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Chapter 8

Signals in the time-domain

In this chapter we present the notion of a signal in its most intuitively natural setting, the time-domain. We begin in Section 8.1 with a description of what we mean by time in various forms. This will provide us with the sorts of sets on which the notion of a signal is defined. We follow our discussion of time with a basic description of signals. In this initial discussion a signal is simply a function. If one talks of signals only as functions with no additional features, then it becomes very difficult to actually do anything in a clear way with signals. Indeed, it is extremely important to be able to describe, in a particular application, the sort of signals one wishes to allow. The set of allowable signals should be sufficiently large that any signals arising in the application are likely to be allowed, but not so large that one cannot say anything useful about the problem. For this reason, we spend a significant portion of this chapter talking in detail about various properties of signals. Some of the most useful of these structures involve a norm (see Chapter 6) that allows us to give signals the notion of size. As we shall see, there are various notions of size, and the one to use in a certain situation is a matter of understanding the problem at hand. With these ideas at one's disposal, it is relatively easy to understand the various classes of signals that will be of interest to us. The discrete-time situation is considered first, in Section 8.2. In Section 8.3 we discuss continuous-time signals. For most of the chapter we focus on signals that are scalar-valued (with scalars being either in $\mathbb{R}$ or $\mathbb{C}$) functions of a single real variable. In Section **??** we introduce the possibility of signals with domains and codomains of dimension greater than one.

Do I need to read this chapter? If you are learning about signals, this is the place to start. •

Contents

Section 8.1

Time, signals, and elementary properties

This section is mainly motivational, and gives only fairly elementary definitions and no deep results. The idea is to develop some ideas about where signals come up, how one represents them, and what simple properties might be used to characterise them. In Section 8.1.7 we motivate the more technical discussion that follows in Sections 8.2 and 8.3. Here we shall see that significant diversions to Chapters 5, 6, and 7 are necessary if one is to really arrive at useful tools for dealing with signals.

Do I need to read this section? This section is mainly light reading, and will hopefully motivate the heavier reading to follow. If you are the type that welcomes lightness before heaviness, this section will be a beneficial read for you. •

8.1.1 Examples and problems involving time-domain signals

Signals in the time-domain are normally mathematically represented in one of two ways: continuous or discrete. A continuous-time representation means that one has assigned a value of the signal for all times. On the other hand, for a discrete-time representation, one only has values for the signal at certain times, typically evenly spaced. There are other attributes one can assign to signals, but we postpone to subsequent sections a detailed discussion of these. Here we mean to merely give a few concrete examples of signals so that we have some idea of what we might mean in practice.

8.1.1 Examples (Time-domain signals)

1. In Figure 8.1 is shown the opening average for the Dow Jones Industrial Average

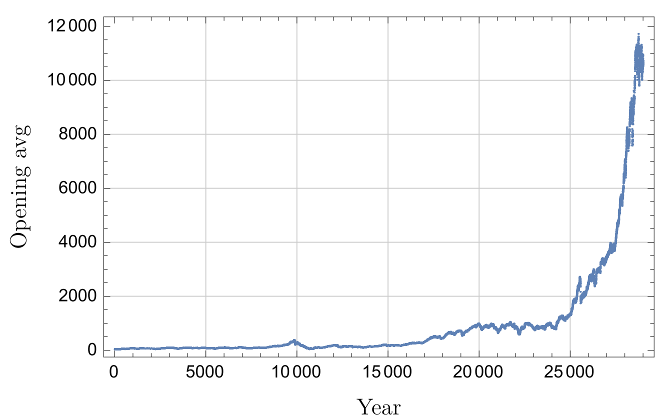


Figure 8.1 Dow Jones Industrial Average opening data from May 26, 1896 to January 26, 2001

over a span of more than one hundred years.[1] This is a discrete-time signal.

2. In Figure 8.2 is shown data representing the yearly average temperature as

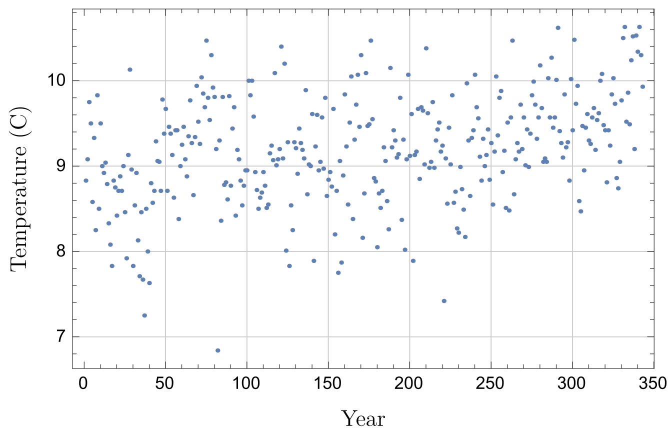


Figure 8.2 Average daily temperature by year in Central England

recorded in Central England since 1659.[2] As with the Dow Jones data, this signal is discrete-time.

3. On June 28, 1991 the author experienced a cordial earthquake measuring 5.5 Mw (moment magnitude). The raw accelerometer data for this quake is shown in Figure 8.3.[3] This is an example of a continuous-time signal.
4. In Figure 8.4 we show a plot of a segment of human speech recorded in the time-domain, with the signal being normalised to have maximum value +1 and minimum value −1. This is a continuous-time representation of a signal.
5. In Figure 8.5 we show the time-domain representations of two musical clips. The clip on the left is the first movement of Mozart's *Eine kleine Nachtmusik* (K525), and that on the right is from the soundtrack of the Darren Aronofsky movie π. These are both continuous-time signals although, when they are pressed onto a CD, the resulting signal becomes a discrete-time signal. •

While the notion of a time-domain signal is not so exotic, what is more exotic is the mathematics behind representing time-domain signals in a useful and general manner. Let us address in a superficial way some of the problems that give rise to the necessity of talking coherently and precisely about *classes* of signals.

1. In Example 8.1.1–5 we mentioned that, when pressing music onto a CD, one converts a continuous-time signal to a discrete-time signal. It then becomes

[1] Data downloaded from `http://www.travismorien.com/FAQ/dow.htm` (link no longer active).

[2] Data downloaded from the Meteorological Office in the United Kingdom, `http://www.met-office.gov.uk/`.

[3] Data downloaded from the United States National Strong-Motion Program, `http://nsmp.wr.usgs.gov/`, the data being compiled by the United States Geological Survey.

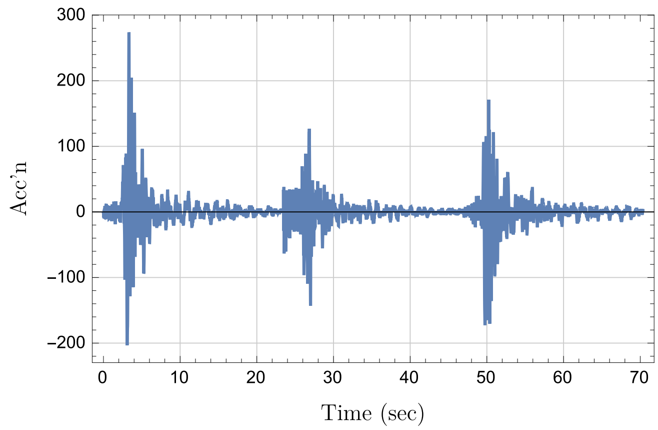


Figure 8.3 Accelerometer data for Sierra Madre earthquake of June 28, 1991

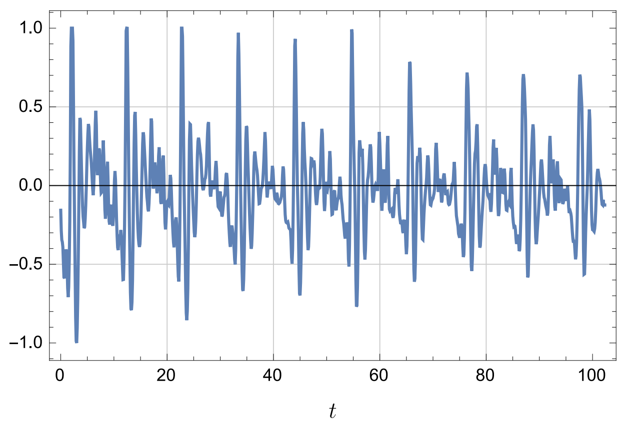
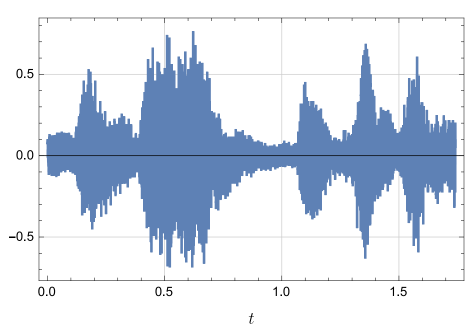


Figure 8.4 Human speech in the time-domain



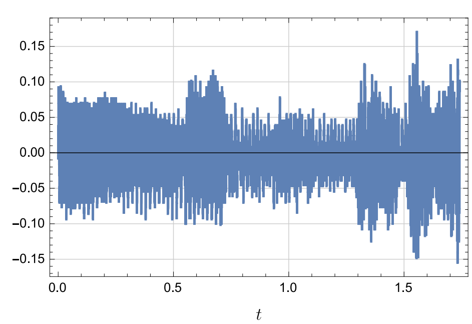


Figure 8.5 A time-domain representation of part of the first movement of Mozart's *Eine Kleine Nachtmusik* (K525) (left) and a portion of the soundtrack of the movie π (right)

interesting to know when the discrete-time representation is faithful, in some sense, to the continuous-time representation. Clearly something is lost. Can one quantify what is lost? Are there signals for which *nothing* is lost?

2. Suppose one wishes to design an algorithm to control a process, and wants to ensure that external disturbances do not too seriously affect the behaviour of the system. For what class of disturbances can one develop a general theory that guarantees good system behaviour?
3. It sometimes arises that one is interested in signals that are not, in fact, actually signals. The prototypical example of this is the so-called Dirac δ-function. This "signal" is intended to model an impulse, by which we mean a large magnitude signal that is defined for a very short period of time. Clearly, given a signal of some large magnitude, and defined for some short time, one can always devise another signal, possessing some larger magnitude and defined for a shorter time. Thus, what one is really after—the highest magnitude signal defined for the shortest time—does not exist. Is there a mathematically precise way to capture the essence of this nonexistent signal?
4. Suppose one wishes to measure a given signal, but that the signal is included in some background noise, i.e., is included as part of a larger signal, the rest of which is of no interest. Is it possible to extract the signal of interest? For what sorts of signals is this possible? For what sorts of noise is this possible?

This volume is devoted to developing the machinery needed to address questions such as these.

8.1.2 Time

For the notion of time as we consider it, it will be helpful to recall the notion of a semigroup from Definition 4.1.2, a group from Definition 4.1.4, and a subgroup from Definition 4.1.9. There are lots of examples of groups, some of which are discussed in Section 4.1. The group of interest to us is $(\mathbb{R}, +)$, the group formed by the set $\mathbb{R}$ of real numbers and the group multiplication given by addition of numbers. We shall be interested in subgroups of this group and also in subsets that are semigroups. Examples of subgroups of $\mathbb{R}$ include

1. the set $\mathbb{Z}$ of integers (see Section **??**),
2. the set $\mathbb{Z}(\Delta) = \{\Delta a \mid a \in \mathbb{Z}\}$ of integer multiples of $\Delta \in \mathbb{R}_{>0}$, and
3. the set $\mathbb{Q}$ of rational numbers (see Definition 2.1.1).

It is an exercise for the reader (see Exercise 4.1.6) to show that these are indeed subgroups. As examples of subsets of $\mathbb{R}$ that are semigroups we include

4. the set $t_0 + \mathbb{Q} = \{t_0 + q \mid q \in \mathbb{Q}\}$ for $t_0 \in \mathbb{R}$ and
5. the set $\mathbb{Z}(t_0, \Delta) = \{t_0 + k\Delta \mid k \in \mathbb{Z}\}$ for $t_0 \in \mathbb{R}$.

If $t_0 \in \mathbb{Q}$ then $t_0 + \mathbb{Q} = \mathbb{Q}$ and if $t_0 \in \mathbb{Z}(\Delta)$ then $\mathbb{Z}(t_0, \Delta) = \mathbb{Z}(\Delta)$. In Exercise 4.1.7 the reader can show that these statements are true.

Now we define the basic collections of times that we will encounter in the text, recalling from Example 2.5.4 the notion of an interval.

8.1.2 Definition (Time-domain) A ***time-domain*** is a subset of $\mathbb{R}$ of the form $\mathbb{S} \cap I$ where $\mathbb{S} \subseteq \mathbb{R}$ is a semigroup in $(\mathbb{R}, +)$ and $I \subseteq \mathbb{R}$ is an interval. A time-domain is

(i) ***continuous*** if $\mathbb{S} = \mathbb{R}$,

(ii) ***discrete*** if $\mathbb{S} = \mathbb{Z}(t_0, \Delta)$ for some $t_0 \in \mathbb{R}$ called the ***origin shift*** and for some $\Delta \in \mathbb{R}_{>0}$ called the ***sampling interval***,

(iii) ***finite*** if $\mathrm{cl}(I)$ is compact,

(iv) ***infinite*** if it is not finite,

(v) ***positively infinite*** if $\sup I = \infty$,

(vi) ***negatively infinite*** if $\inf I = -\infty$, and

(vii) ***totally infinite*** if $I = \mathbb{R}$. •

Let us give some examples, just by means of establishing notation for future use.

8.1.3 Examples (Time-domains)

1. We denote $\mathbb{Z}_{\geq 0}(\Delta) = \mathbb{Z}(\Delta) \cap \mathbb{R}_{\geq 0}$.
2. We denote $\mathbb{Z}_{>0}(\Delta) = \mathbb{Z}(\Delta) \cap \mathbb{R}_{>0}$. •

8.1.4 Remarks (Some commonly made assumptions about time-domains)

1. We shall denote a typical point in a time-domain by t to signify time. However, it is possible that in some applications of our techniques the "time" variable will not be time. Nonetheless, we shall talk as if it were indeed time since this gives access to some intuition.
2. We shall deal almost exclusively with discrete time-domains where $\mathbb{S} = \mathbb{Z}(\Delta)$, i.e., with no origin shift.
3. Note that for continuous time-domains we use the words "finite" and "infinite" not in their usual mathematical way (where finite means "consists of a finite number of points"), but in the common usage of these words as they refer to time.
4. To eliminate the need to deal with trivial cases, we shall tacitly suppose that all time-domains consist of more than one point, unless otherwise stated.
5. We restrict ourselves in the discrete case to signals that are sampled at regular intervals. It can happen that sampling will happen at irregular intervals. However, to generate a useful theory for such signals is difficult, so much so that if one is confronted with an irregularly sampled signal, the first thing to do is convert it to a regularly sampled signal. •

Now let us consider some transformations of time-domains that will be useful to us. We begin with a very general definition.

8.1.5 Definition (Reparameterisation) For time-domains $\mathbb{T}_1$ and $\mathbb{T}_2$, a ***reparameterisation*** of $\mathbb{T}_1$ to $\mathbb{T}_2$ is a bijection $\tau\colon \mathbb{T}_2 \to \mathbb{T}_1$ that is either monotonically increasing or monotonically decreasing. •

It perhaps seems odd why a reparameterisation of $\mathbb{T}_1$ should have $\mathbb{T}_1$ as its codomain, and not its domain. The reason for this will be clear in Section 8.1.4 when we discuss how reparameterisations are used to transform signals. For the moment, let us content ourselves with a few specific sorts of reparameterisations.

8.1.6 Examples (Reparameterisations)

1. For $a \in \mathbb{R}$, the ***shift*** of a time-domain $\mathbb{T}_1$ by a is defined by taking the time-domain
$$\mathbb{T}_2 = \{t + a \mid t \in \mathbb{T}\}$$
and the reparameterisation $\tau_a\colon \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$ defined by $\tau_a(t) = t - a$.
2. For a time-domain $\mathbb{T}_1$, the ***transposition*** of $\mathbb{T}_1$ is defined by taking the time-domain
$$\mathbb{T}_2 = \{-t \mid t \in \mathbb{T}_1\}$$
and the reparameterisation $\sigma\colon \mathbb{T}_2 \to \mathbb{T}_1$ defined by $\sigma(t) = -t$. Often we will use the reparameterisation in the case when $\sigma(\mathbb{T}_1) = \mathbb{T}_1$.
3. For a time-domain $\mathbb{T}_1$ and for $\lambda \in \mathbb{R}_{>0}$, the ***dilation*** of $\mathbb{T}_1$ by λ is defined by taking the time-domain
$$\mathbb{T}_2 = \{\lambda t \mid t \in \mathbb{T}_1\}$$
and the reparameterisation $\rho_\lambda\colon \mathbb{T}_2 \to \mathbb{T}_1$ defined by $\rho_\lambda(t) = \lambda^{-1}t$.
4. Here we take $\mathbb{T}_1 = \mathbb{T}_2 = [0, 1]$ and define a reparameterisation $\tau\colon \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$ by $\tau(t) = \frac{1}{2}(1 - \cos(\pi t))$. We illustrate this reparameterisation in Figure 8.6. •

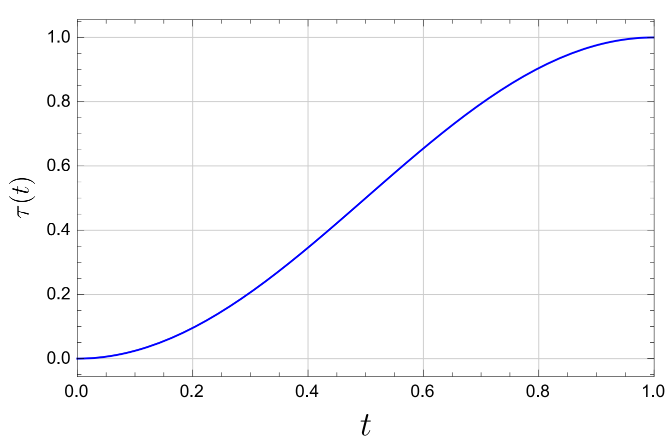


Figure 8.6 A reparameterisation of $[0, 1]$

8.1.3 Time-domain signals: basic definitions

In this section we give the coarsest definition of a signal, along with some examples of signals. This will serve to provide a setting for the more abstract notions of signal spaces to follow in Sections 8.2 and 8.3. Throughout this chapter, and indeed this volume, we will use the symbol $\mathbb{F}$ to stand for either $\mathbb{R}$ or $\mathbb{C}$. We will denote by $|a|$ the absolute value of a if $a \in \mathbb{R}$ and the complex magnitude of a if $a \in \mathbb{C}$. Similarly, $\bar{a} = a$ if $a \in \mathbb{R}$ and $\bar{a}$ is the complex conjugate of a if $a \in \mathbb{C}$.

8.1.7 Definition (Time-domain signal) Let $\mathbb{T} = \mathbb{S} \cap I$ be a time-domain and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An ***$\mathbb{F}$-valued time-domain signal*** on $\mathbb{T}$ is a map $f : \mathbb{T} \to \mathbb{F}$. If $\mathbb{T}$ is continuous then f is a ***continuous-time*** signal and if $\mathbb{T}$ is discrete then f is a ***discrete-time*** signal. •

8.1.8 Notation ("Signal" versus "time-domain signal") Since it is most natural to think of signals in the time-domain—as opposed to in the frequency-domain as we shall discuss in Chapter 9—we shall very often just say "signal" instead of "time-domain signal." •

We next consider the manner in which we shall depict signals in the time-domain. For $\mathbb{R}$-valued signals defined on a continuous time-domain $\mathbb{T}$, the usual depiction is simply the graph of the signal in the sense that we learn in elementary school. However, for $\mathbb{C}$-valued signals or for signals defined on discrete time-domains, there is no such standard depiction. So let us give our rules for this. First of all, let us consider how to depict a $\mathbb{R}$-valued discrete-time signal. We do this as in Figure 8.7, where we also show a depiction of a $\mathbb{R}$-valued continuous-time signal

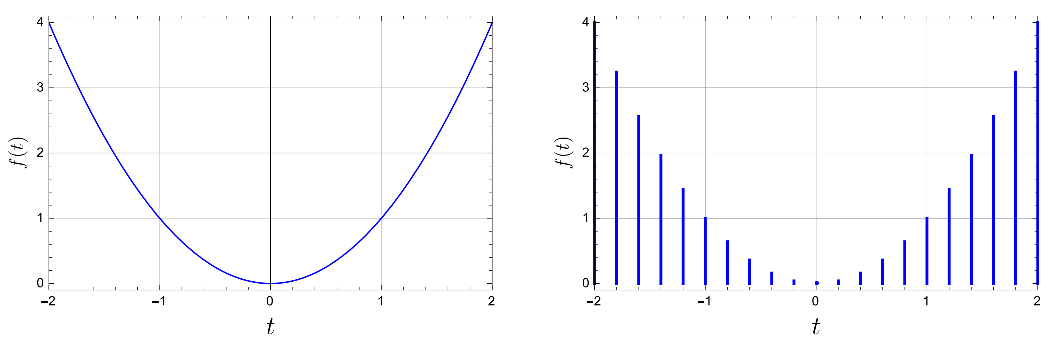


Figure 8.7 The depiction of a $\mathbb{R}$-valued continuous-time signal (left) and discrete-time signal (right)

for contrast. The idea is that one represents a discrete-time signal by placing a line going from $(t, 0)$ to $(t, f(t))$ in the plane. To represent a $\mathbb{C}$-valued signal $f : \mathbb{T} \to \mathbb{C}$ one can proceed in two natural ways. One way is to depict the signal is to plot the two $\mathbb{R}$-valued signals $t \mapsto \mathrm{Re}(f(t))$ and $\mathrm{Im}(f(t))$. One could alternatively plot the two $\mathbb{R}$-valued signals $t \mapsto |f(t)|$ and $t \mapsto \arg(f(t))$. In Figure 8.8 we show these two possible depictions of the complex signal $t \mapsto \mathrm{e}^{\mathrm{i}t}$ on the continuous time-domain $[0, 2\pi]$. Similar plots can be produced for discrete-time complex signals. Note

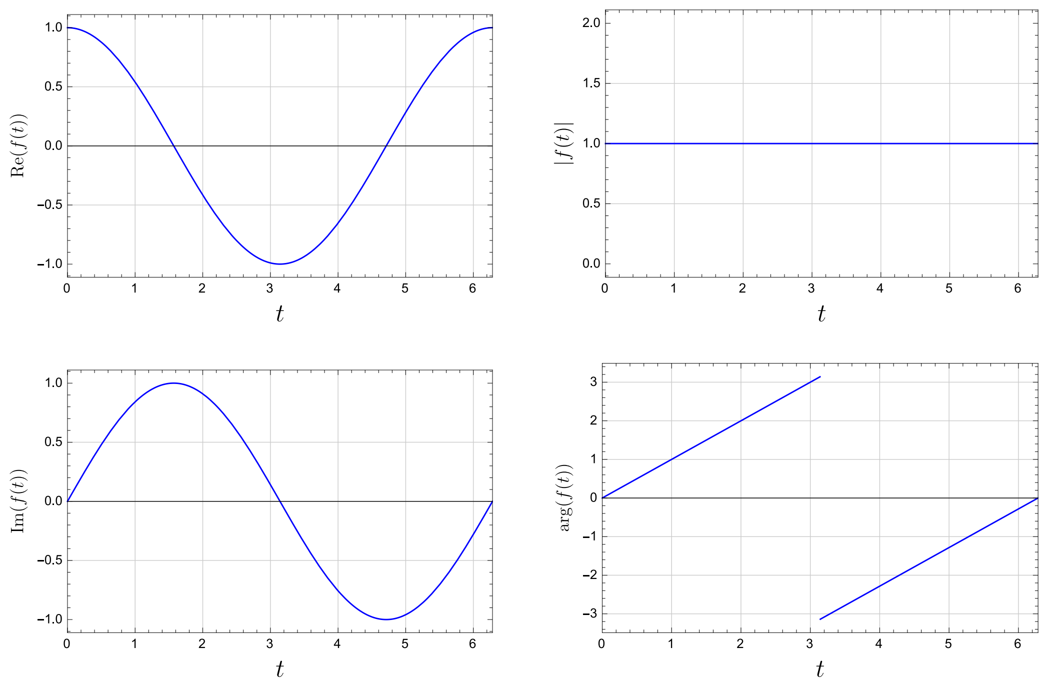


Figure 8.8 The real and imaginary parts (left) and the magnitude and phase (right) for the signal $t \mapsto e^{it}$

that representing a complex signal using magnitude and phase has the potential problem that when the signal has zero magnitude the phase is not well-defined. One could arbitrarily choose, say, to set the phase to be zero at these points, but this is not actually the best thing to do since it may destroy some nice features of the phase. For example, the phase may extend continuously to include points where the magnitude is zero, but this may not be preserved by setting the phase to an arbitrary value. In our examples we shall generally try to sidestep these complications with representing complex-valued signals by considering only real-valued signals.

Let us consider some examples of signals to illustrate the where discrete- and continuous-time signals might naturally arise.

8.1.9 Examples (Signals)

1. The signal

$$\mathbf{1}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is called the ***unit step signal*** and is a continuous-time signal defined on a totally infinite time-domain.

2. The signal

$$\mathsf{R}(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is called the ***unit ramp signal*** and again is a continuous-time signal defined on a totally infinite time-domain.

3. A ***binary data stream*** is a discrete-time signal defined on $\mathbb{T} = \mathbb{Z}$ and taking values in the set $\{0, 1\}$.

4. Consider the special binary data stream $\mathsf{P}\colon \mathbb{Z} \to \{0, 1\}$ defined by

$$\mathsf{P}(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This is called the ***unit pulse***.

5. On $[0, 1]$ define a $\mathbb{R}$-valued signal g by

$$g(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ 0, & t \in (\frac{1}{2}, 1). \end{cases}$$

Now for $a, f \in \mathbb{R}_{>0}$ and $\phi \in \mathbb{R}$ define a signal

$$\square_{a,\nu,\phi}(t) = \sum_{n \in \mathbb{Z}} a g(\nu t + \phi),$$

which we call the ***square wave*** of amplitude a, frequency ν, and phase ϕ. In Figure 8.9 we show the features of this signal. Note that as we have defined it,

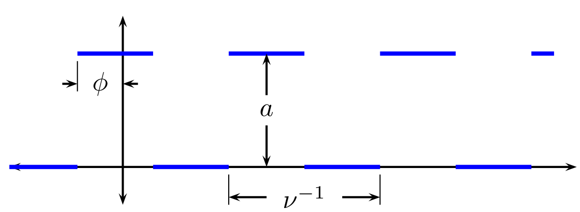


Figure 8.9 The square wave $\square_{a,\nu,\phi}$

$\square_{a,\nu,\phi}$ is a continuous-time signal defined on a totally infinite time-domain.

6. We proceed as in the preceding example, but now take

$$g(t) = \begin{cases} 2t, & t \in [0, \frac{1}{2}], \\ 2 - 2t, & t \in (\frac{1}{2}, 1), \end{cases}$$

and define

$$\triangle_{a,\nu,\phi}(t) = \sum_{n \in \mathbb{Z}} a g(\nu t + \phi),$$

which we call the ***sawtooth*** of amplitude a, frequency ν, and phase shift ϕ. This signal is plotted in Figure 8.10. As with the square wave defined above, this is a continuous-time signal defined on a totally infinite time-domain.

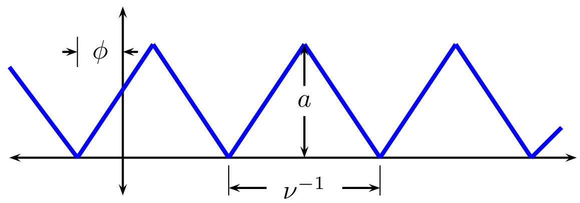


Figure 8.10 The saw tooth $\triangle_{a,\nu,\phi}$

7. The Down Jones Industrial Average opening data depicted in Figure 8.1 is a discrete-time signal defined on a finite time-domain.
8. The Central England yearly average temperature data in Figure 8.2 is an example of a discrete-time signal defined on a finite time-domain.
9. The earthquake data of Figure 8.3 is an example of a continuous-time signal defined on a finite time-domain. •

8.1.4 Elementary transformations of signals

In this section we shall consider ways of producing new signals from existing ones. The idea of a "transformation" of a signal will be important to us in this volume in terms of Fourier analysis. However, the things we discuss now are of a far more elementary nature and are given mainly be means of establishing notation.

We first consider transformations of signals achieved by a manipulation of the codomain. The notation for this is as follows.

8.1.10 Definition (Codomain transformation of a signal) If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if $\mathbb{T}$ is a time-domain, if $f\colon \mathbb{T} \to \mathbb{F}$ is a signal, and if $\phi\colon \mathbb{F} \to \mathbb{F}$ is a map, the ***codomain transformation*** of f by ϕ is the signal $\phi \circ f\colon \mathbb{T} \to \mathbb{F}$. •

This, then, is a simple idea merely given a suggestive name. Let us illustrate this with a few examples.

8.1.11 Examples (Codomain transformations)

1. We define $\phi\colon \mathbb{F} \to \mathbb{F}$ by $\phi(x) = \bar{x}$. Then the codomain transformed signal $\phi \circ f$ we denote by $\bar{f}$.
2. Let $\mathbb{F} = \mathbb{R}$ and define $\phi\colon \mathbb{R} \to \mathbb{R}$ by $\phi(x) = |x|$. Then, for a signal $f\colon \mathbb{T} \to \mathbb{R}$ the codomain transformed signal $\phi \circ f$ is known as the ***full-wave rectification*** of f. This is depicted for a discrete-time signal in Figure 8.11. Of course the same ideas apply to continuous-time signals.
3. We again let $\mathbb{F} = \mathbb{R}$ and now we consider $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} 0, & x < 0, \\ x, & x \geq 0. \end{cases}$$

In this case, for a signal $f\colon \mathbb{T} \to \mathbb{R}$ the codomain transformed signal $\phi \circ f$ is the ***half-wave rectification*** of f and is depicted in Figure 8.11 for a discrete-time signal.

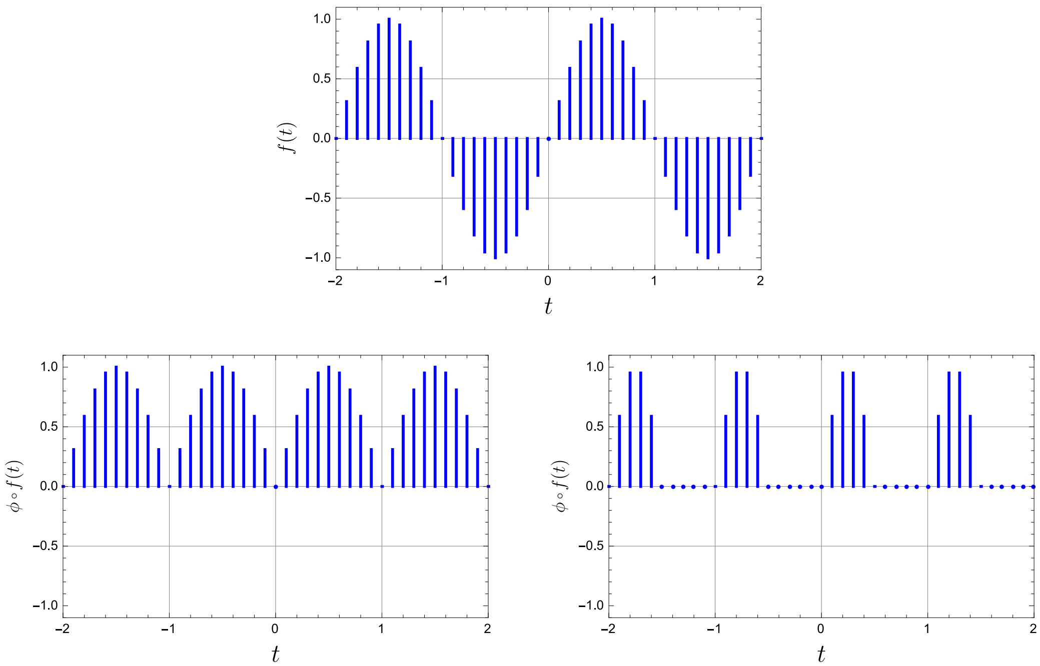


Figure 8.11 Full-wave rectification (bottom left) and half-wave rectification (bottom right) of a discrete-time signal (top)

4. We take $\mathbb{F} = \mathbb{R}$ and for $M \in \mathbb{R}_{>0}$ consider the function $\phi_M \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_M(x) = \begin{cases} x, & x \in [-M, M], \\ -M, & x < -M, \\ M, & x > M. \end{cases}$$

We give the graph of this function on the left in Figure 8.12. The idea of

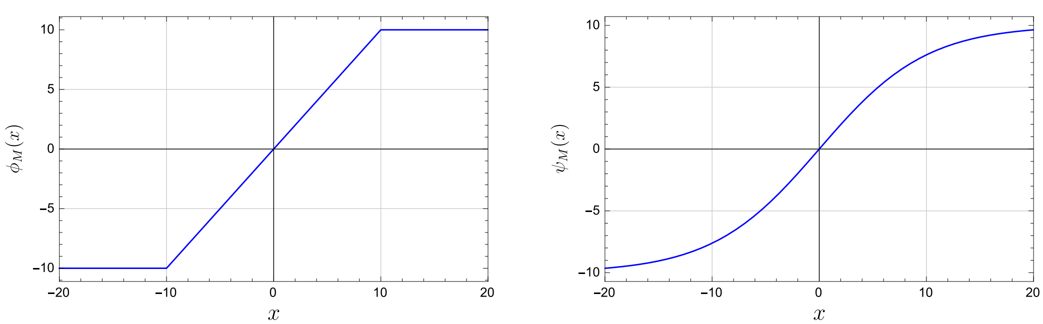


Figure 8.12 Two saturation functions for $M = 10$

this codomain transformation is that it truncates the values of a signal to have a maximum absolute value of M. Such a codomain transformation is called a ***saturation function***. Sometimes it is advisable to use a smooth saturation function, and an example of one such is $\psi_M(x) = M\tanh(\frac{x}{M})$ whose graph we show on the right in Figure 8.12. In Figure 8.13 we show the two saturation

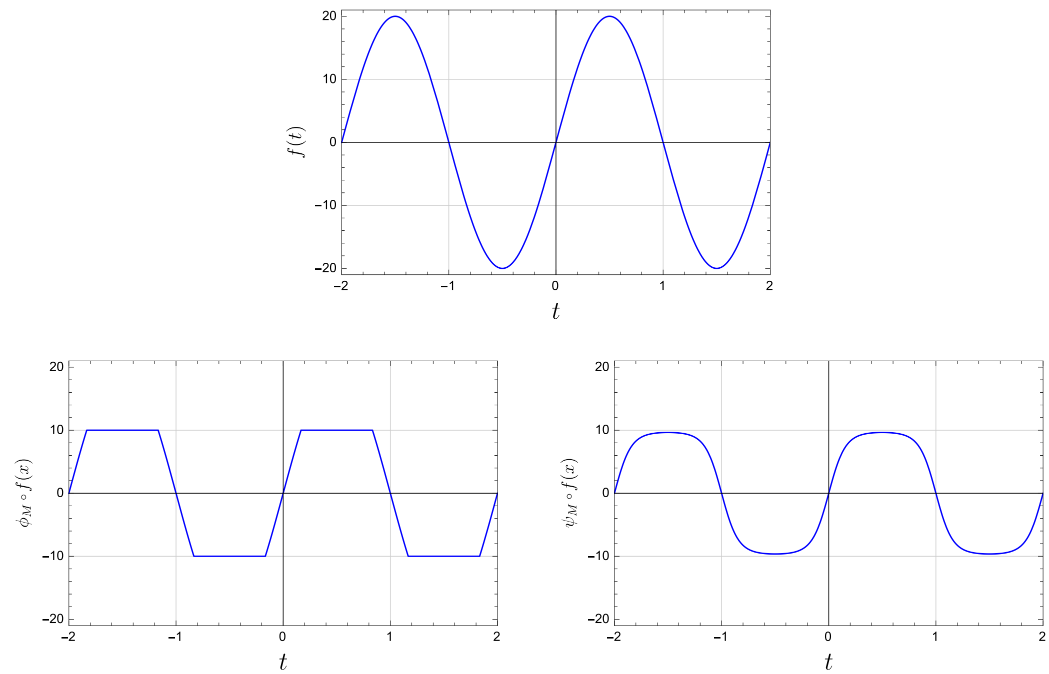


Figure 8.13 The application of the saturation function ϕ_M (bottom left) and the saturation function ψ_M (bottom right) to a continuous-time signal (top)

functions applied to a continuous-time signal. Of course, one can as well apply the idea to a discrete-time signal.

5. Particularly in our world where almost everything is managed by digital computers, signals with continuous values are not often what one deals with in practice. Instead, what one actually has at hand is a signal whose values live in a discrete set. Thus one would like to convert a signal with continuous values to one with discrete values. This general process is known as ***quantisation***. A simple way to quantise a signal is via the codomain transformation $\theta_h \colon \mathbb{R} \to \mathbb{R}$ defined by $\theta_h(x) = h\lceil\frac{x}{h}\rceil$, where we recall from Section 2.2.3 the definition of the ceiling function $x \mapsto \lceil x \rceil$ as giving the largest integer less than or equal to x. The graph of the function is depicted in Figure 2.1. The quantisation θ_h is called the ***uniform* h*-quantisation***. In Figure 8.14 we depict the uniform quantisation of a continuous-time signal. The same idea applies, and indeed is more natural, for discrete-time signals. •

Next we consider transformations of signals achieved by altering the domain of the signal. In Definition 8.1.5 we consider the natural class of domain transformations to consider, calling them reparameterisations. For these we make the

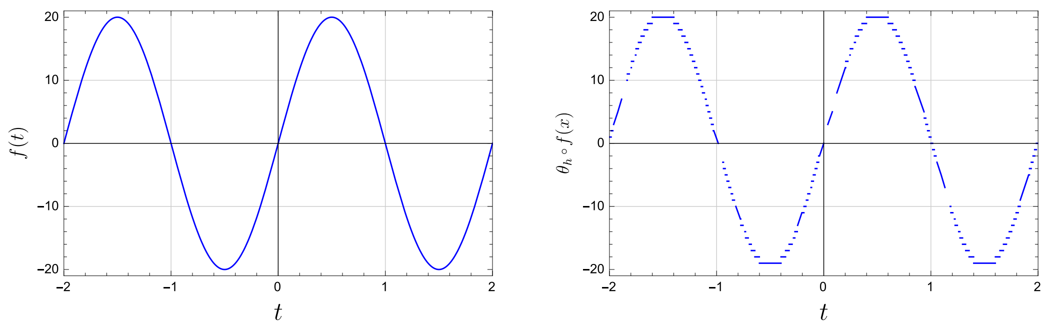


Figure 8.14 The uniform quantisation (right) of a continuous-time signal (left)

following definition.

