\begin{tikzcd} \mathscr{F}(\mathcal{U}) \arrow[r, "\Phi_{\mathcal{U}}"] \arrow[d, "r_{\mathcal{U},\mathcal{V}}"'] & \mathscr{G}(\mathcal{U}) \arrow[d, "r_{\mathcal{U},\mathcal{V}}"] \\ \mathscr{F}(\mathcal{V}) \arrow[r, "\Phi_{\mathcal{V}}"'] & \mathscr{G}(\mathcal{V}) \end{tikzcd} \tag{1.2}$$

commutes for every $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$. We shall often use the abbreviation $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of sets, Φ is called a ***morphism*** of sheaves.

By $\mathrm{Mor}(\mathscr{G}; \mathscr{H})$ we denote the set of morphisms of presheaves of sets.

(ii) If Φ is a morphism from $\mathscr{F}$ to $\mathscr{G}$ and if Ψ is a morphism from $\mathscr{G}$ to $\mathscr{H}$, then we define the ***composition*** of Φ and Ψ to be the morphism $\Psi \circ \Phi$ from $\mathscr{F}$ to $\mathscr{H}$ given by $(\Psi \circ \Phi)_{\mathcal{U}} = \Psi_{\mathcal{U}} \circ \Phi_{\mathcal{U}}$.

(iii) The identity morphism of a presheaf $\mathscr{F}$ is the presheaf morphism from $\mathscr{F}$ to itself defined by $\mathrm{id}_{\mathscr{F}} = (\mathrm{id}_{\mathscr{F}(\mathcal{U})})_{\mathcal{U} \in \mathscr{O}}$.

(iv) A morphism Φ of presheaves $\mathscr{G}$ and $\mathscr{H}$ is an ***isomorphism*** if there exists a morphism Ψ of presheaves $\mathscr{H}$ and $\mathscr{G}$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathscr{H}}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathscr{G}}$. $\bullet$

1.1.57 Definition (Morphism of étalé spaces of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces of sets over $\mathcal{S}$.

(i) An ***étalé morphism*** of $\mathscr{S}$ and $\mathscr{T}$ is a continuous map $\Phi\colon \mathscr{S} \to \mathscr{T}$ such that $\Phi(\mathscr{S}_x) \subseteq \mathscr{T}_x$.

By $\mathrm{Mor}(\mathscr{S}; \mathscr{T})$ we denote the set of étalé morphisms of étalé spaces of sets.

(ii) An étalé morphism $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an ***isomorphism*** if there exists an étalé morphism $\Psi\colon \mathscr{T} \to \mathscr{S}$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathscr{T}}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathscr{S}}$. •

Let us show that the preceding notions are often in natural correspondence. To do so, let us first indicate how to associate an étalé morphism to a morphism of presheaves, and vice versa. First let us build an étalé morphism from a morphism of presheaves. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ be a morphism of presheaves of sets $\mathscr{F}$ and $\mathscr{G}$ over $(\mathcal{S}, \mathscr{O})$. Define a mapping $\mathrm{Et}(\Phi)\colon \mathrm{Et}(\mathscr{F}) \to \mathrm{Et}(\mathscr{G})$ by

$$\mathrm{Et}(\Phi)([s]_x) = [\Phi_{\mathcal{U}}(s)]_x, \tag{1.3}$$

where $\mathcal{U}$ is such that $s \in \mathscr{F}(\mathcal{U})$. We denote by $\mathrm{Et}(\Phi)_x$ the restriction of $\mathrm{Et}(\Phi)$ to $\mathrm{Et}(\mathscr{F})_x$.

1.1.58 Proposition (Étalé morphisms of sets from presheaf morphisms of sets) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{F}$ and $\mathscr{G}$ be presheaves of sets over $\mathcal{S}$. If $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is a morphism of the presheaves $\mathscr{F}$ and $\mathscr{G}$, then $\mathrm{Et}(\Phi)$ is a morphism of the étalé spaces $\mathrm{Et}(\mathscr{F})$ and $\mathrm{Et}(\mathscr{G})$.*

Proof The definition $\mathrm{Et}(\Phi)([s]_x) = [\Phi_{\mathcal{U}}(s)]_x$ is well-defined, i.e., independent of the choice of representative $(s, \mathcal{U})$, by virtue of the commuting of the diagram (1.2). Let us show that $\mathrm{Et}(\Phi)$ is continuous. Let $\beta \in \mathrm{image}(\mathrm{Et}(\Phi))$ and write $\beta = [\Phi_{\mathcal{U}}(s)]_x$. Consider the open set $\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))$ and let

$$[t]_x \in \mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))).$$

Write $t \in \mathscr{F}(\mathcal{V})$. Thus $[\Phi_{\mathcal{V}}(t)]_x = [\Phi_{\mathcal{U}}(s)]_x$ and so $\Phi_{\mathcal{V}}(t)$ and $\Phi_{\mathcal{U}}(s)$ have equal restriction to some $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$. Thus

$$\mathcal{B}(\mathcal{W}, r_{\mathcal{V},\mathcal{W}}(t)) \subseteq \mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))),$$

showing that $\mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s)))$ is open. ■

Now let us construct a presheaf morphism given a morphism of étalé spaces. If $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an étalé morphism of étalé spaces of sets over $(\mathcal{S}, \mathscr{O})$, if $\mathcal{U} \in \mathscr{O}$, and if $\sigma \in \Gamma(\mathcal{U}; \mathscr{S})$, then we define a presheaf morphism $\mathrm{Ps}(\Phi)$ from $\mathrm{Ps}(\mathscr{S})$ to $\mathrm{Ps}(\mathscr{T})$ by requiring that $\mathrm{Ps}(\Phi)_{\mathcal{U}}(\sigma) \in \Gamma(\mathcal{U}; \mathscr{T})$ is given by

$$\mathrm{Ps}(\Phi)_{\mathcal{U}}(\sigma)(x) = \Phi(\sigma(x)). \tag{1.4}$$

It is then fairly easy to show that $\mathrm{Ps}(\Phi)$ is a morphism of presheaves.

1.1.59 Proposition (Presheaf morphisms of sets from étalé morphisms of sets) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces of sets over $\mathcal{S}$. If $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an étalé morphism, then $\mathrm{Ps}(\Phi)$ is a morphism of the presheaves $\mathrm{Ps}(\mathscr{S})$ and $\mathrm{Ps}(\mathscr{T})$.*

Proof This construction is well-defined since Φ is continuous. It is also obvious that $\mathrm{Ps}(\Phi)$ commutes with restrictions. ■

The following property of étalé morphisms is sometimes useful.

1.1.60 Proposition (Étalé morphisms are open) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{S}$ and $\mathscr{T}$ are étalé spaces of sets over $\mathcal{S}$, then, for a mapping $\Phi\colon \mathscr{S} \to \mathscr{T}$, the following statements are equivalent:*

(i) *Φ is an étalé morphism;*

(ii) *Φ is an open mapping and $\Phi(\mathscr{S}_x) \subseteq \mathscr{T}_x$ for every $x \in \mathcal{S}$.*

Proof First suppose that Φ is an étalé morphism. We will show that it is also open. Let $\mathcal{O} \subseteq \mathscr{S}$ be open and, for $[\sigma]_x \in \mathcal{O}$ let $\mathcal{U}$ be a neighbourhood of x such that the basic neighbourhood $\mathcal{B}(\mathcal{U}, \sigma|\mathcal{U})$ is contained in $\mathcal{O}$. Note that, by continuity, Φ maps $\mathcal{B}(\mathcal{U}, \sigma|\mathcal{U})$ to $\mathcal{B}(\mathcal{U}, \Phi\circ\sigma|\mathcal{U})$. Thus this latter neighbourhood is contained in $\Phi(\mathcal{O})$. Moreover, $\Phi(\mathcal{O})$ is the union of these neighbourhood, showing that it is open.

Now suppose that Φ is open and maps the stalk at x to the stalk at x. We will show that Φ is continuous. Let $x \in \mathcal{S}$, let $\beta \in \mathscr{T}_x$, and let $\mathcal{B}(\mathcal{U}, \tau)$ be a basic neighbourhood of β in $\mathscr{T}$. Let $\alpha \in \Phi^{-1}(\tau(x))$ and let $\mathcal{U}_\alpha \subseteq \mathcal{U}$ and $\sigma_\alpha \in \Gamma(\mathcal{U}_\alpha; \mathscr{S})$ be such that $\Phi\circ\sigma_\alpha(x) = \alpha$. Since Φ is open and since σ_α is a homeomorphism from $\mathcal{U}_\alpha$ to $\mathcal{B}(\mathcal{U}_\alpha, \sigma_\alpha)$, $\Phi\circ\sigma(\mathcal{B}(\mathcal{U}_\alpha, \sigma_\alpha)$ is open. By Lemma 1.1.40 we have $\tau|\mathcal{U}_\alpha = \Phi\circ\sigma_\alpha$. Thus $\Phi(\mathcal{B}(\mathcal{U}_\alpha, \sigma_\alpha)) \subseteq \mathcal{B}(\mathcal{U}, \tau)$. Thus we have a neighbourhood $\cup_{\alpha\in\Phi^{-1}(\tau(x))}\mathcal{B}(\mathcal{U}_\alpha, \sigma_\alpha)$ of $\Phi^{-1}(\beta)$ that maps by Φ into $\mathcal{B}(\mathcal{U}, \tau)$, showing that Φ is continuous. ■

Let us give a few examples of morphisms of sheaves.

1.1.61 Examples (Morphisms of sheaves of sets)

1. Let $\mathscr{F}$ be a presheaf of sets over a topological space $(\mathcal{S}, \mathscr{O})$. Then the family $(\beta_{\mathscr{F},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of mappings $\beta_{\mathscr{F},\mathcal{U}}\colon \mathscr{F}(\mathcal{U}) \to \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ defined by $\beta_{\mathscr{F},\mathcal{U}}(s)(x) = [s]_x$ is a morphism of the presheaves $\mathscr{F}$ and $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$, and is an isomorphism if $\mathscr{F}$ is a sheaf. We shall have more to say about this presheaf morphism in Proposition 1.1.82.
2. Let $\mathscr{S}$ be an étalé space of sets over a topological space $(\mathcal{S}, \mathscr{O})$. We then have the étalé morphism $\alpha_{\mathscr{S}}\colon \mathscr{S} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ defined by $\alpha_{\mathscr{S}}(\sigma(x)) = [\sigma]_x$ for a local section σ over a neighbourhood of x. This is an isomorphism of étalé spaces, as we shall show in Proposition 1.1.81. •

Let us now adapt a standard construction from category theorem, one we shall present in Example 2.1.5–4.

1.1.62 Construction (Hom functors for presheaves of sets)

1. We let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf of sets. To another presheaf $\mathscr{G}$ we assign the set $\mathrm{Mor}(\mathscr{F}; \mathscr{G})$ of presheaf morphisms from $\mathscr{F}$ to $\mathscr{G}$. To a presheaf morphism $\Phi = (\Phi_\mathcal{U})_{\mathcal{U}\in\mathscr{O}}$ from a presheaf $\mathscr{G}$ to a presheaf $\mathscr{H}$ we assign a map $\mathrm{Mor}(\mathscr{F}, \Phi)$ from $\mathrm{Mor}(\mathscr{F}; \mathscr{G})$ to $\mathrm{Mor}(\mathscr{F}; \mathscr{H})$ by $\mathrm{Mor}(\mathscr{F}; \Phi)(\Psi) = \Phi\circ\Psi$.
2. We can reverse the arrows in the preceding construction. Thus we again let $\mathscr{F}$ be a fixed presheaf. To a presheaf $\mathscr{G}$ we assign the set $\mathrm{Mor}(\mathscr{G}; \mathscr{F})$ of presheaf morphisms from $\mathscr{G}$ to $\mathscr{F}$. To a presheaf morphism Φ from $\mathscr{G}$ to $\mathscr{H}$ we assign a map $\mathrm{Mor}(\Phi; \mathscr{F})$ from $\mathrm{Mor}(\mathscr{H}; \mathscr{F})$ to $\mathrm{Mor}(\mathscr{G}; \mathscr{F})$ by $\mathrm{Mor}(\Phi; \mathscr{F})(\Psi) = \Psi\circ\Phi$. •

In closing, let us understand the morphisms of sheaves can be themselves organised into a sheaf. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be presheaves of sets over $\mathcal{S}$. For $\mathcal{U} \in \mathscr{O}$ we then have the restrictions $\mathscr{F}|\mathcal{U}$ and $\mathscr{G}|\mathcal{U}$ which are presheaves of sets over $\mathcal{U}$. Let us define a presheaf $\mathscr{M}or(\mathscr{F};\mathscr{G})$ by assigning to $\mathcal{U} \in \mathscr{O}$ the collection of presheaf morphisms from $\mathscr{F}|\mathcal{U}$ to $\mathscr{G}|\mathcal{U}$. Thus a section of $\mathscr{M}or(\mathscr{F};\mathscr{G})$ over $\mathcal{U}$ is a family $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ where $\Phi_{\mathcal{V}}\colon \mathscr{F}(\mathcal{V}) \to \mathscr{G}(\mathcal{V})$. If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ maps the section $(\Phi_{\mathcal{W}})_{\mathcal{U}\supseteq\mathcal{W}\text{ open}}$ over $\mathcal{U}$ to the section $(\Phi_{\mathcal{W}})_{\mathcal{V}\supseteq\mathcal{W}\text{ open}}$ over $\mathcal{V}$. Let us give a useful property of the presheaf $\mathscr{M}or(\mathscr{E};\mathscr{F})$.

1.1.63 Proposition (The presheaf of morphisms of sheaves of sets is a sheaf) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of sets over $\mathcal{S}$. Then $\mathscr{M}or(\mathscr{F};\mathscr{G})$ is a sheaf.*

Proof Let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover of $\mathcal{U}$. Let $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ and $(\Psi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ be sections over $\mathcal{U}$ whose restrictions to each of the open sets $\mathcal{U}_a$, $a \in A$, agree. Let $\mathcal{V} \subseteq \mathcal{U}$ be open and let $s \in \mathscr{F}(\mathcal{V})$. By hypothesis, $\Phi_{\mathcal{V}\cap\mathcal{U}_a}(s_a) = \Psi_{\mathcal{V}\cap\mathcal{U}_a}(s_a)$ for every $a \in A$ and $s_a \in \mathscr{F}(\mathcal{V} \cap \mathcal{U}_a)$. This implies that

$$\Phi_{\mathcal{V}}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s)) = \Psi_{\mathcal{V}}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))$$

for every $a \in A$, and so

$$r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(\Phi_{\mathcal{V}}(s)) = r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(\Psi_{\mathcal{V}}(s))$$

for every $a \in A$. Since $\mathscr{G}$ is separated, this implies that $\Phi_{\mathcal{V}}(s) = \Psi_{\mathcal{V}}(s)$. We conclude, therefore, that $\mathscr{M}or(\mathscr{F};\mathscr{G})$ is separated.

Now again let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$. For each $a \in A$ let $(\Phi_{a,\mathcal{V}})_{\mathcal{U}_a\supseteq\mathcal{V}\text{ open}}$ be a section of $\mathscr{M}or(\mathscr{F};\mathscr{G})$ over $\mathcal{U}_a$ and suppose that the restrictions of the sections over $\mathcal{U}_a$ and $\mathcal{U}_b$ agree on the intersection $\mathcal{U}_a \cap \mathcal{U}_b$ for every $a, b \in A$. Let $\mathcal{V} \subseteq \mathcal{U}$ be open and let $s \in \mathscr{F}(\mathcal{V})$. By hypothesis

$$\Phi_{a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V}\cap\mathcal{U}_a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))) = \Phi_{b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V}\cap\mathcal{U}_b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_b}(s))$$

for every $a, b \in A$. Thus

$$r_{\mathcal{V}\cap\mathcal{U}_a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\Phi_{a,\mathcal{V}\cap\mathcal{U}_a}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))) = r_{\mathcal{V}\cap\mathcal{U}_b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\Phi_{b,\mathcal{V}\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_b}(s)))$$

for every $a, b \in A$. Since $\mathscr{G}$ is has the gluing property, we infer the existence of $t \in \mathscr{G}(\mathcal{V})$ such that

$$r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(t) = \Phi_{a,\mathcal{V}\cap\mathcal{U}_a}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s)) \tag{1.5}$$

for every $a \in A$. We define $\Phi_{\mathcal{V}}$ by asking that $\Phi_{\mathcal{V}}(s) = t$. Thus the section $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ of $\mathscr{H}om(\mathscr{F};\mathscr{G})$ over $\mathcal{U}$ so defined has the property that it restricts to $(\Phi_{a,\mathcal{V}})_{\mathcal{U}_a\supseteq\mathcal{V}\text{ open}}$ for each $a \in A$. We finally must show that the diagram

$$\begin{tikzcd} \mathscr{F}(\mathcal{V}) \arrow[r, "\Phi_{\mathcal{V}}"] \arrow[d] & \mathscr{G}(\mathcal{V}) \arrow[d] \\ \mathscr{F}(\mathcal{W}) \arrow[r, "\Phi_{\mathcal{W}}"'] & \mathscr{G}(\mathcal{W}) \end{tikzcd}$$

commutes for every $\mathcal{V}, \mathcal{W} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, and where the vertical arrows are the restriction maps. Consider the open cover $(\mathcal{V}, \mathcal{W})$ for $\mathcal{V}$. Let $s \in \mathscr{F}(\mathcal{V})$ and note that our construction of Φ, particular (1.5), gives

$$r_{\mathcal{V},\mathcal{W}}(\Phi_{\mathcal{V}}(s)) = \Phi_{\mathcal{W}}(r_{\mathcal{V},\mathcal{W}}(s)),$$

as desired. ∎

In fact, this is much more easily and naturally done for étalé spaces. So suppose that we have a topological space $(\mathcal{S}, \mathscr{O})$ and let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces over $\mathcal{S}$. For $\mathcal{U} \in \mathscr{O}$ we then have (from Definition 1.1.38) the restrictions $\mathrm{Et}(\mathscr{F})|\mathcal{U}$ and $\mathrm{Et}(\mathscr{G})|\mathcal{U}$ as étalé spaces of sets over $\mathcal{U}$. Let us define a presheaf $\mathscr{M}or(\mathscr{S}; \mathscr{T})$ by assigning to $\mathcal{U} \in \mathscr{O}$ the collection of étalé morphisms from $\mathscr{S}|\mathcal{U}$ to $\mathscr{T}|\mathcal{U}$. Thus a section of $\mathscr{M}or(\mathscr{S}; \mathscr{T})$ over $\mathcal{U}$ is a continuous map $\Phi_{\mathcal{U}}\colon \mathscr{S}|\mathcal{U} \to \mathscr{T}|\mathcal{U}$ such that $\Phi_{\mathcal{U}}(\sigma) \in \mathscr{S}_x$ if $\sigma \in \mathscr{T}_x$. If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ is simply standard restriction.

1.1.64 Proposition (The presheaf of morphisms of étalé spaces of sets is a sheaf) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{S}$ and $\mathscr{S}$ be étalé spaces of sets over $\mathcal{S}$. Then $\mathscr{M}or(\mathscr{S}; \mathscr{T})$ is a sheaf.*

Proof By Proposition 1.1.15 $\mathscr{M}or(\mathscr{S}; \mathscr{T})$ is separated. Let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a \in A}$ be an open cover of $\mathcal{U}$. Suppose that, for $a \in A$, we have a morphism $\Phi_a\colon \mathscr{S}|\mathcal{U}_a \to \mathscr{T}|\mathcal{U}_a$ and that Φ_a and Φ_b agree on $\mathcal{U}_a \cap \mathcal{U}_b$ for all $a, b \in A$. Then define $\Phi\colon \mathscr{S}|\mathcal{U} \to \mathscr{T}|\mathcal{U}$ by $\Phi([s]_x) = \Phi_a([s]_x)$ where $a \in A$ is such that $x \in \mathcal{U}_a$. It is clear that Φ is well-defined and that its restriction to $\mathcal{U}_a$ agrees with Φ_a for each $a \in A$. Thus $\mathscr{M}or(\mathscr{S}; \mathscr{T})$ satisfies the gluing property. ■

Morphisms of presheaves and étalé spaces of rings

Next we turn to specialising the constructions and results from the preceding section to sheaves of rings.

1.1.65 Definition (Morphism of presheaves of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings, and let $\mathscr{S}$, and $\mathscr{T}$ be presheaves of rings over $\mathcal{S}$.

(i) A ***morphism*** of the presheaves $\mathscr{R}$ and $\mathscr{S}$ is a morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ of the presheaves of sets $\mathscr{R}$ and $\mathscr{S}$ with the additional condition that $\Phi_{\mathcal{U}}$ is a homomorphism of rings for each $\mathcal{U} \in \mathscr{O}$. If $\mathscr{R}$ and $\mathscr{S}$ are sheaves, Φ is called a ***morphism*** of sheaves of rings.

By $\mathrm{Hom}(\mathscr{R}; \mathscr{S})$ we denote the set of morphisms of presheaves of rings.

(ii) The ***composition*** of morphisms of presheaves of rings is the same as their composition as presheaves of sets, noting that this composition is indeed a morphism of presheaves of rings.

(iii) The identity morphism of a presheaf $\mathscr{R}$ of rings is the same as the identity morphism of $\mathscr{R}$ as a sheaf of sets.

(iv) An ***isomorphism*** of presheaves of rings $\mathscr{R}$ and $\mathscr{S}$ is an isomorphism of presheaves of sets that is a morphism of presheaves of rings. •

1.1.66 Definition (Morphism of étalé spaces of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{S}$.

(i) An ***étalé morphism*** of $\mathscr{A}$ and $\mathscr{B}$ is an étalé morphism of sheaves of sets with the additional condition that $\Phi|\mathscr{A}_x$ is a homomorphism of rings for every $x \in \mathcal{S}$.

By $\mathrm{Hom}(\mathscr{A}; \mathscr{B})$ we denote the set of étalé morphisms of étalé spaces of rings.

(ii) An étalé morphism $\Phi\colon \mathscr{A} \to \mathscr{B}$ of étalé spaces of rings is an ***isomorphism*** if there exists an étalé morphism $\Psi\colon \mathscr{B} \to \mathscr{A}$ of étalé spaces of rings such that $\Phi \circ \Psi = \mathrm{id}_{\mathscr{B}}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathscr{A}}$. •

Let us indicate how one can interchange the two notions of morphisms. Given a morphism Φ of presheaves of rings $\mathscr{R}$ and $\mathscr{S}$, we define a morphism $\mathrm{Et}(\Phi)$ of the corresponding étalé spaces of rings as in (1.3).

1.1.67 Proposition (Étalé morphisms of rings from presheaf morphisms of rings) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of rings over $\mathcal{S}$. If $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of the presheaves $\mathscr{R}$ and $\mathscr{S}$, then $\mathrm{Et}(\Phi)$ is a morphism of the étalé spaces of rings $\mathrm{Et}(\mathscr{R})$ and $\mathrm{Et}(\mathscr{S})$.*

Proof From Proposition 1.1.58 we know that $\mathrm{Et}(\Phi)$ is a well-defined étalé morphism of sets. That $\mathrm{Et}(\Phi)$ is a morphism of rings when restricted to stalks follows from the commuting of the diagram (1.2) and the definition of the ring operation on stalks. ■

Let us also show how étalé morphisms give rise to presheaf morphisms. If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is an étalé morphism of étalé spaces of rings, we can define a presheaf morphism of the presheaves $\mathrm{Ps}(\mathscr{A})$ and $\mathrm{Ps}(\mathscr{B})$ as in (1.4).

1.1.68 Proposition (Presheaf morphisms of rings from étalé morphisms of rings) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{S}$. If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is an étalé morphism of rings, then $\mathrm{Ps}(\Phi)$ is a morphism of the presheaves of rings $\mathrm{Ps}(\mathscr{A})$ and $\mathrm{Ps}(\mathscr{B})$.*

Proof From Proposition 1.1.59 we know that $\mathrm{Ps}(\Phi)$ is a morphism of presheaves of sets. It is clear that it defines a homomorphism of rings on stalks when the étalé space possesses this structures. ■

Let us give a few examples of morphisms of sheaves of rings.

1.1.69 Examples (Morphisms of sheaves of rings)

1. As in Example 1.1.61–1, if $\mathscr{R}$ is a presheaf of rings over a topological space $(\mathcal{S}, \mathscr{O})$, then we have a morphism $\beta_{\mathscr{R}}$ from $\mathscr{R}$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{R}))$, and it is an isomorphism if $\mathscr{R}$ is a sheaf of rings. This map is easily verified to be a morphism of presheaves of rings.
2. As in Example 1.1.61–2, if $\mathscr{A}$ is an étalé space of rings over a topological space $(\mathcal{S}, \mathscr{O})$, then we have an étalé morphism $\alpha_{\mathscr{A}}\colon \mathscr{A} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{A}))$ which is an isomorphism of étalé spaces of rings. •

As with morphisms of sheaves of sets, we can adapt the notion of a Hom functor to sheaves of rings.

1.1.70 Construction (Hom functors for presheaves of rings)

1. We let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings. To another presheaf $\mathscr{S}$ we assign the set $\mathrm{Hom}(\mathscr{R};\mathscr{S})$ of presheaf morphisms from $\mathscr{R}$ to $\mathscr{S}$. To a presheaf morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from a presheaf $\mathscr{S}$ to a presheaf $\mathscr{T}$ we assign

a map $\mathrm{Hom}(\mathscr{R},\Phi)$ from $\mathrm{Hom}(\mathscr{R};\mathscr{S})$ to $\mathrm{Hom}(\mathscr{R};\mathscr{T})$ by $\mathrm{Hom}(\mathscr{R};\Phi)(\Psi) = \Phi\circ\Psi$. Obviously, $\mathrm{Hom}(\mathscr{R};\Phi)(\Psi)$ is indeed a morphism of presheaves of rings.

2. We can reverse the arrows in the preceding construction. Thus we again let $\mathscr{R}$ be a fixed presheaf. To a presheaf $\mathscr{S}$ we assign the set $\mathrm{Hom}(\mathscr{S};\mathscr{R})$ of presheaf morphisms from $\mathscr{S}$ to $\mathscr{R}$. To a presheaf morphism Φ from $\mathscr{S}$ to $\mathscr{T}$ we assign a map $\mathrm{Hom}(\Phi;\mathscr{R})$ from $\mathrm{Hom}(\mathscr{T};\mathscr{R})$ to $\mathrm{Hom}(\mathscr{S};\mathscr{R})$ by $\mathrm{Hom}(\Phi;\mathscr{R})(\Psi) = \Psi\circ\Phi$. Again, it is clear that $\mathrm{Hom}(\Phi;\mathscr{R})(\Psi)$ is a morphism of presheaves of rings. •

The construction of a sheaf of morphisms of sheaves of rings follows exactly as with sheaves of sets. Thus suppose that we have a topological space $(\mathcal{S},\mathscr{O})$ and let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of sets over $\mathcal{S}$. For $\mathcal{U}\in\mathscr{O}$ we then have the restrictions $\mathscr{R}|\mathcal{U}$ and $\mathscr{S}|\mathcal{U}$ which are presheaves of rings over $\mathcal{U}$. Let us define a presheaf $\mathscr{H}om(\mathscr{R};\mathscr{S})$ by assigning to $\mathcal{U}\in\mathscr{O}$ the collection of presheaf morphisms from $\mathscr{R}|\mathcal{U}$ to $\mathscr{S}|\mathcal{U}$. Thus a section of $\mathrm{Hom}(\mathscr{R};\mathscr{S})$ over $\mathcal{U}$ is a family $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\,\mathrm{open}}$ where $\Phi_{\mathcal{V}}\in\mathrm{Hom}(\mathscr{R}(\mathcal{V});\mathscr{S}(\mathcal{V}))$. If $\mathcal{U},\mathcal{V}\in\mathscr{O}$ satisfy $\mathcal{V}\subseteq\mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ maps the section $(\Phi_{\mathcal{W}})_{\mathcal{U}\supseteq\mathcal{W}\,\mathrm{open}}$ over $\mathcal{U}$ to the section $(\Phi_{\mathcal{W}})_{\mathcal{V}\supseteq\mathcal{W}\,\mathrm{open}}$ over $\mathcal{V}$.

1.1.71 Proposition (The presheaf of morphisms of sheaves of rings is a sheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space and let $\mathscr{R}$ and $\mathscr{S}$ be sheaves of rings over $\mathcal{S}$. Then $\mathscr{H}om(\mathscr{R};\mathscr{S})$ is a sheaf.*

Proof This is a consequence of Proposition 1.1.63. ■

We can also mirror the constructions for étalé spaces of rings. Thus suppose that we have a topological space $(\mathcal{S},\mathscr{O})$ and let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{S}$. For $\mathcal{U}\in\mathscr{O}$ the restrictions $\mathscr{A}|\mathcal{U}$ and $\mathscr{B}|\mathcal{U}$ are étalé spaces of rings. Let us define a presheaf $\mathscr{H}om(\mathscr{A};\mathscr{B})$ by assigning to $\mathcal{U}\in\mathscr{O}$ the collection of étalé morphisms from $\mathscr{A}|\mathcal{U}$ to $\mathscr{B}|\mathcal{U}$. Thus a section of $\mathscr{H}om(\mathscr{A};\mathscr{B})$ over $\mathcal{U}$ is a continuous map $\Phi_{\mathcal{U}}\colon\mathscr{A}|\mathcal{U}\to\mathscr{B}|\mathcal{U}$ such that $\Phi_{\mathcal{U}}(\sigma)\in\mathscr{A}_x$ if $\sigma\in\mathscr{B}_x$ and such that the induced map from $\mathscr{A}_x$ to $\mathscr{B}_x$ is a ring homomorphism. If $\mathcal{U},\mathcal{V}\in\mathscr{O}$ satisfy $\mathcal{V}\subseteq\mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ is simply standard restriction.

1.1.72 Proposition (The presheaf of morphisms of étalé spaces of rings is a sheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space and let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{S}$. Then $\mathscr{H}om(\mathscr{A};\mathscr{B})$ is a sheaf.*

Proof This is a consequence of Proposition 1.1.64. ■

Morphisms of presheaves and étalé spaces of modules

We now discuss morphisms of sheaves of modules.

1.1.73 Definition (Morphism of presheaves of modules) Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$, and let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$.

(i) A ***morphism*** of the presheaves $\mathscr{E}$ and $\mathscr{F}$ is a morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of the presheaves of $\mathscr{E}$ and $\mathscr{F}$ with the additional condition that $\Phi_{\mathcal{U}}$ is a homomorphism of $\mathscr{R}(\mathcal{U})$-modules for each $\mathcal{U}\in\mathscr{O}$. If $\mathscr{E}$ and $\mathscr{F}$ are sheaves, Φ is called a ***morphism*** of sheaves of $\mathscr{R}$-modules.

By $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ we denote the set of morphisms of presheaves of $\mathscr{R}$-modules.

(ii) The ***composition*** of morphisms of presheaves of modules is the same as their composition as presheaves of sets, noting that this composition is indeed a morphism of presheaves of modules.

(iii) The identity morphism of a presheaf $\mathscr{E}$ of modules is the same as the identity morphism of $\mathscr{E}$ as a sheaf of sets.

(iv) An ***isomorphism*** of presheaves of modules $\mathscr{E}$ and $\mathscr{F}$ is an isomorphism of presheaves of sets that is a morphism of presheaves of modules. •

1.1.74 Definition (Morphism of étalé spaces of modules) Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings, and let $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of $\mathscr{A}$-modules.

(i) An ***étalé morphism*** of $\mathscr{U}$ and $\mathscr{V}$ is an étalé morphism of sheaves of sets with the additional condition that $\Phi|\mathscr{U}_x$ is a homomorphism of rings for every $x \in \mathcal{S}$.

By $\mathrm{Hom}_{\mathscr{A}}(\mathscr{U};\mathscr{V})$ we denote the set of étalé morphisms of étalé spaces of $\mathscr{A}$-modules.

(ii) An étalé morphism $\Phi\colon \mathscr{U} \to \mathscr{V}$ of étalé spaces of $\mathscr{A}$-modules is an ***isomorphism*** if there exists an étalé morphism $\Psi\colon \mathscr{E} \to \mathscr{F}$ of étalé spaces of $\mathscr{A}$-modules such that $\Phi\circ\Psi = \mathrm{id}_{\mathscr{V}}$ and $\Psi\circ\Phi = \mathrm{id}_{\mathscr{U}}$. •

Let us indicate how one can interchange the two notions of morphisms. Given a morphism Φ of presheaves of $\mathscr{R}$-modules $\mathscr{E}$ and $\mathscr{F}$, we define a morphism $\mathrm{Et}(\Phi)$ of the corresponding étalé spaces of $\mathrm{Et}(\mathscr{R})$-modules as in (1.3).

1.1.75 Proposition (Étalé morphisms of modules from presheaf morphisms of modules) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of rings over $\mathcal{S}$. If $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of the presheaves $\mathscr{E}$ and $\mathscr{F}$, then $\mathrm{Et}(\Phi)$ is a morphism of the étalé spaces of $\mathrm{Et}(\mathscr{R})$-modules $\mathrm{Et}(\mathscr{E})$ and $\mathrm{Et}(\mathscr{F})$.*

Proof From Proposition 1.1.58 we know that $\mathrm{Et}(\Phi)$ is a well-defined étalé morphism of sets. That $\mathrm{Et}(\Phi)$ is a morphism of $\mathrm{Et}(\mathscr{R})$-modules when restricted to stalks follows from the commuting of the diagram (1.2) and the definition of the module operations on stalks. ■

Let us also show how étalé morphisms give rise to presheaf morphisms. If $\Phi\colon \mathscr{E} \to \mathscr{F}$ is an étalé morphism of étalé spaces of $\mathrm{Et}(\mathscr{R})$-modules, we can define a presheaf morphism of the presheaves $\mathrm{Ps}(\mathscr{E})$ and $\mathrm{Ps}(\mathscr{F})$ as in (1.4).

1.1.76 Proposition (Presheaf morphisms of modules from étalé morphisms of modules) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be étalé spaces of rings over $\mathcal{S}$. If $\Phi\colon \mathscr{E} \to \mathscr{F}$ is an étalé morphism of $\mathscr{A}$-modules, then $\mathrm{Ps}(\Phi)$ is a morphism of the presheaves of $\mathscr{A}$-modules $\mathrm{Ps}(\mathscr{E})$ and $\mathrm{Ps}(\mathscr{F})$.*

Proof From Proposition 1.1.59 we know that $\mathrm{Ps}(\Phi)$ is a morphism of presheaves of sets. It is clear that it defines a homomorphism of rings on stalks when the étalé space possesses this structures. ■

Let us give a few examples of morphisms of sheaves of rings.

1.1.77 Examples (Morphisms of sheaves of modules)

1. Let $r \in \{infty, \omega, \mathrm{hol}\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\tau\colon \mathsf{F} \to \mathsf{M}$ be vector bundles of class C^r. If we have a vector bundle mapping $\Phi\colon \mathsf{E} \to \mathsf{F}$ of class C^r over id_{M}, we define a morphism $\hat{\Phi} = (\hat{\Phi})_{\mathcal{U}\,\text{open}}$ between the presheaves $\mathscr{G}^r_{\mathsf{E}}$ and $\mathscr{G}^r_{\mathsf{F}}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules by $\hat{\Phi}_{\mathcal{U}}(\xi)(x) = \Phi \circ \xi(x)$ for $\xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U})$ and $x \in \mathcal{U}$. We shall have more to say about this morphism in Section 1.4.5.
2. Let $r \in \{\infty, \omega\}$ and let M be a smooth or real analytic manifold. Let us consider the sheaf $\mathscr{G}^k_{\bigwedge^r(\mathsf{T}^*\mathsf{M})}$ of germs of sections of the bundle of k-forms. Since the exterior derivative d commutes with restrictions to open sets, d induces a morphism of sheaves:
$$\mathrm{d}\colon \mathscr{G}^r_{\bigwedge^k(\mathsf{T}^*\mathsf{M})} \to \mathscr{G}^r_{\bigwedge^{k+1}(\mathsf{T}^*\mathsf{M})}.$$
This is a morphism of sheaves of $\mathbb{R}$-vector spaces, but not a morphism of sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules, since d is not linear with respect to multiplication by C^r-functions.
3. We let M be a holomorphic manifold and consider the sheaf $\mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{\mathbb{C}}\mathsf{M})}$ of germs of sections of the bundles of forms of bidegree (r,s), $r,s \in \mathbb{Z}_{\ge 0}$. This is a sheaf of $\mathscr{C}^\infty(\mathsf{M};\mathbb{C})$-modules, of course. The mappings ∂ and $\bar{\partial}$ of Section GA1.4.6.2 commute with restrictions to open sets, and so define morphisms of sheaves
$$\partial\colon \mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \to \mathscr{G}^\infty_{\bigwedge^{r+1,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}, \quad \bar{\partial}\colon \mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \to \mathscr{G}^\infty_{\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}.$$
These are morphisms of sheaves of $\mathbb{C}$-vector spaces, but neither of these are morphisms of $\mathscr{C}^\infty(\mathsf{M};\mathbb{C})$-modules, since neither ∂ nor $\bar{\partial}$ are linear with respect to multiplication by smooth functions.
4. If in the preceding example we instead regard $\mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{\mathbb{C}}\mathsf{M})}$ as sheaves of $\mathscr{C}^{\mathrm{hol}}_{\mathsf{M}}$-modules, then, by Proposition GA1.4.6.7(**??**) and because $\bar{\partial}$ annihilates holomorphic functions, $\bar{\partial}$ is a morphism of $\mathscr{C}^{\mathrm{hol}}_{\mathsf{M}}$-modules.
5. Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as appropriate, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. As we saw in Lemma GA1.5.5.3, the bundle of k-jets of sections of E, $\mathsf{J}^k\mathsf{E}$, is a vector bundle of class C^r. We define a morphism $j_k = (j_{k,\mathcal{U}})_{\mathcal{U}\,\text{open}}$ from $\mathscr{G}^r_{\mathsf{E}}$ to $\mathscr{G}^r_{\mathsf{J}^k\mathsf{E}}$ by
$$j_{k,\mathcal{U}}(\xi)(x) = j_k\xi(x), \qquad \xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}),\ x \in \mathcal{U}.$$
This is easily verified to be a morphism of sheaves of $\mathbb{F}$-vector spaces.
6. Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as required, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. Let ∇ be a connection in E of class C^r. We consider two morphisms associated with this connection. To unify notation, we let TM denote the holomorphic tangent bundle in the case that $r = \mathrm{hol}$.
First, we fix X be a vector field of class C^r and define a morphism $\nabla_X = (\nabla_{X,\mathcal{U}})_{\mathcal{U}\,\text{open}}$ from $\mathscr{G}^r_{\mathsf{E}}$ to itself by
$$\nabla_{X,\mathcal{U}}(\xi)(x) = ((\nabla_X|\mathcal{U})\xi)(x), \qquad \xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}),\ x \in \mathcal{U}.$$

This makes sense because, as we saw above with exterior derivative, covariant differentiation commutes with restriction to open sets. The morphism ∇_X is a morphism of presheaves of $\mathbb{F}$-vector spaces.

In similar manner, we can fix $\xi \in \mathscr{G}^r_{\mathsf{E}}(\mathsf{M})$, and define a morphism $\nabla\xi = ((\nabla\xi)_{\mathcal{U}})_{\mathcal{U}\ \mathrm{open}}$ from the sheaf $\mathscr{G}^r_{\mathsf{TM}}$ to $\mathscr{G}^r_{\mathsf{E}}$ by

$$(\nabla\xi)_{\mathcal{U}}(X)(x) = \nabla_X(\xi|\mathcal{U})(x), \qquad X \in \mathscr{G}^t_{\mathsf{TM}}(\mathcal{U}),\ x \in \mathcal{U}.$$

This morphism is one of sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules. •

As with morphisms of sheaves of sets and rings, we can adapt the notion of a Hom functor to sheaves of modules.

1.1.78 Construction (Hom functors for presheaves of modules)

1. We let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules. To another presheaf $\mathscr{F}$ of $\mathscr{R}$-modules we assign the set $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ of presheaf morphisms from $\mathscr{E}$ to $\mathscr{F}$. To a presheaf morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from a presheaf $\mathscr{F}$ to a presheaf $\mathscr{G}$ we assign a map $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E},\Phi)$ from $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ to $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\mathscr{G})$ by $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\Phi)(\Psi) = \Phi\circ\Psi$. Obviously, $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\Phi)(\Psi)$ is indeed a morphism of presheaves of $\mathscr{R}$-modules.
2. We can reverse the arrows in the preceding construction. Thus we again let $\mathscr{R}$ be a presheaf of rings and let $\mathscr{E}$ be a fixed presheaf of $\mathscr{R}$-modules. To a presheaf $\mathscr{F}$ we assign the set $\mathrm{Hom}_{\mathscr{R}}(\mathscr{F};\mathscr{E})$ of presheaf morphisms from $\mathscr{F}$ to $\mathscr{E}$. To a presheaf morphism Φ from $\mathscr{F}$ to $\mathscr{G}$ we assign a map $\mathrm{Hom}_{\mathscr{R}}(\Phi;\mathscr{E})$ from $\mathrm{Hom}_{\mathscr{R}}(\mathscr{G};\mathscr{E})$ to $\mathrm{Hom}_{\mathscr{R}}(\mathscr{F};\mathscr{E})$ by $\mathrm{Hom}_{\mathscr{R}}(\Phi;\mathscr{E})(\Psi) = \Psi\circ\Phi$. Again, it is clear that $\mathrm{Hom}_{\mathscr{R}}(\Phi;\mathscr{E})(\Psi)$ is a morphism of presheaves of $\mathscr{R}$-modules. •

The construction of a sheaf of morphisms of sheaves of modules follows exactly as with sheaves of sets. Thus suppose that we have a topological space $(\mathcal{S}, \mathscr{O})$, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. For $\mathcal{U} \in \mathscr{O}$ we then have the restrictions $\mathscr{E}|\mathcal{U}$ and $\mathscr{F}|\mathcal{U}$ which are presheaves of $\mathscr{R}|\mathcal{U}$-modules over $\mathcal{U}$. Let us define a presheaf $\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ by assigning to $\mathcal{U} \in \mathscr{O}$ the collection of presheaf morphisms from $\mathscr{E}|\mathcal{U}$ to $\mathscr{F}|\mathcal{U}$. Thus a section of $\mathrm{Hom}_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ over $\mathcal{U}$ is a family $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\ \mathrm{open}}$ where $\Phi_{\mathcal{V}} \in \mathrm{Hom}_{\mathscr{R}(\mathcal{U})}(\mathscr{E}(\mathcal{V});\mathscr{F}(\mathcal{V}))$. If $\mathcal{U},\mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ maps the section $(\Phi_{\mathcal{W}})_{\mathcal{U}\supseteq\mathcal{W}\ \mathrm{open}}$ over $\mathcal{U}$ to the section $(\Phi_{\mathcal{W}})_{\mathcal{V}\supseteq\mathcal{W}\ \mathrm{open}}$ over $\mathcal{V}$.

1.1.79 Proposition (The presheaf of morphisms of sheaves of modules is a sheaf) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ and $\mathscr{S}$ be sheaves of rings over $\mathcal{S}$. Then $\mathscr{H}om(\mathscr{R};\mathscr{S})$ is a sheaf.*

Proof This is a consequence of Proposition 1.1.63. ■

The construction of a sheaf of morphisms of étalé spaces of $\mathscr{A}$-modules follows exactly as with sheaves of sets. Thus suppose that we have a topological space $(\mathcal{S}, \mathscr{O})$, let $\mathscr{A}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{M}$ and $\mathscr{N}$ be étalé spaces of $\mathscr{A}$-modules.

For $\mathcal{U} \in \mathscr{O}$ the restrictions $\mathscr{E}|\mathcal{U}$ and $\mathscr{F}|\mathcal{U}$ are étalé spaces of $\mathscr{A}$-modules. Let us define a presheaf $\mathscr{H}om_{\mathscr{A}}(\mathscr{M};\mathscr{N})$ by assigning to $\mathcal{U} \in \mathscr{O}$ the collection of étalé morphisms from $\mathscr{M}|\mathcal{U}$ to $\mathscr{N}|\mathcal{U}$. Thus a section of $\mathscr{H}om_{\mathscr{A}}(\mathscr{M};\mathscr{N})$ over $\mathcal{U}$ is a continuous map $\Phi_{\mathcal{U}}\colon \mathscr{M}|\mathcal{U} \to \mathscr{N}|\mathcal{U}$ such that $\Phi_{\mathcal{U}}(\sigma) \in \mathscr{M}_x$ if $\sigma \in \mathscr{M}_x$ and such that the induced map from $\mathscr{M}_x$ to $\mathscr{N}_x$ is a homomorphism of $\mathscr{A}$-modules. If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ is simply standard restriction.

1.1.80 Proposition (The presheaf of morphisms of étalé spaces of modules is a sheaf) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{M}$ and $\mathscr{N}$ be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$. Then $\mathscr{H}om_{\mathscr{A}}(\mathscr{M};\mathscr{N})$ is a sheaf.*

Proof This is a consequence of Proposition 1.1.64. ■

1.1.6 Correspondences between presheaves and étalé spaces

We have a process of starting with a presheaf $\mathscr{F}$ and constructing another presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$, and also a process of starting with an étalé space $\mathscr{S}$ and constructing another étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. One anticipates that there is a relationship between these objects, and we shall explore this now.

Correspondences between presheaves and étalé spaces of sets

We begin by looking at the situation with sheaves of sets.

1.1.81 Proposition ($\mathrm{Et}(\mathrm{Ps}(\mathscr{S})) \simeq \mathscr{S}$ (set version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{S}$ is an étalé space of sets over $\mathcal{S}$, then the map $\alpha_{\mathscr{S}}\colon \mathscr{S} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ given by $\alpha_{\mathscr{S}}(\sigma(\mathrm{x})) = [\sigma]_{\mathrm{x}}$, where $\sigma\colon \mathcal{U} \to \mathscr{S}$ is a section over $\mathcal{U}$, is an isomorphism of étalé spaces.*

Proof First, let us verify that $\alpha_{\mathscr{S}}$ is well-defined. Suppose that local sections σ and τ of $\mathscr{S}$ agree at x. By Lemma 1.1.40 it follows that σ and τ agree in some neighbourhood of x. But this means that $[\sigma]_x = [\tau]_x$, giving well-definedness of $\alpha_{\mathscr{S}}$. To show that $\alpha_{\mathscr{S}}$ is injective, suppose that $\alpha_{\mathscr{S}}(\sigma(x)) = \alpha_{\mathscr{S}}(\tau(x))$. Thus $[\sigma]_x = [\tau]_x$ and so σ and τ agree on some neighbourhood of x by Lemma 1.1.40. Thus $\sigma(x) = \tau(x)$, giving injectivity. To show that $\alpha_{\mathscr{S}}$ is surjective, let $[\sigma]_x \in \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. Again since sections of $\mathscr{S}$ are local inverses for the étalé projection, it follows that $\alpha_{\mathscr{S}}(\sigma(x)) = [\sigma]_x$, giving surjectivity. It is also clear that $\alpha_{\mathscr{S}}(\mathscr{S}_x) \subseteq \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))_x$. It remains to show that $\alpha_{\mathscr{S}}$ is continuous. Let $[\sigma]_x \in \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ and let $\mathcal{O}$ be a neighbourhood of $[\sigma]_x$ in $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. By Lemma 1.1.40, there exists a neighbourhood $\mathcal{U}$ of x such that $\mathcal{B}(\mathcal{U}, [\sigma])$ is a neighbourhood of x contained in $\mathcal{O}$. Here $[\sigma]$ is the section of $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ over $\mathcal{U}$ given by $[\sigma](y) = [\sigma]_y$. Since $\alpha_{\mathscr{S}}(\sigma(y)) = [\sigma]_y$ for every $y \in \mathcal{U}$, it follows that $\alpha_{\mathscr{S}}(\mathcal{B}(\mathcal{U}, \sigma)) = \mathcal{B}(\mathcal{U}, [\sigma])$, giving continuity as desired. ■

Now let us look at the relationship between a presheaf $\mathscr{F}$ and the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$.

1.1.82 Proposition ($\mathrm{Ps}(\mathrm{Et}(\mathscr{F})) \simeq \mathscr{F}$ if $\mathscr{F}$ is a sheaf (set version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F}$ is a sheaf of sets over $\mathcal{S}$, then the map which assigns to $\mathrm{s} \in \mathscr{F}(\mathcal{U})$ the section $\beta_{\mathscr{F},\mathcal{U}}(\mathrm{s}) \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ given by $\beta_{\mathscr{F},\mathcal{U}}(\mathrm{s})(\mathrm{x}) = [\mathrm{s}]_{\mathrm{x}}$ is an isomorphism of presheaves.*

Proof We must show that $\beta_{\mathscr{F},\mathcal{U}}$ is a bijection for each $\mathcal{U} \in \mathscr{O}$. To see that $\beta_{\mathscr{F},\mathcal{U}}$ is injective, suppose that $\beta_{\mathscr{F},\mathcal{U}}(s) = \beta_{\mathscr{F},\mathcal{U}}(t)$. Then $[s]_x = [t]_x$ for every $x \in \mathcal{U}$. Thus, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{U}_x}(s) = r_{\mathcal{U},\mathcal{U}_x}(t)$. By condition (i) of Definition 1.1.11 it follows that $s = t$. For surjectivity, let $\sigma \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$. Let $x \in \mathcal{U}$ and let $\mathcal{U}_x$ be a neighbourhood of x and $s_x \in \mathscr{F}(\mathcal{U}_x)$ be such that $\sigma(x) = [s_x]_x$. Since sections of $\mathrm{Et}(\mathscr{F})$ are local inverses for the local homeomorphism $\pi_{\mathscr{F}}$ (by definition of the étalé topology), sections of $\mathrm{Et}(\mathscr{F})$ agreeing at x must agree in a neighbourhood of x. In particular, there must exist a neighbourhood of x, $\mathcal{V}_x \subseteq \mathcal{U}_x$, such that $\sigma(y) = [s_x]_y$ for every $y \in \mathcal{V}_x$. It follows from Definition 1.1.11(i), therefore, that

$$r_{\mathcal{V}_{x_1},\mathcal{V}_{x_1}\cap\mathcal{V}_{x_2}}(s_{x_1}) = r_{\mathcal{V}_{x_2},\mathcal{V}_{x_1}\cap\mathcal{V}_{x_2}}(s_{x_2})$$

for every $x_1, x_2 \in \mathcal{U}$. By Definition 1.1.11(ii) it follows that there exists $s_\sigma \in \mathscr{F}(\mathcal{U})$ such that $\sigma(x) = [s_x]_x = [s_\sigma]_x$ for every $x \in \mathcal{U}$, as desired. ■

Thus, one of the nice things about the étalé space is that it allows one to realise a presheaf as a presheaf of sections of something, somehow making the constructions more concrete (although the étalé spaces themselves can be quite difficult to understand). This correspondence between sheaves and étalé spaces leads to a common abuse of notation and terminology, with the frequent and systematic confounding of a sheaf and its étalé space. Moreover, as we shall see in Section 1.3.1, there is a degree of inevitability to this, as some constructions with sheaves lead one naturally to building étalé spaces.

Now we shall show that the processes above for going from morphisms of presheaves and étalé spaces and back commute in situations where such commutativity is expected.