8.1.12 Definition (Domain transformation of a signal) If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if $\mathbb{T}_1$ and $\mathbb{T}_2$ are time-domains, if $f: \mathbb{T}_1 \to \mathbb{F}$ is a signal, and if $\tau: \mathbb{T}_2 \to \mathbb{T}_1$ is a reparameterisation of $\mathbb{T}_1$, the ***domain transformation*** of f by τ is the signal $\tau^* f: \mathbb{T}_2 \to \mathbb{F}$ defined by $\tau^* f(t) = f \circ \tau(t)$. •

The funny notation $\tau^* f$ to denote the composition $f \circ \tau$ is intended to convey the idea that τ transforms the signal f into the new signal $\tau^* f$, an idea that is less easy to see from the notation $f \circ \tau$. We also wee here why it is natural to define a reparameterisation of $\mathbb{T}_1$ as having codomain $\mathbb{T}_1$, not domain $\mathbb{T}_1$.

Let us consider some examples of domain transformations, corresponding to some of the examples of reparameterisations introduced in Example 8.1.6.

8.1.13 Examples (Domain transformations)

1. For $a \in \mathbb{R}$ let us consider the shift $\tau_a: \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$. For a signal $f: \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\tau_a^* f(t) = f(t - a)$ for every $t \in \mathbb{T}_2$.
2. Let us consider the transposition $\sigma: \mathbb{T}_2 \to \mathbb{T}_1$. For a signal $f: \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\sigma^* f(t) = f(-t)$ for every $t \in \mathbb{T}_2$.
3. For $\lambda \in \mathbb{R}_{>0}$, let us consider the dilation $\rho_\lambda: \mathbb{T}_2 \to \mathbb{T}_1$. For a signal $f: \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\rho_\lambda^* f(t) = f(\lambda^{-1} t)$. •

The reader is asked to understand these transformations in Exercise 8.1.4.

The signal transformations considered above all have the feature that the character of the time-domain is preserved. That is to say, a discrete-time (resp. continuous-time) signal is transformed to a discrete-time (resp. continuous-time) signal. However, it is also interesting and important to consider transformations taking continuous-time signals to discrete-time signals, and vice versa. Let us now turn our attention to this.

8.1.14 Definition (Sampling, interpolation) Let $\mathbb{T}_{\mathrm{cont}}$ be a continuous time-domain and let $\mathbb{T}_{\mathrm{disc}}$ be a discrete time-domain such that $\mathbb{T}_{\mathrm{cont}}$ is the smallest continuous time-domain containing $\mathbb{T}_{\mathrm{disc}}$. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

(i) For a signal $f_{\mathrm{cont}}\colon \mathbb{T}_{\mathrm{cont}} \to \mathbb{F}$ define a signal $f_{\mathrm{disc}}\colon \mathbb{T}_{\mathrm{disc}} \to \mathbb{R}$ by $f_{\mathrm{disc}}(t) = f_{\mathrm{cont}}(t)$ for all $t \in \mathbb{T}_{\mathrm{disc}}$. The signal f_{disc} is the ***$\mathbb{T}_{\mathrm{disc}}$-sampled signal*** corresponding to f_{cont}. The map $f_{\mathrm{cont}} \mapsto f_{\mathrm{disc}}$ is called ***sampling***.

(ii) For a signal $f_{\mathrm{disc}}\colon \mathbb{T}_{\mathrm{disc}} \to \mathbb{F}$, an ***interpolant*** of f_{disc} is a signal $f_{\mathrm{cont}}\colon \mathbb{T}_{\mathrm{cont}} \to \mathbb{F}$ with the property that $f_{\mathrm{cont}}(t) = f_{\mathrm{disc}}(t)$ for all $t \in \mathbb{T}_{\mathrm{disc}}$. A rule for assigning to any f_{disc} an interpolant f_{cont} is called ***interpolation***. •

Note that sampling is uniquely defined. However, there are many possible ways in which one may interpolate from a discrete-time signal to a continuous-time signal. Let us consider a few of these.

8.1.15 Examples (Sampling and interpolation)

1. Sampling is easy to understand, and we illustrate it in Figure 8.15.

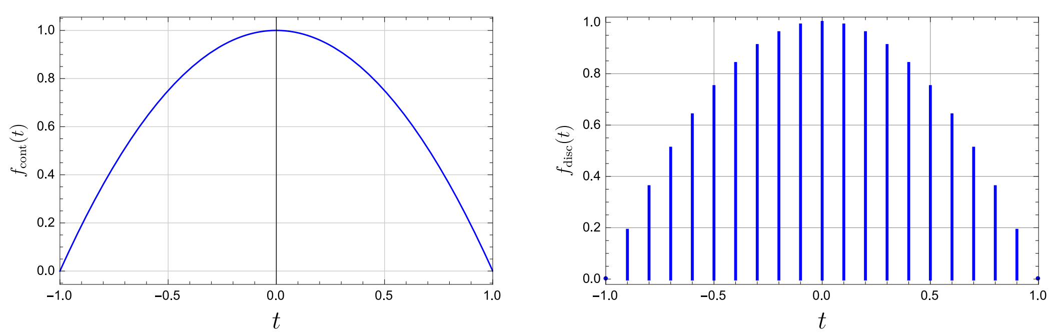


Figure 8.15 Sampling

Consider a discrete time-domain $\mathbb{T}_{\mathrm{disc}}$ with origin shift t_0 and sampling interval Δ and let $\mathbb{T}_{\mathrm{cont}}$ be the smallest continuous time-domain containing $\mathbb{T}_{\mathrm{disc}}$. Note that every point in $\mathbb{T}_{\mathrm{cont}}$ lies in a unique interval of the form $[t_0 + k\Delta, t_0 + (k+1)\Delta)$ for some $k \in \mathbb{Z}_{>0}$. Let us denote this interval by I_k. We take $\mathbb{F} = \mathbb{R}$ and a signal $f_{\mathrm{disc}}\colon \mathbb{T}_{\mathrm{disc}} \to \mathbb{R}$.

2. The interpolation defined by defining $f_{\mathrm{cont}}(t) = f_{\mathrm{disc}}(t_0 + k\Delta)$ if $t \in I_k$ is called the ***zeroth-order hold***. This is depicted in Figure 8.16. This is a simple interpolation method that has the advantage that it can be implemented in real time since it does not rely on knowledge of the value of signal at future times. As we shall see, some other interpolation schemes do not have this feature.

3. The interpolation defined by

$$f_{\mathrm{cont}}(t) = f_{\mathrm{disc}}(t_0 + k\Delta) + \frac{f_{\mathrm{disc}}(t_0 + (k+1)\Delta) - f_{\mathrm{disc}}(t_0 + k\Delta)}{\Delta}(t - (t_0 + k\Delta))$$

when $t \in I_k$ is called the ***first-order hold***. Checking the formulae will convince the reader that the first-order hold linearly interpolates between the values of

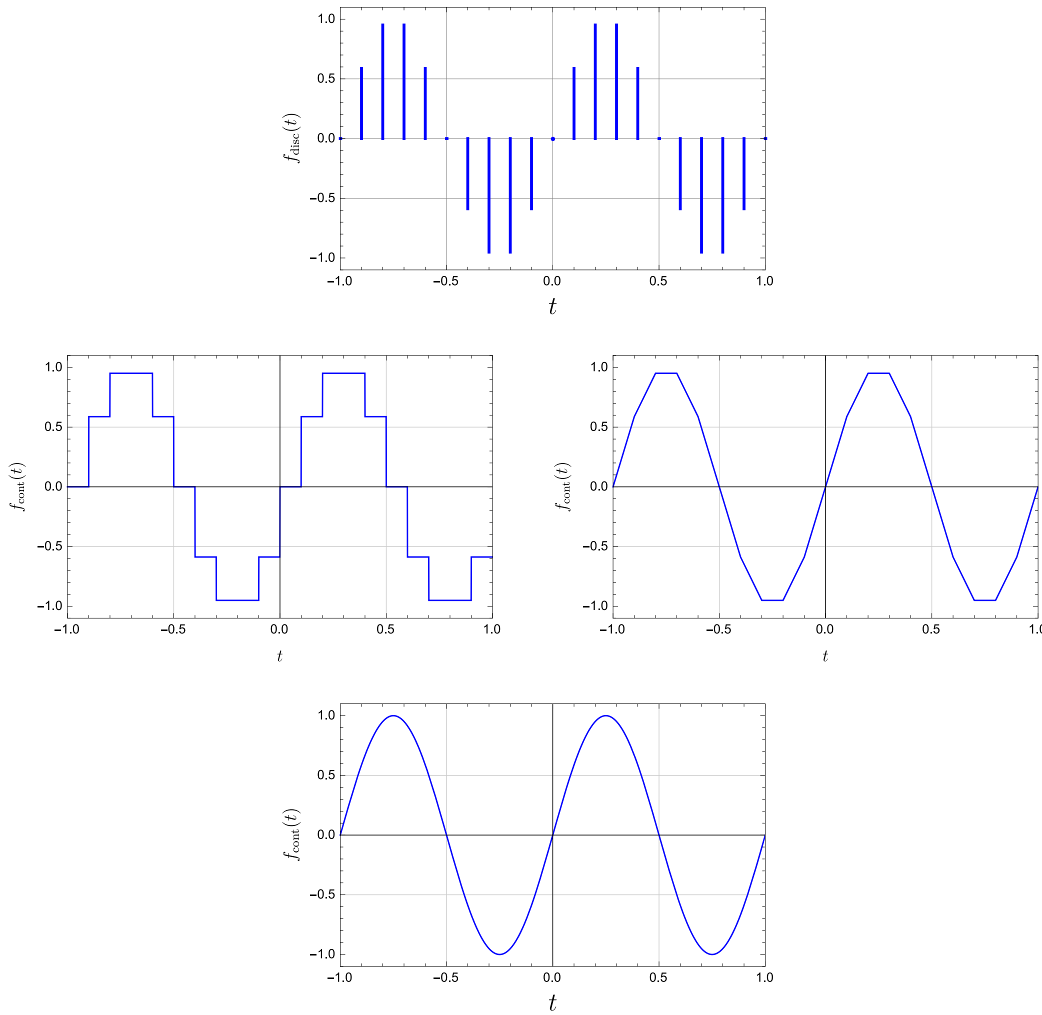


Figure 8.16 Zeroth-order hold (middle left), first-order hold (middle right), and cubic spline (bottom) applied to a discrete-time signal (top)

the discrete-time signal; we illustrate this interpolation in Figure 8.16. While it typically provides a more pleasant continuous-time signal, e.g., one that is continuous, it does require knowledge of the future values of the signal and so must necessarily carry a delay when implemented in real time.

4. Another general scheme for interpolation is the so-called ***spline interpolation***. The topic of splines is a huge one, so we only give a brief discussion. A popular technique of spline interpolation is the ***cubic spline***. Here, on each of the intervals I_k, one asks that f_{cont} be a cubic polynomial function. Thus, if one has N intervals $I_1, \dots, I_N$, one has N cubic polynomials to determine, each with four unknown coefficients. To determine the $4N$ coefficients one imposes conditions on the cubic polynomials. These are:

   (a) $f_{\text{cont}}(t) = f_{\text{disc}}(t)$ at the endpoints of the intervals $I_1, \dots, I_N$ (these are $N + 1$

conditions);

(b) the value at the right endpoint of I_k of the cubic polynomial on I_k should agree with the value at the left endpoint of I_{k+1} of the cubic polynomial on I_{k+1} (these are $N-1$ conditions);

(c) the value at the right endpoint of I_k of the derivative of the cubic polynomial on I_k should agree with the value at the left endpoint of I_{k+1} of the derivative of the cubic polynomial on I_{k+1} (these are $N-1$ conditions);

(d) the value at the right endpoint of I_k of the second derivative of the cubic polynomial on I_k should agree with the value at the left endpoint of I_{k+1} of the second derivative of the cubic polynomial on I_{k+1} (these are $N-1$ conditions).

The above conditions give $N + 1 + 3(N - 1) = 4N - 2$ linear conditions on the $4N$ coefficients, and these may be shown to be linearly independent. One then needs two additional conditions to be able to unambiguously prescribe an interpolation method. These typically involve determining a condition at each of the left endpoint of I_1 and the right endpoint of I_N. One such choice is the ***natural cubic spline*** where one asks that the second derivatives at these points be zero. This is what is shown in Figure 8.16. •

8.1.5 Causal and acausal signals

In "real life" signals occur on finite time-domains. However, it is convenient mathematically to allow infinite time-domains. And apart from mathematical convenience, many useful ideas are best discussed considering what would happen when time goes to infinity. The allowing of signals that are defined for increasingly *negative* times is more difficult to motivate physically. However, such signals can arise during the course of a mathematical treatment, and so it is useful to allow them. In this section we consider carefully the characterisation of signals on the basis of how they are look for infinite and negatively infinite times.

8.1.16 Definition (Causal signal, acausal signal) Let f be a signal on a time-domain $\mathbb{T}$. We say f is

(i) ***causal*** if either

  (a) $\mathbb{T}$ is positively infinite but not negatively infinite or

  (b) $\mathbb{T}$ is totally infinite and there exists $T \in \mathbb{T}$ so that $f(t) = 0$ for all $t < T$;

(ii) ***acausal*** if either

  (a) $\mathbb{T}$ is negatively infinite but not positively infinite or

  (b) $\mathbb{T}$ is totally infinite and there exists $T \in \mathbb{T}$ so that $f(t) = 0$ for all $t > T$. •

Let us visit the examples we provided for signals in the preceding section and consider their causal character.

8.1.17 Examples (Causal and acausal signals)

1. The unit step signal $\mathsf{1}$ is causal.
2. The unit ramp signal R is also causal.
3. A binary data stream is neither causal nor acausal.
4. The unit pulse P is both causal and acausal.
5. The square wave $\square_{a,\nu,\phi}$ is neither causal nor acausal.
6. The sawtooth $\triangle_{a,\nu,\phi}$ is neither causal nor acausal.
7. The Dow Jones Industrial Average opening averages data is neither causal nor acausal.
8. The Central England yearly average temperature data is neither causal nor acausal.
9. The Sierra Madre earthquake data is neither causal nor acausal. •

Note that some of these signals are neither causal nor acausal. There are two reasons why this can happen.

1. In Examples 5 and 6 the signals are nonzero for arbitrarily large positive and negative times. These are examples of signals that are not physical. However, one often wishes to use the square wave and the sawtooth for positive times. In this case, one can proceed in one of two ways: (a) one can leave the signals defined on all of $\mathbb{T} = \mathbb{R}$ and multiply them by the step signal to render them zero for negative times or (b) one can simply restrict them to be defined on $\mathbb{T} = [0, \infty)$. One should be careful to understand that these two ways of making the signal causal, although they appear to be the same, are really different since the time-domains are different. There will be occasions in book where it will be necessary to really understand the time-domain one is using.
2. In Examples 7, 8, and 9 the signals are only defined on a finite time-domain, and this makes them ineligible for being either causal or acausal. If one wishes to realise these signals as causal signals, one can extend them from their current time-domain $\mathbb{T}$ to a time-domain $\overline{\mathbb{T}}$ that is either positively infinite or totally infinite by making the extended signals zero on $\overline{\mathbb{T}} \setminus \mathbb{T}$. As with the preceding case, one should understand that these two extensions are genuinely different because they have different time-domains.

8.1.6 Periodic and harmonic signals

We shall discuss periodic signals in some detail in Sections 8.2.4 and 8.3.4. However, we consider them here, along with harmonic signals, since it is something easy to do before we launch into the mathematical treatment of signals.

8.1.18 Definition (Periodic signal, harmonic signal) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a totally infinite time-domain.

(i) A signal $f\colon \mathbb{T} \to \mathbb{F}$ is ***periodic*** with ***period*** $T \in \mathbb{R}_{>0}$ if $f(t + T) = f(t)$ for all $t \in \mathbb{T}$, i.e., if $\tau^*_{-T} f = f$.

(ii) The ***fundamental period*** of a periodic signal f is the smallest number T_0 for which f has period T_0, provided that this number is nonzero.

(iii) A signal $f\colon \mathbb{T} \to \mathbb{F}$ is ***harmonic*** with ***frequency*** $\nu \in \mathbb{R} \setminus \{0\}$, ***amplitude*** $a \in \mathbb{R}_{>0}$, and ***phase*** $\phi \in \mathbb{R}$ if

$$f(t) = \begin{cases} a\mathrm{e}^{\mathrm{i}(2\pi\nu t+\phi)}, & \mathbb{F} = \mathbb{C}, \\ a\cos(2\pi\nu t + \phi), & \mathbb{F} = \mathbb{R}, \end{cases} \tag{8.1}$$

for all $t \in \mathbb{T}$. The ***angular frequency*** for the harmonic signal is $\omega = 2\pi\nu$. For $\mathbb{R}$-valued signals, the quantity $\mathrm{e}^{\mathrm{i}\phi}$ is the ***phasor*** for the signal. The signal defined by (8.1) is denoted $\mathsf{H}_{a,\nu,\phi}$.

(iv) A ***trigonometric polynomial*** of period T is a finite linear combination of harmonic signals of period T. The ***degree*** of a trigonometric polynomial P the smallest positive integer d such that

$$P(t) = \sum_{n=-d}^{d} c_n \mathrm{e}^{2\pi\mathrm{i}nT^{-1}}$$

for some $c_n \in \mathbb{C}$, $n \in \{-d, \dots, 0, \dots, d\}$. •

8.1.19 Notation (Frequency versus angular frequency) It is worth mentioning the distinction between frequency and angular frequency. This is easiest to understand in terms of the units one uses for each. The units for frequency are s^{-1} or Hz (pronounced "hertz"[4] and the units for angular frequency are rad/s. The convention about which of these frequencies to use is not uniformly established. The distinction does come up with the various flavours of Fourier transform notation that are used. For the Fourier transform we use frequency and not angular frequency. However, there will be other occasions where we will use angular frequency.***missing stuff*** •

Let us record some properties of, and relationships between, periodic and harmonic signals.

8.1.20 Proposition (Properties of periodic and harmonic signals) *Let $\mathbb{T}$ be a totally infinite time-domain. The following statements hold:*

(i) a periodic signal with period T is also a periodic signal with period kT for $\mathrm{k} \in \mathbb{Z}_{>0}$;

(ii) if $\mathbb{T}$ is a continuous time-domain then harmonic signals are periodic;

(iii) if $\mathbb{T}$ is a discrete time-domain with sampling interval Δ then $\mathsf{H}_{\mathrm{a},\nu,\phi}$ is periodic if and only if $\nu\Delta \in \mathbb{Q}$;

(iv) if $\mathbb{T}$ is a discrete time-domain with sampling interval Δ then $\mathsf{H}_{\mathrm{a},\nu,\phi} = \mathsf{H}_{\mathrm{a},\nu+\mathrm{j}\Delta^{-1},\phi}$ for all $\mathrm{j} \in \mathbb{Z}_{>0}$.

[4]Heinrich Rudolf Hertz (1857–1894) was a German physicist who is perhaps most well-known for his contributions to contact in mechanical systems and electromagnetic theory.

Proof (i) This is Exercise 8.1.5.

(ii) This follows directly from the definitions.

(iii) First suppose that $\nu\Delta \in \mathbb{Q}$, so that we have $\nu\Delta = \frac{j}{k}$ for $j, k \in \mathbb{Z}$, and we may as well suppose that j and k are coprime. Then we compute

$$\begin{aligned}\mathsf{H}_{a,\nu,\phi}(t + k\Delta) &= a\mathrm{e}^{\mathrm{i}(2\pi\nu(t+k\Delta)+\phi)} = a\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{2\pi\mathrm{i}\nu t}\mathrm{e}^{2\pi\mathrm{i}\nu k\Delta}\\ &= a\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{2\pi\mathrm{i}\nu t}\mathrm{e}^{2\pi\mathrm{i}j} = a\mathrm{e}^{\mathrm{i}(2\pi\nu t+\phi)} = \mathsf{H}_{a,\nu,\phi}(t),\end{aligned}$$

for all $t \in \mathbb{T}$. Thus $\mathsf{H}_{a,\nu,\phi}$ is periodic with period $k\Delta$. Now suppose that $\mathsf{H}_{a,\nu,\phi}$ is periodic with period T. Then we must have $T = k\Delta$ for some $k \in \mathbb{Z}_{>0}$. Then

$$\begin{aligned} & \mathsf{H}_{a,\nu,\phi}(t + k\Delta) = \mathsf{H}_{a,\nu,\phi}(t), \qquad t \in \mathbb{T}\\ \implies\quad & a\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{2\pi\mathrm{i}\nu(t+k\Delta)} = a\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{2\pi\mathrm{i}\nu t}, \qquad t \in \mathbb{T}\\ \implies\quad & \mathrm{e}^{2\pi\mathrm{i}\nu k\Delta} = 1.\end{aligned}$$

Thus $2\pi\nu k\Delta = 2\pi j$ for some $j \in \mathbb{Z}_{>0}$, giving $\nu\Delta = \frac{j}{k} \in \mathbb{Q}$.

(iv) For $t = \Delta k \in \mathbb{Z}(\Delta)$ we have, in the event that $\mathbb{F} = \mathbb{C}$,

$$\begin{aligned}\mathsf{H}_{a,\nu+j\Delta^{-1},\phi}(t) &= a\mathrm{e}^{\mathrm{i}(2\pi(\nu+j\Delta^{-1})\Delta k+\phi)}\\ &= a\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{2\pi\mathrm{i}\nu\Delta k}\mathrm{e}^{2\pi\mathrm{i}jk} = a\mathrm{e}^{\mathrm{i}(2\pi\nu\Delta k+\phi)} = \mathsf{H}_{a,\nu,\phi}(t).\end{aligned}$$

The idea is exactly the same if $\mathbb{F} = \mathbb{R}$. ■

The phenomenon illustrated by part (iv) of the proposition is an important one in digital signal processing and is called ***aliasing***. The phenomenon is that signals that are different in continuous-time can look the same in discrete-time.

Note that a periodic signal with period T on a time-domain $\mathbb{T}$ is determined uniquely by its values on the set $[0, T] \cap \mathbb{T}$. Indeed, we shall frequently *only* think of such a signal as being defined on $[0, T] \cap \mathbb{T}$. Let us, therefore, give this time-domain a name.

8.1.21 Definition (Fundamental domain of a periodic signal) If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if $\mathbb{T}$ is an infinite time-domain, and if $f : \mathbb{T} \to \mathbb{F}$ is periodic with period T, then the ***fundamental domain*** of f is $[0, T] \cap \mathbb{T}$. •

It is convenient, in fact, to be able to start with a signal defined on $[0, T] \cap \mathbb{T}$ and extend it to a periodic signal. There are a few natural ways to do this. Let us give some useful terminology for this.

8.1.22 Definition (Even and odd signals) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathbb{T}$ be an infinite time-domain with a zero origin shift, and let $f : \mathbb{T} \to \mathbb{F}$. The signal f

(i) is ***even*** if $f(-t) = f(t)$ for each $t \in \mathbb{T}$, i.e., if $\sigma^* f = f$, and

(ii) is ***odd*** if $f(-t) = -f(t)$ for each $t \in \mathbb{T}$, i.e., if $\sigma^* f = -f$. •

We then have the following terminology.

8.1.23 Definition (Periodic extension, even and odd extension) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathbb{T}$ be a time-domain of the form $[a, a+T) \cap \mathbb{S}$, and let $f\colon \mathbb{T} \to$ be a signal. Let $\overline{\mathbb{T}}$ be the unique infinite time-domain for which $\overline{\mathbb{T}} \cap [0, T) = \mathbb{T}$.

(i) the ***T-periodic extension*** of f is the signal $f_{\mathrm{per}}\colon \overline{\mathbb{T}} \to \mathbb{F}$ defined by

$$f_{\mathrm{per}}(t) = f(t - kT), \qquad t \in [a + kT, a + (k+1)T).$$

(ii) if $a = 0$, the ***even extension*** of f is the signal $f_{\mathrm{even}}\colon \overline{\mathbb{T}} \to \mathbb{F}$ that is the $2T$-periodic extension of the signal $\overline{f}\colon [0, 2T) \to \mathbb{F}$ defined by

$$\overline{f}(t) = \begin{cases} f(t), & t \in [0, T), \\ f(2T - t), & t \in [T, 2T). \end{cases}$$

(iii) if $a = 0$, the ***odd extension*** of f is the signal $f_{\mathrm{odd}}\colon \overline{\mathbb{T}} \to \mathbb{F}$ that is the $2T$-periodic extension of the signal $\overline{f}\colon [0, 2T) \to \mathbb{F}$ defined by

$$\overline{f}(t) = \begin{cases} f(t), & t \in [0, T), \\ f(t - T), & t \in [T, 2T). \end{cases}$$ •

This is all trivial as an example makes clear.

8.1.24 Examples (Periodic extension, even and odd extension)

1. The first example we consider is a discrete-time example. Let $\Delta \in \mathbb{R}_{>0}$ and let $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$. From Example 8.1.9–4 recall the unit pulse $\mathsf{P}\colon \mathbb{Z}(\Delta) \to \mathbb{R}$ defined by

$$\mathsf{P}(t) = \begin{cases} 1, & t = 0, \\ 0, & \text{otherwise}. \end{cases}$$

The T-periodic extension of P is the ***T-periodic unit pulse*** $\mathsf{P}_{\mathrm{per},T}\colon \mathbb{Z}(\Delta) \to \mathbb{R}$ defined by

$$\mathsf{P}_{\mathrm{per},T}(t) = \begin{cases} 1, & t = kT \text{ for some } k \in \mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases}$$

2. Consider the signal $f\colon [\frac{1}{3}, \frac{4}{3}] \to \mathbb{R}$ defined by $f(t) = t$. The periodically extended signal $f_{\mathrm{per}}\colon \mathbb{R} \to \mathbb{R}$ is shown in Figure 8.17. Note that the periodic extension is neither even nor odd.
3. We consider the signal $f\colon [0, 1] \to \mathbb{R}$ defined by $f(t) = t$. In Figure 8.18 we give the 1-periodic extension, along with the even and odd extensions. Note that the even and odd extensions do indeed have period 2, and also not that the periodic extension is neither even nor odd. •

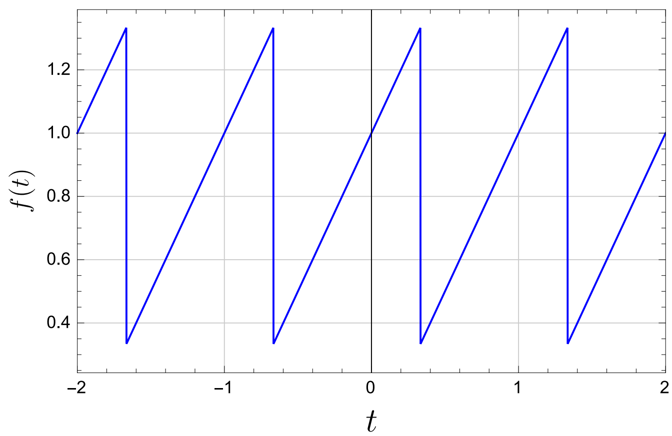


Figure 8.17 Periodic extension of a signal

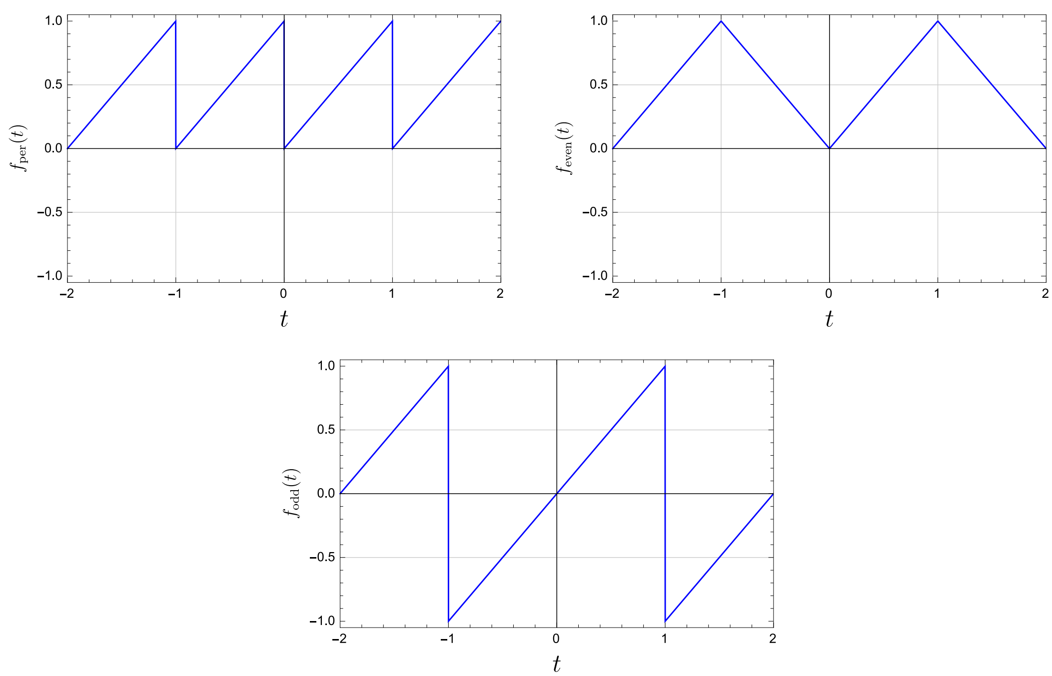


Figure 8.18 Periodic (top left), even (top right), and odd (bottom) extensions

8.1.7 Characterising time-domain signals

The preceding discussion of time-domain signals has a pleasant, breezy, high-level flavour. However, except for the purposes of establishing some language, it is almost devoid of technical value. What one is interested in in most applications is not instances of signals, but *classes* of signals, and classes of signals less vapid than "causal," "acausal," or, "periodic." In this section we address the sorts of properties by which one might organise classes of signals.

Although we will not discuss systems systematically until Volume **??**, it is convenient to use the notion of a system to motivate our discussion of signal properties. A system, in the broadest terms, is a "black box" accepting inputs and returning outputs. We schematically represent this in Figure 8.19. The inputs and

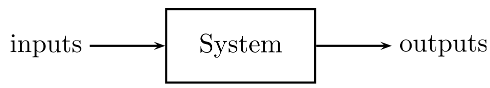


Figure 8.19 A depiction of a system

outputs are both to be thought of, for our purposes, as living in some collection of signals. A very useful and broadly used system property is linearity, the idea being that the output resulting from a linear combination of inputs is the same linear combination of outputs. It, therefore, makes sense to suppose that our signals spaces are vector spaces. System motivations aside, the characteristic of linearity for signal spaces seems quite natural. Thus we shall make free use of vector space concepts, mostly elementary ones, from Section 4.3.

While linearity is a natural property for a system—and the set of signals serving as inputs and outputs—it is simply too "floppy" to have much value *per se*. Moreover, most linear system models derived using physical principles have more structure than mere linearity. Indeed, most systems one encounters in practice have some "continuity" properties that turn out to be of great value. This property is most directly described in the following way: if a sequence $(u_j)_{j\in\mathbb{Z}_{>0}}$ if inputs converges to an input u, then the corresponding sequence of outputs $(y_j)_{j\in\mathbb{Z}_{>0}}$ converges to the output y associated with u. The notion of convergence requires more than mere linearity, and this is especially true for signal spaces since these tend to be infinite-dimensional. To allow us to discuss convergence we shall in this chapter use the notion of a norm, sometimes derived from an inner product. Thus we will make substantial use of material from Chapters 6 and 7. In particular, we will directly use some of the examples from Section 6.7. Indeed, all of the basic signal space structure we introduce in this chapter can be found in Section 6.7.

Readers unacquainted with the details of how the standard signal spaces are developed may be surprised and/or dismayed by how involved some of the constructions are. Indeed, apart from relying on material on Banach and Hilbert spaces from Chapters 6 and 7, we will also see that the Lebesgue integral, developed in Chapter 5, plays an essential rôle in the development. Therefore, it is maybe worth saying a few words about how one may approach all this. The reader may also, at this point, read the preface for this series of texts to guide them in going through this material if they are doing so for the first time.

It is important to keep in mind that it is not that the problems, *per se*, necessarily merit complicated mathematics, but that *general* solutions to the problems do. That is to say, if in a particular instance (say, one wants to know whether one's discrete-time representation of Beethoven's Ninth Symphony will be pleasant to listen to) one wishes to address one of these questions, then it is likely that much of the mathematical sophistication in this volume can be avoided. However, if one wishes to develop a *general* methodology that is *guaranteed* to work (say by one's proving of a theorem), then it is often the case that an astonishing amount of mathematical sophistication quickly becomes necessary. It is very easy to disparage such mathematical sophistication as being so much unnecessary abstraction. An excellent example of this is the famous remark by Richard W. Hamming (1915–1998):

> Does anyone believe that the difference between the Lebesgue and Riemann integrals[5] can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane.

This, however, seems to us to be a confusion of the specific with the general. That is to say, while it is not likely that there will ever exist an aircraft whose flight is literally dependent on the generality of the Lebesgue integral, it may very well be the case that certain aspects of the design of an aircraft are facilitated by general techniques for which the theorems ensuring their validity depend on the Lebesgue integral. For more discussion, see Section 8.1.8.

Moreover, that one can do without a certain degree of mathematical sophistication becomes more dubious when one turns to frequency representations of signals, as we do in Chapters 12, 13, and 14.

8.1.8 Notes

A discussion of the quote by Hamming appearing in Section 8.1.7 has been carried out by **MD/MI:02**. We advise the reader to read this article and develop an opinion on what is discussed there. From our point of view, one of the participants in the discussion is really quite ill-informed about the distinctions between the Riemann and Lebesgue integral, both from the point of view of their theoretical development and their application. We shall allow the reader to decide which author we indict in this way.

Exercises

8.1.1 List ten signals, five continuous-time and five discrete-time, that have affected your life in the past week. Indicate as many of the elementary properties of these signals, in the language of this section, as you can think of.

8.1.2 A subset S of $\mathbb{R}$ is ***discrete*** if there exists $r \in \mathbb{R}_{>0}$ so that for each $t \in S$ we have $\{s \mid |t-s| < r\} \cap S = \{t\}$. Show that the subgroups of $(\mathbb{R}, +)$ that are discrete as sets are of the form $\mathbb{Z}(\Delta)$ for some $\Delta \in \mathbb{R}_{>0}$.

8.1.3 Let $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ be time-domains and let $\tau_1 \colon \mathbb{T}_2 \to \mathbb{T}_1$ and $\tau_2 \colon \mathbb{T}_3 \to \mathbb{T}_2$ be reparameterisations. Show that $\tau_1 \circ \tau_2 \colon \mathbb{T}_3 \to \mathbb{T}_1$ is a reparameterisation.

8.1.4 Let $\mathbb{T} = \mathbb{R}$ and let $f \colon \mathbb{R} \to \mathbb{R}$ be a "general-looking" signal. Sketch the graph of f along with the graphs of the following signals:

   1. $\tau_a^* f$ for $a \in \mathbb{R}_{>0}$;

[5] This has to do with the topic of Lebesgue integration which we cover in Chapter 5.

2. $\tau_a^* f$ for $a \in \mathbb{R}_{<0}$;
3. $\sigma^* f$;
4. $\rho_\lambda^* f$ for $\lambda < 1$;
5. $\rho_\lambda^* f$ for $\lambda > 1$.

Hint: *By a "general-looking" signal we mean one for which all of the signals whose graph you are sketching are different.*

8.1.5 Show that if $f\colon \mathbb{R} \to \mathbb{F}$ has a period T, then it has a period kT for any $k \in \mathbb{Z}_{>0}$.

8.1.6 Show that if $t \mapsto e^{2\pi i \nu t}$ is T-periodic then $\nu = \frac{n}{T}$ for $n \in \mathbb{Z}$.

8.1.7 Give an example of a periodic signal whose fundamental period is not well-defined.

Section 8.2

Spaces of discrete-time signals

In this section we begin our systematic presentation of classes of signals. Since they are simpler, we begin with spaces of discrete-time signals. Much of the background for this section is pulled from Section 6.7.2. We assume that the reader knows what a vector space is and what a norm is, and some of the basic associated ideas. This may well require referring to material in Chapters 6 and 7. We shall try to make the necessary references when needed, but as a bare minimum the reader should know the basic properties of norms from Section 6.1.1, know the definitions of convergence of sequences in normed vector spaces from Section 6.2, and be familiar with the rôle of completeness discussed in Section 6.3.

Do I need to read this section? If you are reading this chapter then read this section. •

8.2.1 The vector space structure of general discrete-time signals

In this brief section we introduce the "big" vector space of discrete-time signals on a given time-domain. The idea is to give ourselves the basic object upon which everything else in this section is derived. The notation here originated in Notation 4.3.45.

8.2.1 Definition ($\mathbb{F}^{\mathbb{T}}$) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a discrete time-domain. We denote by $\mathbb{F}^{\mathbb{T}}$ the set of maps $f\colon \mathbb{T} \to \mathbb{F}$. The $\mathbb{F}$-vector space structure on $\mathbb{F}^{\mathbb{T}}$ is given by

$$(f_1 + f_2)(t) = f_1(t) + f_2(t), \quad (\alpha f)(t) = \alpha(f(t)),$$

for $f, f_1, f_2 \in \mathbb{F}^{\mathbb{T}}$ and for $\alpha \in \mathbb{F}$. We may also use the product of signals $f_1, f_2 \in \mathbb{F}^{\mathbb{T}}$ defined by

$$(f_1 f_2)(t) = f_1(t) f_2(t)$$

which makes $\mathbb{F}^{\mathbb{T}}$ into an $\mathbb{F}$-algebra. •

The case when $\mathbb{F}^{\mathbb{T}}$ is finite-dimensional is particularly simple and easy to characterise.

8.2.2 Proposition (Finite-dimensionality of $\mathbb{F}^{\mathbb{T}}$) *If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and if $\mathbb{T}$ is a discrete time-domain then $\mathbb{F}^{\mathbb{T}}$ is finite-dimensional if and only if $\mathbb{T}$ is finite. Moreover, if $\mathbb{T}$ is finite then $\dim_{\mathbb{F}}(\mathbb{F}^{\mathbb{T}}) = \mathrm{card}(\mathbb{T})$.*

Proof This is Exercise 8.2.1. ■

Note that it is *not* true that $\dim_{\mathbb{F}}(\mathbb{F}^{\mathbb{T}}) = \mathrm{card}(\mathbb{T})$ when $\mathbb{T}$ is infinite. This follows from Proposition **??** and Theorem **??**; indeed, from these results one can deduce that $\dim_{\mathbb{F}}(\mathbb{F}^{\mathbb{T}}) = \mathrm{card}(\mathbb{R})$ for an infinite discrete time-domain $\mathbb{T}$.

8.2.2 Discrete-time signal spaces characterised by their values

Next we consider a few discrete-time signal spaces that are characterised by how the values of the signals behave. Since we have already presented everything here in great detail in already in Section 6.7, we merely present the notation and recall the main properties of the various signal spaces, referring to the proofs that have already been given.

For a discrete time-domain $\mathbb{T}$ the signal spaces we consider here are the these:

$$\begin{aligned}
\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid f(t) = 0 \text{ for all but finitely many } t \in \mathbb{T}\};\\
\mathsf{c}_0(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid \text{ for each } \epsilon \in \mathbb{R}_{>0} \text{ there exists a finite subset } S \subseteq \mathbb{T}\\
&\qquad \text{such that } |f(t)| > \epsilon \text{ iff } t \in S\}.
\end{aligned}$$

If, for $f \in \mathbb{F}^{\mathbb{T}}$, we denote the ***support***

$$\mathrm{supp}(f) = \{t \in \mathbb{T} \mid f(t) \neq 0\}$$

then $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ is the set of signals with ***finite support***. For the purposes of this section, the norm we use on these vector spaces is the ∞-norm. Thus, if $f \in \mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ or $f \in \mathsf{c}_0(\mathbb{T};\mathbb{F})$ then we define

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \mathbb{T}\},$$

noting that the supremum is well-defined. Note that $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ is the generalisation to arbitrary discrete time-domains of the vector space $\mathbb{F}_0^\infty$ (see Example 6.1.3–**??**) and $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ is the generalisation to arbitrary discrete time-domains of the vector space $\mathsf{c}_0(\mathbb{F})$ (see Definition 6.7.11). We shall explore how the generalisation manifests itself as we go along.

Let us now list some facts about $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ and $\mathsf{c}_0(\mathbb{T};\mathbb{F})$.

1. If $\mathbb{T}$ is finite then
$$\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) = \mathsf{c}_0(\mathbb{T};\mathbb{F}) = \mathbb{F}^{\mathbb{T}},$$
and so the vector spaces are all finite-dimensional in this case. Because of this, the use of the norm $\|\cdot\|_\infty$ is not significant in that the topology on the spaces will be the same, no matter what norm is used; this is Theorem 6.1.15.
2. If $\mathbb{T}$ is infinite then
$$\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) \subset \mathsf{c}_0(\mathbb{T};\mathbb{F}) \subset \mathbb{F}^{\mathbb{T}}.$$
This is obvious.
3. If $\mathbb{T}$ is infinite then
$$\mathsf{c}_0(\mathbb{T};\mathbb{F}) = \Big\{f \in \mathbb{F}^{\mathbb{T}} \;\Big|\; \lim_{|t|\to\infty} f(t) = 0\Big\}.$$
4. In Example 6.1.3–**??** we considered the vector space $\mathbb{F}_0^\infty$ which, in our present language, is simply $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F})$. For any infinite discrete time-domain $\mathbb{T}$ there exists an isomorphism of normed vector spaces between $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F}), \|\cdot\|_\infty)$. This is pretty clear, but the reader may wish to verify this in

Exercise 8.2.8. This isomorphism may not really be natural; for example there is no really natural way to construct an isomorphism from $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F})$ to $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z};\mathbb{F})$. However, the mere existence of an isomorphism of normed vector spaces allows us to deduce for $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ certain of the properties we have deduced for $\mathbb{F}_0^\infty$. In particular, if $\mathbb{T}$ is infinite then the normed vector space $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}),\|\cdot\|_\infty)$ is not complete; this is Exercise 6.3.1.

5. In Definition 6.7.11 we defined the vector space $\mathsf{c}_0(\mathbb{F})$ which, in our present notation, is precisely $\mathsf{c}_0(\mathbb{Z}_{>0};\mathbb{F})$. As in the preceding paragraph, there exists an isomorphism of normed vector spaces between $\mathsf{c}_0(\mathbb{Z}_{>0};\mathbb{F})$ and $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ for any infinite discrete time-domain $\mathbb{T}$. Thus certain of the conclusions we have deduced for $\mathsf{c}_0(\mathbb{F})$ hold for $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ in this case. The conclusion of principal interest is this: $(\mathsf{c}_0(\mathbb{T};\mathbb{F}),\|\cdot\|_\infty)$ is a separable $\mathbb{F}$-Banach space and is, moreover, the completion of $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}),\|\cdot\|_\infty)$; this follows from Theorem 6.7.12 and Proposition 6.7.13.

8.2.3 The ℓ^p-spaces

Perhaps the most important discrete-time signal spaces in applications are those characterised by their summability properties. These were discussed at some length in Section 6.7.2, so we again just give the definitions and regurgitate the most useful properties.

For a discrete time-domain $\mathbb{T}$ with sampling interval Δ and for $p \in [1,\infty)$ the spaces we consider are:

$$\ell^\infty(\mathbb{T};\mathbb{F}) = \{f \in \mathbb{F}^{\mathbb{T}} \mid \sup\{|f(t)| \mid t \in \mathbb{T}\} < \infty\};$$
$$\ell^p(\mathbb{T};\mathbb{F}) = \Big\{f \in \mathbb{F}^{\mathbb{T}} \;\Big|\; \sum_{t\in\mathbb{T}} |f(t)|^p < \infty\Big\}.$$

On $\ell^\infty(\mathbb{T};\mathbb{F})$ we use the norm

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \mathbb{T}\}$$

and on $\ell^p(\mathbb{T};\mathbb{F})$ we use the norm

$$\|f\|_p = \Big(\Delta \sum_{t\in\mathbb{T}} |f(t)|^p\Big)^{1/p}.$$

There is a factor of Δ in the definition of the p-norm for $p \in [1,\infty)$ that seems to come from nowhere. It presence is motivated by connections between discrete-time signals and generalised signals that we are not able to explore at this time. The interested reader can refer to ***missing stuff***. We shall see as we go along that if $\mathbb{T}$ is finite then $(\ell^p(\mathbb{T};\mathbb{F}),\|\cdot\|_p)$ are the generalisations of the normed vector spaces $(\mathbb{F}^n,\|\cdot\|_p)$ considered in detail in Section 6.7.1. If $\mathbb{T}$ is infinite then $(\ell^p(\mathbb{T};\mathbb{F}),\|\cdot\|_p)$ are the generalisation of the normed vector spaces $(\ell^p(\mathbb{F});\|\cdot\|_p)$ considered in Section 6.7.2.

Let us list some facts about the ℓ^p-spaces.

1. If $\mathbb{T}$ is finite then
$$\ell^p(\mathbb{T};\mathbb{F}) = \ell^\infty(\mathbb{T};\mathbb{F}) = \mathbb{F}^{\mathbb{T}}$$
for all $p \in [1,\infty)$. Thus, for each $p \in [1,\infty]$, the space $(\ell^p(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ is isomorphic as a normed vector space to the normed vector space $(\mathbb{F}^n, \|\cdot\|_p)$ as discussed in Section 6.7.1. Thus this is the easiest case to consider. Moreover, as far as topology goes, the spaces $\ell^p(\mathbb{T};\mathbb{F})$ are all "the same" whenever $\mathbb{T}$ is finite.
2. If $\mathbb{T}$ is infinite then
$$\ell^p(\mathbb{T};\mathbb{F}) \subset \ell^\infty(\mathbb{T};\mathbb{F}) \subset \mathbb{F}^{\mathbb{T}}.$$
These inclusions are fairly clear, but these issues will be considered in detail in Section 8.2.6.
3. In Definitions 6.7.8 and 6.7.15 we considered the normed vector spaces $(\ell^p(\mathbb{F}), \|\cdot\|_p)$ for $p \in [1,\infty]$. In terms of our present notation we have $\ell^p(\mathbb{F}) = \ell^p(\mathbb{Z}_{>0};\mathbb{F})$. It is not difficult to show that, in fact, the normed vector spaces $(\ell^p(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ and $(\ell^p(\mathbb{Z}_{>0};\mathbb{F}), \|\cdot\|_p)$ are isomorphic (up to a constant factor for the norm in the case of $p \in [1,\infty)$) for any infinite discrete time-domain $\mathbb{T}$. This allows us to draw conclusions for $\ell^p(\mathbb{T};\mathbb{F})$ based on conclusions we have already drawn for $\ell^p(\mathbb{F})$. For example, we have the following facts.
   (a) If $\mathbb{T}$ is infinite then $(\ell^\infty(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is a nonseparable $\mathbb{F}$-Banach space.
   (b) If $\mathbb{T}$ is infinite and if $p \in [1,\infty)$ then $(\ell^p(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ is a separable $\mathbb{F}$-Banach space and, moreover, is the completion of $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$. This Banach space is a Hilbert space if and only if $p = 2$.

8.2.4 Periodic discrete-time signal spaces

An important class of signals in both theory and application are those that are periodic. For discrete time-domains, T-periodic signals have no exotic behaviour. Indeed, since they are determined by their values on the fundamental domain $[0,T) \cap \mathbb{T}$ and since such fundamental domains are necessarily finite, we have the following result.

8.2.3 Proposition (Spaces of periodic discrete-time signals are finite-dimensional) *Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, let $\mathbb{T}$ be an infinite time-domain, and let $\mathrm{T} \in \mathbb{R}_{>0}$. Then the subspace of $\mathbb{F}^{\mathbb{T}}$ consisting of T-periodic signals is finite-dimensional.*

Thus we do not need to discriminate notationally between spaces of periodic discrete-time signals, and, for a discrete time-domain $\mathbb{T}$, we merely denote
$$\ell_{\mathrm{per},T}(\mathbb{T};\mathbb{F}) = \{f \in \mathbb{F}^{\mathbb{T}} \mid f \text{ is } T\text{-periodic}\}.$$
Note, however, that we may use a variety of norms on this space. Indeed, we can use any one of the norms
$$\|f\|_p = \Big(\Delta \sum_{t\in[0,T)\cap\mathbb{T}} |f(t)|^p\Big)^{1/p}, \qquad p \in [1,\infty),$$
$$\|f\|_\infty = \max\{|f(t)| \mid t \in [0,T) \cap \mathbb{T}\}.$$

We leave it to the reader as Exercise 8.2.4 to make the elementary verification that these are norms. Note that for $p \in [1, \infty)$ these are *not* the norms for the signals on $\mathbb{T}$, but take into account the periodicity of the signals.

8.2.5 Other characteristics of discrete-time signals

In this section we give some characteristic of signals that are often useful in practice. Some of these are simply renaming of norms we have provided above. Some of the properties, however, are not related to the norms, but are still useful.

8.2.4 Definition (Signal characteristics) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a discrete time-domain and let $f: \mathbb{T} \to \mathbb{F}$ be an $\mathbb{F}$-valued signal on $\mathbb{T}$. If $\mathbb{T}$ has sampling interval Δ then we define $N_{\min}$ and $N_{\max}$ by asking that $\Delta N_{\min} = \inf \mathbb{T}$ and $\Delta N_{\max} = \sup \mathbb{T}$. We allow either or both of $N_{\min}$ and $N_{\max}$ to be infinite in magnitude.

(i) If $f \in \ell^1(\mathbb{T}; \mathbb{F})$ then $\|f\|_1$ is the ***action*** of f.

(ii) If $f \in \ell^2(\mathbb{T}; \mathbb{F})$ then $\|f\|_2^2$ is the ***energy*** of f.

(iii) If $f \in \ell^\infty(\mathbb{T}; \mathbb{F})$ then $\|f\|_\infty$ is the ***amplitude*** of f.

(iv) If the limit

$$\lim_{\substack{N_- \to N_{\min} \\ N_+ \to N_{\max}}} \frac{1}{N_+ - N_- + 1} \sum_{j=N_-}^{N_+} |f(j\Delta)|^2$$

exists we denote it by $\mathrm{pow}(f)$ and call it the ***average power*** of f. The set of signals whose average power exists are called ***power signals*** and the set of these is denoted by $\ell^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$.

(v) If $f \in \ell^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$ then $\mathrm{rms}(f) = \sqrt{\mathrm{pow}(f)}$ is the ***root mean square value*** (***rms value***) of f.

(vi) The ***mean*** of f is given by

$$\mathrm{mean}(f) = \lim_{\substack{N_- \to N_{\min} \\ N_+ \to N_{\max}}} \frac{1}{N_+ - N_- + 1} \sum_{j=N_-}^{N_+} f(j\Delta),$$

if the limit is defined. •

We shall not have much occasion to use the set of power signals. Indeed, mathematically this is not so useful a collection of signals, at least for infinite discrete time-domains; for finite discrete time-domains the average power is simply a scaled version of the energy. The motivation for the definition of average power is that it should give some sort of integral (in this case, summation) measure of a signal that does not decay to zero at infinity.

8.2.5 Remark (The importance of p ∈ {1, 2, ∞}) Note that special names are given to the p-norms for $p \in \{1, 2, \infty\}$. This suggests that these cases are somehow especially important. This is indeed the case, so let us explain this a little. The importance of the ∞-norm and of signals with finite ∞-norm is more or less clear. The importance of the 1-norm and of signals with finite 1-norm is less easy to see at this point.

However, the fact that the 1-norm of a sequence being finite corresponds to the sequence being absolutely summable is important in Fourier analysis. We shall see specific instances of this in Theorems 12.2.33 and 14.1.7. The case of $p = 2$ is perhaps even more difficult to image the importance of. Nonetheless, in some applications the only sequence space even discussed is ℓ^2. Often physical justifications are given for this. However, the real reason for the importance of $p = 2$ is that in this case ℓ^2 is a Hilbert space, and not a general Banach space. This allows the use of special Hilbert space tools, particularly Hilbert bases, in the analysis of sequences in ℓ^2. We shall see the importance of this in, for example, Sections 12.3 and 14.1.6. •

Let us give an example where we work out some of the quantities defined above.

8.2.6 Example (Signal characteristics) Let $\mathbb{T} = \mathbb{Z}_{>0}$ and define $f\colon \mathbb{T} \to \mathbb{R}$ by $f(j) = \frac{1}{j^2}$. Then we compute:

1. the action of f is $\frac{\pi^2}{6}$;
2. the energy of f is $\frac{\pi^4}{90}$;
3. the amplitude of f is 1;
4. f is a power signal and $\mathrm{pow}(f) = 0$;
5. the rms value of f is $\mathrm{rms}(f) = 0$;
6. the mean of f is defined and $\mathrm{mean}(f) = 0$.

Generally speaking, of course, it will be impossible to determine explicit expressions for many of these properties, except in exceptional cases. We have used the computer to determine certain of the sums above. In previous centuries one might have looked these up in a table or (gasp!) tried to figure them out somehow. •

8.2.6 Inclusions of discrete-time signal spaces

We have already alluded above to some of the inclusion relations that exist between the various discrete-time signal spaces. Here we discuss this more thoroughly and prove some facts about these inclusions.

8.2.7 Theorem (Inclusions between discrete-time signal spaces) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T} \subseteq \mathbb{R}$ be a discrete time-domain. The following statements hold:*

(i) if $\mathbb{T}$ is finite then

$$\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) = \mathsf{c}_0(\mathbb{T};\mathbb{F}) = \ell^{\mathrm{p}}(\mathbb{T};\mathbb{F}) = \ell^{\mathrm{pow}}(\mathbb{T};\mathbb{F}) = \mathbb{F}^{\mathbb{T}}$$

for all $\mathrm{p} \in [1, \infty]$;

(ii) if $\mathbb{T}$ is infinite then $\ell^\infty(\mathbb{T};\mathbb{F}) \subset \ell^{\mathrm{pow}}(\mathbb{T};\mathbb{F})$;

(iii) if $\mathbb{T}$ is infinite then $\mathsf{c}_0(\mathbb{T};\mathbb{F}) \subset \ell^\infty(\mathbb{T};\mathbb{F})$;

(iv) if $\mathbb{T}$ is infinite then $\ell^{\mathrm{p}}(\mathbb{T};\mathbb{F}) \subset \mathsf{c}_0(\mathbb{T};\mathbb{F})$ for all $\mathrm{p} \in [1, \infty)$;

(v) if $\mathbb{T}$ is infinite and if $\mathrm{p}, \mathrm{q} \in [1, \infty]$ then $\ell^{\mathrm{p}}(\mathbb{T};\mathbb{F}) \subset \ell^{\mathrm{q}}(\mathbb{T};\mathbb{F})$ if and only if $\mathrm{p} < \mathrm{q}$.

Moreover, the inclusions in parts (iii), (iv), and (v) are continuous.

Proof (i) This is obvious.

(ii) For simplicity we consider the case of $\mathbb{T} = \mathbb{Z}_{>0}$; the case of an arbitrary infinite discrete time-domain follows from this (why?). Let $f \in \ell^\infty(\mathbb{T};\mathbb{F})$ and denote $M = \|f\|_\infty$. Then

$$\frac{1}{N+1}\sum_{j=1}^{N}|f(j)|^2 \le \frac{M^2}{N+1}\sum_{j=1}^{N}1 = \frac{M^2N}{N+1}.$$

Thus $\mathrm{pow}(f) \le M^2$ and so f is a power signal. That the inclusion is strict is left to the reader to verify as Exercise 8.2.11.

(iii) Let $f \in \mathsf{c}_0(\mathbb{T};\mathbb{F})$. By definition of $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ there exists a finite subset $S \subseteq \mathbb{T}$ such that $|f(t)| > 1$ if and only if $t \in S$. Therefore,

$$\|f\|_\infty = \max\{1, \max\{|f(t)|| \ t \in S\}\} < \infty.$$

The signal $f(t) = 1$ for every $t \in \mathbb{T}$ is in $\ell^\infty(\mathbb{T};\mathbb{F})$ but not in $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ and so the inclusion is strict. To show that the inclusion of $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ in $\ell^\infty(\mathbb{T};\mathbb{F})$ is continuous, we note merely that it is obviously norm-preserving.

(iv) Let $p \in [1,\infty)$ and let Δ be the sampling interval for $\mathbb{T}$. By Proposition 2.4.7, it follows that if $f \in \ell^p(\mathbb{T};\mathbb{F})$ then $\lim_{|t|\to\infty}|f(t)|^p = 0$. Thus $f \in \mathsf{c}_0(\mathbb{T};\mathbb{F})$. In Exercise 8.2.9 the reader can show that the inclusion is strict. To see that the inclusion is continuous, let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\ell^p(\mathbb{T};\mathbb{F})$ converging to f. Let $\epsilon \in \mathbb{R}_{>0}$ and let $N \in \mathbb{Z}_{>0}$ be such that $\|f - f_j\|_p^p < \Delta\epsilon^p$. Then, for each $t_0 \in \mathbb{T}$ and for each $j \ge N$,

$$\Delta|f(t_0) - f_j(t_0)|^p < \Delta\sum_{t\in\mathbb{T}}|f(t) - f_j(t)|^p = \|f - f_j\|_p^p < \Delta\epsilon^p.$$

Thus $\|f - f_j\|_\infty < \epsilon$ for every $j \ge N$, showing that $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\ell^\infty(\mathbb{T};\mathbb{F})$. Continuity of the inclusion now follows from Theorem 6.5.2.

(v) By parts (iii) and (iv) it follows that $\ell^p(\mathbb{T};\mathbb{F}) \subset \ell^\infty(\mathbb{T};\mathbb{F})$ for every $p \in [1,\infty)$. Moreover, by Proposition 3.6.10 we have $|a|^p > |a|^q$ for $a \in \mathbb{F}$ satisfying $|a| \in (0,1)$ and for $p,q \in [1,\infty)$ satisfying $p < q$. This shows that

$$\sum_{t\in\mathbb{T}}|f(t)|^q = \sum_{t\in\mathbb{T}}|f(t)|^p|f(t)|^{q-p} \le \|f\|_\infty^{q-p}\sum_{t\in\mathbb{T}}|f(t)|^p = \|f\|_\infty^{q-p}\|f\|^p.$$

Thus $\ell^p(\mathbb{T};\mathbb{F}) \subseteq \ell^q(\mathbb{T};\mathbb{F})$. That the inclusion is strict we leave the reader to show in Exercise 8.2.10. To show continuity of the inclusion, let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\ell^p(\mathbb{T};\mathbb{F})$ converging to f. For $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be such that $\|f - f_j\|_p^p < \epsilon^q$ for $j \ge N$. Then, for $j \ge N$ we have

$$\|f - f_j\|_q^q = \Delta\sum_{t\in\mathbb{T}}|f(t) - f_j(t)|^q \le \Delta\sum_{t\in\mathbb{T}}|f(t) - f_j(t)|^p = \|f - f_j\|_p^p < \epsilon^q.$$

Thus the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\ell^q(\mathbb{T};\mathbb{F})$, giving continuity of the inclusion by Theorem 6.5.2. ■

The Venn diagrams for these relationships are shown in Figure 8.20. The following examples of signals provide representatives for all regions of the Venn diagram for discrete-time signals defined on infinite time-domains. The "shape" of these diagrams follow from Theorem 8.2.7 and the exercises referred to in the proof.

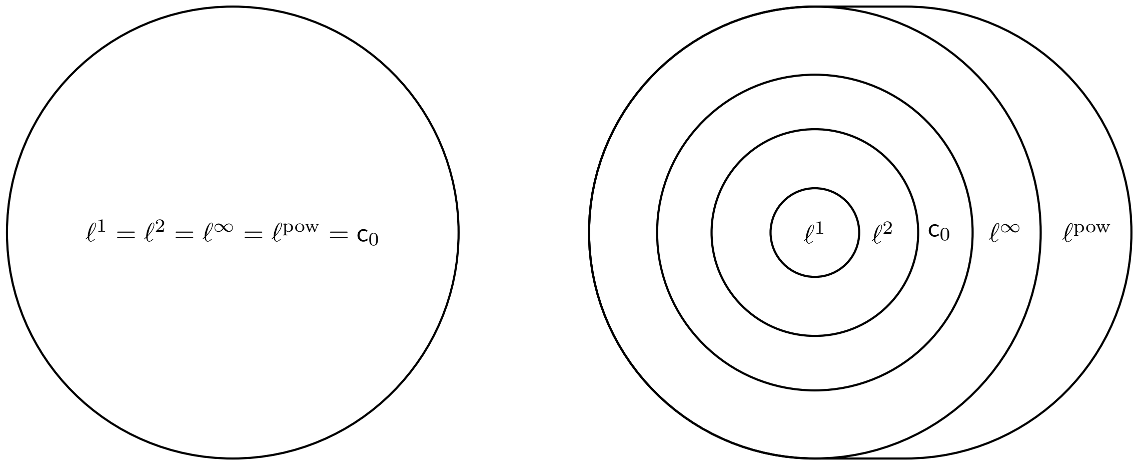


Figure 8.20 Venn diagrams illustrating inclusions of signal spaces for discrete time-domains: the finite case (left) and the infinite case (right)

Exercises

8.2.1 Prove Proposition 8.2.2.

8.2.2 Let $\Delta \in \mathbb{R}_{>0}$ and consider the finite discrete time-domain $\mathbb{T} = \{j\Delta \mid j \in \{0, 1, \ldots, n-1\}\}$. Show that $\{e^{2\pi i \Delta mn} \mid m \in \{0, 1, \ldots, n-1\}\}$ is an orthogonal basis for $\ell^2(\mathbb{T}; \mathbb{C})$.

The matter of determining when a signal is in one of the ℓ^p-spaces can be a little problematic. Certainly one does not want to rely on being able to explicitly compute the p-norm; counting on one's ability to compute infinite sums is an activity doomed to failure. In the following exercise you will provide some conditions that, while simple, are often enough to ascertain when a given signal is in ℓ^p. It is enough to consider the case of $\mathbb{T} = \mathbb{Z}_{>0}$.

8.2.3 Prove the following result.

Proposition *If* $\mathrm{f} \in \mathbb{F}^{\mathbb{Z}_{>0}}$ *and if* $\lim_{\mathrm{t}\to\infty} \frac{|\mathrm{f(t)}|}{\mathrm{t^a}} = 0$ *for some* $\mathrm{a} < -\frac{1}{\mathrm{p}}$ *then* $\mathrm{f} \in \ell^{\mathrm{p}}(\mathbb{Z}_{>0}; \mathbb{F})$.

8.2.4 Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be an infinite discrete time-domain. Show that $\|\cdot\|_p$, $p \in [1, \infty]$, is a norm on $\ell_{\mathrm{per},T}(\mathbb{T}; \mathbb{F})$.

8.2.5 Show that, for any discrete time-domain $\mathbb{T}$, if $f, g \in \ell^2(\mathbb{T}; \mathbb{F})$ then $fg \in \ell^1(\mathbb{T}; \mathbb{F})$ and $\|fg\|_1 \leq \|f\|_2 \|g\|_2$.

8.2.6 For the following discrete-time signals defined on $\mathbb{T} = \mathbb{Z}_{>0}$, compute their action, energy, amplitude, average power, rms value, and mean:

(a) $f(t) = \cos(\pi t)$;

(b) $f(t) = \cos(\pi t) + 1$;

(c) $f(t) = \frac{1}{t}$.

8.2.7 For the following discrete-time signals defined on $\mathbb{T} = \{1, \dots, N\}$, compute their action, energy, amplitude, average power, rms value, and mean:

(a) $f(t) = \cos(\pi t)$;

(b) $f(t) = \cos(\pi t) + 1$;

(c) $f(t) = \frac{1}{t}$.

8.2.8 Show that, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for an infinite time-domain $\mathbb{T}$,

(a) there exists an isomorphism of normed vector spaces between $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T}; \mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0}; \mathbb{F}), \|\cdot\|_\infty)$,

(b) there exists an isomorphism of normed vector spaces between $(\mathsf{c}_0(\mathbb{T}; \mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{c}_0(\mathbb{Z}_{>0}; \mathbb{F}), \|\cdot\|_\infty)$, and

(c) there exists an isomorphism of normed vector spaces between $(\ell^p(\mathbb{T}; \mathbb{F}), \|\cdot\|_p)$ and $(\ell^p(\mathbb{Z}_{>0}; \mathbb{F}), \|\cdot\|_p)$ for each $p \in [1, \infty]$, with the consideration of a constant factor for the norm in the case of $p \in [1, \infty)$.

8.2.9 For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for an infinite discrete time-domain $\mathbb{T}$, show that $\ell^p(\mathbb{T}; \mathbb{F})$ is a *strict* subspace of $\mathsf{c}_0(\mathbb{T}; \mathbb{F})$ for each $p \in [1, \infty)$. Does there exist $f \in \mathsf{c}_0(\mathbb{T}; \mathbb{F})$ such that $f \notin \ell^p(\mathbb{T}; \mathbb{F})$ for *every* $p \in [1, \infty)$?

8.2.10 For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, for an infinite discrete time-domain $\mathbb{T}$, and for $p, q \in [1, \infty)$ satisfying $p < q$, show that $\ell^p(\mathbb{T}; \mathbb{F})$ is a *strict* subspace of $\ell^q(\mathbb{T}; \mathbb{F})$.

8.2.11 For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for an infinite discrete time-domain $\mathbb{T}$, show that $\ell^\infty(\mathbb{T}; \mathbb{F})$ is a *strict* subset of $\ell^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$.

Section 8.3

Spaces of continuous-time signals

In this section we present the classes of continuous-time signals that will arise in these volumes. As with our treatment of discrete-time signal spaces, we will refer back to material from Section 6.7, mainly from Sections 6.7.4 and 6.7.7. Spaces of continuous-time signals are significantly more complicated to deal with than their discrete-time counterparts. There are two reasons for this.

1. For discrete-time signals there is no (nontrivial) notion of continuity. For continuous-time signals, continuity is a property of which one wishes to keep track. That is to say, one wants to include in one's classes of signals those that are continuous, possibly with other properties. But one also wishes to allow for discontinuous signals, both because they arise in practice and because they arise as limits of sequences of continuous signals. By allowing discontinuous signals we open ourselves to the question, "How discontinuous must a signal be before we are justified in ignoring it?" Our answer is that we restrict our attention to signals that are measurable with respect to the Lebesgue measure. This is an extremely large class, actually, and serves our purposes well.
2. Another reason for the complication associated with continuous-time signals is that the simple sums used to characterise the discrete-time ℓ^p-signals get replaced with integrals. If we want our spaces to be Banach spaces, and we do, this precludes the use of the Riemann integral since it behaves badly with respect to limits. Thus we must resort to the Lebesgue integral to get the completeness we need to do any of the useful analysis we present in these volumes.

The upshot of the preceding discussion is that we must add to our list of prerequisite material from Section 8.2 the prerequisite of measure theory from Chapter 5. A reader who wishes to can, at least initially, sidestep the discussion of the Lebesgue integral, pretending that we are only interested in Riemann integrable functions. However, be aware that in doing this, many important theorems referred to in this chapter, and presented later in this volume, are actually invalid. Thus the Lebesgue integral really is essential, even if you wish it were not.

Do I need to read this section? If you are reading this chapter then read this section. •

8.3.1 The vector space structure of general continuous-time signals

As we did with discrete-time signals, we get started by looking at a big space of signals in which all of our continuous-time signal spaces will sit as subspaces.

8.3.1 Definition ($\mathbb{F}^{\mathbb{T}}$) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a continuous time-domain. We denote by $\mathbb{F}^{\mathbb{T}}$ the set of maps $f\colon \mathbb{T} \to \mathbb{F}$. The $\mathbb{F}$-vector space structure on $\mathbb{F}^{\mathbb{T}}$ is given by

$$(f_1 + f_2)(t) = f_1(t) + f_2(t), \quad (\alpha f)(t) = \alpha(f(t)),$$

for $f, f_1, f_2 \in \mathbb{F}^{\mathbb{T}}$ and for $\alpha \in \mathbb{F}$. We may also use the product of signals $f_1, f_2 \in \mathbb{F}^{\mathbb{T}}$ defined by

$$(f_1 f_2)(t) = f_1(t) f_2(t)$$

which makes $\mathbb{F}^{\mathbb{T}}$ into an $\mathbb{F}$-algebra. •

Of course, unless $\mathbb{T}$ is a mere point, the vector space $\mathbb{F}^{\mathbb{T}}$ has a very large dimension. Indeed, using Proposition **??** and Theorem **??** one may deduce that $\dim_{\mathbb{F}}(\mathbb{F}^{\mathbb{T}}) = 2^{\operatorname{card}(\mathbb{R})}$. But, in fact, the vector space $\mathbb{F}^{\mathbb{T}}$ is far too large to be useful, and we shall restrict ourselves to spaces with a substantial amount of structure. Even so, all classes of continuous-times signals we encounter will be infinite-dimensional. The following result indicates why this is so. In the statement of the result we recall the notion of the support of a continuous function from Definition 6.7.28(**??**).***missing stuff***

8.3.2 Proposition (Infinite-dimensionality of continuous-time signal spaces) *Let* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and let* $\mathbb{T}$ *be a continuous time-domain with nonempty interior. If* $[\mathrm{a}, \mathrm{b}] \subseteq \operatorname{int}(\mathbb{T})$ *and if* V *is any subspace of* $\mathbb{F}^{\mathbb{T}}$ *containing the continuous functions whose support is contained in* $[\mathrm{a}, \mathrm{b}]$*, then* V *is infinite-dimensional.*

Proof For simplicity we consider the special case where $[a, b] = [0, 1] \subseteq \operatorname{int}(\mathbb{T})$. A simple adaptation of the argument gives the general case. For $j \in \mathbb{Z}_{>0}$ we define

$$f_j(t) = \begin{cases} \sin(j\pi t), & t \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that the collection of signals $\{f_j\}_{j \in \mathbb{Z}_{>0}}$ is linearly independent. Indeed, consider any finite subset $\{f_{j_1}, \dots, f_{j_k}\}$ and suppose that there are constants $c_1, \dots, c_k \in \mathbb{R}$ so that

$$c_1 \sin(j_1 \pi t) + \cdots + c_k \sin(j_k \pi t) = 0, \quad t \in [0, 1]. \tag{8.2}$$

This means that for any $l \in \{1, \dots, k\}$ we have

$$\begin{aligned} & \sin(j_l \pi t)(c_1 \sin(j_1 \pi t) + \cdots + c_k \sin(j_k \pi t)) = 0, \quad t \in [0, 1] \\ \implies \quad & \int_0^1 \Big(c_1 \sin(j_1 \pi t) \sin(j_l \pi t) + \cdots + c_j \sin^2(j_l \pi t) + \dots \\ & + c_k \sin(j_k \pi t) \sin(j_l \pi t)\Big) \, \mathrm{d}t = 0 \\ \implies \quad & \frac{1}{2} c_l = 0. \end{aligned}$$

Here we have used the readily verified equality

$$\int_0^1 \sin(j\pi t) \sin(k\pi t) \, \mathrm{d}t = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}, & j = k, \end{cases}$$

for any $j, k \in \mathbb{Z}$. In any event, we have shown that if (8.2) holds, then $c_l = 0$, $l \in \{1, \dots, k\}$. This shows that the signals with support contained in $[0, 1]$ is infinite-dimensional. Thus V is also infinite-dimensional. ■

One might be inclined to say that, even with severe restrictions to the classes of continuous-time signals we consider in $\mathbb{F}^{\mathbb{T}}$, any reasonable class of such signals will be much larger than any discrete-time signal space. As we shall see, this is actually not the case in general. Some hint about these matters can be seen from the fact that many of the normed vector spaces from Sections 6.7.4 and 6.7.7 are separable.

8.3.2 Spaces of continuous signals

Let us first consider spaces comprised of continuous signals. The material here has been gone through thoroughly in Section 6.7.4, and so we will mainly reproduce the definitions and summarise the important results. The reader is strongly encouraged to go through the material in Section 6.7.4 to really see how everything fits together.

We let $\mathbb{F} \in \{\mathbb{R}; \mathbb{C}\}$ and let $\mathbb{T}$ be a continuous time-domain. The spaces of continuous signals we consider are these:

$$\begin{aligned} \mathsf{C}^0(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid f \text{ is continuous}\};\\ \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \mid f \text{ has compact support}\};\\ \mathsf{C}^0_0(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \mid \text{ for every } \epsilon \in \mathbb{R}_{>0} \text{ there exists a compact set } K \subseteq \mathbb{T}\\ &\qquad \text{such that } \{t \in \mathbb{T} \mid |f(t)| \geq \epsilon\} \subseteq K\};\\ \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \mid \text{ there exists } M \in \mathbb{R}_{>0} \text{ such that } |f(t)| \leq M\\ &\qquad \text{for all } t \in \mathbb{T}\}. \end{aligned}$$

The norm we consider for all of these spaces of signals is the ∞-norm:

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \mathbb{T}\}.$$

The supremum in the definition always exists for f in $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$, $\mathsf{C}^0_0(\mathbb{T};\mathbb{F})$, or $\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F})$. If $f \in \mathsf{C}^0(\mathbb{T};\mathbb{F})$ then $\|f\|_\infty$ is generally only defined when $\mathbb{T}$ is compact. Therefore, we will not deal much with $\mathsf{C}^0(\mathbb{T};\mathbb{F})$ except in this compact case.

Let us reproduce some of the more important facts about these spaces of continuous functions.

1. If $\mathbb{T}$ is compact then

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0_0(\mathbb{T};\mathbb{F}) = \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0(\mathbb{T};\mathbb{F}).$$

   The case of a compact time-domain is an important one.

2. If $\mathbb{T}$ is not compact then

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0_0(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0(\mathbb{T};\mathbb{F}).$$

   This is not difficult to see, but should be thought about to be comprehended thoroughly. The reader can engage in this sort of thought in Exercises 8.3.2 and 8.3.3. It is also worth making sure to understand that there is no useful relationship between spaces of continuous functions defined on a bounded but not compact time-domain $\mathbb{T}$ and on its closure $\mathrm{cl}(\mathbb{T})$ which is compact. This can be explored in Exercise 8.3.6. Note that the analogous behaviour is not seen for discrete-time signals since bounded discrete time-domains are finite. The reason for this behaviour for continuous-time signals is that an open end of a bounded interval is, in a topological sense, at infinity. The reader can get some insight into this in Exercise 8.3.5.

3. If $\mathbb{T}$ is closed and infinite then
$$\mathsf{C}^0_0(\mathbb{T};\mathbb{F}) = \left\{ f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \;\middle|\; \lim_{|t|\to\infty} f(t) = 0 \right\}.$$
4. The normed vector space $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is a Banach space if and only if $\mathbb{T}$ is compact. This is proved as Proposition 6.7.38.
5. The normed vector space $(\mathsf{C}^0_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is a separable $\mathbb{F}$-Banach space and is, moreover, the completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$. This is proved as Theorem 6.7.40.
6. The normed vector space $(\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is a $\mathbb{F}$-Banach space and is separable if and only if $\mathbb{T}$ is compact. These facts are proved in Theorem 6.7.39.

The reader may wish to think about the analogies presented in Table 8.1, which

Table 8.1 The relationships between the discrete-time signal spaces in the left column are analogous to the relationships between the continuous-time signal spaces in the right column; the discrete time-domain $\mathbb{T}_{\mathrm{d}}$ is infinite in order that the analogies hold

Discrete-time signal space	Continuous-time signal space
$\mathsf{c}_{\mathrm{fin}}(\mathbb{T}_{\mathrm{d}};\mathbb{F})$	$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T}_{\mathrm{c}};\mathbb{F})$
$\ell^\infty(\mathbb{T}_{\mathrm{d}};\mathbb{F})$	$\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T}_{\mathrm{c}};\mathbb{F})$
$\mathsf{c}_0(\mathbb{T}_{\mathrm{d}};\mathbb{F})$	$\mathsf{C}^0_0(\mathbb{T}_{\mathrm{c}};\mathbb{F})$

is essentially a reproduction of Table 6.1, in order to understand the relationships between the various discrete-time signal spaces and their continuous-time analogues.

8.3.3 The L^p-spaces

Next we turn to spaces of continuous-time signals characterised by their integrals. This has the desirable effect of allowing us to naturally consider signals that are possibly discontinuous. The price to be paid for this (absolutely necessary, from a practical point of view) generality is that it now becomes difficult to characterise these spaces of signals. This should not be surprising, really, as sets of possibly discontinuous signals can be expected to be pretty crazy. In this section we shall give a summary of how the L^p-spaces of Section 6.7.7 were constructed. The details are omitted here as the reader can refer back for these. As we go through our summary we will also present the main results for these spaces.