1.1.83 Proposition (Consistency of morphism constructions (set version)) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{F}$ and $\mathscr{G}$ be presheaves of rings, and let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces of sets over $\mathcal{S}$. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a morphism of the presheaves $\mathscr{F}$ and $\mathscr{G}$ and let $\Psi\colon \mathscr{S} \to \mathscr{T}$ be an étalé morphism. Then the diagrams*

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\alpha_{\mathscr{S}}"] \arrow[d, "\Psi"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{S})) \arrow[d, "\mathrm{Et}(\mathrm{Ps}(\Psi))"] \\ \mathscr{T} \arrow[r, "\alpha_{\mathscr{T}}"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{T})) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F} \arrow[r, "\beta_{\mathscr{F}}"] \arrow[d, "\Phi"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{F})) \arrow[d, "\mathrm{Ps}(\mathrm{Et}(\Phi))"] \\ \mathscr{G} \arrow[r, "\beta_{\mathscr{G}}"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{G})) \end{tikzcd}$$

commute.

Proof If $\sigma \in \Gamma(\mathcal{U}; \mathscr{S})$ then $\alpha_{\mathscr{S}}(\sigma(x)) = [\sigma]_x$. Note that $\mathrm{Ps}(\Psi)_{\mathcal{U}}(\sigma)(x) = \Psi(\sigma(x))$ for $x \in \mathcal{U}$ and so

$$\mathrm{Et}(\mathrm{Ps}(\Psi))(\alpha_{\mathscr{S}}(\sigma(x))) = [\Psi(\sigma)]_x = \alpha_{\mathscr{T}}(\Psi(\sigma(x))),$$

giving the commutativity of the left diagram.

For the right diagram, let $s \in \mathscr{F}(\mathcal{U})$ so that $\beta_{\mathscr{F}}(s) \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ is defined by $\beta_{\mathscr{F}}(s)(x) = [s]_x$ for $x \in \mathcal{U}$. Also, $\mathrm{Et}(\Phi)([s]_x) = [\Phi(s)]_x$ and so

$$\mathrm{Ps}(\mathrm{Et}(\Phi))(\beta_{\mathscr{F}}(s))(x) = [\Phi(s)]_x = \beta_{\mathscr{G}}(\Phi(s))(x)$$

as desired. ■

Correspondences between presheaves and étalé spaces of rings

Next we consider the correspondence between presheaves and étalé spaces of rings.

1.1.84 Proposition ($\mathrm{Et}(\mathrm{Ps}(\mathscr{A})) \simeq \mathscr{A}$ (ring version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, then the map $\alpha_{\mathscr{A}}\colon \mathscr{A} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{A}))$ given by $\alpha_{\mathscr{A}}(\sigma(\mathrm{x})) = [\sigma]_{\mathrm{x}}$, where $\sigma\colon \mathcal{U} \to \mathscr{A}$ is a section over $\mathcal{U}$, is an isomorphism of étalé spaces of rings.*

Proof By Proposition 1.1.81 we know that $\alpha_{\mathscr{A}}$ is an isomorphism of sheaves of sets. Let us verify that the ring operations are preserved by $\alpha_{\mathscr{A}}$. The definition of the ring operation on stalks of $\mathrm{Et}(\mathrm{Ps}(\mathscr{A}))$ ensures that

$$\alpha_{\mathscr{A}}(\sigma(x) + \tau(x)) = [\sigma + \tau]_x = [\sigma]_x + [\tau]_x = \alpha_{\mathscr{A}}(\sigma(x)) + \alpha_{\mathscr{A}}(\tau(x))$$

and

$$\alpha_{\mathscr{A}}(\sigma(x) \cdot \tau(x)) = [\sigma \cdot \tau]_x = [\sigma]_x \cdot [\tau]_x = \alpha_{\mathscr{A}}(\sigma(x)) \cdot \alpha_{\mathscr{A}}(\tau(x)),$$

i.e., $\alpha_{\mathscr{A}}$ is a ring homomorphism of stalks. ■

Now let us look at the relationship between a presheaf $\mathscr{R}$ of rings and the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{R}))$.

1.1.85 Proposition ($\mathrm{Ps}(\mathrm{Et}(\mathscr{R})) \simeq \mathscr{R}$ if $\mathscr{R}$ is a sheaf of rings) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, then the map which assigns to $\mathrm{f} \in \mathscr{R}(\mathcal{U})$ the section $\beta_{\mathscr{R},\mathcal{U}}(\mathrm{f}) \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{R}))$ given by $\beta_{\mathscr{R},\mathcal{U}}(\mathrm{f})(\mathrm{x}) = [\mathrm{f}]_{\mathrm{x}}$ is an isomorphism of presheaves of rings.*

Proof By Proposition 1.1.82 we know that $\beta_{\mathscr{R},\mathcal{U}}$ is an isomorphism of presheaves of sets. Here we prove that $\beta_{\mathscr{R},\mathcal{U}}$ is a ring homomorphism. Indeed,

$$\beta_{\mathscr{R},\mathcal{U}}(f + g)(x) = [f + g]_x = [f]_x + [g]_x = \beta_{\mathscr{R},\mathcal{U}}(f)(x) + \beta_{\mathscr{R},\mathcal{U}}(g)(x)$$

and

$$\beta_{\mathscr{R},\mathcal{U}}(f \cdot g)(x) = [f \cdot g]_x = [f]_x \cdot [g]_x = (\beta_{\mathscr{R},\mathcal{U}}(f)(x)) \cdot (\beta_{\mathscr{R},\mathcal{U}}(g)(x)),$$

showing that $\beta_{\mathscr{R},\mathcal{U}}$ is indeed a homomorphism of rings. ■

Now we shall show that the processes above for going from morphisms of presheaves and étalé spaces and back commute in situations where such commutativity is expected.

1.1.86 Proposition (Consistency of morphism constructions (ring version)) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of rings, and let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{S}$. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ be a morphism of the presheaves of rings $\mathscr{R}$ and $\mathscr{S}$ and let $\Psi\colon \mathscr{A} \to \mathscr{B}$ be an étalé morphism of étalé spaces of rings. Then the diagrams*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\alpha_{\mathscr{A}}"] \arrow[d, "\Psi"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{A})) \arrow[d, "\mathrm{Et}(\mathrm{Ps}(\Psi))"] \\ \mathscr{B} \arrow[r, "\alpha_{\mathscr{B}}"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{B})) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{R} \arrow[r, "\beta_{\mathscr{R}}"] \arrow[d, "\Phi"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{R})) \arrow[d, "\mathrm{Ps}(\mathrm{Et}(\Phi))"] \\ \mathscr{S} \arrow[r, "\beta_{\mathscr{S}}"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{S})) \end{tikzcd}$$

commute.

Proof This follows from Proposition 1.1.83. ■

Correspondences between presheaves and étalé spaces of modules

Now let us look at the relationship between a presheaf of modules and étalé spaces of modules.

1.1.87 Proposition ($\mathrm{Et}(\mathrm{Ps}(\mathscr{U})) \simeq \mathscr{U}$ (module version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{A}$ is an étalé space of rings, and if $\mathscr{E}$ is an étalé space of $\mathscr{A}$-modules over $\mathcal{S}$, then the map $\alpha_{\mathscr{U}}\colon \mathscr{U} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{U}))$ given by $\alpha_{\mathscr{U}}(\sigma(x)) = [\sigma]_x$, where $\sigma\colon \mathcal{U} \to \mathscr{U}$ is a section over $\mathcal{U}$, is an isomorphism of étalé spaces of Abelian groups with respect to module addition and for which the diagram*

$$\begin{tikzcd}
\mathscr{A}_x \times \mathscr{U}_x \arrow[r] \arrow[d, "\alpha_{\mathscr{A}}\times\alpha_{\mathscr{U}}"'] & \mathscr{U}_x \arrow[d, "\alpha_{\mathscr{U}}"] \\
\mathrm{Et}(\mathrm{Ps}(\mathscr{A}))_x \times \mathrm{Et}(\mathrm{Ps}(\mathscr{U}))_x \arrow[r] & \mathrm{Et}(\mathrm{Ps}(\mathscr{U}))_x
\end{tikzcd}$$

commutes for each $x \in \mathcal{S}$, where the horizontal arrows are module multiplication.

Proof By Proposition 1.1.81 we know that $\alpha_{\mathscr{U}}$ is an isomorphism of sheaves of sets. The verification that it preserves the Abelian group structure of module addition is straightforward. To verify that the diagram commutes, we calculate

$$\alpha_{\mathscr{U}}(f(x)\cdot\sigma(x)) = [f\cdot\sigma]_x = [f]_x\cdot[\sigma]_x = \alpha_{\mathscr{A}}(f(x))\cdot\alpha_{\mathscr{U}}(\sigma(x))$$

for local sections f and σ of $\mathscr{A}$ and $\mathscr{U}$ defined in some neighbourhood of x. This gives the desired conclusion. ■

1.1.88 Proposition ($\mathrm{Ps}(\mathrm{Et}(\mathscr{E})) \simeq \mathscr{E}$ if $\mathscr{E}$ is a sheaf of modules) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, and if $\mathscr{E}$ is a sheaf of $\mathscr{R}$-modules, then the map which assigns to $s \in \mathscr{E}(\mathcal{U})$ the section $\beta_{\mathscr{E},\mathcal{U}}(s) \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{E}))$ given by $\beta_{\mathscr{E},\mathcal{U}}(s)(x) = [s]_x$ defines an isomorphism of presheaves of Abelian groups with respect to module addition and for which the diagram*

$$\begin{tikzcd}
\mathscr{R}(\mathcal{U}) \times \mathscr{E}(\mathcal{U}) \arrow[r] \arrow[d, "\beta_{\mathscr{R},\mathcal{U}}\times\beta_{\mathscr{E},\mathcal{U}}"'] & \mathscr{E}(\mathcal{U}) \arrow[d, "\beta_{\mathscr{E},\mathcal{U}}"] \\
\mathrm{Ps}(\mathrm{Et}(\mathscr{R}))(\mathcal{U}) \times \mathrm{Ps}(\mathrm{Et}(\mathscr{E}))(\mathcal{U}) \arrow[r] & \mathrm{Ps}(\mathrm{Et}(\mathscr{E}))(\mathcal{U})
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication.

Proof By Proposition 1.1.82 we know that $\beta_{\mathscr{E},\mathcal{U}}$ is an isomorphism of presheaves of sets. It is straightforward to verify that the morphisms preserve the Abelian group structure of module addition. To verify the commuting of the diagram, we compute

$$\beta_{\mathscr{E},\mathcal{U}}(f\cdot s)(x) = [f\cdot s]_x = [f]_x[s]_x = \beta_{\mathscr{R},\mathcal{U}}(f)(x)\cdot\beta_{\mathscr{E},\mathcal{U}}(s)(x),$$

as desired. ■

Now we shall show that the processes above for going from morphisms of presheaves and étalé spaces and back commute in situations where such commutativity is expected.

1.1.89 Proposition (Consistency of morphism constructions (module version)) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a morphism of the presheaves of $\mathscr{R}$-modules $\mathscr{E}$ and $\mathscr{F}$ and let $\Psi\colon \mathscr{U} \to \mathscr{V}$ be an étalé morphism of étalé spaces of $\mathscr{A}$-modules. Then the diagrams*

$$\begin{tikzcd} \mathscr{U} \arrow[r, "\alpha_{\mathscr{U}}"] \arrow[d, "\Psi"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{U})) \arrow[d, "\mathrm{Et}(\mathrm{Ps}(\Psi))"] \\ \mathscr{V} \arrow[r, "\alpha_{\mathscr{V}}"'] & \mathrm{Et}(\mathrm{Ps}(\mathscr{V})) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E} \arrow[r, "\beta_{\mathscr{E}}"] \arrow[d, "\Phi"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{E})) \arrow[d, "\mathrm{Ps}(\mathrm{Et}(\Phi))"] \\ \mathscr{F} \arrow[r, "\beta_{\mathscr{F}}"'] & \mathrm{Ps}(\mathrm{Et}(\mathscr{F})) \end{tikzcd}$$

commute.

Proof This follows from Proposition 1.1.83. ■

1.1.7 Subpresheaves and étalé subspaces

We wish to talk about some standard algebraic constructions in the sheaf setting, and this requires that we know what a subsheaf is.

Subpresheaves and étalé subspaces of sets

We begin with subsheaves of sets.

1.1.90 Definition (Subpresheaf of sets, étalé subspace of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{F}$ and $\mathscr{G}$ be presheaves of sets over $\mathcal{S}$, and let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces of sets over $\mathcal{S}$.

(i) The presheaf $\mathscr{F}$ is a ***subpresheaf*** of $\mathscr{G}$ if, for each $\mathcal{U} \in \mathscr{O}$, $\mathscr{F}(\mathcal{U}) \subseteq \mathscr{G}(\mathcal{U})$ and if the inclusion maps $i_{\mathscr{F},\mathcal{U}}\colon \mathscr{F}(\mathcal{U}) \to \mathscr{G}(\mathcal{U})$, $\mathcal{U} \in \mathscr{O}$, define a morphism $i_{\mathscr{F}} = (i_{\mathscr{F},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves of sets. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we say that $\mathscr{F}$ is a ***subsheaf*** of $\mathscr{G}$.

(ii) The étalé space $\mathscr{S}$ is an ***étalé subspace*** of $\mathscr{T}$ if $\mathscr{S}_x \subseteq \mathscr{T}_x$ and if the inclusion map from $\mathscr{S}$ into $\mathscr{T}$ is a étalé morphism of étalé spaces of sets. •

As with morphisms, we can often freely go between subpresheaves and étalé subspaces. Let us spell this out. Suppose that $\mathscr{F}$ is a subpresheaf of $\mathscr{G}$. The commuting of the diagram (1.2) ensures that the mapping $\mathrm{Et}(i_{\mathscr{F}})\colon [s]_x \mapsto [i_{\mathscr{F},\mathcal{U}}(s)]_x$ from $\mathrm{Et}(\mathscr{F})_x$ to $\mathrm{Et}(\mathscr{G})_x$ is injective, with $\mathcal{U}$ being such that $s \in \mathscr{F}(\mathcal{U})$. As we saw in Proposition 1.1.58, this injection of $\mathrm{Et}(\mathscr{F})$ into $\mathrm{Et}(\mathscr{G})$ is an étalé morphism, and so $\mathrm{Et}(\mathscr{F})$ is a étalé subspace of $\mathrm{Et}(\mathscr{G})$. Conversely, if $\mathscr{S}$ is an étalé subspace of $\mathscr{T}$, then we obviously have $\Gamma(\mathcal{U}; \mathscr{S}) \subseteq \Gamma(\mathcal{U}; \mathscr{T})$. We can see that $(\Gamma(\mathcal{U}; \mathscr{S}))_{\mathcal{U}\in\mathscr{O}}$ is a subpresheaf of $(\Gamma(\mathcal{U}; \mathscr{T}))_{\mathcal{U}\in\mathscr{O}}$ by Proposition 1.1.59.

As for the passing to and from these constructions, Proposition 1.1.83 ensures that, when $\mathscr{F}$ and $\mathscr{G}$ are sheaves, the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ corresponds, under the restriction of $\beta_{\mathscr{G}}$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$, to the subpresheaf $\mathscr{F}$. Conversely, the étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ always corresponds, under the restriction of $\alpha_{\mathscr{S}}$ to $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$, to $\mathscr{S}$.

In order to illustrate that the preceding discussion has some content, let us give an explicit example showing when one has to exercise some care.

1.1.91 Example (Distinct presubsheaves with the same stalks) Let us consider the presheaf $\mathscr{C}^r_{\mathbb{R}}$ of functions of class C^r on $\mathbb{R}$. This is obviously a subpresheaf of itself. Moreover, in Example 1.1.19–1 we considered the subpresheaf $\mathscr{C}^R_{\mathrm{bdd}}(\mathbb{R})$ of bounded functions of class C^r. These étalé subspaces have the same stalks since the condition of boundedness places no restrictions on the germs. However, the presheaves are different. Thus the character of a presubsheaf is only ensured to be characterised by its stalks when the presheaf and the presubsheaf are sheaves. •

The following characterisation of étalé subspaces is sometimes useful.

1.1.92 Proposition (Étalé subspaces of sets are open sets) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{T}$ is an étalé space of sets over $\mathcal{S}$, and if $\mathscr{S} \subseteq \mathscr{T}$ is such that $\mathscr{S}_{\mathrm{x}} \triangleq \mathscr{S} \cap \mathscr{T}_{\mathrm{x}} \neq \emptyset$ for each $\mathrm{x} \in \mathcal{S}$, then the following statements are equivalent:*

(i) $\mathscr{S}$ is an étalé subspace of sets of $\mathscr{T}$;

(ii) $\mathscr{S}$ is an open subset of $\mathscr{T}$.

Proof The implication (i) $\Longrightarrow$ (ii) follows from Proposition 1.1.60. For the converse implication, we need only show that the inclusion of $\mathscr{S}$ into $\mathscr{T}$ is continuous. Let $[s]_x \in \mathscr{S}$ and let $\mathcal{O}$ be a neighbourhood of $[s]_x$ in $\mathscr{T}$. Let $\mathcal{U}$ be a neighbourhood of x such that $\mathcal{B}(\mathcal{U}, s)$ is contained in $\mathcal{O}$. Since $\mathcal{B}(\mathcal{U}, s)$ is a neighbourhood of $[s]_x$ in $\mathscr{S}$ the continuity of the inclusion follows. ■

Subpresheaves and étalé subspaces of rings

Next we turn to subsheaves and subspaces of rings.

1.1.93 Definition (Subpresheaf of rings, étalé subspace of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of rings over $\mathcal{S}$, and let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of sets over $\mathcal{S}$.

(i) The presheaf $\mathscr{R}$ is a ***subpresheaf*** of $\mathscr{S}$ if, for each $\mathcal{U} \in \mathscr{O}$, $\mathscr{R}(\mathcal{U}) \subseteq \mathscr{S}(\mathcal{U})$ and if the inclusion maps $i_{\mathscr{R},\mathcal{U}} \colon \mathscr{R}(\mathcal{U}) \to \mathscr{S}(\mathcal{U})$, $\mathcal{U} \in \mathscr{O}$, define a morphism $i_{\mathscr{R}} = (i_{\mathscr{R},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves of rings. If $\mathscr{R}$ and $\mathscr{S}$ are sheaves, we say that $\mathscr{R}$ is a ***subsheaf*** of $\mathscr{S}$.

(ii) The étalé space $\mathscr{A}$ is an ***étalé subspace*** of $\mathscr{B}$ if $\mathscr{A}_x \subseteq \mathscr{B}_x$ and if the inclusion map from $\mathscr{A}$ into $\mathscr{B}$ is a étalé morphism of étalé spaces of rings. •

As with morphisms, we can often freely go between subpresheaves and étalé subspaces of rings. Let us spell this out. Suppose that $\mathscr{R}$ is a subpresheaf of rings of $\mathscr{S}$. The commuting of the diagram (1.2) ensures that the mapping $\mathrm{Et}(i_{\mathscr{R}}) \colon [f]_x \mapsto [i_{\mathscr{R},\mathcal{U}}(f)]_x$ from $\mathrm{Et}(\mathscr{R})_x$ to $\mathrm{Et}(\mathscr{S})_x$ is injective, with $\mathcal{U}$ being such that $f \in \mathscr{R}(\mathcal{U})$. As we saw in Proposition 1.1.67, this injection of $\mathrm{Et}(\mathscr{R})$ into $\mathrm{Et}(\mathscr{S})$ is an étalé morphism of étalé spaces of rings, and so $\mathrm{Et}(\mathscr{R})$ is a étalé subspace of rings of $\mathrm{Et}(\mathscr{S})$. Conversely, if $\mathscr{A}$ is an étalé subspace of rings of $\mathscr{B}$, then we obviously have $\Gamma(\mathcal{U}; \mathscr{A}) \subseteq \Gamma(\mathcal{U}; \mathscr{B})$. We can see that $(\Gamma(\mathcal{U}; \mathscr{A}))_{\mathcal{U}\in\mathscr{O}}$ is a subpresheaf of rings of $(\Gamma(\mathcal{U}; \mathscr{B}))_{\mathcal{U}\in\mathscr{O}}$ by Proposition 1.1.68.

As for the passing to and from these constructions, Proposition 1.1.86 ensures that, when $\mathscr{R}$ and $\mathscr{S}$ are sheaves of rings, the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{R}))$ corresponds, under the restriction of $\beta_{\mathscr{S}}$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{R}))$, to the subpresheaf $\mathscr{R}$. Conversely, the étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{A}))$ always corresponds, under the restriction of $\alpha_{\mathscr{B}}$ to $\mathrm{Et}(\mathrm{Ps}(\mathscr{A}))$, to $\mathscr{A}$.

The following characterisation of étalé subspaces is sometimes useful.

1.1.94 Proposition (Étalé subspaces of rings are open sets) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{B}$ is an étalé space of rings over $\mathcal{S}$, and if $\mathscr{A} \subseteq \mathscr{B}$ is such that $\mathscr{A}_{\mathrm{x}} \triangleq \mathscr{A} \cap \mathscr{B}_{\mathrm{x}} \neq \emptyset$ for each $\mathrm{x} \in \mathcal{S}$, then the following statements are equivalent:*

(i) $\mathscr{A}$ is an étalé subspace of rings of $\mathscr{B}$;

(ii) $\mathscr{A}$ is an open subset of $\mathscr{B}$ and $\mathscr{A}_{\mathrm{x}}$ is a subring of $\mathscr{B}_{\mathrm{x}}$ for each $\mathrm{x} \in \mathcal{S}$.

Proof This follows from Proposition 1.1.92, with the obvious additional necessary and sufficient condition that $\mathscr{A}_x$ should be a subring of $\mathscr{B}_x$. ■

Subpresheaves and étalé subspaces of modules

Next we turn to subsheaves and subspaces of modules.

1.1.95 Definition (Subpresheaf of modules, étalé subspace of modules) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of $\mathscr{A}$-modules.

(i) The presheaf $\mathscr{E}$ is a ***subpresheaf*** of $\mathscr{F}$ if, for each $\mathcal{U} \in \mathscr{O}$, $\mathscr{E}(\mathcal{U}) \subseteq \mathscr{F}(\mathcal{U})$ and if the inclusion maps $i_{\mathscr{E},\mathcal{U}} \colon \mathscr{E}(\mathcal{U}) \to \mathscr{F}(\mathcal{U})$, $\mathcal{U} \in \mathscr{O}$, define a morphism $i_{\mathscr{E}} = (i_{\mathscr{E},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves of $\mathscr{R}$-modules. If $\mathscr{E}$ and $\mathscr{F}$ are sheaves, we say that $\mathscr{E}$ is a ***subsheaf*** of $\mathscr{F}$.

(ii) The étalé space $\mathscr{U}$ is an ***étalé subspace*** of $\mathscr{V}$ if $\mathscr{U}_x \subseteq \mathscr{V}_x$ and if the inclusion map from $\mathscr{U}$ into $\mathscr{V}$ is a étalé morphism of étalé spaces of $\mathscr{A}$-modules. •

As with morphisms, we can often freely go between subpresheaves and étalé subspaces of modules. Let us spell this out. Suppose that $\mathscr{E}$ is a subpresheaf of $\mathscr{R}$-modules of $\mathscr{F}$. The commuting of the diagram (1.2) ensures that the mapping $\mathrm{Et}(i_{\mathscr{E}}) \colon [s]_x \mapsto [i_{\mathscr{E},\mathcal{U}}(s)]_x$ from $\mathrm{Et}(\mathscr{E})_x$ to $\mathrm{Et}(\mathscr{F})_x$ is injective, with $\mathcal{U}$ being such that $s \in \mathscr{E}(\mathcal{U})$. As we saw in Proposition 1.1.75, this injection of $\mathrm{Et}(\mathscr{E})$ into $\mathrm{Et}(\mathscr{F})$ is an étalé morphism of étalé spaces of rings, and so $\mathrm{Et}(\mathscr{E})$ is a étalé subspace of $\mathrm{Et}(\mathscr{F})$. Conversely, if $\mathscr{U}$ is an étalé subspace of $\mathscr{A}$-modules of $\mathscr{V}$, then we obviously have $\Gamma(\mathcal{U}; \mathscr{U}) \subseteq \Gamma(\mathcal{U}; \mathscr{V})$. We can see that $(\Gamma(\mathcal{U}; \mathscr{U}))_{\mathcal{U}\in\mathscr{O}}$ is a subpresheaf of $\mathrm{Ps}(\mathscr{A})$-modules of $(\Gamma(\mathcal{U}; \mathscr{V}))_{\mathcal{U}\in\mathscr{O}}$ by Proposition 1.1.76.

As for the passing to and from these constructions, Proposition 1.1.89 ensures that, when $\mathscr{E}$ and $\mathscr{F}$ are sheaves of $\mathscr{R}$-modules, the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{E}))$ corresponds, under the restriction of $\beta_{\mathscr{F}}$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{E}))$, to the subpresheaf $\mathscr{E}$. Conversely, the étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{U}))$ always corresponds, under the restriction of $\alpha_{\mathscr{V}}$ to $\mathrm{Et}(\mathrm{Ps}(\mathscr{U}))$, to $\mathscr{U}$.

The following characterisation of étalé subspaces is sometimes useful.

1.1.96 Proposition (Étalé subspaces of modules are open sets) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, if $\mathscr{V}$ is an étalé space of $\mathscr{A}$-modules over $\mathcal{S}$, and if $\mathscr{U} \subseteq \mathscr{V}$ is such that $\mathscr{U}_{\mathrm{x}} \triangleq \mathscr{U} \cap \mathscr{V}_{\mathrm{x}} \neq \emptyset$ for each $\mathrm{x} \in \mathcal{S}$, then the following statements are equivalent:*

(i) $\mathscr{U}$ is an étalé subspace of $\mathscr{A}$-modules of $\mathscr{V}$;

(ii) $\mathscr{U}$ is an open subset of $\mathscr{V}$ and $\mathscr{U}_{\mathrm{x}}$ is an $\mathscr{A}_{\mathrm{x}}$-submodule of $\mathscr{V}_{\mathrm{x}}$ for each $\mathrm{x} \in \mathcal{S}$.

Proof This follows from Proposition 1.1.92, with the obvious additional necessary and sufficient condition that $\mathscr{U}_x$ should be a submodule of $\mathscr{V}_x$. ■

1.1.8 The sheafification of a presheaf

While it is true that many of the presheaves we will encounter are sheaves, cf. Proposition 1.1.18, it is also the case that some presheaves are not sheaves, and we saw some natural and not so natural example of this in Examples Example 1.1.14 and Example 1.1.19. As we saw in those examples, a presheaf may fail to be a sheaf for two reasons: (1) the local behaviour of restrictions of sections does not accurately represent the local behaviour of sections (failure of the presheaf to be separated); (2) there are characteristics of global sections that are not represented by local characteristics (failure of the presheaf to satisfy the gluing conditions). The process of sheafification seeks to repair these defects by shrinking or enlarging the sets of sections as required by the sheaf axioms.

The sheafification of a presheaf of sets

The construction is as follows for presheaves of sets.

1.1.97 Definition (Sheafification of presheaves of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$. The ***sheafification*** of $\mathscr{F}$ is the presheaf $\mathscr{F}^+$ such that an element of $\mathscr{F}^+(\mathcal{U})$ is comprised of the (not necessarily continuous) maps $\sigma\colon \mathcal{U} \to \mathrm{Et}(\mathscr{F})$ such that

(i) $\pi_{\mathscr{F}} \circ \sigma = \mathrm{id}_{\mathcal{U}}$,

(ii) for each $x \in \mathcal{U}$ there is a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and $s \in \mathscr{F}(\mathcal{V})$ such that $\sigma(y) = r_{\mathcal{V},y}(s)$ for every $y \in \mathcal{V}$, and

(iii) if $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, then the restriction map $r^+_{\mathcal{U},\mathcal{V}}$ is defined by

$$r^+_{\mathcal{U},\mathcal{V}}(\sigma)(x) = \sigma(x)$$

for each $x \in \mathcal{V}$. •

As one hopes, the sheafification of a presheaf is a sheaf. This is true, as we record in the following result, along with some other properties of sheafification.

1.1.98 Proposition (Properties of the sheafification of a presheaf of sets) *If $(\mathcal{S}, \mathscr{O})$ is topological space and if $\mathscr{F}$ is a presheaf of sets over $\mathcal{S}$, then*

(i) $\mathscr{F}^+ = \mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$,

(ii) the sheafification $\mathscr{F}^+$ is a sheaf, and

(iii) if $\mathrm{x} \in \mathcal{S}$, the map $\iota_{\mathrm{x}} \colon \mathscr{F}_{\mathrm{x}} \to \mathscr{F}^+_{\mathrm{x}}$ defined by $\iota_{\mathrm{x}}([\mathrm{s}]_{\mathrm{x}}) = [\sigma_{\mathrm{s}}]_{\mathrm{x}}$ where $\sigma_{\mathrm{s}}(\mathrm{y}) = [\mathrm{s}]_{\mathrm{y}}$ for y in some neighbourhood of x, is a bijection.

Proof (i) It is clear that we have an inclusion from $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ into $\mathscr{F}^+$, just by definition of $\mathscr{F}^+$. We shall show that this inclusion is a surjective mapping of presheaves. For surjectivity of the natural inclusion, let $\mathcal{U} \in \mathscr{O}$ and let $\tau \in \mathscr{F}^+(\mathcal{U})$. For $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}$ of x and $s_x \in \mathscr{F}(\mathcal{U}_x)$ such that $\tau(y) = [s_x]_y$ for each $y \in \mathcal{U}_x$. Define $\sigma_x \in \Gamma(\mathcal{U}_x; \mathrm{Et}(\mathscr{F}))$ by $\sigma_x(y) = [s_x]_y$. Thus we have an open cover $(\mathcal{U}_x)_{x\in\mathcal{U}}$ of $\mathcal{U}$ and a corresponding family $(\sigma_x)_{x\in\mathcal{U}}$ of sections of $\mathrm{Et}(\mathscr{F})$. Since $\mathrm{Et}(\mathscr{F})$ is separated, it follows that

$$r_{\mathcal{U}_{x_1},\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}}(\sigma_{x_1}) = r_{\mathcal{U}_{x_2},\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}}(\sigma_{x_2}),$$

cf. the proof of surjectivity for Proposition 1.1.82. Now we use the gluing property of $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ to assert the existence of $\sigma \in \mathscr{F}(\mathcal{U})$ such that $r_{\mathcal{U},\mathcal{U}_x}(\sigma) = \sigma_x$ for every $x \in \mathcal{U}$. We clearly have $\sigma(x) = \tau(x)$ for every $x \in \mathcal{U}$, giving surjectivity.

(ii) This follows from the previous part of the result along with Proposition 1.1.45.

(iii) To prove injectivity of the map, suppose that $\iota_x([s]_x) = \iota_x([t]_x)$. Then there exists a neighbourhood $\mathcal{U}$ of x such that s and t restrict to $\mathcal{U}$ and agree on $\mathcal{U}$. Thus $\sigma_s = \sigma_t$ on $\mathcal{U}$. For surjectivity, let $[\sigma]_x \in \mathscr{F}^+_x$. Then there exists a neighbourhood $\mathcal{V}$ of x such that σ is defined on $\mathcal{V}$ and a section $s \in \mathscr{F}(\mathcal{V})$ such that $\sigma(x) = [s]_x$. Thus $\iota_x([s]_x) = \sigma_s(x) = \sigma(x)$, giving surjectivity. ■

The sheafification has an important "universality" property.

1.1.99 Proposition (Universality of the sheafification (set version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F}$ is a presheaf of sets over $\mathcal{S}$, then there exists a morphism of presheaves $(\iota_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{F}^+$ such that, if $\mathscr{G}$ is a sheaf of sets over $\mathcal{S}$ and if $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves of sets from $\mathscr{F}$ to $\mathscr{G}$, then there exists a unique morphism of presheaves of sets $(\Phi^+_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}^+$ to $\mathscr{G}$ satisfying $\Phi_{\mathcal{U}} = \Phi^+_{\mathcal{U}} \circ \iota_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$.*

Moreover, if $\hat{\mathscr{F}}$ is a sheaf of sets and if $(\hat{\iota}_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves of sets from $\mathscr{F}$ to $\hat{\mathscr{F}}$ having the above property, then there exists a unique isomorphism of presheaves of sets from $\hat{\mathscr{F}}$ to $\mathscr{F}^+$.

Proof Let us define $\iota_{\mathcal{U}} \colon \mathscr{F}(\mathcal{U}) \to \mathscr{F}^+(\mathcal{U})$ by $\iota_{\mathcal{U}}(s)(x) = [s]_x$. Now, given a morphism $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves from $\mathscr{F}$ to $\mathscr{G}$, define a morphism $(\Phi'^+_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves from $\mathscr{F}^+$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{G}))$ by

$$\Phi'^+_{\mathcal{U}}([s]_x) = [\Phi_{\mathcal{U}}(s)]_x.$$

We should show that this definition is independent of s. That is to say, we should show that if $[s]_x = [t]_x$ for every $x \in \mathcal{U}$ then $\Phi_{\mathcal{U}}(s) = \Phi_{\mathcal{U}}(t)$. Since $[s]_x = [t]_x$ for every $x \in \mathcal{U}$, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x$ such that $r_{\mathcal{U},\mathcal{U}_x}(s) = r_{\mathcal{U},\mathcal{U}_x}(t)$. Since $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves, we have

$$r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s)) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(t)).$$

Since $\mathscr{G}$ is separable, we infer that $\Phi_{\mathcal{U}}(t) = \Phi_{\mathcal{U}}(s)$, as desired.

Recall from Example 1.1.61–1 the mapping $\beta_{\mathscr{G},\mathcal{U}}$ from $\mathscr{G}(\mathcal{U})$ to $\Gamma(\mathcal{U};\mathrm{Et}(\mathscr{G}))$ and that the family of mappings $(\beta_{\mathscr{G},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ defines a presheaf isomorphism by virtue of $\mathscr{G}$ being a sheaf. Sorting through the definitions gives $\Phi_{\mathcal{U}}(s) = \beta^{-1}_{\mathscr{G},\mathcal{U}} \circ \Phi'^{+}_{\mathcal{U}} \circ \iota_{\mathcal{U}}$, which gives the existence part of the first assertion by taking $\Phi^{+}_{\mathcal{U}} = \beta^{-1}_{\mathscr{G},\mathcal{U}} \circ \Phi'^{+}_{\mathcal{U}}$. For the uniqueness part of the assertion, note that the requirement that $\Phi_{\mathcal{U}}(s) = \Phi^{+}_{\mathcal{U}} \circ \iota_{\mathcal{U}}(s)$ implies that

$$\Phi^{+}_{\mathcal{U}}([s]_x) = \Phi_{\mathcal{U}}(s)(x) = \beta^{-1}_{\mathscr{G},\mathcal{U}} \circ \Phi'^{+}_{\mathcal{U}}([s]_x),$$

as desired.

Now we turn to the second assertion. Thus $\hat{\mathscr{F}}$ is a sheaf and for each $\mathcal{U} \in \mathscr{O}$ we have a mapping $\hat{\iota}_{\mathcal{U}} \colon \mathscr{F}(\mathcal{U}) \to \hat{\mathscr{F}}(\mathcal{U})$ such that, for any presheaf morphism $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{G}$, there exists a unique presheaf morphism $(\hat{\Phi}_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\hat{\mathscr{F}}$ to $\mathscr{G}$ such that $\Phi_{\mathcal{U}} = \hat{\Phi}_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. Applying this hypothesis to the presheaf morphism $(\iota_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{F}^{+}$ gives a unique presheaf morphism $(\kappa_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\hat{\mathscr{F}}$ to $\mathscr{F}^{+}$ such that $\iota_{\mathcal{U}} = \kappa_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. We claim that, for every $\mathcal{U} \in \mathscr{O}$, $\kappa_{\mathcal{U}}$ is a bijection from $\hat{\mathscr{F}}(\mathcal{U})$ to $\mathscr{F}^{+}(\mathcal{U})$. Fix $\mathcal{U} \in \mathscr{O}$. In the same manner as we deduced the existence of $\kappa_{\mathcal{U}}$, we have a mapping $\hat{\kappa}_{\mathcal{U}} \colon \mathscr{F}^{+}(\mathcal{U}) \to \hat{\mathscr{F}}(\mathcal{U})$ such that $\hat{\iota}_{\mathcal{U}} = \hat{\kappa}_{\mathcal{U}} \circ \iota_{\mathcal{U}}$. Thus $\hat{\iota}_{\mathcal{U}} = \hat{\kappa}_{\mathcal{U}} \circ \kappa_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$. However, we also have $\hat{\iota}_{\mathcal{U}} = \mathrm{id}_{\hat{\mathscr{F}}(\mathcal{U})} \circ \hat{\iota}_{\mathcal{U}}$ and so, by the uniqueness part of the first part of the proposition, we have $\hat{\kappa}_{\mathcal{U}} \circ \kappa_{\mathcal{U}} = \mathrm{id}_{\hat{\mathscr{F}}(\mathcal{U})}$. In like manner, $\kappa_{\mathcal{U}} \circ \hat{\kappa}_{\mathcal{U}} = \mathrm{id}_{\hat{\mathscr{F}}(\mathcal{U})}$, giving that $\hat{\kappa}_{\mathcal{U}}$ is the inverse of $\kappa_{\mathcal{U}}$. ■

To better get a handle on the sheafification of a presheaf, let us consider the sheafification of the presheaves from Examples 1.1.14.

1.1.100 Examples (Sheafification of presheaves of sets)

1. Let us determine the sheafification $\mathscr{F}^{+}_{X}$ of the constant presheaf $\mathscr{F}_X$ over a topological space $(\mathcal{S}, \mathscr{O})$ associated with a set X. As in Example 1.1.27 we have $\mathscr{F}^{+}_{X} \simeq \mathcal{S} \times X$ and so, first of all, sections of $\mathscr{F}^{+}_{X}$ over $\mathcal{U} \in \mathscr{O}$ are identified with maps from $\mathcal{U}$ to X. Let $\sigma \colon \mathcal{U} \to X$ be a section of $\mathscr{F}^{+}_{X}$ under this identification and let $x \in \mathcal{U}$. By definition of $\mathscr{F}^{+}_{X}$ there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and $s \in \mathscr{F}_X(\mathcal{V})$ such that $\sigma(y) = s$ for every $y \in \mathcal{V}$. Thus σ is locally constant. Since any section of $\mathrm{Et}(\mathscr{F}_X)$ is, by our construction of the étalé topology on $\mathrm{Et}(\mathscr{F}_X)$ in Example 1.1.27 and by our definition of the constant étalé space $\mathscr{S}_X$ in Example 1.1.41, locally constant, the sheafification of $\mathscr{F}_X$ is exactly $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}_X))$, as per Proposition 1.1.102(i).
   Note that, if $\mathcal{U} \in \mathscr{O}$ is connected, then sections of $\mathscr{F}^{+}_{X}$ are not just locally constant, but constant. Thus we can identify $\mathscr{F}^{+}_{X}(\mathcal{U})$ with X in an obvious way, i.e., the constant local section $x \mapsto (x, s)$ is identified with $s \in X$. However, if $\mathcal{U}$ is not connected—say $\mathcal{U}$ has connected components $(\mathcal{U}_a)_{a\in A}$, then $\mathscr{F}^{+}_{X}(\mathcal{U})$ cannot be identified naturally with X since a section over $\mathcal{U}$ will generally take different values, depending on the connected $\mathcal{U}_a$, i.e., a section will take the value $s_a \in X$ on $\mathcal{U}_a$. This sheaf is called the ***constant sheaf***.
2. Here we consider the case of Example 1.1.14–1 where $\mathcal{S} = \{0, 1\}$. Here, because of the discrete topology on $\mathcal{S}$ and because of the character of the restriction maps for the presheaf $\mathscr{F}$ under consideration, we have $\mathscr{F}_0 = [0_{\{0\}}]_0$ and $\mathscr{F}_1 = [0_{\{1\}}]_1$. Thus the

sheafification $\mathscr{F}^+$ has zero stalks. In this case the presheaf has to shrink to obtain the sheafification, in order to account for the fact that the germs are trivial. •

Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$. Suppose that we have a presheaf $\mathscr{E}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules, where M is a smooth, real analytic manifold or holomorphic manifold, as required. In these cases, the sheafification of $\mathscr{E}$ is also a presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules by virtue of Proposition 1.1.106(iii).

The sheafification of a presheaf of rings

Now we turn to the sheafification of a presheaf of rings.

1.1.101 Definition (Sheafification of presheaves of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$. The ***sheafification*** of $\mathscr{R}$ is the sheafification $\mathscr{R}^+$ of $\mathscr{R}$ as a presheaf of sets, with the additional property that we define ring operations on $\mathscr{R}^+(\mathcal{U})$ by

$$[f]_x + [g]_x = [f + g]_x, \quad [f]_x \cdot [g]_x = [f \cdot g]_x,$$

where $[f]_x, [g]_x \in \mathscr{F}(\mathcal{V})$ for some sufficiently small neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and for $x \in \mathcal{U}$. •

The basic properties of sheafification of presheaves of sets apply also to presheaves of rings.

1.1.102 Proposition (Properties of the sheafification of a presheaf of rings) *If $(\mathcal{S}, \mathscr{O})$ is topological space and if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, then*

(i) $\mathscr{R}^+ = \mathrm{Ps}(\mathrm{Et}(\mathscr{R}))$,

(ii) the sheafification $\mathscr{R}^+$ is a sheaf, and

(iii) if $\mathsf{x} \in \mathcal{S}$, the map $\iota_{\mathsf{x}} \colon \mathscr{R}_{\mathsf{x}} \to \mathscr{R}^+_{\mathsf{x}}$ defined by $\iota_{\mathsf{x}}([\mathsf{f}]_{\mathsf{x}}) = [\sigma_{\mathsf{f}}]_{\mathsf{x}}$, where $\sigma_{\mathsf{f}}(\mathsf{y}) = [\mathsf{f}]_{\mathsf{y}}$ for y in some neighbourhood of x, is an isomorphism of rings.

Proof The only part of the result that does not follow from Proposition 1.1.98 is the verification that the map ι_x in part (iii) is a ring isomorphism. This, however, follows from the definition of the ring operations for the sheafification. ■

The "universality" property of sheafification also applies to sheaves of rings.

1.1.103 Proposition (Universality of the sheafification (ring version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, then there exists a morphism of presheaves of rings $(\iota_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ from $\mathscr{R}$ to $\mathscr{R}^+$ such that, if $\mathscr{S}$ is a sheaf of rings over $\mathcal{S}$ and if $(\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is a morphism of presheaves of rings from $\mathscr{R}$ to $\mathscr{S}$, then there exists a unique morphism of presheaves of rings $(\Phi^+_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ from $\mathscr{R}^+$ to $\mathscr{S}$ satisfying $\Phi_{\mathcal{U}} = \Phi^+_{\mathcal{U}} \circ \iota_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$.*

Moreover, if $\hat{\mathscr{R}}$ is a sheaf of rings and if $(\hat{\iota}_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is a morphism of presheaves of rings from $\mathscr{R}$ to $\hat{\mathscr{R}}$ having the above property, then there exists a unique isomorphism of presheaves of rings from $\hat{\mathscr{R}}$ to $\mathscr{R}^+$.

Proof The only aspect of the result that does not follow from Proposition 1.1.99 is the verification that all morphisms preserve the ring structure. We shall simply make reference to the appropriate morphisms from the proof of Proposition 1.1.99, leaving to the reader the straightforward filling in of the small gaps.

First of all, the mapping $\iota_{\mathcal{U}}\colon \mathscr{R}(\mathcal{U}) \to \mathscr{R}^+(\mathcal{U})$ is easily seen to be a homomorphism of rings, simply by the definition of the ring operations. Similarly, the mapping $\Phi'^{+}_{\mathcal{U}}\colon \mathscr{R}^+(\mathcal{U}) \to \Gamma(\mathcal{U};\mathrm{Et}(\mathscr{S}))$ is easily verified to be a ring homomorphism. From Proposition 1.1.85 we know that $\beta_{\mathscr{S},\mathcal{U}}$ is a ring homomorphism, and from this we deduce that $\Phi^{+}_{\mathcal{U}}$ is a ring homomorphism. For the second assertion of the proposition, referring to the proof of Proposition 1.1.99, we see that we need to show that $\kappa_{\mathcal{U}}\colon \hat{\mathscr{R}}(\mathcal{U}) \to \mathscr{R}^+(\mathcal{U})$ and $\hat{\kappa}_{\mathcal{U}}\colon \mathscr{R}^+(\mathcal{U}) \to \hat{\mathscr{R}}(\mathcal{U})$ are ring homomorphisms. This, however, follows from the conclusions of the first part of the theorem, as we can see by how these mappings are constructed in the proof of Proposition 1.1.99. ■

To better get a handle on the sheafification of a presheaf, let us consider the sheafification of the presheaves from Examples 1.1.19.

1.1.104 Examples (Sheafification of presheaves of rings)

1. We revisit Example 1.1.19–1 where we consider the presheaf $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ of functions of class C^r on $\mathsf{M} = \mathbb{R}$ that were bounded on their domains. Here we claim that the sheafification of $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ is simply $\mathrm{Ps}(\mathrm{Et}(\mathscr{C}^r_{\mathbb{R}}))$. By Proposition 1.1.102(i) we have $\mathrm{Ps}(\mathrm{Et}(\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R}))) = (\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R}))^+$. It is also clear that $\mathrm{Et}(\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})) = \mathrm{Et}(\mathscr{C}^r_{\mathbb{R}})$ since the restriction of a function being bounded does not restrict stalks, and so we have our desired conclusion.
2. Another interesting example of sheafification comes from the presheaf of integrable functions considered in Example 1.1.7–6. We let $\mathcal{W} \subseteq \mathbb{R}^n$ be open and take the presheaf $\mathscr{L}^1_{\mathcal{W}} = (\mathrm{L}^1(\mathcal{U};\mathbb{R}))_{\mathcal{U}\subseteq\mathcal{W}\text{ open}}$ of integrable functions. In Example 1.1.19–2 we showed that $\mathscr{L}^1_{\mathcal{W}}$ is not a sheaf. One can readily see that its sheafification is the sheaf $\mathscr{L}^1_{\mathrm{loc},\mathcal{W}} = (\mathrm{L}^1_{\mathrm{loc}}(\mathcal{U};\mathbb{R}))_{\mathcal{U}\subseteq\mathcal{W}\text{ open}}$ which assigns to an open set $\mathcal{U}$ the set of locally integrable functions on $\mathcal{U}$. Indeed, the definition of sheafification shows us that a section of the sheafification over $\mathcal{U}$ has the property that it is integrable in some neighbourhood of any point in $\mathcal{U}$. A simple argument then shows that the restriction of such a section to any compact set is integrable, i.e., that the section is locally integrable. •

The sheafification of a presheaf of modules

Now we turn to the sheafification of a presheaf of modules.

1.1.105 Definition (Sheafification of presheaves of modules) Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules. The ***sheafification*** of $\mathscr{E}$ is the sheafification $\mathscr{E}^+$ of $\mathscr{E}$ as a presheaf of sets, with the additional property that we make $\mathscr{E}^+$ into a sheaf of $\mathscr{R}^+$-modules by defining $\mathscr{R}^+(\mathcal{U})$-module operations on $\mathscr{E}^+(\mathcal{U})$ by

$$[s]_x + [t]_x = [s+t]_x, \quad [f]_x \cdot [s]_x = [f\cdot s]_x,$$

where $[s]_x, [t]_x \in \mathscr{E}(\mathcal{V})$ and $[f]_x \in \mathscr{R}(\mathcal{V})$ for some sufficiently small neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and for $x \in \mathcal{U}$. •

Let us record some basic properties of sheafification of modules.

1.1.106 Proposition (Properties of the sheafification of a presheaf of modules) *If $(\mathcal{S}, \mathscr{O})$ is topological space, if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, and if $\mathscr{E}$ is a presheaf of $\mathscr{R}$-modules over $\mathcal{S}$, then*

(i) $\mathscr{E}^+ = \mathrm{Ps}(\mathrm{Et}(\mathscr{E}))$,

(ii) *the sheafification $\mathscr{E}^+$ is a sheaf, and*

(iii) *if $x \in \mathcal{S}$, the maps $\iota_x^{\mathscr{R}}\colon \mathscr{R}_x \to \mathscr{R}_x^+$ and $\iota_x^{\mathscr{E}}\colon \mathscr{E}_x \to \mathscr{E}_x^+$ defined by $\iota_x^{\mathscr{R}}([f]_x) = [\sigma_f]_x$ and $\iota_x^{\mathscr{E}}([s]_x) = [\sigma_s]_x$, respectively, where $\sigma_f(y) = [f]_y$ and $\sigma_s(y) = [s]_y$ for y in some neighbourhood of x, is a morphism of Abelian groups with respect to module addition and has the property that the diagram*

$$\begin{tikzcd}
\mathscr{R}_x \times \mathscr{E}_x \arrow[r] \arrow[d, "\iota_x^{\mathscr{R}} \times \iota_x^{\mathscr{E}}"'] & \mathscr{E}_x \arrow[d, "\iota_x^{\mathscr{E}}"] \\
\mathscr{R}_x^+ \times \mathscr{E}_x^+ \arrow[r] & \mathscr{E}_x^+
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication.

Proof Except for the final assertion, the result follows from Proposition 1.1.106. The final assertion follows from the computations in the proof of Proposition 1.1.88, along with the defining properties of restriction maps for presheaves of modules. ■

The universality property of sheafification also applies to modules.

1.1.107 Proposition (Universality of the sheafification (module version)) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, and if $\mathscr{E}$ is a presheaf of $\mathscr{R}$-modules, then there exists a morphism of presheaves of $\mathscr{R}$-modules $(\iota_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{E}$ to $\mathscr{E}^+$ such that, if $\mathscr{F}$ is a sheaf of $\mathscr{R}$-modules over $\mathcal{S}$ and if $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves of $\mathscr{R}$-modules from $\mathscr{E}$ to $\mathscr{F}$, then there exists a unique morphism of presheaves of $\mathscr{R}$-modules $(\Phi_{\mathcal{U}}^+)_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{E}^+$ to $\mathscr{F}$ satisfying $\Phi_{\mathcal{U}} = \Phi_{\mathcal{U}}^+ \circ \iota_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$.*

Moreover, if $\hat{\mathscr{E}}$ is a sheaf of $\mathscr{R}$-modules and if $(\hat{\iota}_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves of $\mathscr{R}$-modules from $\mathscr{E}$ to $\hat{\mathscr{E}}$ having the above property, then there exists a unique isomorphism of presheaves of $\mathscr{R}$-modules from $\hat{\mathscr{E}}$ to $\mathscr{E}^+$.

Proof The only aspect of the result that does not follow from Proposition 1.1.99 is the verification that all morphisms preserve the $\mathscr{R}$-module structure. We shall simply make reference to the appropriate morphisms from the proof of Proposition 1.1.99, leaving to the reader the straightforward filling in of the small gaps.

First of all, the mapping $\iota_{\mathcal{U}}\colon \mathscr{E}(\mathcal{U}) \to \mathscr{E}^+(\mathcal{U})$ is easily seen to be a homomorphism of $\mathscr{R}(\mathcal{U})$-modules, simply by the definition of the ring operations. Similarly, the mapping $\Phi_{\mathcal{U}}'^{+}\colon \mathscr{E}^+(\mathcal{U}) \to \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ is easily verified to be a homomorphism of $\mathscr{R}(\mathcal{U})$-modules. From Proposition 1.1.88 we know that $\beta_{\mathscr{F},\mathcal{U}}$ is a homomorphism of $\mathscr{R}(\mathcal{U})$-modules (keeping in mind that $\mathscr{R}$ is a sheaf here), and from this we deduce that $\Phi_{\mathcal{U}}^+$ is a homomorphism of

$\mathscr{R}(\mathcal{U})$-modules. For the second assertion of the proposition, referring to the proof of Proposition 1.1.99, we see that we need to show that $\kappa_{\mathcal{U}}\colon \hat{\mathscr{E}}(\mathcal{U}) \to \mathscr{E}^+(\mathcal{U})$ and $\hat{\kappa}_{\mathcal{U}}\colon \mathscr{E}^+(\mathcal{U}) \to \hat{\mathscr{E}}(\mathcal{U})$ are homomorphisms of $\mathscr{R}(\mathcal{U})$-modules. This, however, follows from the conclusions of the first part of the theorem, as we can see by how these mappings are constructed in the proof of Proposition 1.1.99. ■

1.2 Direct and inverse images of sheaves

We have thus far only considered morphisms of sheaves defined over the same space. In this section we consider two ways in which a sheaf can be transferred from one topological space to another by use of a continuous map. The operations are quite involved and interconnected in intricate ways.

1.2.1 Direct and inverse images of presheaves

We begin by making our constructions with presheaves. As with our presentation in Section 1.1, it will be advantageous at times to break the presentation into constructions for sheaves of sets, rings, and modules.

Direct and inverse images of presheaves of sets

In order to make one of the constructions, we will generalise to arbitrary sets the notion of a germ. Thus we let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf over $\mathcal{S}$. Let $A \subseteq \mathcal{S}$. Let $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ be neighbourhoods of A. Sections $s \in \mathscr{F}(\mathcal{U})$ and $t \in \mathscr{F}(\mathcal{V})$ are ***equivalent*** if there exists a neighbourhood $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ of A such that $r_{\mathcal{U},\mathcal{W}}(s) = r_{\mathcal{V},\mathcal{W}}(t)$. Let $\mathscr{F}_A$ denote the set of equivalence classes under this equivalence relation. Let us denote an equivalence class by $[(s, \mathcal{U})]_A$ or by $[s]_A$ if the subset $\mathcal{U}$ is of no consequence. Restriction maps can be defined between such sets of equivalence classes as well. Thus we let $A, B \subseteq \mathcal{S}$ be subsets for which $A \subseteq B$. If $[(s, \mathcal{U})]_B \in \mathscr{F}_B$ then, since $\mathcal{U}$ is also a neighbourhood of A, $[(s, \mathcal{U})]_B \in \mathscr{F}_A$, and we denote by $r_{B,A}([(s, \mathcal{U})]_B)$ the equivalence class in $\mathscr{F}_A$. One can readily verify that these restriction maps are well-defined.

1.2.1 Definition (Direct image and inverse image presheaves) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, and let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$ and $\mathscr{G}$ be a presheaf of sets over $\mathcal{T}$.

(i) The ***direct image presheaf*** of $\mathscr{F}$ by Φ is the presheaf $\Phi_{\mathrm{pre},*}\mathscr{F}$ on $\mathcal{T}$ given by $\Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V}) = \mathscr{F}(\Phi^{-1}(\mathcal{V}))$ for $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$. If $r_{\mathcal{U},\mathcal{V}}$ denote the restriction maps for $\mathscr{F}$, the restriction maps $\Phi_{\mathrm{pre},*}r_{\mathcal{U},\mathcal{V}}$ for $\Phi_{\mathrm{pre},*}\mathscr{F}$ satisfy, for $\mathcal{U}, \mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ with $\mathcal{V} \subseteq \mathcal{U}$,

$$\Phi_{\mathrm{pre},*}r_{\mathcal{U},\mathcal{V}}(s) = r_{\Phi^{-1}(\mathcal{U}),\Phi^{-1}(\mathcal{V})}(s)$$

for $s \in \Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{U}) = \mathscr{F}(\Phi^{-1}(\mathcal{U}))$.

(ii) The ***inverse image presheaf*** of $\mathscr{F}$ by Φ is the presheaf $\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$ over $\mathcal{S}$ defined by $(\Phi^{-1}_{\mathrm{pre}}\mathscr{F})(\mathcal{U}) = \mathscr{F}_{\Phi(\mathcal{U})}$. The restriction maps for $\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$ are defined by $\Phi^{-1}_{\mathrm{pre}}r_{\mathcal{U},\mathcal{V}}([s]) = r_{\Phi(\mathcal{U}),\Phi(\mathcal{V})}([s])$. •

Before we get to the specific properties of direct and inverse images, let us give some examples to give some context to the discussion. Let us begin with examples of the direct image.

1.2.2 Examples (Direct image)

1. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathcal{T} = \{\mathrm{pt}\}$ be the topological space with one point. If $\Phi \in \mathrm{C}^0(\mathcal{S}; \{\mathrm{pt}\})$ then clearly Φ is the constant mapping defined by $\Phi(x) = \mathrm{pt}$. Thus, if $\mathscr{F}$ is a presheaf over $\mathcal{S}$, then there is only one possible direct image in this case, and it is given by $\Phi_{\mathrm{pre},*}\mathscr{F}(\{\mathrm{pt}\}) = F(\mathcal{S})$, i.e., the global sections of $\mathscr{F}$.
2. Let $\{\mathrm{pt}\}$ be a one point set, and suppose that $(\mathcal{T}, \mathscr{O})$ is a topological space for which points are closed sets. Then any map $\Phi\colon \{\mathrm{pt}\} \to \mathcal{T}$ is continuous. In Example 1.1.4–1 we described the presheaves (which are indeed sheaves) over $\{\mathrm{pt}\}$. Letting $\mathscr{F}$ be such a sheaf, we can easily see that $\Phi_{\mathrm{pre},*}\mathscr{F}$ is a skyscraper sheaf at $\Phi(\mathrm{pt})$, cf. Example 1.1.4–2.
3. We let $\mathcal{S} = \mathcal{T} = \mathbb{S}^1$ and define $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ by $\Phi(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{2\mathrm{i}\theta}$. Thus Φ is to be thought of as the projection from the double cover of $\mathbb{S}^1$ to $\mathbb{S}^1$. Suppose that $\mathscr{F}_X$ is a constant presheaf over $\mathcal{S}$ defined by $\mathscr{F}_X(\mathcal{U}) = X$ for some set X. If $\mathcal{V} \subseteq \mathcal{T}$ then $\Phi_{\mathrm{pre},*}\mathscr{F}_X(\mathcal{V}) = \mathscr{F}_X(\Phi^{-1}(\mathcal{V})) = X$. Thus $\Phi_{\mathrm{pre},*}\mathscr{F}_X$ is a constant presheaf over $\mathcal{T}$.
4. Consider now the previous example, but consider the sheaf $\mathscr{G}_X = \mathrm{Ps}(\mathrm{Et}(\mathscr{F}_X))$, i.e., the sheafification of $\mathscr{F}_X$. Suppose that $\mathcal{V} \subseteq \mathcal{T}$ is connected and is such that $\mathcal{U} = \Phi^{-1}(\mathcal{V})$ consists of two disjoint open sets, e.g., $\mathcal{V}$ is a small connected neighbourhood of some point in $\mathcal{T}$. Then $\Phi_{\mathrm{pre},*}\mathscr{G}_X(\mathcal{V}) = \mathscr{G}_X(\mathcal{U})$ consists of maps $s\colon \mathcal{V} \to X \times X$ of the form $s(\mathrm{e}^{\mathrm{i}\theta}) = (x_1, x_2)$ for some $x_1, x_2 \in X$, i.e., constant maps from $\mathcal{V}$ to $X \times X$. We can thus write $\Phi_{\mathrm{pre},*}\mathscr{G}_X(\mathcal{V}) = X \times X$ for such open sets $\mathcal{V}$. We conclude that the stalks of $\Phi_{\mathrm{pre},*}\mathscr{G}_X$ are $X \times X$. Note, however, that $\Phi_{\mathrm{pre},*}\mathscr{G}_X(\mathcal{T}) = X$, so $\Phi_{\mathrm{pre},*}\mathscr{G}_X$ is not a constant presheaf.
5. Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ otherwise. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a $\mathbb{F}$-vector bundle of class C^r and consider the presheaf, indeed sheaf, $\mathscr{G}^r_{\mathsf{E}}$ of sections of E. Let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$ be a map into another C^r-manifold N. Then we have the direct image presheaf $\Phi_{\mathrm{pre},*}\mathscr{G}^r_{\mathsf{E}}$ over N. Let us think informally about whether this might be a presheaf of sections of a vector bundle. Let $\mathcal{V} \subseteq \mathsf{N}$ be open. Then $\Phi_{\mathrm{pre},*}\mathscr{G}^r_{\mathsf{E}}(\mathcal{V}) = \Gamma^r(\mathsf{E}|\Phi^{-1}(\mathcal{V}))$. Since a point $y \in \mathcal{V}$ may be the image of multiple points, even infinitely many points, in M that are not close, it is problematic to think about any vector bundle over N whose sections are sections of $\Phi_{\mathrm{pre},*}\mathscr{G}^r_{\mathsf{E}}(\mathcal{V})$. Indeed, as we shall see as we go along, this is a general problem with the direct image presheaf; its stalks are difficult to describe. •

Let us give some examples of inverse images.

1.2.3 Examples (Inverse image)

1. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathcal{U} \in \mathscr{O}$. Let $\mathscr{F}$ be a presheaf over $\mathcal{S}$. We wish to examine the inverse image of $\mathscr{F}$ by the inclusion map $\iota_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{S}$. Let $\mathcal{V} \subseteq \mathcal{U}$ be open. Then $\iota_{\mathcal{U}}(\mathcal{V})$ is open in $\mathcal{S}$ and so $\Phi^{-1}_{\mathrm{pre}}\mathscr{F}(\mathcal{V}) = \mathscr{F}(\mathcal{V})$. Thus $\iota_{\mathcal{U}}\mathscr{F} = \mathscr{F}|\mathcal{U}$, the restriction of $\mathscr{F}$ to $\mathcal{U}$ (see Definition 1.1.3).

2. The preceding example suggests that we can extend the notion of restriction to arbitrary subsets. Indeed, let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F}$ be a presheaf over $\mathcal{S}$. For a subset $A \subseteq \mathcal{S}$, equipped with the subspace topology, we have the continuous inclusion map $\iota_A \colon A \to \mathcal{S}$. The ***restriction*** of $\mathscr{F}$ to A we can then define to be $\iota_{\mathrm{pre},A}^{-1}\mathscr{F}$.
3. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $x \in \mathcal{S}$. Let $\iota_x \colon \{x\} \to \mathcal{S}$ be the inclusion map and let $\mathscr{F}$ be a presheaf over $\mathcal{S}$. It follows, more or less immediately from the definition, that $\iota_{\mathrm{pre},x}^{-1}\mathscr{F} = \mathscr{F}_x$, the stalk of $\mathscr{F}$ at x.
4. Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ otherwise. Let $\tau \colon \mathsf{F} \to \mathsf{N}$ be a $\mathbb{F}$-vector bundle of class C^r and consider the presheaf, indeed sheaf, $\mathscr{G}_{\mathsf{F}}^r$ of sections of F. Let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$ be a map from another C^r-manifold M. Then we have the inverse image presheaf $\Phi_{\mathrm{pre}}^{-1}\mathscr{G}_{\mathsf{F}}^r$ over M. Let us think informally about whether this might be a presheaf of sections of a vector bundle. Let $x \in \mathsf{M}$ and note that the stalk of $\Phi_{\mathrm{pre}}^{-1}\mathscr{G}_{\mathsf{F}}^r$ at x depends only on the value of sections of F in a neighbourhood of $\Phi(x)$. (We shall make this precise in Proposition 1.2.6.) It seems not implausible, therefore, that $\Phi_{\mathrm{pre}}^{-1}\mathscr{G}_{\mathsf{M}}^r$ is the sheaf of sections of the pull-back vector bundle of Section GA1.4.3.6. •

Let us understand the stalks of the direct and inverse images of presheaves. The stalks of the direct image presheaf are quite difficult to understand in any general way. However, what we can say is the following.

1.2.4 Proposition (Maps on stalks of the direct image presheaf of sets) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$, and let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$. For $\mathrm{x} \in \mathcal{S}$, there exists a natural mapping from $(\Phi_{\mathrm{pre},*}\mathscr{F})_{\Phi(\mathrm{x})}$ to $\mathscr{F}_{\mathrm{x}}$.*

Proof Let $y = \Phi(x)$. Let $[(s, \mathcal{V})]_y \in (\Phi_{\mathrm{pre},*}\mathscr{F})_y$. Thus $s \in \mathscr{F}(\Phi^{-1}(\mathcal{V}))$, and so we have $[(s, \Phi^{-1}(\mathcal{V}))]_x \in \mathscr{F}_x$. One then readily verifies that the map

$$[(s, \mathcal{V})]_y \mapsto [(s, \Phi^{-1}(\mathcal{V}))]_x$$

is well-defined, and so gives the desired mapping. ■

In general, the mapping of stalks from the preceding result has no nice properties.

1.2.5 Examples (Maps on stalks of direct image presheaf)

1. We revisit Example 1.2.2–3. Thus we let $\mathcal{S} = \mathcal{T} = \mathbb{S}^1$ and consider the mapping $\Phi(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{2\mathrm{i}\theta}$. We take the constant sheaf $\mathscr{F}_X^+ = \mathrm{Ps}(\mathrm{Et}(\mathscr{F}_X))$, i.e., the sheafification $\mathscr{F}_X^+$ of the constant sheaf $\mathscr{F}_X$. Let $\mathrm{e}^{\mathrm{i}\theta} \in \mathcal{S}$ so $\Phi(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{2\mathrm{i}\theta}$. As we saw in Example 1.2.2–3, $\mathscr{F}_{X,\mathrm{e}^{\mathrm{i}\theta}}^+ = X$ and $(\Phi_{\mathrm{pre},*}\mathscr{F}_X^+)_{\Phi(\mathrm{e}^{\mathrm{i}\theta})} = X \times X$. Note that $\Phi^{-1}(\Phi(\mathrm{e}^{\mathrm{i}\theta})) = \{\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}(\theta+\pi)}\}$. For a small connected neighbourhood $\mathcal{V}$ of $\Phi(\mathrm{e}^{\mathrm{i}\theta})$ we have $\Phi^{-1}(\mathcal{V}) = \mathcal{U}_1 \cup \mathcal{U}_2$ for disjoint connected neighbourhoods $\mathcal{U}_1$ of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathcal{U}_2$ of $\mathrm{e}^{\mathrm{i}(\theta+\pi)}$. Thus a section s of $\Phi^{-1}(\mathcal{V})$ has the form

$$s(\mathrm{e}^{\mathrm{i}\phi}) = \begin{cases} x_1, & \mathrm{e}^{\mathrm{i}\phi} \in \mathcal{U}_1, \\ x_2, & \mathrm{e}^{\mathrm{i}\phi} \in \mathcal{U}_2, \end{cases}$$

for $x_1, x_2 \in X$. It follows that $[(s, \Phi^{-1}(\mathcal{V}))]_{e^{i\theta}} = x_1$. Thus the map between stalks is

$$X \times X \ni (x_1, x_2) \mapsto x_1 \in X,$$

showing that the natural map on stalks is generally not injective.

2. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathcal{T} = \{\mathrm{pt}\}$ be a one-point topological space. Let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be the constant map and let $\mathscr{F}$ be a presheaf over $\mathcal{S}$. In Example 1.2.2–1 we saw that $\Phi_{\mathrm{pre},*}\mathscr{F} = F(\mathcal{S})$. Let $x \in \mathcal{S}$ so that $\Phi(x) = \mathrm{pt}$. The map of stalks from $(\Phi_{\mathrm{pre},*}\mathscr{F})_{\{\mathrm{pt}\}}$ to $\mathscr{F}_x$ then maps a global section s of $\mathscr{F}$ to its germ at x. Generally this map is not surjective. For example, if $\mathcal{S}$ is a compact connected holomorphic manifold of positive dimension and $\mathscr{F}$ is the sheaf of holomorphic functions, then the global sections are constant functions (by Corollary GA1.4.2.11), while there are germs that are not constant. •

The stalks of the inverse image presheaf, on the other hand, are comparatively easy to describe.

1.2.6 Proposition (Stalks of the inverse image presheaf of sets) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathscr{G}$ be a presheaf of sets over $\mathcal{T}$, and let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$. For $x \in \mathcal{S}$, the map $[(t, \mathcal{V})]_{\Phi(x)} \mapsto [[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x$ is a bijection of stalks $\mathscr{G}_{\Phi(x)}$ and $(\Phi^{-1}_{\mathrm{pre}}\mathscr{G})_x$.*

Proof First let us show that the map is well-defined. Suppose that $[(t, \mathcal{V})]_{\Phi(x)} = [(t', \mathcal{V}')]_{\Phi(x)}$ so that there exists a neighbourhood $\mathcal{V}''$ of x with $\mathcal{V}'' \subseteq \mathcal{V} \cap \mathcal{V}'$ such that $r_{\mathcal{V},\mathcal{V}''}(t) = r_{\mathcal{V}',\mathcal{V}''}(t')$. Let $\mathcal{U}$, $\mathcal{U}'$, and $\mathcal{U}''$ be neighbourhoods of x such that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$, $\Phi(\mathcal{U}')$, $\Phi(\mathcal{U}'') \subseteq \mathcal{V}''$, and $\mathcal{U}'' \subseteq \mathcal{U} \cap \mathcal{U}'$. Then

$$r_{\Phi(\mathcal{U}),\Phi(\mathcal{U}'')}(t) = r_{\Phi(\mathcal{U}'),\Phi(\mathcal{U}'')}(t')$$

since $r_{\mathcal{V},\mathcal{V}''}(t) = r_{\mathcal{V}',\mathcal{V}''}(t')$ and since $\Phi(\mathcal{U}'') \subseteq \mathcal{V}''$. From this it follows that

$$[[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x = [[(t', \mathcal{V}')]_{\Phi(\mathcal{U}')}, \mathcal{U}']_x,$$

giving well-definedness.

Next we prove that the map $[(t, \mathcal{V})]_{\Phi(x)} \mapsto [[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x$ is injective. Suppose that

$$[[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x = [[(t', \mathcal{V}')]_{\Phi(\mathcal{U}')}, \mathcal{U}']_x.$$

Then there exists a neighbourhood $\mathcal{U}''$ of x such that $\mathcal{U}'' \subseteq \mathcal{U} \cap \mathcal{U}'$ and such that

$$r_{\Phi(\mathcal{U}),\Phi(\mathcal{U}'')}(t) = r_{\Phi(\mathcal{U}'),\Phi(\mathcal{U}'')}(t').$$

Thus there exists a neighbourhood $\mathcal{V}''$ of $\Phi(\mathcal{U}'')$ for which $r_{\mathcal{V},\mathcal{V}''}(t) = r_{\mathcal{V}',\mathcal{V}''}(t')$. Since $\mathcal{V}''$ is a neighbourhood of $\Phi(x)$, it follows that $[(t, \mathcal{V})]_{\Phi(x)} = [(t', \mathcal{V}')]_{\Phi(x)}$, giving the desired injectivity.

Next we show the surjectivity of the map $[(t, \mathcal{V})]_{\Phi(x)} \mapsto [[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x$. Let $\mathcal{U}$ be a neighbourhood of x and let $[(t, \mathcal{V})]_{\Phi(\mathcal{U})}$ be a section of $\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$ over $\mathcal{U}$. It is then clear that $[(t, \mathcal{V})]_{\Phi(x)}$ maps to $[[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x$. ■

The following result gives an important connection between the direct and inverse images.

1.2.7 Proposition (Relationships between direct and inverse images of presheaves of sets) *Let $(\mathcal{S}, \mathscr{O}_\mathcal{S})$ and $(\mathcal{T}, \mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$ and $\mathscr{G}$ be a presheaf of sets over $\mathcal{T}$. Then there exist canonical morphisms of presheaves $\mathrm{j}_\mathscr{F} \colon \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{F} \to \mathscr{F}$ and $\mathrm{i}_\mathscr{G} \colon \mathscr{G} \to \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$.*

Proof Let $\mathcal{U} \in \mathscr{O}_\mathcal{S}$ and let $\mathcal{V}$ be a neighbourhood of $\Phi(\mathcal{U})$. Thus $\Phi^{-1}(\mathcal{V})$ is a neighbourhood of $\mathcal{U}$ and so $r^\mathscr{F}_{\phi^{-1}(\mathcal{V}),\mathcal{U}}$ is a mapping from $\mathscr{F}(\Phi^{-1}(\mathcal{V})) = \Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V})$ to $\mathscr{F}(\mathcal{U})$. Moreover, if $\mathcal{V}$ and $\mathcal{V}'$ are neighbourhoods of $\Phi(\mathcal{U})$ for which $\mathcal{V}' \subseteq \mathcal{V}$, then the diagram

$$\begin{tikzcd}
\Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V}) \arrow[d] \arrow[dr, "r^\mathscr{F}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}"] & \\
& \mathscr{F}(\mathcal{U}) \\
\Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V}') \arrow[ur, swap, "r^\mathscr{F}_{\Phi^{-1}(\mathcal{V}'),\mathcal{U}}"] &
\end{tikzcd}$$

commutes, where the vertical arrow is the restriction map for $\Phi_{\mathrm{pre},*}$. From this and the definition of the inverse image presheaf, we infer the existence of a mapping from $\Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{U})$ to $\mathscr{F}(\mathcal{U})$, this then defining $j_\mathscr{F}$. Explicitly, we have

$$j_{\mathscr{F},\mathcal{U}}([(s, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})}) = r^\mathscr{F}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(s).$$

Let $\mathcal{V} \in \mathscr{O}_\mathcal{T}$. If $t \in \mathscr{G}(\mathcal{V})$ then, since $\mathcal{V}$ is a neighbourhood of $\Phi(\Phi^{-1}(\mathcal{V}))$, we have

$$[(t, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))} \in \Phi^{-1}_{\mathrm{pre}}\mathscr{G}(\Phi^{-1}(\mathcal{V})) = \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{G}(\mathcal{V}).$$

Thus we have a map

$$\mathscr{G}(\mathcal{V}) \ni t \mapsto [(t, \mathcal{V})]_\mathcal{V} = [(t, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))} \in \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{G}(\mathcal{V}),$$

and we can verify that this map commutes with restrictions, so we have the desired presheaf morphism $i_\mathscr{G}$. ■

Direct and inverse images of presheaves of rings

Presheaves of rings are presheaves of sets, of course, and so a presheaf of rings has its direct and inverse image defined in the same manner as for presheaves of sets. One must verify that these operations interact well with the ring structure.

1.2.8 Proposition (Ring structure of direct and inverse images of rings) *Let $(\mathcal{S}, \mathscr{O}_\mathcal{S})$ and $(\mathcal{T}, \mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{S}$ be a presheaf of rings over $\mathcal{T}$. Then $\Phi_{\mathrm{pre},*}\mathscr{R}$ and $\Phi^{-1}_{\mathrm{pre}}\mathscr{S}$ are presheaves of rings.*

Proof The statement for direct images follow immediately from the definitions, so we will only explicitly prove that $\Phi^{-1}_{\mathrm{pre}}\mathscr{S}$ is a presheaf of rings. To prescribe the ring structure of $\Phi^{-1}_{\mathrm{pre}}\mathscr{S}$, let $\mathcal{U} \in \mathscr{O}_{\mathcal{S}}$, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be neighbourhoods of $\Phi(\mathcal{U})$, and let $[(f_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})}, [(f_2, \mathcal{V}_2)]_{\Phi(\mathcal{U})} \in \Phi^{-1}_{\mathrm{pre}}\mathscr{S}(\mathcal{U})$. We can then define

$$[(f_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})} + [(f_2, \mathcal{V}_2)]_{\Phi(\mathcal{U})} = [(r_{\mathcal{V}_1,\mathcal{V}_1\cap\mathcal{V}_2}(f_1) + r_{\mathcal{V}_2,\mathcal{V}_1\cap\mathcal{V}_2}(f_2), \mathcal{V}_1 \cap \mathcal{V}_2)]_{\Phi(\mathcal{U})}$$

and

$$[(f_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})} \cdot [(f_2, \mathcal{V}_2)]_{\Phi(\mathcal{U})} = [(r_{\mathcal{V}_1,\mathcal{V}_1\cap\mathcal{V}_2}(f_1) \cdot r_{\mathcal{V}_2,\mathcal{V}_1\cap\mathcal{V}_2}(f_2), \mathcal{V}_1 \cap \mathcal{V}_2)]_{\Phi(\mathcal{U})}.$$

It is easy to verify that these operations are well-defined, in the sense that they are independent of representatives, and that they make $\Phi^{-1}_{\mathrm{pre}}\mathscr{S}(\mathcal{U})$ into a ring. ■

The verification of the independence on germs of constructions involving the inverse image amounts to the fact that the algebraic structure in question is preserved under direct limits; see . what?

Let us consider the canonical maps on stalks for presheaves of rings. For stalks of direct images, the result is the following.

1.2.9 Proposition (Maps on stalks of the direct image presheaf of rings) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$. For $\mathrm{x} \in \mathcal{S}$, there exists a natural ring homomorphism from $(\Phi_{\mathrm{pre},*}\mathscr{F})_{\Phi(\mathrm{x})}$ to $\mathscr{F}_{\mathrm{x}}$.*

Proof This is a simple matter of verifying that the mapping constructed in the proof of Proposition 1.2.4 is a ring homomorphism. ■

For inverse images, the result is the following.

1.2.10 Proposition (Stalks of the inverse image presheaf of rings) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathscr{S}$ be a presheaf of rings over $\mathcal{T}$, and let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$. For $\mathrm{x} \in \mathcal{S}$, the map $[(\mathrm{f}, \mathcal{V})]_{\Phi(\mathrm{x})} \mapsto [[(\mathrm{f}, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_{\mathrm{x}}$ is an isomorphism of the rings $\mathscr{G}_{\Phi(\mathrm{x})}$ and $(\Phi^{-1}_{\mathrm{pre}}\mathscr{G})_{\mathrm{x}}$.*

Proof This is a direct verification, and amounts to the fact that direct limits preserve the ring structure. ■

Finally, we verify that the natural relationships between the direct and inverse images also preserve the ring structure.

1.2.11 Proposition (Relationships between direct and inverse images of presheaves of rings) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$ and $\mathscr{S}$ be a presheaf of rings over $\mathcal{T}$. Then there exist canonical morphisms of presheaves of rings $\mathrm{j}_{\mathscr{R}} \colon \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{R} \to \mathscr{R}$ and $\mathrm{i}_{\mathscr{S}} \colon \mathscr{S} \to \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{S}$.*

Proof In the proof of Proposition 1.2.7 we showed that

$$j_{\mathscr{R},\mathcal{U}}([(f, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})}) = r^{\mathscr{R}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(f)$$

and

$$i_{\mathscr{S},\mathcal{V}}(g) = [(g, \mathcal{V})]_{\mathcal{V}}.$$

From these definitions, we easily verify that the presheaf morphisms are morphisms of presheaves of rings. ■

Direct and inverse images of presheaves of modules

Let us now investigate how the direct and inverse images interact with module structures.

1.2.12 Proposition (Module structure of direct and inverse images of modules) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$ and $\mathscr{S}$ be a presheaf of rings over $\mathcal{T}$, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules and $\mathscr{F}$ be a presheaf of $\mathscr{S}$-modules over $\mathcal{T}$. Then the following statements hold:*

(i) $\Phi_{\mathrm{pre},*}\mathscr{E}$ is a presheaf of $\Phi_{\mathrm{pre},*}\mathscr{R}$-modules and $\Phi_{\mathrm{pre}}^{-1}\mathscr{F}$ is a presheaf of $\Phi_{\mathrm{pre}}^{-1}\mathscr{S}$-modules;

(ii) if $\mathscr{G}$ is a presheaf of $\Phi_{\mathrm{pre}}^{-1}\mathscr{S}$-modules, then $\Phi_{\mathrm{pre},*}\mathscr{G}$ is a presheaf of $\mathscr{S}$-modules.

Proof (i) The statement for direct images follows immediately from the definitions. To prescribe the $\Phi_{\mathrm{pre}}^{-1}\mathscr{S}$-module structure of $\Phi_{\mathrm{pre}}^{-1}\mathscr{F}$, let $\mathcal{U} \in \mathscr{O}_{\mathcal{S}}$, let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}$ be neighbourhoods of $\Phi(\mathcal{U})$, and let $[(s_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})}, [(s_2, \mathcal{V}_2)]_{\Phi(\mathcal{U})} \in \Phi_{\mathrm{pre}}^{-1}\mathscr{F}(\mathcal{U})$ and $[(f, \mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi_{\mathrm{pre}}^{-1}\mathscr{S}(\mathcal{U})$. We can then define

$$[(s_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})} + [(s_2, \mathcal{V}_2)]_{\Phi(\mathcal{U})} = [(r_{\mathcal{V}_1, \mathcal{V}_1 \cap \mathcal{V}_2}(s_1) + r_{\mathcal{V}_2, \mathcal{V}_1 \cap \mathcal{V}_2}(s_2), \mathcal{V}_1 \cap \mathcal{V}_2)]_{\Phi(\mathcal{U})}$$

and

$$[(f, \mathcal{V})]_{\Phi(\mathcal{U})} \cdot [(s_1, \mathcal{V}_1)]_{\Phi(\mathcal{U})} = [(r_{\mathcal{V}, \mathcal{V} \cap \mathcal{V}_1}(f) \cdot r_{\mathcal{V}_1, \mathcal{V} \cap \mathcal{V}_1}(s_1), \mathcal{V} \cap \mathcal{V}_1)]_{\Phi(\mathcal{U})}.$$

It is easy to verify that these operations are well-defined, in the sense that they are independent of representatives, and that they make $\Phi_{\mathrm{pre}}^{-1}\mathscr{E}(\mathcal{U})$ into a $\Phi_{\mathrm{pre}}^{-1}\mathscr{S}(\mathcal{U})$-module.

(ii) Let $\mathcal{V}$ be open and let $t \in \Phi_{\mathrm{pre},*}\mathscr{G}(\mathcal{V}) = \mathscr{G}(\Phi^{-1}(\mathcal{V}))$. Since $\Phi(\Phi^{-1}(\mathcal{V})) = \mathcal{V}$ is open, we identify $[(g, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))} \in \Phi_{\mathrm{pre}}^{-1}\mathscr{S}(\Phi^{-1}(\mathcal{V}))$ with $g \in \mathscr{S}(\mathcal{V})$. We then define

$$g \cdot t = [(g, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))} \cdot t, \tag{1.6}$$

which we easily verify is well-defined. ■

Note that one "obvious" implication is missing from Proposition 1.2.12, and this is because it is not true. Let us be clear. If $\mathscr{H}$ is a presheaf of $\Phi_{\mathrm{pre},*}\mathscr{R}$-modules, then $\Phi_{\mathrm{pre}}^{-1}\mathscr{H}$ is generally *not* a presheaf of $\mathscr{R}$-modules in any useful way. This is because sections of the base ring, $\Phi_{\mathrm{pre},*}\mathscr{R}$, over $\mathcal{V}$ are sections of $\mathscr{R}$ over $\Phi^{-1}(\mathcal{V})$. If $\mathcal{V}$ if a neighbourhood of $\Phi(\mathcal{U})$, $\Phi^{-1}(\mathcal{V})$ may be an open set with points far away from $\mathcal{U}$. If the restriction from such open sets to $\mathcal{U}$ is not surjective, then there is no way of defining multiplication by elements of $\mathscr{R}(\mathcal{U})$. This general non-surjectivity of the restriction maps for a presheaf often causes problems for the direct image, cf. Example 1.2.5.

Let us consider the canonical maps on stalks for presheaves of modules. For stalks of direct images, the result is the following.

1.2.13 Proposition (Maps on stalks of the direct image presheaf of modules) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules, and let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$. For $x \in \mathcal{S}$, the canonical bijection of Proposition 1.2.4 from*

$(\Phi_{\mathrm{pre},*}\mathscr{F})_{\Phi(x)}$ *to* $\mathscr{F}_x$ *is a morphism of Abelian groups with respect to module addition and has the property that the diagram*

$$\begin{tikzcd}
(\Phi_{\mathrm{pre},*}\mathscr{R})_{\Phi(x)} \times (\Phi_{\mathrm{pre},*}\mathscr{E})_{\Phi(x)} \arrow[r] \arrow[d] & (\Phi_{\mathrm{pre},*}\mathscr{E})_{\Phi(x)} \arrow[d] \\
\mathscr{R}_x \times \mathscr{E}_x \arrow[r] & \mathscr{E}_x
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication and the vertical arrows are the canonical mappings on stalks.

Proof Let $[(f, \mathcal{V}_1)]_{\Phi(x)} \in (\Phi_{\mathrm{pre},*}\mathscr{R})_{\Phi(x)}$ and let $[(s, \mathcal{V}_2)]_{\Phi(x)} \in (\Phi_{\mathrm{pre},*}\mathscr{E})_{\Phi(x)}$. For simplicity, and without loss of generality by restriction, suppose that $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$. The verification that the canonical mapping preserves the Abelian group structure is immediate. We also note that

$$[(f \cdot s, \Phi^{-1}(\mathcal{V}))]_x = [(f, \Phi^{-1}(\mathcal{V}))]_x \cdot [(f, \Phi^{-1}(\mathcal{V}))]_x,$$

which amounts to the commuting of the diagram. ■

For inverse images, the result is the following.

1.2.14 Proposition (Stalks of the inverse image presheaf of modules) *Let* $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ *and* $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ *be topological spaces, let* $\mathscr{S}$ *be a presheaf of rings over* $\mathcal{T}$*, let* $\mathscr{F}$ *be a presheaf of* $\mathscr{S}$*-modules, and let* $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$*. For* $x \in \mathcal{S}$*, the canonical map of Proposition 1.2.6 from* $\mathscr{F}_{\Phi(x)}$ *and* $(\Phi^{-1}_{\mathrm{pre}}\mathscr{F})_x$ *is a morphism of Abelian groups with respect to module addition and has the property that the diagram*

$$\begin{tikzcd}
\mathscr{S}_{\Phi(x)} \times \mathscr{F}_{\Phi(x)} \arrow[r] \arrow[d] & \mathscr{F}_{\Phi(x)} \arrow[d] \\
(\Phi^{-1}_{\mathrm{pre}}\mathscr{S})_x \times (\Phi^{-1}_{\mathrm{pre}}\mathscr{F})_x \arrow[r] & (\Phi^{-1}_{\mathrm{pre}}\mathscr{F})_x
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication and the vertical arrows are the canonical mappings.

Proof From Proposition 1.2.6 the canonical map on stalks in this case is

$$[(t, \mathcal{V})]_{\Phi(x)} \mapsto [[(t, \mathcal{V})]_{\Phi(\mathcal{U})}, \mathcal{U}]_x.$$

If $\mathcal{V}, \mathcal{V}' \in \mathscr{O}_{\mathcal{T}}$ satisfy $\mathcal{V}' \subseteq \mathcal{V}$ and if $\mathcal{U}, \mathcal{U}' \in \mathscr{O}_{\mathcal{S}}$ satisfy $\Phi(\mathcal{U}) \subseteq \mathcal{V}$, $\Phi(\mathcal{U}') \subseteq \mathcal{V}'$, and $\mathcal{U}' \subseteq \mathcal{U}$, then we have maps

$$S(\mathcal{V}) \ni g \mapsto [(g, \mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi^{-1}_{\mathrm{pre}}\mathscr{S}(\mathcal{U}), \qquad \mathscr{F}(\mathcal{V}) \ni t \mapsto [(t, \mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi^{-1}_{\mathrm{pre}}\mathscr{F}(\mathcal{U}), \tag{1.7}$$

with similar maps for the "primed" expressions. One can then verify, using the properties of multiplication in presheaves of modules, that the diagram

$$\begin{tikzcd}
& \mathscr{S}(\mathcal{V}')\times\mathscr{F}(\mathcal{V}') \arrow[rr] \arrow[dd] & & \mathscr{F}(\mathcal{V}') \arrow[dd] \\
\mathscr{S}(\mathcal{V})\times\mathscr{F}(\mathcal{V}) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathscr{F}(\mathcal{V}) \arrow[ur] \arrow[dd] & \\
& \Phi^{-1}_{\mathrm{pre}}\mathscr{S}(\mathcal{U}')\times\Phi^{-1}_{\mathrm{pre}}\mathscr{E}(\mathcal{U}') \arrow[rr] & & \Phi^{-1}_{\mathrm{pre}}\mathscr{E}(\mathcal{U}') \\
\Phi^{-1}_{\mathrm{pre}}\mathscr{S}(\mathcal{U})\times\Phi^{-1}_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) \arrow[ur] \arrow[rr] & & \Phi^{-1}_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) \arrow[ur] &
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication, the diagonal arrows are restrictions, and the vertical arrows are the maps (1.7). Taking direct limits along the diagonal arrows, i.e., by letting the neighbourhoods $\mathcal{V}$, and correspondingly $\mathcal{U}$, shrink, one gets the diagram in the statement of the proposition. It is clear that the canonical map preserves the Abelian group structure of module addition. ■

Let us now turn to the relationships between direct and inverse limits of presheaves of modules, i.e., to the module version of Propositions 1.2.7 and 1.2.11. In this case, the story is a little more subtle, as one has to carefully account for the proliferation of module structures present. Nonetheless, one *does* have the morphism of presheaves of sets $i_{\mathscr{F}}\colon \mathscr{F} \to \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$ associated with a presheaf $\mathscr{F}$ of $\mathscr{S}$-modules over a topological space $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ and a continuous map $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$. In this case, we have the following property of this morphism of presheaves of sets.

1.2.15 Proposition (Relationships between direct and inverse images of presheaves of modules I) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$, let $\mathscr{S}$ be a presheaf of rings over $\mathcal{T}$, and let $\mathscr{F}$ be a presheaf of $\mathscr{S}$-modules. Then the canonical morphism $\mathrm{i}_{\mathscr{F}}$ of presheaves of sets from Proposition 1.2.7 is a morphism of presheaves of $\mathscr{S}$-modules.*

Proof We should first be sure we understand the $\mathscr{S}$-module structure on $\Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$. First of all, by Proposition 1.2.12(i) we have that $\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$ is a presheaf of $\Phi^{-1}_{\mathrm{pre}}\mathscr{S}$-modules. Then, by Proposition 1.2.12(ii) we have that $\Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$ is a presheaf of $\mathscr{S}$-modules. In the proof of Proposition 1.2.7 we showed that

$$i_{\mathscr{F},\mathcal{V}}(t) = [(t, \mathcal{V})]_{\mathcal{V}}.$$

It is clear that this map is a morphism of Abelian groups with respect to module addition. To verify that the morphism also preserves module multiplication, let $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ and let $g \in \mathscr{S}(\mathcal{V})$ and $t \in \mathscr{F}(\mathcal{V})$. We then have

$$i_{\mathscr{F},\mathcal{V}}(g \cdot t) = [(g \cdot t, \mathcal{V})]_{\mathcal{V}} = [(g, \mathcal{V})]_{\mathcal{V}} \cdot [(t, \mathcal{V})]_{\mathcal{V}} = g \cdot i_{\mathscr{F},\mathcal{V}}(t),$$

recalling the definition (1.6) of the $\mathscr{S}$-module structure on $\Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{F}$. ■

We also have the following, slightly different, result when one reverse the order of composition of direct and inverse image.

1.2.16 Proposition (Relationships between direct and inverse images of presheaves of modules II) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ be a presheaf of $\mathscr{R}$-modules. Then the canonical morphism $\mathrm{j}_{\mathscr{E}}$ of presheaves of sets from Proposition 1.2.7 is a morphism of presheaves of Abelian groups with respect to module addition and has the property that the diagram*

$$\begin{tikzcd}
\Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{R}(\mathcal{U}) \times \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{E}(\mathcal{U}) \arrow[r] \arrow[d] & \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{E}(\mathcal{U}) \arrow[d] \\
\mathscr{R}(\mathcal{U}) \times \mathscr{E}(\mathcal{U}) \arrow[r] & \mathscr{E}(\mathcal{U})
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication and the vertical arrows are the canonical maps.

Proof Let $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ and let $\mathcal{U} \in \mathscr{O}_{\mathcal{S}}$ be such that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$. From the proof of Proposition 1.2.7 we have

$$j_{\mathscr{E},\mathcal{U}}([(s, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})}) = r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(s).$$

Thus, if $f \in \Phi^{-1}(\mathcal{V})$ and $s \in \Phi^{-1}(\mathcal{V})$ (without loss of generality, by restricting if necessary, we suppose that these local sections are defined over the same open set), then we have

$$\begin{aligned}
j_{\mathscr{E},\mathcal{U}}([(f \cdot s, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})} &= r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(f \cdot s) = r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(f) \cdot r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}(s) \\
&= j_{\mathscr{R},\mathcal{U}}([(f, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})} \cdot j_{\mathscr{E},\mathcal{U}}([(s, \Phi^{-1}(\mathcal{V}))]_{\Phi(\mathcal{U})},
\end{aligned}$$

using module multiplication as defined in Proposition 1.2.12. The preservation of the Abelian group structure associated with module addition is easily verified. ■

1.2.2 Direct and inverse images of sheaves

Next let us examine whether the direct and inverse images are sheaves. We shall see here that there is an essential difference in the properties of direct and inverse images.

Direct and inverse images of sheaves of sets

We begin with a result for direct images of sheaves of sets.

1.2.17 Proposition (The direct image of a sheaf of sets is a sheaf) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, and let $\mathscr{F}$ be a presheaf of sets over $\mathcal{S}$. If $\mathscr{F}$ is a sheaf then so is $\Phi_{\mathrm{pre},*}\mathscr{F}$.*

Proof Let $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ and let $(\mathcal{V}_a)_{a\in A}$ be an open cover of $\mathcal{V}$. Suppose that $s, t \in \Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V})$ satisfy $\Phi_{\mathrm{pre},*}r_{\mathcal{V},\mathcal{V}_a}(s) = \Phi_{\mathrm{pre},*}r_{\mathcal{V},\mathcal{V}_a}(s)$ for every $a \in A$. This means that

$$r_{\Phi^{-1}(\mathcal{V}),\Phi^{-1}(\mathcal{V}_a)}(s) = r_{\Phi^{-1}(\mathcal{V}),\Phi^{-1}(\mathcal{V}_a)}(t).$$

Since $(\Phi^{-1}(\mathcal{V}_a))_{a\in A}$ is an open cover for $\Phi^{-1}(\mathcal{V})$ and since $\mathscr{F}$ is separable, this implies that $s = t$. Next suppose that $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ and that $(\mathcal{V}_a)_{a\in A}$ is an open cover of $\mathcal{V}$ and that

$s_a \in \Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V}_a)$, $a \in A$, satisfy $\Phi_{\mathrm{pre},*}r_{\mathcal{V}_a,\mathcal{V}_a\cap\mathcal{V}_b}(s_a) = \Phi_{\mathrm{pre},*}r_{\mathcal{V}_b,\mathcal{V}_a\cap\mathcal{V}_b}(s_b)$ for every $a, b \in A$. This means that

$$r_{\Phi^{-1}(\mathcal{V}_a),\Phi^{-1}(\mathcal{V}_a)\cap\Phi^{-1}(\mathcal{V}_b)}(s_a) = r_{\Phi^{-1}(\mathcal{V}_b),\Phi^{-1}(\mathcal{V}_a)\cap\Phi^{-1}(\mathcal{V}_b)}(s_b)$$

for every $a, b \in A$. Since $\mathscr{F}$ has the gluing property, there exists $s \in \mathscr{F}(\Phi^{-1}(\mathcal{V}))$ such that

$$r_{\Phi^{-1}(\mathcal{V}),\Phi^{-1}(\mathcal{V}_a)}(s) = s_a, \qquad a \in A.$$

Clearly, then

$$\Phi_{\mathrm{pre},*}r_{\mathcal{V},\mathcal{V}_a}(s) = s_a, \qquad a \in A,$$

showing that $\Phi_{\mathrm{pre},*}\mathscr{F}$ has the gluing property. ■

One the other hand, the inverse image of a sheaf is not necessarily a sheaf.

1.2.18 Example (The inverse image of a sheaf is not always a sheaf) We let $\mathcal{S} = \{x_1, \dots, x_n\}$ be a finite topological space equipped with the discrete topology, let $\mathcal{T} = \{\mathrm{pt}\}$ be a one-point space, and note that the map $\Phi\colon \mathcal{S} \to \mathcal{T}$ defined by $\Phi(x_j) = \mathrm{pt}$ is continuous. On $\mathcal{T}$ we consider the constant sheaf $\mathscr{F}_X$ associated with the set X; see Example 1.1.4–3. Note that, because $\mathcal{T}$ is a one-point set, the constant presheaf is also the constant sheaf. By definition, the inverse image presheaf is the constant presheaf associated with the set X. However, this presheaf is not a sheaf as we saw in Example 1.1.100–1. •

Thus, while $\mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre},*}\mathscr{F}))$ is isomorphic to $\Phi_{\mathrm{pre},*}\mathscr{F}$ if $\mathscr{F}$ is a sheaf (see Proposition 1.1.82), it is not the case that $\mathrm{Ps}(\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}))$ is isomorphic to $\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$, even when $\mathscr{G}$ is a sheaf. To rectify this, we make the following definition that puts the direct and inverse images on the same footing, in some sense.

1.2.19 Definition (Direct image and inverse image of sheaves of sets) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, and let $\mathscr{F}$ be a sheaf of sets over $\mathcal{S}$ and $\mathscr{G}$ be a sheaf of sets over $\mathcal{T}$.

(i) The ***direct image*** of $\mathscr{F}$ by Φ is the sheaf $\Phi_*\mathscr{F} = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre},*}\mathscr{F}))$ over $\mathcal{T}$.

(ii) The ***inverse image*** of $\mathscr{G}$ by Φ is the sheaf $\Phi^{-1}\mathscr{G} = \mathrm{Ps}(\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}))$ over $\mathcal{S}$. •

As with their presheaf counterparts, there are relationships between the direct and inverse images of sheaves.

1.2.20 Proposition (Relationships between direct and inverse images of sheaves of sets) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{F}$ be a sheaf of sets over $\mathcal{S}$ and $\mathscr{G}$ be a sheaf of sets over $\mathcal{T}$. Then there exist canonical morphisms of sheaves $\mathrm{j}_{\mathscr{F}}\colon \Phi^{-1}\Phi_*\mathscr{F} \to \mathscr{F}$ and $\mathrm{i}_{\mathscr{G}}\colon \mathscr{G} \to \Phi_*\Phi^{-1}\mathscr{G}$.*

Proof By Proposition 1.2.7 we have morphisms

$$j_{\mathscr{F}}\colon \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{F} \to \mathscr{F}, \quad i_{\mathscr{G}}\colon \mathscr{G} \to \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{G}.$$

By Propositions 1.1.58 and 1.1.59 this induces morphisms

$$\mathrm{Ps}(\mathrm{Et}(j_{\mathscr{F}}))\colon \Phi^{-1}\Phi_*\mathscr{F} \to \mathrm{Ps}(\mathrm{Et}(\mathscr{F})), \quad \mathrm{Ps}(\mathrm{Et}(i_{\mathscr{G}}))\colon \mathrm{Ps}(\mathrm{Et}(\mathscr{G})) \to \Phi_*\Phi^{-1}\mathscr{G}.$$

Since $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ is isomorphic to $\mathscr{F}$ and $\mathrm{Ps}(\mathrm{Et}(\mathscr{G}))$ is isomorphic to $\mathscr{G}$ since $\mathscr{F}$ and $\mathscr{G}$ are sheaves, the result follows (noting that we are abusing notation with $j_{\mathscr{F}}$ and $i_{\mathscr{G}}$). ■

Direct and inverse images of sheaves of rings

Since sheafification of presheaves of rings gives rise to a sheaf of rings, we can simply go ahead and make the following definition.

1.2.21 Definition (Direct image and inverse image of sheaves of rings) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, and let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$ and $\mathscr{S}$ be a sheaf of rings over $\mathcal{T}$.

(i) The ***direct image*** of $\mathscr{R}$ by Φ is the sheaf $\Phi_*\mathscr{R} = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre},*}\mathscr{R}))$ over $\mathcal{T}$.

(ii) The ***inverse image*** of $\mathscr{S}$ by Φ is the sheaf $\Phi^{-1}\mathscr{S} = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre}}^{-1}\mathscr{S}))$ over $\mathcal{S}$. •

By Propositions 1.1.84 and 1.1.85 it follows that $\Phi_*\mathscr{R}$ and $\Phi^{-1}\mathscr{S}$ are sheaves of rings.

We have the following relationships between direct and inverse images of sheaves.

1.2.22 Proposition (Relationships between direct and inverse images of sheaves of rings) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$ and $\mathscr{S}$ be a sheaf of rings over $\mathcal{T}$. Then there exist canonical morphisms of sheaves of rings $\mathrm{j}_{\mathscr{R}}\colon \Phi^{-1}\Phi_*\mathscr{R} \to \mathscr{R}$ and $\mathrm{i}_{\mathscr{S}}\colon \mathscr{S} \to \Phi_*\Phi^{-1}\mathscr{S}$.*

Proof This follows in the same manner as Proposition 1.2.20, but using Propositions 1.1.67 and 1.1.68. ■

Direct and inverse images of sheaves of modules

Now we turn to direct and inverse images of sheaves of modules.

1.2.23 Definition (Direct image and inverse image of sheaves of modules) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$ and $\mathscr{S}$ be a sheaf of rings over $\mathcal{T}$, and let $\mathscr{E}$ be a sheaf of $\mathscr{R}$-modules and $\mathscr{F}$ be a sheaf of $\mathscr{S}$-modules.

(i) The ***direct image*** of $\mathscr{E}$ by Φ is the sheaf $\Phi_*\mathscr{E} = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre},*}\mathscr{E}))$ over $\mathcal{T}$.

(ii) The ***inverse image*** of $\mathscr{F}$ by Φ is the sheaf $\Phi^{-1}\mathscr{F} = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre}}^{-1}\mathscr{F}))$ over $\mathcal{S}$. •

By Proposition 1.2.12 and by Propositions 1.1.87 and 1.1.88 we have that $\Phi_*\mathscr{E}$ is a sheaf of $\Phi_*\mathscr{R}$-modules and that $\Phi^{-1}\mathscr{F}$ is a sheaf of $\Phi^{-1}\mathscr{S}$-modules.

As with presheaves of modules, there are relationships between direct and inverse images of sheaves of modules.