We let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a continuous time-domain. Since the constructions differ for L^∞ and L^p, $p \in [0, \infty)$, we present them separately, starting with $\mathsf{L}^\infty(\mathbb{T}; \mathbb{F})$. Recall that for a measurable function $\phi\colon \mathbb{T} \to \mathbb{F}$ we define

$$\operatorname{ess\,sup}\{\phi(t) \mid t \in \mathbb{T}\} = \inf\{M \in \mathbb{R}_{\geq 0} \mid \lambda(\{t \in \mathbb{T} \mid \phi(t) > M\} = 0)\}.$$

Then we define

$$\mathsf{L}^{(\infty)}(\mathbb{T}; \mathbb{F}) = \{f\colon \mathbb{T} \to \mathbb{F} \mid f \text{ is measurable and } \operatorname{ess\,sup}\{|f(t)| \mid t \in \mathbb{T}\} < \infty\}.$$

On $\mathsf{L}^{(\infty)}(\mathbb{T}; \mathbb{F})$ we define a seminorm $\|\cdot\|_\infty$ by

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(t)| \mid t \in \mathbb{T}\}.$$

In Proposition 6.7.45 we verify that $(\mathsf{L}^{(\infty)}(\mathbb{T}; \mathbb{F}), \|\cdot\|_\infty)$ is a seminormed $\mathbb{F}$-vector space. The signals of zero norm are precisely

$$Z^\infty(\mathbb{T}; \mathbb{F}) = \{f \in \mathsf{L}^{(\infty)}(\mathbb{T}; \mathbb{F}) \mid \lambda(\{t \in \mathbb{T} \mid f(t) \neq 0\}) = 0\}.$$

Thus signals in $Z^\infty(\mathbb{T}; \mathbb{F})$ are those that are zero almost everywhere. We then define

$$\mathsf{L}^\infty(\mathbb{T}; \mathbb{F}) = \mathsf{L}^{(\infty)}(\mathbb{T}; \mathbb{F})/Z^\infty(\mathbb{T}; \mathbb{F}).$$

Thus elements of $\mathsf{L}^\infty(\mathbb{T}; \mathbb{F})$ are to be thought of as equivalence classes of signals, where two signals f and g are declared to be equivalent if their difference $f - g$ is almost everywhere zero; that is f and g are equal almost everywhere. As we indicate in Notation 6.7.48, we shall make the convenient abuse of writing an equivalence class in $\mathsf{L}^\infty(\mathbb{T}; \mathbb{F})$ as f, with the understanding that in doing so we are really consider f and all signals that agree with it almost everywhere. In Theorem 6.7.47 we show that $(\mathsf{L}^\infty(\mathbb{T}; \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space, and in Proposition 6.7.49 that it is only separable in the (useless) case when $\mathbb{T} = \{a\}$ for some $a \in \mathbb{R}$.

Now let us turn to the construction of the spaces L^p for $p \in [1, \infty)$. We define

$$\mathsf{L}^{(p)}(\mathbb{T}; \mathbb{F}) = \left\{f\colon \mathbb{T} \to \mathbb{F} \;\middle|\; f \text{ is measurable and } \int_{\mathbb{T}} |f|^p \,\mathrm{d}\lambda < \infty\right\}.$$

The integral one must use for the results in this section to be valid is the Lebesgue integral, not the Riemann integral. Conceptually, at least for a finite duration, it is not too dangerous to sidestep this technical matter. However, we do recommend that the reader at some point put in the slight effort needed to understand the Lebesgue integral, and why it, and not the Riemann integral, is suited to our needs here.[6] In any case, the norm we use here is the p-norm:

$$\|f(t)\|_p = \Big(\int_{\mathbb{T}} |f|^p \,\mathrm{d}\lambda\Big)^{1/p}.$$

[6] We have mentioned this elsewhere, but the idea is so simple and important that we will repeat it again here. The big advantage of the Lebesgue integral over the Riemann integral is that there are limit theorems that hold for the former that do not hold for the latter. Most crucially, the Dominated Convergence Theorem for the two integrals have a completely different character. It is really this, and not other stuff that you may read about, that gives the Lebesgue integral its power.

In Proposition 6.7.54 we show, using the Minkowski inequality, that $(\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F});\|\cdot\|_p)$ is a seminormed $\mathbb{F}$-vector space for every $p \in [1,\infty)$. The signals in $\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ that have zero norm are

$$Z^p(I;\mathbb{F}) = \{f \in \mathsf{L}^{(p)}(I;\mathbb{F}) \mid \lambda(\{t \in \mathbb{T} \mid f(t) \neq 0\}) = 0\},$$

i.e., those signals that are almost everywhere zero. We then define

$$\mathsf{L}^p(\mathbb{T};\mathbb{F}) = \mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})/Z^p(\mathbb{T};\mathbb{F})$$

for each $p \in [1,\infty)$. As we described with $\mathsf{L}^\infty(\mathbb{T};\mathbb{F})$ above, we shall denote elements of $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ as is they were signals and not equivalence classes of signals. In Theorem 6.7.56 we show that $(\mathsf{L}^p(\mathbb{T};\mathbb{F}),\|\cdot\|_p)$ is a $\mathbb{F}$-Banach space, and is, moreover, the completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}),\|\cdot\|_p)$ for each $p \in [1,\infty)$. In Proposition 6.7.58 we show that $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ is separable.

8.3.3 Remark (Signals versus equivalence classes of signals) We shall very often be concerned with signals in $\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ rather than equivalence classes of signals in $\mathsf{L}^p(\mathbb{T};\mathbb{F})$. However, there will also be occasions when it is essential to think of equivalence classes of signals because we wish to utilise the Banach or Hilbert space structure of the spaces $\mathsf{L}^p(\mathbb{T};\mathbb{F})$. We shall generally try to be careful just which space, $\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ or $\mathsf{L}^p(\mathbb{T};\mathbb{F})$, we mean. There are, however, occasions when it is really not so important whether we are thinking about signals or equivalence classes of signals, e.g., in cases when we are concerned with the signal only inasmuch as we are concerned with its integral. Therefore, we may be a little careless with our notation at times. This should not cause any problems. Indeed, standard practice is simply to *not* distinguish between signals and equivalence classes of signals, and to simply use the notation $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ in all cases. However, in the interests of being sufficiently pedantic, we shall make this distinction in these volumes. •

In Table 8.2 we depict the interrelationships of various continuous-time signal

Table 8.2 The relationships between the discrete-time signal spaces in the left column are analogous to the relationships between the continuous-time signal spaces in the right column; the discrete time-domain $\mathbb{T}_{\mathrm{d}}$ is infinite in order that the analogies hold

Discrete-time signal space	Continuous-time signal space
$\mathsf{c}_{\mathrm{fin}}(\mathbb{T}_{\mathrm{d}};\mathbb{F})$	$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T}_{\mathrm{c}};\mathbb{F})$
$\ell^p(\mathbb{T}_{\mathrm{d}};\mathbb{F})$	$\mathsf{L}^p(\mathbb{T}_{\mathrm{c}};\mathbb{F})$

spaces to their discrete-time counterparts. We comment that $\mathsf{L}^\infty(\mathbb{T};\mathbb{F})$ does not appear in this table, essentially as a consequence of its not being the completion of any space of continuous continuous-time signals.

8.3.4 Periodic continuous-time signal spaces

Unlike the situation for discrete-time signals, there are nontrivial things one can say about spaces of periodic continuous-time signals.

We let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $T \in \mathbb{R}_{>0}$. We begin by defining

$$\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}, \mathbb{F}) = \{f \in \mathsf{C}^0(\mathbb{R}; \mathbb{F}) \mid f \text{ is } T\text{-periodic}\}.$$

The natural norm to use on this space is

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in [0, T]\}.$$

One can verify that $(\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}; \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

8.3.4 Proposition (($\mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R}; \mathbb{F}), \|\cdot\|_\infty$) is a Banach space) *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for $\mathrm{T} \in \mathbb{R}_{>0}$, $(\mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R}; \mathbb{F}), \|\cdot\|_\infty)$ is a separable Banach space.*

Proof Since $\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ is a subspace of $\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R}; \mathbb{F})$, if $(f_j)_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ this sequence converges to $f \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R}; \mathbb{C})$ by Theorem 6.7.31. We will show that $f \in \mathsf{C}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$. Let $t \in \mathbb{R}$. Then the sequences $(f_j(t))_{j\in\mathbb{Z}_{>0}}$ and $(f(t+T))_{j\in\mathbb{Z}_{>0}}$ are identical. Thus, since they converge to $f(t)$ and $f(t+T)$, respectively, we must have $f(t) = f(t+T)$ and so f is T-periodic. Separability of $\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ follows from Corollary 6.7.37 along with the fact that the map $f \mapsto f|[0, T]$ is an injective map from $\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ into a subspace of $\mathsf{C}^0([0, T]; \mathbb{F})$. ■

Now we turn to adaptations of the various L^p-spaces to periodic signals. Here one has to contend with the fact that the spaces of signals are really spaces of equivalence classes of signals. Let us first clarify how this equivalence relation interacts with periodicity. Recall that the equivalence relation is that two signals $f, g\colon \mathbb{R} \to \mathbb{F}$ are equivalent if $(f - g)(t) = 0$ for almost every $t \in \mathbb{R}$.

8.3.5 Lemma (Periodicity and equivalence classes of signals) *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, for $\mathrm{T} \in \mathbb{R}_{>0}$, and for a measurable signal $\mathrm{f}\colon \mathbb{R} \to \mathbb{F}$ the following statements are equivalent:*

(i) there exists a T-periodic measurable signal $\mathrm{g}\colon \mathbb{R} \to \mathbb{F}$ such that $(\mathrm{f} - \mathrm{g})(\mathrm{t}) = 0$ for almost every $\mathrm{t} \in \mathbb{R}$;

(ii) $\mathrm{f}(\mathrm{t} + \mathrm{T}) = \mathrm{f}(\mathrm{t})$ for almost every $\mathrm{t} \in \mathbb{R}$.

Proof (i) $\implies$ (ii) Let $j \in \mathbb{Z}_{>0}$ and let

$$Z_{j,1} = \{t \in [jT, (j+1)T) \mid f(t) \neq g(t)\},$$
$$Z_{j,2} = \{t \in [jT, (j+1)T) \mid f(t+T) \neq g(t+T)\}.$$

Both $Z_{j,1}$ and $Z_{j,2}$ have measure zero by hypothesis. If $t \in [jT, (j+1)T) \setminus (Z_{j,1} \cup Z_{j,2})$ then

$$f(t) = g(t) = g(t+T) = f(t+T).$$

Thus, taking $Z_j = Z_{j,1} \cup Z_{j,2}$ and $A_j = [jT, (j+1)T) \setminus Z_j$ we see that $f(t) = f(t+T)$ for every $t \in A_j$. Thus $f(t) = f(t+T)$ for every $t \in \cup_{j\in\mathbb{Z}} A_j$ and since $\mathbb{R} \setminus \cup_{j\in\mathbb{Z}} A_j = \cup_{j\in\mathbb{Z}} Z_j$ has measure zero, our assertion is established.

(ii) $\Longrightarrow$ (i) For $j \in \mathbb{Z}$ define

$$Z_j = \{t \in [0, T) \mid f(t + jT) \neq f(t)\}.$$

We claim that Z_j has measure zero. Let us verify this for $j \in \mathbb{Z}_{>0}$, the situation for $j \in \mathbb{Z}_{<0}$ being entirely analogous. For $j \in \mathbb{Z}_{>0}$ we prove our claim by induction on j. The claim is true by hypothesis for $j = 1$. Suppose it true for $j \in \{1, \dots, k\}$. Then define

$$N_k = \{t \in [0, T) \mid f(t + kT) \neq f(t + (k + 1)T)\},$$

noting that N_k has measure zero by hypothesis. If $t \in [0, T) \setminus (N_k \cup Z_k)$ then

$$f(t + (k + 1)T) = f(t + kT) = f(t).$$

Thus $Z_{k+1} \subseteq N_k \cup Z_k$ and so Z_{k+1} has measure zero.

Now define $h\colon [0, T) \to \mathbb{F}$ by

$$h(t) = \begin{cases} 0, & t \in \cup_{j \in \mathbb{Z}} Z_j, \\ f(t), & \text{otherwise.} \end{cases}$$

Note that $\cup_{j\in\mathbb{Z}} Z_j$ has zero measure. Therefore, by Proposition 5.6.10 it follows that h is measurable. Clearly $(f - h)(t) = 0$ for almost every $t \in [0, T)$. Now define $g\colon \mathbb{R} \to \mathbb{F}$ to be the T-periodic extension of h to give this part of the lemma. ■

We may now sensibly define what we mean by a periodic equivalence class of signals. Following what we did above in our construction of the L^p-spaces, we define

$$Z(\mathbb{R}; \mathbb{F}) = \{f\colon \mathbb{R} \to \mathbb{F} \mid f \text{ is measurable and } f(t) = 0 \text{ for almost every } t \in \mathbb{R}\}.$$

We then make the following definition.

8.3.6 Definition (Periodic equivalence classes of signals) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $T \in \mathbb{R}_{>0}$. For a measurable signal $f\colon \mathbb{R} \to \mathbb{F}$, the equivalence class $f + Z(\mathbb{R}; \mathbb{F})$ is ***T-periodic*** if there exists a T-periodic signal g such that

$$f + Z(\mathbb{R}; \mathbb{F}) = g + Z(\mathbb{R}; \mathbb{F}). \qquad \bullet$$

One way to read Lemma 8.3.5 is to say, "Equivalence classes of periodic signals are in 1–1 correspondence with periodic equivalence classes of signals."

With the above annoying technicalities out of the way, we can now proceed to define L^p-spaces of periodic signals. We let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $T \in \mathbb{R}_{>0}$. We first define

$$\mathsf{L}^{(\infty)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F}) = \{f \mid f \text{ measurable, } f\ T\text{-periodic, and } \operatorname{ess\,sup}\{|f(t)| \mid t \in [0, T)\} < \infty\}$$

and then define

$$\mathsf{L}^{\infty}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F}) = \mathsf{L}^{(\infty)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})/Z(\mathbb{R}; \mathbb{F}).$$

On $\mathsf{L}^{\infty}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we use the norm

$$\|f\| = \operatorname{ess\,sup}\{|f(t)| \mid t \in [0,T)\}.$$

For $p \in [1,\infty)$ we define

$$\mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) = \Big\{f \;\Big|\; f \text{ measurable, } f\ T\text{-periodic, and } \int_{[0,T)} |f(t)|^p \,\mathrm{d}\lambda < \infty\Big\}$$

and then define

$$\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) = \mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})/Z(\mathbb{R};\mathbb{F}).$$

On $\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we use the norm

$$\|f\|_p = \Big(\int_{[0,T)} |f(t)|^p \,\mathrm{d}\lambda\Big)^{1/p}.$$

By Lemma 8.3.5 it follows that, for each $p \in [1,\infty]$, the map

$$f + Z^p(\mathbb{R};\mathbb{F}) \mapsto f|[0,T) + Z^p([0,T);\mathbb{F})$$

is a norm-preserving isomorphism from $\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ to $\mathsf{L}^p([0,T);\mathbb{F})$. From this we conclude that $(\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{F}), \|\cdot\|_p)$ is a Banach space, and is separable if and only if $p \in [1,\infty)$.

8.3.5 Other characteristics of continuous-time signals

Let us provide for continuous-time signals the analogous definitions from Definition 8.2.4.

8.3.7 Definition (Signal characteristics) Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ and let $\mathbb{T}$ be a continuous time-domain and let $f\colon \mathbb{T} \to \mathbb{F}$ be an $\mathbb{F}$-valued signal on $\mathbb{T}$. Define $T_{\min} = \inf \mathbb{T}$ and $T_{\max} = \sup \mathbb{T}$. We allow either or both of $T_{\min}$ and $T_{\max}$ to be infinite in magnitude.

(i) If $f \in \mathsf{L}^{(1)}(\mathbb{T};\mathbb{F})$ then $\|f\|_1$ is the ***action*** of f.

(ii) If $f \in \mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$ then $\|f\|_2^2$ is the ***energy*** of f.

(iii) If $f \in \mathsf{L}^{(\infty)}(\mathbb{T};\mathbb{F})$ then $\|f\|_\infty$ is the ***amplitude*** of f.

(iv) If the limit

$$\lim_{\substack{T_- \to T_{\min}\\ T_+ \to T_{\max}}} \frac{1}{T_+ - T_-} \int_{T_-}^{T_+} |f(t)|^2 \,\mathrm{d}t$$

exists we denote it by $\mathrm{pow}(f)$ and call it the ***average power*** of f. The set of signals whose average power exists are called ***power signals*** and the set of these is denoted by $\mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F})$.

(v) If $f \in \mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F})$ then $\mathrm{rms}(f) = \sqrt{\mathrm{pow}(f)}$ is the ***root mean square value*** (***rms value***) of f.

(vi) The ***mean*** of f is given by

$$\mathrm{mean}(f) = \lim_{\substack{T_- \to T_{\min} \\ T_+ \to T_{\max}}} \frac{1}{T_+ - T_-} \int_{T_-}^{T_+} f(t)\,\mathrm{d}t,$$

if the limit is defined. •

8.3.8 Remark (The importance of p ∈ {1, 2, ∞}) As we remarked on in Remark 8.2.5, the cases of $\mathsf{L}^{(p)}$-spaces for $p \in \{1, 2, \infty\}$ are distinguished in applications. For the continuous time versions of these results, this will be borne out in Sections 12.3 and 13.3 in a general way. •

Let us give a couple of examples illustrating the above ideas.

8.3.9 Examples (Signal characteristics)

1. Let us consider the unshifted square wave $\square_{a,\nu,0}$ of amplitude a and frequency ν defined on $\mathbb{T} = \mathbb{R}$. It is straightforward to see that $\|\square_{a,\nu,0}\|_\infty = a$ and that $\square_{a,\nu,0}$ has undefined action and energy. To compute the average power we note that for sufficiently large T we have

$$\frac{a^2\lfloor T\rfloor}{2T} \le \frac{1}{2T}\int_{-T}^{T} |\square_{a,\nu,0}(t)|^2\,\mathrm{d}t \le \frac{a^2\lceil T\rceil}{2T},$$

by direct calculation. Therefore, taking the limit as $T \to \infty$, we get $\mathrm{pow}(\square_{a,\nu,0}) = \frac{1}{2}a^2$. This therefore immediately gives $\mathrm{rms}(\square_{a,\nu,0}) = \frac{1}{\sqrt{2}}a$. For the mean of the signal we have, for sufficiently large T,

$$\frac{a\lfloor T\rfloor}{2T} \le \frac{1}{2T}\int_{-T}^{T} \square_{a,\nu,0}(t)\,\mathrm{d}t \le \frac{a\lceil T\rceil}{2T},$$

therefore giving $\mathrm{mean}(\square_{a,\nu,0}) = \frac{1}{2}a$.

2. Next let us consider the unshifted sawtooth $\triangle_{a,\nu,0}$ of amplitude a and frequency ν defined on $\mathbb{T} = \mathbb{R}$. Here we compute

$$\frac{a^2\lfloor T\rfloor}{3T} \le \frac{1}{2T}\int_{-T}^{T} |\triangle_{a,\nu,0}(t)|^2\,\mathrm{d}t \le \frac{a^2\lceil T\rceil}{3T},$$

giving $\mathrm{pow}(\triangle_{a,\nu,0}) = \frac{1}{3}a^2$ and $\mathrm{rms}(\triangle_{a,\nu,0}) = \frac{1}{\sqrt{3}}a$. For the mean we compute

$$\frac{a\lfloor T\rfloor}{2T} \le \frac{1}{2T}\int_{-T}^{T} \triangle_{a,\nu,0}(t)\,\mathrm{d}t \le \frac{a^2\lceil T\rceil}{2T},$$

giving $\mathrm{mean}(\triangle_{a,\nu,0}) = \frac{1}{2}a$. •

The set of power signals is often not given much discussion. However, this class of signals is a little unusual if one digs into its mathematical structure. For example, we have the following result.

8.3.10 Proposition (The set of power signals is not a vector space) *If* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and if* $\mathbb{T}$ *is an infinite continuous time-domain then* $\mathsf{L}^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$ *is not a subspace of* $\mathbb{F}^{\mathbb{T}}$*.*

Proof We shall provide a counterexample to the subspace structure in the case where $\mathbb{T} = [0, \infty)$, and the case of a general infinite time-domain follows by simple manipulations of this example.

Define signals

$$f_1(t) = \begin{cases} 1, & t \in [n, n+1),\ n \text{ even and positive,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_2(t) = \begin{cases} 1, & t \in [2j+1, 2j+2) \subseteq [n!, (n+1)!),\ n \geq 2 \text{ and even, and } j \in \mathbb{Z}, \\ -1, & t \in [2j, 2j+1) \subseteq [n!, (n+1)!),\ n \geq 2 \text{ and odd, and } j \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Computations give

$$\begin{aligned} \frac{\lfloor T \rfloor}{2T} &\leq \frac{1}{T} \int_0^T |f_1(t)|^2 \, \mathrm{d}t \leq \frac{\lceil T \rceil + 1}{2T}, \\ \frac{\lfloor T \rfloor - 3}{2T} &\leq \frac{1}{T} \int_0^T |f_2(t)|^2 \, \mathrm{d}t \leq \frac{\lceil T \rceil + 1}{2T}, \end{aligned}$$

from which we ascertain that $\mathrm{pow}(f_1) = \mathrm{pow}(f_2) = \frac{1}{2}$. In particular, $f_1, f_2 \in \mathsf{L}^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$. However, we claim that $f_1 + f_2 \notin \mathsf{L}^{\mathrm{pow}}(\mathbb{T}; \mathbb{F})$. Indeed, if n is a large even integer we may compute

$$\begin{aligned} \frac{1}{n!} \int_0^{n!} |f_1(t) + f_2(t)|^2 \, \mathrm{d}t &= \frac{(n-1)! - (n-2)! + \cdots + 1}{n!} \\ &= \frac{(n-1)! + (n-3)!(1 - n - 2) + \cdots + 1(1-2)}{n!} \\ &\leq \frac{(n-1)!}{n!}, \end{aligned}$$

and if n is a large odd integer we compute

$$\begin{aligned} \frac{1}{n!} \int_0^{n!} |f_1(t) + f_2(t)|^2 \, \mathrm{d}t &= \frac{n! - (n-1)! + \cdots + 1}{n!} \\ &= \frac{(n-1)!(n-1) + 2!(3-1) + 1}{n!} \\ &\geq \frac{(n-1)!(n-1)}{n!}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \lim_{\substack{n \to \infty \\ n \text{ even}}} \frac{1}{n!} \int_0^{n!} |f_1(t) + f_2(t)|^2 \, \mathrm{d}t = 0 \\ \lim_{\substack{n \to \infty \\ n \text{ odd}}} \frac{1}{n!} \int_0^{n!} |f_1(t) + f_2(t)|^2 \, \mathrm{d}t = 1, \end{aligned}$$

implying that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f_1(t) + f_2(t)|^2\,\mathrm{d}t$$

does not exist, as desired. ■

As we shall see in Proposition 8.3.12, if $\mathbb{T}$ is finite then $\mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F})$ is an $\mathbb{F}$-vector space.

8.3.6 Inclusions of continuous-time signal spaces

In this section we explore the relationships between the various continuous-time signal spaces. As we shall see, the relationships are more or less simple to understand for finite time-domains, although not as simple as for bounded discrete time-domains. For infinite continuous time-domains, the story is complicated, perhaps surprisingly so.

First of all, recall from Section 8.3.2 that if $\mathbb{T}$ is compact then

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0_0(\mathbb{T};\mathbb{F}) = \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0(\mathbb{T};\mathbb{F})$$

and if $\mathbb{T}$ is not compact then

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0_0(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) \subset \mathsf{C}^0(\mathbb{T};\mathbb{F}).$$

These inclusions are easy to understand so we do not dwell on them. Instead we focus on the relationships between the L^p-spaces.

8.3.11 Theorem (Inclusions between continuous-time signal spaces) *Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ and let $\mathbb{T} \subseteq \mathbb{R}$ be a continuous time-domain. The following statements hold:*

(i) $\mathsf{L}^{(1)}(\mathbb{T};\mathbb{F}) \cap \mathsf{L}^{(\infty)}(\mathbb{T};\mathbb{F}) \subseteq \mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$;

(ii) $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ *is dense in* $\mathsf{L}^{\mathrm{p}}(\mathbb{T};\mathbb{F})$ *for all* $\mathrm{p} \in [1,\infty)$;

(iii) if $\mathbb{T}$ is bounded then $\mathsf{L}^{(\infty)}(\mathbb{T};\mathbb{F}) \subseteq \mathsf{L}^{(\mathrm{p})}(\mathbb{T};\mathbb{F})$ *for any* $\mathrm{p} \in [1,\infty)$;

(iv) if $\mathbb{T}$ is bounded then $\mathsf{L}^{(\mathrm{p})}(\mathbb{T};\mathbb{F}) \subseteq \mathsf{L}^{(\mathrm{q})}(\mathbb{T};\mathbb{F})$ *for* $\mathrm{q} < \mathrm{p} \in [1,\infty)$.

Moreover, the inclusions in parts (iii) and (iv) are continuous.

Proof (i) We have

$$\begin{aligned}\|f\|_2^2 = \int_{\mathbb{T}} |f(t)|^2\,\mathrm{d}t &= \int_{\mathbb{T}} |f(t)||f(t)|\,\mathrm{d}t\\ &\leq \|f\|_\infty \int_{\mathbb{T}} |f(t)|\,\mathrm{d}t = \|f\|_\infty \|f\|_1,\end{aligned}$$

as desired.

(ii) This was proved as Theorem 6.7.56.

(iii) The inclusion is simple:

$$\|f\|_p^p = \int_{\mathbb{T}} |f(t)|^p\,\mathrm{d}t \leq \|f\|_\infty^p \int_{\mathbb{T}} \mathrm{d}t < \infty.$$

To show that the inclusion of $\ell^\infty(\mathbb{T};\mathbb{F})$ is continuous, let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\ell^\infty(\mathbb{T};\mathbb{F})$ converging to f. Then, for $\epsilon\in\mathbb{R}_{>0}$, there exists $N\in\mathbb{Z}_{>0}$ such that

$$\operatorname{ess\,sup}\{|f(t)-f_j(t)|^p \mid t\in\mathbb{T}\} < \tfrac{\epsilon^p}{\lambda(\mathbb{T})}$$

for $j\geq N$. Then

$$\int_{\mathbb{T}} |f(t)-f_j(t)|^p\,\mathrm{d}t \leq \epsilon^p,$$

and so $\|f-f_j\|_p<\epsilon$ for every $j\geq N$. Thus the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ and so the inclusion of $\mathsf{L}^\infty(\mathbb{T};\mathbb{F})$ in $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ is continuous by Theorem 6.5.2.

(iv) Let $q<p$ and let $f\in\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$. Let

$$A=\{t\in\mathbb{T} \mid |f(t)|\geq 1\}.$$

We then have

$$\begin{aligned}\int_{\mathbb{T}} |f(t)|^q\,\mathrm{d}t &= \int_{\mathbb{T}\setminus A} |f(t)|^q\,\mathrm{d}t + \int_A |f(t)|^q\,\mathrm{d}t\\ &\leq \int_{\mathbb{T}\setminus A} |f(t)|^q\,\mathrm{d}t + \int_A |f(t)|^p\,\mathrm{d}t\\ &\leq \int_{\mathbb{T}}\mathrm{d}t + \int_{\mathbb{T}} |f(t)|^p\,\mathrm{d}t = \|f\|_p+\lambda(\mathbb{T})<\infty,\end{aligned}$$

so giving $f\in\mathsf{L}^{(q)}(\mathbb{T};\mathbb{F})$. To show that the inclusion of $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ in $\mathsf{L}^q(\mathbb{T};\mathbb{F})$ is continuous for $p>q$, define $r=\frac{p}{q}$ so that $1<r\leq p$. Note that if $f\in\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ then

$$(|f(t)|^q)^r=|f(t)|^p,$$

implying that $|f|^q\in\mathsf{L}^{(r)}(\mathbb{T};\mathbb{F})$. Define $g(t)=1$ for $t\in\mathbb{T}$. Using Hölder's inequality, Lemma 6.7.51, for $|f|^p\in\mathsf{L}^{(r)}(\mathbb{T};\mathbb{F})$ and for $g\in\mathsf{L}^{(r')}(\mathbb{T};\mathbb{F})$ we have

$$\begin{aligned}\int_{\mathbb{T}} |f(t)|^q\,\mathrm{d}t = \int_{\mathbb{T}} |f(t)|^q|g(t)|\,\mathrm{d}t &\leq \Big(\int_{\mathbb{T}}(|f(t)|^q)^r\Big)^{1/r}\Big(\int_{\mathbb{T}}1\,\mathrm{d}t\Big)^{1/r'}\\ &\leq \Big(\int_{\mathbb{T}}|f(t)|^p\Big)^{q/p}\lambda(\mathbb{T})^{(p-q)/p}.\end{aligned}$$

Thus, for $f\in\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$, we have

$$\|f\|_q\leq\|f\|_p\lambda(\mathbb{T})^{\frac{1}{q}-\frac{1}{p}}.$$

Now let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ converging to f. Let $\epsilon\in\mathbb{R}_{>0}$ and let $N\in\mathbb{Z}_{>0}$ be such that

$$\|f-f_j\|_p\leq\frac{\epsilon}{\lambda(\mathbb{T})^{\frac{1}{q}-\frac{1}{p}}}$$

for $j\geq N$. Then $\|f-f_j\|_q\leq\epsilon$ for $j\geq N$ and so the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^q(\mathbb{T};\mathbb{F})$. Continuity of the inclusion now follows from Theorem 6.5.2. ■

For power signals, we have the following correspondences.

8.3.12 Proposition (Inclusions involving continuous-time power signals) *Let $\mathbb{T}$ be a continuous time-domain. The following statements hold:*

(i) if $\mathbb{T}$ is finite then $\mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F}) = \mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$;

(ii) if $\mathbb{T}$ is infinite and $f \in \mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$ then $\mathrm{pow}(f) = 0$;

(iii) if $f \in \mathsf{L}^{(\infty)}(\mathbb{T};\mathbb{F})$ is a power signal then $\mathrm{pow}(f) \le \|f\|_\infty^2$;

(iv) if $\mathbb{T}$ is infinite then $\mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F}) \subseteq \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$.

Proof (i) This follows immediately from the definitions.

(ii) We shall show that for $\mathbb{T} = [0,\infty)$, and the other cases can be deduced from this easily. For $T > 0$ we have

$$\int_0^T |f(t)|^2\,\mathrm{d}t \le \|f\|_2^2 \quad \Longrightarrow \quad \frac{1}{T}\int_0^T |f(t)|^2\,\mathrm{d}t \le \frac{1}{T}\|f\|_2^2.$$

The result follows since as $T \to \infty$, the right-hand side goes to zero.

(iii) In the case where $\mathbb{T}$ is finite with length L we have

$$\mathrm{pow}(f) = \frac{1}{L}\int_{\mathbb{T}} |f(t)|^2\,\mathrm{d}t \le \frac{1}{L}\|f\|_\infty^2 \int_{\mathbb{T}} \mathrm{d}t = \|f\|_\infty^2.$$

For the infinite case we consider $\mathbb{T} = [0,\infty)$, noting that other infinite time-domain cases follow from this. We compute

$$\mathrm{pow}(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |f(t)|^2\,\mathrm{d}t \le \|f\|_\infty^2 \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t = \|f\|_\infty^2,$$

as desired.

(iv) Let $[a,b]$ be a compact subinterval of $\mathbb{T}$. Provided that $T > 0$ is sufficiently large that $[a,b] \subseteq [-T,T]$ we have

$$\int_a^b |f(t)|^2\,\mathrm{d}t \le \int_{-T}^T |f(t)|^2\,\mathrm{d}t.$$

In the limit as $T \to \infty$ the quantity

$$\frac{1}{T}\int_{-T}^T |f(t)|^2\,\mathrm{d}t$$

is finite. Thus there exists a sufficiently large T_0 so that the preceding quantity is finite when $T \ge T_0$. From this it follows that

$$T_0 \int_a^b |f(t)|^2\,\mathrm{d}t$$

is finite, showing that $f|[a,b] \in \mathsf{L}^{(2)}([a,b];\mathbb{F})$. The result now follows from part (iv) of Theorem 8.3.11. ■

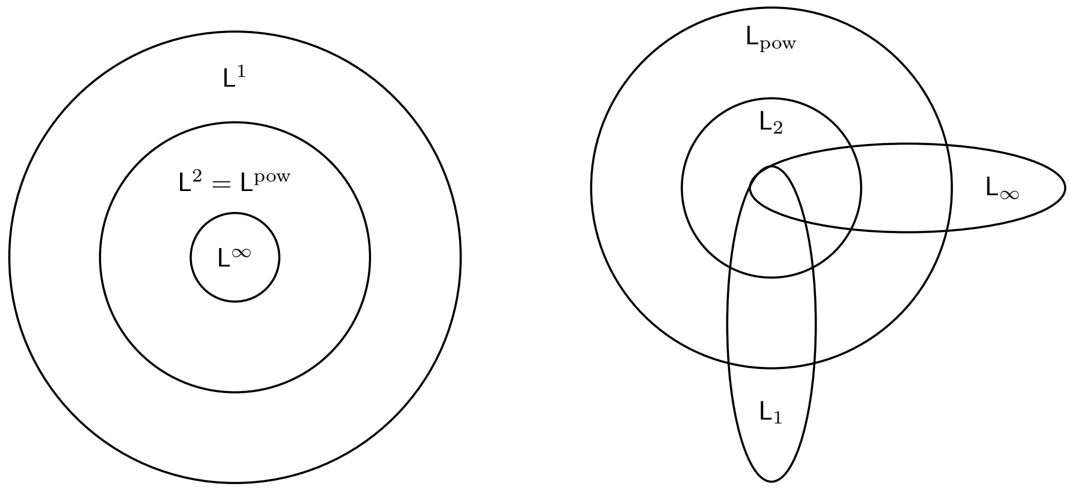


Figure 8.21 Venn diagrams illustrating inclusions of signal spaces for continuous time-domains: the finite case (left) and the infinite case (right)

8.3.13 Remark Suppose that $\mathbb{T}$ is infinite. Since there exists nonzero signals in $\mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$ it follows that there are nonzero signals f for which $\mathrm{pow}(f)$ is not zero. Thus $\mathrm{pow}(\cdot)$ is not a norm (even if the $\mathsf{L}^{\mathrm{pow}}(\mathbb{T};\mathbb{F})$ were a vector space, which it is not for infinite intervals, by Proposition 8.3.10). •

The Venn diagrams of Figure 8.21 show the relationships between the common types of signals for both finite and infinite continuous time-domains. For finite time-domains, the inclusions are straightforward, or follow from result proved above. For infinite time-domains, the following examples complete the Venn diagram characterisation, when combined with the results already established above.

8.3.14 Examples (Continuous-time signal space inclusions)

1. The signal $f_1(t) = \cos t$ is in $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$, but is in none of the spaces $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ for $1 \le p < \infty$.
2. The signal $f_2(t) = 1(t)\frac{1}{1+t}$ is not in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$, although it is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$; one computes $\|f\|_2 = 1$.
3. The signal $f_3(t) = 1$ is in $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{\mathrm{pow}}(\mathbb{R};\mathbb{F})$ but not in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$ or $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$.
4. The signal
$$f_4(t) = \begin{cases} \sqrt{\frac{1}{t}}, & t \in (0,1], \\ 0, & \text{otherwise} \end{cases}$$
is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ but not in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ for $p \in \{2, \infty, \mathrm{pow}\}$.
5. The signal
$$f_5(t) = \begin{cases} \log t, & t \in (0,1], \\ 0, & \text{otherwise} \end{cases}$$
is in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ for $p \in [1,\infty)$, but is not in $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$.

6. The signal

$$f_6(t) = \begin{cases} \frac{1}{1+t} + \log t, & t \in (0,1], \\ \frac{1}{1+t}, & t > 1, \\ 0, & \text{otherwise} \end{cases}$$

is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$ but in neither $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ nor $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$.

7. The signal

$$f_7(t) = \begin{cases} 1 + \log t, & t \in (0,1], \\ 1, & t > 1, \\ 0, & \text{otherwise} \end{cases}$$

is in $\mathsf{L}^{\mathrm{pow}}(\mathbb{R};\mathbb{F})$ but not in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ for $p \in [1,\infty]$.

8. For $j \in \mathbb{Z}_{>0}$ define

$$g_j = \begin{cases} j, & t \in (j, j + j^{-3}), \\ 0, & \text{otherwise.} \end{cases}$$

Then one checks that the signal

$$f_8(t) = \sum_{j=1}^{\infty} g_j(t)$$

is in $\mathsf{L}^{\mathrm{pow}}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ but in neither $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$ nor $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$.

9. For $j \in \mathbb{Z}_{>0}$ define

$$g_j = \begin{cases} 1, & t \in (2^{2j}, 2^{2j+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Then one checks that the signal

$$f_9(t) = \sum_{j=1}^{\infty} g_j(t)$$

is in $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ but not in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ for $p \in [1,\infty) \cup \{\mathrm{pow}\}$. •

8.3.7 Some other useful classes of continuous-time signals

In this section we provide some notation for certain spaces of signals that we will make use of in this and later volumes. We will not discuss in detail any of the detailed properties of these spaces of signals. Some are discussed elsewhere in these volumes in detail and others we will simply not bother to investigate because it is either unnecessary or uninteresting.

1. It is convenient to have some notation for the differentiable counterparts of the continuous signals considered in Section 8.3.2. Thus, for a continuous

time-domain and for $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we denote

$$\begin{aligned}
\mathsf{C}^r(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid f \text{ is } r \text{ times continuously differentiable}\};\\
\mathsf{C}^r_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^r(\mathbb{T};\mathbb{F}) \mid f \text{ has compact support}\};\\
\mathsf{C}^r_0(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^r(f;\mathbb{F}) \mid \text{ for every } \epsilon \in \mathbb{R}_{>0} \text{ there exists a compact set}\\
&\qquad K \subseteq I \text{ such that } \{t \in \mathbb{T} \mid |f(t)| \geq \epsilon\} \subseteq K\};\\
\mathsf{C}^r_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathsf{C}^r(\mathbb{T};\mathbb{F}) \mid \text{ there exists } M \in \mathbb{R}_{>0} \text{ such that } |f(t)| \leq M\\
&\qquad \text{for all } t \in \mathbb{T}\},\\
\mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) &= \{f \in \mathsf{C}^r(\mathbb{R};\mathbb{F}) \mid f \text{ is } T\text{-periodic}\}.
\end{aligned}$$

Note that none of these spaces when equipped with the ∞-norm are Banach spaces, cf. Example 6.6.25–**??**

2. There are also periodic analogues of the above classes of differentiable signal spaces. Thus, for $T \in \mathbb{R}_{>0}$ and for $r \in \mathbb{Z}_{>0}$, we define

$$\mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) = \{f \in \mathsf{C}^0(\mathbb{R};\mathbb{F}) \mid f \text{ is } T\text{-periodic}\}.$$

3. In Sections 3.1.7 and 3.2.7***missing stuff*** we encountered the notion of piecewise continuous and piecewise differentiable signals on compact continuous time-domains. We can define these notions on a more general continuous time-domain $\mathbb{T}$ by saying that $f \in \mathbb{F}^{\mathbb{T}}$ is piecewise continuous if, for each compact time-domain $\mathbb{S} \subseteq \mathbb{T}$, $f|\mathbb{S}$ is piecewise continuous. We denote the set of piecewise continuous signals on $\mathbb{T}$ by $\mathsf{C}^0_{\mathrm{pw}}(\mathbb{T};\mathbb{F})$. In like manner we define the set of piecewise differentiable signals on $\mathbb{T}$, and denote this by $\mathsf{C}^1_{\mathrm{pw}}(\mathbb{T};\mathbb{F})$.
The space of piecewise continuous or piecewise differentiable signals is not very useful. That is to say, we will use these spaces when we are asking that a given signal have the properties of piecewise continuity or differentiability.