1.2.24 Proposition (Relationships between direct and inverse images of sheaves of modules I) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, let $\mathscr{S}$ be a sheaf of rings over $\mathcal{T}$, and let $\mathscr{F}$ be a sheaf of $\mathscr{S}$-modules. Then the canonical morphism $\mathrm{i}_{\mathscr{F}}$ of sheaves of sets from Proposition 1.2.20 is a morphism of presheaves of $\mathscr{S}$-modules.*

Proof This follows by an argument entirely like that in the proof of Proposition 1.2.22. ■

1.2.25 Proposition (Relationships between direct and inverse images of sheaves of modules II) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ be a sheaf of $\mathscr{R}$-modules. Then the canonical morphism $\mathrm{j}_{\mathscr{E}}$ of presheaves of sets from Proposition 1.2.20 is a morphism of sheaves of Abelian groups with respect to module addition and has the property that the diagram*

$$\begin{tikzcd}
\Phi^{-1}\Phi_*\mathscr{R}(\mathcal{U}) \times \Phi^{-1}\Phi_*\mathscr{E}(\mathcal{U}) \arrow[r] \arrow[d] & \Phi^{-1}\Phi_*\mathscr{E}(\mathcal{U}) \arrow[d] \\
\mathscr{R}(\mathcal{U}) \times \mathscr{E}(\mathcal{U}) \arrow[r] & \mathscr{E}(\mathcal{U})
\end{tikzcd}$$

commutes, where the horizontal arrows are module multiplication and the vertical arrows are the canonical maps.

Proof For the commuting of the diagram, we can apply $\mathrm{Ps}(\mathrm{Et}(\cdot))$ to the diagram from Proposition 1.2.16 and use the fact that $\mathrm{Ps}(\mathrm{Et}(\mathscr{R})) \simeq \mathscr{R}$ and $\mathrm{Ps}(\mathrm{Et}(\mathscr{E})) \simeq \mathscr{E}$. That $j_{\mathscr{E}}$ is a morphism of sheaves of Abelian groups follows from Proposition 1.2.16 by a similar argument. ■

1.2.3 Direct and inverse images of étalé spaces

It is also possible to define direct and inverse images for étalé spaces. Indeed, as we shall see, for the inverse image this definition is substantially simpler than the presheaf definition.

Direct and inverse images of étalé spaces of sets

We first define direct and inverse images for étalé spaces of sets.

1.2.26 Definition (Direct image and inverse image of étalé spaces of sets) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{S}$ be an étalé space of sets over $\mathcal{S}$, and let $\mathscr{T}$ be an étalé space of sets over $\mathcal{T}$.

(i) The ***direct image*** of $\mathscr{S}$ by Φ is $\Phi_*\mathscr{S} = \mathrm{Et}(\Phi_*\mathrm{Ps}(\mathscr{S}))$.

(ii) The ***inverse image*** of $\mathscr{T}$ by Φ is the étalé space $\Phi^{-1}\mathscr{T}$ defined by

$$\Phi^{-1}\mathscr{T} = \{(x, \sigma) \in \mathcal{S} \times \mathscr{T} \mid \Phi(x) = \tau(\sigma)\}$$

with the projection $\Phi^{-1}\tau \colon \Phi^{-1}\mathscr{T} \to \mathcal{S}$ given by $\Phi^{-1}(x, \sigma) = x$, and with the topology being that induced by the product topology on $\mathcal{S} \times \mathscr{T}$. •

As we saw in Example 1.2.5, we cannot expect, in general, that there will be a nice description of the direct image of $\mathrm{Et}(\mathscr{F})$ for a sheaf $\mathscr{F}$. Also, our description of the inverse image of an étalé space is not immediately connected with our previous constructions with the inverse image. We should, therefore, repair this gap. We begin by verifying that the inverse image is indeed an étalé space.

1.2.27 Proposition (The inverse image of an étalé space of sets is an étalé space of sets) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\tau \colon \mathscr{T} \to \mathcal{S}$ be an étalé space of sets over $\mathcal{T}$. Then $\Phi^{-1}\mathscr{T}$ is an étalé space of sets.*

Proof Let $(x, \sigma) \in \Phi^{-1}\mathscr{T}$, let $\mathcal{V}$ be a neighbourhood of $\Phi(x)$, and let $s \in \Gamma(\mathcal{V}; \mathscr{T})$ be such that $s(\Phi(x)) = \sigma$. Note that $(\Phi^{-1}(\mathcal{V}) \times s(\mathcal{V})) \cap \Phi^{-1}\mathscr{T}$ is a neighbourhood of (x, σ) in the topology of $\Phi^{-1}\mathscr{T}$. Moreover,

$$\begin{aligned}(\Phi^{-1}(\mathcal{V}) \times s(\mathcal{V})) \cap \Phi^{-1}\mathscr{T} &= \{(x', \sigma') \in \mathscr{T} \mid \Phi(x') \in \mathcal{V},\ \sigma' = s \circ \Phi(x')\}\\ &= \{(x', s \circ \Phi(x')) \mid x' \in \Phi^{-1}(\mathcal{V})\}.\end{aligned}$$

Thus the neighbourhood $(\Phi^{-1}(\mathcal{V}) \times s(\mathcal{V})) \cap \Phi^{-1}\mathscr{T}$ of (x, σ) is mapped homeomorphically onto the neighbourhood $\Phi^{-1}(\mathcal{V})$ of x. ■

Now we can connect the two notions of inverse image. For symmetry, we include the statement for the direct image, although the assertion here is less profound.

1.2.28 Proposition (The direct and inverse image of an étalé space of sets is the étalé space of the direct and inverse image) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{S}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$, let $\mathscr{F}$ be a sheaf of sets over $\mathcal{S}$, and let $\mathscr{G}$ be a sheaf of sets over $\mathcal{T}$. Then $\Phi_*\mathrm{Et}(\mathscr{F})$ is isomorphic to $\mathrm{Et}(\Phi_{\mathrm{pre}}\mathscr{F})$ and $\Phi^{-1}\mathrm{Et}(\mathscr{G})$ is isomorphic to $\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G})$.*

Proof We have

$$\Phi_*\mathrm{Et}(\mathscr{F}) = \mathrm{Et}(\Phi_*\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))) \simeq \mathrm{Et}(\Phi_*\mathscr{F}) \simeq \mathrm{Et}(\Phi_{\mathrm{pre},*}\mathscr{F}),$$

giving the result for the direct image. For the inverse image, the argument is as follows. Since $\mathscr{G}$ is a sheaf, we can and do identify $\mathscr{G}$ with $\mathrm{Ps}(\mathrm{Et}(\mathscr{G}))$. Then we consider the morphism from $\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$ to $\mathrm{Ps}(\Phi^{-1}\mathrm{Et}(\mathscr{G}))$ defined by assigning to $[(t, \mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi^{-1}_{\mathrm{pre}}(\mathcal{U})$ the local section of $\mathrm{Et}(\Phi^{-1}\mathrm{Et}(\mathscr{G}))$ over $\mathcal{U}$ given by $x \mapsto (\Phi(x), t \circ \Phi(x))$. Since this map is an isomorphism on stalks by Proposition 1.2.6 and by the definition of $\Phi^{-1}\mathrm{Et}(\mathscr{G})$, it follows from Proposition 1.1.99 that we get the induced isomorphism from $\Phi^{-1}\mathscr{G}$ to $\mathrm{Ps}(\Phi^{-1}\mathrm{Et}(\mathscr{G}))$, this in turn inducing the desired isomorphism of the result. ■

Direct and inverse images of étalé spaces of rings

Étalé spaces of rings being étalé spaces of sets, we can define their direct and inverse images as in Definition 1.2.26. We should show, however, that the resulting étalé spaces are spaces of rings.

1.2.29 Proposition (The direct and inverse image of an étalé space of rings is an étalé space of rings) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{B}$ be an étalé space of rings over $\mathcal{T}$. Then $\Phi_*\mathscr{A}$ and $\Phi^{-1}\mathscr{B}$ are étalé spaces of rings.*

Proof For the direct image, the result is simply Proposition 1.1.48. For the inverse image we first should define the ring operations. To do this, we let $(x, \alpha), (x, \beta) \in (\Phi^{-1}\mathscr{B})_x$ and define

$$(x, \alpha) + (x, \beta) = (x, \alpha + \beta), \quad (x, \alpha) \cdot (x, \beta) = (x, \alpha \cdot \beta).$$

We must also show that these operations are continuous. Let $(x,\alpha_1),(x,\alpha_2)\in\Phi^{-1}\mathscr{T}$ and, as in the proof of Proposition 1.2.27, let $\mathcal{V}$ be a neighbourhood of $\Phi(x)$, and let $a_1,a_2\in\Gamma(\mathcal{V};\mathscr{B})$ be such that $a_j(\Phi(x))=\alpha_j$, $j\in\{1,2\}$. Then

$$(\Phi^{-1}(\mathcal{V})\times a_1(\mathcal{V}))\cap\Phi^{-1}\mathscr{B},\quad (\Phi^{-1}(\mathcal{V})\times a_2(\mathcal{V}))\cap\Phi^{-1}\mathscr{B},\quad (\Phi^{-1}(\mathcal{V})\times(a_1+a_2)(\mathcal{V}))\cap\Phi^{-1}\mathscr{B}$$

are neighbourhoods of (x,α_1), (x,α_2), and $(x,\alpha_1+\alpha_2)$, respectively. The continuity of addition in $\Phi^{-1}\mathscr{B}$ now follows immediately from that for addition in $\mathscr{B}$. An entirely similar argument, replacing addition with multiplication, shows that ring multiplication is also continuous. ■

As with sets, our construction of the direct and inverse image of an étalé space of rings corresponds to our construction above with sheaves.

1.2.30 Proposition (The direct and inverse image of an étalé space of rings is the étalé space of the direct and inverse image) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{S},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi\in C^0(\mathcal{S};\mathcal{T})$, let $\mathscr{A}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{B}$ be a sheaf of rings over $\mathcal{T}$. Then $\Phi_*\mathrm{Et}(\mathscr{A})$ is isomorphic to $\mathrm{Et}(\Phi_{\mathrm{pre}}\mathscr{A})$ and $\Phi^{-1}\mathrm{Et}(\mathscr{B})$ is isomorphic to $\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\mathscr{B})$.*

Proof This follows as does Proposition 1.2.28, now using Propositions 1.2.10 and 1.1.103. ■

Direct and inverse images of étalé spaces of modules

Now let us extend the preceding constructions to étalé spaces of modules.

1.2.31 Proposition (The direct and inverse image of an étalé space of modules is an étalé space of modules) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{T},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi\in C^0(\mathcal{S};\mathcal{T})$ be a continuous map, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $\mathscr{B}$ be an étalé space of rings over $\mathcal{T}$, let $\mathscr{U}$ be an étalé space of $\mathscr{A}$-modules, and let $\mathscr{V}$ be an étalé space of $\mathscr{B}$-modules. Then $\Phi_*\mathscr{U}$ is an étalé space of $\Phi_*\mathscr{A}$-modules and $\Phi^{-1}\mathscr{V}$ is an étalé space of $\Phi^{-1}\mathscr{B}$-modules.*

Proof For the direct image, this follows from Proposition 1.2.12(i). For the inverse image, we define first the module operations. We let $(x,\alpha)\in(\Phi^{-1}\mathscr{B})_x$ and $(x,\sigma),(x,\tau)\in(\Phi^{-1}\mathscr{V})_x$, and define

$$(x,\sigma)+(x,\tau)=(x,\sigma+\tau),\quad (x,\alpha)\cdot(x,\sigma)=(x,\alpha\cdot\sigma).$$

The manner by which one proves the continuity of these operations mirrors the corresponding part of the proof from Proposition 1.2.29. ■

As with sets, our construction of the direct and inverse image of an étalé space of rings corresponds to our construction above with sheaves.

1.2.32 Proposition (The direct and inverse image of the étalé space of modules is the étalé space of the direct and inverse image) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{S},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi\in C^0(\mathcal{S};\mathcal{T})$, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $\mathscr{B}$ be an étalé space of rings over $\mathcal{T}$, let $\mathscr{U}$ be an étalé space of $\mathscr{A}$-modules, and let $\mathscr{V}$ be an étalé space of $\mathscr{B}$-modules. Then $\Phi_*\mathrm{Et}(\mathscr{U})$ is isomorphic to $\mathrm{Et}(\Phi_{\mathrm{pre}}\mathscr{U})$ and $\Phi^{-1}\mathrm{Et}(\mathscr{V})$ is isomorphic to $\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\mathscr{V})$.*

Proof This follows as does Proposition 1.2.28, now using Propositions 1.2.14 and 1.1.107. ■

1.2.4 Morphisms and direct and inverse images

Let us now discuss morphisms in the context of direct and inverse images. As we shall see, it is in the context of morphisms that one gets the clearest understanding of the relationships between the direct and inverse image.

Morphisms and direct and inverse images of sets

We begin by considering presheaves, sheaves, and étalé spaces of sets. We begin with presheaf morphisms.

1.2.33 Definition (Direct and inverse images of presheaf morphisms (set version)) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of sets over $\mathcal{S}$, let $\mathscr{G}$ and $\mathscr{H}$ be presheaves of sets over $\mathcal{T}$, let $\phi = (\phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}_{\mathcal{S}}}$ be a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, and let $\psi = (\psi_{\mathcal{V}})_{\mathcal{V} \in \mathscr{O}_{\mathcal{T}}}$ be a presheaf morphism from $\mathscr{G}$ to $\mathscr{H}$.

(i) The ***direct image*** of ϕ is the presheaf morphism $\Phi_{\mathrm{pre},*}\phi$ from $\Phi_{\mathrm{pre},*}\mathscr{E}$ to $\Phi_{\mathrm{pre},*}\mathscr{F}$ given by

$$(\Phi_{\mathrm{pre},*}\phi)_{\mathcal{V}}(f) = \phi_{\Phi^{-1}(\mathcal{V})}(f) \in \mathscr{F}(\Phi^{-1}(\mathcal{V})) = \Phi_{\mathrm{pre},*}\mathscr{E}(\mathcal{V}),$$

for $f \in \Phi_{\mathrm{pre},*}\mathscr{E}(\mathcal{V}) = E(\Phi^{-1}(\mathcal{V}))$ and $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$.

(ii) The ***inverse image*** of ψ is the presheaf morphism $\Phi^{-1}_{\mathrm{pre}}\psi$ from $\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$ to $\Phi^{-1}_{\mathrm{pre}}\mathscr{H}$ given by

$$(\Phi^{-1}_{\mathrm{pre}}\psi)_{\mathcal{U}}[(g, \mathcal{V})]_{\Phi(\mathcal{U})} = [\psi_{\mathcal{V}}(g), \mathcal{V})]_{\Phi(\mathcal{U})},$$

for $g \in \mathscr{G}(\mathcal{V})$ and where $\Phi(\mathcal{U}) \subseteq \mathcal{V}$. •

The usual sorts of arguments may be applied to show that the inverse image of a morphism is well-defined, in that it is independent of representative of germ.

The extension to sheaves takes the expected form.

1.2.34 Definition (Direct and inverse images of sheaf morphisms (set version)) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of sets over $\mathcal{S}$, let $\mathscr{G}$ and $\mathscr{H}$ be sheaves of sets over $\mathcal{T}$, let $\phi = (\phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}_{\mathcal{S}}}$ be a sheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, and let $\psi = (\psi_{\mathcal{V}})_{\mathcal{V} \in \mathscr{O}_{\mathcal{T}}}$ be a sheaf morphism from $\mathscr{G}$ to $\mathscr{H}$.

(i) The ***direct image*** of ϕ is the sheaf morphism $\Phi_*\phi = \mathrm{Ps}(\mathrm{Et}(\Phi_{\mathrm{pre},*}\phi))$ from $\Phi_*\mathscr{E}$ to $\Phi_*\mathscr{F}$.

(ii) The ***inverse image*** of ψ is the presheaf morphism $\Phi^{-1}\Phi = \mathrm{Ps}(\mathrm{Et}(\Phi^{-1}_{\mathrm{pre}}\psi))$ from $\Phi^{-1}\mathscr{G}$ to $\Phi^{-1}\mathscr{H}$. •

We may also define the inverse and direct image for étalé morphisms.

1.2.35 Definition (Direct and inverse images of étalé morphisms (set version)) Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{S}$ and $\mathscr{T}$ be étalé spaces of sets over $\mathcal{S}$, let $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of sets over $\mathcal{T}$, and let $\phi\colon \mathscr{S} \to \mathscr{T}$ and $\psi\colon \mathscr{U} \to \mathscr{V}$ be étalé morphisms.

(i) The ***direct image*** of ϕ is the étalé morphism $\Phi_*\phi\colon \Phi_*\mathscr{S} \to \Phi_*\mathscr{T}$ given by $\Phi_*\phi = \mathrm{Et}(\Phi_{\mathrm{pre},*}\mathrm{Ps}(\phi))$.

(ii) The ***inverse image*** of ψ is the étalé morphism $\Phi^{-1}\psi\colon \Phi^{-1}\mathscr{U} \to \Phi^{-1}\mathscr{V}$ given by $\Phi^{-1}\psi(x, \sigma) = (x, \psi(\sigma))$. •

Of course, one should verify that the direct and inverse images of étalé morphisms are étalé morphisms. For the direct image, this is clear. We shall prove this for the inverse image in Proposition 1.2.40 below when we prove that étalé morphisms of étalé spaces of rings are continuous.

To further elucidate the notions of direct and inverse image, and to understand the relationship between them, we have the following result.

1.2.36 Theorem (Adjoint relationship between direct and inverse image (presheaf set version)) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of sets over $\mathcal{S}$ and $\mathscr{G}$ and $\mathscr{H}$ be presheaves of sets over $\mathcal{T}$. Then there exists a bijection $\Phi_{\mathscr{E},\mathscr{G}}$ between $\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}; \mathscr{E})$ and $\mathrm{Mor}(\mathscr{G}; \Phi_{\mathrm{pre},*}\mathscr{E})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{H}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{H}}"] \arrow[d, "\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\phi;\mathscr{E})"'] & \mathrm{Mor}(\mathscr{H}; \Phi_{\mathrm{pre},*}\mathscr{E}) \arrow[d, "\mathrm{Mor}(\phi;\Phi_{\mathrm{pre},*}\mathscr{E})"] \\
\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{G}}"'] & \mathrm{Mor}(\mathscr{G}; \Phi_{\mathrm{pre},*}\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{G}}"] \arrow[d, "\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G};\psi)"'] & \mathrm{Mor}(\mathscr{G}; \Phi_{\mathrm{pre},*}\mathscr{E}) \arrow[d, "\mathrm{Mor}(\mathscr{G};\Phi_{\mathrm{pre},*}\psi)"] \\
\mathrm{Mor}(\Phi^{-1}_{\mathrm{pre}}\mathscr{G}; \mathscr{F}) \arrow[r, "\Phi_{\mathscr{F},\mathscr{G}}"'] & \mathrm{Mor}(\mathscr{G}; \Phi_{\mathrm{pre},*}\mathscr{F})
\end{tikzcd}$$

commute for any morphisms ϕ from $\mathscr{G}$ to $\mathscr{H}$ and ψ from $\mathscr{E}$ to $\mathscr{F}$, and where we recall Construction 1.1.62.

Proof We recall from Proposition 1.2.7 the morphisms

$$j_{\mathscr{E}}\colon \Phi^{-1}_{\mathrm{pre}}\Phi_{\mathrm{pre},*}\mathscr{E} \to \mathscr{E}, \quad i_{\mathscr{G}}\colon \mathscr{G} \to \Phi_{\mathrm{pre},*}\Phi^{-1}_{\mathrm{pre}}\mathscr{G}.$$

Let $\alpha = (\alpha_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}_{\mathcal{S}}}$ be a morphism from $\Phi^{-1}_{\mathrm{pre}}\mathscr{G}$ to $\mathscr{E}$. We then have that $\Phi_{\mathrm{pre},*}\alpha \circ i_{\mathscr{G}}$ is a presheaf morphism from $\mathscr{G}$ to $\Phi_{\mathrm{pre},*}\mathscr{E}$. Thus we define $\Phi_{\mathscr{E},\mathscr{G}}(\alpha) = \Phi_{\mathrm{pre},*}\alpha \circ i_{\mathscr{G}}$. To verify that $\Phi_{\mathscr{E},\mathscr{G}}$ is a bijection, we demonstrate an inverse. Let $\beta = (\beta_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ be a morphism from $\mathscr{G}$ to $\Phi_{\mathrm{pre},*}\mathscr{E}$. We then define $\Psi_{\mathscr{E},\mathscr{G}}(\beta) = j_{\mathscr{E}} \circ \Phi^{-1}_{\mathrm{pre}}\beta$, and claim that $\Psi_{\mathscr{E},\mathscr{G}}$ is the inverse of $\Phi_{\mathscr{E},\mathscr{G}}$. We have

$$(\Phi_{\mathrm{pre},*}\alpha)_{\mathcal{V}}([(t, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))}) = \alpha_{\Phi^{-1}(\mathcal{V})}([(t, \mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))})$$

and

$$(\Phi_{\mathrm{pre}}^{-1}\beta)_{\mathcal{U}}([(t,\mathcal{V})]_{\Phi(\mathcal{U})}) = [(\beta_{\mathcal{V}}(t),\mathcal{V})]_{\Phi(\mathcal{U})}.$$

Thus we have

$$(\Phi_{\mathscr{E},\mathscr{G}}(\alpha))_{\mathcal{V}}(t) = \alpha_{\Phi^{-1}(\mathcal{V})}([(t,\mathcal{V})]_{\Phi(\Phi^{-1}(\mathcal{V}))})$$

and

$$(\Psi_{\mathscr{E},\mathscr{G}}(\beta))_{\mathcal{U}}([(t,\mathcal{V})]_{\Phi(\mathcal{U})}) = r_{\Phi^{-1}(\mathcal{V}),\mathcal{U}} \circ \beta_{\mathcal{V}}(t). \tag{1.8}$$

With these formulae, we readily verify that

$$\Psi_{\mathscr{E},\mathscr{G}} \circ \Phi_{\mathscr{E},\mathscr{G}}(\alpha) = \alpha, \qquad \Phi_{\mathscr{E},\mathscr{G}} \circ \Psi_{\mathscr{E},\mathscr{G}}(\alpha) = \beta,$$

as desired.

To complete the proof by showing that the diagrams in the statement of the theorem commute, we shall write down the formulae needed, and leave the then direct verifications to the reader. If $\alpha = (\alpha_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}_{\mathcal{S}}}$ is a morphism from $\Phi_{\mathrm{pre}}^{-1}\mathscr{H}$ to $\mathscr{E}$, then $\mathrm{Mor}(\Phi_{\mathrm{pre}}^{-1}\phi;\mathscr{E})(\alpha)$ is the morphism $\alpha \circ \Phi_{\mathrm{pre}}^{-1}\phi$, i.e., the morphism induced by the presheaf morphism

$$\Phi_{\mathrm{pre}}^{-1}\mathscr{G}(\mathcal{U}) \ni [(t,\mathcal{V})]_{\Phi(\mathcal{U})} \mapsto \alpha_{\mathcal{U}}([(\phi_{\mathcal{V}}(t),\mathcal{V})]_{\Phi_{\mathcal{U}}}) \in \mathscr{E}(\mathcal{U}).$$

If $\beta = (\beta_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ is a morphism from $\mathscr{H}$ to $\Phi_{\mathrm{pre},*}\mathscr{E}$, then $\mathrm{Mor}(\phi,\Phi_{\mathrm{pre},*}\mathscr{E})$ is the morphism $\beta\circ\phi$, i.e., the morphism induced by the presheaf morphism

$$\mathscr{G}(\mathcal{V}) \ni t \mapsto \beta_{\mathcal{V}} \circ \phi_{\mathcal{V}}(t) \in \Phi_{\mathrm{pre},*}\mathscr{E}(\mathcal{V}).$$

With these formulae, we can verify the first diagram in the statement of the theorem. In like manner, if $\alpha = (\alpha_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}_{\mathcal{S}}}$ is a morphism from $\Phi_{\mathrm{pre}}^{-1}\mathscr{G}$ to $\mathscr{E}$, then $\mathrm{Mor}(\Phi_{\mathrm{pre}}^{-1}\mathscr{G},\psi)$ is the morphism $\psi\circ\alpha$, i.e., the morphism induced by the presheaf morphism

$$\Phi_{\mathrm{pre}}^{-1}\mathscr{G}(\mathcal{U}) \ni [(t,\mathcal{V})]_{\Phi(\mathcal{U})} \mapsto \psi_{\mathcal{U}} \circ \alpha_{\mathcal{U}}([(t,\mathcal{V})]_{\Phi(\mathcal{U})}) \in \mathscr{F}(\mathcal{U}).$$

If $\beta = (\beta_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ is a morphism from $\mathscr{G}$ to $\Phi_{\mathrm{pre},*}\mathscr{E}$, then $\mathrm{Mor}(\mathscr{G},\Phi_{\mathrm{pre},*}\psi)$ is the morphism $\Phi_{\mathrm{pre},*}\psi\circ\beta$, i.e., the morphism induced by the presheaf morphism

$$\mathscr{G}(\mathcal{V}) \ni t \mapsto \phi_{\Phi^{-1}(\mathcal{V})}(\beta_{\mathcal{V}}(t)) \in \Phi_{\mathrm{pre},*}\mathscr{F}(\mathcal{V}).$$

One can use these formulae to verify that the second diagram in the statement of the theorem commutes. ■

The result also has an analogue with sheaves.

1.2.37 Theorem (Adjoint relationship between direct and inverse image (sheaf set version)) *Let $(\mathcal{S},\mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T},\mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$, and let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of sets over $\mathcal{S}$ and $\mathscr{G}$ and $\mathscr{H}$ be sheaves of sets over $\mathcal{T}$. Then there exists a bijection $\Phi_{\mathscr{E},\mathscr{G}}$ between $\mathrm{Mor}(\Phi^{-1}\mathscr{G};\mathscr{E})$ and $\mathrm{Mor}(\mathscr{G};\Phi_*\mathscr{E})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Mor}(\Phi^{-1}\mathscr{H};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{H}}"] \arrow[d, "\mathrm{Mor}(\Phi^{-1}\phi;\mathscr{E})"'] & \mathrm{Mor}(\mathscr{H};\Phi_*\mathscr{E}) \arrow[d, "\mathrm{Mor}(\phi;\Phi_*\mathscr{E})"] \\
\mathrm{Mor}(\Phi^{-1}\mathscr{G};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{G}}"'] & \mathrm{Mor}(\mathscr{G};\Phi_*\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Mor}(\Phi^{-1}\mathscr{G};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{G}}"] \arrow[d, "\mathrm{Mor}(\Phi^{-1}\mathscr{G};\psi)"'] & \mathrm{Mor}(\mathscr{G};\Phi_*\mathscr{E}) \arrow[d, "\mathrm{Mor}(\mathscr{G};\Phi_*\psi)"] \\ \mathrm{Mor}(\Phi^{-1}\mathscr{G};\mathscr{F}) \arrow[r, "\Phi_{\mathscr{F},\mathscr{G}}"'] & \mathrm{Mor}(\mathscr{G};\Phi_*\mathscr{F}) \end{tikzcd}$$

commute for any morphisms ϕ from $\mathscr{G}$ to $\mathscr{H}$ and ψ from $\mathscr{E}$ to $\mathscr{F}$, and where we recall Construction 1.1.62.

Proof The result follows in the same manner as Theorem 1.2.36, now using Proposition 1.2.20 and the definition of the direct and inverse images of sheaves, rather than presheaves. ■

Morphisms and direct and inverse images of rings

Now we turn to direct and inverse images of rings, again beginning with presheaf morphisms. As presheaves of rings are presheaves of sets, we can define the direct and inverse images of morphisms of presheaves of rings as in Definition 1.2.33, but we should be sure that these preserve the ring structure.

1.2.38 Proposition (Direct and inverse images of presheaf morphisms (ring version)) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{T},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ be a continuous map, let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of rings over $\mathcal{S}$, let $\mathscr{A}$ and $\mathscr{B}$ be presheaves of rings over $\mathcal{T}$, let $\phi = (\phi_\mathcal{U})_{\mathcal{U}\in\mathscr{O}_\mathcal{S}}$ be a presheaf morphism of rings from $\mathscr{R}$ to $\mathscr{S}$, and let $\psi = (\psi_\mathcal{V})_{\mathcal{V}\in\mathscr{O}_\mathcal{T}}$ be a presheaf morphism of rings from $\mathscr{A}$ to $\mathscr{B}$. Then $\Phi_{\mathrm{pre},*}\phi$ and $\Phi^{-1}_{\mathrm{pre}}\psi$ are morphisms of presheaves of rings.*

Proof The task of verifying the preservation of the ring operations is elementary, and we leave it to the reader. ■

A similar result holds for sheaves of modules.

1.2.39 Proposition (Direct and inverse images of sheaf morphisms (ring version)) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{T},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ be a continuous map, let $\mathscr{R}$ and $\mathscr{S}$ be sheaves of rings over $\mathcal{S}$, let $\mathscr{A}$ and $\mathscr{B}$ be sheaves of rings over $\mathcal{T}$, let $\phi = (\phi_\mathcal{U})_{\mathcal{U}\in\mathscr{O}_\mathcal{S}}$ be a sheaf morphism of rings from $\mathscr{R}$ to $\mathscr{S}$, and let $\psi = (\psi_\mathcal{V})_{\mathcal{V}\in\mathscr{O}_\mathcal{T}}$ be a sheaf morphism of rings from $\mathscr{A}$ to $\mathscr{B}$. Then $\Phi_*\phi$ and $\Phi^{-1}\psi$ are morphisms of sheaves of rings.*

Proof As with the preceding result, the task of verifying the preservation of the ring operations is elementary, and we leave it to the reader. ■

We may also define the inverse and direct image for étalé morphisms.

1.2.40 Proposition (Direct and inverse images of étalé morphisms (ring version)) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{T},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ be a continuous map, let $\mathscr{R}$ and $\mathscr{S}$ be étalé spaces of rings over $\mathcal{S}$, let $\mathscr{A}$ and $\mathscr{B}$ be étalé spaces of rings over $\mathcal{T}$, and let $\phi\colon \mathscr{R} \to \mathscr{S}$ and $\psi\colon \mathscr{A} \to \mathscr{B}$ be étalé morphisms. Then $\Phi_*\phi$ and $\Phi^{-1}\psi$ are morphisms of étalé spaces of rings.*

Proof For the direct image, note that $\mathrm{Ps}(\phi)$ is a morphism of presheaves of rings by Proposition 1.1.68. Then $\Phi_*(\phi)$ is a morphism of presheaves of rings by Proposition 1.2.38, and so $\Phi_*\phi$ is a morphism of étalé spaces of rings by Proposition 1.1.67. For the inverse image, the fact that $\Phi^{-1}\psi$ is a homomorphism of rings on stalks because ψ has this property, and by definition. We must show that $\Phi^{-1}\psi$ is continuous. For this, let $(x,\alpha)\in\Phi^{-1}\mathscr{A}$ and let $\mathcal{U}$ be a neighbourhood of x in $\mathcal{S}$. Let $\mathcal{V}$ be a neighbourhood of $\Phi(x)$ and let $a\in\Gamma(\mathcal{V};\mathscr{B})$ be such that $a(\Phi(x))=\alpha$. Suppose that $\mathcal{U}\subseteq\Phi^{-1}(\mathcal{V})$. Then

$$(\Phi^{-1}(\mathcal{V})\times s(\mathcal{V}))\cap\pi^{-1}(\mathcal{U})$$

is a neighbourhood of (x,α) which projects homeomorphically onto $\mathcal{U}$, where π is the étalé projection for $\mathscr{A}$; see the proof of Proposition 1.2.27. Since ψ is an étalé morphism, $\psi\circ s\in\Gamma(\mathcal{V};\mathscr{B})$. We then conclude that

$$(\Phi^{-1}(\mathcal{V})\times\psi\circ s(\mathcal{V}))\cap\pi^{-1}(\mathcal{U})$$

is a neighbourhood of $(x,\psi(\alpha))$ which projects homeomorphically onto $\mathcal{U}$. By definition of $\Phi^{-1}\psi$ this means that

$$\Phi^{-1}\psi((\Phi^{-1}(\mathcal{V})\times s(\mathcal{V}))\cap\pi^{-1}(\mathcal{U})),$$

implying that $\Phi^{-1}\psi$ is an open map, and so an étalé morphism by Proposition 1.1.60. ■

To further elucidate the notions of direct and inverse image, and to understand the relationship between them, we have the following result.

1.2.41 Theorem (Adjoint relationship between direct and inverse image (presheaf ring version)) *Let $(\mathcal{S},\mathscr{O}_\mathcal{S})$ and $(\mathcal{T},\mathscr{O}_\mathcal{T})$ be topological spaces, let $\Phi\in\mathrm{C}^0(\mathcal{S};\mathcal{T})$, and let $\mathscr{R}$ and $\mathscr{S}$ be presheaves of rings over $\mathcal{S}$ and $\mathscr{A}$ and $\mathscr{B}$ be presheaves of rings over $\mathcal{T}$. Then there exists a bijection $\Phi_{\mathscr{R},\mathscr{A}}$ between $\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{A};\mathscr{R})$ and $\mathrm{Hom}(\mathscr{A};\Phi_{\mathrm{pre},*}\mathscr{R})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{B};\mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{B}}"] \arrow[d, "\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\phi;\mathscr{R})"'] & \mathrm{Hom}(\mathscr{B};\Phi_{\mathrm{pre},*}\mathscr{R}) \arrow[d, "\mathrm{Hom}(\phi;\Phi_{\mathrm{pre},*}\mathscr{R})"] \\
\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{A};\mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{A}}"'] & \mathrm{Hom}(\mathscr{A};\Phi_{\mathrm{pre},*}\mathscr{R})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{A};\mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{A}}"] \arrow[d, "\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{A};\psi)"'] & \mathrm{Hom}(\mathscr{A};\Phi_{\mathrm{pre},*}\mathscr{R}) \arrow[d, "\mathrm{Hom}(\mathscr{A};\Phi_{\mathrm{pre},*}\psi)"] \\
\mathrm{Hom}(\Phi^{-1}_{\mathrm{pre}}\mathscr{A};\mathscr{S}) \arrow[r, "\Phi_{\mathscr{S},\mathscr{A}}"'] & \mathrm{Hom}(\mathscr{A};\Phi_{\mathrm{pre},*}\mathscr{S})
\end{tikzcd}$$

commute for any morphisms ϕ from $\mathscr{A}$ to $\mathscr{B}$ and ψ from $\mathscr{R}$ to $\mathscr{S}$, and where we recall Construction 1.1.70.

Proof The result follows in the same manner as Theorem 1.2.36, now using Proposition 1.2.11 and noting that all morphisms are morphisms of presheaves of rings. ■

The result also has an analogue with sheaves.

1.2.42 Theorem (Adjoint relationship between direct and inverse image (sheaf ring version)) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$, and let $\mathscr{R}$ and $\mathscr{S}$ be sheaves of rings over $\mathcal{S}$ and $\mathscr{A}$ and $\mathscr{B}$ be sheaves of sets over $\mathcal{T}$. Then there exists a bijection $\Phi_{\mathscr{R},\mathscr{A}}$ between $\mathrm{Hom}(\Phi^{-1}\mathscr{A}; \mathscr{R})$ and $\mathrm{Hom}(\mathscr{A}; \Phi_*\mathscr{R})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}(\Phi^{-1}\mathscr{B}; \mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{B}}"] \arrow[d, "\mathrm{Hom}(\Phi^{-1}\phi;\mathscr{R})"'] & \mathrm{Hom}(\mathscr{B}; \Phi_*\mathscr{R}) \arrow[d, "\mathrm{Hom}(\phi;\Phi_*\mathscr{R})"] \\
\mathrm{Hom}(\Phi^{-1}\mathscr{A}; \mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{A}}"'] & \mathrm{Hom}(\mathscr{A}; \Phi_*\mathscr{R})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}(\Phi^{-1}\mathscr{A}; \mathscr{R}) \arrow[r, "\Phi_{\mathscr{R},\mathscr{A}}"] \arrow[d, "\mathrm{Hom}(\Phi^{-1}\mathscr{A};\psi)"'] & \mathrm{Hom}(\mathscr{A}; \Phi_*\mathscr{R}) \arrow[d, "\mathrm{Hom}(\mathscr{A};\Phi_*\psi)"] \\
\mathrm{Hom}(\Phi^{-1}\mathscr{A}; \mathscr{S}) \arrow[r, "\Phi_{\mathscr{S},\mathscr{A}}"'] & \mathrm{Hom}(\mathscr{A}; \Phi_*\mathscr{S})
\end{tikzcd}$$

commute for any morphisms ϕ from $\mathscr{A}$ to $\mathscr{B}$ and ψ from $\mathscr{R}$ to $\mathscr{S}$, and where we recall Construction 1.1.70.

Proof The result follows in the same manner as Theorem 1.2.37, now using Proposition 1.2.22 and noting that all morphisms are morphisms of sheaves of rings. ■

Morphisms and direct and inverse images of modules

Let us now consider presheaves, sheaves, and étalé spaces of modules. Her we shall encounter some little subtleties with which we will have to be a little careful. Let us begin with stating how direct and inverse images morphisms of sets respect the module structure.

1.2.43 Proposition (Direct and inverse images of presheaf morphisms (module version)) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{R}$ a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules, let $\mathscr{A}$ be a presheaf of rings over $\mathcal{T}$, let $\mathscr{C}$ and $\mathscr{D}$ be presheaves of $\mathscr{A}$-modules, let $\phi = (\phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}_{\mathcal{S}}}$ be a presheaf morphism of $\mathscr{R}$-modules from $\mathscr{E}$ to $\mathscr{F}$, and let $\psi = (\psi_{\mathcal{V}})_{\mathcal{V} \in \mathscr{O}_{\mathcal{T}}}$ be a presheaf morphism of $\mathscr{A}$-modules from $\mathscr{C}$ to $\mathscr{D}$. Then $\Phi_{\mathrm{pre},*}\phi$ and $\Phi^{-1}_{\mathrm{pre}}\psi$ are morphisms of presheaves of $\Phi_{\mathrm{pre},*}\mathscr{R}$- and $\Phi^{-1}_{\mathrm{pre}}\mathscr{A}$-modules, respectively.*

Proof The task of verifying the preservation of the module operations is elementary, and we leave it to the reader. ■

A similar result holds for sheaves of modules.

1.2.44 Proposition (Direct and inverse images of sheaf morphisms (module version)) *Let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$ be a continuous map, let $\mathscr{R}$ a sheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\mathscr{R}$-modules, let $\mathscr{A}$ be a sheaf of rings over $\mathcal{T}$, let $\mathscr{C}$ and $\mathscr{D}$ be sheaves of $\mathscr{A}$-modules, let $\phi = (\phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}_{\mathcal{S}}}$ be a sheaf morphism of $\mathscr{R}$-modules* *from* $\mathscr{E}$ *to* $\mathscr{F}$*, and let* $\psi = (\psi_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ *be a sheaf morphism of* $\mathscr{A}$*-modules from* $\mathscr{C}$ *to* $\mathscr{D}$*. Then* $\Phi_*\phi$ *and* $\Phi^{-1}\psi$ *are morphisms of sheaves of* $\Phi_*\mathscr{R}$*- and* $\Phi^{-1}\mathscr{A}$*-modules, respectively.*

Proof As with the preceding result, the task of verifying the preservation of the ring operations is elementary, and we leave it to the reader. ■

We may also define the inverse and direct image for étalé morphisms.

1.2.45 Proposition (Direct and inverse images of étalé morphisms (module version)) *Let* $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ *and* $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ *be topological spaces, let* $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ *be a continuous map, let* $\mathscr{R}$ *be an étalé of rings over* $\mathcal{S}$*, let* $\mathscr{U}$ *and* $\mathscr{V}$ *be étalé spaces of* $\mathscr{R}$*-modules, let* $\mathscr{A}$ *be an étalé spaces of rings over* $\mathcal{T}$*, let* $\mathscr{M}$ *and* $\mathscr{N}$ *be étalé spaces of* $\mathscr{A}$*-modules, and let* $\phi\colon \mathscr{U} \to \mathscr{V}$ *and* $\psi\colon \mathscr{M} \to \mathscr{N}$ *be étalé morphisms of* $\mathscr{R}$*- and* $\mathscr{A}$*-modules, respectively. Then* $\Phi_*\phi$ *and* $\Phi^{-1}\psi$ *are morphisms of étalé spaces of* $\Phi_*\mathscr{R}$*- and* $\Phi^{-1}\mathscr{A}$*-modules, respectively.*

Proof For the direct image, note that $\mathrm{Ps}(\phi)$ is a morphism of presheaves of $\mathrm{Ps}(\mathscr{R})$-modules by Proposition 1.1.76. Then $\Phi_*(\phi)$ is a morphism of presheaves of $\Phi_*\mathrm{Ps}(\mathscr{R})$-modules by Proposition 1.2.43, and so $\Phi_*\phi$ is a morphism of étalé spaces of $\mathrm{Et}(\Phi_*\mathrm{Ps}(\mathscr{R}))$-modules by Proposition 1.1.75, noting that $\mathrm{Et}(\Phi_*\mathrm{Ps}(\mathscr{R})) \simeq \mathscr{R}$ by Proposition 1.2.30. For the inverse image, the argument goes like that in Proposition 1.2.40. ■

Next we investigate the relationships between sets of morphisms related to the direct and inverse image.

1.2.46 Theorem (Adjoint relationship between direct and inverse image (presheaf module version)) *Let* $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ *and* $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ *be topological spaces, let* $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ *be a continuous map, let* $\mathscr{A}$ *be a presheaf of rings over* $\mathcal{T}$*, let* $\mathscr{E}$ *and* $\mathscr{F}$ *be presheaves of* $\Phi^{-1}\mathscr{A}$*-modules, let* $\mathscr{C}$ *and* $\mathscr{D}$ *be presheaves of* $\mathscr{A}$*-modules, let* $\phi = (\phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}_{\mathcal{S}}}$ *be a presheaf morphism of* $\Phi^{-1}_{\mathrm{pre}}\mathscr{A}$*-modules from* $\mathscr{E}$ *to* $\mathscr{F}$*, and let* $\psi = (\psi_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ *be a presheaf morphism of* $\mathscr{A}$*-modules from* $\mathscr{C}$ *to* $\mathscr{D}$*. Then there exists a bijection* $\Phi_{\mathscr{E},\mathscr{M}}$ *between* $\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{M};\mathscr{E})$ *and* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})$ *for which the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{N};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{N}}"] \arrow[d, "\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\phi;\mathscr{E})"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{N};\Phi_{\mathrm{pre},*}\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(\phi;\Phi_{\mathrm{pre},*}\mathscr{E})"] \\
\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{M};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{M};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"] \arrow[d, "\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{M};\psi)"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\psi)"] \\
\mathrm{Hom}_{\Phi^{-1}_{\mathrm{pre}}\mathscr{A}}(\Phi^{-1}_{\mathrm{pre}}\mathscr{M};\mathscr{F}) \arrow[r, "\Phi_{\mathscr{F},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{F})
\end{tikzcd}$$

commute for any morphisms ϕ *of the* $\mathscr{A}$*-modules* $\mathscr{M}$ *and* $\mathscr{N}$*, and* ψ *of the* $\Phi^{-1}\mathscr{A}$*-modules* $\mathscr{E}$ *and* $\mathscr{F}$*, and where we recall Construction 1.1.78. (Note that we regard* $\mathscr{E}$ *as an* $\mathscr{A}$*-module by Proposition 1.2.12(ii).)*

Proof In Theorem 1.2.36 we constructed a bijection from $\mathrm{Mor}(\Phi_{\mathrm{pre}}^{-1}\mathscr{M};\mathscr{E})$ to $\mathrm{Mor}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})$, and so here we must prove that this bijection maps $\mathrm{Hom}_{\Phi_{\mathrm{pre}}^{-1}\mathscr{A}}(\Phi_{\mathrm{pre}}^{-1}\mathscr{M};\mathscr{E})$ bijectively onto $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})$. We recall that this bijection is given by mapping the morphism $\alpha \in \mathrm{Hom}_{\Phi_{\mathrm{pre}}^{-1}\mathscr{A}}(\Phi_{\mathrm{pre}}^{-1}\mathscr{M};\mathscr{E})$ to $\Phi_{\mathrm{pre},*}\alpha \circ i_{\mathscr{M}}$. From Proposition 1.2.15 we know that $i_{\mathscr{M}}$ is a morphism of presheaves of $\mathscr{A}$-modules. It follows from Proposition 1.2.43, therefore, that $\Phi_{\mathrm{pre},*}\alpha \circ i_{\mathscr{M}}$ is a morphism of presheaves of $\Phi^{-1}\mathscr{A}$-modules. We should also verify that the inverse of $\Phi_{\mathscr{E},\mathscr{M}}$ maps $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})$ to $\mathrm{Hom}_{\Phi_{\mathrm{pre}}^{-1}\mathscr{A}}(\Phi_{\mathrm{pre}}^{-1}\mathscr{M};\mathscr{E})$. In Theorem 1.2.36 we showed that the inverse $\Psi_{\mathscr{E},\mathscr{M}}$ of $\Phi_{\mathscr{E},\mathscr{M}}$ is defined by $\beta \mapsto j_{\mathscr{E}} \circ \Phi_{\mathrm{pre}}^{-1}\beta$ for $\beta \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_{\mathrm{pre},*}\mathscr{E})$. By that same result, we have that $\Psi_{\mathscr{E},\mathscr{M}}(\beta) \in \mathrm{Mor}(\Phi_{\mathrm{pre}}^{-1}\mathscr{M};\mathscr{E})$. Since $j_{\mathscr{E}}$ and $\Phi_{\mathrm{pre}}^{-1}\beta$ are morphisms of presheaves of Abelian groups, so is $\Psi_{\mathscr{E},\mathscr{M}}(\beta)$. It remains to show that $\Psi_{\mathscr{E},\mathscr{M}}(\beta)$ respects the $\Phi_{\mathrm{pre}}^{-1}\mathscr{A}$-module structure. Let $\mathcal{U} \in \mathscr{O}_{\mathcal{S}}$ and let $\mathcal{V} \in \mathscr{O}_{\mathcal{T}}$ be such that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$. Let $[(g,\mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi_{\mathrm{pre}}^{-1}\mathscr{A}(\mathcal{U})$ and $[(t,\mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi_{\mathrm{pre}}^{-1}\mathscr{M}(\mathcal{U})$. We compute, using (1.8),

$$\begin{aligned}(\Psi_{\mathscr{E},\mathscr{M}}(\beta))_{\mathcal{U}}([(g,\mathcal{V})]_{\Phi(\mathcal{U})} \cdot [(t,\mathcal{V})]_{\Phi(\mathcal{U})}) &= r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}} \circ \beta_{\mathcal{V}}(g \cdot t) \\ &= r^{\Phi_{\mathrm{pre}}^{-1}\mathscr{A}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}}([(g,\mathcal{V})]_{\Phi(\mathcal{U})}) \cdot r^{\mathscr{E}}_{\Phi^{-1}(\mathcal{V}),\mathcal{U}} \circ \beta_{\mathcal{V}}(t) \\ &= [(g,\mathcal{V})]_{\Phi(\mathcal{U})} \cdot (\Psi_{\mathscr{E},\mathscr{M}}(\beta))_{\mathcal{U}}([(t,\mathcal{V})]_{\Phi(\mathcal{U})}),\end{aligned}$$

using the definition of the restriction map for $\Phi_{\mathrm{pre}}^{-1}\mathscr{A}$ in the last step. This shows that, indeed, $\Psi_{\mathscr{E},\mathscr{M}}(\beta)$ is a morphism of $\Phi_{\mathrm{pre}}^{-1}\mathscr{A}$-modules, as desired. ■

The result also has an analogue with sheaves.

1.2.47 Theorem (Adjoint relationship between direct and inverse image (sheaf module version)) *Let $(\mathcal{S},\mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T},\mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ be a continuous map, let $\mathscr{A}$ be a sheaf of rings over $\mathcal{T}$, let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\Phi^{-1}\mathscr{A}$-modules, let $\mathscr{C}$ and $\mathscr{D}$ be sheaves of $\mathscr{A}$-modules, let $\phi = (\phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}_{\mathcal{S}}}$ be a sheaf morphism of* Phi$^{-1}\mathscr{A}$*-modules from $\mathscr{E}$ to $\mathscr{F}$, and let $\psi = (\psi_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ be a sheaf morphism of $\mathscr{A}$-modules from $\mathscr{C}$ to $\mathscr{D}$. Then there exists a bijection $\Phi_{\mathscr{E},\mathscr{M}}$ between $\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{M};\mathscr{E})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\mathscr{E})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{N};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{N}}"] \arrow[d, "\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\phi;\mathscr{E})"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{N};\Phi_*\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(\phi;\Phi_*\mathscr{E})"] \\
\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{M};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{M};\mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"] \arrow[d, "\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{M};\psi)"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\psi)"] \\
\mathrm{Hom}_{\Phi^{-1}\mathscr{A}}(\Phi^{-1}\mathscr{M};\mathscr{F}) \arrow[r, "\Phi_{\mathscr{F},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\mathscr{F})
\end{tikzcd}$$

commute for any morphisms ϕ of the $\mathscr{A}$-modules $\mathscr{M}$ and $\mathscr{N}$, and ψ of the $\Phi_{\mathrm{pre}}^{-1}\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$, and where we recall Construction 1.1.78. (Note that we regard $\mathscr{E}$ as an $\mathscr{A}$-module by Proposition 1.2.12(ii).)

Proof The result follows in the same manner as Theorem 1.2.46, now using Theorem 1.2.37 and Proposition 1.2.24. ■

Of course, the preceding two results are not quite what one wants. Ideally, one would like to have sheaves of rings $\mathscr{R}$ on $\mathcal{S}$ and $\mathscr{A}$ on $\mathcal{T}$ with $\mathscr{E}$ a sheaf of $\mathscr{R}$-modules and $\mathscr{M}$ a sheaf of $\mathscr{A}$-modules, and then have a bijection between $\mathrm{Hom}_{\mathscr{R}}(\Phi^{-1}\mathscr{M};\mathscr{E})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M};\Phi_*\mathscr{E})$. However, this does not even make sense since $\Phi^{-1}\mathscr{M}$ is not an $\mathscr{R}$-module and $\Phi_*\mathscr{E}$ is not an $\mathscr{A}$-module (it is, actually, but only after a few minutes thought). We shall rectify this right now in the context of ringed spaces.

1.2.5 Ringed spaces and morphisms of ringed spaces

In this section we investigate further the structure associated with morphisms between sheaves of rings and sheaves of modules over topological spaces. The ring setting can be used to describe generalisations of manifolds, and we pursue this facet of ringed spaces in Chapter 6. The module setting is to be thought of as a generalisation of sections of vector bundles as modules over ringed spaces with vector bundle mappings inducing mappings on sections. This setting itself can then be generalised to provide a notion of a vector bundle over more general spaces.

We begin with the notion of a ringed space, which is to be thought of as the specification of a space of functions on a topological space.