4. It is sometimes useful to allow for signals on infinite time-domains that are not in L^p, but which are in L^p on every compact interval. Thus, for a time-domain $\mathbb{T}$ and for $p \in [1,\infty]$ we denote by $\mathsf{L}^{(p)}_{\mathrm{loc}}(\mathbb{T};\mathbb{F})$ the collection of signals $f \in \mathbb{F}^{\mathbb{T}}$ for which $f|\mathbb{S} \in \mathsf{L}^{(p)}(\mathbb{S};\mathbb{F})$ for every compact continuous time-domain $\mathbb{S} \subseteq \mathbb{T}$. Following Definition 5.9.19 we call these signals ***locally* $\mathsf{L}^{\mathbf{p}}$**. Most often we are interested in the case when $p = 1$, and signals in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{T};\mathbb{F})$ are called ***locally integrable***, as we have previously seen.
As we have seen in Section 5.9.6, $\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{T}}$.

5. We denote by $Z(\mathbb{T};\mathbb{F})$ the subspace of $\mathbb{F}^{\mathbb{T}}$ consisting of signals that are almost everywhere zero. We then define

$$\mathsf{L}^p_{\mathrm{loc}}(\mathbb{T};\mathbb{F}) = \mathsf{L}^{(p)}(\mathbb{S};\mathbb{F})/Z(\mathbb{T};\mathbb{F}).$$

We shall sometimes call equivalence classes of signals in $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{T};\mathbb{F})$ ***locally* $\mathsf{L}^{\mathbf{p}}$**, accepting a slight abuse of notation.

6. We denote by $\mathsf{BV}(\mathbb{T};\mathbb{F})$ the set of $\mathbb{F}$-valued signals on $\mathbb{T}$ that have bounded variation. For infinite time-domains, we denote by $\mathrm{TV}(\mathbb{T};\mathbb{F})$ the set of $\mathbb{F}$-valued signals on I that have finite variation. These signals are discussed in Sections 3.3 and **??**.
7. If $\mathbb{T}$ is compact, we denote by $\mathsf{AC}(\mathbb{T};\mathbb{F})$ the signals on $\mathbb{T}$ that are absolutely continuous. For a noncompact time-domain $\mathbb{T}$ the locally absolutely continuous signals on $\mathbb{T}$ are denoted by $\mathsf{AC}_{\mathrm{loc}}(\mathbb{T};\mathbb{F})$. These signals are discussed in Sections 5.9.6 and **??**.

8.3.8 Notes

Proposition 8.3.10 is proved by **JM:96**.
Many of the signals from Example 8.3.14 are given by **JCD/BAF/ART:90**.

Exercises

8.3.1 For the $\mathbb{F}$-vector space $\mathsf{C}^0([0,1];\mathbb{F})$ of continuous $\mathbb{F}$-valued functions on $[0,1]$, consider the vectors defined by the functions $f_j\colon t \mapsto t^j$, $j \in \mathbb{Z}_{\geq 0}$. Show that the set $\{f_j \mid j \in \mathbb{Z}_{\geq 0}\}$ is linearly independent.

8.3.2 By means of examples, show that the inclusions

$$\mathsf{C}^0_{\mathrm{cpt}}((0,1];\mathbb{R}) \subset \mathsf{C}^0_0((0,1];\mathbb{R}) \subset \mathsf{C}^0_{\mathrm{bdd}}((0,1];\mathbb{R}) \subset \mathsf{C}^0((0,1];\mathbb{R})$$

are strict.

8.3.3 By means of examples, show that the inclusions

$$\mathsf{C}^0_{\mathrm{cpt}}([0,\infty);\mathbb{R}) \subset \mathsf{C}^0_0([0,\infty);\mathbb{R}) \subset \mathsf{C}^0_{\mathrm{bdd}}([0,\infty);\mathbb{R}) \subset \mathsf{C}^0([0,\infty);\mathbb{R})$$

are strict.

8.3.4 For each of the following five signals $f\colon (0,1] \to \mathbb{R}$, answer the following questions with concise explanations:

1. is $f \in \mathsf{C}^0_{\mathrm{cpt}}((0,1];\mathbb{R})$?
2. is $f \in \mathsf{C}^0_0((0,1];\mathbb{R})$?
3. is $f \in \mathsf{C}^0_{\mathrm{bdd}}((0,1];\mathbb{R})$?
4. is $f \in \mathsf{C}^0((0,1];\mathbb{R})$?
5. is $f \in \mathsf{L}^2((0,1];\mathbb{R})$?
6. *if possible*, find a sequence in $\mathsf{C}^0_{\mathrm{cpt}}((0,1];\mathbb{R})$ converging to f in $(\mathsf{C}^0_0((0,1];\mathbb{R}), \|\cdot\|_\infty)$;
7. *if possible*, find a sequence in $\mathsf{C}^0_{\mathrm{cpt}}((0,1];\mathbb{R})$ converging to f in $(\mathsf{L}^2((0,1];\mathbb{R}), \|\cdot\|_2)$.

Here are the functions:

(a) $f(t) = \begin{cases} 0, & t \in (0,\frac{1}{2}], \\ t - \frac{1}{2}, & t \in (\frac{1}{2},1]; \end{cases}$

(b) $f(t) = t^{-1/4}$;

(c) $f(t) = \begin{cases} 0, & t \in (0, \frac{1}{2}], \\ 1, & t \in (\frac{1}{2}, 1]; \end{cases}$

(d) $f(t) = t^{1/2}$;

(e) $f(t) = 1 + t$.

8.3.5 Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Show that for each pair of normed vector spaces below, there exists between them a norm-preserving isomorphism:

(a) $(\mathsf{C}^0_{\mathrm{bdd}}([0,1);\mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{C}^0_{\mathrm{bdd}}([0,\infty), \|\cdot\|_\infty)$;

(b) $(\mathsf{C}^0_{\mathrm{bdd}}((-1,0];\mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{C}^0_{\mathrm{bdd}}((-\infty,0], \|\cdot\|_\infty)$;

(c) $(\mathsf{C}^0_{\mathrm{bdd}}((0,1);\mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R}, \|\cdot\|_\infty)$.

8.3.6 Find a signal $f \in \mathsf{C}^0_{\mathrm{bdd}}((0,1];\mathbb{R})$ for which there does not exist a signal $\hat{f} \in \mathsf{C}^0([0,1];\mathbb{R})$ such that $f = \hat{f}|(0,1]$.

The reader should compare the conclusions of the following exercise with those of Exercise 8.3.6.

8.3.7 Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a continuous time-domain. For $f \in \mathbb{F}^{\mathbb{T}}$ define $\hat{f} \in \mathbb{F}^{\mathrm{cl}(\mathbb{T})}$ by

$$\hat{f}(t) = \begin{cases} f(t), & t \in \mathbb{T}, \\ 0, & t \in \mathrm{cl}(\mathbb{T}) \setminus \mathbb{T}. \end{cases}$$

For $p \in [1,\infty]$ show that the map $f \mapsto \hat{f}$ is an isomorphism of the normed vector spaces $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ and $\mathsf{L}^p(\mathrm{cl}(\mathbb{T});\mathbb{F})$.

8.3.8 For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, for a continuous time-domain $\mathbb{T}$, and for $p, q \in [1,\infty)$ with $p < q$, show that $\mathsf{L}^{(q)}_{\mathrm{loc}}(\mathbb{T};\mathbb{F}) \subseteq \mathsf{L}^{(p)}_{\mathrm{loc}}(\mathbb{T};\mathbb{F})$.

The matter of determining when a signal is in one of the L^p-spaces can be a little problematic. Certainly one does not want to rely on being able to explicitly compute the p-norm; counting on one's ability to compute integrals is an activity doomed to failure. In the following exercise you will provide some conditions that, while simple, are often enough to ascertain when a given signal is in L^p. We concentrate on the cases of $\mathbb{T} = \mathbb{R}$ since a signal on any time-domain $\mathbb{T}$ can be extended to $\mathbb{R}$ by asking that it be zero on $\mathbb{R} \setminus \mathbb{T}$.

8.3.9 Answer the following questions.

(a) Prove the following result.

Proposition *If* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, *if* $\mathrm{f} \in \mathbb{F}^{\mathbb{R}}$ *is measurable and satisfies*

(i) $|\mathrm{f}|^{\mathrm{p}} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *and*

(ii) $\lim_{|\mathrm{t}|\to\infty} \frac{|\mathrm{f(t)}|}{|\mathrm{t}|^{\mathrm{a}}} = 0$ *for some* $\mathrm{a} < -\frac{1}{\mathrm{p}}$

then $\mathrm{f} \in \mathsf{L}^{(\mathrm{p})}(\mathbb{R};\mathbb{F})$.

(b) Is the assumption that $|f|^p \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ necessary for f to be in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$?

(c) Is the assumption that $\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^a} = 0$ for some $a < -\frac{1}{p}$ necessary for f to be in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$?

8.3.10 Show that, for any continuous time-domain $\mathbb{T}$, if $f, g \in \mathsf{L}^{(2)}(\mathbb{T};\mathbb{F})$ then $fg \in \mathsf{L}^{(1)}(\mathbb{T};\mathbb{F}$ and $\|fg\|_1 \leq \|f\|_2\|g\|_2$.

8.3.11 Let $f\colon \mathbb{R} \to \mathbb{F}$ be a continuous-time T-periodic signal with the property that

$$\int_0^T |f(t)|^2 \, \mathrm{d}t < \infty.$$

Show that f is a power signal and that

$$\mathrm{pow}(f) = \frac{1}{T}\int_0^T |f(t)|^2 \, \mathrm{d}t.$$

8.3.12 For the following continuous-time signals defined on $\mathbb{T} = [0,\infty)$, compute their action, energy, amplitude, average power, rms value, and mean:

(a) $f(t) = \sin(2\pi t)$;

(b) $f(t) = \sin(2\pi t) + 1$;

(c) $f(t) = \frac{1}{1+t}$.

8.3.13 For the following continuous-time signals defined on $\mathbb{T} = [0,1]$, compute their action, energy, amplitude, average power, rms value, and mean:

(a) $f(t) = \sin(2\pi t)$;

(b) $f(t) = \sin(2\pi t) + 1$;

(c) $f(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{\sqrt{t}}, & t \in (0,1]. \end{cases}$

8.3.14 Let f be a continuous-time power signal defined on $\mathbb{T} = \mathbb{R}$. Show that the signal f_λ defined by $f_\lambda(t) = f(\lambda t)$ is a power signal and that $\mathrm{pow}(f_\lambda) = \mathrm{pow}(f)$.

8.3.15 Let $\mathbb{T} = [0,\infty)$.

(a) Find a continuous signal $f\colon \mathbb{T} \to \mathbb{R}$ so that $f \in \mathsf{L}^{(1)}(\mathbb{T};\mathbb{R})$ and $f \notin \mathsf{L}^{(2)}(\mathbb{T};\mathbb{R})$.

(b) Find a continuous signal $f\colon \mathbb{T} \to \mathbb{R}$ so that $f \in \mathsf{L}^{(2)}(\mathbb{T};\mathbb{R})$ and $f \notin \mathsf{L}^{(1)}(\mathbb{T};\mathbb{R})$.

8.3.16 Show that $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{R}) \cap \mathsf{C}^0_0(\mathbb{R};\mathbb{R}) \subseteq \mathsf{L}^{(2)}(\mathbb{R};\mathbb{R})$.

8.3.17 Let $p \in [1,\infty)$. Is it true that

$$\mathsf{L}^{(p)}([0,\infty);\mathbb{F}) \cap \mathsf{C}^0([0,\infty);\mathbb{F}) \subseteq \mathsf{C}^0_0([0,\infty);\mathbb{F})?$$

If this is true, prove it. If it is not true, demonstrate this with a counterexample.

8.3.18 Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, let $p \in [1,\infty)$, let $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, let $C \in \mathbb{F}$, and let $g_{C,f}\colon \mathbb{R} \to \mathbb{F}$ be defined by

$$g_{C,f}(t) = C + \int_0^t f(\tau) \, \mathrm{d}\tau.$$

Do the following.

(a) Show that the limits $\lim_{t\to-\infty} g_{C,f}(t)$ and $\lim_{t\to\infty} g_{C,f}(t)$ exist.
(b) If $g_{C,f} \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, show that $\lim_{|t|\to\infty} g_{C,f}(t) = 0$.

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Chapter 9

Signals in the frequency-domain

This chapter provides the reader with some motivation for considering frequency-domain representations of signals. The idea of a time-domain representation of a signal described in Chapter 8 does not require much motivation since it is this representation that we regard as being the "real one," in that we believe we experience the world as time, not frequency, unfolds. Nonetheless, frequency-domain representations are extremely useful in practice, and are in many cases a more natural method for representing data. However, to really make sense of frequency-domain representations, one needs to precisely define the correspondences between the time- and frequency-domain. These correspondences are non-trivial, actually, and indeed comprise Chapters 12, 13, and 14. Thus our task in this chapter is a difficult one: to discuss frequency-domain representations of signals without actually being able to say what we really mean. Difficult and murky this may be, but it is perhaps useful for readers unfamiliar with the frequency-domain to possess this motivation before we get rigorous in the sequel.

Do I need to read this chapter? If you just want to get to the Fourier transforms and their properties, then maybe you can bypass this. But hopefully this chapter will at least be interesting reading, even if it has little technical content. •

Contents

Section 9.1

Frequency and substance

Our objective in this section is to convince the reader that frequency is all around us, and to do so at the scientific level of a reader who has, say, seen several episodes of *Star Trek*.

A "spectrum" is, very roughly speaking, a range of frequencies. Frequency here is to be thought of as in Definition 8.1.18 when we discussed periodic and harmonic signals in the time-domain. When representing a signal in the frequency-domain, one is indicating how much "energy" the signal possesses at certain frequencies. Thus one decomposes the signal (in a manner that is by no means clear at this point) into its constituent frequencies. The history of spectral decomposition can be traced back at least to Newton whose experiment of passing light through a prism illustrated that light could be decomposed. The decomposition that Newton saw was a decomposition based on the frequency content in the light being passed through the prism.

Do I need to read this section? If you thought that reading this chapter seemed like a good idea, then presumably reading this section seems like a good idea as well. •

9.1.1 The electromagnetic spectrum

Light and energy is observed in nature occurring in a broad frequency range that is called the ***electromagnetic spectrum***. The frequency along the electromagnetic spectrum can be measured in various units, including m (the "physical" wavelength), s^{-1} or Hz (the temporal frequency), or joules (the energy of a photon at this frequency). It is common in the physics/chemistry literature to see m^{-1} used. However, we shall use Hz, since in many of the applications we present, this is the most natural way of thinking of things. However, the two unit systems are related in the following way. If one is "travelling with the wave"[1] and one measures the period of the waveform as a distance, this is the ***wavelength*** measured (say) in m. However, electromagnetic waves move through space (or whatever medium) at the speed of light, denoted c. Thus a stationary observer of the wave will see a single wavelength pass in time equal to the wavelength divided by c, which is the temporal period of the waveform. The ***frequency*** is the inverse of this period. Thus we have

$$\text{frequency } (\mathrm{s}^{-1}) = \frac{c\ (\mathrm{m/s})}{\text{wavelength (m)}}.$$

[1]This interpretation must be taken with a grain of salt, since electromagnetic waves do not travel through space in the same manner as do ripples across the surface of a pond on a calm day. However, there is a sense in which such an interpretation is at least useful, and we restrict ourselves to this.

The speed of light in a vacuum is $c = 299,792,458\text{m/s}$, and it is common to ignore the difference between the speed of light in a vacuum and its speed in other media.

The electromagnetic spectrum is roughly divided into seven regions and these are displayed in Table 9.1. An idea of the relative portions of the spectra occupied

Table 9.1 The frequency bands of the electromagnetic spectrum

Name	Frequency range	Physical phenomenon
Radio	$\nu < 3 \times 10^9\text{Hz}$	AM/FM radio, television, shortwave, produced by oscillatory movement of charged particles, able to pass through atmosphere
Microwave	$3 \times 10^9\text{Hz} < \nu < 3 \times 10^{11}\text{Hz}$	kitchen appliance, satellite communication, radar, study of galactic structure, able to pass through atmosphere
Infrared	$3 \times 10^{11}\text{Hz} < \nu < 4 \times 10^{14}\text{Hz}$	useful for studying galactic dust, very little passes through atmosphere
Visible	$4 \times 10^{14}\text{Hz} < \nu < 7.5 \times 10^{14}\text{Hz}$	responsible for color as we know it, passes through atmosphere
Ultraviolet	$7.5 \times 10^{14}\text{Hz} < \nu < 3 \times 10^{16}\text{Hz}$	emitted by hot stars, mostly blocked by atmosphere
X-ray	$3 \times 10^{16}\text{Hz} < \nu < 3 \times 10^{19}\text{Hz}$	medical use, emitted by hot gas
Gamma-ray	$3 \times 10^{19}\text{Hz} < \nu$	emitted by radioactive material

by the various categories are shown in Figure 9.1. Note that the visible portion

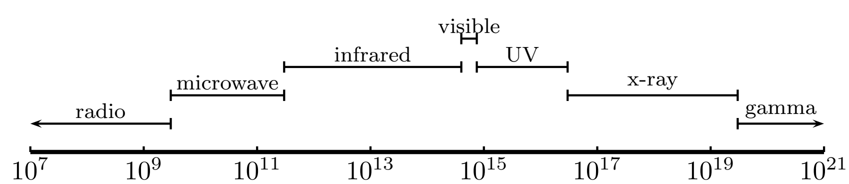


Figure 9.1 The electromagnetic spectrum

of the spectrum is quite small, so that one must really go beyond visual means to understand all spectral features of a physical phenomenon. In Figure 9.2 we show the visible portion of the spectrum in terms of the color emitted by physical phenomenon.

Figure 9.2 The visible spectrum from red to yellow to green to cyan to blue to violet

9.1.2 Emission and absorption spectra

First let us consider a simple experiment, one commonly performed by physics and chemistry undergraduates. Take a tube filled with hydrogen gas and heat it by passing through it an electric charge. The light emitted by the tube is passed through a spectrograph, old-fashioned versions of which work much like Newton's prism, but newer versions of which are based on the diffraction grating. The spectrograph will decompose the light into certain of its spectral components. Typically a single spectrograph will only be able to reproduce certain parts of the electromagnetic spectrum. In a simple spectrograph that works a lot like Newton's prism in that the spectrum is transmitted onto a physical surface, what one will see is, pretty much by definition, the visible part of the electromagnetic spectrum of the light emitted by hydrogen. This is called the ***Balmer spectrum*** of hydrogen after Johann Balmer who discovered this part of the spectrum in 1885. The colours of the visible spectrum for hydrogen occur at

$$4.56676\times10^{14}\text{Hz},\quad 6.16512\times10^{14}\text{Hz},\quad 6.90493\times10^{14}\text{Hz},\quad 7.30681\times10^{14}\text{Hz}.$$

The lines of the emission spectrum are shown in Figure 9.3, and represent what an

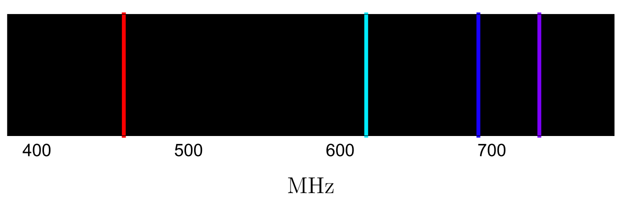


Figure 9.3 The visible emission spectrum for hydrogen

undergraduate performing this experiment might see in the lab.

Other parts of the hydrogen spectrum were located by the researchers Theodore Lyman (ultraviolet vacuum, 1906-1914), Louis Paschen (infrared, 1908), Frederick Brackett (visible and infrared, 1922), and August Pfund (infrared, 1924), and these researchers have their name attached to those parts of the hydrogen atom spectrum they identified. Remarkably, the wavelength ℓ of these spectral lines can be determined with great accuracy by a simple formula:

$$\frac{1}{\ell} = R_H\Big(\frac{1}{n_f^2} - \frac{1}{n_i^2}\Big), \tag{9.1}$$

where R_H is ***Rydberg's constant*** for hydrogen, which has the numerical value $R_H = 1.09678 \times 10^7 \mathrm{m}^{-1}$. The Balmer spectrum corresponds to $(n_f, n_i) \in \{(2,3),(2,4),(2,5),(2,6)\}$, and Balmer actually knew the formula (9.1) in these cases. However, it was not until Neils Bohr developed his model for the hydrogen atom in 1913 that there was a somewhat satisfactory theoretical explanation for the relationship between (9.1) and the emission spectrum for hydrogen. And even then, Bohr's model left something to be desired in terms of its extension to other emission spectra, and in terms of the theory having certain aspects that were without proper motivation. However, this takes us beyond both the scope of this book, and the expertise of its author.

The experiment described above can be performed with any of a number of substances, and with variations on how the experiment is setup. This leads to ***Kirchhoff's laws of spectral formation***, which tell us the sort of spectrum we can expect to see. These laws are as follows.

1. A hot opaque body, such as a hot, dense gas produces a ***continuous*** spectrum, by which we mean a continuous spectrum of light, as for example produced by a rainbow.
2. A hot, transparent gas produces an ***emission line*** spectrum, by which we mean a discrete set of frequencies will be produced.
3. A cool, transparent (dilute) gas in front of a source of continuous emission produces an ***absorption line*** spectrum. This is essentially the "opposite" of an emission line spectrum, in that frequencies are omitted from the spectrum rather than produced in it.

The scenario is depicted cartoon-style in Figure 9.4.

9.1.3 Determining the composition of a distant star

Let us now consider a problem that we shall not be able to be as concrete with, at least for the moment. The brightest star in the sky is Sirius. Suppose one wishes to ask, "Of what is Sirius made?" Obviously, it is difficult to travel to Sirius since it is about nine light years distant. Nonetheless, we *are* able to say a great deal about the physical composition of stars like Sirius. How is this done? The idea is that one points a radio telescope at Sirius that receives a signal. To analyse this signal, one compares it to the signal one might get from known physical elements, like for example the emission spectrum for hydrogen described in the preceding section. By understanding the absorption and/or emission lines for the spectrum, certain elements can be identified as being present in the star. In Figure 9.5 is shown a record of the emission events for a white dwarf. While Sirius is not a white dwarf, it *is* accompanied by a white dwarf (a small very dense star) discovered by Alvan Clark in 1862 while testing a new telescope. When observing Sirius, he observed

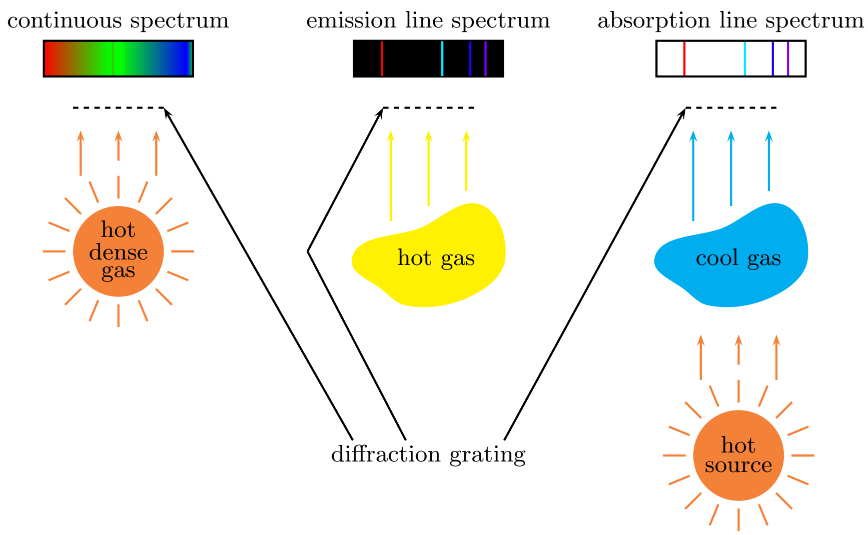


Figure 9.4 The Kirchhoff laws of spectral formation

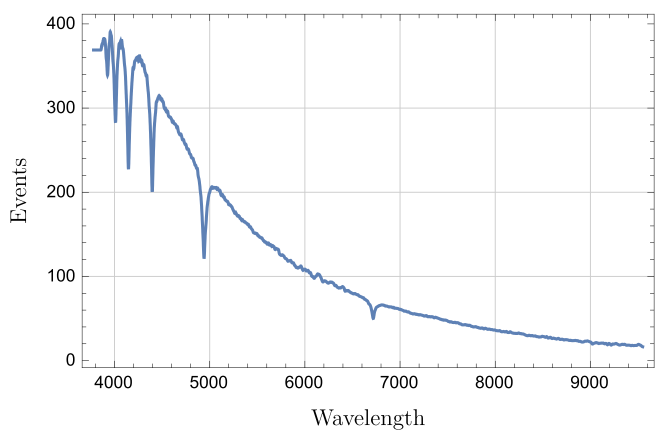


Figure 9.5 Events recorded for a prescribed wavelength for a white dwarf

a "wobble" which was caused by the presence of another, barely visible star. This was the first white dwarf discovered, and is called Sirius B.

Section 9.2

Sound and music

The preceding section dealt with natural "frequencies" that are observed in nature, and which travel with "waves" moving with the speed of light. In this section we shall see that frequency comes naturally in other settings as well.

Do I need to read this section? If you were amused by the preceding section, you may well be amused by this one as well. •

9.2.1 Sound identification using frequency

Sound is made by the generation of a wave that travels through the air, much as a ripple travels over the surface of a pond on a calm day. The displacement of the air carries with it a pressure difference, and it is this pressure difference that activates the mechanism in the ear, causing us to hear something. One can measure this pressure differential as a function of time, and in doing so one will end of with a signal in the time-domain, as discussed in Chapter 8. Recall the speech signal depicted in Figure 8.4. This time-domain representation would be how such a signal would be normally recorded. However, it is a little difficult to know what to do with it. For example, suppose that one wished to ascertain who the speaker was. This is not easy to do by, say, comparing the given signal with a comparison signal from a person who you think might have made the sound. It turns out that a good way to analyse speech data is by determining the energy present in the signal at various frequencies, and comparing *that* to known data for a candidate speech-maker. In Figure 9.6 we show the frequency content of the time-domain

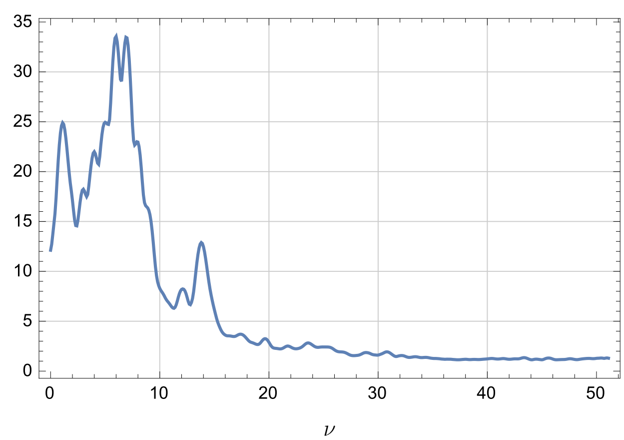


Figure 9.6 Frequency content of speech signal from Figure 8.4

speech signal. Of course, at this point we are not saying how we come up with this—this is exactly the point of the subsequent several chapters of the book.

9.2.2 A little music theory

One can apply the ideas expressed in the preceding section to music. Recall from Figure 8.5 the time-domain representation of two musical clips. In Figure 9.7 we show both the time-domain and frequency-domain representations of two musical clips.

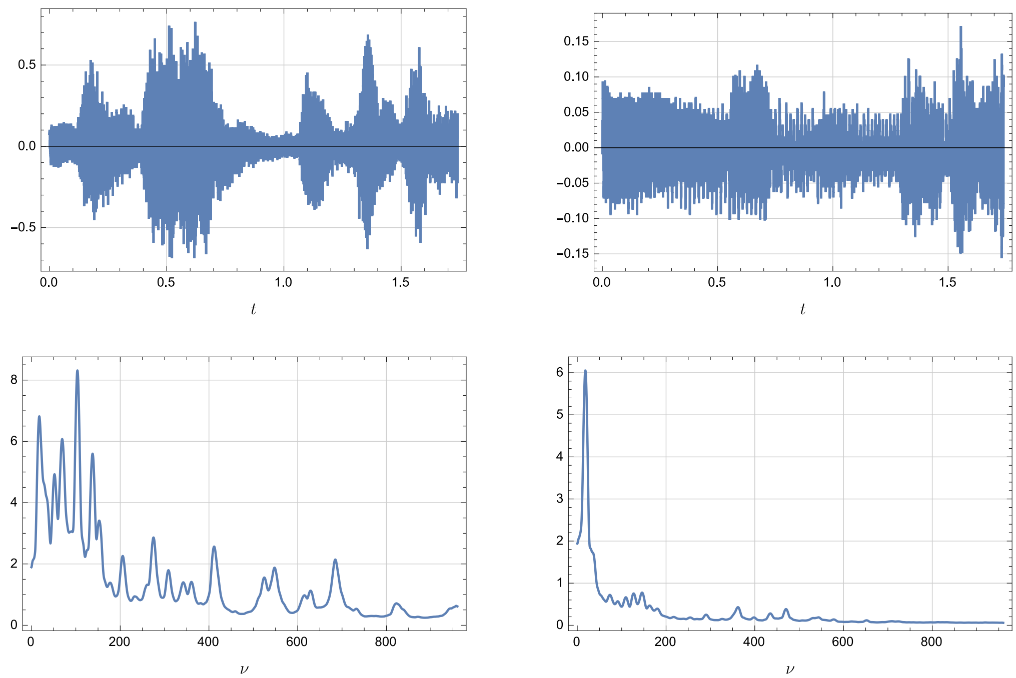


Figure 9.7 A time-domain (top) and frequency-domain (bottom) representation of part of the first movement of Mozart's *Eine Kleine Nachtmusik* (K525) (left) and a portion of the soundtrack of the movie π (right)

sical clips. The clip on the left is from the first movement of Mozart's *Eine kleine Nachtmusik* (K525), and that on the right is from the soundtrack of the Darren Aronofsky movie π. While the frequency-domain signals admittedly do not look all that coherent, it is nonetheless true that there are many powerful techniques for analysing signals that rely on representing a signal in terms of frequency, and not time.

Since we have mentioned Mozart, perhaps it is interesting to say a few words about musical notes as they relate to frequency. A ***note*** is a sound consisting of a single frequency ν. Thus a note in the time-domain is a harmonic signal $\mu\colon \mathbb{R} \to \mathbb{R}$. Let us denote $\nu(\mu)$ as the frequency of a note. A note μ_4 is an ***octave higher*** than a note μ_1 if $\frac{\nu(\mu_4)}{\nu(\mu_1)} = 2$. Thus an increase by an octave is a doubling of frequency. It was observed by the Greeks[2] that the human brain seems to identify notes that differ by an octave as belonging to the same "family." Physically, an octave arises (loosely) in the following way. Consider a guitar string of length ℓ plucked so that it emits a note.[3] Now suppose that one places a clamp in exactly the middle of the string, effectively making it two strings of length $\frac{\ell}{2}$. When plucked, the string will now produce a note with frequency double that of the original string.

So that's an octave. The problem now arises of how does one "ascend" from a note μ to the note one octave higher. A ***scale*** is a division of an octave into a finite number of intervals. There is no "correct" way to define a scale. We shall describe two closely related scales that are common in western music. Both are based on the division of the octave into twelve parts. First of all, why twelve? Well, it turns out that it is possible to say a great deal about why twelve is the right number. Historically, probably the best explanation is that one wishes to go from 1 to 2 by rational numbers whose numerators and denominators are not too large. The reason for this "not too large" restriction is that notes related by rational numbers with not too large numerators and denominators sound "nice." For example, a note with frequency $\frac{3}{2}\nu(\mu)$ played alongside the note μ seems pleasing to the ear. However, the note with frequency $\frac{7919}{6997}\nu(\mu)$ played alongside μ will not sound as pleasing. For various reasons that can be understood fairly well mathematically (see the notes in Section 9.2.3), the octave scale can be well divided into twelve (not equal, as we shall see) parts with frequencies rationally related to the bottom frequency by rational numbers that have numerators and denominators that are not too large.

Now, the first scale we define is called the ***chromatic just temperament***, and it produces a scale with rationally related frequencies, as described above. This scale was first expounded upon thoroughly by Bartolomeo Ramos de Pareja (1440-1491?) in 1482, although many had contributed pieces of it prior to de Pareja. To be concrete, we start with a specific note which we denote $\mu^j_{C_4}$ which has a frequency of $\nu^j_{A_4} = 220\mathrm{Hz}$. The note one octave higher will then be denoted $\mu^j_{A_5}$ which has frequency $\nu^j_{A_5} = 440\mathrm{Hz}$. To construct the just temperament we define notes $\mu^j_{A^\sharp_4} = \mu^j_{B^\flat_4}$, $\mu^j_{B_4}$, $\mu^j_{C_4}$, $\mu^j_{C^\sharp_4} = \mu^j_{D^\flat_4}$, $\mu^j_{D_4}$, $\mu^j_{D^\sharp_4} = \mu^j_{E^\flat_4}$, $\mu^j_{E_4}$, $\mu^j_{F_4}$, $\mu^j_{F^\sharp_4} = \mu^j_{G^\flat_4}$, $\mu^j_{G_4}$, and $\mu^j_{G^\sharp_4} = \mu^j_{A^\flat_4}$ according to the rules laid out in Table 9.2. The naming of (say) the second element in the table as $\mu^j_{A^\sharp_4}$ or $\mu^j_{B^\flat_4}$ is arbitrary, and largely a matter of convenience or convention. But, though the name may be different, the sound is the same. Note that the division is by even parts on a scale that is logarithmic with base 2. Two adjacent notes are separated by an ***semitone***. Note that while the ratios formed by the notes with the frequency of the original note are nice rational numbers with small numerator and denominator (this is one of the objectives of chromatic just temperament), the problem arises that there are four distinct semitones. In

[2] In the west, the first musical scale seems to have been developed by the Pythagoreans.

[3] A guitar string when plucked normally does not emit a note, but a sound that is a sum of many notes. But that is another course.