1.2.48 Definition (Ringed space, morphism of ringed space) A ***ringed space*** is a pair $(\mathcal{S},\mathscr{R}_{\mathcal{S}})$ where $(\mathcal{S},\mathscr{O})$ is a topological space and where $\mathscr{R}_{\mathcal{S}}$ is a sheaf of rings over $\mathcal{S}$. If $(\mathcal{S},\mathscr{R}_{\mathcal{S}})$ and $(\mathcal{T},\mathscr{R}_{\mathcal{T}})$ are ringed spaces, a ***morphism*** from $(\mathcal{S},\mathscr{R}_{\mathcal{S}})$ to $(\mathcal{T},\mathscr{R}_{\mathcal{T}})$ is a pair $(\Phi,\Phi^\sharp)$ where $\Phi\in \mathrm{C}^0(\mathcal{S};\mathcal{T})$ and where $\Phi^\sharp$ is a morphism from $\mathscr{R}_{\mathcal{T}}$ to $\Phi_*\mathscr{R}_{\mathcal{S}}$. •

Note that, according to Theorem 1.2.42, corresponding to $\Phi^\sharp\in\mathrm{Hom}(\mathscr{R}_{\mathcal{T}};\Phi_*\mathscr{R}_{\mathcal{S}})$, a uniquely defined morphism $\Phi^\flat\in\mathrm{Hom}(\Phi^{-1}\mathscr{R}_{\mathcal{T}};\mathscr{R}_{\mathcal{S}})$. Indeed, one can equivalently define a morphism of ringed spaces by prescribing such a morphism $\Phi^\flat$.

The typical example one should have in mind is the following.

1.2.49 Example (Morphism between spaces of continuous functions) Let $(\mathcal{S},\mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T},\mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, and let $\mathscr{C}^0(\mathcal{S};\mathbb{F})$ and $\mathscr{C}^0(\mathcal{T};\mathbb{F})$ be the sheaves of continuous $\mathbb{F}$-valued functions. If $\Phi\in\mathrm{C}^0(\mathcal{S};\mathcal{T})$ then we can define $\Phi^\sharp=(\Phi^\sharp_{\mathcal{V}})_{\mathcal{V}\in\mathscr{O}_{\mathcal{T}}}$ by

$$\Phi^\sharp_{\mathcal{V}}(g)=g\circ(\Phi|\Phi^{-1}(\mathcal{V})).$$

Thus, if $g\in\mathrm{C}^0(\mathcal{V};\mathbb{F})$ then $\Phi^\sharp_{\mathcal{V}}(g)\in\mathrm{C}^0(\Phi^{-1}(\mathcal{V});\mathbb{F})=\Phi_*\mathscr{C}^0(\mathcal{S};\mathbb{F})(\mathcal{V})$. If $[(g,\mathcal{V})]_{\Phi(\mathcal{U})}\in\Phi^{-1}_{\mathrm{pre}}\mathscr{C}^0(\mathcal{T};\mathbb{F})(\mathcal{U})$ for $\mathcal{V}\in\mathscr{O}_{\mathcal{T}}$ and $\mathcal{U}\in\mathscr{O}_{\mathcal{S}}$ such that $\Phi(\mathcal{U})\subseteq\mathcal{V}$, then we have

$$\Phi^\flat_{\mathcal{U}}([(g,\mathcal{V})]_{\Phi(\mathcal{U})})=g\circ(\Phi|\mathcal{U}).$$

We can see that $\Phi^\sharp$ and $\Phi^\flat$ are “the same thing,” up to appropriate restriction. •

The preceding example can obviously be generalised to smooth, real analytic, or holomorphic mappings. Because, in these examples, the sheaf of rings is somehow fundamental to the structure of the space, the sheaf of rings $\mathscr{R}_{\mathcal{S}}$ for a ringed space $(\mathcal{S}, \mathscr{R}_{\mathcal{S}})$ is often called the ***structure sheaf***. This point of view of spaces is pursued with some vigour in Chapter 6.

Now let us talk about sheaves of modules on ringed spaces. To do this in such a manner as to preserve the structure sheaves in each case, we need to define appropriate module structures. The key to doing this are the following general algebraic constructions. Let R and S be commutative rings with unit, let A be an R-module, and let C be an S-module. Suppose we have a ring homomorphism $\phi\colon \mathsf{R} \to \mathsf{S}$. We can then define an R-module structure on C by using the existing module addition along with the R-module multiplication

$$r \cdot y \triangleq \phi(r) \cdot y, \qquad r \in \mathsf{R},\ y \in \mathsf{C}.$$

It is an elementary verification to show that C is an R-module with this multiplication. This is the ***restriction by ϕ*** of C to R, and we denote this R-module by C_{R}. Now note that, in a similar manner, S is an R-module with the existing addition and module multiplication defined by

$$r \cdot s \triangleq \phi(r) \cdot s, \qquad r \in \mathsf{R},\ s \in \mathsf{S}.$$

Thus we can form the tensor product $\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}$ which we regard as an S-module by

$$s' \cdot (s \otimes_{\mathsf{R}} x) = (s' \cdot s) \otimes x, \qquad s, s' \in \mathsf{S},\ x \in \mathsf{A}.$$

This S-module is called the ***change of base by ϕ*** of A to S. Associated with these constructions is the following result.

1.2.50 Lemma (A relationship between restriction and change of base) *Let R and S be commutative rings with unit, let A be an R-module, let C be an S-module, and suppose that we have a ring homomorphism $\phi\colon \mathsf{R} \to \mathsf{S}$. Then there exists a canonical bijection from $\mathrm{Hom}_{\mathsf{R}}(\mathsf{A}; \mathsf{C}_{\mathsf{R}})$ to $\mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}; \mathsf{C})$.*

Proof To $\alpha \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A}; \mathsf{C}_{\mathsf{R}})$ we associate $\alpha' \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}; \mathsf{C})$ by

$$\alpha'(s \otimes_{\mathsf{R}} x) = s \cdot \alpha(x), \qquad s \in \mathsf{S},\ x \in \mathsf{A}.$$

Since the map $(s, x) \mapsto s \cdot \alpha(x)$ is bilinear as a map of Abelian groups, it follows that α' is a well-defined map of the Abelian groups $\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}$ and C. Moreover, it is also clearly linear with respect to multiplication by elements of S, and so α' is indeed an element of $\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}$.

To show that the assignment $\alpha \mapsto \alpha'$ is a bijection, let us define an inverse. To $\beta \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}; \mathsf{C})$ we assign $\beta' \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A}; \mathsf{C}_{\mathsf{R}})$ by

$$\beta'(x) = \beta(1 \otimes_{\mathsf{R}} x), \qquad x \in \mathsf{A}.$$

We have

$$\beta'(r \cdot x) = \beta(1 \otimes_{\mathsf{R}} (r \cdot x)) = \beta((r \cdot 1) \otimes_{\mathsf{R}} x) = \beta(\phi(r) \otimes_{\mathsf{R}} x) = \phi(r)\beta(1 \otimes_{\mathsf{R}} x),$$

showing that β' is indeed an element of $\mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}})$.

To verify that the assignment $\beta \mapsto \beta'$ is the inverse of the assignment $\alpha \mapsto \alpha'$, let $\alpha \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}})$ and compute, with $\beta = \alpha'$,

$$\beta'(x) = \alpha'(1 \otimes_{\mathsf{R}} x) = \alpha(x).$$

Similarly, for $\beta \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A};\mathsf{C})$, take $\alpha = \beta'$ and compute

$$\alpha'(s \otimes_{\mathsf{R}} x) = s \cdot \beta'(x) = s \cdot \beta(1 \otimes_{\mathsf{R}} x) = \beta(s \otimes x),$$

as desired. ■

Restriction and change of base can also be applied to homomorphisms of modules. As above, let R and S be commutative rings with unit, let A and B be R-modules, and let C and D be S-modules. Suppose that we have a ring homomorphism $\phi\colon \mathsf{R} \to \mathsf{S}$, so defining the R-modules C_{R} and D_{R} and the S-modules $\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A}$ and $\mathsf{S} \otimes_{\mathsf{R}} \mathsf{B}$. Given homomorphisms $\sigma \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{B})$ and $\tau \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{C};\mathsf{D})$, we define homomorphisms $\overline{\sigma} \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A};\mathsf{S} \otimes_{\mathsf{R}} \mathsf{B})$ and $\underline{\tau} \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{C}_{\mathsf{R}};\mathsf{D}_{\mathsf{R}})$ by

$$\overline{\sigma}(s \otimes_{\mathsf{R}} x) = s \otimes_{\mathsf{R}} \sigma(x), \quad \underline{\tau}(x) = \tau(x).$$

One readily verifies that these are homomorphisms relative to the given module structures. These constructions with homomorphisms also interact well with the correspondence of Lemma 1.2.50.

1.2.51 Lemma (Restriction, change of base, and homomorphisms) *Let R and S be commutative rings with unit, let A and B be R-modules, and let C and D be S-modules. Suppose that we have a ring homomorphism $\phi\colon \mathsf{R} \to \mathsf{S}$ and homomorphisms $\sigma \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{B})$ and $\tau \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{C};\mathsf{D})$. Then the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{B};\mathsf{C}) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathsf{S}}(\overline{\sigma};\mathsf{C})"'] & \mathrm{Hom}_{\mathsf{R}}(\mathsf{B};\mathsf{C}_{\mathsf{R}}) \arrow[d, "\mathrm{Hom}_{\mathsf{R}}(\sigma;\mathsf{C}_{\mathsf{R}})"] \\
\mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A};\mathsf{C}) \arrow[r] & \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A};\mathsf{C}) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathsf{S}}(\mathsf{S}\otimes_{\mathsf{R}}\mathsf{A};\tau)"'] & \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}}) \arrow[d, "\mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\underline{\tau})"] \\
\mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{A};\mathsf{D}) \arrow[r] & \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{D}_{\mathsf{R}})
\end{tikzcd}$$

commute, where we recall (the appropriate variation of) Construction 1.1.78.

Proof For the first diagram, let $\beta \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S} \otimes_{\mathsf{R}} \mathsf{B};\mathsf{C})$ with $\beta' \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{B};\mathsf{C}_{\mathsf{R}})$ the homomorphism from Lemma 1.2.50. We have

$$\mathrm{Hom}_{\mathsf{R}}(\sigma;\mathsf{C}_{\mathsf{R}})(\beta') = \beta' \circ \sigma.$$

We also have

$$\mathrm{Hom}_{\mathsf{S}}(\overline{\sigma};\mathsf{C})(\beta) = \beta \circ \overline{\sigma}.$$

Let us denote by $(\beta\circ\overline{\sigma})' \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}})$ the homomorphism from Lemma 1.2.50. We now compute

$$\beta'\circ\sigma(x) = \beta(1\otimes_{\mathsf{R}}\sigma(x)).$$

We also have

$$(\beta\circ\overline{\sigma})'(x) = \beta\circ\overline{\sigma}(1\otimes_{\mathsf{R}} x) = \beta(1\otimes_{\mathsf{R}}\sigma(x)).$$

This gives the desired commutativity of the first diagram.

For the second diagram, let $\beta \in \mathrm{Hom}_{\mathsf{S}}(\mathsf{S}\otimes_{\mathsf{R}}\mathsf{A};\mathsf{C})$ and let $\beta' \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\mathsf{C}_{\mathsf{R}})$ be the homomorphism of Lemma 1.2.50. We have

$$\mathrm{Hom}_{\mathsf{S}}(\mathsf{S}\otimes_{\mathsf{R}}\mathsf{A};\tau) = \tau\circ\beta$$

and

$$\mathrm{Hom}_{\mathsf{R}}(\mathsf{A};\underline{\tau})(\beta') = \underline{\tau}\circ\beta'.$$

Let $(\tau\circ\beta)'$ be the homomorphism corresponding to Lemma 1.2.50. We compute

$$(\tau\circ\beta)'(x) = \tau\circ\beta(1\otimes_{\mathsf{R}} x)$$

and

$$\underline{\tau}\circ\beta'(x) = \underline{\tau}\circ\beta(1\otimes_{\mathsf{R}} s) = \tau\circ\beta(1\otimes_{\mathsf{R}} x),$$

as desired. ■

With these notions, we can make the following definition.

1.2.52 Definition (Direct and inverse image of sheaves by morphisms of ringed spaces) Let $(\mathcal{S},\mathscr{R}_{\mathcal{S}})$ and $(\mathcal{T},\mathscr{R}_{\mathcal{T}})$ be ringed spaces and let $(\Phi,\Phi^{\sharp})$ be a morphism of these ringed spaces. Let $\mathscr{E}$ be a sheaf of $\mathscr{R}_{\mathcal{S}}$-modules and $\mathscr{M}$ a sheaf of $\mathscr{R}_{\mathcal{T}}$-modules.

(i) The ***direct image*** of $\mathscr{E}$ by $(\Phi,\Phi^{\sharp})$ is the sheaf of sets $\Phi_*\mathscr{E}$ with the $\mathscr{R}_{\mathcal{T}}$-module structure obtained by restriction by the morphism $\Phi^{\sharp}\colon \mathscr{R}_{\mathcal{T}}\to\Phi_*\mathscr{R}_{\mathcal{S}}$. To be precise, the module structure is defined by

$$g\cdot s = \Phi^{\sharp}(s)\cdot g, \qquad g\in\mathscr{R}_{\mathcal{T}}(\mathcal{V}),\ s\in\mathscr{E}(\Phi^{-1}(\mathcal{V})).$$

(ii) The ***inverse image*** of $\mathscr{M}$ by $(\Phi,\Phi^{\sharp})$ is the sheaf $\Phi^*\mathscr{M}$ of $\mathscr{R}_{\mathcal{S}}$-modules obtained by change of base by the morphism from $\Phi^{\flat}\colon \Phi^{-1}\mathscr{R}_{\mathcal{T}}\to\mathscr{R}_{\mathcal{S}}$. To be precise

$$\Phi^*\mathscr{M}(\mathcal{U}) = \mathscr{R}_{\mathcal{S}}(\mathcal{U})\otimes_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}(\mathcal{U})}\Phi^{-1}\mathscr{M}(\mathcal{U}).$$ •

The definitions can be extended to morphisms.

1.2.53 Definition (Direct and inverse image of sheaf morphisms by morphisms of ringed spaces) Let $(\mathcal{S}, \mathscr{R}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{R}_{\mathcal{T}})$ be ringed spaces and let $(\Phi, \Phi^{\sharp})$ be a morphism of these ringed spaces. Let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\mathscr{R}_{\mathcal{S}}$-modules, let $\mathscr{M}$ and $\mathscr{N}$ be sheaves of $\mathscr{R}_{\mathcal{T}}$-modules, let ϕ be an $\mathscr{R}_{\mathcal{S}}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, and let ψ be an $\mathscr{R}_{\mathcal{T}}$-module morphism from $\mathscr{M}$ to $\mathscr{N}$. Keeping in mind the morphisms $\Phi^{\sharp}$ from $\mathscr{R}_{\mathcal{S}}$ to $\Phi_*\mathscr{R}_{\mathcal{T}}$ and $\Phi^{\flat}$ from $\Phi^{-1}\mathscr{R}_{\mathcal{T}}$ to $\mathscr{R}_{\mathcal{S}}$ and the constructions above with homomorphisms associated with restriction and change of base,

(i) $\Phi_*\phi$ is the $\mathscr{R}_{\mathcal{T}}$-module morphism $\underline{\Phi_*\phi}$ (abuse of notation alert) from $\Phi_*\mathscr{E}$ to $\Phi_*\mathscr{F}$ and

(ii) $\Phi^*\psi$ is the $\mathscr{R}_{\mathcal{S}}$-module morphism $\overline{\Phi^{-1}\psi}$ from $\Phi^*\mathscr{M}$ to $\Phi^*\mathscr{N}$. •

We can now prove the final useful characterisation of direct and inverse images.

1.2.54 Theorem (Adjoint relationship between direct and inverse image (ringed space version)) *Let $(\mathcal{S}, \mathscr{R}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{R}_{\mathcal{T}})$ be ringed spaces and let $(\Phi, \Phi^{\sharp})$ be a morphism of these ringed spaces. Let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\mathscr{R}_{\mathcal{S}}$-modules and let $\mathscr{M}$ and $\mathscr{N}$ be sheaves of $\mathscr{R}_{\mathcal{T}}$-modules. Then there exists a bijection $\Phi_{\mathscr{E},\mathscr{M}}$ between $\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \mathscr{E})$ and $\mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}; \Phi_*\mathscr{E})$ for which the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{N}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{N}}"] \arrow[d, "\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\phi; \mathscr{E})"'] & \mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{N}; \Phi_*\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\phi; \Phi_*\mathscr{E})"] \\
\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}; \Phi_*\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \mathscr{E}) \arrow[r, "\Phi_{\mathscr{E},\mathscr{M}}"] \arrow[d, "\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \psi)"'] & \mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}; \Phi_*\mathscr{E}) \arrow[d, "\mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}; \Phi_*\psi)"] \\
\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \mathscr{F}) \arrow[r, "\Phi_{\mathscr{F},\mathscr{M}}"'] & \mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}; \Phi_*\mathscr{F})
\end{tikzcd}$$

commute for any morphisms ϕ of the $\mathscr{R}_{\mathcal{T}}$-modules $\mathscr{M}$ and $\mathscr{N}$, and ψ of the $\mathscr{R}_{\mathcal{S}}$-modules $\mathscr{E}$ and $\mathscr{F}$, and where we recall Construction 1.1.78.

Proof By Lemma 1.2.50, keeping in mind the morphism $\Phi^{\flat}$ from $\Phi^{-1}\mathscr{R}_{\mathcal{T}}$ to $\mathscr{R}_{\mathcal{S}}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\Phi^*\mathscr{M}; \mathscr{E}) = \mathrm{Hom}_{\mathscr{R}_{\mathcal{S}}}(\mathscr{R}_{\mathcal{S}} \otimes_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}} \Phi^{-1}\mathscr{M}; \mathscr{E}) \simeq \mathrm{Hom}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}(\Phi^{-1}\mathscr{M}; \mathscr{E}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}),$$

where $\mathscr{E}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}$ is the sheaf defined by

$$\mathscr{E}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}(\mathcal{U}) = \mathscr{E}(\mathcal{U})_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}(\mathcal{U})},$$

i.e., it is the restriction by $\Phi^{\flat}$ of $\mathscr{E}$ to $\Phi^{-1}\mathscr{R}_{\mathcal{T}}$. Now we apply Theorem 1.2.47 to arrive at a canonical bijection

$$\mathrm{Hom}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}(\Phi^{-1}\mathscr{M}; \mathscr{E}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}) \simeq \mathrm{Hom}_{\mathscr{R}_{\mathcal{T}}}(\mathscr{M}, \Phi_*\mathscr{E}_{\Phi^{-1}\mathscr{R}_{\mathcal{T}}}).$$

Note that

$$\Phi_*\mathscr{E}_{\Phi^{-1}\mathscr{R}_\mathcal{T}}(\mathcal{V}) = \mathscr{E}_{\Phi^{-1}\mathscr{R}_\mathcal{T}}(\Phi^{-1}(\mathcal{V})) = \mathscr{E}(\Phi^{-1}(\mathcal{V}))_{\Phi^{-1}\mathscr{R}_\mathcal{T}(\Phi^{-1}(\mathcal{V}))},$$

from which we conclude that $\Phi_*\mathscr{E}_{\Phi^{-1}\mathscr{R}_\mathcal{T}}$ is the direct image of $\mathscr{E}$ by $(\Phi, \Phi^\sharp)$, and this gives the existence of the correspondence $\Phi_{\mathscr{E},\mathscr{M}}$.

The commuting of the diagrams in the statement of the theorem follows from Theorem 1.2.47 and Lemma 1.2.51. ■

Let us consider a fairly concrete instance of some of the rather abstract constructions in this section.

1.2.55 Example (Pull-back bundles and inverse images) Let $r \in \{\infty, \omega, \mathrm{hol}\}$. Let $\pi\colon \mathsf{F} \to \mathsf{N}$ be a C^r-vector bundle and let $\Phi \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$. We claim that there is an $\mathscr{C}^r_\mathsf{M}$-module isomorphism from $\mathscr{G}^r_{\Phi^*\mathsf{F}}$ to $\Phi^*\mathscr{G}^r_\mathsf{F}$, where $\Phi^*\mathsf{F}$ is the pull-back bundle (see Section GA1.4.3.6). Let $\mathcal{V} \subseteq \mathsf{N}$ be open and let $\mathcal{U} \subseteq \mathsf{M}$ be open with $\Phi(\mathcal{U}) \subseteq \mathcal{V}$. Let $\eta \in \mathscr{G}^r_\mathsf{F}(\mathcal{V})$ and note that $x \mapsto (x, \eta\circ\Phi(x))$ is a section of $\Phi^*\mathsf{F}$ over $\mathcal{U}$; let us denote this section by $\Phi^*\eta$. More or less as we saw in Proposition 1.2.27, the map

$$\Phi^{-1}\mathscr{G}^r_\mathsf{F}(\mathcal{U}) \ni [(\eta, \mathcal{V})]_{\Phi(\mathcal{U})} \mapsto \Phi^*\eta \in \mathscr{G}^r_{\Phi^*\mathsf{F}}(\mathcal{U})$$

defines a bijection. Let us see that it preserves the appropriate module structure. Let $[(g, \mathcal{V})]_{\Phi(\mathcal{U})} \in \Phi^{-1}\mathscr{C}^r_\mathsf{M}(\mathcal{U})$ and note that

$$\Phi^*(g\cdot\eta)(x) = (x, (g\circ\Phi(x))\cdot(\eta\circ\Phi(x))) = (g\circ\Phi(x))\cdot(\Phi^*\eta(x)), \qquad x \in \mathcal{U}.$$

Thus, noting the definition of $\Phi^\flat$ from Example 1.2.49, we have

$$\Phi^*(g\cdot\eta) = \Phi^\flat(g)\cdot\Phi^*\eta,$$

which is the desired linearity with respect to multiplication. •

1.3 Algebraic constructions with presheaves, sheaves, and étalé spaces

In the preceding sections we provided the basic constructions for presheaves and sheaves, but the constructions for the most part emphasised set-theoretic and topological properties. In this section we focus more on the algebraic constructions that are possible with sheaves. Specifically, we carefully study sheaves of modules. It is not uncommon to see much of what we talk about here presented in the context of sheaves of Abelian groups. One should keep in mind that sheaves of Abelian groups are sheaves of modules over the constant sheaf with values in $\mathbb{Z}$.

1.3.1 Kernel, image, etc., of presheaf morphisms

One can expect that it is possible to assign the usual algebraic constructions of kernels, images, quotients, etc., to morphisms of presheaves and étalé spaces. The story turns out to have some hidden dangers that one must carefully account for. In this section we work with presheaves of modules over a prescribed sheaf of rings.

1.3.1 Definition (Kernel, image, quotient, cokernel, coimage presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$.

(i) The ***kernel presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\ker_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \ker(\Phi_{\mathcal{U}}).$$

(ii) The ***image presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{image}(\Phi_{\mathcal{U}}).$$

(iii) If $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, the ***quotient presheaf*** of $\mathscr{F}$ by $\mathscr{E}$ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) = \mathscr{F}(\mathcal{U})/\mathscr{E}(\mathcal{U}).$$

(iv) The ***cokernel presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{coker}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{coker}(\Phi_{\mathcal{U}}) = \mathscr{F}(\mathcal{U})/\,\mathrm{image}(\Phi_{\mathcal{U}}).$$

(v) The ***coimage presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{coimage}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{coimage}(\Phi_{\mathcal{U}}) = \mathscr{E}(\mathcal{U})/\ker(\Phi_{\mathcal{U}}).$$

In all cases, the restriction maps are the obvious ones, induced by the restriction maps $r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}$ and $r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}$ for $\mathscr{E}$ and $\mathscr{F}$, respectively. Thus, for example, the restriction map for $\ker(\Phi)$ is

$$\ker_{\mathrm{pre}}(\Phi)(\mathcal{U}) \ni s \mapsto r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(s) \in \ker_{\mathrm{pre}}(\Phi)(\mathcal{V}),$$

the restriction map for $\mathrm{image}_{\mathrm{pre}}(\Phi)$ is

$$\mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{U}) \ni t \mapsto r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(t) \in \mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{V}),$$

and the restriction map for $\mathscr{F}/_{\mathrm{pre}}\mathscr{E}$ is

$$\mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) \ni s + \mathscr{E}(\mathcal{U}) \mapsto r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(s) + \mathscr{E}(\mathcal{V}) \in \mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{V}). \qquad \bullet$$

Using the properties of $\mathscr{R}$-module morphisms and subpresheaves, one readily verifies that the given definitions of the restrictions maps make sense.

Let us first see that the stalks of the presheaves just defined are what one expects.

1.3.2 Proposition (Stalks of algebraic constructions are algebraic constructions of stalks) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$. Then the following statements hold:*

(i) $\mathrm{ker}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{ker}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(ii) $\mathrm{image}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{image}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(iii) if $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, then $\mathrm{Et}(\mathscr{F}/_{\mathrm{pre}}\mathscr{E})_{\mathrm{x}} = \mathrm{Et}(\mathscr{F})_{\mathrm{x}}/\mathrm{Et}(\mathscr{E})_{\mathrm{x}}$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(iv) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{coker}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(v) $\mathrm{coimage}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{coimage}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*.*

Proof (i) Note that $\alpha \in \mathrm{ker}_{\mathrm{pre}}(\Phi)_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $s \in \mathrm{ker}(\Phi_{\mathcal{U}})$ such that $\alpha = r_{\mathcal{U},x}(s)$. Since $\mathrm{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$ we conclude that $\alpha \in \mathrm{ker}_{\mathrm{pre}}(\Phi)_x$ if and only if $\mathrm{Et}(\Phi)_x(\alpha) = 0$.

(ii) Note that $\beta \in \mathrm{image}_{\mathrm{pre}}(\Phi)_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $s \in \mathscr{E}(\mathcal{U})$ such that $\beta = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$. Let $\alpha = r_{\mathcal{U},x}(s)$. Since $\mathrm{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$ we conclude that $\beta \in \mathrm{image}_{\mathrm{pre}}(\Phi)_x$ if and only if $\beta \in \mathrm{image}(\mathrm{Et}(Phi)_x)$.

(iii) We have $\beta \in \mathrm{Et}(\mathscr{F}/_{\mathrm{pre}}\mathscr{E})_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $t \in \mathscr{F}(\mathcal{U})$ such that $\beta = r_{\mathcal{U},x}(t+\mathscr{E}(\mathcal{U}))$. Since the restriction maps are group homomorphisms, one directly verifies that

$$r_{\mathcal{U},x}(t + \mathscr{E}(\mathcal{U})) = r_{\mathcal{U},x}(t) + r_{\mathcal{U},x}(\mathscr{E}(\mathcal{U}))$$

and since $r_{\mathcal{U},x}(\mathscr{E}(\mathcal{U})) = \mathrm{Et}(\mathscr{E})_x$ (again, this is directly verified), this part of the result follows.

(iv) and (v) follow from the first three assertions. ■

As we are about to see, not all parts of the preceding definition are on an equal footing. In fact, what we shall see is that the kernel presheaf is pretty nicely behaved, while the other constructions need more care if one is to give them the interpretations one normally gives to these sorts of algebraic constructions.

1.3.2 The kernel, image, etc., of sheaf morphisms

While the constructions of the preceding section are natural and valid, they are only a starting point for talking about morphisms between sheaves of modules. The beginning of the rest of the story begins with the following nice property of the kernel presheaf.

1.3.3 Proposition (The kernel presheaf is often a sheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$. Then* $\mathrm{ker}_{\mathrm{pre}}(\Phi)$ *is a sheaf.*

Proof Let $\mathcal{U} \in \mathscr{O}$, let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$, let $s, t \in \mathrm{ker}_{\mathrm{pre}}(\Phi)(\mathcal{U})$, and suppose that $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(s) = r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(t)$ for every $a \in A$. Since $\mathscr{E}$ is a sheaf, $s = t$, and so $\mathrm{ker}_{\mathrm{pre}}(\Phi)$ is separated.

Next let $\mathcal{U} \in \mathscr{O}$, let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$, let $s_a \in \mathrm{ker}_{\mathrm{pre}}(\Phi)(\mathcal{U}_a)$, $a \in A$, satisfy $r^{\mathscr{E}}_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}) = r^{\mathscr{E}}_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_1}}(s_{a_2})$ for every $a \in A$. Since $\mathscr{E}$ is a sheaf, there exists $s \in \mathscr{E}(\mathcal{U})$ such that $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(s) = s_a$ for each $a \in A$. Moreover,

$$r^{\mathscr{F}}_{\mathcal{U},\mathcal{U}_a}(\Phi_{\mathcal{U}}(s)) = \Phi_{\mathcal{U}_a}(s_a) = 0,$$

and since $\mathscr{F}$ is separated we have $\Phi_{\mathcal{U}}(s) = 0$ and so $s \in \ker_{\mathrm{pre}}(\Phi)(\mathcal{U})$, as desired. ■

By example, let us illustrate that the image presheaf is not generally a sheaf, even when the domain and range are sheaves.

1.3.4 Examples (The image presheaf of a presheaf morphism may not be a sheaf)

1. Let $\mathcal{S} = \mathbb{S}^1$, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathscr{E} = \mathscr{C}^r_{\mathbb{S}^1}$ be the sheaf of functions of class C^r on $\mathbb{S}^1$, and let $\mathscr{F}$ be the presheaf of nowhere zero $\mathbb{C}$-valued functions of class C^r on $\mathbb{S}^1$. We consider both $\mathscr{E}$ and $\mathscr{F}$ to be presheaves of $\mathbb{C}$-vector spaces, with the group structure being addition in the former case and multiplication in the latter case. One may verify easily that $\mathscr{F}$ is also a sheaf. Let us consider the sheaf morphism exp from $\mathscr{E}$ to $\mathscr{F}$ defined by asking that
$$\exp_{\mathcal{U}}(f)(x, y) = \mathrm{e}^{2\pi \mathrm{i} f(x,y)}, \qquad (x, y) \in \mathcal{U}.$$
Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the open subsets covering $\mathbb{S}^1$ defined by
$$\mathcal{U}_1 = \{(x, y) \in \mathbb{S}^1 \mid y < \tfrac{1}{\sqrt{2}}\}, \qquad \mathcal{U}_2 = \{(x, y) \in \mathbb{S}^1 \mid y > -\tfrac{1}{\sqrt{2}}\}.$$
Let $f_1 \in \mathrm{C}^r(\mathcal{U}_1)$ be defined by asking that $f_1(x, y)$ be the angle of the point (x, y) from the positive x-axis; thus $f_1(x, y) \in (-\frac{5\pi}{4}, \frac{\pi}{4})$. In like manner, let $f_2 \in \mathrm{C}^r(\mathcal{U}_2)$ be the function defined by asking that $f_2(x, y)$ be the angle of the point (x, y) measured from the positive x-axis; thus $f_2(x, y) \in (-\frac{\pi}{4}, \frac{5\pi}{4})$. Note that $\exp_{\mathcal{U}_1}(f_1)$ and $\exp_{\mathcal{U}_2}(f_2)$ agree on $\mathcal{U}_1 \cap \mathcal{U}_2$. However, there exists no $f \in \mathrm{C}^r(\mathbb{S}^1)$ such that $\exp_{\mathbb{S}^1}(f)$ agrees with $\exp_{\mathcal{U}_1}(f_1)$ on $\mathcal{U}_1$ and with $\exp_{\mathcal{U}_2}(f_2)$ on $\mathcal{U}_2$. Thus $\mathrm{image}_{\mathrm{pre}}(\exp)$ is not a sheaf.
2. Here we consider one-dimensional complex projective space $\mathbb{CP}^1$. Let us define a holomorphic vector field X on $\mathbb{CP}^1$ by writing its local representatives in the charts $(\mathcal{U}_+, \psi_+)$ and $(\mathcal{U}_-, \psi_-)$ introduced in Example GA1.4.5.14. Thus we ask that the local representative of X in $(\mathcal{U}_+, \phi_+)$ be $z_+ \mapsto (z_+, z_+)$ and in $(\mathcal{U}_-, \phi_-)$ to be $z_- \mapsto -z_-$. According to the transition functions for $\mathsf{T}^{1,0}\mathbb{CP}^1$ from Example GA1.4.5.14, this gives a well-defined vector field X on $\mathbb{CP}^1$. We now define a morphism m_X of the $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$-modules $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ and $\mathscr{G}^{\mathrm{hol}}_{\mathsf{T}^{1,0}\mathbb{CP}^1}$ by $m_{X,\mathcal{U}}(f) = f \cdot (X|\mathcal{U})$, i.e., multiplication of X by f. We claim that $\mathrm{image}_{\mathrm{pre}}(m_X)$ is not a sheaf. To see this, let $f_+ \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}_+)$ and $f_- \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}_-)$ be defined by $f_+(z_+) = z_+$ and $f_-(z_-) = -z_-^{-1}$, making a slight abuse of notation and writing points in $\mathcal{U}_+$ and $\mathcal{U}_-$ as z_+ and z_-, using the chart maps ψ_+ and ψ_-. Then, making similar abuses of notation, we have
$$m_{X,\mathcal{U}_+}(f_+)(z_+) = (z_+, z_+^2), \quad m_{X,\mathcal{U}_-}(f_-)(z_-) = (z_-, 1).$$
One can verify from Example GA1.4.5.14 that $m_{X,\mathcal{U}_+}(f_+)$ and $m_{X,\mathcal{U}_-}(f_-)$ agree on $\mathcal{U}_+ \cap \mathcal{U}_-$. However, there cannot be a function f on $\mathcal{U}_+ \cup \mathcal{U}_- = \mathbb{CP}^1$ for which $m_{X,\mathbb{CP}^1}(f)$ agrees with both $m_{X,\mathcal{U}_+}(f_+)$ and $m_{X,\mathcal{U}_-}(f_-)$ upon restriction. Indeed, if $f \in \mathrm{C}^{\mathrm{hol}}(\mathbb{CP}^1)$ then f is constant by Corollary GA1.4.2.11. This means that $m_{X,\mathbb{CP}^1}(f)$ must be a constant multiple of X, which is not the case for either $m_{X,\mathcal{U}_+}(f_+)$ or $m_{X,\mathcal{U}_-}(f_-)$. •

Other algebraic constructions on sheaves also fail to give rise to sheaves. Let us show this for quotients.

1.3.5 Example (Quotients of sheaves may not be a sheaf) We consider the holomorphic manifold $\mathbb{CP}^1 \simeq \mathbb{S}^2$ with z_+ the north pole and z_- the south pole. Let $S = \{z_+, z_-\}$ and let $\mathscr{I}_S$ be the subpresheaf of $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ defined by

$$\mathscr{I}_S(\mathcal{U}) = \{f \in \mathrm{C}^{\mathrm{hol}}(\mathcal{U}) \mid f(z) = 0 \text{ for all } z \in S\}.$$

One readily verifies that $\mathscr{I}_S$ is a subsheaf. We claim that the presheaf $\mathscr{Q}_S \triangleq \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}/_{\mathrm{pre}}\mathscr{I}_S$ is not a sheaf. First, let $\mathcal{U} \subseteq \mathbb{CP}^1$ be an open set for which $\mathcal{U} \cap S = \emptyset$. Then $\mathscr{I}_S(\mathcal{U}) = \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ and so $\mathscr{Q}_S(\mathcal{U}) = 0$. Now let $(\mathcal{U}_+, \phi_+)$ and $(\mathcal{U}_-, \phi_-)$ be the charts for $\mathbb{CP}^1$ from Example GA1.4.3.5–**??**. Let $f_+ \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}_+)$ and $f_- \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}_-)$ with $f_+ + \mathscr{I}_S(\mathcal{U}_+)$ and $f_- + \mathscr{I}_S(\mathcal{U}_-)$ the representatives in $\mathscr{Q}_S(\mathcal{U}_+)$ and $\mathscr{Q}_S(\mathcal{U}_-)$, respectively. Note that

$$r_{\mathcal{U}_+, \mathcal{U}_+ \cap \mathcal{U}_-}(f_+ + \mathscr{I}_S(\mathcal{U}_+)) = r_{\mathcal{U}_+, \mathcal{U}_+ \cap \mathcal{U}_-}(f_- + \mathscr{I}_S(\mathcal{U}_-)); \tag{1.9}$$

indeed, both are zero since $\mathscr{Q}_S(\mathcal{U}_+ \cap \mathcal{U}_-) = 0$. However, since $\mathbb{CP}^1 = \mathcal{U}_+ \cup \mathcal{U}_-$, the only functions in $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}_+ \cup \mathcal{U}_-)$ are constant by Corollary GA1.4.2.11. Thus $\mathscr{I}_S(\mathbb{CP}^1) = 0$ and so $\mathscr{Q}_S(\mathbb{CP}^1) = \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathbb{CP}^1)$. This implies that there exists $f + \mathscr{I}_S(\mathbb{CP}^1) \in \mathscr{Q}_S(\mathbb{CP}^1)$ such that

$$r_{\mathbb{CP}^1, \mathcal{U}_+}(f + \mathscr{I}_S(\mathbb{CP}^1)) = f_+ + \mathscr{I}_S(\mathbb{CP}^1), \quad r_{\mathbb{CP}^1, \mathcal{U}_-}(f + \mathscr{I}_S(\mathbb{CP}^1)) = f_- + \mathscr{I}_S(\mathbb{CP}^1)$$

if and only if $f_+(z_+) = f_-(z_-)$. However, there is no such restriction on f_+ or f_- to satisfy (1.9), and so such $f + \mathscr{I}_S(\mathbb{CP}^1) \in \mathscr{Q}_S(\mathbb{CP}^1)$ need not exist. Thus $\mathscr{Q}_S$ is indeed not a sheaf. •

The examples show that, in order to achieve a useful theory, we need to modify our definitions to make sure we are dealing with objects where the stalks capture the behaviour of the presheaf. The following definition illustrates how to do this.

1.3.6 Definition (Kernel, image, quotient, cokernel, coimage for sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$.

(i) The ***kernel*** of Φ is the sheaf $\ker(\Phi) = \mathrm{Ps}(\mathrm{Et}(\ker_{\mathrm{pre}}(\Phi)))$.

(ii) The ***image*** of Φ is the sheaf $\mathrm{image}(\Phi) = \mathrm{Ps}(\mathrm{Et}(\mathrm{image}_{\mathrm{pre}}(\Phi)))$.

(iii) If $\mathscr{E}$ is a subsheaf of $\mathscr{F}$, the ***quotient*** of $\mathscr{F}$ by $\mathscr{E}$ is the sheaf $\mathscr{F}/\mathscr{E} = \mathrm{Ps}(\mathrm{Et}(\mathscr{F}/_{\mathrm{pre}}\mathscr{E}))$.

(iv) The ***cokernel*** of Φ is the sheaf $\mathrm{coker}(\Phi) = \mathrm{Ps}(\mathrm{Et}(\mathrm{coker}_{\mathrm{pre}}(\Phi)))$.

(v) The ***coimage*** of Φ is the sheaf $\mathrm{coimage}(\Phi) = \mathrm{Ps}(\mathrm{Et}(\mathrm{coimage}_{\mathrm{pre}}(\Phi)))$. •

Let us look at how these constructions manifest themselves in our preceding examples of presheaf morphisms whose images are not a sheaves.

1.3.7 Examples (Sheafification of image presheaves)

1. We resume Example 1.3.4–1. The morphism, recall was defined for $r \in \{\infty, \omega\}$ from $\mathscr{C}^r(\mathbb{S}^1;\mathbb{C};)$ to the sheaf $\mathscr{F}$ of nowhere zero $\mathbb{C}$-valued functions of class C^r on $\mathbb{S}^1$. Explicitly, if $\mathcal{U} \subseteq \mathbb{S}^1$ is open, then the morphism is

$$\exp_{\mathcal{U}}(f)(x,y) = \mathrm{e}^{2\pi\mathrm{i}f(x,y)}, \qquad (x,y) \in \mathcal{U}.$$

In Example 4.1.4–1 we shall show in a more general setting that the morphism exp is surjective on stalks. Thus the image sheaf image(exp), i.e., the sheafification of $\mathrm{image}_{\mathrm{pre}}(\exp)$, is equal to $\mathscr{F}$.

2. Here we continue with Example 1.3.4–2, where we considered the morphism m_X from $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ to $\mathscr{G}^{\mathrm{hol}}_{\mathsf{T}^{1,0}\mathbb{CP}^1}$ given by multiplication of a fixed vector field X by a function. The vector field X vanishes at the north pole z_+ and south pole z_-. One easily sees, following the arguments from Example 1.3.4–2, that the image sheaf $\mathrm{image}(m_X)$, i.e., the sheafification of $\mathrm{image}_{\mathrm{pre}}(m_X)$, is the subsheaf of $\mathscr{G}^{\mathrm{hol}}_{\mathsf{T}^{1,0}\mathbb{CP}^1}$ defined by

$$\mathrm{image}(m_X)_z = \begin{cases} \mathscr{G}^{\mathrm{hol}}_{z,\mathbb{CP}^1}, & z \notin \{z_+, z_-\}, \\ \mathfrak{m}_z, & z \in \{z_+, z_-\}, \end{cases}$$

where $\mathfrak{m}_z$ is the unique maximal ideal consisting of germs of functions at z that vanish at z. •

Let us do the same for our quotient example.

1.3.8 Example (Sheafification of quotient presheaves) Let us carry on with Example 1.3.5. Note that since $\mathscr{I}_{S,z_+} = \mathfrak{m}_{z_+}$ and $\mathscr{I}_{S,z_-} = \mathfrak{m}_{z_-}$, with $\mathfrak{m}_z$ denoting the unique maximal ideal in $\mathscr{C}^{\mathrm{hol}}_{z,\mathbb{CP}^1}$ consisting of germs of functions vanishing at z, as in Theorem GA1.2.3.1. Thus $\mathscr{Q}_{S,z_+} \simeq \mathbb{C}$ and $\mathscr{Q}_{S,z_-} \simeq \mathbb{C}$, the isomorphisms being given by

$$[f_+]_{z_+} + \mathscr{I}_{S,z_+} \mapsto f_+(z_+), \quad [f_-]_{z_-} + \mathscr{I}_{S,z_-} \mapsto f_-(z_-),$$

respectively. Thus, with $\mathscr{Q}^+_S$ denoting the sheafification,

$$\mathscr{Q}^+_S(\mathcal{U}) = \begin{cases} 0, & \mathcal{U} \cap S = \emptyset, \\ \mathbb{C}, & \mathcal{U} \cap S = \{z_+\} \text{ or } \mathcal{U} \cap S = \{z_-\}, \\ \mathbb{C} \oplus \mathbb{C}, & S \subseteq \mathcal{U}. \end{cases}$$

This is some sort of skyscraper sheaf. •

Note that, if Φ is a morphism of sheaves of $\mathscr{R}$-modules, $\ker(\Phi)$ and $\ker_{\mathrm{pre}}(\Phi)$ are in natural correspondence by Propositions 1.3.3 and 1.1.88. We think of $\ker(\Phi)$ as the sheaf of sections of the étalé space of $\ker_{\mathrm{pre}}(\Phi)$ in order to be consistent with the other algebraic constructions. While these algebraic constructions involve a distracting use of sheafification, it is important to note that, at the stalk level, the constructions have the hoped for properties.

1.3.9 Proposition (Agreement of stalks of algebraic constructions) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements hold:*

(i) $\ker_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \ker(\Phi)_{\mathrm{x}}$;

(ii) $\mathrm{image}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \mathrm{image}(\Phi)_{\mathrm{x}}$;

(iii) if $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, then $(\mathscr{F}/_{\mathrm{pre}}\mathscr{E})_{\mathrm{x}} \simeq (\mathscr{F}/\mathscr{E})_{\mathrm{x}}$;

(iv) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \mathrm{coker}(\Phi)_{\mathrm{x}}$;

(v) $\mathrm{coimage}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \mathrm{coimage}(\Phi)_{\mathrm{x}}$.

(In all cases, "$\simeq$" stands for the isomorphism from a presheaf to its sheafification from part (iii) of Proposition 1.1.106.)

Proof All of these assertions follow from Proposition 1.1.106(iii) and Proposition 1.3.2. ■

While the image presheaf $\mathrm{image}_{\mathrm{pre}}(\Phi)$ of a morphism of sheaves of $\mathscr{R}$-modules $\mathscr{E}$ and $\mathscr{F}$ is not necessarily a sheaf, it is still a subpresheaf of $\mathscr{F}$. One might expect that this attribute could be lost upon sheafification, but thankfully it is not.

1.3.10 Proposition (The image sheaf is a subsheaf of the codomain) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then there exists a natural injective $\mathscr{R}$-module morphism from* $\mathrm{image}(\Phi)$ *into $\mathscr{F}$.*

Proof By Proposition 1.1.107, since we have an inclusion $i_\Phi = (i_\Phi(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ of $\mathrm{image}_{\mathrm{pre}}(\Phi)$ in $\mathscr{F}$, we have a natural induced morphism $i_\Phi^+ = (i_{\Phi,\mathcal{U}}^+)_{\mathcal{U}\in\mathscr{O}}$ of sheaves from $\mathrm{image}(\Phi)$ into $\mathscr{F}$. We need only show that this induced morphism is injective. To do this, we recall the notation from the proof of Proposition 1.1.107. Thus we have $i_{\Phi,\mathcal{U}}^+ = \beta_{\mathcal{U}}^{-1} \circ i'^{+}_{\Phi,\mathcal{U}}$, where $\beta_{\mathcal{U}}$ is as in Proposition 1.1.88 (for the sheaf $\mathscr{F}$) and where

$$i'^{+}_{\Phi,\mathcal{U}}([s]_x) = [i_{\Phi,\mathcal{U}}(s)]_x.$$

Since $\mathscr{F}$ is a sheaf, $\beta_{\mathcal{U}}$ is an isomorphism, and so is injective. So we need only show that $i'^{+}_{\Phi,\mathcal{U}}$ is injective. Suppose that $[i_{\Phi,\mathcal{U}}(s)]_x = 0$. Thus there exists a neighbourhood $\mathcal{V}$ of x such that

$$r_{\mathcal{U},\mathcal{V}}(i_{\Phi,\mathcal{U}}(s)) = i_{\Phi,\mathcal{V}}(r_{\mathcal{U},\mathcal{V}}(s)) = 0,$$

using the commuting diagram (1.2). Injectivity of $i_{\Phi,\mathcal{V}}$ gives $r_{\mathcal{U},\mathcal{V}}(s) = 0$ and so $[s]_x = 0$, which gives the desired injectivity of $i'^{+}_{\Phi,\mathcal{U}}$. ■

1.3.3 Kernel, image, etc., of étalé morphisms

We now turn our attention to algebraic constructions associated to étalé morphisms of étalé spaces of modules.

1.3.11 Definition (Kernel, image, quotient, cokernel, coimage for étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$, and let $\Phi\colon \mathscr{U} \to \mathscr{V}$ be an étalé morphism of $\mathscr{A}$-modules.

(i) The ***kernel*** of Φ is the étalé subspace $\ker(\Phi)$ of $\mathscr{U}$ given by $\ker(\Phi)_x = \ker(\Phi|\mathscr{U}_x)$.

(ii) The ***image*** of Φ is the étalé subspace $\mathrm{image}(\Phi)$ of $\mathscr{V}$ given by $\mathrm{image}(\Phi)_x = \mathrm{image}(\Phi|\mathscr{V}_x)$.

(iii) If $\mathscr{U}$ is a étalé subspace of $\mathscr{V}$, the ***quotient*** of $\mathscr{V}$ by $\mathscr{U}$ is the étalé space $\mathscr{V}/\mathscr{U}$ over $\mathcal{S}$ given by $(\mathscr{V}/\mathscr{U})_x = \mathscr{V}_x/\mathscr{U}_x$, with the quotient topology induced by the projection from $\mathscr{V}$ to $\mathscr{V}/\mathscr{U}$.

(iv) The ***cokernel*** of Φ is the étalé space $\mathrm{coker}(\Phi) = \mathscr{V}/\mathrm{image}(\Phi)$.

(v) The ***coimage*** of Φ is the étalé space $\mathrm{coimage}(\Phi) = \mathscr{U}/\ker(\Phi)$. •

Let us verify that the above étalé spaces are indeed étalé spaces.

1.3.12 Proposition (Kernels, images, and quotients of étalé spaces are étalé spaces) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{A}$ is an étalé space of rings over $\mathcal{S}$, if $\mathscr{U}$ and $\mathscr{V}$ be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$, and if $\Phi\colon \mathscr{U} \to \mathscr{V}$ is an étalé morphism, then the following statements hold:*

(i) $\ker(\Phi)$ *is an étalé subspace of* $\mathscr{U}$*;*

(ii) $\mathrm{image}(\Phi)$ *is an étalé subspace of* $\mathscr{V}$*;*

(iii) if $\mathscr{U}$ is a étalé subspace of $\mathscr{V}$, then $\mathscr{V}/\mathscr{U}$ is an étalé space;

(iv) $\mathrm{coker}(\Phi)$ *is an étalé space;*

(v) $\mathrm{coimage}(\Phi)$ *is an étalé space.*

Proof (i) Let $\zeta\colon \mathcal{S} \to \mathscr{U}$ be the zero section. Thus $\zeta(x)$ is the zero element in $\mathscr{U}_x$. We claim that ζ is continuous. Let $\mathcal{O}$ be a neighbourhood of $\zeta(x)$. Since the group operation is continuous and since $\zeta(x) + \zeta(x) = \zeta(x)$, there exist neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\zeta(x)$ such that

$$\{\alpha + \beta \mid (\alpha, \beta) \in \mathcal{O}_1 \times \mathcal{O}_2 \cap \mathscr{U} \times_{\mathcal{S}} \mathscr{U}\} \subseteq \mathcal{O}.$$

Let $\mathcal{P} = \mathcal{O} \cap \mathcal{O}_2 \cap \mathcal{O}_2$, noting that $\mathcal{P}$ is a neighbourhood of $\zeta(x)$. By shrinking $\mathcal{O}_1$ and $\mathcal{O}_2$ if necessary, we may suppose that $\pi|\mathcal{P}$ is a homeomorphism onto $\pi(\mathcal{P})$. Let $\alpha \in \mathcal{P}$ and let $y = \pi(\alpha)$. Note that $\pi(\alpha + \alpha) = \pi(\alpha) = y$, and since $\pi|\mathcal{P}$ is a homeomorphism we have $\alpha + \alpha = \alpha$, giving $\alpha = \zeta(y)$. Thus $\mathcal{P} = \zeta(\pi(\mathcal{P}))$, showing that $\zeta(\mathcal{P}) \subseteq \mathcal{O}$, giving the desired continuity of ζ. Since sections are local homeomorphisms (they are locally inverses of the étalé projection), it follows that $\mathrm{image}(\zeta)$ is open. Since Φ is continuous, $\ker(\Phi) = \Phi^{-1}(\mathrm{image}(\zeta))$ is open and by Proposition 1.1.96 it follows that $\ker(\Phi)$ is a étalé subspace.

(ii) This follows from Propositions 1.1.60 and 1.1.96.

(iii) Let us denote by $\pi_{\mathscr{U}}\colon \mathscr{V} \to \mathscr{V}/\mathscr{U}$ the mapping which, when restricted to fibres, is the canonical projection and let us denote by $\rho_{\mathscr{U}}\colon \mathscr{V}/\mathscr{U} \to \mathcal{S}$ the canonical projection. We must show that $\rho_{\mathscr{U}}$ is a local homeomorphism. Since $\rho_{\mathscr{U}} = \rho \circ \pi_{\mathscr{U}}$ and since compositions of local homeomorphisms are local homeomorphisms (this is directly verified), it suffices to show that $\pi_{\mathscr{U}}$ is a local homeomorphism. Clearly $\pi_{\mathscr{U}}$ is continuous by the definition of the quotient topology. We claim that $\pi_{\mathscr{U}}$ is also open. Let $\mathcal{B}(\mathcal{U},\tau)$ be a basic neighbourhood in $\mathscr{V}$. Note that

$$\pi_{\mathscr{U}}(\mathcal{B}(\mathcal{U},\tau)) = \mathcal{B}(\mathcal{U},\tau+\mathscr{U}),$$

where $\tau+\mathscr{U}$ means the section (not necessarily continuous, since we are still trying to understand this) of $\mathscr{V}/\mathscr{U}$ over $\mathcal{U}$ given by $(\tau+\mathscr{U})(x) = \tau(x)+\mathscr{U}_x$. Thus a typical point in $\pi_{\mathscr{U}}^{-1}(\pi_{\mathscr{U}}(\mathcal{B}(\mathcal{U},\tau)))$ has the form $\tau(x)+\sigma(x)$ for $x\in\mathcal{U}$ and where σ is a section of $\mathscr{U}$ defined on some neighbourhood $\mathcal{V}\subseteq\mathcal{U}$ of x. Thus $\mathcal{B}(\mathcal{V},\tau|\mathcal{V}+\sigma)$ is a basic neighbourhood of $\tau(x)+\sigma(x)$ in $\pi_{\mathscr{U}}^{-1}(\pi_{\mathscr{U}}(\mathcal{B}(\mathcal{U},\tau)))$ showing that the latter set is open, and hence $\pi_{\mathscr{U}}(\mathcal{B}(\mathcal{U},\tau))$ is open in the quotient topology. This shows that basic open sets in $\mathscr{V}$ are mapped to open sets in $\mathscr{V}/\mathscr{U}$, showing that $\pi_{\mathscr{U}}$ is open, as claimed. To complete this part of the proof it suffices to show that $\pi_{\mathscr{U}}|\mathcal{B}(\mathcal{U},\tau)$ is a bijection. For injectivity, suppose that $\tau(x)+\mathscr{U}_x = \tau(y)+\mathscr{U}_y$ for $x,y\in\mathcal{U}$. Clearly this implies that $x=y$, giving injectivity. Surjectivity is equally clear.

Parts (iv) and (v) follow from the first three parts. ■

1.3.4 Monomorphisms and epimorphisms

In this section we consider the relationships between kernels and injectivity, and cokernels and surjectivity. Let us begin with presheaves.

1.3.13 Proposition (Characterisations of the kernel presheaf) *If $(\mathcal{S},\mathscr{O})$ is a topological space, if $\mathscr{R}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are presheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\ker_{\mathrm{pre}}(\Phi)(\mathcal{U})$ *is the zero section of* $\mathscr{E}(\mathcal{U})$ *for each* $\mathcal{U}\in\mathscr{O}$*;*

(ii) $\Phi_{\mathcal{U}}$ *is injective for each* $\mathcal{U}\in\mathscr{O}$*.*

Furthermore, the preceding conditions imply that

(iii) $\mathrm{Et}(\Phi)_{\mathrm{x}}$ *is injective for every* $\mathrm{x}\in\mathcal{S}$*,*

and this last condition implies the first two if $\mathscr{E}$ *is separated.*

Proof The equivalence of (i) and (ii) is an immediate consequence of the usual statement that a morphism of modules is injective if and only if it has trivial kernel.

(ii) $\Longrightarrow$ (iii) Let $\alpha\in\mathrm{Et}(\mathscr{E})_x$ and suppose that $\mathrm{Et}(\Phi)_x(\alpha)=0$. Suppose that $\alpha = r_{\mathcal{U},x}(s)$ for some neighbourhood $\mathcal{U}$ of x. It follows from Lemma 1.1.40 that there exists a neighbourhood $\mathcal{V}\subseteq\mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{V}}(\Phi_{\mathcal{U}}(s))=0$. Using the commuting of the diagram (1.2) and the hypothesis that $\Phi_{\mathcal{V}}$ is injective we conclude that $r_{\mathcal{U},\mathcal{V}}(s)=0$, giving $\alpha=0$.

(iii) $\Longrightarrow$ (ii) Here we need to make the additional assumption that $\mathscr{E}$ is separated. Suppose that $s\in\mathscr{E}(\mathcal{U})$ is such that $\Phi_{\mathcal{U}}(s)$ is the zero section of $\mathscr{F}(\mathcal{U})$. Thus

$$\mathrm{Et}(\Phi)_x(r_{\mathcal{U},x}(s)) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s)) = 0$$

for every $x\in\mathcal{U}$ and so by hypothesis we have $r_{\mathcal{U},x}(s)=0$ for every $x\in\mathcal{U}$. By Lemma 1.1.40, for each $x\in\mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x\subseteq\mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{U}_x}(s)=0$, and an application of the fact that $\mathscr{E}$ is separated gives $s=0$. ■

The same sort of thing can be carried out for cokernels, but with one important difference.

1.3.14 Proposition (Characterisations of the cokernel presheaf) *If $(\mathbb{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a presheaf of rings over $\mathbb{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are presheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\operatorname{coker}_{\mathrm{pre}}(\Phi)(\mathcal{U})$ *is the zero section of* $\mathscr{F}(\mathcal{U})$ *for each* $\mathcal{U} \in \mathscr{O}$*;*

(ii) $\Phi_{\mathcal{U}}$ *is surjective for each* $\mathcal{U} \in \mathscr{O}$*.*

Furthermore, the preceding conditions imply that

(iii) $\mathrm{Et}(\Phi)_x$ *is surjective for every* $x \in \mathbb{S}$*.*

Proof The equivalence of (i) and (ii) follows from the usual assertion that a morphism of modules is an epimorphism if and only if its cokernel is trivial. We shall prove that (ii) implies (iii). Let $\beta \in \mathrm{Et}(\mathscr{F})_x$ and write $\beta = r_{\mathcal{U},x}(t)$ for $t \in \mathscr{F}(\mathcal{U})$. The hypothesised surjectivity of $\Phi_{\mathcal{U}}$ ensures that $t = \Phi_{\mathcal{U}}(s)$ for some $s \in \mathscr{E}(\mathcal{U})$. Thus

$$\beta = r_{\mathcal{U},x}(t) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s)) = \mathrm{Et}(\Phi)_x(r_{\mathcal{U},x}(s)),$$

which gives the result. ■

The important distinction to make here, compared to the corresponding result for kernels, is that the third assertion is not equivalent to the first two, even when $\mathscr{E}$ and $\mathscr{F}$ are sheaves. Let us give an example to illustrate this.

1.3.15 Examples (Surjectivity on stalks does not imply surjectivity)

1. Let $r \in \{\infty, \omega\}$. We shall work with the manifold $\mathbb{S}^1$. Note that we have a canonical one-form, which we denote by $\mathrm{d}\theta$, on $\mathbb{S}^1$ arising from the trivialisation $\mathsf{T}^*\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R}$. Moreover, any C^r-one-form α on an open subset $\mathcal{U} \subseteq \mathbb{S}^1$ can be written as $\alpha = g\mathrm{d}\theta|\mathcal{U}$ for some C^r-function g on $\mathcal{U}$, and so we identify C^r-one-forms with C^r-functions. We consider the sheaf $\mathscr{C}^r_{\mathbb{S}^1}$ of functions of class C^r on $\mathbb{S}^1$. For $f \in \mathscr{C}^r_{\mathbb{S}^1}(\mathcal{U})$ let $\mathrm{d}f = f'\mathrm{d}\theta|\mathcal{U}$. We let Φ be the presheaf morphism from $\mathscr{C}^r_{\mathbb{S}^1}$ to $\mathscr{C}^r_{\mathbb{S}^1}$ defined by $\Phi_{\mathcal{U}}(f) = f'$ for $f \in \mathrm{C}^r(\mathcal{U})$. (Here we are thinking of $\mathscr{C}^r_{\mathbb{S}^1}$ as being a sheaf of $\mathbb{R}$-vector spaces.) We claim that the induced map on stalks is surjective. Indeed, if $(x, y) \in \mathbb{S}^1$, if $\mathcal{U}$ is a connected and simply connected neighbourhood of (x, y) in $\mathbb{S}^1$, and if $g \in \mathrm{C}^r(\mathcal{U})$, we can define $f \in \mathrm{C}^r(\mathcal{U})$ such that $\mathrm{d}f = g$ by taking f to be the indefinite integral of g, with the variable of integration being the usual angle variable. Since the germ $\mathrm{Et}(\mathscr{C}^r_{\mathbb{S}^1})_{(x,y)}$ is determined by the value of functions on connected and simply connected neighbourhoods of (x, y), it follows that $\mathrm{Et}(\Phi)_{(x,y)}$ is surjective. However, $\Phi_{\mathbb{S}^1}$ is not surjective since, for example, $\mathrm{d}\theta \notin \operatorname{image}(\Phi_{\mathbb{S}^1})$.
2. Let us consider the morphism m_X of $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$-modules from Example 1.3.4–2. Let us modify the codomain of m_X to our present needs. Note that the vector field X defined in Example 1.3.4–2 vanishes at the north and south pole of $\mathbb{CP}^1 \simeq \mathbb{S}^2$, but is nonzero everywhere else. Thus we let $\mathscr{E}$ be the subsheaf of $\mathscr{G}^{\mathrm{hol}}_{\mathsf{T}^{1,0}\mathbb{CP}^1}$ consisting of those holomorphic vector fields vanishing at the north and south pole. One can readily check that $\mathscr{E}$ is a sheaf. Moreover, we can think of m_X as a morphism of the $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$-modules $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ and $\mathscr{E}$. We claim that m_X is surjective on stalks, but not surjective on open sets.

Let us first see why m_X is surjective on stalks. Let $z \in \mathbb{CP}^1$. Suppose first that z is neither the north nor south pole of $\mathbb{CP}^1$. Let $\mathcal{U}$ be a connected neighbourhood of z on which X does not vanish, this being possible since X does not vanish at z. Then, if $Y \in \mathscr{E}(\mathcal{U})$, we can write $Y = f \cdot (X|\mathcal{U})$ for $f \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U})$. Thus $Y \in \mathrm{image}_{\mathrm{pre}}(m_X)(\mathcal{U})$ and we conclude that $m_{X,z} \colon \mathscr{C}^{\mathrm{hol}}_{z,\mathbb{CP}^1} \to \mathscr{E}_x$ is surjective. If z is either the north nor south pole—for specificity let us work with the south pole so $z \in \mathcal{U}_+$—let $\mathcal{U}$ be a connected neighbourhood of z and let $Y \in \mathscr{E}(\mathcal{U})$. In some neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of z we can write the local representative of Y in the chart $(\mathcal{U}_+\psi_+)$ as

$$z_+ \mapsto (z_+, z_+P(z_+))$$

for some power series P in z_+ since Y vanishes at z. From this we infer that $Y|\mathcal{U}' = f \cdot (X|\mathcal{U}')$ for some $f \in \mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}(\mathcal{U}')$, and so $m_{X,z} \colon \mathscr{C}^{\mathrm{hol}}_{z,\mathbb{CP}^1} \to \mathscr{E}_x$ is again surjective.

To see that m_X is not surjective on open sets, we will show that $m_{X,\mathbb{CP}^1}$ is not surjective. Indeed, since holomorphic functions on $\mathbb{CP}^1$ are constant by Corollary GA1.4.2.11, it follows that $\mathrm{image}_{\mathrm{pre}}(m_X)(\mathbb{CP}^1)$ consists of vector fields that are constant multiples of X. Since there are holomorphic vector fields on $\mathbb{CP}^1$ that are not constant multiples of X (see Example GA1.4.5.20), it follows that $m_{X,\mathbb{CP}^1}$ is not surjective, as claimed. •

The preceding examples notwithstanding, it is true that surjectivity on stalks, combined with injectivity on stalks, does imply surjectivity globally.

1.3.16 Proposition (Correspondence of isomorphisms and stalk-wise isomorphisms) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are sheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\Phi_{\mathcal{U}} \colon \mathscr{E}(\mathcal{U}) \to \mathscr{F}(\mathcal{U})$ *is an isomorphism for every* $\mathcal{U} \in \mathscr{O}$*;*

(ii) $\mathrm{Et}(\Phi)_{\mathrm{x}} \colon \mathrm{Et}(\mathscr{E})_{\mathrm{x}} \to \mathrm{Et}(\mathscr{F})_{\mathrm{x}}$ *is an isomorphism for every* $\mathrm{x} \in \mathcal{S}$*.*

Proof That (i) implies (ii) follows from Propositions 1.3.13 and 1.3.14. It follows from Proposition 1.3.13 that injectivity of $\mathrm{Et}(\Phi)_x$ for each $x \in \mathcal{S}$ implies injectivity of $\Phi_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. So suppose that $\mathrm{Et}(\Phi)_x$ is bijective for every $x \in \mathcal{S}$. Let $\mathcal{U} \in \mathscr{O}$ and let $t \in \mathscr{F}(\mathcal{U})$. For $x \in \mathcal{U}$ let $\alpha \in \mathrm{Et}(\mathscr{E})_x$ be such that $\mathrm{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(t)$. Let $\alpha = r_{\mathcal{U},x}(s_x)$ for some $s_x \in \mathscr{E}(\mathcal{U})$. By Lemma 1.1.40 let $\mathcal{U}_x \subseteq \mathcal{U}$ be a neighbourhood of x such that $r_{\mathcal{U},\mathcal{U}_x}(t) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s_x))$. Now let $x, y \in \mathcal{U}$ and note that

$$\Phi_{\mathcal{U}_x \cap \mathcal{U}_y}(r_{\mathcal{U}_x,\mathcal{U}_x \cap \mathcal{U}_y}(s_x)) = \Phi_{\mathcal{U}_x \cap \mathcal{U}_y}(r_{\mathcal{U}_y,\mathcal{U}_x \cap \mathcal{U}_y}(s_y)),$$

since both expressions are equal to $r_{\mathcal{U},\mathcal{U}_x \cap \mathcal{U}_y}(t)$. By injectivity of $\Phi_{\mathcal{U}_x \cap \mathcal{U}_y}$ (which follows since we are assuming that $\mathrm{Et}(\Phi)_x$ is injective for every $x \in \mathcal{S}$), it follows that

$$r_{\mathcal{U}_x,\mathcal{U}_x \cap \mathcal{U}_y}(s_x) = r_{\mathcal{U}_y,\mathcal{U}_x \cap \mathcal{U}_y}(s_y).$$

Thus, since $\mathscr{E}$ is a sheaf, there exists $s \in \mathscr{E}(\mathcal{U})$ such that $r_{\mathcal{U},\mathcal{U}_x}(s) = s_x$ for every $x \in \mathcal{U}$. Finally, we claim that $\phi(s) = t$. This follows from separability of $\mathscr{F}$ since we have $r_{\mathcal{U},\mathcal{U}_x}(t) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s_x))$ for every $x \in \mathcal{U}$. ■

The preceding three results and example indicate that surjectivity of morphisms of presheaves and étalé spaces will not necessarily correspond. We will be interested mainly in looking at things at the level of stalks, so let us consider carefully the implications of properties holding at the stalk level.

1.3.17 Proposition (Characterisations of the kernel) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are sheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\mathrm{image}(\mathrm{Et}(\Phi))$ *is the zero section of* $\mathrm{Et}(\mathscr{F})$ *over* $\mathcal{S}$*;*

(ii) $\ker_{\mathrm{pre}}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iii) $\ker(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iv) $\Phi_{\mathcal{U}}$ *is injective for every* $\mathcal{U} \in \mathscr{O}$*;*

(v) $\mathrm{Et}(\Phi)_x$ *is injective for every* $x \in \mathcal{S}$*;*

(vi) $\mathrm{Et}(\Phi)$ *is injective.*

Proof These equivalences were either already proved, or follow immediately from definitions. ■

The same sort of thing can be carried out for cokernels, but with one important difference.

1.3.18 Proposition (Characterisations of cokernel) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, if $\mathscr{E}$ and $\mathscr{F}$ are sheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U} \in \mathscr{O}}$ is an $\mathscr{R}$-module morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\mathrm{image}(\mathrm{Et}(\Phi)) = \mathrm{Et}(\mathscr{F})$*;*

(ii) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iii) $\mathrm{coker}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iv) $\mathrm{Et}(\Phi)_x$ *is surjective for every* $x \in \mathcal{S}$*;*

(v) $\mathrm{Et}(\Phi)$ *is surjective.*

Proof As with the preceding result, these equivalences were either already proved, or follow immediately from definitions. ■

Once again, we point out the missing assertion from the statement about cokernels as compared to the statement about kernels.

1.3.5 Direct sums and direct products

Now we turn our attention to a few standard algebraic constructions on sheaves, beginning with direct sums and tensor products.

Direct sums and direct products of presheaves

We begin by considering presheaves.

1.3.19 Definition (Direct sums and direct products of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $(\mathscr{E}_a)_{a\in A}$, be a family of presheaves of $\mathscr{R}$-modules over $\mathcal{S}$.

(i) The ***direct product presheaf*** of the presheaves $\mathscr{E}_a$, $a \in A$, is the presheaf $\prod_{a\in A} \mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\Big(\prod_{a\in A}{}^{\mathrm{pre}}\mathscr{E}_a\Big)(\mathcal{U}) = \prod_{a\in A} \mathscr{E}_a(\mathcal{U}) = \Big\{\phi\colon A \to \cup_{a\in A}\mathscr{E}_a(\mathcal{U}) \;\Big|\; \phi(a) \in \mathscr{E}_a(\mathcal{U}) \text{ for all } a \in A\Big\}.$$

If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$ the restriction map $r_{\mathcal{U},\mathcal{V}}$ for $\oplus^{\mathrm{pre}}_{a\in A}\mathscr{E}_a$ is defined by $r_{\mathcal{U},\mathcal{V}}(\phi)(a) = r^a_{\mathcal{U},\mathcal{V}}(\phi(a))$, where $r^a_{\mathcal{U},\mathcal{V}}$ is the restriction map for $\mathscr{E}_a$, $a \in A$.

(ii) The ***direct sum presheaf*** of the presheaves $\mathscr{E}_a$, $a \in A$, is the presheaf $\oplus^{\mathrm{pre}}_{a\in A}\mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\begin{aligned}\Big(\bigoplus_{a\in A}{}^{\mathrm{pre}}\mathscr{E}_a\Big)(\mathcal{U}) &= \oplus_{a\in A}\, \mathscr{E}_a(\mathcal{U})\\ &= \Big\{\phi \in \prod_{a\in A} \mathscr{E}_a(\mathcal{U}) \;\Big|\; \phi(a) = 0 \text{ for all but finitely many } a \in A\Big\}.\end{aligned}$$

The restriction maps are the same as for the direct product. •

Let us record a basic property of direct sums and products.

1.3.20 Proposition (Stalks of direct sums and direct products) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $(\mathscr{E}_{\mathrm{a}})_{\mathrm{a}\in\mathrm{A}}$, be a family of presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Then*

(i) there is a natural mapping of $(\prod^{\mathrm{pre}}_{\mathrm{a}\in\mathrm{A}} \mathscr{E}_{\mathrm{a}})_{\mathrm{x}}$ in $\prod_{\mathrm{a}\in\mathrm{A}} \mathscr{E}_{\mathrm{a,x}}$ for each $\mathrm{x} \in \mathcal{S}$ and

(ii) $(\oplus^{\mathrm{pre}}_{\mathrm{a}\in\mathrm{A}}\mathscr{E}_{\mathrm{a}})_{\mathrm{x}} = \oplus_{\mathrm{a}\in\mathrm{A}}\mathscr{E}_{\mathrm{a,x}}$ for each $\mathrm{x} \in \mathcal{S}$.

Proof (i) The mapping in question is

$$\Big(\prod_{a\in A}{}^{\mathrm{pre}}\mathscr{E}_a\Big)_x \ni [(\phi, \mathcal{U})]_x \mapsto [\phi]_x \in \prod_{a\in A} \mathscr{E}_{a,x},$$

where $[\phi]_x\colon A \to \cup_{a\in A}\mathscr{E}_{a,x}$ is given by $[\phi]_x(a) = [(\phi(a), \mathcal{U})]_x$.

(ii) First let $[\phi]_x \in (\oplus_{a\in A}\mathscr{E}_a)_x$. Then there exists a neighbourhood $\mathcal{U}$ of x and $a_1, \dots, a_k \in A$ such that ϕ is a section over $\mathcal{U}$ and $\phi(a) \neq 0$ if and only if $a \in \{a_1, \dots, a_k\}$. Thus $[\phi]_x$, as a map from A to $\cup_{a\in A}\mathscr{E}_{a,x}$, is given by $[\phi]_x(a) = [\phi(a)]_x$ and so is an element of $\oplus_{a\in A}\mathscr{E}_{a,x}$. Conversely, if $[\phi]_x \in \oplus_{a\in A}\mathscr{E}_{a,x}$ then there exists a neighbourhood $\mathcal{U}$ of x and $a_1, \dots, a_k \in A$ such that ϕ is a section over $\mathcal{U}$ and $\phi(a) \neq 0$ if and only if $a \in \{a_1, \dots, a_k\}$. Thus $[\phi]_x \in (\oplus_{a\in A}\mathscr{E}_a)_x$. ■

Direct sums and direct products of sheaves

Let us turn to direct sums and products of sheaves. First we consider when these operations produce sheaves from sheaves.

1.3.21 Proposition (When direct products and sums of sheaves are sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $(\mathscr{E}_{\mathrm{a}})_{\mathrm{a}\in\mathrm{A}}$, be a family of presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Then*

(i) $\prod^{\mathrm{pre}}_{\mathrm{a}\in\mathrm{A}} \mathscr{E}_{\mathrm{a}}$ *is a sheaf if* $\mathscr{E}_{\mathrm{a}}$ *is a sheaf for each* $\mathrm{a} \in \mathrm{A}$ *and*

(ii) $\oplus^{\mathrm{pre}}_{\mathrm{a}\in\mathrm{A}} \mathscr{E}_{\mathrm{a}}$ *is a sheaf if* A *is finite and if* $\mathscr{E}_{\mathrm{a}}$ *is a sheaf for each* $\mathrm{a} \in \mathrm{A}$.

Proof Let us first consider direct products. Let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_b)_{b\in B}$ be an open cover for $\mathcal{U}$. Suppose that $\phi, \phi' \in (\prod^{\mathrm{pre}}_{a\in A} \mathscr{E}_a)(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_b}(\phi) = r_{\mathcal{U},\mathcal{U}_b}(\phi')$ for each $b \in B$. Then, by definition of the restriction maps, $r^a_{\mathcal{U},\mathcal{U}_b}(\phi(a)) = r^a_{\mathcal{U},\mathcal{U}_b}(\phi'(a))$ for each $a \in A$ and $b \in B$. From this we deduce that $\phi(a) = \phi'(a)$ for each $a \in A$, giving separatedness of $\prod^{\mathrm{pre}}_{a\in A} \mathscr{E}_a$. Next suppose that we have $\phi_b \in \prod^{\mathrm{pre}}_{a\in A} \mathscr{E}_a(\mathcal{U}_b)$ for each $b \in B$ satisfying

$$r_{\mathcal{U}_{b_1},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_1}) = r_{\mathcal{U}_{b_2},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_2})$$

for every $b_1, b_2 \in B$. This implies that

$$r^a_{\mathcal{U}_{b_1},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_1}(a)) = r^a_{\mathcal{U}_{b_2},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_2}(a))$$

for every $a \in A$ and $b_1, b_2 \in B$. Thus, for each $a \in A$, there exists $\phi_a \in \mathscr{E}_a$ such that

$$r^a_{\mathcal{U},\mathcal{U}_b}(\phi_a) = \phi_b(a)$$

for each $b \in B$. Now define $\phi\colon A \to \cup_{a\in A}\mathscr{E}_a$ by $\phi(a) = \phi_a$, and note that $\phi \in \prod^{\mathrm{pre}}_{a\in A} \mathscr{E}_a$.

For finite direct sums, the same argument holds, especially noting in the last step that $\phi \in \oplus^{\mathrm{pre}}_{a\in A}\mathscr{E}_a$ since A is finite. ■

Note that in the second statement of the previous result, it is generally necessary that A be finite as the following example shows.

1.3.22 Example (Infinite direct sums of sheaves are not generally sheaves) Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We take $\mathsf{M} = \mathbb{F}$ and consider the presheaf $\oplus^{\mathrm{pre}}_{k\in\mathbb{Z}_{>0}} \mathscr{C}^r_{\mathbb{F}}$. We claim that this is not a presheaf. Let

$$\mathcal{U} = \mathbb{F} \setminus \cup_{j\in\mathbb{Z}_{\geq 0}}\{x \in \mathbb{F} \mid |x| = j\}$$

and let

$$\mathcal{U}_j = \mathcal{U} \cap (\mathsf{D}^1(0, j) \setminus \overline{\mathsf{D}}^1(0, j-1)), \qquad j \in \mathbb{Z}_{>0},$$

so that $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ is an open cover for $\mathcal{U}$. For $j \in \mathbb{Z}_{>0}$ define $\phi_j \in \oplus_{k\in\mathbb{Z}_{>0}}\mathrm{C}^r(\mathcal{U}_j)$ by

$$\phi_j(k)(x) = \begin{cases} 1, & k \in \{1, \dots, j\}, \\ 0, & \text{otherwise} \end{cases}$$

for $x \in \mathcal{U}_j$. Note, however, that there is no section $\phi \in \oplus_{k\in\mathbb{Z}_{>0}}\mathrm{C}^r(\mathcal{U})$ which restricts to ϕ_j for each $j \in \mathbb{Z}_{>0}$ since any such section ϕ has the property that, for any $k \in \mathbb{Z}_{>0}$, $\phi(k)$ is nonzero, being nonzero restricted to $\mathcal{U}_k$. •

All of the above lead us to the following definition.

1.3.23 Definition (Direct sums and direct products of sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $(\mathscr{E}_a)_{a\in A}$, be a family of sheaves of $\mathscr{R}$-modules over $\mathcal{S}$.

(i) The ***direct product sheaf*** of the sheaves $\mathscr{E}_a$, $a \in A$, is the sheaf $\prod_{a\in A} \mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\prod_{a\in A} \mathscr{E}_a = \mathrm{Ps}\Big(\mathrm{Et}\Big(\prod_{a\in A} {}^{\mathrm{pre}}\mathscr{E}_a\Big)\Big).$$

(ii) The ***direct sum sheaf*** of the sheaves $\mathscr{E}_a$, $a \in A$, is the sheaf $\prod_{a\in A} \mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\bigoplus_{a\in A} \mathscr{E}_a = \mathrm{Ps}\Big(\mathrm{Et}\Big(\bigoplus_{a\in A} {}^{\mathrm{pre}}\mathscr{E}_a\Big)\Big).$$ •

Direct sums and direct products of étalé spaces

We turn now to étalé spaces.

1.3.24 Definition (Direct sums and direct products of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\pi_a \colon \mathscr{U}_a \to \mathcal{S}$, $a \in A$, be a family of étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$.

(i) The ***direct product*** of the étalé spaces $\mathscr{U}_a$, $a \in A$, is the set $\prod_{a\in A} \mathscr{U}_a$ defined by

$$\prod_{a\in A} \mathscr{U}_a = \Big\{\phi \colon A \to \cup_{a\in A}\mathscr{U}_a \Big|\ \phi(a) \in \mathscr{U}_a \text{ for all } a \in A \text{ and}$$
$$\pi_{a_1}(\phi(a_1)) = \pi_{a_2}(\phi(a_2)) \text{ for all } a_1, a_2 \in A\Big\},$$

together with the étalé projection Π defined by $\Pi(\phi) = \pi_a(\phi(a))$ for some (and so for all) $a \in A$.

(ii) The ***direct sum*** of the étalé spaces $\mathscr{U}_a$, $a \in A$, is the subset $\oplus_{a\in A}\mathscr{U}_a$ of $\prod_{a\in A} \mathscr{U}_a$ defined by

$$\oplus_{a\in A}\mathscr{U}_a = \Big\{\phi \in \prod_{a\in A} \mathscr{U}_a \ \Big|\ \phi(a) = \{0\} \text{ for all but finitely many } a \in A\Big\},$$

and with the étalé projection being the restriction of that for the direct product. •

In order for the definition of the direct sum of étalé spaces to be itself an étalé space, we need to assign an appropriate topology to the set. This is more or less easily done. Recall that the product topology on $\prod_{a\in A} \mathscr{U}_a$ is that topology generated by sets of the form $\prod_{a\in A} \mathcal{O}_a$, where the set

$$\{a \in A \mid \ \mathcal{O}_a \neq \mathscr{U}_a\}$$

is finite. The product topology is the initial topology associated with the family of canonical projections $\mathrm{pr}_a \colon \prod_{a'\in A} \mathscr{U}_{a'} \to \mathscr{U}_a$, i.e., the coarsest topology for which all of
what? these projections is continuous (see below). The topology on $\oplus_{a\in A}\mathscr{U}_a$ is that induced by

the product topology on $\prod_{a\in A}\mathscr{U}_a$. One concludes that sections of $\prod_{a\in A}\mathscr{U}_a$ over $\mathcal{U}$ are precisely the maps $\sigma\colon \mathcal{U}\to\prod_{a\in A}\mathscr{U}_a$ such that $\mathrm{pr}_a\circ\sigma$ is a section of $\mathscr{U}_a$ over $\mathcal{U}$ for each $a\in A$. Sections σ of $\oplus_{a\in A}\mathscr{U}_a$ over $\mathcal{U}$ have the property that there exists $a_1,\dots,a_k\in A$ and sections $\sigma_1,\dots,\sigma_k$ of $\mathscr{U}_{a_1},\dots,\mathscr{U}_{a_k}$, respectively, such that

$$\mathrm{pr}_a\circ\sigma(x)=\begin{cases}\sigma_{a_j}(x), & a=a_j\in\{a_1,\dots,a_k\},\\ 0, & a\notin\{a_1,\dots,a_k\}\end{cases}$$

for each $x\in\mathcal{U}$.

1.3.6 Tensor products

The next algebraic operation we consider is tensor product.

Tensor products of presheaves

Now we turn to tensor products, starting with presheaves.

1.3.25 Definition (Tensor products of presheaves) Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_a$, $a\in\{1,2\}$, be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. The ***tensor product presheaf*** of the presheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ is the presheaf $\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2=(\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ defined by

$$\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2(\mathcal{U})=\mathscr{E}_1(\mathcal{U})\otimes\mathscr{E}_2(\mathcal{U}),$$

the tensor product on the right being of $\mathscr{R}(\mathcal{U})$-modules. If $\mathcal{U},\mathcal{V}\in\mathscr{O}$ satisfy $\mathcal{V}\subseteq\mathcal{U}$ the restriction map $r_{\mathcal{U},\mathcal{V}}$ for $\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2$ is defined by

$$r_{\mathcal{U},\mathcal{V}}(s_1\otimes_{\mathrm{pre}}s_2)=r^1_{\mathcal{U},\mathcal{V}}(s_1)\otimes r^2_{\mathcal{U},\mathcal{V}}(s_2),$$

where $r^a_{\mathcal{U},\mathcal{V}}$ is the restriction map for $\mathscr{E}_a$, $a\in\{1,2\}$, and where $s_a\in\mathscr{E}_a(\mathcal{U})$, $a\in\{1,2\}$. •

Let us understand the stalks of the tensor product presheaf.

1.3.26 Proposition (Stalks of tensor product presheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_\mathrm{a}$, $\mathrm{a}\in\{1,2\}$, be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Then we have an isomorphism*

$$(\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2)_\mathrm{x}\simeq\mathscr{E}_{1,\mathrm{x}}\otimes\mathscr{E}_{2,\mathrm{x}}$$

of $\mathscr{R}_\mathrm{x}$-modules for each $\mathrm{x}\in\mathcal{S}$, the tensor product on the right being on $\mathscr{R}_\mathrm{x}$-modules.

Proof Consider the mapping

$$(\mathscr{E}_1\otimes_{\mathrm{pre}}\mathscr{E}_2)_x\ni[s_1\otimes s_2,\mathcal{U}]_x\mapsto[(s_1,\mathcal{U})]_x\otimes[(s_2,\mathcal{U})]_x\in\mathscr{E}_{1,x}\otimes\mathscr{E}_{2,x}.$$

It is a routine exercise to verify that this induces the desired isomorphism. ■

The natural way in which one defines tensor products of homomorphisms carries over to morphisms of presheaves.

1.3.27 Definition (Morphisms defined on tensor products of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E}_a$ and $\mathscr{F}_a$, $a \in \{1, 2\}$, be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi_a = (\Phi_{a,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}_a$ to $\mathscr{F}_a$, $a \in \{1, 2\}$. The ***tensor product*** of Φ_1 and Φ_2 is the $\mathscr{R}$-module morphism $\Phi_1 \otimes_{\mathrm{pre}} \Phi_1$ from $\mathscr{E}_1 \otimes_{\mathrm{pre}} \mathscr{E}_2$ to $\mathscr{F}_1 \otimes_{\mathrm{pre}} \mathscr{F}_2$ defined by

$$(\Phi_1 \otimes_{\mathrm{pre}} \Phi_2)_{\mathcal{U}}(s_1 \otimes s_2) = \Phi_{1,\mathcal{U}}(s_1) \otimes \Phi_{2,\mathcal{U}}(s_2)$$

for $s_a \in \mathscr{E}_a(\mathcal{U})$, $a \in \{1, 2\}$. •

Tensor products of sheaves

Now we turn to sheaves, first noting that taking tensor products does not preserve sheaves.

1.3.28 Examples (Tensor products of sheaves may not be sheaves)

1. Let $\mathcal{X} = [0, 1] \times \mathbb{Z}$ and define an equivalence relation $\sim_1$ in $\mathcal{X}$ by declaring that $(x_1, k_1) \sim_1 (x_2, k_2)$ if either
   (a) $(x_1, k_1) = (x_2, k_2)$ and $x_1, x_2 \notin \{0, 1\}$,
   (b) $x_1 = 0$, $x_2 = 1$, and $k_1 = -k_2$, or
   (c) $x_1 = 1$, $x_2 = 0$, and $k_1 = -k_2$.

   We also let $\mathcal{A} = [0, 1]$ and define an equivalence relation $\sim_0$ in $\mathcal{A}$ by declaring that $x_1 \sim_0 x_2$ if either
   (a) $x_1 = x_2$ and $x_1, x_2 \notin \{0, 1\}$,
   (b) $x_1 = 0$ and $x_2 = 1$, or
   (c) $x_1 = 1$ and $x_2 = 0$.

   We denote $\mathcal{Y} = \mathcal{X}/\sim_1$ and $\mathcal{B} = \mathcal{A}/\sim_0$ and denote by $\pi_1 \colon \mathcal{X} \to \mathcal{Y}$ and $\pi_0 \colon \mathcal{A} \to \mathcal{B}$ the canonical projections. We equip $\mathcal{A}$ with its natural topology as a subset of $\mathbb{R}$, we equip $\mathcal{X}$ with the product topology, and we equip $\mathcal{Y}$ and $\mathcal{Y}$ with their quotient topology. Define a projection $\pi \colon \mathcal{Y} \to \mathcal{B}$ by $\pi([(x, k)]) = [x]$. This can be thought of as a discrete version of the Möbius vector bundle. By $\Gamma(\mathcal{B})$ we denote the presheaf over $\mathcal{B}$ whose sections over $\mathcal{U} \subseteq \mathcal{B}$ are continuous sections of $\pi \colon \mathcal{Y} \to \mathcal{B}$ over $\mathcal{U}$. A local section over $\mathcal{U}$, $s \in \Gamma(\mathcal{U}; \mathcal{B})$, has the form $s([x]) = [(x, \hat{s}([x]))]$ for a function $\hat{s} \colon \mathcal{U} \to \mathbb{Z}$. Continuity requires that $\hat{s}([1]) = \hat{s}[0])$. If $\mathcal{U} = \mathcal{Y}$ then this mandates that $\hat{s}([x]) = 0$ for every $[x] \in \mathcal{U}$. This presheaf can be easily verified to be a sheaf of Abelian groups with the group structure defined pointwise by $[(x, \hat{s}_1([x]))] + [(x, \hat{s}_2([x]))] = [(x, \hat{s}_1([x]) + \hat{s}_2([x]))]$. We shall be interested in the tensor product of this sheaf with itself. To this end, if $s_1, s_2 \in \Gamma(\mathcal{U}; \mathcal{B})$ are local sections over the open set $\mathcal{U} \subseteq \mathcal{Y}$, then

   $$(s_1 \otimes s_2)(x) = [(x, \hat{s}_1([x])\hat{s}_2([x]))],$$

   i.e., tensor product is integer multiplication.

We claim that $\Gamma(\mathcal{B}) \otimes_{\mathrm{pre}} \Gamma(\mathcal{B})$ is not a sheaf. Define $\mathcal{U}_1, \mathcal{U}_2$ by

$$\mathcal{U}_1 = \pi_0((\tfrac{1}{8}, \tfrac{7}{8})), \quad \mathcal{U}_2 = \pi_0([0, \tfrac{1}{4}) \cup (\tfrac{3}{4}, 1]).$$

Define sections $s_1, t_1 \in \Gamma(\mathcal{U}_1; \mathcal{B})$ by $s_1([x]) = [(x, 1)]$ and $t_1([x]) = [(x, -1)]$. Define sections $s_2, t_2 \in \Gamma(\mathcal{U}_2; \mathcal{B})$ by

$$s_2([x]) = \begin{cases} [(x, 1)], & x \in [0, \frac{1}{8}), \\ [(x, -1)], & x \in (\frac{7}{8}, 1] \end{cases}$$

and

$$t_2([x]) = \begin{cases} [(x, -1)], & x \in [0, \frac{1}{8}), \\ [(x, 1)], & x \in (\frac{7}{8}, 1]. \end{cases}$$

For $x \in (\frac{1}{8}, \frac{1}{4}) \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ we have

$$s_1 \otimes t_1([x]) = [(x, 1 \cdot (-1))] = [(x, (-1) \cdot 1)] = s_2 \otimes t_2([x])$$

and for $x \in (\frac{3}{4}, \frac{7}{8}) \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ we have

$$(s_1 \otimes t_1)([x]) = [(x, 1 \cdot (-1))] = [(x, (-1) \cdot 1)] = s_2 \otimes t_2([x]).$$

Note that $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{Y}$ and that the only continuous section over $\mathcal{Y}$ is the zero section. Thus there can be no sections $s, t \in \Gamma(\mathcal{Y}; \mathcal{B})$ such that $r_{\mathcal{Y},\mathcal{U}_1}(s \otimes t) = s_1 \otimes t_1$ and $r_{\mathcal{Y},\mathcal{U}_2}(s \otimes t) = s_2 \otimes t_2$. Thus $\Gamma(\mathcal{B}) \otimes_{\mathrm{pre}} \Gamma(\mathcal{B})$ is not a sheaf, as claimed.

2. We consider the sheaves $\mathscr{O}_{\mathbb{CP}^1}(d)$, $d \in \mathbb{Z}$, of $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$-modules from Example 1.3.4–2. We claim that the tensor product presheaf $\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1)$ is not a sheaf. To see this, we use the standard open cover $(\mathcal{U}_+, \mathcal{U}_-)$ described in Example GA1.4.3.5–**??**, and associated with $\mathbb{C}$-charts whose coordinates we denote by z_+ and z_-, respectively. Let us consider the local sections

$$\xi_+ \in \mathscr{O}_{\mathbb{CP}^1}(-1)(\mathcal{U}_+), \quad \alpha_+ \in \mathscr{O}_{\mathbb{CP}^1}(1)(\mathcal{U}_+), \quad \xi_- \in \mathscr{O}_{\mathbb{CP}^1}(-1)(\mathcal{U}_-), \quad \alpha_- \in \mathscr{O}_{\mathbb{CP}^1}(1)(\mathcal{U}_-)$$

with local representatives

$$z_+ \mapsto (z_+, z_+^{-1}), \quad z_+ \mapsto (z_+, z_+), \quad z_- \mapsto (z_+, z_-^{-1}), \quad z_+ \mapsto (z_+, z_-),$$

respectively. We then have that

$$\xi_+ \otimes_{\mathrm{pre}} \alpha_+ \in (\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1))(\mathcal{U}_+), \quad \xi_- \otimes_{\mathrm{pre}} \alpha_- \in (\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1))(\mathcal{U}_-)$$

have the local representatives

$$z_+ \mapsto (z_+, 1), \quad z_- \mapsto (z_-, 1),$$

respectively. Thus we have local sections of $\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1)$ defined over the two open sets $\mathcal{U}_+$ and $\mathcal{U}_-$ which agree on $\mathcal{U}_+ \cap \mathcal{U}_-$. However, there can be no section of $\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1)$ over $\mathcal{U}_+ \cup \mathcal{U}_- = \mathbb{CP}^1$ that restricts to the local sections on both $\mathcal{U}_+$ and $\mathcal{U}_-$. Indeed, since every global section of $\mathscr{O}_{\mathbb{CP}^1}(-1)$ is zero as we saw in Example GA1.4.3.14, it follows that every global section of $\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1)$ is also zero. •

The example suggest that, once again, we must turn to sheafification to make the tensor product of sheaves coherent.

1.3.29 Definition (Tensor products of sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_a$, $a \in \{1,2\}$, be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$. The ***tensor product sheaf*** of the sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ is $\mathscr{E}_1 \otimes \mathscr{E}_2 = \mathrm{Ps}(\mathrm{Et}(\mathscr{E}_1 \otimes_{\mathrm{pre}} \mathscr{E}_2))$. •

The notion of morphisms of tensor products can be adapted to sheaves.

1.3.30 Definition (Morphisms defined on tensor products of sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, let $\mathscr{E}_a$ and $\mathscr{F}_a$, $a \in \{1,2\}$, be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi_a = (\Phi_{a,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}_a$ to $\mathscr{F}_a$, $a \in \{1,2\}$. The ***tensor product*** of Φ_1 and Φ_2 is the $\mathscr{R}$-module morphism $\Phi_1 \otimes \Phi_1$ from $\mathscr{E}_1 \otimes \mathscr{E}_2$ to $\mathscr{F}_1 \otimes \mathscr{F}_2$ defined by $\Phi_1 \otimes \Phi_2 = \mathrm{Ps}(\mathrm{Et}(\Phi_1 \otimes_{\mathrm{pre}} \Phi_2))$. •

Let us look at the tensor product sheaf in the cases above where the tensor product is not a sheaf.

1.3.31 Examples (Sheafification of tensor products)

1. We revisit Example 1.3.28–1 where we considered the sheaf $\Gamma(\mathcal{B})$ of continuous sections of the discrete Möbius vector bundle, thought of as a sheaf of Abelian groups. We claim that $\Gamma(\mathcal{B}) \otimes \Gamma(\mathcal{B})$ is isomorphic to $\Gamma(\mathcal{B})$. By Proposition 1.1.107 we have the canonical mapping ι from the presheaf $\Gamma(\mathcal{B}) \otimes_{\mathrm{pre}} \Gamma(\mathcal{B})$ to its sheafification $\Gamma(\mathcal{B}) \otimes \Gamma(\mathcal{B})$ given by
$$\iota_{\mathcal{U}}(s_1 \otimes s_2)(x) = [s_1]_x \otimes [s_2]_x.$$
We also have the morphism from $\Gamma(\mathcal{B}) \otimes \Gamma(\mathcal{B})$ to $\Gamma(\mathcal{B})$ which maps the local section $\iota_{\mathcal{U}}(s_1 \otimes s_2)$ to the local section
$$[x] \mapsto [(x, \hat{s}_1(x)\hat{s}_2(x))].$$
Since $\mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z}$, this latter map is an isomorphism on stalks, and so an isomorphism by Proposition 1.3.16.
2. Now we continue with Example 1.3.28–2. In this case, since $\mathsf{O}_{\mathbb{CP}^1}(-1) \otimes \mathsf{O}_{\mathbb{CP}^1}(1)$ is isomorphic to the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}$ by Example GA1.4.3.20, it follows that the sheafification of $\mathscr{O}_{\mathbb{CP}^1}(-1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1)$ is isomorphic to $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$.
3. In both of the above example, the presheaf tensor product was not a sheaf by virtue of not satisfying the gluing property. It can also happen that the tensor product of two sheaves is not separated, which we illustrate by the following example. We again take $\mathbb{CP}^1$ and now we consider the sheaf $\mathscr{O}_{\mathbb{CP}^1}(1)$ of sections of the hyperplane line bundle. We note that, by Example GA1.4.3.14, $\dim_{\mathbb{C}}(\mathscr{O}_{\mathbb{CP}^1}(1)(\mathbb{CP}^1)) = 2$. Therefore,
$$\dim_{\mathbb{C}}((\mathscr{O}_{\mathbb{CP}^1}(1) \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}(1))(\mathbb{CP}^1)) = 4.$$
However, $\mathsf{O}_{\mathbb{CP}^1}(1) \otimes \mathsf{O}_{\mathbb{CP}^1}(1) = \mathsf{O}_{\mathbb{CP}^1}(2)$, as we saw in Example GA1.4.3.20. Therefore,
$$\dim_{\mathbb{C}}(\mathscr{O}_{\mathbb{CP}^1}(1) \otimes \mathscr{O}_{\mathbb{CP}^1}(1))(\mathbb{CP}^1) = 3,$$

again by Example GA1.4.3.14. Thus there are too many global sections for the presheaf $\mathscr{O}_{\mathbb{CP}^1} \otimes_{\mathrm{pre}} \mathscr{O}_{\mathbb{CP}^1}$, meaning this presheaf is not separated. •

Tensor products of étalé spaces

Next we consider étalé spaces.

1.3.32 Definition (Tensor products of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{U}_a$, $a \in \{1,2\}$, be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$. The ***tensor product*** of the étalé spaces $\mathscr{U}_1$ and $\mathscr{U}_2$ is $\mathscr{U}_1 \otimes \mathscr{U}_2 = \mathrm{Et}(\mathrm{Ps}(\mathscr{U}_1) \otimes_{\mathrm{pre}} \mathrm{Ps}(\mathscr{U}_2))$. •

Note that we have side stepped the issue of topologising tensor products of étalé spaces by our going to presheaves and back to étalé spaces. In any case, the stalks of the tensor product have the expected form.

1.3.33 Proposition (Stalks of the tensor product of étalé spaces) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, and let $\mathscr{U}_{\mathrm{a}}$, $\mathrm{a} \in \{1,2\}$, be étalé spaces of $\mathscr{A}$-modules over $\mathcal{S}$. Then we have an isomorphism*

$$(\mathscr{U}_1 \otimes \mathscr{U}_2)_{\mathrm{x}} \simeq \mathscr{U}_{1,\mathrm{x}} \otimes \mathscr{U}_{2,\mathrm{x}}$$

of $\mathscr{A}_{\mathrm{x}}$-modules for each $\mathrm{x} \in \mathcal{S}$, where the tensor product on the right is of $\mathscr{A}_{\mathrm{x}}$-modules.

Proof This follows from one application of Proposition 1.3.26 and two applications of Proposition 1.1.53. ■

Finally, we can define the tensor product of morphisms for étalé spaces.

1.3.34 Definition (Morphisms defined on tensor products of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $\mathscr{U}_a$ and $\mathscr{V}_a$, $a \in \{1,2\}$, be sheaves of $\mathscr{A}$-modules over $\mathcal{S}$, and let $\Phi_a \colon \mathscr{E}_a \to \mathscr{F}_a$ be a morphism étalé spaces of $\mathscr{A}$-modules, $a \in \{1,2\}$. The ***tensor product*** of Φ_1 and Φ_2 is the morphism of étalé spaces of $\mathscr{A}$-module $\Phi_1 \otimes \Phi_1$ from $\mathscr{E}_1 \otimes \mathscr{E}_2$ to $\mathscr{F}_1 \otimes \mathscr{F}_2$ defined by $\Phi_1 \otimes \Phi_2 = \mathrm{Et}(\mathrm{Ps}(\Phi_1) \otimes_{\mathrm{pre}} \mathrm{Ps}(\Phi_2))$. •

1.3.7 Exact sequences

A detailed understanding of exact sequences is an essential part of the study of sheaves. When we look at cohomology in Chapter 4, we shall develop the necessary ideas in some depth. Here we simply provide the definitions.

Exact sequences of presheaves

We are interested in looking at exact sequences of presheaves and étalé spaces. Let us give the definitions so that we first know what we are talking about.

1.3.35 Definition (Exact sequence of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$, let $J \subseteq \mathbb{Z}$ be of one of the following forms:

$$J = \{0, 1, \dots, n\}, \quad J = \mathbb{Z}_{\geq 0}, \quad J = \mathbb{Z},$$

let $(\mathscr{E}_j)_{j\in J}$, be a family of presheaves of $\mathscr{R}$-modules, and let $\Phi_j = (\Phi_{j,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}_j$ to $\mathscr{E}_{j+1}$, whenever $j, j+1 \in J$. If $j_0 \in J$ is such that $j_0 - 1, j_0, j_0 + 1 \in J$ then the sequence

$$\cdots \longrightarrow \mathscr{E}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{E}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{E}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

is ***exact*** at j_0 if $\ker(\Phi_{j_0,\mathcal{U}}) = \operatorname{image}(\Phi_{j_0-1,\mathcal{U}})$ for every $\mathcal{U} \in \mathscr{O}$. •

Of particular interest are so-called short exact sequences.

1.3.36 Definition (Short exact sequence of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$. A ***short exact sequence*** of presheaves is an exact sequence of the form

$$0 \longrightarrow \mathscr{E} \xrightarrow{\Phi} \mathscr{F} \xrightarrow{\Psi} \mathscr{G} \longrightarrow 0$$

for presheaves $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ of $\mathscr{R}$-modules, and $\mathscr{R}$-module morphisms $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ and $\Psi = (\Psi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{E}$ to $\mathscr{F}$ and $\mathscr{F}$ to $\mathscr{G}$, respectively. •

Exact sequences of sheaves

This notion of exactness of presheaves is natural. However, what one often knows in practice is only exactness of sequences of stalks. Since it is sheaves that are determined by their stalks, one often refers to this notion as exactness as sequences of sheaves.

1.3.37 Definition (Exact sequence of sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, let $J \subseteq \mathbb{Z}$ be of one of the following forms:

$$J = \{0, 1, \dots, n\}, \quad J = \mathbb{Z}_{\geq 0}, \quad J = \mathbb{Z},$$

let $(\mathscr{E}_j)_{j\in J}$, be a family of sheaves of $\mathscr{R}$-modules, and let $\Phi_j = (\Phi_{j,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be an $\mathscr{R}$-module morphism from $\mathscr{E}_j$ to $\mathscr{E}_{j+1}$, whenever $j, j+1 \in J$. If $j_0 \in J$ is such that $j_0 - 1, j_0, j_0 + 1 \in J$ then the sequence

$$\cdots \longrightarrow \mathscr{E}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{E}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{E}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

is ***exact*** at j_0 if $\ker(\Phi_{j_0,x}) = \operatorname{image}(\Phi_{j_0-1,x})$ for every $x \in \mathcal{S}$. •

We confess to the potential source of confusion in the language here. In practice, however, this is not a problem.