Note	Frequency ratio	Ratio with next step	Frequency
$\mu^j_{A_4}$	$\frac{\nu^j_{A_4}}{\nu^j_{A_4}} = 1$	$\frac{\nu^j_{A^\sharp_4}}{\nu^j_{A_4}} = \frac{27}{25}$	220Hz
$\mu^j_{A^\sharp_4} = \mu^j_{B^\flat_4}$	$\frac{\nu^j_{A^\sharp_4}}{\nu^j_{A_4}} = \frac{27}{25}$	$\frac{\nu^j_{B_4}}{\nu^j_{A^\sharp_4}} = \frac{25}{24}$	237.6Hz
$\mu^j_{B_4}$	$\frac{\nu^j_{B_4}}{\nu^j_{A_4}} = \frac{9}{8}$	$\frac{\nu^j_{C_4}}{\nu^j_{B_4}} = \frac{16}{15}$	247.5Hz
$\mu^j_{C_4}$	$\frac{\nu^j_{C_4}}{\nu^j_{A_4}} = \frac{6}{5}$	$\frac{\nu^j_{C^\sharp_4}}{\nu^j_{C_4}} = \frac{16}{15}$	264Hz
$\mu^j_{C^\sharp_4} = \mu^j_{D^\flat_4}$	$\frac{\nu^j_{C^\sharp_4}}{\nu^j_{A_4}} = \frac{32}{25}$	$\frac{\nu^j_{D_4}}{\nu^j_{C^\sharp_4}} = \frac{135}{128}$	281.6Hz
$\mu^j_{D_4}$	$\frac{\nu^j_{D_4}}{\nu^j_{A_4}} = \frac{27}{20}$	$\frac{\nu^j_{D^\sharp_4}}{\nu^j_{D_4}} = \frac{16}{15}$	297Hz
$\mu^j_{D^\sharp_4} = \mu^j_{E^\flat_4}$	$\frac{\nu^j_{D^\sharp_4}}{\nu^j_{A_4}} = \frac{36}{25}$	$\frac{\nu^j_{E_4}}{\nu^j_{D^\sharp_4}} = \frac{25}{24}$	316.8Hz
$\mu^j_{E_4}$	$\frac{\nu^j_{E_4}}{\nu^j_{A_4}} = \frac{3}{2}$	$\frac{\nu^j_{F_4}}{\nu^j_{E_4}} = \frac{16}{15}$	330Hz
$\mu^j_{F_4}$	$\frac{\nu^j_{F_4}}{\nu^j_{A_4}} = \frac{8}{5}$	$\frac{\nu^j_{F^\sharp_4}}{\nu^j_{F_4}} = \frac{135}{128}$	352Hz
$\mu^j_{F^\sharp_4} = \mu^j_{G^\flat_4}$	$\frac{\nu^j_{F^\sharp_4}}{\nu^j_{A_4}} = \frac{27}{16}$	$\frac{\nu^j_{G_4}}{\nu^j_{F^\sharp_4}} = \frac{16}{15}$	371.26Hz
$\mu^j_{G_4}$	$\frac{\nu^j_{G_4}}{\nu^j_{A_4}} = \frac{9}{5}$	$\frac{\nu^j_{G^\sharp_4}}{\nu^j_{G_4}} = \frac{16}{15}$	396Hz
$\mu^j_{G^\sharp_4} = \mu^j_{A^\flat_5}$	$\frac{\nu^j_{G^\sharp_4}}{\nu^j_{A_4}} = \frac{48}{25}$	$\frac{\nu^j_{A_5}}{\nu^j_{G^\sharp_4}} = \frac{25}{24}$	422.4Hz
$\mu^j_{A_5}$	$\frac{\nu^j_{A_5}}{\nu^j_{A_4}} = 2$	$\frac{\nu^j_{A^\sharp_5}}{\nu^j_{A_5}} = \frac{27}{25}$	440Hz

Table 9.2 The division of an octave by chromatic just temperament

Table 9.2 these gaps are described by the ratio with the next frequency in the table to facilitate comparison with Table 9.3 below. To further facilitate this comparison, we compute the base 2 logarithm of the ratios defining the semitones:

$$\begin{aligned} \log_2 \tfrac{25}{24} &\approx 0.0588937, \quad \log_2 \tfrac{135}{128} \approx 0.0768156 \\ \log_2 \tfrac{16}{15} &\approx 0.0931094, \quad \log_2 \tfrac{27}{25} \approx 0.111031. \end{aligned} \tag{9.2}$$

The use of the next scale we present became solidified around the time of J. S. Bach. This is the ***tempered scale***, and divides the octave into twelve parts, with the division being regular in a certain sense. To illustrate, let us again choose the note $\mu^e_{A_4}\colon \mathbb{R} \to \mathbb{R}$ with frequency $\nu^e_{A_4} = \nu(\mu^e_{A_4}) = 220\text{Hz}$, and denote by $\mu^e_{A_5}$ the note one octave higher than $\mu^e_{A_4}$. We then define eleven notes (some with multiple names) $\mu^e_{A^\sharp_4} = \mu^e_{B^\flat_4}$, $\mu^e_{B_4}$, $\mu^e_{C_4}$, $\mu^e_{C^\sharp_4} = \mu^e_{D^\flat_4}$, $\mu^e_{D_4}$, $\mu^e_{D^\sharp_4} = \mu^e_{E^\flat_4}$, $\mu^e_{E_4}$, $\mu^e_{F_4}$, $\mu^e_{F^\sharp_4} = \mu^e_{G^\flat_4}$, $\mu^e_{G_4}$, and $\mu^e_{G^\sharp_4} = \mu^e_{A^\flat_4}$ according to the rules laid out in Table 9.3. Note here that all semitones

Note	Frequency ratio	Ratio with next step	Frequency (approx)
$\mu^e_{A_4}$	$\nu^e_{A_4}$	$\log_2 \frac{\nu^e_{A^\sharp_4}}{\nu^e_{A_4}} = \frac{1}{12}$	220Hz
$\mu^e_{A^\sharp_4} = \mu^e_{B^\flat_4}$	$\log_2 \frac{\nu^e_{A^\sharp_4}}{\nu^e_{A_4}} = \frac{1}{12}$	$\log_2 \frac{\nu^e_{B_4}}{\nu^e_{A^\sharp_4}} = \frac{1}{12}$	233.082Hz
$\mu^e_{B_4}$	$\log_4 \frac{\nu^e_{B_4}}{\nu^e_{A_4}} = \frac{1}{6}$	$\log_2 \frac{\nu^e_{C_4}}{\nu^e_{B_4}} = \frac{1}{12}$	246.942Hz
$\mu^e_{C_4}$	$\log_2 \frac{\nu^e_{C_4}}{\nu^e_{A_4}} = \frac{1}{4}$	$\log_2 \frac{\nu^e_{C^\sharp_4}}{\nu^e_{C_4}} = \frac{1}{12}$	261.626Hz
$\mu^e_{C^\sharp_4} = \mu^e_{D^\flat_4}$	$\log_2 \frac{\nu^e_{C^\sharp_4}}{\nu^e_{A_4}} = \frac{1}{3}$	$\log_2 \frac{\nu^e_{D_4}}{\nu^e_{C^\sharp_4}} = \frac{1}{12}$	277.183Hz
$\mu^e_{D_4}$	$\log_2 \frac{\nu^e_{D_4}}{\nu^e_{A_4}} = \frac{5}{12}$	$\log_2 \frac{\nu^e_{D^\sharp_4}}{\nu^e_{D_4}} = \frac{1}{12}$	293.665Hz
$\mu^e_{D^\sharp_4} = \mu^e_{E^\flat_4}$	$\log_2 \frac{\nu^e_{D^\sharp_4}}{\nu^e_{A_4}} = \frac{1}{2}$	$\log_2 \frac{\nu^e_{E_4}}{\nu^e_{D^\sharp_4}} = \frac{1}{12}$	311.127Hz
$\mu^e_{E_4}$	$\log_2 \frac{\nu^e_{E_4}}{\nu^e_{A_4}} = \frac{7}{12}$	$\log_2 \frac{\nu^e_{F_4}}{\nu^e_{E_4}} = \frac{1}{12}$	329.628Hz
$\mu^e_{F_4}$	$\log_2 \frac{\nu^e_{F_4}}{\nu^e_{A_4}} = \frac{2}{3}$	$\log_2 \frac{\nu^e_{F^\sharp_4}}{\nu^e_{F_4}} = \frac{1}{12}$	349.228Hz
$\mu^e_{F^\sharp_4} = \mu^e_{G^\flat_4}$	$\log_2 \frac{\nu^e_{F^\sharp_4}}{\nu^e_{A_4}} = \frac{3}{4}$	$\log_2 \frac{\nu^e_{G_4}}{\nu^e_{F^\sharp_4}} = \frac{1}{12}$	369.994Hz
$\mu^e_{G_4}$	$\log_2 \frac{\nu^e_{G_4}}{\nu^e_{A_4}} = \frac{5}{6}$	$\log_2 \frac{\nu^e_{G^\sharp_4}}{\nu^e_{G_4}} = \frac{1}{12}$	391.995Hz
$\mu^e_{G^\sharp_4} = \mu^e_{A^\flat_5}$	$\log_2 \frac{\nu^e_{G^\sharp_4}}{\nu^e_{A_4}} = \frac{11}{12}$	$\log_2 \frac{\nu^e_{A_5}}{\nu^e_{G^\sharp_4}} = \frac{1}{12}$	415.305Hz
$\mu^e_{A_5}$	$\log_2 \frac{\nu^e_{A_5}}{\nu^e_{A_4}} = 2$	$\log_2 \frac{\nu^e_{A^\sharp_4}}{\nu^e_{A_5}} = \frac{1}{12}$	440Hz

Table 9.3 The division of an octave by even temperament

are equal and have the numerical value $\frac{1}{12} = 0.8\bar{3}$, which can be compared to the four semitone values for the just Chromatic scale as given in (9.2).

The convention of assigning to $\mu^e_{A_4}$ the frequency of 220Hz has been in place since 1939, and is the convention now used for fixing the notes in the musical scale used for (for example) tuning a piano. The note $\mu^e_{C_4}$ in Table 9.3 is ***middle C*** on the piano. With this convention, the lowest note on the piano keyboard is μ_{C_1}.

9.2.3 Notes

Some interesting discussion of the mathematics of ascending an octave can be found in the paper of [**JD/RE/AM:92**]. Readers interested in a scientific discussion of music are referred to the classic book of **JJ:68**.

Exercises

9.2.1 Compute the frequencies of the following notes:

(a) $\mu^e_{A_0}$;
(b) $\mu^e_{C_1}$;
(c) $\mu^e_{G^\sharp_5}$;
(d) $\mu^e_{A^\flat_6}$;
(e) $\mu^e_{B^\flat_2}$;
(f) $\mu^e_{D^\sharp_4}$.

9.2.2 For the following pairs of notes, describe the degree and quality of the interval between them:

(a) $(\mu^e_{A_0}, \mu^e_{G^\sharp_5})$;
(b) $(\mu^e_{D^\sharp_4}, \mu^e_{G^\sharp_5})$;
(c) $(\mu^e_{A_0}, \mu^e_{C^\flat_0})$;
(d) $(\mu^e_{A_0}, \mu^e_{B^\sharp_0})$;
(e) $(\mu^e_{A_0}, \mu^e_{F_0})$;
(f) $(\mu^e_{A_0}, \mu^e_{E^\sharp_0})$.

Section 9.3

Signal transmission schemes

It may be supposed that we are all familiar with the terms "AM" and "FM."[4] Perhaps we may also suppose that we all know that these are abbreviations for "amplitude modulation" and "frequency modulation." Maybe a less familiar expression is "phase modulation," although we have all probably used devices that make use of phase modulation technology. In this section we will review some ideas related to these techniques for signal transmission. We do not attempt to go into the details of what makes one scheme better in which circumstances to the other two, but merely content ourselves with identifying the rôle of frequency in each of the schemes.

Do I need to read this section? If the reader does not already know about the details of these techniques, then this section might make for an interesting introduction. •

9.3.1 A general communication problem

The techniques in this section are used to solve the following problem. One has some signal $t \mapsto s_1(t)$ one wishes to transmit. However, someone else has a signal $t \mapsto s_2(t)$ that they wish to transmit. If both signals are transmitted at the same time, then a receiver will see $s_1 + s_2$ plus any other junk floating around. If s_1 and s_2 are similar (say both are transmissions of the human voice), it will not be possible to separate s_1 and s_2 from $s_1 + s_2$. Thus the problem is to find a way to transmit a signal so that it is in some way distinguished from all of the other signals flying about that are similar to it. The three modulation schemes we discuss here are designed to achieve exactly this objective.

This problem is an example of a general communication problem, a schematic for which is given in Figure 9.8. The idea in this schematic is that a signal enters

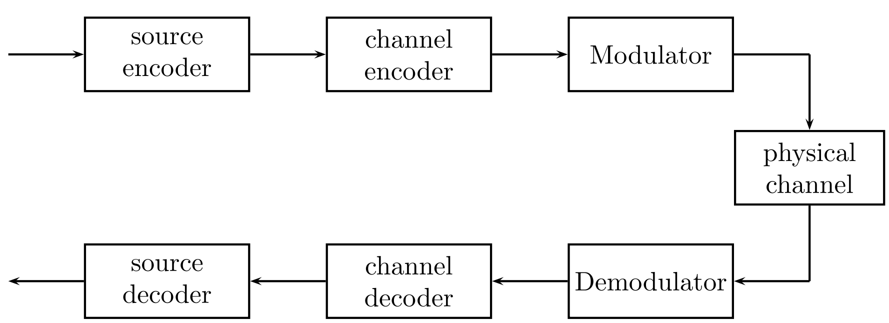


Figure 9.8 The general communication problem

[4]This supposition will probably cease to be valid in the not too distant future.

at the top left. The source encoder processes the signal in some way to make it amenable for transmission. For example, perhaps this step involves some data compression. The channel encoder manipulates the data to provide robustness. At this step one may, for example, perform some error correction. The modulator then makes the data ready for transmission. The physical channel is the medium over which the data is transmitted; this is air for radio broadcasting. The demodulator then retrieves the actual data from the transmitted data, the channel decoder, knowing what was done by the channel encoder, produces an accurate representation of what came out of the course encoder, and the source decoder reverts the data to its final usable form.

In this section we are interested in the modulation/demodulation parts of the general scheme.

9.3.2 Amplitude modulation

Historically, the technique of amplitude modulation was employed to transmit telephone signals along electric power lines. The idea was that the existing electric power "carried along" the audio signal from the telephone. This idea was adapted to radio transmission, but in radio transmission one does not hitch a ride on a preexisting signal, but the whole signal is constructed, including the carrier.

In ***amplitude modulation*** (***AM***) one starts with a ***carrier signal*** which is a harmonic signal, say

$$c(t) = A_c \sin(\omega_c t + \phi_c),$$

where A_c, ω_c and ϕ_c are the ***amplitude***, ***frequency***, and ***phase*** of the carrier signal. One wishes to transmit the signal $s(t)$, and to do so one instead transmits the signal $M^{\mathrm{am}}_{c,s}(t) = (A + s(t))c(t)$, called the ***amplitude modulated signal***. The quantity $\frac{\|s\|_\infty}{A}$ is called the ***modulation index***. This quantity can be selected to achieve various effects.

Let us get some idea of how this works by looking at a special case. We take $c(t) = A_c \sin(\omega_c t)$ and $s(t) = A_s \cos(\omega_s t)$. We then have the amplitude modulated signal

$$\begin{aligned} M^{\mathrm{am}}_{c,s}(t) &= A_c(A + A_s \cos(\omega_s t)) \sin(\omega_c t) \\ &= AA_c \sin \omega_c t + \tfrac{1}{2} A_c A_s \sin((\omega_c + \omega_s)t) + \tfrac{1}{2} A_c A_s \sin((\omega_c - \omega_s)t), \end{aligned}$$

after some trigonometric identities have been applied. The following observations can be made.

1. The amplitude modulated has components at three frequencies, ω_c, $\omega_c + \omega_s$, and $\omega_c - \omega_s$. The frequency component at frequency ω_c is just a scaled version of the carrier signal. The effects of the amplitude modulation appear in the other two components. It turns out that the same thing happens for a general signal. One gets the effects of the carrier signal at the frequency ω_c and the frequency content of the transmitted signal appear shifted both by ω_c and $-\omega_c$. In practice one chooses ω_c to be larger than the largest frequency contained in the transmitted

signal. This prevents overlapping in the two shifted portions of the spectrum. These shifted portions of the spectrum are called ***sidebands***.

The point is that the problem of signal transmission by two transmitters is solved by each transmitter choosing a different carrier frequency ω_c. This is the frequency to which you tune your radio dial in the unlikely event that you listen to AM radio.

2. The information contained at the shifted frequency $\omega_c - \omega_s$ is already present in the information contained at the shifted frequency $\omega_c + \omega_s$. Thus it would be more efficient to be able to have only one of the two sidebands. There is a technique for achieving this and it is called ***single sideband*** (***SSB***) transmission.
3. In this example the modulation index is $\frac{A_s}{A}$. In some schemes for amplitude modulation it is required that the modulation index not exceed 1, which means that the amplitude must remain positive. In Figure 9.9 we show the amplitude

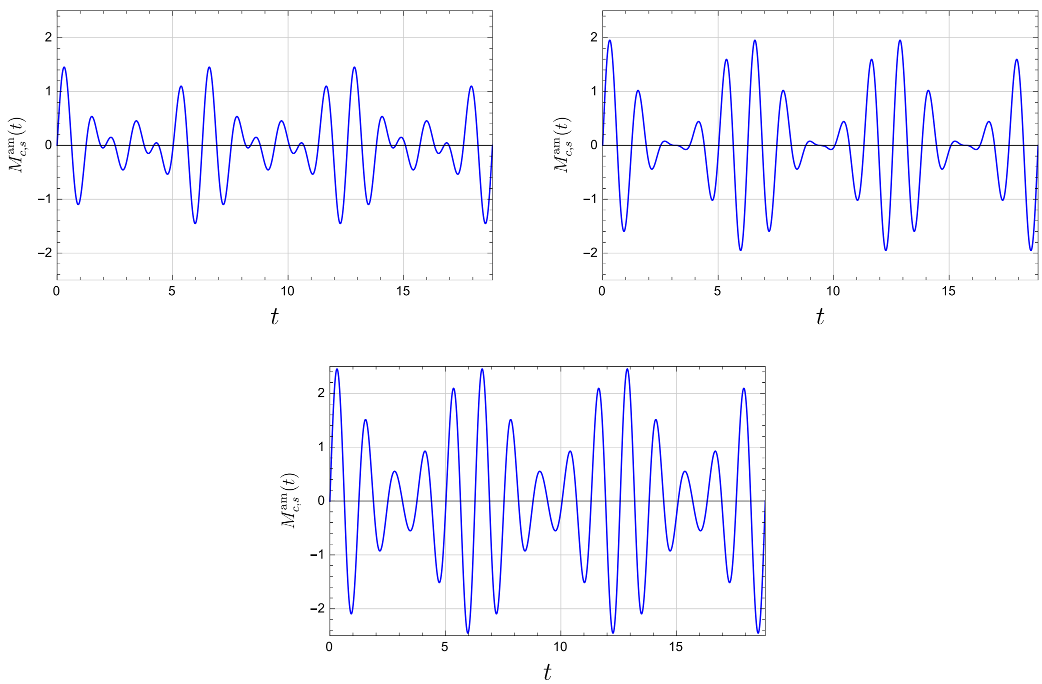


Figure 9.9 Amplitude modulation with modulation index 0.5 (top left), 1 (top right), and 1.5 (bottom); in all cases we have $\omega_c = 5$, $\omega_s = 1$, $A_c = 1$, and $A_s = 1$

modulated signal for the example with various modulation indexes.

4. In order to recover s from $M^{\mathrm{am}}_{c,s}$, the latter signal must undergo ***demodulation***. This is relatively easy to understand if one knows a little Fourier transform theory, see ***missing stuff***.

9.3.3 Frequency modulation

The idea for using frequency modulation came from Edwin Armstrong in 1935, and began seeing widespread use in the 1940's.

For frequency modulation, the idea is sort of the same as amplitude modulation, but the details are quite different. One starts out with a sinusoidal ***carrier signal***

$$c(t) = A_c \sin(\omega_c t + \phi_c)$$

with ***amplitude*** A_c, ***frequency*** ω_c, and ***phase*** ϕ_c. One wishes, again, to transmit the signal $s(t)$. To do so one defines

$$\omega_{c,s}(t) = \int_0^t (\omega_c + \Omega s(\tau))\,\mathrm{d}\tau$$

and, using this ***instantaneous frequency***, the ***frequency modulated signal***

$$M^{\mathrm{fm}}_{c,s}(t) = A_c \sin(\omega_{c,s}(t) + \phi_c).$$

For frequency modulation the ***modulation index*** is $\frac{\Omega\|s\|_\infty}{\omega_s}$ where ω_s is the frequency of the signal s to be transmitted. If s is not harmonic then ω_s is not well-defined, and one may use an average or some such thing.

To get some idea of what is going on with frequency modulation, let us consider again the special case of $c(t) = A_c \sin(\omega_c t)$ and $s(t) = A_s \cos(\omega_s t)$. For amplitude modulation in this case we could determine the modulated signal easily using simple trigonometry. For the frequency modulated signal, things are more complicated. Nonetheless, after some work, one can compute

$$M^{\mathrm{fm}}_{c,s}(t) = A_c \sin(\omega_c t + \tfrac{A_s\Omega}{\omega_s}\sin(\omega_s t)) = \sum_{k=-\infty}^{\infty} J_k(\tfrac{A_s\Omega}{\omega_s}) \sin((\omega_c + k\omega_s)t),$$

where J_k is the Bessel's function of the first kind of index k:

$$J_k(x) = \begin{cases} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+k)!}\left(\frac{x}{2}\right)^{2m+k}, & k \in \mathbb{Z}_{>0}, \\ (-1)^k \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-k)!}\left(\frac{x}{2}\right)^{2m-k}, & k \in \mathbb{Z}_{<0}. \end{cases}$$

The exact form of the frequency modulated signal is not so important as the observation that the signal is a sum of harmonic signals of frequencies shifted from the carrier frequency ω_c by integer multiples of the transmitted signal frequency ω_s. Thus, for frequency modulation, there are infinitely many sidebands. The point is that, just as with amplitude modulation, the modulated signal is most easily interpreted in terms of the frequencies at which the signal possess harmonics.

In Figure 9.10 we show the frequency modulated signal for various modulation indexes.

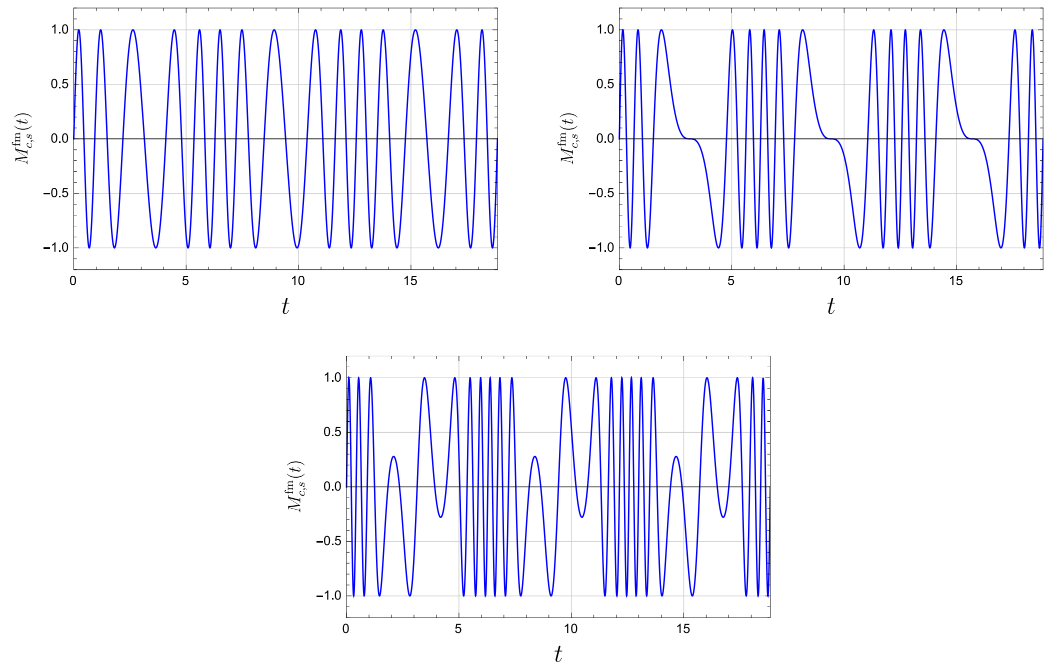


Figure 9.10 Frequency modulation with modulation index 2 (top left), 5 (top right, and 10 (bottom); in all cases we have $\omega_c = 5$, $\omega_s = 1$, $A_c = 1$, and $A_s = 1$

9.3.4 Phase modulation

Phase modulation looks a lot like frequency modulation, so we go through the development quickly. One again starts with a sinusoidal ***carrier signal***

$$c(t) = A_c \sin(\omega_c t + \phi_c)$$

with ***amplitude*** A_c, ***frequency*** ω_c, and ***phase*** ϕ_c. One wishes, again, to transmit the signal $s(t)$. To do so one defines the ***instantaneous phase***

$$\phi_{c,s}(t) = \phi_c + \Phi s(t)$$

and then the ***phase modulated signal***

$$M^{\mathrm{pm}}_{c,s}(t) = A_c(\omega_c t + \phi_{c,s}(t)).$$

The ***modulation index*** is $\|\Phi s\|_\infty$. One often sees it written that one or the other of frequency and phase modulation is a special case of the other. This is not quite true. What is true is that the set of frequency modulated signals is, up to a constant phase, a subset of the phase modulated signals. However, this is a rather different statement than frequency modulation being a special case of phase modulation.

Let us again consider the special case $c(t) = A_c \sin(\omega_c t)$ and $s(t) = A_s \cos(\omega_s t)$. Here we have

$$M^{\mathrm{pm}}_{c,s}(t) = A_c \sin(\omega_c t + \Phi A_s \cos(\omega_s t)).$$

One may determine that

$$M_{c,s}^{\mathrm{pm}}(t) = \sum_{k=-\infty}^{\infty} J_k(A_s\Phi)\cos((\omega_c + k\omega_s)t + \tfrac{k-1}{2}\pi).$$

Again we see that the phase modulated signal is a sum of harmonics with frequencies being the carrier frequency shifted by integer multiples of the signal frequency.

In Figure 9.11 we show the phase modulated signal for various modulation indexes.

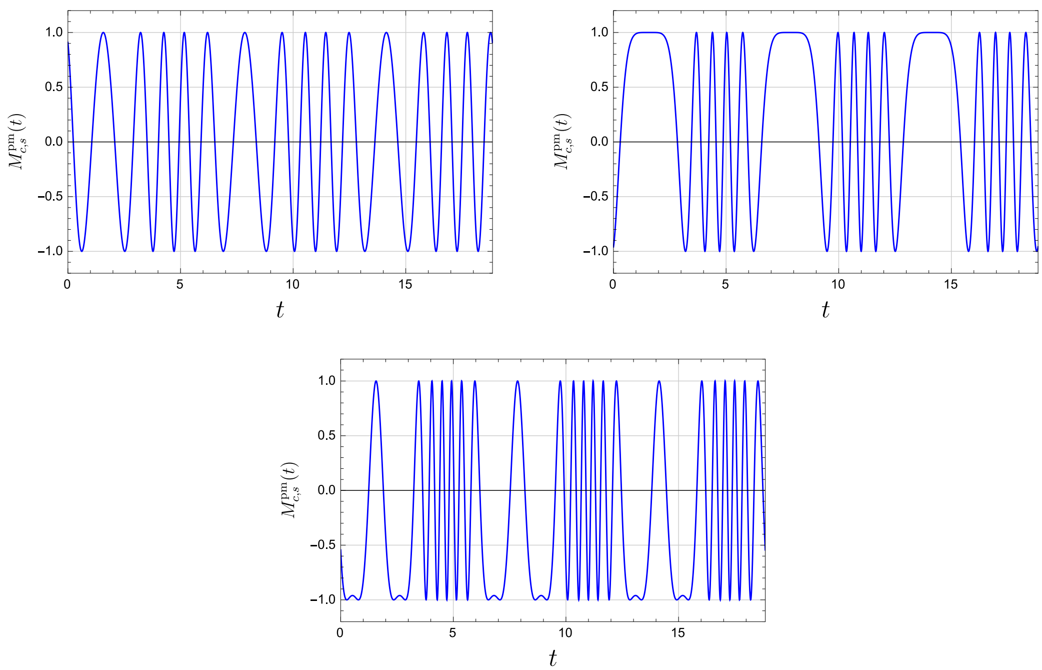


Figure 9.11 Phase modulation with modulation index 2 (top left), 5 (top right, and 10 (bottom); in all cases we have $\omega_c = 5$, $\omega_s = 1$, $A_c = 1$, and $A_s = 1$

Exercises

9.3.1 Explain the statement, "The set of frequency modulated signals is, up to a constant phase, a subset of the phase modulated signals."

Section 9.4

System identification using frequency response

We continue in this section with some ideas to motivate the use of frequency rather than time as a means of characterising signals. While system identification, the topic of this section, more clearly lies in the realm of system theory (as will be discussed in detail in Volume **??**), it is possible to say a few helpful things here that relate clearly to the notion of frequency.

Do I need to read this section? If the preceding three sections did not satiate your need to motivate the usefulness of frequency, then go ahead, read this section too. •

9.4.1 System modelling

When making a model of a system, there are possibly three strategies one might employ.

White-box modelling White-box modelling refers to modelling from first principles. Such first principles might include the principles of Newtonian mechanics, electromagnetics, fluid mechanics, thermodynamics, chemistry, quantum mechanics, etc. This is the strategy one might employ if the system one is modelling is well enough understood. Many systems are simply not of the sort that admit first principle modelling. Many biological, economic, social, etc., systems, for example, are not presently sufficiently well understood to allow them to be modelled in any "principled" way. Thus sometimes white-box modelling is just not possible. Moreover, even when it *is* possible, sometimes white-box modelling is not advisable. Indeed, a white-box model of an extremely complex system might just be too difficult to manage.

Grey-box modelling In grey-box modelling one has a form of a model, or maybe a rough form of a model at hand based on some knowledge of the system. However, there are parameters in the model that are not determinable from first principles, but must be determined in some way. In this case one might use some strategy for determining the values of the undetermined parameters. This is some form of system identification.

Black-box modelling In black-box modelling the premise is that one is so ignorant of one's system that the entire model has to be conjured in some way. As mentioned above, such systems are frequently encountered in biology, economics, social sciences, etc. It might also be the case that one has a system that is modellable, but one wishes to instead produce a more manageable model. The field of system identification typically deals with systems such as these.

9.4.2 A simple example

Consider the pair of coupled masses shown in Figure 9.12. The three springs

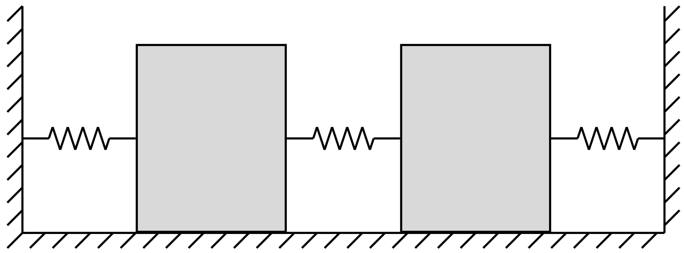


Figure 9.12 A coupled mass system

have the same spring constant k and the masses are also equal with mass m. Let us understand the behaviour of this system first before we start to pretend we do not understand it. In Figure 9.13 we depict the natural modes of vibration for the

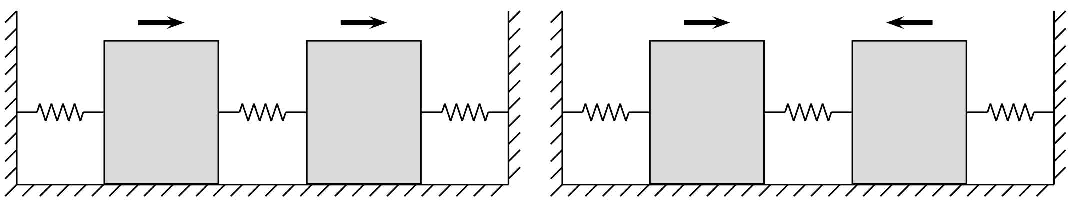


Figure 9.13 The two natural modes of vibration for the coupled mass system: (1) the masses oscillate with equal amplitude in the same direction (left) and (2) the masses oscillate with equal amplitude in the opposite direction

system. The frequency of the mode on the left is $\sqrt{\frac{k}{m}}\mathrm{rad}/s$ and the frequency of the mode on the right is $\sqrt{\frac{3k}{m}}\mathrm{rad}/s$. Now let us forget that we know about this system, but suppose instead that we are given a box with the system inside as shown in Figure 9.14. The idea is that we use the lever on top to actuate the system and we measure the response from the rod sticking out the right side of the box. Our task is to try to understand what is inside the box by manipulating the lever appropriately. A natural way to do this is to provide a harmonic input to the lever. If the system is linear, one is ensured that a harmonic output will result. By measuring the amplitude of the output at various input frequencies one might hope to be able to deduce something about what is in the box. For example, if one provides inputs at or near the natural frequencies $\sqrt{\frac{k}{m}}\mathrm{rad}/s$ and $\sqrt{\frac{3k}{m}}\mathrm{rad}/s$, then one might expect the output to be larger than that for input frequencies that are far from these natural frequencies.

The details of this are not the point. The point is that varying frequency inputs to a system can be a useful way of understanding its behaviour.

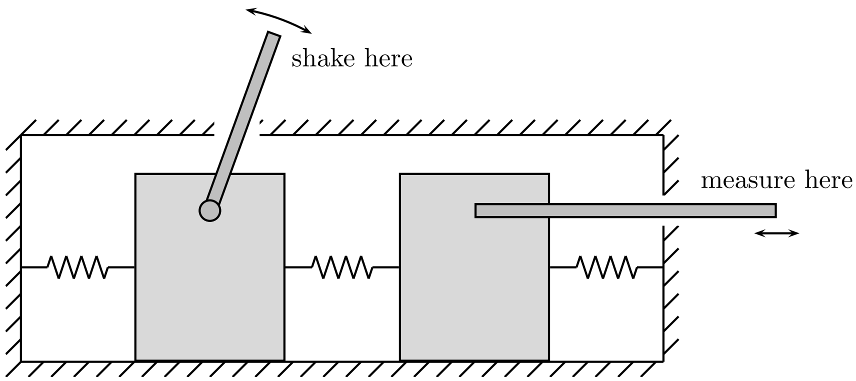


Figure 9.14 A box with the coupled mass system inside

9.4.3 A not-so-simple example

The coupled mass system from the previous section is cute, but lacks a little substance. But the same sorts of ideas apply to far more complicated systems. In Figure 9.15 we depict a building in an earthquake. Naturally, one would like for the building to move around as little as possible during the earthquake so as to minimise the possibility of structural failure. If one knows something about the typical frequency characteristics of the ground movement during an earthquake, one can reasonably ask that the building not exhibit a lot of motion when subject to harmonic signals with frequencies in those of the range present during an earthquake.

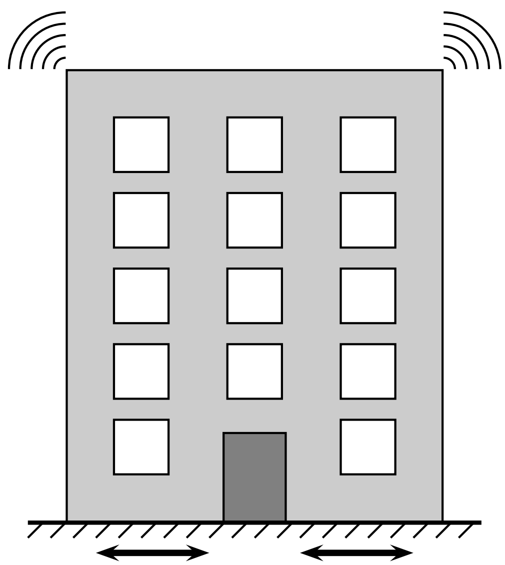


Figure 9.15 Earthquake! Get under a table!

One would, therefore, like to be able to model a building in such a way as to ascertain how it responds to signals of certain frequencies. Now, this is an example of a system for which a white-box model is possible to derive. We know enough about the behaviour of materials that we could, in principle, produce a model from physical principles. However, such a model would be very complicated, probably more so than would be needed to achieve the desired objectives. A technique that is used, both on real buildings and on scaled laboratory models for buildings, is to put sensors at various points on the building and provide input forces at various points on the building. By measuring the outputs for various inputs, one attempts to devise a simplified model that captures the desired facets of the problem. As with our toy example with two coupled masses, a common way to arrive at a model is to use harmonic inputs of varying frequency.

9.4.4 Notes

There is a famous instance where the issues discussed in Section 9.4.3 were revealed in a spectacular way. On November 7, 1940, approximately four months after it opened, the bridge across the Tacoma Narrows in Puget Sound in Washington collapsed. The collapse was preceded by a period of about an hour where the bridge oscillated wildly at a frequency of about 0.2Hz. This oscillation was induced by aerodynamic effect caused by the wind conditions in the Sound. While the wind speed was steady, vortex-shedding effects were responsible for the harmonic excitation of the bridge.

Section 9.5

Frequency-domains and signals in the frequency-domain

After having spent the preceding four***missing stuff*** sections motivating the meaning and usefulness of frequency-domain representations, in this section we present some language and notation concerning frequency-domains and signals as they might be represented in the frequency-domain. For time-domain representations of signals, the characteristics we presented in Sections 8.1.2 and 8.1.3 are fairly easy to understand. For frequency-domain representations the meaning of the various frequency-domains and properties of frequency-domain signals may not be so clear. However, it will be useful to have the terminology here in the sequel.

The approach here, and the technical aspects of what we say, follow our approach of Sections 8.1, 8.2, and 8.3. Therefore, our discussion here will be a little abbreviated since we will assume that the reader is familiar with our developments in the time-domain.

Do I need to read this section? We shall present notation and terminology in this section that we will freely use in the sequel. Thus the reader ought to read this section in order to be familiar with this. •

9.5.1 Frequency

As with time-domains, our definitions of frequency-domains rely on the notion of subgroups and semigroups in the group $(\mathbb{R}, +)$ of real numbers with addition.

9.5.1 Definition (Frequency-domain) A ***frequency-domain*** is a subset of $\mathbb{R}$ of the form $\mathbb{V} \cap I$ where $\mathbb{V} \subseteq \mathbb{R}$ is a semigroup in $(\mathbb{R}, +)$ and $I \subseteq \mathbb{R}$ is an interval. A frequency-domain is

(i) ***continuous*** if $\mathbb{V} = \mathbb{R}$,

(ii) ***discrete*** if $\mathbb{V} = \mathbb{Z}(\nu_0, \Omega)$ for some $\nu_0 \in \mathbb{R}$ called the ***origin shift*** and for some $\Omega > 0$ called the ***fundamental frequency***,

(iii) ***bounded*** if $\mathrm{cl}(I)$ is compact,

(iv) ***unbounded*** if it is not finite. •

9.5.2 Remarks (Some commonly made assumptions about frequency-domains)

1. We shall denote a typical point in a frequency-domain by ν or ω, depending on whether we mean to use frequency or angular frequency, respectively. Just as we generally think of the independent variable for time-domain signals as representing time in the usual sense, we shall think of frequency as being in units of Hz or rad/s. However, if time is not really time but something else (say, spatial distance) then the units of frequency will also be altered (to, say, wavelength).

2. As with time-domain signals, we will deal almost exclusively with discrete frequency-domains that are not shifted. In practice there are rules for converting shifted frequency-domains to unshifted ones.***missing stuff***
3. Also as with time-domains, we shall assume that discrete frequency-domains are regularly spaced, i.e., that the fundamental frequency is well-defined. •

9.5.2 Frequency-domain signals: basic definitions and properties

Next we define what we mean by a signal in the frequency-domain.

9.5.3 Definition (Frequency-domain signal) Let $\mathbb{W} = \mathbb{V} \cap I$ be a frequency-domain and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An ***$\mathbb{F}$-valued frequency-domain signal*** on $\mathbb{W}$ is a map $F\colon \mathbb{W} \to \mathbb{F}$. If $\mathbb{W}$ is continuous then F is a ***continuous-frequency*** signal and if $\mathbb{W}$ is discrete then F is a ***discrete-frequency*** signal. •

9.5.4 Notation ("Frequency-domain representation" versus "frequency-domain signal") Since it is more natural to think of signals as "happening" in the time-domain, we shall often refer to a frequency-domain signal as a "frequency-domain representation" of a signal, it being implied that the signal "really lives" in the time-domain, but we represent it (by means as yet unknown) in the frequency domain. •

The manner in which we graphically represent frequency-domain signals is the same as we used for time-domain signals. We refer to Figures 8.7 and 8.8, and the surrounding discussion, for the details of this. One thing to point out, however, is that in the frequency-domain it is far more natural to arrive at signals that are complex-valued, even when the corresponding time-domain signal is real. We shall see this in the examples below.

Let us next consider some examples of frequency-domain representations of signals. While we have not yet said how to make this correspondence, for each frequency-domain representation we will also indicate what is the time-domain signal. This will hopefully make it easier to understand our Fourier transform theory that follows in subsequent chapters.

9.5.5 Examples (Frequency-domain representations of signals)

1. Let us take $\mathbb{W} = \mathbb{Z}$ and define $F\colon \mathbb{W} \to \mathbb{C}$ by

$$F(\nu) = \begin{cases} \frac{1}{2\mathrm{i}}, & \nu = 1, \\ -\frac{1}{2\mathrm{i}}, & \nu = -1, \\ 0, & \text{otherwise.} \end{cases}$$

The way one constructs the time-domain signal from this is as follows. Corresponding to $F(1) = \frac{1}{2\mathrm{i}}$ we have the time-domain signal $\frac{1}{2\mathrm{i}}\mathrm{e}^{2\pi\mathrm{i}t}$ and corresponding to $F(-11) = -\frac{1}{2\mathrm{i}}$ we have the time-domain signal $-\frac{1}{2\mathrm{i}}\mathrm{e}^{-2\pi\mathrm{i}t}$. The time-domain signal corresponding to F is then

$$f(t) = F(1)\mathrm{e}^{2\pi\mathrm{i}t} + F(-1)\mathrm{e}^{-2\pi\mathrm{i}t} = \frac{1}{2\mathrm{i}}(\mathrm{e}^{2\pi\mathrm{i}t} - \mathrm{e}^{-2\pi\mathrm{i}t}) = \sin(2\pi t).$$

2. Let us take $\mathbb{W} = \mathbb{Z}$ and define $F\colon \mathbb{W} \to \mathbb{R}$ by

$$F(\nu) = \begin{cases} \frac{1}{2}, & \nu = 1, \\ -\frac{1}{2}, & \nu = -1, \\ 0, & \text{otherwise.} \end{cases}$$

The time-domain signal corresponding to this frequency-domain representation is

$$f(t) = F(1)e^{2\pi i t} + F(-1)e^{-2\pi i t} = \tfrac{1}{2}(e^{2\pi i t} - e^{-2\pi i t}) = \cos(2\pi t).$$

3. We generalise the preceding two examples by again taking $\mathbb{W} = \mathbb{Z}$ and now taking $F\colon \mathbb{W} \to \mathbb{F}$ to be any frequency-domain signal such that $\{\nu \in \mathbb{W} \mid F(\nu) \neq 0\}$ is finite. Then the corresponding time-domain signal is defined to be

$$f(t) = \sum_{\nu \in \mathbb{W}} F(\nu)e^{2\pi i \nu t},$$

this sum making sense since it is finite. The idea is that $F(\nu)$ in the frequency-domain represents $F(\nu)e^{2\pi i\nu t}$ in the time-domain. To get the entire signal in the time-domain, one sums over all frequencies.

4. Take the frequency-domain $\mathbb{W} = \mathbb{R}$ and define $F\colon \mathbb{W} \to \mathbb{R}$ by

$$F(\nu) = \begin{cases} 1, & \nu \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

As with the preceding examples, we have not actually said how one determines the time-domain signal corresponding to this frequency-domain representation. However, we can generalise the preceding example where we sum over the frequencies in the frequency domain multiplied by a complex harmonic at that frequency. In this case of a continuous frequency-domain the adaptation of this idea gives

$$f(t) = \int_{\mathbb{R}} F(\nu)e^{2\pi i\nu t}\,d\nu = \int_{-1}^{1} e^{2\pi i\nu t}\,d\nu = \frac{1}{2\pi t}(e^{2\pi i t} - e^{-2\pi i t}) = \frac{\sin(2\pi t)}{\pi t},$$

with the understanding that at $t = 0$ we use L'Hôpital's Rule to get $f(t) = 2$ (which also agrees with the integral computation). If you are new to the idea of a frequency-domain representation, this example will probably just seem strange and arbitrary at this point.

5. Let us turn the previous example around. Thus we define $\mathbb{W} = \mathbb{R}$ and $F\colon \mathbb{W} \to \mathbb{R}$ by $F(\nu) = \frac{\sin(2\pi\nu)}{\pi\nu}$. Were we to follow the above recipe for determining the corresponding time-domain signal then we would have

$$f(t) \text{ “ = ” } \int_{\mathbb{R}} F(\nu)e^{2\pi i\nu t}\,d\nu.$$

Note, however, that $\nu \mapsto F(\nu)e^{2\pi i\nu t}$ is actually not integrable. Therefore, it is not clear at all that one can use this idea of "summing over frequencies" to retrieve the time-domain signal. However, there is a sense, in fact, where this *does* work, and in this sense the corresponding time-domain signal is precisely

$$f(t) = \begin{cases} 1, & t \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

As it turns out, the reason that this computation can be made is that $F \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{R})$. Again, this likely seems merely mysterious if this is all new to you.

6. We again take the frequency-domain $\mathbb{W} = \mathbb{R}$ and now take $F(\nu) = 1$ for all $\nu \in \mathbb{R}$. Again, using the "summing over frequency" idea of determining the time-domain signal, we would have

$$f(t) \text{ `` = '' } \int_{\mathbb{R}} F(\nu)e^{2\pi i\nu t}\,d\nu = \int_{\mathbb{R}} e^{2\pi i\nu t}\,d\nu.$$

Now the function $\nu \mapsto e^{2\pi i\nu t}$ is *really* not integrable. For example, this function is not in $\mathsf{L}^{(p)}(\mathbb{R}, \mathbb{C})$ for any $p \in [1, \infty)$. Nonetheless, there is a sense in which the above integral can be computed. However, upon doing do what one gets is not a function in the usual sense. Indeed, what one gets is the Dirac delta-signal at $t = 0$, typically denoted δ_0. •

A few comments corresponding to these examples are in order.

1. We should reiterate that at this point we have simply not indicated how one systematically comes up in the examples above with the time-domain signals corresponding to the given frequency-domain representations. Instead, we are just presenting a slightly reasonable prescription for how one might do this in the examples we consider.

   The precise ideas behind these examples are presented in Chapters 12 and 13. Demystification, probably preceded by further mystification, will only occur at this time.

2. The situation in Example 5 above is explained in Section 13.3.

3. The situation in Example 6 above is explained in Sections 13.4 and **??**.

Now let us consider some attributes that a frequency-domain signal might possess. The support $\mathrm{supp}(F)$ of a frequency-domain signal F can be defined as for a regular function as in Definition 6.7.28(**??**).***missing stuff***

9.5.6 Definition (Band-limited, periodic frequency-domain signals) Let $\mathbb{W}$ be a frequency-domain, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $F\colon \mathbb{W} \to \mathbb{F}$ be a frequency-domain signal.

(i) The frequency-domain signal is ***band-limited*** if $\mathrm{supp}(F)$ is bounded.

(ii) The frequency-domain signal is ***periodic*** with ***period*** $W \in \mathbb{R}_{>0}$ if $F(\nu+W) = F(\nu)$ for all $\nu \in \mathbb{W}$.

(iii) The ***fundamental period*** of a periodic frequency-domain signal F is the smallest number W_0 for which F has period W_0, provided that this number is nonzero. •

The interpretations of these sorts of properties are not so easily made for frequency-domain signals, so these are to be merely thought of as providing terminology for later access.

9.5.3 Spaces of discrete frequency-domain signals

The spaces of signals we consider in the frequency-domain are, it turns out, the same as those for time-domain signals. Since we have discussed these in detail in Sections 8.2 and 8.3, with appropriate references to material in Chapters 5, 6, and 7, we only provide the notation here. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for a frequency-domain $\mathbb{W}$ all of the frequency-domain signal spaces we consider are subspaces of the $\mathbb{F}$-vector space $\mathbb{F}^{\mathbb{W}}$. The first batch of subspaces we consider are

$$\begin{aligned}
\mathsf{c}_{\mathrm{fin}}(\mathbb{W};\mathbb{F}) &= \{F \in \mathbb{F}^{\mathbb{W}} \mid F(\nu) = 0 \text{ for all but finitely many } \nu \in \mathbb{W}\};\\
\mathsf{c}_{0}(\mathbb{W};\mathbb{F}) &= \{F \in \mathbb{F}^{\mathbb{W}} \mid \text{ for each } \epsilon \in \mathbb{R}_{>0} \text{ there exists a finite subset } S \subseteq \mathbb{W}\\
&\qquad \text{such that } |F(\nu)| > \epsilon \text{ iff } \nu \in S\};\\
\ell^{\infty}(\mathbb{W};\mathbb{F}) &= \{F \in \mathbb{F}^{\mathbb{W}} \mid \sup\{|F(\nu)| \mid \nu \in \mathbb{W}\} < \infty\}.
\end{aligned}$$

For all of these $\mathbb{F}$-vector spaces we use the norm

$$\|F\|_{\infty} = \sup\{|F(\nu)| \mid \nu \in \mathbb{W}\}.$$

We also use the vector space

$$\ell^{p}(\mathbb{W};\mathbb{F}) = \Big\{F \in \mathbb{F}^{\mathbb{W}} \Bigm| \sum_{\nu \in \mathbb{W}} |F(\nu)|^{p} < \infty\Big\}$$

with the norm

$$\|F\|_{p} = \Big(\Omega \sum_{\nu \in \mathbb{W}} |f(\nu)|^{p}\Big)^{1/p}.$$

The properties of these frequency-domain signal spaces, and the relationships between them are discussed in Sections 8.2.2 and 8.2.3. The inclusion relations for them are discussed in Section 8.2.6.

We may also consider periodic frequency-domain signals, although the significance of these is less transparent than for periodic time-domain signals. That is, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, for an infinite frequency-domain $\mathbb{W}$, and for $\Omega \in \mathbb{R}_{>0}$, we define

$$\ell^{p}_{\mathrm{per},W}(\mathbb{W};\mathbb{F}) = \{F \in \mathbb{F}^{\mathbb{W}} \mid F \text{ is } W\text{-periodic}\}.$$

Recall that things are particularly simple in the discrete case since these spaces are actually finite-dimensional and independent of p. The norms considered on

$\ell^p_{\mathrm{per},W}(\mathbb{W};\mathbb{F})$ are

$$\|F\|_p = \Big(\Omega \sum_{\nu\in[0,W)\cap\mathbb{W}} |F(\nu)|^p\Big)^{1/p}, \qquad p\in[1,\infty),$$
$$\|F\|_\infty = \max\{|F(\nu)| \mid \nu\in[0,W)\cap\mathbb{W}\}.$$

9.5.4 Spaces of continuous frequency-domain signals

Let us quickly remind the reader of the notation for continuous frequency-domain signals; as in the discrete-frequency case, the notation is borrowed directly from the time-domain; the reader will want to read Section 8.3 carefully to remember the precise definitions of these spaces and some of their attributes. We let $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ and let $\mathbb{W}$ be a continuous frequency-domain. All frequency-domain signal spaces considered here are subspaces of $\mathbb{F}^{\mathbb{W}}$.

First we recall the spaces

$$\begin{aligned}
\mathsf{C}^0(\mathbb{W};\mathbb{F}) &= \{F\in\mathbb{F}^{\mathbb{W}} \mid F \text{ is continuous}\};\\
\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{W};\mathbb{F}) &= \{F\in\mathsf{C}^0(\mathbb{W};\mathbb{F}) \mid F \text{ has compact support}\};\\
\mathsf{C}^0_0(\mathbb{W};\mathbb{F}) &= \{F\in\mathsf{C}^0(\mathbb{W};\mathbb{F}) \mid \text{ for every } \epsilon\in\mathbb{R}_{>0} \text{ there exists a compact set}\\
&\qquad K\subseteq\mathbb{W} \text{ such that } \{\nu\in\mathbb{W} \mid |F(\nu)|\geq\epsilon\}\subseteq K\};\\
\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{W};\mathbb{F}) &= \{F\in\mathsf{C}^0(\mathbb{W};\mathbb{F}) \mid \text{ there exists } M\in\mathbb{R}_{>0} \text{ such that } |F(\nu)|\leq M\\
&\qquad \text{for all } \nu\in\mathbb{W}\}.
\end{aligned}$$

On all of these subspaces the norm we use is

$$\|F\|_\infty = \sup\{|F(\nu)| \mid \nu\in\mathbb{W}\},$$

noting that this norm is always defined by F in $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{W};\mathbb{F})$, $\mathsf{C}^0_0(\mathbb{W};\mathbb{F})$, or $\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{W};\mathbb{F})$.

We also have the spaces $\mathsf{L}^{(p)}(\mathbb{W};\mathbb{F})$ and $\mathsf{L}^p(\mathbb{W};\mathbb{F})$, $p\in[1,\infty]$, that are defined exactly as they are in the time-domain. The norms are

$$\|F\|_p = \Big(\int_{\mathbb{W}} |F|^p \,\mathrm{d}\lambda\Big)^{1/p}, \qquad p\in[1,\infty),$$
$$\|F\|_\infty = \operatorname{ess\,sup} |F(\nu)| \nu\in\mathbb{W}.$$

We refer the reader to Section 8.3.3 for the details of these constructions.

One also has the spaces of periodic continuous-frequency signals $\mathsf{C}^0_{\mathrm{per},\Omega}(\mathbb{R};\mathbb{F})$, $\mathsf{L}^{(p)}_{\mathrm{per},\Omega}(\mathbb{R};\mathbb{F})$, and $\mathsf{L}^p_{\mathrm{per},\Omega}(\mathbb{R};\mathbb{F})$, $p\in[1,\infty]$. We refer to Section 8.3.4 for details.

Finally, we may also make reference to some of the classes of signals discussed in the time-domain in Section 8.3.7, but in the frequency-domain.

Section 9.6

An heuristic introduction to the four Fourier transforms

In this section, with the preceding sections as motivation, we provide preliminary definitions of some of the transforms we will introduce. We do this so as to acquaint the reader with some of the issues that go into the definitions. We will be neither rigorous nor complete. The rough introduction we give will (hopefully) provide some motivation for the presentation in Chapters 12, 13, and 14 where we discuss the mathematical tools necessary to talk about frequency-domain representations in a rigorous way. As with all heuristic approaches, there will be a matter of taste involved. We make no claim that our heuristics are better than any others. Indeed, we permit, and even encourage, the reader to look through as many alternate points of view as possible since these will all contribute something. Also, we do not advise the reader to take the remainder of this section too much to heart. Everything done here will be done at great length and with great care in subsequent chapters.

Do I need to read this section? There is no significant technical content in this section, but perhaps it might be insightful to some readers. Moreover, while few of our computations are rigorous, we do arrive at correct formulae for all of the frequency-domain transforms that we will encounter in Chapters 12, 13, and 14. This too may be helpful. •

9.6.1 Periodic continuous-time signals

We begin with the situation that is most easily motivated. The situation is that of a signal $f\colon \mathbb{R} \to \mathbb{C}$ that is T-periodic. For simplicity, we consider $\mathbb{C}$-valued signals since this includes $\mathbb{R}$-valued signals as a special case. We wish to write f as a possibly infinite linear combination of "simple" T-periodic signals. The simplest sort of T-periodic signal are those that are harmonic, $t \mapsto \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$, $n \in \mathbb{Z}$. So let us crazily suppose that our objective is to write

$$f(t) = \sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} \tag{9.3}$$

for appropriate coefficients $c_n(f)$, $n \in \mathbb{Z}$. This is sometimes called a ***harmonic expansion*** of f. Now, there is no reason whatsoever to expect a strategy like this will succeed. However, as we shall see in Chapter 12, this actually *is* a reasonable strategy. In any event, we will proceed as if this makes sense. One must determine the coefficients $c_n(f)$, $n \in \mathbb{Z}$. Let us do this in a rather relaxed way. Using the (easily verified) relation

$$\int_0^T \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = \begin{cases} T, & m = n, \\ 0, & m \neq n, \end{cases}$$

we compute

$$\begin{aligned}
&\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} c_n(f)\int_0^T \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = Tc_m(f)\\
\implies\quad & c_m(f) = \frac{1}{T}\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t,
\end{aligned}$$

making the assumption that the infinite sum can be swapped with the integral (generally, it cannot be).

At this point in our discussion we merely think of the preceding formulae as coming from an attempt to make sense of the attempt to write f as an infinite sum of harmonics of period T. Let us now take an alternative point of view towards this. The signal f is a T-periodic time-domain signal. The coefficient $c_n(f)$ is the coefficient of the harmonic of frequency nT^{-1} (and angular frequency $2\pi nT^{-1}$). Thus we might think of $c_n(f)$ as being the "amount" of the signal f at the frequency nT^{-1}. This points to the frequency-domain representation of f as being the signal $nT^{-1}\mapsto c_n(f)$ on $\mathbb{Z}(T^{-1})$. Thus we have a mapping $\mathscr{F}_{\mathrm{CD}}$ from T-periodic continuous-time signals to discrete-frequency signals with fundamental frequency T^{-1}. Explicitly we have

$$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t.$$

(Note that we have lost a factor of $\frac{1}{T}$. This is of no consequence and is really a matter of convention.) The subscript "CD" is intended to signify the fact that the mapping takes a continuous-time signal and returns a discrete-frequency signal. Now the expression (9.3) can be thought of as an inverse to $\mathscr{F}_{\mathrm{CD}}$ in that it takes the frequency-domain signal (represented by the coefficients $c_n(f)$, $n\in\mathbb{Z}$) and returns the time-domain signal. More generally, if we have $F\colon \mathbb{Z}(T^{-1})\to\mathbb{C}$ we may define

$$\mathscr{F}_{\mathrm{CD}}^{-1}(F)(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} F(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

(Note that we have recovered the factor of $\frac{1}{T}$ here.) Summarising:

$$\boxed{\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) &= \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t,\\
\mathscr{F}_{\mathrm{CD}}^{-1}(F)(t) &= \frac{1}{T}\sum_{n\in\mathbb{Z}} F(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.
\end{aligned}} \tag{9.4}$$

There are lots of interesting questions here. For example, the following questions naturally arise.

1. Can our machinations be made to make sense?

2. Is it true that $\mathscr{F}_{\mathrm{CD}}^{-1}$ is really the inverse of $\mathscr{F}_{\mathrm{CD}}$? That is to say, given a T-periodic signal f is it true that
$$\mathscr{F}_{\mathrm{CD}}^{-1} \circ \mathscr{F}_{\mathrm{CD}}(f)(t) = f(t)$$
for almost every t?
3. Are there useful relationships between f and $\mathscr{F}_{\mathrm{CD}}(f)$?
4. Are there possibilities for choosing the coefficients in the expression (9.3) other than the one we give?

9.6.2 Aperiodic continuous-time signals

In this section we adapt the analysis of the preceding section to signals $f\colon \mathbb{R} \to \mathbb{C}$ that are not necessarily periodic. The ideas here are not as easy to motivate as they are in the periodic case. In the periodic case, if you squint your eyes you might be able to convince yourself that writing a periodic signal as an infinite sum of harmonic is feasible. The corresponding statement for aperiodic signals is not so easy to dream up and, moreover, the final answer seems decidedly less believable than the already unbelievable situation in the periodic case. Nonetheless, we shall proceed apace, our idea being to use the development from the preceding section as a starting point, and using some bogus limiting argument. If the development in the preceding section was a little sloppy, in this section, it will be downright outrageous. Nevertheless, it is worthwhile to consider the limiting approach we take here for signals that are not periodic to see the connection between the discrete frequency representation in the preceding section and what will turn out to be a continuous frequency representation.

We consider a signal $f\colon \mathbb{R} \to \mathbb{C}$ that is not necessarily periodic. We do, however, assume that f is integrable. Moreover, we suppose for simplicity that $f(t)$ decays to zero as $t \to \infty$. We adopt the following approach to attempt to derive the frequency representation for f, using as a starting point the development of the preceding section. We will restrict f to $[-\frac{T}{2}, \frac{T}{2}]$ and consider the T-periodic signal f_T that is equal to f on $[-\frac{T}{2}, \frac{T}{2}]$ (see Figure 9.16). For f_T we write
$$f_T(t) = \sum_{n \in \mathbb{Z}} c_n(f_T) \mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}},$$
with the equals sign being taken with an appropriate degree of skepticism, and with
$$c_n(f_T) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \mathrm{e}^{-2\pi \mathrm{i} n \frac{t}{T}} \, \mathrm{d}t.$$
Note that we have made some adjustments to the formulae in the preceding section to take into account the fact that the interval is $[-\frac{T}{2}, \frac{T}{2}]$ and not $[0, T]$. This will be

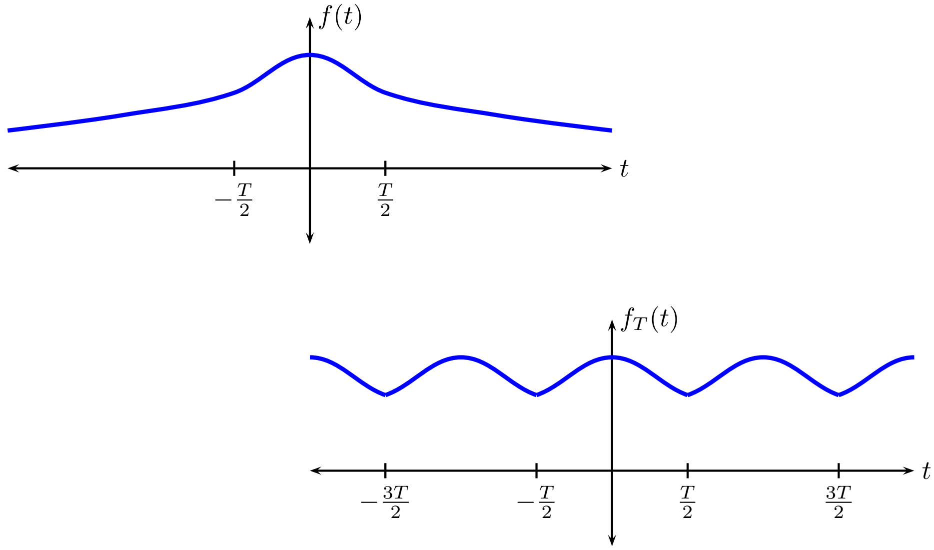


Figure 9.16 The signal f_T constructed from f

done in a systematic way in Chapter 12. We define $\Delta\nu = T^{-1}$ so that we may write

$$\begin{aligned} f_T(t) &= \sum_{n\in\mathbb{Z}} c_n(f_T)\mathrm{e}^{2\pi\mathrm{i}n\Delta\nu t}, \\ c_n(f_T) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu t}\,\mathrm{d}t. \end{aligned} \tag{9.5}$$

Now fix $\nu \in \mathbb{R}$ and let $n_\nu \in \mathbb{Z}$ have the property that $\nu \in [n_\nu\Delta\nu, (n_\nu + 1)\Delta\nu)$. Then define

$$\mathscr{F}(f;T)(\nu) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\mathrm{e}^{-2\pi\mathrm{i}n_\nu\Delta\nu t}\,\mathrm{d}t.$$

Note that as $T \to \infty$ we have $n_\nu\Delta\nu \to \nu$, and so we may define

$$\mathscr{F}(f)(\nu) = \lim_{T\to\infty} \mathscr{F}(f;T)(\nu) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t.$$

In a similar manner, for $n \in \mathbb{Z}$ let ν_n have the property that $\nu_n = n\Delta\nu$. We then have

$$\sum_{n\in\mathbb{Z}} c_n(f_T)\mathrm{e}^{2\pi\mathrm{i}n\Delta\nu t} = \sum_{n\in\mathbb{Z}} \frac{1}{T}\mathscr{F}(f;T)(\nu_n)\mathrm{e}^{\mathrm{i}\nu_n t} = \sum_{n\in\mathbb{Z}} \Delta\nu\mathscr{F}(f;T)(\nu_n)\mathrm{e}^{2\pi\mathrm{i}\nu_n t}.$$

Taking the limit as $T \to \infty$, or equivalently as $\Delta\nu \to 0$, the sum becomes an integral and we have

$$\lim_{T\to\infty} \sum_{n\in\mathbb{Z}} c_n(f_T)\mathrm{e}^{2\pi\mathrm{i}n\Delta\nu t} = \int_{-\mathbb{R}} \mathscr{F}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu.$$

Summarising, we have the relationships

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} \mathscr{F}(f)(\nu) \mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu, \\ \mathscr{F}(f)(\nu) &= \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t, \end{aligned} \tag{9.6}$$

which tell us what the relationships (9.5) look like when $T \to \infty$. Again, the equals signs should be regarded with extreme suspicion.

Let us now develop a "transform" point of view of the preceding discussion, just as we did in the preceding section. Again, the idea is that $\mathscr{F}(f)(\nu)$ tells us the "frequency content" of f at the frequency ν. Thus we think of the mapping $\mathscr{F}_{\mathrm{CC}}$ that sends a time-domain signal f to its frequency-domain representation by the formula

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t.$$

The subscript "CC" indicates that the transform sends a continuous-time signal to a continuous-frequency signal. The "inverse" of $\mathscr{F}_{\mathrm{CC}}$ then takes a frequency-domain signal and returns a time-domain signal by the formula

$$\mathscr{F}_{\mathrm{CC}}^{-1}(F)(t) = \int_{\mathbb{R}} F(\nu) \mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu.$$

Summarising:

$$\boxed{\begin{aligned} \mathscr{F}_{\mathrm{CC}}(f)(\nu) &= \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t, \\ \mathscr{F}_{\mathrm{CC}}^{-1}(F)(t) &= \int_{\mathbb{R}} F(\nu) \mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu. \end{aligned}} \tag{9.7}$$

The reader should stare for at the formula alongside (9.4) for sufficiently long that they can come to see the relationship between the ideas being expressed in each case.

The interesting questions here include the following.

1. Can our machinations be made to make sense?
2. Is it true that $\mathscr{F}_{\mathrm{CC}}^{-1}$ is really the inverse of $\mathscr{F}_{\mathrm{CC}}$? That is to say, given a signal f is it true that
$$\mathscr{F}_{\mathrm{CC}}^{-1} \circ \mathscr{F}_{\mathrm{CC}}(f)(t) = f(t)$$
for almost every t?
3. Are there useful relationships between f and $\mathscr{F}_{\mathrm{CC}}(f)$?
4. Are there possibilities for the expression (9.6) other than the one we give?

We shall study $\mathscr{F}_{\mathrm{CC}}$ and its inverse in detail in Chapter 13.

9.6.3 Periodic discrete-time signals

We now mimic the above procedure, but for discrete-time signals. First we consider the periodic case. Thus we suppose that we have a signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined on the discrete-time domain with sampling interval Δ. We assume that the signal is periodic with period $N\Delta$; thus $f(t + N\Delta) = f(t)$ for all $t \in \mathbb{Z}(\Delta)$. For our heuristic introduction we shall attempt to make use of the preceding discussion about continuous-time signals. To do this, we think of the discrete-time signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ as being equivalent to the continuous-time generalised signal

$$g_f(t) = \sum_{j\in\mathbb{Z}} f(j\Delta)\delta_{j\Delta}.$$

The idea is that a discrete-time signal gives an "impulse" at each of its discrete points. This possibly seems reasonable, but even if it does not we proceed as if it does. Motivated by our methodology of Section 9.6.1, we seek constants $c_n(f)$ with the property that

$$g_f(t) = \sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{N\Delta}}.$$

Proceeding as in Section 9.6.1 we have

$$\begin{aligned} c_n(f) &= \frac{1}{N\Delta}\int_0^{N\Delta} g_f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{N\Delta}}\,\mathrm{d}t \\ &= \frac{1}{N\Delta}\int_0^{N\Delta} \Big(\sum_{j\in\mathbb{Z}} f(j\Delta)\delta_{j\Delta}\Big)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{N\Delta}}\,\mathrm{d}t \\ &= \frac{1}{N\Delta}\sum_{j=0}^{N-1} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\frac{j}{N}}, \end{aligned}$$

using the definition of the generalised signals $\delta_{j\Delta}$. Note that in computing the integral using the properties of $\delta_{j\Delta}$ we have included δ_0 but not $\delta_{N\Delta}$. This can be justified by noting that the fundamental domain of g_f is $[0, N\Delta)$, i.e., the right endpoint is not included in the fundamental domain. One can readily check that $c_{n+N}(f) = c_n(f)$ for all $n \in \mathbb{Z}$. Thus we have the fundamental relations

$$\begin{aligned} g_f(t) &= \sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{N\Delta}}, \\ c_n(f) &= \frac{1}{N\Delta}\sum_{j=0}^{N-1} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\frac{j}{N}}. \end{aligned}$$

Now we wish to recover a formula for f, not g_f, from the first of these formulae. There is a little magic to this that will only be justified in ***missing stuff***. The first observation is that periodicity of the coefficients $c_n(f)$—the fact that $c_{n+N}(f) = c_n(f)$ for all $n \in \mathbb{Z}$—implies that

$$\sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{N\Delta}} = \sum_{n=0}^{N-1} c_n(f)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{N\Delta}}\Big(\sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}k\frac{t}{\Delta}}\Big).$$

In Example **??–??** we shall see that

$$\sum_{k\in\mathbb{Z}} e^{2\pi i k\frac{t}{\Delta}} = \sum_{k\in\mathbb{Z}_{>0}} \delta_{k\Delta}.$$

Note that the left-hand side is clearly senseless as a function, but the right-hand side says that it is a distribution in any case. Forgetting the possibility of our doing anything senseless, we simply have

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{2\pi i n\frac{t}{N\Delta}} = \sum_{n=0}^{N-1} c_n(f)e^{2\pi i n\frac{t}{N\Delta}}\Big(\sum_{k\in\mathbb{Z}_{>0}} \delta_{k\Delta}\Big).$$

"Evaluating" this at $t = j\Delta$ gives

$$g_f(j\Delta) = f(j\Delta) = \sum_{n=0}^{N-1} c_n(f)e^{2\pi i n\frac{j}{N}}.$$

This gives us our desired representation of the periodic discrete-time signal f.

Now let us apply the transform point of view to the discussion. In this case we note that we have mapped a $N\Delta$-periodic discrete-time signal defined on $\mathbb{Z}(\Delta)$ to a Δ^{-1}-periodic frequency-domain signal defined on $\mathbb{Z}(\frac{1}{N\Delta})$. Thus the mapping is one between two N-dimensional vector spaces. We denote the time-domain to frequency-domain map by $\mathscr{F}_{\mathrm{DD}}$ and its inverse by $\mathscr{F}_{\mathrm{DD}}^{-1}$. Explicitly we have

$$\boxed{\begin{aligned} \mathscr{F}_{\mathrm{DD}}(f)(\tfrac{n}{N\Delta}) &= \Delta\sum_{j=0}^{N-1} f(j\Delta)e^{-2\pi i n\frac{j}{N}},\\ \mathscr{F}_{\mathrm{DD}}^{-1}(F)(j\Delta) &= \frac{1}{N\Delta}\sum_{n=0}^{N-1} F(\tfrac{n}{N\Delta})e^{2\pi i\frac{n}{N}j}. \end{aligned}} \tag{9.8}$$

(Note that the factor of $\frac{1}{N\Delta}$ is moved about freely, just as was the factor for $\frac{1}{T}$ for $\mathscr{F}_{\mathrm{CD}}$.) The subscript "DD" indicates that the transform is from a discrete-time signal to a discrete-frequency signal.

Let us make some observations about the formulae (9.8).

1. Can our machinations using generalised distributions be made to make sense?
2. Is it true that $\mathscr{F}_{\mathrm{DD}}^{-1}$ is really the inverse of $\mathscr{F}_{\mathrm{DD}}$? That is to say, given an $N\Delta$-periodic discrete-time signal f is it true that

$$\mathscr{F}_{\mathrm{DD}}^{-1}\circ\mathscr{F}_{\mathrm{DD}}(f)(j\Delta) = f(j\Delta)$$

   for every $j \in \{0, 1, \dots, N-1\}$? Note that this is merely a question in finite-dimensional linear algebra, whereas this question is actually extremely involved for $\mathscr{F}_{\mathrm{CD}}$ and $\mathscr{F}_{\mathrm{CC}}$.
3. Since the coefficients $\{f_j\}_{j\in\mathbb{Z}}$ and $\{c_n(f)\}_{n\in\mathbb{Z}}$ are periodic with period N, the relationships between them are actually simple computationally since all sums are finite. Is there an efficient way to perform these computations?

   The transform $\mathscr{F}_{\mathrm{DD}}$ will be studied in detail in Section 14.2.

9.6.4 Aperiodic discrete-time signals

Now we consider again a discrete-time signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$, but now we do not assume that f is periodic. We still convert f to a continuous-time generalised signal g_f and write

$$g_f(t) = \sum_{j\in\mathbb{Z}} f(j\Delta)\delta_{j\Delta}.$$

Analogous with the construction of Section 9.6.2, we assume that the sequence $\{f(j\Delta)\}_{j\in\mathbb{Z}_{>0}}$ is absolutely summable. Thus we may compute, as in Section 9.6.2,

$$\mathscr{F}(f)(\nu) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \sum_{j\in\mathbb{Z}} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}\nu j\Delta}.$$

Note the similarity between this formula as a function of ν and the expression (9.3) for a periodic signal as a function of t. Thus we see that in (9.3) t is replaced with ν and that T is replaced with Δ^{-1}, and there is also a sign change in the exponential. In any case, at least if we are ruthless with our limits, we can compute

$$\int_0^{\Delta^{-1}} \mathscr{F}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu k\Delta}\,\mathrm{d}\nu = \sum_{j\in\mathbb{Z}} f(j\Delta)\int_0^{\Delta^{-1}} \mathrm{e}^{2\pi\mathrm{i}\nu k\Delta}\mathrm{e}^{-2\pi\mathrm{i}\nu j\Delta}\,\mathrm{d}\nu = \Delta^{-1}f(k\Delta)$$

$$\implies \quad f(k\Delta) = \Delta\int_0^{\Delta^{-1}} \mathscr{F}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu k\Delta}\,\mathrm{d}\nu.$$

Now let us give our transform interpretation of the preceding computations. Our frequency-domain representation of the discrete time-domain signal f is a continuous-frequency signal. Thus we denote the corresponding transform by $\mathscr{F}_{\mathrm{DC}}$ with inverse $\mathscr{F}_{\mathrm{DC}}^{-1}$. Our computations have shown that

$$\boxed{\begin{aligned}\mathscr{F}_{\mathrm{DC}}(f)(\nu) &= \Delta\sum_{j\in\mathbb{Z}} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}j\Delta\nu},\\ \mathscr{F}_{\mathrm{DC}}^{-1}(F)(j\Delta) &= \int_0^{\Delta^{-1}} F(\nu)\mathrm{e}^{2\pi\mathrm{i}j\Delta\nu}\,\mathrm{d}\nu.\end{aligned}} \tag{9.9}$$

(The factor of Δ is manipulated freely as is the factor of $\frac{1}{T}$ in the definition of $\mathscr{F}_{\mathrm{CD}}$.) As usual, there are questions to be asked here, including these.

1. Can our machinations using generalised distributions be made to make sense?
2. Is it true that $\mathscr{F}_{\mathrm{DC}}^{-1}$ is really the inverse of $\mathscr{F}_{\mathrm{DC}}$? That is to say, given a discrete-time signal f is it true that

   $$\mathscr{F}_{\mathrm{DC}}^{-1}\circ\mathscr{F}_{\mathrm{DC}}(f)(j\Delta) = f(j\Delta)$$

   for every $j \in \mathbb{Z}_{>0}$?
3. Are there useful relationships between f and $\mathscr{F}_{\mathrm{DC}}(f)$?

   We shall study the transform $\mathscr{F}_{\mathrm{DC}}$ in detail in Section 14.1.

9.6.5 Notes

The idea of considering the four transforms presented in Chapters 12, 13, and 14 as being different versions of the same idea we take from [**HK/RS:91**]. While this idea is certainly known and understood by everyone who works with these things, the unified presentation of these, and the unified notation $\mathscr{F}_{\mathrm{CD}}$, $\mathscr{F}_{\mathrm{CC}}$, $\mathscr{F}_{\mathrm{DD}}$, and $\mathscr{F}_{\mathrm{DC}}$ we borrow from **HK/RS:91**. This way of presenting the subject seems to us to simply be the correct one, and we wish to acknowledge **HK/RS:91** for making this clear.

The ideas we describe in Section 9.6.1 form the beginning of the subject of Fourier series, and was proposed by **JBJF:22*missing stuff*** (1768–1830) in the course of the study of heat conduction in solids. Fourier was trying to understand the temperature distribution in a rod as depicted in Figure 9.17. Fourier's idea was to

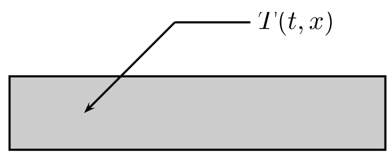


Figure 9.17 Temperature distribution in a rod

write the temperature in the rod in the form

$$T(t,x) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty}\Big(a_n(t)\cos(\tfrac{2\pi n x}{\ell}) + b_n(t)\sin(\tfrac{2\pi n x}{\ell})\Big),$$

where ℓ is the length of the rod. To many his colleagues at the time this seemed a hopeless idea. One of the reasons for this was that it was thought to be infeasible to write an arbitrary function as a series of continuous functions, the thinking being that a convergent series of continuous functions should converge to a continuous function. Since physically there seemed no reason to suppose that the temperature distribution in the rod was continuous, Fourier's idea was thought to be doomed. However, Fourier has since been vindicated, and indeed his idea has spawned harmonic analysis, one of the most important areas in mathematics and applied mathematics.

Exercises

9.6.1 Show that the following four sets of signals are linearly independent:

$$\begin{aligned}
&\{t \mapsto \mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}} \mid n \in \mathbb{Z}\};\\
&\{t \mapsto \mathrm{e}^{2\pi \mathrm{i} \nu t} \mid \nu \in \mathbb{R}\};\\
&\{k\Delta \mapsto \mathrm{e}^{2\pi \mathrm{i} n \frac{k}{N}} \mid n \in \mathbb{Z}\};\\
&\{k\Delta \mapsto \mathrm{e}^{2\pi \mathrm{i} \nu k\Delta} \mid \nu \in \mathbb{R}\}.
\end{aligned}$$

The first two sets are comprised of continuous-time signals and the second two sets of signals are comprised of discrete-time signals.
Hint:

1. *Note that it suffices to show linear independence of the second and fourth sets. Why?*
2. *Note that it suffices for the second and fourth sets to consider $\nu \in \mathbb{Q}$. Why?*

These observations allow one to show linear independence of the sets

$$\{t \mapsto e^{2\pi i \nu t} \mid \nu \in \mathbb{Q}\},$$
$$\{k\Delta \mapsto e^{2\pi i \nu k\Delta} \mid \nu \in \mathbb{Q}\}.$$

Chapter 10

Distributions in the time-domain

It is surprisingly often that one naturally encounters signals that are not really signals, but limits of signals in a certain sense that is not covered by the theory of continuous-time signals spaces developed in the preceding chapter. The way we will deal with these signals is through the use of distributions or, as they are sometimes known, "generalised signals." In these volumes we shall provide a fairly complete presentation of how distribution theory arises in transform theory and system theory. It *is* possible to do many things without knowing the details of the theory of distributions. But the fact of the matter is that there comes a time when it is harder to *not* use distributions than it is to use them. Therefore, we elect to use them and to understand how they interact with more mundane matters.

In this chapter we shall encounter a number of different varieties of distributions, and these classes are related to one another in sometimes nontrivial ways. The first class of distributions we consider, those we simply call "distributions," has some more or less straightforward motivation that we provide in Section 10.1. However, the other classes of distributions might seem fairly unmotivated when we first encounter them. This is because these classes of distributions are designed to work with various sorts of Fourier transforms that we will encounter in subsequent chapters, mainly Chapters 12, 13, and 14. Therefore, while we present the properties of these classes of distributions in this chapter, we will not understand the utility of some of these until we get to transform theory. The reader is advised, then, to maybe read some of the sections in this chapter as preliminary to the associated Fourier transform application.

In this chapter we give a self-contained treatment of distributions in the time-domain. However, there is nothing that links distributions with time, *per se*, and so one can also talk about generalised frequency-domain signals.

Do I need to read this chapter? It is a healthy thing to at least know what the delta-signal is. So read enough to understand this as a bare minimum. •

Contents

Section 10.1

Motivation for distributions

The most difficult aspect of the theory of distributions is not how to use distributions, but to understand what you are doing when you use them. Part of arriving at an understanding of what distributions are and are not involves understanding why the definition of a distribution is as it is. In this section we do this by providing some situations where the need for distributions reveals itself in a natural way.