1.3.38 Terminology For a family $(\mathscr{E}_j)_{j\in J}$ of sheaves of $\mathscr{R}$-modules and a corresponding sequence

$$\cdots \longrightarrow \mathscr{E}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{E}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{E}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

we have two notions of exactness of this sequence at j_0, one according to Definition 1.3.35 and one according to Definition 1.3.37. We will *always* mean the stalkwise exactness of Definition 1.3.37 when we talk about exactness. When we mean exactness of presheaves according to Definition 1.3.35, we will write

$$\cdots \longrightarrow \mathscr{E}_{j_0-1}(\mathcal{U}) \xrightarrow{\Phi_{j_0-1,\mathcal{U}}} \mathscr{E}_{j_0}(\mathcal{U}) \xrightarrow{\Phi_{j_0,\mathcal{U}}} \mathscr{E}_{j_0+1}(\mathcal{U}) \xrightarrow{\Phi_{j_0+1,\mathcal{U}}} \cdots$$

explicitly indicating the open set $\mathcal{U}$. •

Of course, we also have short exact sequences of sheaves.

1.3.39 Definition (Short exact sequence of sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$. A ***short exact sequence*** of sheaves is an exact sequence of the form

$$0 \longrightarrow \mathscr{E} \xrightarrow{\Phi} \mathscr{F} \xrightarrow{\Psi} \mathscr{G} \longrightarrow 0$$

for sheaves $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ of $\mathscr{R}$-modules, and $\mathscr{R}$-module morphisms $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ and $\Psi = (\Psi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{E}$ to $\mathscr{F}$ and $\mathscr{F}$ to $\mathscr{G}$, respectively. •

We refer to the Terminology 1.3.38 for how we deal with the ambiguity of language here. Along these lines, let us make a connection between exact sequences of presheaves and exact sequences of sheaves.

1.3.40 Proposition (Short exact sequences of presheaves are short exact sequences of sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be sheaves of $\mathscr{R}$-modules. Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ and $\Psi = (\Psi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be $\mathscr{R}$-module morphisms from $\mathscr{E}$ to $\mathscr{F}$ and from $\mathscr{F}$ to $\mathscr{G}$, respectively. If the sequence*

$$0 \longrightarrow \mathscr{E}(\mathcal{U}) \xrightarrow{\Phi_{\mathcal{U}}} \mathscr{F}(\mathcal{U}) \xrightarrow{\Psi_{\mathcal{U}}} \mathscr{G}(\mathcal{U}) \longrightarrow 0$$

is exact for every $\mathcal{U} \in \mathscr{O}$, then the sequence

$$0 \longrightarrow \mathscr{E} \xrightarrow{\Phi} \mathscr{F} \xrightarrow{\Psi} \mathscr{G} \longrightarrow 0$$

is a short exact sequence of sheaves.

Proof Let $x \in \mathcal{S}$.

Suppose that $\Phi_x([s,\mathcal{U}]_x) = 0$. This means that there exists a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ for which $\Phi_{\mathcal{U}'}(r_{\mathcal{U},\mathcal{U}'}(s)) = 0$. Since $\Phi_{\mathcal{U}'}$ is injective, $r_{\mathcal{U},\mathcal{U}'}(s) = 0$ and so $[s]_x = 0$. Thus Φ_x is injective.

Let $[(u,\mathcal{U})]_x \in \mathscr{G}_x$. Since $\Psi_{\mathcal{U}}$ is surjective, there exists $t \in \mathscr{F}(\mathcal{U})$ such that $\Phi_{\mathcal{U}}(t) = u$. Then $\Phi_x([t]_x) = [u]_x$, and so Ψ_x is surjective.

Next let $[(t,\mathcal{U})]_x \in \operatorname{image}(\Phi_x)$. Then there exists a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ such that $r_{\mathcal{U},\mathcal{U}'}(t) \in \operatorname{image}(\Phi_{\mathcal{U}'}) = \ker(\Psi_{\mathcal{U}'})$. Thus $\Psi_{\mathcal{U}'}(r_{\mathcal{U},\mathcal{U}'}(t)) = 0$ and so $\Psi_x([t]_x) = 0$. Thus $\operatorname{image}(\Phi_x) \subseteq \ker(\Psi_x)$.

Finally, let $[(t,\mathcal{U})]_x \in \ker(\Psi_x)$. Then there exists a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ such that $\Psi_{\mathcal{U}'}(r_{\mathcal{U},\mathcal{U}'}(t)) = 0$. Thus there exists $s \in \mathscr{E}(\mathcal{U}')$ such that $r_{\mathcal{U},\mathcal{U}'}(t) = \Phi_{\mathcal{U}'}(s)$, and so $\Phi_x([s]_x) = [t]_x$, showing that $\ker(\Psi_x) \subseteq \operatorname{image}(\Phi_x)$. ■

The converse assertion, that a sequence that is short exact on stalks is short exact on open sets, is false. However, this is a point of departure for sheaf cohomology, so we leave this for Chapter 4, particularly to Section 4.1.2.

Exact sequences of étalé spaces

The corresponding notion can also be developed for étalé spaces.

1.3.41 Definition (Exact sequence of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$, let $J \subseteq \mathbb{Z}$ be of the form

$$J = \{0, 1, \dots, n\}, \quad J = \mathbb{Z}_{\geq 0}, \quad J = \mathbb{Z},$$

let $(\mathscr{U}_j)_{j\in J}$, be a family of étalé spaces of $\mathscr{A}$-modules, and let $\Phi_j\colon \mathscr{U}_j \to \mathscr{U}_{j+1}$ be an étalé morphism, whenever $j, j+1 \in J$. If $j_0 \in J$ is such that $j_0 - 1, j_0, j_0 + 1 \in J$ then the sequence

$$\cdots \longrightarrow \mathscr{U}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{U}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{U}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

is ***exact*** at j_0 if $\ker(\Phi_{j_0}) = \mathrm{image}(\Phi_{j_0-1})$. •

Of course, keeping in mind the Terminology 1.3.38, a sequence

$$\cdots \longrightarrow \mathscr{E}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{E}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{E}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots \tag{1.10}$$

is exact at j_0 if and only if

$$\cdots \longrightarrow \mathrm{Et}(\mathscr{E}_{j_0-1}) \xrightarrow{\mathrm{Et}(\Phi_{j_0-1})} \mathrm{Et}(\mathscr{E}_{j_0}) \xrightarrow{\mathrm{Et}(\Phi_{j_0})} \mathrm{Et}(\mathscr{E}_{j_0+1}) \xrightarrow{\mathrm{Et}(\Phi_{j_0+1})} \cdots \tag{1.11}$$

is exact at j_0. Moreover, this exactness is implied by the exactness of the sequence

$$\cdots \longrightarrow \mathscr{U}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{U}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{U}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots \tag{1.12}$$

at j_0 for every $\mathcal{U} \in \mathscr{O}$. However, exactness of the sequences (1.10) and (1.11) does not imply the exactness of the sequence (1.12) for every $\mathcal{U}$. This is really the subject of sheaf cohomology, which we discuss in Chapter 4.

Finally, we have short exact sequences of étalé spaces.

1.3.42 Definition (Short exact sequence of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$. A ***short exact sequence*** of étalé spaces is an exact sequence of the form

$$0 \longrightarrow \mathscr{U} \xrightarrow{\Phi} \mathscr{V} \xrightarrow{\Psi} \mathscr{W} \longrightarrow 0$$

for étalé spaces $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$ of $\mathscr{A}$-modules, and étalé morphisms of $\mathscr{A}$-modules $\Phi\colon \mathscr{R} \to \mathscr{S}$ and $\Psi\colon \mathscr{S} \to \mathscr{T}$, respectively. •

1.3.8 Operations on sheaves and short exact sequences

An important issue when dealing with sheaves is understanding how various operations interact with exact sequences. The proper setting for dealing with this is via the use of functors, but we shall only consider this formally in . Our presentation here will be just a little awkward for this reason.

what?

1.3.43 Definition (Exact, left exact, right exact operations on presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a presheaf of rings over $\mathcal{S}$. Let F be a rule assigning to each presheaf of $\mathscr{R}$-modules $\mathscr{F}$ over $\mathcal{S}$ a presheaf of $\mathscr{R}$-modules $F(\mathscr{F})$ over $\mathcal{S}$ and to every $\mathscr{R}$-module morphism Φ of presheaves $\mathscr{F}$ and $\mathscr{G}$ an $\mathscr{R}$-module morphism $F(\Phi)$ of $F(\mathscr{F})$ and $F(\mathscr{G})$. The assignment F is:

(i) ***left exact*** if the sequence

$$0 \longrightarrow F(\mathscr{F}) \xrightarrow{F(\Phi)} F(\mathscr{G}) \xrightarrow{F(\Psi)} F(\mathscr{H})$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{H} \longrightarrow 0;$$

(ii) ***right exact*** if the sequence

$$F(\mathscr{F}) \xrightarrow{F(\Phi)} F(\mathscr{G}) \xrightarrow{F(\Psi)} F(\mathscr{H}) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{H} \longrightarrow 0;$$

(iii) ***exact*** if it is left and right exact. •

For sheaves, we have the following notion.

1.3.44 Definition (Exact, left exact, right exact operations on sheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$. Let F be a rule assigning to each sheaf of $\mathscr{R}$-modules $\mathscr{F}$ over $\mathcal{S}$ a sheaf of $\mathscr{R}$-modules $F(\mathscr{F})$ over $\mathcal{S}$ and to every $\mathscr{R}$-module morphism Φ of sheaves $\mathscr{F}$ and $\mathscr{G}$ an $\mathscr{R}$-module morphism $F(\Phi)$ of $F(\mathscr{F})$ and $F(\mathscr{G})$. The assignment F is:

(i) ***left exact*** if the sequence

$$0 \longrightarrow F(\mathscr{F}) \xrightarrow{F(\Phi)} F(\mathscr{G}) \xrightarrow{F(\Psi)} F(\mathscr{H})$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{H} \longrightarrow 0;$$

(ii) ***right exact*** if the sequence

$$F(\mathscr{F}) \xrightarrow{F(\Phi)} F(\mathscr{G}) \xrightarrow{F(\Psi)} F(\mathscr{H}) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{H} \longrightarrow 0;$$

(iii) ***exact*** if it is left and right exact. •

Finally, we have the corresponding notion for étalé spaces.

1.3.45 Definition (Exact, left exact, right exact operations on étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{A}$ be an étalé space of rings over $\mathcal{S}$. Let F be a rule assigning to each étalé space of $\mathscr{A}$-modules $\mathscr{U}$ over $\mathcal{S}$ an étalé space $F(\mathscr{U})$ and to every étalé morphism of $\mathscr{A}$-modules $\Phi\colon \mathscr{U} \to \mathscr{V}$ an étalé morphism of $\mathscr{A}$-modules $F(\Phi)\colon F(\mathscr{U}) \to F(\mathscr{V})$. The assignment F is:

(i) ***left exact*** if the sequence

$$0 \longrightarrow F(\mathscr{U}) \xrightarrow{F(\Phi)} F(\mathscr{V}) \xrightarrow{F(\Psi)} F(\mathscr{W})$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{U} \xrightarrow{\Phi} \mathscr{V} \xrightarrow{\Psi} \mathscr{W} \longrightarrow 0;$$

(ii) ***right exact*** if the sequence

$$F(\mathscr{U}) \xrightarrow{F(\Phi)} F(\mathscr{V}) \xrightarrow{F(\Psi)} F(\mathscr{W}) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow \mathscr{U} \xrightarrow{\Phi} \mathscr{V} \xrightarrow{\Psi} \mathscr{W} \longrightarrow 0;$$

(iii) ***exact*** if it is left and right exact. •

Let us describe the operations with which we shall be concerned. We let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ be a topological space, let $\mathscr{R}$ be a sheaf, or étalé space of rings over $\mathcal{S}$, and let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be sheaves or étalé spaces, respectively, of $\mathscr{R}$-modules over $\mathcal{S}$.

1. *Taking morphisms from a given sheaf or étalé space I:* We fix the $\mathscr{R}$-module $\mathscr{E}$. To an $\mathscr{R}$-module $\mathscr{F}$ we assign the $\mathscr{R}$-module $\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\mathscr{F})$. To a morphism $\Phi \in \mathrm{Hom}_{\mathscr{R}}(\mathscr{F};\mathscr{G})$ we assign the morphism $\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\Phi)$ from $\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\mathscr{F})$ to $\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\mathscr{G})$ by

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{E};\Phi)_{\mathcal{U}}(\Psi) = \Phi_{\mathcal{U}} \circ \Psi$$

for $\Psi \in \mathscr{H}om_{\mathscr{R}}(\mathscr{F};\mathscr{G})(\mathcal{U})$.

2. *Taking morphisms from a given sheaf or étalé space II:* We fix the $\mathscr{R}$-module $\mathscr{E}$. To an $\mathscr{R}$-module $\mathscr{F}$ we assign the $\mathscr{R}$-module $\mathscr{H}om_{\mathscr{R}}(\mathscr{F};\mathscr{E})$. To a morphism $\Phi \in \mathrm{Hom}_{\mathscr{R}}(\mathscr{F};\mathscr{G})$ we assign the morphism $\mathscr{H}om_{\mathscr{R}}(\Phi;\mathscr{E})$ from $\mathscr{H}om_{\mathscr{R}}(\mathscr{G};\mathscr{E})$ to $\mathscr{H}om_{\mathscr{R}}(\mathscr{F};\mathscr{E})$ by

$$\mathscr{H}om_{\mathscr{R}}(\Phi;\mathscr{E})_{\mathcal{U}}(\Psi) = \Psi \circ \Phi_{\mathcal{U}}$$

for $\Psi \in \mathscr{H}om_{\mathscr{R}}(\mathscr{F};\mathscr{G})(\mathcal{U})$.

3. *Taking tensor products with a given presheaf, sheaf, or étalé space:* We fix an $\mathscr{R}$-module $\mathscr{E}$. To an $\mathscr{R}$-module $\mathscr{F}$ we assign the $\mathscr{R}$-module $\mathscr{E} \otimes \mathscr{F}$. To a morphism Φ from $\mathscr{F}$ to $\mathscr{G}$ we assign the morphism $\mathrm{id}_{\mathscr{E}} \otimes \Phi$ from $\mathscr{E} \otimes \mathscr{F}$ to $\mathscr{E} \otimes \mathscr{G}$.

4. *Taking direct images of presheaves, sheaves, or étalé spaces:* We consider a topological space $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ with $\mathscr{S}$ a sheaf or étalé space of rings over $\mathcal{T}$. We let $(\Phi, \Phi^\sharp)$ be a morphism of the ringed spaces $(\mathcal{S}, \mathscr{R})$ and $(\mathcal{T}, \mathscr{S})$. To an $\mathscr{R}$-module $\mathscr{E}$ over $\mathcal{S}$ we assign the $\mathscr{S}$-module $\Phi_*\mathscr{E}$ and to a morphism ϕ of $\mathscr{R}$-modules $\mathscr{E}$ and $\mathscr{F}$ we assign the morphism $\Phi_*\phi$ of $\mathscr{S}$-modules $\Phi_*\mathscr{E}$ and $\Phi_*\mathscr{F}$.
5. *Taking inverse images of presheaves, sheaves, or étalé spaces:* We consider a topological space $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ with $\mathscr{S}$ a sheaf or étalé space of rings over $\mathcal{T}$. We let $(\Phi, \Phi^\sharp)$ be a morphism of the ringed spaces $(\mathcal{S}, \mathscr{R})$ and $(\mathcal{T}, \mathscr{S})$. To an $\mathscr{S}$-module $\mathscr{L}$ over $\mathcal{T}$ we assign the $\mathscr{R}$-module $\Phi^*\mathscr{L}$ and to a morphism ϕ of $\mathscr{S}$-modules $\mathscr{L}$ and $\mathscr{M}$ we assign the morphism $\Phi^*\phi$ of $\mathscr{R}$-modules $\Phi^*\mathscr{L}$ and $\Phi^*\mathscr{M}$.

Let us state how these operations interact with short exact sequences. In all of the results, we let $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ and $(\mathcal{T}, \mathscr{O}_{\mathcal{T}})$ be topological spaces, let $\Phi \in C^0(\mathcal{S}; \mathcal{T})$, we let $\mathscr{R}$ be a sheaf or étalé space of rings over $\mathcal{S}$, let $\mathscr{S}$ be a sheaf or étalé space of rings over $\mathcal{T}$, let $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{R}$-modules, and let $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ be $\mathscr{S}$-modules.

1.3.46 Proposition (Morphisms and short exact sequences I) *If the sequence*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \mathscr{F}) \xrightarrow{\mathcal{H}om_{\mathscr{R}}(\mathscr{E};\phi)} \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \mathscr{G}) \xrightarrow{\mathcal{H}om_{\mathscr{R}}(\mathscr{E};\psi)} \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \mathscr{H})$$

is exact, i.e., $\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; -)$ *is left exact.*

Proof Let $x \in \mathcal{S}$. Suppose that

$$\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi)(\alpha) = \phi \circ \alpha = 0$$

and so $\phi_x \circ \alpha_x([s]_x) = 0$ for every $[s]_x \in \mathscr{E}_x$. Since ϕ_x is injective we have $\alpha_x([s]_x) = 0$ for every $[s]_x \in \mathscr{E}_x$, i.e., $\alpha_x = 0$ and so $\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi)$ is injective since x is arbitrary.

Since $\mathrm{image}(\phi) \subseteq \ker(\psi)$ we have $\psi \circ \phi = 0$ and so $\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \psi \circ \phi) = 0$. Since

$$\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \psi \circ \phi) = \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \psi) \circ \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi)$$

(as is easily verified), it follows that $\mathrm{image}(\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \psi)) \subseteq \ker(\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi))$.

Let $x \in \mathcal{S}$. Finally, let $\beta \in \ker(\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \psi))$, meaning that $\psi_x \circ \beta_x = 0$, meaning that $\mathrm{image}(\beta_x) \subseteq \ker(\psi_x) = \mathrm{image}(\phi_x)$. Define $\alpha \in \mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \mathscr{F})$ by asking that $\alpha_x([s]_x)$ be the unique element $[t]_x \in \mathscr{F}_x$ for which $\beta_x([s]_x) = \phi_x([t]_x)$, this making sense since ϕ_x is injective and since $\mathrm{image}(\beta_x) \subseteq \mathrm{image}(\phi_x)$. Note that

$$\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi)_x(\alpha_x)([s]_x) = \phi_x \circ \alpha_x([s]_x) = \beta_x([s]_x),$$

showing that $\beta \in \mathrm{image}(\mathcal{H}om_{\mathscr{R}}(\mathscr{E}; \phi))$ since x is arbitrary. ■

1.3.47 Proposition (Morphisms and short exact sequences II) *If the sequence*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow \mathcal{H}om_{\mathscr{R}}(\mathscr{H};\mathscr{E}) \xrightarrow{\mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E})} \mathcal{H}om_{\mathscr{R}}(\mathscr{G};\mathscr{E}) \xrightarrow{\mathrm{Hom}_{\mathscr{R}}(\phi;\mathscr{E})} \mathcal{H}om_{\mathscr{R}}(\mathscr{F};\mathscr{E})$$

is exact, i.e., $\mathcal{H}om_{\mathscr{R}}(-;\mathscr{E})$ is contravariant and left exact.

Proof Let $x \in \mathcal{S}$. Suppose that

$$\mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E})(\beta) = \beta \circ \psi = 0.$$

Thus $\beta_x \circ \psi_x([t]_x) = 0$ for every $[t]_x \in \mathscr{G}_x$. Since ψ_x is surjective, $\beta_x([u]_x) = 0$ for every $[u]_x \in \mathscr{H}_x$, showing that $\beta_x = 0$. Injectivity of $\mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E}) = 0$ follows since x is arbitrary.

Since $\mathrm{image}(\phi) \subseteq \ker(\psi)$ we have $\psi \circ \phi = 0$ and, since

$$0 = \mathcal{H}om_{\mathscr{R}}(\psi \circ \phi;\mathscr{E}) = \mathcal{H}om_{\mathscr{R}}(\phi;\mathscr{E}) \circ \mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E})$$

(as is easily verified), we conclude that $\mathrm{image}(\mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E})) \subseteq \ker(\mathcal{H}om_{\mathscr{R}}(\phi;\mathscr{E}))$.

Let $x \in \mathcal{S}$. Next let $\alpha \in \ker(\mathcal{H}om_{\mathscr{R}}(\phi;\mathscr{E}))$. Thus $\alpha_x \circ \phi_x = 0$ and so $\alpha_x|\,\mathrm{image}(\phi_x) = \alpha_x|\ker(\psi_x) = 0$. Therefore, there exists $\overline{\alpha}_x \in \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{G}_x/\ker(\psi_x);\mathscr{E}_x)$ such that

$$\overline{\alpha}_x([t]_x + \ker(\psi_x)) = \alpha_x([t]_x), \qquad [t]_x \in \mathscr{G}_x.$$

There is also an isomorphism $\overline{\psi}_x \in \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{G}_x/\ker(\psi_x);\mathscr{H}_x)$ such that

$$\overline{\psi}_x([t]_x + \ker(\psi_x)) = \psi_x([t]_x), \qquad [t]_x \in \mathscr{G}_x.$$

Let $\beta \in \mathcal{H}om_{\mathscr{R}}(\mathscr{H};\mathscr{E})$ be defined by $\beta_x = \overline{\alpha}_x \circ \overline{\psi}_x^{-1}$. Note that

$$\mathcal{H}om_{\mathscr{R}}(\psi;\mathscr{E})_x(\beta_x)([t]_x) = \beta_x \circ \psi_x([t]_x) = \overline{\alpha}_x \circ \overline{\psi}_x^{-1} \circ \psi_x([t]_x) = \alpha([t]_x),$$

as desired. ■

1.3.48 Proposition (Tensor products and short exact sequences) *If the sequence*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact, then the sequence

$$\mathscr{E} \otimes \mathscr{F} \xrightarrow{\mathrm{id}_{\mathscr{E}} \otimes \phi} \mathscr{E} \otimes \mathscr{G} \xrightarrow{\mathrm{id}_{\mathscr{E}} \otimes \psi} \mathscr{E} \otimes \mathscr{H} \longrightarrow 0$$

is exact, i.e., tensor product is right exact.

Proof Let $x \in \mathcal{S}$.

Let $[s]_x \otimes [u]_x \in \mathscr{E}_x \otimes \mathscr{H}_x$. Since ψ_x is surjective there exists $[t]_x \in \mathscr{G}_x$ such that

$$[s]_x \otimes [u]_x = [s]_x \otimes \psi_x([t]_x) = \mathrm{id}_{\mathscr{E},x} \otimes \psi_x([s]_x \otimes [t]_x).$$

Since $\mathscr{E}_x \otimes \mathscr{H}_x$ is generated by elements of the form $[s]_x \otimes [u]_x$, we conclude that $\mathrm{id}_{\mathscr{E},x} \otimes \psi_x$ is surjective.

Note that $\psi_x \circ \phi_x = 0$. One easily verifies that

$$(\mathrm{id}_{\mathscr{E}} \otimes \psi) \circ (\mathrm{id}_{\mathscr{E}} \otimes \phi) = \mathrm{id}_{\mathscr{E}} \otimes (\psi \circ \phi),$$

from which we conclude that $\mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x) \subseteq \ker(\mathrm{id}_{\mathscr{E},x} \otimes \psi_x)$.

From the preceding paragraph, there exists a homomorphism

$$\chi \in \mathscr{H}om_{\mathscr{R}}((\mathscr{E} \otimes \mathscr{G}) / \mathrm{image}(\mathrm{id}_{\mathscr{E}} \otimes \phi); \mathscr{E} \otimes \mathscr{H})$$

such that

$$\chi_x([s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x)) = \mathrm{id}_{\mathscr{E},x} \otimes \psi_x([s]_x \otimes [t]_x) = [s]_x \otimes \psi_x([t]_x)$$

for every $[s]_x \in \mathscr{E}_x$ and $[t]_x \in \mathscr{G}_x$. We claim that χ_x is an isomorphism.

To see this, define a bilinear map

$$\begin{aligned} \beta_x \colon \mathscr{E}_x \times \mathscr{H}_x &\to (\mathscr{E}_x \otimes \mathscr{G}_x) / \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x) \\ ([s]_x, [u]_x) &\mapsto [s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x), \end{aligned}$$

where $\psi_x([t]_x) = [u]_x$, this being possible since ψ_x is surjective. Let us show that β_x is well-defined, i.e., independent of the choice of $[t]_x$. So suppose that $[t]_x, [t']_x \in \mathscr{G}_x$ satisfy $\psi_x([t]_x) = \psi_x([t']_x) = [u]_x$. Then $[t]_x - [t']_x \in \ker(\psi_x) = \mathrm{image}(\phi_x)$ and so we can write $[t]_x - [t']_x = \phi_x([v]_x)$ for $[v]_x \in \mathscr{F}_x$. We then have

$$\begin{aligned} [s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x) &= [s]_x \otimes ([t']_x + \phi_x([v]_x)) + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x) \\ &= [s]_x \otimes [t']_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x), \end{aligned}$$

giving the desired well-definedness. Since β is bilinear we have an induced linear map

$$\hat{\beta}_x \in \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{E}_x \otimes \mathscr{H}_x; (\mathscr{E}_x \otimes \mathscr{G}_x) / \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x))$$

satisfying

$$\hat{\beta}_x([s]_x \otimes [u]_x) = [s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x),$$

where $[t]_x \in \mathscr{G}_x$ is such that $\psi_x([t]_x) = [u]_x$. Now note that

$$\chi_x \circ \hat{\beta}_x([s]_x, [u]_x) = \chi_x([s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x)) = [s]_x \otimes \chi_x([t]_x) = [s]_x \otimes [u]_x,$$

where, of course, $[t]_x \in \mathscr{G}_x$ is such that $\psi_x([t]_x) = [u]_x$. Also,

$$\hat{\beta}_x \circ \chi_x([s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x)) = \hat{\beta}_x([s]_x \otimes \psi_x([t]_x)) = [s]_x \otimes [t]_x + \mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x),$$

and so $\hat{\beta}_x$ is the inverse of χ_x.

To complete the proof, we note that the appropriate isomorphism theorem (e.g., [Hungerford 1980, Theorem IV.1.7]), along with the fact that χ_x is an isomorphism, implies that

$$\mathrm{image}(\mathrm{id}_{\mathscr{E},x} \otimes \phi_x) = \ker(\mathrm{id}_{\mathscr{E},x} \otimes \psi_x).$$

■

1.3.49 Proposition (Direct images and short exact sequences) *If the sequence*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow \Phi_*\mathscr{F} \xrightarrow{\Phi_*\phi} \Phi_*\mathscr{G} \xrightarrow{\Phi_*\psi} \Phi_*\mathscr{H}$$

is exact, i.e., direct image is left exact.

Proof Let $\mathcal{V} \in \mathscr{O}_{\mathfrak{T}}$.

Suppose that $s \in \ker((\Phi_*\phi)_{\mathcal{V}})$. Thus $s \in \mathscr{F}(\Phi^{-1}(\mathcal{V}))$ and $\phi_{\Phi^{-1}(\mathcal{V})}(s) = 0$. By Proposition 1.3.17 we conclude that $s = 0$, and so $\Phi_*\phi$ is injective by the arguments from the proof of Proposition 1.3.40.

Let $t \in \ker((\Phi_*\psi)_{\mathcal{V}})$. Thus $t \in \mathscr{G}(\Phi^{-1}(\mathcal{V}))$ and $\psi_{\Phi^{-1}(\mathcal{V})}(t) = 0$. By Lemma 4.1.3 below (specifically, applying the lemma to the restriction of the sheaves to $\Phi^{-1}(\mathscr{V})$), it follows that $t \in \operatorname{image}(\phi_{\Phi^{-1}(\mathcal{V})})$, and so $\ker((\Phi_*\psi)_{\mathcal{V}}) \subseteq \operatorname{image}((\Phi_*\phi)_{\mathcal{V}})$, and, by the arguments from the proof of Proposition 1.3.40, we conclude that $\ker(\Phi_*\psi) \subseteq \operatorname{image}(\Phi_*\phi)$.

Finally, let $t \in \operatorname{image}((\Phi_*\phi)_{\mathcal{V}})$. As in the previous step of the proof, we conclude that $t \in \ker((\Phi_*\psi)_{\mathcal{V}})$, and so conclude that $\operatorname{image}(\Phi_*\phi) \subseteq \ker(\Phi_*\psi)$. ■

1.3.50 Proposition (Inverse images and short exact sequences) *If the sequence*

$$0 \longrightarrow \mathscr{L} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\psi} \mathscr{N} \longrightarrow 0$$

is exact, then the sequence

$$\Phi^*\mathscr{L} \xrightarrow{\Phi^*\phi} \Phi^*\mathscr{M} \xrightarrow{\Phi^*\psi} \Phi^*\mathscr{N} \longrightarrow 0$$

is exact, i.e., inverse image is right exact.

Proof Here it is most convenient to work with étalé spaces directly. By the definition of $\Phi^{-1}\phi$ and $\Phi^{-1}\psi$ (see Definition 1.2.35 and observe Proposition 1.2.45) we have an exact sequence

$$0 \longrightarrow \Phi^{-1}\mathscr{L} \xrightarrow{\Phi^{-1}\phi} \Phi^{-1}\mathscr{M} \xrightarrow{\Phi^{-1}\psi} \Phi^{-1}\mathscr{N} \longrightarrow 0$$

of étalé spaces of $\Phi^{-1}\mathscr{S}$-modules. Since $\Phi^*\mathscr{L}$, $\Phi^*\mathscr{M}$, and $\Phi^*\mathscr{N}$ are obtained by taking tensor products as $\Phi^{-1}\mathscr{S}$-modules with $\mathscr{R}$, and since the morphisms $\Phi^*\phi$ and $\Phi^*\psi$ are the usual morphisms associated with tensor products, the result now follows from Proposition 1.3.48. ■

1.4 Vector bundles and sheaves

In this section we consider some relationships between vector bundles and sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules. The purpose of studying these relationships is twofold. On the one hand, one gets some useful intuition about sheaves of modules by understanding how they relate to vector bundles. On the other hand, the tools of sheaf theory provide a means to say some useful, and sometimes nontrivial, things about vector bundles.

1.4.1 Nakayama's Lemma and its consequences

Some of the constructions we make in this section will benefit from some general initial discussion of commutative algebra.

Recall that if R is a commutative unit ring, if $\mathsf{I} \subseteq \mathsf{R}$ is an ideal, and if A is a unital R-module, IA is the submodule of A generated by elements of the form rv where $r \in \mathsf{I}$ and $v \in \mathsf{A}$. We state Nakayama's Lemma.

1.4.1 Proposition (Nakayama's Lemma) *Let* R *be a commutative ring with unit, let* I *be an ideal of* R*, and let* A *be a finitely generated* R*-module. If* $\mathsf{A} = \mathsf{IA}$ *then there exists* $\mathrm{r} \in \mathsf{R}$ *such that*

(i) $\mathrm{r} \in 1 + \mathsf{I}$ *and*

(ii) $\mathrm{r}\mathsf{A} = 0$.

Proof Let $v_1, \dots, v_n \in \mathsf{A}$ be generators for A. First let $\phi \in \mathrm{End}_{\mathsf{R}}(\mathsf{A})$ satisfy $\mathrm{image}(\phi) \in \mathsf{IM}$ and write

$$\phi(v_j) = \sum_{k=1}^{n} a_j^k v_k$$

for some $a_j^k \in \mathsf{I}$, $j, k \in \{1, \dots, n\}$. Thus we have the identity

$$\sum_{k=1}^{n} (\delta_j^k \phi - a_j^k \,\mathrm{id}_{\mathsf{A}}) v_k = 0.$$

Let us denote by $\boldsymbol{M}$ the $n \times n$ matrix with entries in the ring $\mathrm{End}_{\mathsf{R}}(\mathsf{A})$ by

$$M_j^k = \delta_j^k \phi - a_j^k \,\mathrm{id}_{\mathsf{A}}\,.$$

If we denote by $\mathrm{adj}(\boldsymbol{M})$ the adjugate of $\boldsymbol{M}$, i.e., the matrix for which $\mathrm{adj}(\boldsymbol{M})\boldsymbol{M} = \det \boldsymbol{M}\boldsymbol{I}_n$ (with $\boldsymbol{I}_n$ the $n \times n$ identity matrix with entries in $\mathrm{End}_{\mathsf{R}}(\mathsf{A})$), then we have

$$0 = \sum_{k,l=1}^{n} \mathrm{adj}(\boldsymbol{M})_k^l (\delta_j^k \phi - a_j^k \,\mathrm{id}_{\mathsf{A}}) v_k = \det \boldsymbol{M} \sum_{k=1}^{n} \delta_j^k v_k, \qquad j \in \{1, \dots, n\}.$$

We conclude that $\det \boldsymbol{M}$ is the zero endomorphism. Expanding the determinant gives

$$\phi^n + p_{n-1}\phi^{n-1} + \dots + p_1 \phi + p_0 \,\mathrm{id}_{\mathsf{A}} = 0,$$

noting that $p_j \in \mathsf{I}$, $j \in \{0, 1, \dots, n-1\}$.

Applying the above argument to $\phi = \mathrm{id}_{\mathsf{A}}$ and taking

$$r = 1 + p_{n-1} + \dots + p_1 + p_0$$

gives

$$rv = (\mathrm{id}_{\mathsf{A}}^n + p_{n-1} \,\mathrm{id}_{\mathsf{A}}^{n-1} + \dots + p_1 \,\mathrm{id}_{\mathsf{A}} + p_0 \,\mathrm{id}_{\mathsf{A}}) v = 0$$

for every $v \in \mathsf{A}$. Clearly we also have $r = 1 + s$ for $s \in \mathsf{I}$. ■

We recall that, given a commutative unit ring R, the ***Jacobson radical*** of R is the intersection of all maximal ideals of R. With this notion at hand, the following corollaries are interesting for us.

1.4.2 Corollary (Consequences of Nakayama's Lemma) *Let* R *be a commutative ring with unit, let* I *be an ideal of* R*, and let* A *be a finitely generated* R*-module. Then the following statements hold:*

(i) *if* $\mathsf{I} \subseteq \mathrm{J}(\mathsf{R})$ *is an ideal and if* $\mathsf{A} = \mathsf{I}\mathsf{A}$ *then* $\mathsf{A} = 0$*;*

(ii) *if* $\mathsf{B} \subseteq \mathsf{A}$ *is a submodule and if* $\mathsf{I} \subseteq \mathrm{J}(\mathsf{R})$ *is an ideal such that* $\mathsf{A} = \mathsf{I}\mathsf{A} + \mathsf{B}$*, then* $\mathsf{B} = \mathsf{A}$*;*

(iii) *if* $\mathsf{I} \subseteq \mathrm{J}(\mathsf{R})$ *is an ideal and if* $\mathrm{v}_1 + \mathsf{I}\mathsf{A}, \dots, \mathrm{v}_n + \mathsf{I}\mathsf{A}$ *generate the* R*-module* $\mathsf{A}/\mathsf{I}\mathsf{A}$*, then* $\mathrm{v}_1, \dots, \mathrm{v}_n$ *generate* A*.*

Proof (i) Let us choose r as in Nakayama's Lemma, noting that $1 - r \in \mathsf{I} \subseteq J(\mathsf{R})$. We claim that r is a unit. Indeed, if r were not a unit, then the ideal (r) is proper and so contained in a maximal ideal J. Since $1 - r$ is an element of every maximal ideal, we have $1 - r \in \mathsf{J}$. This gives $1 \in \mathsf{J}$, contradicting maximality of J. Now, since r is a unit and since $rv = 0$ for every $v \in \mathsf{A}$, we conclude that $\mathsf{A} = 0$.

(ii) Note that

$$\begin{aligned}\mathsf{I}(\mathsf{A}/\mathsf{B}) &= \{a_1(v_1 + \mathsf{B}) + \dots + a_k(v_k + \mathsf{B}) \mid a_j \in \mathsf{I},\ v_j \in \mathsf{A},\ j \in \{1, \dots, k\},\ k \in \mathbb{Z}_{>0}\}\\ &= \{(a_1v_1 + u_1 + \mathsf{B}) + \dots + (a_kv_k + u_k + \mathsf{B}) \mid a_j \in \mathsf{I},\ v_j \in \mathsf{A},\ u_j \in \mathsf{B},\ j \in \{1, \dots, k\},\ k \in \mathbb{Z}_{>0}\}\\ &= (\mathsf{I}\mathsf{A} + \mathsf{B})/\mathsf{B}.\end{aligned}$$

Then note that, by hypothesis,

$$\mathsf{I}(\mathsf{A}/\mathsf{B}) = (\mathsf{I}\mathsf{A} + \mathsf{B})/\mathsf{B} = \mathsf{A}/\mathsf{B}.$$

From part (i) it follows that $\mathsf{A} = \mathsf{B}$.

(iii) Let B be the submodule generated by $v_1, \dots, v_n$. By hypothesis, if $v + \mathsf{I}\mathsf{A}$ then we can write

$$v + \mathsf{I}\mathsf{A} = r_1v_1 + \mathsf{I}\mathsf{A} + \dots + r_nv_n + \mathsf{I}\mathsf{A}$$

for $r_1, \dots, r_n \in \mathsf{R}$. Thus $v \in \mathsf{B} + \mathsf{I}\mathsf{A}$ and so $\mathsf{A} = \mathsf{B} + \mathsf{I}\mathsf{A}$. By part (ii), $\mathsf{B} = \mathsf{A}$, as desired. ■

In the case of local rings, Nakayama's Lemma contributes to the following result.

1.4.3 Proposition (Vector spaces from modules over local rings) *Let* R *be a commutative unit ring that is local, i.e., possesses a unique maximal ideal* $\mathfrak{m}$*, and let* A *be a unital* R*-module. Then* $\mathsf{A}/\mathfrak{m}\mathsf{A}$ *is a vector space over* $\mathsf{R}/\mathfrak{m}$*, and as a vector space is naturally isomorphic to* $(\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$*. Moreover, if* A *is finitely generated as an* R*-module, then the minimal number of generators for* A *is* $\dim_{\mathsf{R}/\mathfrak{m}}(\mathsf{A}/\mathfrak{m}\mathsf{A})$*.*

Proof We first prove that $\mathsf{R}/\mathfrak{m}$ is a field. Denote by $\pi_{\mathfrak{m}} \colon \mathsf{R} \to \mathsf{R}/\mathfrak{m}$ the canonical projection. Let $\mathsf{I} \subseteq \mathsf{R}/\mathfrak{m}$ be an ideal. We claim that

$$\tilde{\mathsf{I}} = \{r \in \mathsf{R} \mid \pi_{\mathfrak{m}}(r) \in \mathsf{I}\}$$

is an ideal in R. Indeed, let $r_1, r_2 \in \tilde{\mathsf{I}}$ and note that $\pi_{\mathfrak{m}}(r_1 - r_2) = \pi_{\mathfrak{m}}(r_1) - \pi_{\mathfrak{m}}(r_2) \in \mathsf{I}$ since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since I is an ideal. Thus $r_1 - r_2 \in \tilde{\mathsf{I}}$. Now let $r \in \tilde{\mathsf{I}}$ and $s \in \mathsf{R}$ and note that $\pi_{\mathfrak{m}}(sr) = \pi_{\mathfrak{m}}(s)\pi_{\mathfrak{m}}(r) \in \mathsf{I}$, again since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since I is an ideal. Thus $\tilde{\mathsf{I}}$ is an ideal. Clearly $\mathfrak{m} \subseteq \tilde{\mathsf{I}}$ so that either $\tilde{\mathsf{I}} = \mathfrak{m}$ or $\tilde{\mathsf{I}} = \mathsf{R}$. In the first case $\mathsf{I} = \{0_{\mathsf{R}} + \mathfrak{m}\}$ and in the second case $\mathsf{I} = \mathsf{R}/\mathfrak{m}$. Thus the only ideals of $\mathsf{R}/\mathfrak{m}$ are $\{0_{\mathsf{R}} + \mathfrak{m}\}$ and

$\mathsf{R}/\mathfrak{m}$. To see that this implies that $\mathsf{R}/\mathfrak{m}$ is a field, let $r + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$ be nonzero and consider the ideal $(r + \mathfrak{m})$. Since $(r + \mathfrak{m})$ is nontrivial we must have $(r + \mathfrak{m}) = \mathsf{R}/\mathfrak{m}$. In particular, $1 = (r + \mathfrak{m})(s + \mathfrak{m})$ for some $s + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$, and so $r + \mathfrak{m}$ is a unit.

Now we show that $\mathsf{A}/\mathfrak{m}\mathsf{A}$ is a vector space over $\mathsf{R}/\mathfrak{m}$. This amounts to showing that the natural vector space operations

$$(u + \mathfrak{m}\mathsf{A}) + (v + \mathfrak{m}\mathsf{A}) = u + v + \mathfrak{m}\mathsf{A}, \quad (r + \mathfrak{m})(u + \mathfrak{m}\mathsf{A}) = ru + \mathfrak{m}\mathsf{A}$$

make sense. The only possible issue is with scalar multiplication, so suppose that

$$r + \mathfrak{m} = s + \mathfrak{m}, \quad u + \mathfrak{m}\mathsf{A} = v + \mathfrak{m}\mathsf{A}$$

so that $s = r + a$ for $a \in \mathfrak{m}$ and $v = u + w$ for $w \in \mathfrak{m}\mathsf{A}$. Then

$$sv = (r + a)(u + w) = ru + au + rw + aw,$$

and we observe that $au, rw, aw \in \mathfrak{m}\mathsf{A}$, and so the sensibility of scalar multiplication is proved.

For the penultimate assertion, note that we have the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathsf{R} \longrightarrow \mathsf{R}/\mathfrak{m} \longrightarrow 0$$

By right exactness of the tensor product [Hungerford 1980, Proposition IV.5.4] this gives the exact sequence

$$\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow \mathsf{A} \longrightarrow (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow 0$$

noting that $\mathsf{R} \otimes_{\mathsf{R}} \mathsf{A} \simeq \mathsf{A}$. By this isomorphism, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in A is simply generated by elements of the form rv for $r \in \mathfrak{m}$ and $v \in \mathsf{A}$. That is to say, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in A is simply $\mathfrak{m}\mathsf{A}$. Thus we have the induced commutative diagram

$$\begin{tikzcd}
& \mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} \arrow[r] \arrow[d] & \mathsf{A} \arrow[r] \arrow[d] & (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathfrak{m}\mathsf{A} \arrow[r] & \mathsf{A} \arrow[r] & \mathsf{A}/\mathfrak{m}\mathsf{A} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We claim that there is an induced mapping as indicated by the dashed arrow, and that this mapping is an isomorphism. To define the mapping, let $\alpha \in (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$ and let $v \in \mathsf{A}$ project to α. The image of β is then taken to be $v + \mathfrak{m}\mathsf{A}$. It is a straightforward exercise to show that this mapping is well-defined and is an isomorphism, using exactness of the diagram.

The final assertion of the proposition follows from part (iii) of Corollary 1.4.2 since $\mathsf{A}/\mathfrak{m}\mathsf{A}$ is a vector space, and so has a well-defined number of generators. ■

1.4.2 From stalks of a sheaf to fibres

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. As we have seen in Example 1.1.10–5, this gives rise in a natural way to a sheaf, the sheaf $\mathscr{G}^r_{\mathsf{E}}$ of sections of E. The stalk of this sheaf at $x \in \mathsf{M}$ is the set $\mathscr{G}^r_{x,\mathsf{E}}$ of germs of sections which is a module over the ring $\mathscr{C}^r_{x,\mathsf{M}}$ of germs of functions. The stalk is *not* the same as the fibre E_x, however, the fibre can be obtained from the stalk, and in this section we see how this is done. The basic result is the following.

1.4.4 Proposition (From stalks to fibres) *Let* $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $r \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $r = \mathrm{hol}$. *Let* $\pi \colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^r. *For* $x \in \mathsf{M}$ *let* $\mathfrak{m}_x$ *denote the unique maximal ideal in* $\mathscr{C}^r_{x,\mathsf{M}}$. *Then the following statements hold:*

(i) the field $\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x$ *is isomorphic to* $\mathbb{F}$ *via the isomorphism*

$$[f]_x + \mathfrak{m}_x \mapsto f(x);$$

(ii) the $\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x$*-vector space* $\mathscr{G}^r_{x,\mathsf{E}}/\mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ *is isomorphic to* E_x *via the isomorphism*

$$[\xi]_x + \mathfrak{m}_k\mathscr{G}^r_{x,\mathsf{E}} \mapsto \xi(x);$$

(iii) the map from $(\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x) \otimes_{\mathscr{C}^r_{x,\mathsf{M}}} \mathscr{G}^r_{x,\mathsf{E}}$ *to* E_x *defined by*

$$([f]_x + \mathfrak{m}_x) \otimes [\xi]_x \mapsto f(x)\xi(x)$$

is an isomorphism of $\mathbb{F}$*-vector spaces.*

Moreover, if $(v_1, \dots, v_k)$ *is a basis for* E_x *and if* $[\xi_1]_x, \dots, [\xi_k]_x \in \mathscr{G}^r_{x,\mathsf{E}}$ *are such that* $[\xi_j]_k + \mathfrak{m}_x$ *maps to* v_j, $j \in \{1, \dots, k\}$, *under the isomorphism from part (ii), then* $[\xi_1]_x, \dots, [\xi_k]_x$ *generate* $\mathscr{G}^r_{x,\mathsf{E}}$.

Proof (i) The map is clearly a homomorphism of fields. To show that it is surjective, if $a \in \mathbb{F}$ then a is the image of $[f]_x + \mathfrak{m}_x$ for any germ $[f]_x$ for which $f(x) = a$. To show injectivity, if $[f]_x + \mathfrak{m}_x$ maps to 0 then clearly $f(x) = 0$ and so $f \in \mathfrak{m}_x$.

(ii) The map is clearly linear, so we verify that it is an isomorphism. Let $v_x \in \mathsf{E}_x$. Then v_x is the image of $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ for any germ $[\xi]_x$ for which $\xi(x) = v_x$. Also suppose that $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ maps to zero. Then $\xi(x) = 0$. Since $\mathscr{G}^r_{\mathsf{E}}$ is locally free (see the next section in case the meaning here is not patently obvious), it follows that we can write

$$\xi(y) = f_1(y)\eta_1(y) + \cdots + f_m(y)\eta_m(y)$$

for sections $\eta_1, \dots, \eta_m$ of class C^r in a neighbourhood of x and for functions $f_1, \dots, f_m$ of class C^r in a neighbourhood of x. Moreover, the sections may be chosen such that $(\eta_1(y), \dots, \eta_m(y))$ is a basis for E_y for every y in some suitably small neighbourhood of x. Thus

$$\xi(x) = 0 \quad\implies\quad f_1(x) = \cdots = f_m(x) = 0,$$

giving $\xi \in \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$, as desired.

(iii) The $\mathbb{F}$-linearity of the stated map is clear, and the fact that the map is an isomorphism follows from the final assertion of Proposition 1.4.3.

The final assertion of the result follows from the final assertion of Proposition 1.4.3. ■

This result relates stalks to fibres. In the next section, specifically in Theorem 1.4.12, we shall take a more global view towards relating vector bundles and sheaves.

In the preceding result we were able to rebuild the fibre of a vector bundle from the germs of sections. There is nothing keeping one from making this construction for a general sheaf.

1.4.5 Definition (Fibres for sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, and let $\mathscr{F}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. The ***fibre*** of $\mathscr{F}$ at $x \in \mathsf{M}$ is the $\mathbb{F}$-vector space $\mathsf{E}(\mathscr{F})_x = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$ and the ***rank*** of $\mathscr{F}$ at x is $\dim_{\mathbb{F}}(\mathsf{E}(\mathscr{F})_x)$. We let $\mathrm{rank}_{\mathscr{F}}\colon \mathsf{M} \to \mathbb{Z}_{\geq 0}$ be the function returning the rank of $\mathscr{F}$. •

This definition of fibre agrees (or more precisely is isomorphic to), of course, with the usual notion of the fibre of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ when $\mathscr{F} = \mathscr{G}^r_{\mathsf{E}}$; this is the content of the proof of Proposition 1.4.4.

We have the following general result concerning the behaviour of rank. We refer to Definition 1.4.8 below for the notion of a locally finitely generated sheaf of modules.

1.4.6 Lemma (Upper semicontinuity of rank) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *If* M *is a manifold of class* C^{r}, *and if* $\mathscr{F}$ *is a locally finitely generated sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules, then* $\mathrm{rank}_{\mathscr{F}}$ *is upper semicontinuous.*

Proof By the final assertion of Proposition 1.4.3, $\dim_{\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x}(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x)$ is the smallest number of generators of $\mathscr{F}_x$ as a $\mathscr{C}^r_{x,\mathsf{M}}$-module. By Lemma 1.4.9 below the minimal number of generators for the stalks of $\mathscr{F}$ at points in a neighbourhood of x is bounded above by the minimal number generators for $\mathscr{F}_x$. In other words, there is a neighbourhood $\mathcal{U}$ of x such that $\dim_{\mathbb{F}}(\mathsf{E}(\mathscr{F})_y) \leq \dim_{\mathbb{F}}(\mathsf{E}(\mathscr{F})_x)$ for $y \in \mathcal{U}$. From this, on a mere moment's reflection, we deduce the desired upper semicontinuity. ■

Let us look at a case of a sheaf which is not equivalent to a vector bundle in this sense.

1.4.7 Example (Fibres for a non-vector bundle sheaf) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let us take $\mathsf{M} = \mathbb{F}$ and define a presheaf $\mathscr{I}^r_0$ by

$$\mathscr{I}^r_0(\mathcal{U}) = \begin{cases} \mathrm{C}^r(\mathcal{U}), & 0 \notin \mathcal{U}, \\ \{f \in \mathrm{C}^r(\mathcal{U}) \mid f(0) = 0\}, & 0 \in \mathcal{U}. \end{cases}$$

One directly verifies that $\mathscr{I}^r_0$ is a sheaf. Moreover, $\mathscr{I}^r_0$ is a sheaf of $\mathscr{C}^r_{\mathbb{F}}$-modules; this too is easily verified. Let us compute the fibres associated with this sheaf. The germs of this sheaf at $x \in \mathbb{F}$ are readily seen to be given by

$$\mathscr{I}^r_{0,x} = \begin{cases} \mathscr{C}^r_{x,\mathbb{F}}, & x \neq 0, \\ \mathfrak{m}_0 = \{[f]_0 \in \mathscr{C}^r_{0,\mathbb{F}} \mid f(x) = 0\}, & x = 0. \end{cases}$$

Thus we have

$$\mathsf{E}(\mathscr{I}^r_0)_x = \begin{cases} \mathscr{C}^r_{x,\mathbb{F}}/\mathfrak{m}_x\mathscr{C}^r_{x,\mathbb{F}} \simeq \mathbb{F}, & x \neq 0, \\ \mathfrak{m}_0/\mathfrak{m}_0^2 \simeq \mathbb{F}, & x = 0. \end{cases}$$

Note that the fibre at 0 is "bigger" than we expect it to be. We shall address this shortly.

Let us expand on this example a little further. Let us consider the morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\text{ open}}$ of $\mathscr{C}^r_{\mathbb{F}}$-modules given by

$$\Phi_{\mathcal{U}}(f)(x) = xf(x),$$

i.e., Φ is multiplication by the function "x." We claim that $\mathscr{I}_0^r$ is the image presheaf of Φ. This claim is justified considering two cases.

1. $r = \infty$: As we showed in the proof of Lemma **??** from the proof of Proposition GA1.4.5.4, if f is a function defined in a neighbourhood $\mathcal{U}$ of 0 and vanishing at 0, we can write $f(x) = xg(x)$ for some $g \in \mathrm{C}^r(\mathcal{U})$.
2. $r \in \{\omega, \mathrm{hol}\}$: In this case, in a neighbourhood $\mathcal{U}$ of 0 we can write $f(x) = xg(x)$ for some $g \in \mathrm{C}^r(\mathcal{U})$ simply by factoring the Taylor series for f, noting that the zeroth order coefficient is zero since $f(0) = 0$.

Now, by Proposition 1.3.3 the kernel presheaf for Φ is a sheaf. If $g \in \ker(\Phi_{\mathcal{U}})$ then it is clear that $g(x) = 0$ for $x \neq 0$, and then continuity requires that $g(x) = 0$ for $x = 0$. That is to say, $\ker(\Phi)$ is the zero sheaf and so the fibres are also zero. •

1.4.3 Locally finitely generated sheaves

In this section we consider the important property of finite generation. This notion is especially important for holomorphic and real analytic sheaves, where it allows us to prove important global existence theorems from local hypotheses.

1.4.8 Definition (Locally finitely generated sheaf, locally finitely presented sheaf, locally free sheaf) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{F}$ be a sheaf of $\mathscr{R}$-modules.

(i) The sheaf $\mathscr{F}$ is ***locally finitely generated*** if, for each $x_0 \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x_0 and sections $s_1, \dots, s_k \in \mathscr{F}(\mathcal{U})$ such that $[s_1]_x, \dots, [s_k]_x$ generate the $\mathscr{R}_x$-module $\mathscr{F}_x$ for every $x \in \mathcal{U}$.

(ii) The sheaf $\mathscr{F}$ is ***locally finitely presented*** if, for each $x_0 \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x_0, $k \in \mathbb{Z}_{>0}$, and a morphism $\Phi\colon \mathscr{R}^k|\mathcal{U} \to \mathscr{F}$ whose kernel is a finitely generated sheaf of $\mathscr{R}$-modules.

(iii) The sheaf $\mathscr{F}$ is ***locally free*** if, for each $x_0 \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\mathscr{F}|\mathcal{U}$ is isomorphic to a direct sum $\oplus_{a\in A}(\mathscr{R}|\mathcal{U})$. •

An immediate consequence of the definitions is that $\mathscr{F}$ is a locally finitely generated sheaf of modules over the sheaf of rings $\mathscr{R}$ if and only if, for each $x \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x, $k \in \mathbb{Z}_{>0}$, and a morphism $\Phi\colon \mathscr{R}^k|\mathcal{U} \to \mathscr{F}|\mathcal{U}$ such that the diagram

$$\mathscr{R}^k|\mathcal{U} \xrightarrow{\Phi} \mathscr{F}|\mathcal{U} \longrightarrow 0$$

is exact, i.e., Φ is surjective. Similarly, $\mathscr{F}$ is locally finitely presented if and only if, for each $x \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x, $k, m \in \mathbb{Z}_{>0}$, and morphisms $\Phi\colon \mathscr{R}^k|\mathcal{U} \to \mathscr{F}|\mathcal{U}$ and $\Psi\colon \mathscr{R}^m|\mathcal{U} \to \mathscr{R}^k|\mathcal{U}$ such that the diagram

$$\mathscr{R}^m|\mathcal{U} \xrightarrow{\Psi} \mathscr{R}^k|\mathcal{U} \xrightarrow{\Phi} \mathscr{F}|\mathcal{U} \longrightarrow 0$$

is exact.

Let us explore these definitions a little. First, the following elementary result shows that, in the locally finitely generated case, the local generators can be selected from the generators for a particular stalk.

1.4.9 Lemma (Local generators for locally finitely generated sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{F}$ be a locally finitely generated sheaf of $\mathscr{R}$-modules. If, for $\mathsf{x}_0 \in \mathcal{S}$, $[\mathsf{s}_1]_{\mathsf{x}_0}, \dots, [\mathsf{s}_k]_{\mathsf{x}_0}$ are generators for the $\mathscr{R}_{\mathsf{x}_0}$-module $\mathscr{F}_{\mathsf{x}_0}$, then there exists a neighbourhood $\mathcal{U}$ of x_0 such that $[\mathsf{s}_1]_{\mathsf{x}}, \dots, [\mathsf{s}_k]_{\mathsf{x}}$ are generators for $\mathscr{F}_{\mathsf{x}}$ for each $\mathsf{x} \in \mathcal{U}$.*

Proof By hypothesis, there exists a neighbourhood $\mathcal{V}$ of x_0 and sections $t_1, \dots, t_k \in \mathscr{F}(\mathcal{V})$ such that $[t_1]_x, \dots, [t_m]_x$ generate $\mathscr{F}_x$ for all $x \in \mathcal{V}$. Since $[s_1]_{x_0}, \dots, [s_k]_{x_0}$ generate $\mathscr{F}_{x_0}$,

$$[t_l]_{x_0} = \sum_{j=1}^{k} [a_l^j]_{x_0} [s_j]_{x_0}, \qquad l \in \{1, \dots, m\},$$

for germs $[a_l^j]_{x_0} \in \mathscr{R}_{x_0}$. We can assume, possibly by shrinking $\mathcal{V}$, that $s_1, \dots, s_k \in \mathscr{F}(\mathcal{V})$ and $a_l^j \in \mathscr{R}(\mathcal{V})$, $j \in \{1, \dots, k\}$, $l \in \{1, \dots, m\}$. By definition of germ, there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{V}$ of x_0 such that

$$r_{\mathcal{V},\mathcal{U}}(t_l) = \sum_{j=1}^{k} r_{\mathcal{V},\mathcal{U}}(a_l^j) r_{\mathcal{V},\mathcal{U}}(s_j), \qquad l \in \{1, \dots, m\}.$$

Taking germs at $x \in \mathcal{U}$ shows that the generators $[t_1]_x, \dots, [t]_m$ of $\mathscr{F}_x$ are linear combinations of $[s_1]_x, \dots, [s_k]_x$, as desired. ■

Note that the property of being locally finitely generated is one about stalks, not one about local sections. Let us begin to explore this by look at some examples of sheaves that are *not* locally finitely generated. We shall subsequently see large classes of natural examples that *are* locally finitely generated, so it is the lack of this property that one should understand to properly contextualise it.

1.4.10 Examples (Sheaves that are not locally finitely generated)

1. Let $r \in \{\omega, \mathrm{hol}\}$ and take $\mathbb{F} = \mathbb{R}$ if $r = \omega$ and $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We consider $\mathsf{M} = \mathbb{F}$ and let
$$S = \{\tfrac{1}{j} \mid j \in \mathbb{Z}_{>0}\} \cup \{0\}.$$
Consider the presheaf $\mathscr{I}_S$ of $\mathscr{C}^r_{\mathsf{M}}$-modules given by
$$\mathscr{I}_S(\mathcal{U}) = \{f \in \mathrm{C}^r(\mathcal{U}) \mid f(x) = 0 \text{ for } x \in \mathcal{U} \cap S\}.$$
One can easily verify that $\mathscr{I}_S$ is a sheaf. We claim $\mathscr{I}_S$ is not locally finitely generated. The easiest way to see this is through the following observation. Note that $\mathscr{I}_{S,0} = \{0\}$ since any function of class C^r in a connected neighbourhood of 0 and vanishing on S must be zero by Proposition GA1.1.1.19. However, note that if $x \neq 0$ then $\mathscr{I}_{S,x} \neq \{0\}$ and so, by Lemma 1.4.9, it follows that $\mathscr{I}_S$ cannot be locally finitely generated.

2. Let us now give an example of a smooth sheaf of modules that is not locally finitely generated. We take $\mathsf{M} = \mathbb{R}$ and let $S = (-\infty, 0]$. We let $\mathscr{I}_S$ of $\mathscr{C}^r_{\mathsf{M}}$-modules given by

$$\mathscr{I}_S(\mathcal{U}) = \{f \in \mathrm{C}^r(\mathcal{U}) \mid f(x) = 0 \text{ for } x \in \mathcal{U} \cap S\}.$$

We shall employ a rather circuitous argument to show that $\mathscr{I}_S$ is not locally finitely generated. First, we identify functions on M with vector fields on M via the identification $f \mapsto f\frac{\partial}{\partial x}$. Upon making this identification, we have a distribution D on M given by

$$\mathsf{D}_x = \operatorname{span}_{\mathbb{R}}(f(x) \mid f \in \mathscr{I}_{S,x}).$$

Thus

$$\mathsf{D}_x = \begin{cases} \mathsf{T}_x\mathsf{M}, & x > 0, \\ \{0\}, & x \leq 0. \end{cases}$$

With this identification, $\mathscr{I}_S$ is thought of as the subsheaf $\mathscr{C}^\infty_{\mathsf{D}}$ of the sheaf of smooth vector fields consisting of those vector fields taking values in D. We assume the reader knows about involutive and integrable distributions, and refer to [Lewis 2013, Section 5.6] for the required background. This being understood, it is clear that D is involutive since D_x is either zero- or one-dimensional. However, D is not integrable since there is no integral manifold for D through 0. This prohibits $\mathscr{C}^\infty_{\mathsf{D}}$ from being locally finitely generated, since there is a one-to-one correspondence between involutive and integrable distributions in the case when the sheaf of sections of the distribution is locally finitely generated; see [Lewis 2013, Theorem 5.6.6(ii)].

3. Let $\mathsf{M} = \mathbb{R}^3$ and define $f \in \mathrm{C}^\omega(\mathbb{R}^3)$ be defined by

$$f(x_1, x_2, x_3) = x_3(x_1^2 + x_2^2) - x_2^3.$$

We take $\mathsf{S} = f^{-1}(0)$ and let $\mathscr{I}_{\mathsf{S}}$ be the sheaf defined by

$$\mathscr{I}_{\mathsf{S}}(\mathcal{U}) = \{g \in \mathscr{C}^\omega_{\mathbb{R}^3}(\mathcal{U}) \mid g(\boldsymbol{x}) = 0 \text{ for all } \boldsymbol{x} \in \mathsf{S} \cap \mathcal{U}\}.$$

We shall examine this sheaf in more detail in Example 6.3.1. For the moment let us point out the salient facts.

(a) The germ $\mathscr{I}_{\mathsf{S},\mathbf{0}}$ is generated over $\mathscr{C}^\omega_{\mathbf{0},\mathbb{R}^3}$ by $[f]_{\mathbf{0}}$.

(b) Let $\xi_1, \xi_2 \in \mathrm{C}^\omega(\mathbb{R}^3)$ be given by $\xi_a(x_1, x_2, x_3) = x_a$, $a \in \{1, 2\}$. Then, for any $(0, 0, x_3) \in \mathbb{R}^3$ with $x_3 \neq 0$, the germ $\mathscr{I}_{\mathsf{S},(0,0,x_3)}$ is generated over $\mathscr{C}^\omega_{(0,0,x_3)}\mathbb{R}^3$ by $[\xi_1]_{(0,0,x_3)}$ and $[\xi_2]_{(0,0,x_3)}$.

It then follows from Lemma 1.4.9 that $\mathscr{I}_{\mathsf{S}}$ is not locally finitely generated in any neighbourhood of $\mathbf{0}$. Note, however, that the $\mathrm{C}^\omega(\mathbb{R}^3)$-module $\mathscr{I}_{\mathsf{S}}(\mathbb{R}^3)$ is generated by f. •

In the smooth case and often in the real analytic case, there actually *is* a correspondence between locally finitely generated sheaves and sheaves with finitely generated spaces of local sections. As we see in the proof of the next result, this is a consequence of the vanishing of the cohomology groups of the sheaves in these cases.

1.4.11 Proposition (Local finite generation of modules of sections) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let M be a manifold of class C^r, let $\mathscr{F}$ be a finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. Assume one of the two cases:*

(i) $r = \infty$;

(ii) $r \in \{\omega, \mathrm{hol}\}$ *and* $\mathscr{F}$ *is coherent;*

Let $x_0 \in \mathsf{M}$. If $[(s_1, \mathcal{U})]_{x_0}, \dots, [(s_k, \mathcal{U})]_{x_0}$ generate $\mathscr{F}_{x_0}$, then there exists a neighbourhood $\mathcal{W} \subseteq \mathcal{U}$ of x_0 such that $r_{\mathcal{U},\mathcal{V}}(s_1), \dots, r_{\mathcal{U},\mathcal{V}}(s_k)$ generate $\mathscr{F}(\mathcal{V})$ for every (Stein, if $r = \mathrm{hol}$) open set $\mathcal{V} \subseteq \mathcal{W}$.

Proof From the proof of Lemma 1.4.9 we see that there exists a neighbourhood $\mathcal{W}$ of x_0 such that $([s_1]_x, \dots, [s_k]_x)$ generate $\mathscr{F}_x$ for every $x \in \mathcal{W}$. If $r = \mathrm{hol}$ then we can suppose that $\mathcal{W}$ is a Stein neighbourhood, e.g., by taking $\mathcal{W}$ to be a polydisk in a $\mathbb{C}$-chart about x_0. If $\mathcal{V} \subseteq \mathcal{W}$ (assuming $\mathcal{V}$ Stein in the case when $r = \mathrm{hol}$), we then have a presheaf morphism $\Phi = (\Phi_{\mathcal{V}'})_{\mathcal{V}' \subseteq \mathcal{V} \text{ open}}$ from $(\mathscr{C}^r_{\mathcal{V}})^k$ to $\mathscr{F}|\mathcal{V}$ given by

$$\Phi_{\mathcal{V}'}(f^1, \dots, f^k) = f^1 r_{\mathcal{U},\mathcal{V}'}(s_1) + \dots + f^k r_{\mathcal{U},\mathcal{V}'}(s_k),$$

Note that the sequence

$$(\mathscr{C}^r_{\mathcal{V}})^k \xrightarrow{\Phi} \mathscr{F}|\mathcal{V} \longrightarrow 0 \tag{1.13}$$

is exact, i.e., is exact on stalks, by hypothesis. If $s \in \mathscr{F}(\mathcal{V})$, exactness of (1.13) implies that, for $x \in \mathcal{V}$,

$$[s]_x = [g^1]_x[s_1]_x + \dots + [g^k]_x[s_k]_x$$

for $[g^1]_1, \dots, [g^k]_x \in \mathscr{C}^r_{x,\mathsf{M}}$. Since the preceding expression involves only a finite number of germs, there exists a neighbourhood $\mathcal{V}_x \subseteq \mathcal{V}$ of x such that

$$r_{\mathcal{V},\mathcal{V}_x}(s) = g^1_x r_{\mathcal{U},\mathcal{V}_x}(s_1) + \dots + g^k_x r_{\mathcal{U},\mathcal{V}_x}(s_k)$$

for $g^1_x, \dots, g^k_x \in C^r(\mathcal{V}_x)$. Let $\mathscr{V} = (\mathcal{V}_x)_{x \in \mathcal{V}}$. If $\mathcal{V}_x \cap \mathcal{V}_y \neq \emptyset$, define $g^j_{xy} \in C^r(\mathcal{V}_x \cap \mathcal{V}_y)$ by

$$g^j_{xy} = g^j_x|\mathcal{V}_x \cap \mathcal{V}_y - g^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1, \dots, k\},$$

and note that $((g^1_{xy}, \dots, g^k_{xy}))_{x,y \in \mathcal{V}} \in Z^1(\mathscr{V}, \ker(\Phi))$. We now note the following (where we make reference to notation from Section 4.4 and results from Sections 4.5 and 5.3):

1. if $r = \infty$ then $H^1(\mathscr{V}; \ker(\Phi)) = 0$ by Theorem 4.5.1;
2. if $r \in \{\omega, \mathrm{hol}\}$ then $\ker(\Phi)$ is coherent by Proposition 5.1.6(iii), and so $H^1(\mathscr{V}; \ker(\Phi)) = 0$ by Theorem 5.3.2.

Since $H^1(\mathscr{V}; \ker(\Phi)) = 0$, there exists $((h^1_x, \dots, h^k_x))_{x \in \mathcal{V}_x} \in C^1(\mathscr{V}; \ker(\Phi))$ such that

$$h^j_y|\mathcal{V}_x \cap \mathcal{V}_y - h^j_x|\mathcal{V}_x \cap \mathcal{V}_y = g^j_{xy} = g^j_x|\mathcal{V}_x \cap \mathcal{V}_y - g^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1, \dots, k\}.$$

Define $f^j_x \in C^r(\mathcal{V}_x)$ by $f^j_x = g^j_x + h^j_x$, and note that

$$f^j_x|\mathcal{V}_x \cap \mathcal{V}_y = f^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1, \dots, k\}.$$

Thus there exists $f^j \in \mathrm{C}^r(\mathcal{V})$ such that $f^j|\mathcal{V}_x = f^j_x$ for each $j \in \{1, \dots, k\}$ and $x \in \mathcal{V}$. Moreover, since

$$h^1_x r_{\mathcal{U},\mathcal{V}_x}(s_1) + \cdots + h^k_x r_{\mathcal{U},\mathcal{V}_x}(s_k) = 0,$$

we have

$$f^1 r_{\mathcal{U},\mathcal{V}}(s_1) + \cdots + f^k r_{\mathcal{U},\mathcal{V}}(s_k) = s,$$

as desired. ■

1.4.4 Locally finitely generated and locally free sheaves

Let us consider locally free, locally finitely generated sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules, as these correspond to something familiar to us.

1.4.12 Theorem (Correspondence between vector bundles and locally free, locally finitely generated sheaves) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r}*. Then* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ *is a locally free, locally finitely generated sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules.*

Conversely, if $\mathscr{F}$ *is a locally free, locally finitely generated sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules, then there exists a vector bundle* $\pi\colon \mathsf{E} \to \mathsf{M}$ *of class* C^{r} *such that* $\mathscr{F}$ *is isomorphic to* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$*.*

Proof First let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r and let $x_0 \in \mathsf{M}$. Let $(\mathcal{V}, \psi)$ be a vector bundle chart such that the corresponding chart $(\mathcal{U}, \phi)$ for M contains x_0. Suppose that $\psi(\mathcal{V}) = \phi(\mathcal{U}) \times \mathbb{F}^m$ and let $\eta_1, \dots, \eta_m \in \Gamma^r(\mathsf{E}|\mathcal{U})$ satisfy $\psi(\eta_j(x)) = (\phi(x), \boldsymbol{e}_j)$ for $x \in \mathcal{U}$ and $j \in \{1, \dots, m\}$. Let us arrange the components η^k_j, $j, k \in \{1, \dots, m\}$, of the sections $\eta_1, \dots, \eta_m$ in an $m \times m$ matrix:

$$\boldsymbol{\eta}(x) = \begin{bmatrix} \eta^1_1(x) & \cdots & \eta^1_m(x) \\ \vdots & \ddots & \vdots \\ \eta^m_1(x) & \cdots & \eta^m_m(x) \end{bmatrix}.$$

Now let $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$, let the components of ξ be ξ^k, $k \in \{1, \dots, k\}$, and arrange the components in a vector

$$\boldsymbol{\xi}(x) = \begin{bmatrix} \xi^1(x) \\ \vdots \\ \xi^m(x) \end{bmatrix}.$$

Now fix $x \in \mathcal{U}$. We wish to solve the equation

$$\xi(x) = f^1(x)\eta_1(x) + \cdots + f^m(x)\eta_m(x)$$

for $f^1(x), \dots, f^m(x) \in \mathbb{F}$. Let us write

$$\boldsymbol{f}(x) = \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix}.$$

Writing the equation we wish to solve as a matrix equation we have

$$\boldsymbol{\xi}(x) = \boldsymbol{\eta}(x)\boldsymbol{f}(x).$$

Therefore,

$$\boldsymbol{f}(x) = \boldsymbol{\eta}^{-1}(x)\boldsymbol{\xi}(x),$$

noting that $\boldsymbol{\eta}(x)$ is invertible since the vectors $\eta_1(x), \dots, \eta_m(x)$ are linearly independent. By Cramer's Rule, or some such, the components of $\boldsymbol{\eta}^{-1}$ are C^r-functions of $x \in \mathcal{U}$, and so ξ is a $\mathrm{C}^r(\mathcal{U})$-linear combination of $\eta_1, \dots, \eta_m$, showing that $\Gamma^r(\mathsf{E}|\mathcal{U})$ is finitely generated. To show that this module is free, it suffices to show that $(\eta_1, \dots, \eta_m)$ is linearly independent over $\mathrm{C}^r(\mathcal{U})$. Suppose that there exists $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{U})$ such that

$$f^1\eta_1 + \dots + f^m\eta_m = 0_{\Gamma^r(\mathsf{E})}.$$

Then, for every $x \in \mathcal{U}$,

$$f^1(x)\eta_1(x) + \dots + f^m(x)\eta_m(x) = 0_x \quad \Longrightarrow \quad f^1(x) = \dots = f^m(x) = 0,$$

giving the desired linear independence.

Next suppose that $\mathscr{F}$ is a locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. Let us first define the total space of our vector bundle. For $x \in \mathsf{M}$ define

$$\mathsf{E}_x = \mathscr{F}_x / \mathfrak{m}_x \mathscr{F}_x.$$

By Propositions 1.4.3 and 1.4.4, E_x is a $\mathbb{F}$-vector space. We take $\mathsf{E} = \mathring{\cup}_{x \in \mathsf{M}} \mathsf{E}_x$. Let $x \in \mathsf{M}$ and let $\mathcal{U}_x$ be a neighbourhood of x such that $\mathscr{F}(\mathcal{U}_x)$ is a free $\mathrm{C}^r(\mathcal{U}_x)$-module. By shrinking $\mathcal{U}_x$ if necessary, we suppose that it is the domain of a coordinate chart $(\mathcal{U}_x, \phi_x)$. Let $s_1, \dots, s_m \in \mathscr{F}(\mathcal{U}_x)$ be such that $(s_1, \dots, s_m)$ is a basis for $\mathscr{F}(\mathcal{U}_x)$. Note that $([s_1]_y, \dots, [s_m]_x)$ is a basis for $\mathscr{F}_y$ for each $y \in \mathcal{U}_x$. It is straightforward to show that

$$([s_1]_y + \mathfrak{m}_y\mathscr{F}_y, \dots, [s_m]_y + \mathfrak{m}_y\mathscr{F}_y)$$

is then a basis for E_y. For $y \in \mathcal{U}_x$ the map

$$a^1([s_1]_y + \mathfrak{m}_y) + \dots + a^m([s_m]_y + \mathfrak{m}_y) \mapsto (a^1, \dots, a^m)$$

is clearly an isomorphism. Now define $\mathcal{V}_x = \mathring{\cup}_{y \in \mathcal{U}_x} \mathsf{E}_y$ and define $\psi_x \colon \mathcal{V}_x \to \phi_x(\mathcal{U}_x) \times \mathbb{F}^m$ by

$$\psi_x(a^1([s_1]_y + \mathfrak{m}_y) + \dots + a^m([s_m]_y + \mathfrak{m}_y)) = (\psi_x(y), (a^1, \dots, a^m)).$$

This is clearly a vector bundle chart for E. Moreover, this construction furnishes a covering of E by vector bundle charts.

It remains to show that two overlapping vector bundle charts satisfy the appropriate overlap condition. Thus let $x, y \in \mathsf{M}$ be such that $\mathcal{U}_x \cap \mathcal{U}_y$ is nonempty. Let $(s_1, \dots, s_m)$ and $(t_1, \dots, t_m)$ be bases for $\mathscr{F}(\mathcal{U}_x)$ and $\mathscr{F}(\mathcal{U}_y)$, respectively. (Note that the cardinality of these bases agrees since, for $z \in \mathcal{U}_x \cap \mathcal{U}_y$, $([s_1]_z, \dots, [s_m]_z)$ and $([t_1]_z, \dots, [t_m]_z)$ are both bases for $\mathscr{F}_z$, cf. [Hungerford 1980, Corollary IV.2.12].) Note that

$$r_{\mathcal{U}_x, \mathcal{U}_x \cap \mathcal{U}_y}(s_j) = \sum_{k=1}^m f_j^k r_{\mathcal{U}_y, \mathcal{U}_x \cap \mathcal{U}_y}(t_k)$$

for $f_j^k \in C^r(\mathcal{U}_x \cap \mathcal{U}_y)$, $j,k \in \{1,\dots,m\}$. At the stalk level we have

$$[s_j]_z = \sum_{k=1}^m [f_j^k]_z [t_k]_z,$$

from which we conclude that

$$([s_j]_z + \mathfrak{m}_z \mathscr{F}_z) = \sum_{k=1}^m f_j^k(z)([t_k]_z + \mathfrak{m}_z \mathscr{F}_z),$$

From this we conclude that the matrix

$$\boldsymbol{f}(z) = \begin{bmatrix} f_1^1(z) & \cdots & f_m^1(z) \\ \vdots & \ddots & \vdots \\ f_1^m(z) & \cdots & f_m^m(z) \end{bmatrix}$$

is invertible, being the change of basis matrix for the two bases for E_z. Moreover, the change of basis formula gives

$$\psi_y \circ \psi_x^{-1}(\boldsymbol{z}, (a^1, \dots, a^m)) = \Big(\phi_y \circ \phi_x^{-1}(\boldsymbol{z}), \Big(\sum_{j=1}^m a^j f_j^1(z), \dots, \sum_{j=1}^m a^j f_j^m(z)\Big)\Big)$$

for every $z \in \mathcal{U}_x \cap \mathcal{U}_y$, where $\boldsymbol{z} = \phi_x(z)$. Thus we see that the covering by vector bundle charts has the proper overlap condition to define a vector bundle structure for E.

It remains to show that $\mathscr{G}^r_{\mathsf{E}}$ is isomorphic to $\mathscr{F}$. Let $\mathcal{U} \subseteq \mathsf{M}$ be open and define $\Phi_{\mathcal{U}} \colon \mathscr{F}(\mathcal{U}) \to \Gamma^r(\mathsf{E}|\mathcal{U})$ by

$$\Phi_{\mathcal{U}}(s)(x) = [s]_x + \mathfrak{m}_x \mathscr{F}_x.$$

For this definition to make sense, we must show that $\Phi_{\mathcal{U}}(s)$ is of class C^r. Let $y \in \mathcal{U}$ and, using the above constructions, let $(s_1, \dots, s_m)$ be a basis for $\mathscr{F}(\mathcal{U}_y)$. Let us abbreviate $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_y$. Note that $(r_{\mathcal{U},\mathcal{V}}(s_1), \dots, r_{\mathcal{U},\mathcal{V}}(s_m))$ is a basis for $\mathscr{F}(\mathcal{V})$. (To see that this is so, one can identify $\mathscr{F}(\mathcal{U})$ with $\Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ using Proposition 1.1.88, and having done this the assertion is clear.) We thus write

$$r_{\mathcal{U},\mathcal{V}}(s) = f^1 r_{\mathcal{U},\mathcal{V}}(s_1) + \dots + f^m r_{\mathcal{U},\mathcal{V}}(s_m).$$

In terms of stalks we thus have

$$[s]_z = [f^1]_z [s_1]_z + \dots + [f^m]_z [s_m]_z$$

for each $z \in \mathcal{V}$. Therefore,

$$\Phi_{\mathcal{U}}(s)(z) = f^1(z)([s_1]_z + \mathfrak{m}_z \mathscr{F}_z) + \dots + f^m(z)([s_m]_z + \mathfrak{m}_z \mathscr{F}_z),$$

which (recalling that $\mathcal{U}_y$, and so also $\mathcal{V}$, is a chart domain) gives the local representative of $\Phi_{\mathcal{U}}(s)$ on $\mathcal{V}$ as

$$\boldsymbol{z} \mapsto (\boldsymbol{z}, (f^1 \circ \phi_y^{-1}(\boldsymbol{z}), \dots, f^m \circ \phi_y^{-1}(\boldsymbol{z}))).$$

Since this local representative is of class C^r and since this construction can be made for any $y \in \mathcal{U}$, we conclude that $\Phi_{\mathcal{U}}(s)$ is of class C^r.

Now, to show that the family of mappings $(\Phi_{\mathcal{U}})_{\mathcal{U} \text{ open}}$ is an isomorphism, by Proposition 1.3.16 it suffices to show that the induced mapping on stalks is an isomorphism. Let us denote the mapping of stalks at x by Φ_x. We again use our constructions from the first part of this part of the proof and let $(s_1, \dots, s_m)$ be a basis for $\mathscr{F}(\mathcal{U}_x)$. Let us show that Φ_x is surjective. Let $[\xi]_x \in \mathscr{G}^r_{x,\mathsf{M}}$, supposing that $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$. Let $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_x$. Let the local representative of ξ on $\mathcal{V}$ in the chart $(\mathcal{V}_x, \psi_x)$ be given by

$$\boldsymbol{y} \mapsto (\boldsymbol{y}, (f^1 \circ \phi_x^{-1}(\boldsymbol{y}), \dots, f^m \circ \phi_x^{-1}(\boldsymbol{y})))$$

for $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{V})$. Then, if

$$[s]_x = [f^1]_x[s_1]_x + \dots + [f^m]_x[s_m]_x,$$

we have $\Phi_x([s]_x) = [\xi]_x$. To prove injectivity of Φ_x, suppose that $\Phi_x([s_x]) = 0_x$. This means that $\Phi_x([s]_x)$ is the germ of a section of E over some neighbourhood $\mathcal{U}$ of x that is identically zero. We may without loss of generality assume that $\mathcal{U} \subseteq \mathcal{U}_x$. We also assume without loss of generality (by restriction of necessary) that $s \in \mathscr{F}(\mathcal{U})$. We thus have

$$\Phi_{\mathcal{U}}(s)(y) = 0, \qquad y \in \mathcal{U}.$$

Since $(r_{\mathcal{U}_x,\mathcal{U}}(s_1), \dots, r_{\mathcal{U}_x,\mathcal{U}}(s_m))$ is a basis for $\mathscr{F}(\mathcal{U})$ we write

$$s = f^1 r_{\mathcal{U}_x,\mathcal{U}}(s_1) + \dots + f^m r_{\mathcal{U}_x,\mathcal{U}}(s_m).$$

for some uniquely defined $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{U})$. We have

$$\Phi_{\mathcal{U}}(s)(y) = f^1(y)([s_1]_y + \mathfrak{m}_y\mathscr{F}_y) + \dots + f^m(y)([s_m]_y + \mathfrak{m}_y\mathscr{F}_y)$$

for each $y \in \mathcal{U}$. Since

$$([s_1]_y + \mathfrak{m}_y\mathscr{F}_y, \dots, [s_m]_y + \mathfrak{m}_y\mathscr{F}_y)$$

is a basis for E_y, we must have $f^1(y) = \dots = f^m(y) = 0$ for each $y \in \mathcal{U}$, giving $[s]_x = 0$. ■

One should be careful about what the theorem does *not* say. It does not say that every locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules is the sheaf of sections of a vector bundle, only that it is isomorphic to such a sheaf of sections. An example clarifies this distinction.

1.4.13 Example (A locally free, locally finitely generated sheaf that is not a sheaf of sections of a vector bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We let $\mathsf{M} = \mathbb{F}$ and take $\mathsf{E} = \mathbb{F} \times \mathbb{F}$ with vector bundle projection $\pi\colon \mathsf{E} \to \mathsf{M}$ being projection onto the first factor. We let $\mathscr{F}$ be the sheaf of C^r sections of E that vanish at $0 \in \mathbb{R}$, cf. Example 1.4.7. As we saw in Example 1.4.7, every section ξ that vanishes at 0 can be written as $\xi(x) = x\eta(x)$ for a nonvanishing section η. •

1.4.5 Sheaf morphisms and vector bundle mappings

Having seen how sheaves and vector bundles are related, let us consider how mappings of vector bundles give rise to morphisms of the corresponding sheaves. We begin by considering the situation of morphisms of vector bundles. Thus we let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as required, and consider $\mathbb{F}$-vector bundles $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\tau\colon \mathsf{F} \to \mathsf{M}$ of class C^r. We let $\Phi\colon \mathsf{E} \to \mathsf{F}$ be a vector bundle mapping over id_{M}. Thus $\Phi(\mathsf{E}_x) \subseteq \mathsf{F}_x$ and $\Phi|\mathsf{E}_x$ is $\mathbb{F}$-linear for each $x \in \mathsf{M}$. We *do not* require that the rank of Φ be locally constant as some authors do. But we do recall from Proposition GA1.4.3.19 that the rank of Φ is locally constant if and only if $\ker(\Phi)$ is a subbundle of E if and only if $\mathrm{image}(\Phi)$ is a subbundle of F. We define a morphism $\hat{\Phi}$ of presheaves of the $\mathscr{C}^r_{\mathsf{M}}$-modules $\mathscr{G}^r_{\mathsf{E}}$ and $\mathscr{G}^r_{\mathsf{F}}$ by defining $\hat{\Phi}_{\mathcal{U}}\colon \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}) \to \mathscr{G}^r_{\mathsf{F}}(\mathcal{U})$ by

$$\hat{\Phi}_{\mathcal{U}}(\xi)(x) = \Phi \circ \xi(x), \qquad x \in \mathcal{U}.$$

That $\hat{\Phi}_{\mathcal{U}}$ is $\mathscr{C}^r_{\mathsf{M}}(\mathcal{U})$-linear is clear from fibre linearity of Φ. And, if $\mathcal{V} \subseteq \mathcal{U}$ is open, it is clear that $\hat{\Phi}_{\mathcal{U}}$ and $\hat{\Phi}_{\mathcal{V}}$ appropriately commute with the restriction maps. In short, $(\hat{\Phi}_{\mathcal{U}})_{\mathcal{U}\,\mathrm{open}}$ defines a morphism of $\mathscr{C}^r_{\mathsf{M}}$-modules.

Conversely, given a morphism Ψ of the sheaves $\mathscr{G}^r_{\mathsf{E}}$ and $\mathscr{G}^r_{\mathsf{F}}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules, we can associate a vector bundle mapping $\check{\Psi}\colon \mathsf{E} \to \mathsf{F}$ of class C^r over id_{M} by

$$\check{\Psi}(e_x) = \Psi(r_{\mathcal{U},x}(s))(x),$$

where $s \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U})$ is such that $s(x) = e_x$ and $\mathcal{U}$ is a neighbourhood of x. Such a local section s exists, for example, by constructing it in a vector bundle chart about x. One can also easily verify that this vector bundle mapping is well-defined, independently of the choice of s. We should show, however, that it is also of class C^r. We do this in the following result, and something more.

1.4.14 Theorem (Correspondences between vector bundle mappings and sheaf morphisms) *Let $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $\mathrm{r} \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $\mathrm{r} = \mathrm{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\tau\colon \mathsf{F} \to \mathsf{M}$ be vector bundles of class C^{r}. The following statements hold:*

(i) if $\Phi\colon \mathsf{E} \to \mathsf{F}$ is a vector bundle of class C^{r}, then $\hat{\Phi} = (\Phi_{\mathcal{U}})_{\mathcal{U}\,open}$ is a morphism of the $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$-modules $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ and $\mathscr{G}^{\mathrm{r}}_{\mathsf{F}}$;

(ii) if $(\Psi_{\mathcal{U}})_{\mathcal{U}\,open}$ is a morphism of $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$-modules $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ and $\mathscr{G}^{\mathrm{r}}_{\mathsf{F}}$, then $\check{\Psi}\colon \mathsf{E} \to \mathsf{F}$ is a vector bundle map of class C^{r}.

Moreover, the assignment $\Phi \mapsto \hat{\Phi}$ is a bijection with inverse $\Psi \mapsto \check{\Psi}$.

Proof Except for the verification that $\check{\Psi}$ is of class C^r, the first two assertions are clear. So let $(\mathcal{V}, \psi)$ and $(\mathcal{W}, \chi)$ be vector bundle charts for E and F, respectively, about x, and suppose that both of these vector bundle charts induce the same chart $(\mathcal{U}, \phi)$ for M. The local representative of $\Phi \triangleq \check{\Psi}$ is then

$$(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{u})$$

for a function $\boldsymbol{x} \mapsto \boldsymbol{\Phi}(\boldsymbol{x}) \in \mathbb{F}^{r \times s}$ for suitable $r, s \in \mathbb{Z}_{>0}$. Let $\xi_1, \dots, \xi_s$ be the local sections of E defined by $\xi_j(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{e}_j)$, where $\boldsymbol{e}_j$ is the jth standard basis vector for $\mathbb{F}^s$, $j \in \{1, \dots, s\}$.

Then, if the columns of the matrix $\boldsymbol{\Phi}$ are denoted by $\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_r$, the local representative of $\hat{\Phi}(\xi_j)$ is $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}_j(\boldsymbol{x}))$. By hypothesis, $\hat{\Phi}(\xi_j)$ is of class C^r, from which we deduce that $\boldsymbol{\Phi}_j$ is of class C^r, from which we deduce that $\boldsymbol{\Phi}$ is of class C^r. Thus Φ is of class C^r as desired.

The final assertion follows from the directly verified equalities

$$\check{\hat{\Phi}}(e_x) = \Phi(e_x), \quad \hat{\check{\Psi}}_{\mathcal{U}}(\xi) = \Psi_{\mathcal{U}}(\xi),$$

for $e_x \in \mathsf{E}$, $\mathcal{U} \subseteq \mathsf{M}$ open, and $\xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U})$. ■

One way if understanding the theorem is that, "A vector bundle mapping is uniquely determined by what it does to sections."

Let us look at the kernel and image presheaves of the sheaf morphism associated to a vector bundle map. For the kernel, we have the following result.

1.4.15 Proposition (The kernel presheaf associated to a vector bundle mapping is a sheaf) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *and* $\tau\colon \mathsf{F} \to \mathsf{M}$ *be vector bundles of class* C^{r}, *and let* $\Phi\colon \mathsf{E} \to \mathsf{F}$ *be a vector bundle mapping of class* C^{r} *over* id_{M}. *If* $\hat{\Phi} = (\hat{\Phi}_{\mathcal{U}})_{\mathcal{U}\ \mathit{open}}$ *is the associated mapping of presheaves, then the kernel presheaf of* $\hat{\Phi}$ *is a sheaf.*

Proof This is a consequence of Proposition 1.3.3. ■

As we saw in Example 1.3.4, the image presheaf of a morphism of sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules may not be a sheaf. This is true even when the morphism arises from a vector bundle mapping.

1.4.16 Example (The image presheaf associated with a vector bundle mapping may not be a sheaf) A recollection of Example 1.3.4–2 suffices here, since the morphism in that example, in fact, is a morphism arising from a vector bundle map. Specifically, let X be the holomorphic vector field on $\mathbb{CP}^1$ from Example 1.3.4–2 let E be the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}$ and let F be the holomorphic tangent bundle $\mathsf{T}^{1,0}\mathbb{CP}^1$. We then have the vector bundle mapping $\mu_X\colon \mathsf{E} \to \mathsf{F}$ defined by $\mu_X(z, \alpha) = \alpha X(z)$. Understanding that $\mathscr{C}^{\mathrm{hol}}_{\mathsf{E}}$ is identified with $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$ in a natural way, the morphism of sheaves of $\mathscr{C}^{\mathrm{hol}}_{\mathbb{CP}^1}$-modules associated with μ_X is identified with the morphism m_X from Example 1.3.4–2. As we saw in that example, the image presheaf is not a sheaf. •

Note that, even though the image presheaf associated to a vector bundle mapping may not be a sheaf, it can be sheafified, and by Proposition 1.3.10 this sheafification is canonically identified with a subsheaf of $\mathscr{G}^\infty_r\mathsf{F}$. Moreover, in a large number of cases the image presheaf is indeed a sheaf.

1.4.17 Proposition (The image presheaf associated with a vector bundle morphism is sometimes a sheaf) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$, *let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *and* $\tau\colon \mathsf{F} \to \mathsf{M}$ *be* C^{r}*-vector bundles, and let* $\Phi\colon \mathsf{E} \to \mathsf{F}$ *be a* C^{r}*-vector bundle mapping over* id_{M}. *Then the following statements hold:*

(i) if $\mathrm{r} \in \{\infty, \omega\}$, *then* $\mathrm{image}_{\mathrm{pre}}(\hat{\Phi})$ *is a sheaf;*

(ii) *if* $\mathrm{r} = \mathrm{hol}$*, if* $\mathcal{U} \subseteq \mathsf{M}$ *is a Stein open set, if* $(\mathcal{U}_{\mathrm{a}})_{\mathrm{a}\in\mathrm{A}}$ *is an open cover of* $\mathcal{U}$*, and if* $\eta_{\mathrm{a}} \in \mathrm{image}_{\mathrm{pre}}(\hat{\Phi})(\mathcal{U}_{\mathrm{a}})$, $\mathrm{a} \in \mathrm{A}$*, satisfy* $\eta_{\mathrm{a}}|\mathcal{U}_{\mathrm{a}} \cap \mathcal{U}_{\mathrm{b}} = \eta_{\mathrm{b}}|\mathcal{U}_{\mathrm{a}} \cap \mathcal{U}_{\mathrm{b}}$ *for each* $\mathrm{a}, \mathrm{b} \in \mathrm{A}$*, then there exists* $\eta \in \mathrm{image}_{\mathrm{pre}}(\hat{\Phi})(\mathcal{U})$ *such that* $\eta|\mathcal{U}_{\mathrm{a}} = \eta_{\mathrm{a}}$ *for each* $\mathrm{a} \in \mathrm{A}$*.*

Proof Let $\mathcal{U} \subseteq \mathsf{M}$ be an open set, supposing it to be Stein if $r = \mathrm{hol}$. Let $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$ and let $\eta_a \in \mathrm{image}_{\mathrm{pre}}(\hat{\Phi})(\mathcal{U}_a)$, $a \in A$, satisfy $\eta_a|\mathcal{U}_a \cap \mathcal{U}_b = \eta_b|\mathcal{U}_a \cap \mathcal{U}_b$ for each $a, b \in A$. Since $\mathscr{G}^r_{\mathsf{F}}$ is a sheaf, there exists $\eta \in \mathscr{G}^r_{\mathsf{F}}(\mathcal{U})$ such that $\eta|\mathcal{U}_a = \eta_a$ for each $a \in A$. We must show that $\eta \in \mathrm{image}_{\mathrm{pre}}(\hat{\Phi})(\mathcal{U})$.

Let $\gamma_a \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}_a)$ be such that $\hat{\Phi}_{\mathcal{U}_a}(\gamma_a) = \eta_a$, $a \in A$. For $a, b \in A$ for which $\mathcal{U}_a \cap \mathcal{A}_b \neq \emptyset$ denote

$$\gamma_{ab} = \gamma_a|\mathcal{U}_a \cap \mathcal{U}_b - \gamma_b|\mathcal{U}_a \cap \mathcal{U}_b.$$

Note that $\hat{\Phi}_{\mathcal{U}_a\cap\mathcal{U}_b}(\gamma_{ab}) = 0$ and so $(\gamma_{ab})_{a,b\in A} \in \mathrm{Z}^1(\mathscr{U}, \ker(\hat{\Phi}))$. We now note the following (where we make reference to notation from Section 4.4 and results from Sections 4.5 and 5.3):

1. if $r = \infty$ then $\mathrm{H}^1(\mathscr{U}; \ker(\hat{\Phi})) = 0$ by Theorem 4.5.1;
2. if $r \in \{\omega, \mathrm{hol}\}$ then $\ker(\hat{\Phi})$ is coherent by Proposition 5.1.6(iii), and so $\mathrm{H}^1(\mathscr{U}; \ker(\hat{\Phi})) = 0$ by Theorem 5.3.2.

In both cases, we conclude that there exists $(\beta_a)_{a\in A}$ such that

$$\gamma_{ab} = \beta_b|\mathcal{U}_a \cap \mathcal{U}_b - \beta_b|\mathcal{U}_a \cap \mathcal{U}_b$$

for $a, b \in A$ for which $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$. Define $\xi_a \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}_a)$ by $\xi_a = \gamma_a + \beta_a$, $a \in A$. One directly verifies that

$$\xi_a|\mathcal{U}_a \cap \mathcal{U}_b = \xi_b|\mathcal{U}_a \cap \mathcal{U}_b.$$

Since $\mathscr{G}^r_{\mathsf{E}}$ is a sheaf there exists $\xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U})$ such that $\xi|\mathcal{U}_a = \xi_a$. We claim that $\hat{\Phi}_{\mathcal{U}}(\xi) = \eta$. Indeed, let $x \in \mathcal{U}$, let $a \in A$ be such that $x \in \mathcal{U}_a$, and compute

$$\begin{aligned}\hat{\Phi}_{\mathcal{U}}(\xi)(x) = (\hat{\Phi}_{\mathcal{U}}(\xi)|\mathcal{U}_a)(x) = \hat{\Phi}_{\mathcal{U}_a}(\xi_a)(x) &= \hat{\Phi}_{\mathcal{U}}(\gamma_a)(x) + \hat{\Phi}_{\mathcal{U}_a}(\beta_a)(x)\\ &= \hat{\Phi}_{\mathcal{U}}(\gamma_a)(x) = \eta_a(x) = (\eta|\mathcal{U}_a)(x) = \eta(x),\end{aligned}$$

as desired. ■

Let us now understand how attributes of sheaf morphisms and vector bundle morphisms are related. We begin by considering the attribute of being surjective on fibres.

1.4.18 Proposition (Surjectivity on fibres and surjectivity on stalks) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$*. Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *and* $\tau\colon \mathsf{F} \to \mathsf{M}$ *be vector bundles of class* C^{r} *and let* $\mathrm{x} \in \mathsf{M}$*. For a vector bundle map* $\Phi\colon \mathsf{E} \to \mathsf{F}$ *of class* C^{r} *with* $\hat{\Phi}$ *the corresponding* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-module morphism from* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ *to* $\mathscr{G}^{\mathrm{r}}_{\mathsf{F}}$*, the following statements are equivalent:*

(i) $\Phi_{\mathrm{x}} = \Phi|\mathsf{E}_{\mathrm{x}}\colon \mathsf{E}_{\mathrm{x}} \to \mathsf{F}_{\mathrm{x}}$ *is surjective;*

(ii) $\hat{\Phi}_{\mathrm{x}}\colon \mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{E}} \to \mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{F}}$ *is surjective.*

Proof Let $(\mathcal{V}, \psi)$ and $(\mathcal{W}, \chi)$ be vector bundle charts for E and F, respectively, about x, and suppose that both of these vector bundle charts induce the same chart $(\mathcal{U}, \phi)$ for M. The local representative of Φ is then

$$(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{u})$$

for a function $\boldsymbol{x} \mapsto \boldsymbol{\Phi}(\boldsymbol{x}) \in \mathbb{F}^{r \times s}$ for suitable $r, s \in \mathbb{Z}_{>0}$. Let $\mathcal{U}' \subseteq \mathcal{U}$ be a neighbourhood of x and let $\xi \in \mathscr{G}^r_{\mathsf{M}}(\mathcal{U}')$ have local representative $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$. The local representative of $\hat{\Phi}_{\mathcal{U}'}(\xi)$ is given by

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{\xi}(\boldsymbol{x})).$$

Suppose that $\hat{\Phi}_x$ is surjective and let $f_x \in \mathsf{F}_x$. Let η be a local section of $\mathscr{G}^r_{\mathsf{F}}$ over a neighbourhood of x for which $\eta(x) = f_x$. Since $\hat{\Phi}_x$ is surjective there exists a local section ξ of $\mathscr{G}^r_{\mathsf{E}}$ over a neighbourhood of x for which $\hat{\Phi}_x([\xi]_x) = [\eta]_x$. By definition of $\hat{\Phi}$ this means that $\Phi_x(\xi(x)) = f_x$.

Now suppose that Φ_x is surjective. This implies that there are columns $\boldsymbol{\Phi}^{j_1}, \dots, \boldsymbol{\Phi}^{j_r}$ of $\boldsymbol{\Phi}$ for which

$$(\boldsymbol{\Phi}_{j_1}(\phi(x)), \dots, \boldsymbol{\Phi}_{j_r}(\phi(x)))$$

are linearly independent. These same columns are linearly independent in a neighbourhood of $\phi(x)$. Now suppose that η is a local section of $\mathscr{G}^r_{\mathsf{F}}$ over a neighbourhood of x and that $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\eta}(\boldsymbol{x}))$ is the local representative of η. We can then write

$$\boldsymbol{\eta}(\boldsymbol{x}) = f^1(\boldsymbol{x})\boldsymbol{\Phi}_{j_1}(\boldsymbol{x}) + \dots + f^r(\boldsymbol{x})\boldsymbol{\Phi}_{j_r}(\boldsymbol{x})$$

for $\boldsymbol{x}$ in a neighbourhood of $\phi(x)$. If we let ξ be the local section of $\mathscr{G}^r_{\mathsf{E}}$ over a neighbourhood of x whose local representative $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$ is defined by

$$\xi^k = \begin{cases} f^{j_l}, & k = j_l \text{ for some } l \in \{1, \dots, r\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is a direct verification that $\hat{\Phi}_x([\xi]_x) = [\eta]_x$. ■

For the attribute of injectivity, the corresponding assertion is generally false.

1.4.19 Example (Injectivity on stalks does not imply injectivity of fibres) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\mathsf{M} = \mathbb{F}$ and let $\mathsf{E} = \mathsf{F} = \mathbb{F} \times \mathbb{F}$ be the trivial bundles with the projection $\pi(x, v) = x$. Define a C^r-vector bundle map $\Phi\colon \mathsf{E} \to \mathsf{F}$ by $\Phi(x, v) = (x, xv)$. We claim that $\ker_{\mathrm{pre}}(\hat{\Phi}) = 0$. Indeed, let $\mathcal{U} \subseteq \mathbb{F}$ be open, let $\xi \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U})$ satisfy $\hat{\Phi}_{\mathcal{U}}(\xi) = 0$. This means that $x\xi(x) = 0$ for every $x \in \mathcal{U}$. If $x \neq 0$ we infer that $\xi(x) = 0$. If $x = 0 \in \mathcal{U}$ then we have $\xi(x) = 0$ by continuity, and so ξ is indeed the zero section. By Proposition 1.3.13 we infer that $\hat{\Phi}$ is injective, as claimed. However, Φ_0 is certainly not injective. •

Bibliography

Bredon, G. E. [1997] *Sheaf Theory*, second edition, number 170 in Graduate Texts in Mathematics, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-94905-4.

Godement, R. [1958] *Topologie algébrique et théorie des faisceaux*, number 13 in Publications de l'Institut de mathématique de l'Université de Strasbourg, Hermann, Paris.

Hungerford, T. W. [1980] *Algebra*, number 73 in Graduate Texts in Mathematics, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-90518-9.

Kashiwara, M. and Schapira, P. [1990] *Sheaves on Manifolds*, number 292 in Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 3-540-51861-4.

Lewis, A. D. [2013] *Foundations of Geometric Control Theory*, To be determined.

Ramanan, S. [2005] *Global Calculus*, number 65 in Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, ISBN 0-8218-3702-8.

Tennison, B. R. [1976] *Sheaf Theory*, number 20 in London Mathematical Society Lecture Note Series, Cambridge University Press, New York/Port Chester/Melbourne/Sydney, ISBN 978-0-521-20784-3.

Warner, F. W. [1983] *Foundations of Differentiable Manifolds and Lie Groups*, second edition, number 94 in Graduate Texts in Mathematics, Springer-Verlag, New York/Heidelberg/Berlin, ISBN 0-387-90894-3.

Willard, S. [1970] *General Topology*, Addison Wesley, Reading, MA, reprint: [Willard 2004].

— [2004] *General Topology*, Dover Publications, Inc., New York, ISBN 0-486-43479-6, reprint of 1970 Addison-Wesley edition.