Do I need to read this section? If you are reading this chapter, then this section may be interesting, although it does not contain much in the way of technical information. •

10.1.1 The impulse

There are a variety of ways one can motivate the introduction of distributions, or as they are sometimes called, generalised signals. One of the most natural ways to do this is through their use in differential equations. We do this in a concrete context.

Consider a mass m oscillating on a spring as shown in Figure 10.1. Suppose that

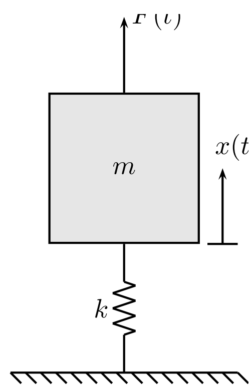


Figure 10.1 A mass on a spring

we measure the displacement of the mass which we denote by y. The governing equations for the system are then

$$m\ddot{x}(t) + kx(t) = F(t), \qquad y(t) = x(t),$$

where m is the mass, k is the stiffness constant of the spring (we assume a linear spring), and F is a force applied to the mass as indicated in the figure. We now consider a special sort of force F. Prior to time $t = 0$ we suppose the system to be in equilibrium. At $t = 0$ we apply a constant force M for duration Δ and thereafter the force is zero. That is, we consider the force $F_{M,\Delta}$ defined by

$$F_{M,\Delta}(t) = \begin{cases} M, & t \in [0, \Delta], \\ 0, & \text{otherwise.} \end{cases}$$

The action of the force $F_{M,\Delta}$ is

$$\|F_{M,\Delta}\|_1 = \int_{\mathbb{R}} |F_{M,\Delta}(t)| \, \mathrm{d}t = M\Delta.$$

Now let $\mathscr{F}_A$ be the collection of all signals with action A. Thus, for any $F_{M,\Delta} \in \mathscr{F}_A$ we have $M\Delta = A$. Now consider the sequence $(F_j)_{j\in\mathbb{Z}}$ of forces in $\mathscr{F}_A$ such that $F_j = F_{Aj,\frac{1}{j}}$. Thus, as $j \to \infty$, these forces are applied for shorter time, but have larger magnitude, subject to the constraint that they have equal action. As a sequence of measurable signals taking values in $\overline{\mathbb{R}}$ we have

$$F_\infty(t) \triangleq \lim_{j\to\infty} F_j(t) = \begin{cases} \infty, & t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that there is no problem in defining this limit as a pointwise limit of measurable $\overline{\mathbb{R}}$-valued signals. Also, each of the signals F_j is a perfectly nice signal that will give rise to a response $t \mapsto x_j(t)$ of the mass.

The question we wish to ask is this. Is $x_\infty \triangleq \lim_{j\to\infty} x_j$ the response of the system to the force $\lim_{j\to\infty} F_j$? Without actually going through the details (the reader can do this in Exercises 10.1.1 and 10.2.12), let us see if we can say something about the two things we are trying to compare.

1. *The limit of the responses:* Although the force F_j is being applied for a duration tending to zero as $j \to \infty$, the magnitude of F_j during its application tends to infinity. Thus it is not clear what wins the race between shorter ad shorter duration and larger and larger magnitude. In fact, we have cooked things so that there is a tie. Were the duration to shrink at a faster rate than the magnitude grew, then the response would tend to zero as $j \to \infty$. Were the magnitude to grow at a faster rate that the duration shrunk, then the response would blow up as $j \to \infty$. It turns out that in this case the limit is well-defined and is nonzero. The reader can explore this as Exercise 10.1.1.
2. *The response of the limit:* This is easier to be clear about. We have an inhomogeneous linear scalar differential equation whose right-hand side is zero except on a set of measure zero. Solutions to such differential equations are obtained by integrating the right-hand side, cf. Exercises 10.1.1 and 10.2.12, and so the resulting response will be zero.

The point is this: The limit response x_∞ is not the response to the limit force F_∞.

The problem, it turns out, is the way we take the limit $\lim_{j\to\infty} F_j$. If we take this limit in the right space (for example, *not* in the space of measurable signals), then it turns out that the limit of the responses *is* the response of the limit forces.

However, the price you pay is that the space in which one works is not a space of signals in the usual sense, but is the space of distributions or generalised signals.

Supposing $A = 1$ for concreteness, the generalised signal $\lim_{j\to\infty} F_j$ is the ubiquitous "delta-signal" which we denote by δ_0. We shall say more about δ_0 in Section 10.1.4.

10.1.2 Differentiating nondifferentiable signals

In the preceding section we saw that something that was not a signal arose in a natural way as a limit of signals. In this section, using the same example, we will see that something that is not a signal can arise by a natural desire to differentiate something that is not differentiable.

We again consider the mass/spring system depicted in Figure 10.1. Now we suppose that we measure the velocity of the mass, and we denote this by y. Then the equations governing the system are

$$m\ddot{x}(t) + kx(t) = F(t), \quad y(t) = \dot{x}(t).$$

For a given force F, one can obtain y by first solving for x and then differentiating to get y, However, it is also natural to directly obtain a differential equation for y. To arrive at this, we differentiate the x-equation to get

$$m\ddot{y}(t) + ky(t) = \dot{F}(t).$$

Now suppose that for $t < 0$ the mass is at rest and we take $F(t) = 1(t)$. We are then confronted with understanding what one might mean by $\dot{1}$. Since 1 is not differentiable at $t = 0$ one does not have recourse to the usual notion of differentiation as described in Section 3.2. To try to understand how to differentiate 1 at $t = 0$ we adopt an alternate property of the derivative. Suppose that $\phi \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ is continuously differentiable with a derivative also in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$. We then might speculate, using integration by parts, that were we able to define $\dot{1}$ it would satisfy

$$\begin{aligned}\int_{\mathbb{R}} \dot{1}(t)\phi(t)\,\mathrm{d}t &= 1(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} 1(t)\dot{\phi}(t)\,\mathrm{d}t \\ &= \phi(\infty) - \int_0^\infty \dot{\phi}(t)\,\mathrm{d}t \\ &= \phi(\infty) - \phi(\infty) + \phi(0) = \phi(0).\end{aligned}$$

(Here we have used the fact that since $\phi, \dot{\phi} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$, we have $\lim_{|t|\to\infty} \phi(t) = 0$ by Exercise 8.3.18.) That is to say, we might take as the *definition* of $\dot{1}$ the signal having the property that for any continuously differentiable signal $\phi\colon \mathbb{R} \to \mathbb{R}$ with the property that it and its derivative are integrable, we have

$$\int_{\mathbb{R}} \dot{1}(t)\phi(t)\,\mathrm{d}t = \phi(0).$$

Now one might ask whether there is an integrable signal having this property. It is actually not difficult to show that the existence of such a signal is an impossibility. We recall from Definition 5.9.19 the definition of a locally integrable signal.

10.1.1 Proposition (Nonexistence of a signal having the properties of the delta-signal) *There exists no locally integrable signal* $\delta_0\colon \mathbb{R} \to \mathbb{R}$ *such that* $\int_{\mathbb{R}} \delta_0(t)f(t)\,dt = f(0)$ *for every* $f \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{R})$.

Proof Suppose that δ_0 is such a signal and let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be the sequence of signals

$$f_j(t) = \begin{cases} 1 + jt, & t \in [-\frac{1}{j}, 0], \\ 1 - jt, & t \in (0, \frac{1}{j}], \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathbb{R}} \delta_0(t) f_j(t)\,\mathrm{d}t = 1, \qquad j \in \mathbb{Z}_{>0}.$$

However, by the Dominated Convergence Theorem,

$$\lim_{j\to\infty} \int_{\mathbb{R}} \delta_0(t) f_j(t)\,\mathrm{d}t = \int_{\mathbb{R}} \delta_0(t) \lim_{j\to\infty} f_j(t)\,\mathrm{d}t = 0,$$

and so we arrive at a contradiction. ■

In Section 10.1.4 we shall have more to say about this object δ_0, and in Section 10.5.7 we shall be rigorous about just what δ_0 is, and ways to understand it.

10.1.3 What should a good theory of distributions achieve?

Having now motivated why things that are not signals can arise in a natural way, we are now in a position to wonder whether there is in fact a larger class of mathematical objects one should be considering other than signals. Obviously there is (e.g., the "mathematical universe"), so we should try to make our objective well-defined. What properties might we want our fantasy "super signal theory" to have?

Here is a possible wish list.

1. The set of "super signals" should be a vector space in a natural way. The justifications for this are just as for they were for signal spaces in Section 8.1.7.
2. The set of "super signals" should contain all reasonable signals. Now, what should reasonable mean? Well, clearly any signal from any of the spaces $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, $p \in [1,\infty]$, should be in our set. But these are far from enough. For example, one would want to include signals like $t \mapsto t^2$ if possible. More generally, one might like to have $\mathsf{C}^0(\mathbb{R};\mathbb{F})$ be included in our set of super signals. Actually, what we *really* want to allow are all locally integrable signals. These seem like a pretty reasonable and reasonably large class of signals. Thus, let us demand that $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ be in our set of "super signals."
3. Our set should contain the object corresponding to the impulse from Section 10.1.1. That is to say, our theory should allow us to define an object and a differential equation theory for using that object that would allow us to retrieve the limit response from Section 10.1.1 as a limit of the forces from that section.

4. In Section 10.1.2 we saw that it might be useful to be able to differentiate (in some sense) nondifferentiable signals. Thus we ask that all of our "super signals" be differentiable (in some sense). Heck, we may as well ask that they be infinitely differentiable!
5. For spaces of signals, we argued in Section 8.1.7 that we should expect there to be some manner of defining convergent sequences. We shall ask the same for our space of "super signals."

10.1.4 Some caveats about what is written about the delta-signal

The reader wishing to load up on intellectual rubbish need do no more than do a Google search for "delta function." What will result is a clear exhibition of the confusion in much of the scientific community regarding just what the delta-signal "is." Some will actually give the definition of the delta-signal as

$$\delta_0(t) = \begin{cases} \infty, & t = 0, \\ 0, & \text{otherwise.} \end{cases} \qquad \textbf{\textit{(This is not a real equation!)}}$$

Then it will be pointed out that this is not really a signal. However, it *is* a perfectly well-defined signal, albeit one taking values in $\overline{\mathbb{R}}$. But there is still nothing really wrong with it: it is in the same equivalence class as the zero signal under equivalence of signals which differ on sets of measure zero. Then there will follow some rules for using the delta-signal. The details of these rules will vary. But all such descriptions suffer from ambiguity to the extent that it is very easy to use them to perform wrong computations. For folks who think of the delta-signal in this way, they use it as a convenient tool. If they make an error using it, this typically shows up by obtaining results that are incoherent for the problem. Then the idea is that one goes back to the manipulations of the delta-signal and says, "Oh, this step must have been wrong," even though the step is in accord with the rules laid out. This is embarrassingly unscientific!

Why not instead really learn what the delta-signal is and how to really use it. It is not that difficult!

10.1.5 Notes

[LS:50]

Exercises

10.1.1 Let us revisit the mass/spring example of Section 10.1. The governing differential equation is

$$m\ddot{x}(t) + kx(t) = F(t). \tag{10.1}$$

For simplicity, take $m = k = 1$.

(a) Show by direct substitution that the solution to (10.1) is given by

$$x(t) = \int_0^t \sin(t - \tau)F(\tau)\,\mathrm{d}\tau$$

if the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
Hint: *First show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t,\tau)\,\mathrm{d}\tau = f(t,t) + \int_0^t \frac{\partial f(t,\tau)}{\partial t}\,\mathrm{d}\tau,$$

provided that all operations make sense.

Define a sequence $(F_j)_{j\in\mathbb{Z}_{>0}}$ of forces by

$$F_j(t) = \begin{cases} j^\alpha, & t \in [0, \frac{1}{j^\beta}], \\ 0, & \text{otherwise} \end{cases}$$

for $\alpha, \beta \in \mathbb{R}_{>0}$.

(b) Compute the response x_j associated to the force F_j with initial conditions $x_j(0) = 0$ and $\dot{x}_j(0) = 0$.

(c) What are the conditions on α and β that guarantee a bounded, nonzero response in the limit as $j \to \infty$?

Section 10.2

Distributions

We begin in this section by providing a theory for generalised signals—we shall call them distributions—that satisfies the objectives of Section 10.1.3. With the theory of distributions we must forgo the comfort of thinking of signals as being functions of time. Indeed, the point of Proposition 10.1.1 is that signals as a function of time are simply not able to capture the features we need from generalised signals. What, then, should distributions *be*? It turns out that a useful definition is to make a distribution a scalar valued function on a certain set of so-called "test functions." Thus distributions are functions of functions. Indeed, distributions are the topological dual of the set of test functions, and we shall explore these topological questions in Section **??**. One does not need to understand these sorts of issues, however, to understand distributions and how to use them.

We prove in this section many of the basic properties of distributions, and so the discussion at times gets technical. However, it is also the case that distributions are extremely easy to use in practice. The thing to keep in mind at all times is that a distribution is a function on the set of test functions. If one does this, one can never stray far.

Do I need to read this section? If you are reading this chapter, then the technical matter starts here. •

10.2.1 Test signals

In this chapter we adopt the convention of using the symbol $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

In our motivation in Section 10.1.2 of the delta-signal as the derivative of the step signal, we introduced the idea of defining a signal based on integrating its product with a signal having certain properties, specifically having the property of being integrable and having an integrable derivative. When one does this, the class of signals used in the integration are called "test signals." They may depend in nature on just what one is doing. In this section we introduce the first our of class of test signals.

10.2.1 Definition (Test signal) A ***test signal*** on $\mathbb{R}$ is a signal $\phi\colon \mathbb{R} \to \mathbb{F}$ with the properties that

(i) ϕ is infinitely differentiable and

(ii) ϕ has compact support.

The set of test signals is denoted $\mathscr{D}(\mathbb{R}; \mathbb{F})$. •

10.2.2 Remark ($\mathscr{D}(\mathbb{R}; \mathbb{F})$ is a vector space) One can easily check that $\mathscr{D}(\mathbb{R}; \mathbb{F})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{F}^{\mathbb{R}}$. •

The set $\mathscr{D}(\mathbb{R};\mathbb{F})$ we have previously denoted by $\mathsf{C}^{\infty}_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$. The notation we use here is the traditional notation used for the test functions.

A good question to ask is, "Are there any nonzero test function?" In case you think this is a stupid question, note that if we replace "infinitely differentiable" with "analytic" in the definition of a test signal, then there are actually no nonzero test signals. However, it turns out that there are nonzero test signals. Most examples of test signals are constructed using the following signal as their basis.

10.2.3 Example (An element of $\mathscr{D}(\mathbb{R};\mathbb{F})$) Define

$$\curlywedge(t) = \begin{cases} \frac{1}{c}\exp(-\frac{1}{1-t^2}), & |t| < 1, \\ 0, & |t| \geq 1 \end{cases}$$

where $c = \int_{-1}^{1} \exp(-\frac{1}{1-t^2})\,\mathrm{d}t$. The signal is plotted in Figure 10.2. This signal is

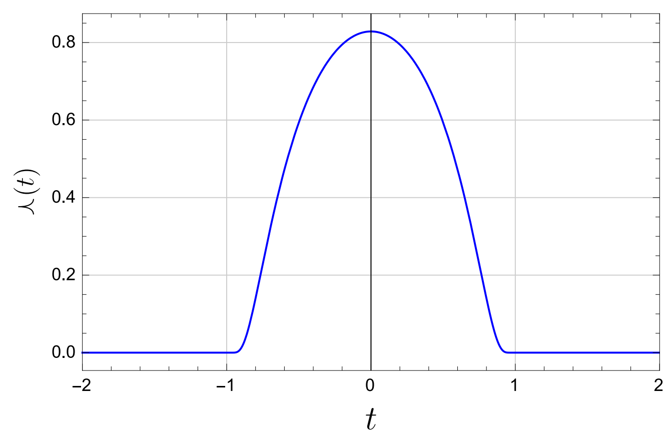


Figure 10.2 The test signal $\curlywedge$

often called a ***bump signal***, for obvious reasons. Clearly $\curlywedge$ has compact support and is infinitely differentiable except at ± 1. To verify that $\curlywedge$ is actually infinitely differentiable, one may show that $\curlywedge^{(k)}(t) = \rho(t)\curlywedge(t)$ for $t \in (-1,1)$, where ρ is a rational function of t having a pole at ± 1 of order $2k$ (cf. Example **??–??**). Therefore,

$$\lim_{t\uparrow 1} \curlywedge^{(k)}(t) = \lim_{t\downarrow -1} \curlywedge^{(k)}(t) = 0,$$

since the exponential decays faster than the rational function blows up. •

Note that the set of test signals forms a vector space since the sum of two test signals is also a test signal, and any scalar multiple of a test signal is also a test signal. Thus $\mathscr{D}(\mathbb{R};\mathbb{F})$ is an infinite-dimensional vector space. If $\mathbb{T} \subseteq \mathbb{R}$ is a closed continuous time-domain of finite length, then $\mathscr{D}(\mathbb{T};\mathbb{F})$ denotes the subspace of $\mathscr{D}(\mathbb{R};\mathbb{F})$ consisting of those test signals ϕ for which $\mathrm{supp}(\phi) \subseteq \mathbb{T}$.

Let us define the notion of convergence in the vector space $\mathscr{D}(\mathbb{R};\mathbb{F})$, and associated to this the notion of continuity for linear maps. Reader not having seen the notion of a linear map may refer back to Definition 4.3.4.

10.2.4 Definition (Convergence in $\mathscr{D}(\mathbb{R};\mathbb{F})$) A sequence of test signals $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ ***converges to zero*** if

(i) there exists a compact continuous time-domain $\mathbb{T}$ so that $\mathrm{supp}(\phi_j) \subseteq \mathbb{T}$ for all $j \in \mathbb{Z}_{>0}$ and,

(ii) for each $k \in \mathbb{Z}_{\geq 0}$, the sequence of signals $(\phi_j^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to the zero signal.

A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ ***converges*** to $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ if the sequence $(\phi_j - \phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

Note that the notion of convergence in the space of signals $\mathscr{D}(\mathbb{R};\mathbb{F})$ is *not* defined using a norm. An interesting question to ask is, "Is there a norm on $\mathscr{D}(\mathbb{R};\mathbb{F})$ for which convergence using that norm is equivalent to convergence as we have defined it?" The answer, it turns out, is, "No." We discuss this in ***missing stuff***.

Let us consider some examples that illustrate what convergence is and is not in $\mathscr{D}(\mathbb{R};\mathbb{F})$.

10.2.5 Examples (Convergence in $\mathscr{D}(\mathbb{R};\mathbb{F})$)

1. If $(a_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathbb{F}$ for which $\lim_{j\to\infty}|a_j| = 0$ then we claim that the sequence $(a_j\wedge)_{j\in\mathbb{Z}_{>0}}$ of test signals converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. This follows since for each $k \in \mathbb{Z}_{\geq 0}$, each of the sequences of signals $(a_j\wedge^{(k)})_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathsf{C}^0([-1,1];\mathbb{F})$ and so converges by Theorem **??**.
2. Next we let $(r_j)_{j\in\mathbb{Z}_{>0}}$ be an increasing sequence of positive real numbers for which $\lim_{j\to\infty} r_j = \infty$. If we define $\wedge_{r_j}(t) = \wedge(\frac{t}{r_j} - 1)$, then we claim that the sequence $(\wedge_{r_j})_{j\in\mathbb{Z}_{>0}}$ *does not* converge to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. While it is true that the sequence of signals and their derivatives converge to zero in the sense that for each $k \in \mathbb{Z}_{\geq 0}$ the sequence $(\wedge_{r_j}^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges pointwise to the zero signal, this convergence is not uniform, as can be gleaned from Figure 10.3. •

Associated with convergence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ is a corresponding notion of continuity.

10.2.6 Definition (Continuous linear maps on $\mathscr{D}(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ of numbers converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ of test signals converging to zero. •

10.2.7 Remark (The rôle of test signals) The reader may be a little perplexed by our introducing the space $\mathscr{D}(\mathbb{R};\mathbb{F})$. Indeed, this is a space of signals that seems to only contain quite strange signals. However, the thing to keep in mind is that the space of test signals is only of interest to us since they will form the domain for the things we are actually interested in in the next section. That is to say, we are not necessarily interested in test signals *per se*, but only as a means of getting at what we are really interested in.

The reader can refer to Remark 10.2.14 and Remark 10.2.28 for the justification of choosing $\mathscr{D}(\mathbb{R};\mathbb{F})$ in the (maybe seemingly strange) way we did. •

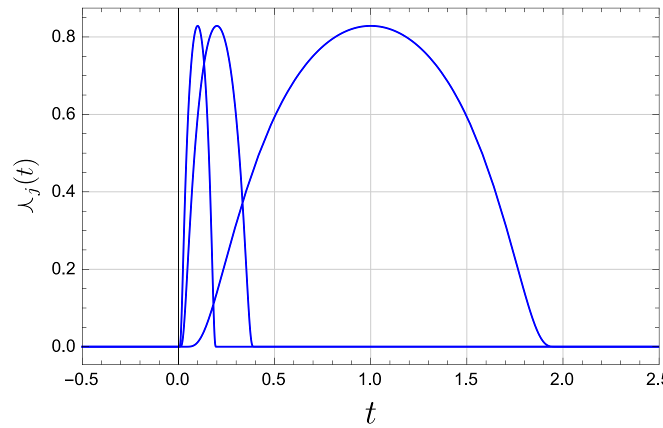


Figure 10.3 The 1st, 5th, and 10th terms in the sequence $(\curlywedge_{r_j})_{j\in\mathbb{Z}_{>0}}$ for $r_j = \frac{1}{j}$

10.2.2 Definition of distributions

Now we define what we mean by a distribution.

10.2.8 Definition (Distribution) A ***distribution***, or a ***generalised signal***, is a continuous linear map from $\mathscr{D}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of distributions is denoted $\mathscr{D}'(\mathbb{R};\mathbb{F})$. •

10.2.9 Notation (Applying a distribution to a test signal) Sometimes it will be convenient to write $\langle\theta;\phi\rangle$ for $\theta(\phi)$. Apart from notation convenience, this is consistent with our notation for the application of an element of the dual of a vector space to an element of the vector space; see Notation **??**. Indeed, a distribution is, by definition, a continuous element of the algebraic dual. •

10.2.10 Remark ($\mathscr{D}'(\mathbb{R};\mathbb{F})$ is vector space) One can easily verify that $\mathscr{D}'(\mathbb{R};\mathbb{F})$ is a $\mathbb{F}$ vector space with the vector space operations

$$(\theta_1+\theta_2)(\phi)=\theta_1(\phi)+\theta_2(\phi),\quad (a\theta)(\phi)=a(\theta(\phi)),$$

for $\theta,\theta_1,\theta_2\in\mathscr{D}'(\mathbb{R};\mathbb{F})$, $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})$, and $a\in\mathbb{F}$. •

Let us consider some elementary constructions with distributions.

10.2.11 Examples (Distributions)

1. If $\theta_1,\theta_2\in\mathscr{D}'(\mathbb{R};\mathbb{F})$ then we define $\theta_1+\theta_2\in\mathscr{D}'(\mathbb{R};\mathbb{F})$ by $(\theta_1+\theta_2)(\phi)=\theta_1(\phi)+\theta_2(\phi)$. Similarly, if $\theta\in\mathscr{D}'(\mathbb{R};\mathbb{F})$ and $a\in\mathbb{F}$ then we define $a\theta\in\mathscr{D}'(\mathbb{R};\mathbb{F})$ by $(a\theta)(\phi)=a(\theta(\phi))$. These operations of vector addition and scalar multiplication may readily be seen to make $\mathscr{D}'(\mathbb{R};\mathbb{F})$ into an $\mathbb{F}$-vector space.
2. Signals can be multiplied pointwise to recover new signals. This is not generally true of distributions (Exercise 10.2.8). However, we claim that if $\phi_0\colon\mathbb{R}\to\mathbb{F}$

is infinitely differentiable and if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then we may define the product of ϕ_0 with θ to obtain a new distribution that we denote by $\phi_0\theta$. This new distribution is defined by

$$\phi_0\theta(\phi) = \theta(\phi_0\phi),$$

noting that $\phi_0\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. To show that $\phi_0\theta$ is indeed a distribution we should show that for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ of test signals converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, the sequence $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$ also converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Since there exists a compact set K such that $\operatorname{supp}(\phi_j) \subseteq K$ for every $j \in \mathbb{Z}_{>0}$, we also have $\operatorname{supp}(\phi_0\phi_j) \subseteq K$. Moreover, we note that

$$(\phi_0\phi_j)^{(\ell)} = \sum_{k=0}^{\ell} \phi_0^{(k)}\phi_j^{(\ell-k)},$$

using the product rule, Proposition 3.2.11. Therefore,

$$\|(\phi_0\phi_j)^{(\ell)}\|_\infty \le \ell \max\{\|\phi_0\|_\infty, \|\phi_0^{(1)}\|_\infty, \dots, \|\phi_0^{(\ell)}\|_\infty\} \cdot \max\{\|\phi_j\|_\infty, \|\phi_j^{(1)}\|_\infty, \dots, \|\phi_j^{(\ell)}\|_\infty\}.$$

Since the right-hand side goes to zero as $j \to \infty$, so too does the left-hand side, giving uniform convergence of $(\phi_0\phi_j)^{(\ell)}$ to zero for $\ell \in \mathbb{Z}_{\ge 0}$, and so giving convergence to zero of $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$.

3. Let us show that our motivational example from Sections 10.1.1 and 10.1.2 does indeed fit into our general framework. Consider the linear map $\delta_{t_0}\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ defined by $\delta_{t_0}(\phi) = \phi(t_0)$. We claim that $\delta_{t_0} \in \mathscr{D}'(\mathbb{R};\mathbb{F})$. Clearly δ_{t_0} is linear. Now let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of test signals converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Then clearly we have $\lim_{j\to\infty}\phi_j(t_0) = 0$, giving continuity of δ_{t_0}. We call δ_{t_0} the ***delta-signal at*** $\mathbf{t_0}$, observing that it is in fact not itself a signal as we showed in Proposition 10.1.1. •

10.2.3 Locally integrable signals are distributions

As we indicated in our wish list from Section 10.1.3, it would be useful allow all locally integrable signals (see Definition 5.9.19) as distributions. In this section we indicate that this is possible.

First we prove a preliminary result.

10.2.12 Proposition (Distributions from locally integrable signals) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *and define* $\theta_{\mathsf{f}}\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ *by*

$$\theta_{\mathsf{f}}(\phi) = \int_{\mathbb{R}} \mathsf{f}(\mathsf{t})\phi(\mathsf{t})\,\mathrm{d}\mathsf{t}.$$

Then the following statements hold:

(i) $\theta_{\mathsf{f}} \in \mathscr{D}'(\mathbb{R};\mathbb{F})$;

(ii) if $\theta_{\mathsf{f}_1} = \theta_{\mathsf{f}_2}$ *for* $\mathsf{f}_1, \mathsf{f}_2 \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$*, then* $\mathsf{f}_1(\mathsf{t}) = \mathsf{f}_2(\mathsf{t})$ *for almost every* $\mathsf{t} \in \mathbb{R}$.

Proof (i) First of all, we note that the integral is always defined. Indeed,

$$\int_{\mathbb{R}} |f(t)\phi(t)|\,\mathrm{d}t = \int_{\mathrm{supp}(\phi)} |f(t)\phi(t)|\,\mathrm{d}t \le \|\phi\|_\infty \int_{\mathrm{supp}(\phi)} |f(t)|\,\mathrm{d}t < \infty.$$

Also, the map $\phi \mapsto \theta_f(\phi)$ is clearly a linear map on $\mathscr{D}(\mathbb{R};\mathbb{F})$. Now let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of test signals converging to zero, and let $\mathbb{T}$ be a compact continuous time-domain for which $\mathrm{supp}(\phi_j) \subseteq \mathbb{T}$. We then have

$$\begin{aligned}\lim_{j\to\infty} \theta_f(\phi_j) &= \lim_{j\to\infty} \int_{\mathbb{T}} f(t)\phi_j(t)\,\mathrm{d}t \\ &\le \lim_{j\to\infty} \|\phi_j\|_\infty \int_{\mathbb{T}} |f(t)|\,\mathrm{d}t = 0,\end{aligned}$$

since $\lim_{j\to\infty}\|\phi_j\|_\infty = 0$ and f is integrable on $\mathbb{T}$. This shows that $\theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{F})$.

(ii) By linearity it suffices to show that if $\theta_f = 0$ then $f(t) = 0$ for almost every $t \in \mathbb{R}$. Thus suppose that

$$\int_{\mathbb{R}} f(t)\phi(t) = 0 \tag{10.2}$$

for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. For $a < b$ define $\psi_{a,b} \in \mathscr{D}(\mathbb{R};\mathbb{F})$ by

$$\psi_{a,b}(t) = \begin{cases} \exp\!\left(-\frac{1}{t-a} - \frac{1}{b-t}\right), & t \in (a,b), \\ 0, & \text{otherwise.}\end{cases}$$

Note that

1. $\mathrm{supp}(\psi_{a,b}) = [a,b]$,
2. $\psi_{a,b}(t) \in \mathbb{R}_{>0}$ for $t \in (a,b)$,
3. $\lim_{n\to\infty} \psi_{a,b}(t)^{1/n} = 1$ for $t \in (a,b)$, and
4. there exists $M \in \mathbb{R}_{>0}$ so that for all $n \in \mathbb{Z}_{>0}$ and $t \in (a,b)$ we have $\psi_{a,b}(t)^{1/n} < M$.

Therefore, for $n \in \mathbb{Z}_{>0}$ we have

$$\int_{\mathbb{R}} f(t)\psi_{a,b}(t)^{1/n}\,\mathrm{d}t = \int_a^b f(t)\psi_{a,b}(t)^{1/n}\,\mathrm{d}t = 0.$$

Since f is locally integrable and since $\psi_{a,b}^{1/n}(t)$ is uniformly bounded in n we have

$$0 = \lim_{n\to\infty} \int_a^b f(t)\psi_{a,b}(t)^{1/n}\,\mathrm{d}t = \int_a^b f(t) \lim_{n\to\infty} \psi_{a,b}(t)^{1/n}\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t,$$

by the Dominated Convergence Theorem. This implies that $f|[a,b]$ is zero almost everywhere. Since this holds for any $a < b$ the result follows. ■

With the preceding result as justification, we make the following definition.

10.2.13 Definition (Distribution associated to a locally integrable signal) For $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ the distribution associated to f is $\theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ defined by

$$\theta_f(\phi) = \int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t. \qquad \bullet$$

The essential meaning of Proposition 10.2.12 is simply that the map $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{F}) \ni f \mapsto \theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ is injective, and so $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ sits as a subspace of $\mathscr{D}'(\mathbb{R};\mathbb{F})$. Of course, this also means that the map $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F}) \ni f \mapsto \theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, just that the map is not injective.

Note that the preceding definition justifies one of the properties of the set $\mathscr{D}(\mathbb{R};\mathbb{F})$ being chosen as it was.

10.2.14 Remark (Justification for signals in $\mathscr{D}(\mathbb{R};\mathbb{F})$ having compact support) Our desire to have locally integrable signals included as generalised signals accounts for the set $\mathscr{D}(\mathbb{R};\mathbb{F})$ of test signals having compact support. Indeed, note that a locally integrable signal can blow up at infinity as fast as one likes. Thus it is not possible to choose a set of test functions with non-compact support for which the integral

$$\int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t$$

will exist for every locally integrable signal f and for every test signal ϕ. Thus we are forced to have our test signals have compact support if we are to have $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$. •

As we saw in Proposition 10.1.1, there exist distributions that are not associated to locally integrable signals as in the preceding definition. This motivates the following definition.

10.2.15 Definition (Regular distribution, singular distribution) A distribution of the form θ_f for $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ is called ***regular***. A distribution that is not regular is called ***singular***. •

10.2.4 The support and singular support of a distribution

Consider the definition of δ_0:

$$\delta_0(\phi) = \phi(0).$$

Although we cannot evaluate δ_0 at a point $t \in \mathbb{R}$, we nonetheless imagine that $t = 0$ is somehow distinguished in the definition of δ_0. We wish to understand how to make this precise.

10.2.16 Definition (Support of a distribution)

(i) A distribution θ ***vanishes*** on an open subset $O \subseteq \mathbb{R}$ if $\theta(\phi) = 0$ for all $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ for which $\mathrm{supp}(\phi) \subseteq O$.

(ii) The ***support*** of $\theta \in \mathscr{D}(\mathbb{R};\mathbb{F})$ is the subset of $\mathbb{R}$ defined by

$$\operatorname{supp}(\theta) = \mathbb{R} \setminus \Big(\cup\{O \subseteq \mathbb{R} \mid O \text{ is open and } \theta \text{ vanishes on } O\}\Big) \qquad \bullet$$

Since $\operatorname{supp}(\theta)$ is the complement in $\mathbb{R}$ of a union of open sets, it is a closed subset of $\mathbb{R}$.

Corresponding to the same notions for signals, we have the following characteristics of distributions.

10.2.17 Definition (Causal, acausal distribution) A distribution θ is ***causal*** if $\operatorname{supp}(\theta) \subseteq [a,\infty)$ for some $a \in \mathbb{R}$ and is ***acausal*** if $\operatorname{supp}(\theta) \subseteq (-\infty,b]$ for some $b \in \mathbb{R}$. The set of causal distributions is denoted $\mathscr{D}'_+(\mathbb{R};\mathbb{F})$ and the set of acausal distributions by $\mathscr{D}'_-(\mathbb{R};\mathbb{F})$. •

Let us consider some examples of distributions where we can give the form of the support.

10.2.18 Examples (Support of a distribution)

1. If $\theta = \theta_f$ for $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ then one readily checks that $\operatorname{supp}(\theta) = \operatorname{supp}(f)$, recalling from ***missing stuff*** the notion of the support of a measurable function.
2. We claim that if $\theta_1,\theta_2 \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then $\operatorname{supp}(\theta_1+\theta_2) = \operatorname{supp}(\theta_1) \cup \operatorname{supp}(\theta_2)$. Indeed, let $O \subseteq \mathbb{R} \setminus \big(\operatorname{supp}(\theta_1) \cup \operatorname{supp}(\theta_2)\big)$ be open and let $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ have support in O. Then $\theta_1(\phi)+\theta_2(\phi) = 0$. This shows that $O \subseteq \mathbb{R} \setminus \big(\operatorname{supp}(\theta_1) \cup \operatorname{supp}(\theta_2)\big)$. To show the converse implication, suppose that O is an open subset of $\mathbb{R}$ for which $\big(\operatorname{supp}(\theta_1) \cup \operatorname{supp}(\theta_2)\big) \cap O \neq \emptyset$. Then we must have $O \cap \operatorname{supp}(\theta_1) \neq \emptyset$ and/or $O \cap \operatorname{supp}(\theta_2) \neq \emptyset$. In either case, there exists a test signal ϕ with support in O so that $\theta(\phi) \neq 0$, thus giving the desired conclusion.
3. If $\phi_0\colon \mathbb{R} \to \mathbb{F}$ is infinitely differentiable and if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, it is straightforward to check that $\operatorname{supp}(\phi_0\theta) = \operatorname{supp}(\theta) \cap \operatorname{supp}(\phi_0)$.
4. We claim that $\operatorname{supp}(\delta_{t_0}) = \{t_0\}$. Indeed, it is clear that it $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ has $t_0 \notin \operatorname{supp}(\phi)$ that $\delta_{t_0}(\phi) = 0$. Therefore $\operatorname{supp}(\delta_{t_0}) \subseteq \{t_0\}$, and since δ_{t_0} is not zero, our claim is verified. •

It is possible that a distribution can be regular on some parts of $\mathbb{R}$ and singular on others. In order to make sense of this, we need to be able to say what it means for two distributions to agree on a subset.

10.2.19 Definition (Singular support of a distribution)

(i) Distributions $\theta_1,\theta_2 \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ ***agree*** on an open subset $O \subseteq \mathbb{R}$ if $\theta_1(\phi) = \theta_2(\phi)$ for each ϕ for which $\operatorname{supp}(\phi) \subseteq O$.

(ii) The ***singular support*** of $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ is the largest closed set $\operatorname{sing}(\theta) \subseteq \mathbb{R}$ with the property that on $\mathbb{R} \setminus \operatorname{sing}(\theta)$ there exists $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ so that θ agrees with θ_f. •

Note that $\operatorname{sing}(\theta) \subseteq \operatorname{supp}(\theta)$.

Let us consider some examples to illustrate the notion of singular support.

10.2.20 Examples (Singular support of a distribution)

1. Clearly $\mathrm{sing}(\theta_f) = \emptyset$ for any $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$. Also, if $\mathrm{sing}(\theta) = \emptyset$ then $\theta = \theta_f$ for some $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$. Thus we have the grammatically convenient statement, "A distribution is singular if and only if it has nonempty singular support."
2. Since $\delta_0(\phi) = 0$ for any ϕ for which $0 \notin \mathrm{int}(\mathrm{supp}(\phi))$ we have $\mathrm{sing}(\delta_0) = \{0\}$. •

10.2.5 Convergence of distributions

In our wish list for distributions in Section 10.1.3 we indicated that we would like for the set of generalised signals to have some useful properties for defining convergence. In this section we shall consider a natural notion of convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$; it may not be clear at this point that it is useful, but as we go along we shall see that it does do some things for us that might merit its being called "useful."

First the definition.

10.2.21 Definition (Convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ is

(i) a ***Cauchy sequence*** if $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$, and

(ii) ***converges*** to a distribution θ if for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$, the sequence of numbers $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ converges to $\theta(\phi)$. •

Note that our definition of convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ is "indirect" in that it relies on what distributions do to test functions. Generally, this sort of convergence is known as ***weak convergence***. The interested reader can read more about this in ***missing stuff***.

Having defined the two notions of a Cauchy and a convergent sequence, the natural issue arising next is the relationship between these? Note that these notions are not just corresponding to a norm, so the matter is not quite equivalent to the way we regard Cauchy sequences in, say, $\mathsf{L}^p(\mathbb{T};\mathbb{F})$. The general framework giving rise to the notion of Cauchy and convergent sequences is considered in ***missing stuff***. Here, let us content ourselves with showing that $\mathscr{D}'(\mathbb{R};\mathbb{F})$ is "complete" in that Cauchy sequences and convergent sequences agree.

10.2.22 Theorem (Cauchy sequences in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ converge) *A sequence $(\theta_\mathrm{j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ converges to some $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ if and only if it is Cauchy.*

Proof Clearly if $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ converges to some $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then it is a Cauchy sequence since convergent sequences in $\mathbb{F}$ are convergent. So we prove the converse.

Define a map $\theta\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by $\theta(\phi) = \lim_{j\to\infty} \theta_j(\phi)$. This certainly makes sense, but we have to show that θ is a distribution, i.e., that it is linear and continuous. Linearity is trivial. For continuity, let $(\phi_k)_{k\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero, and suppose that the sequence $(\theta(\phi_k))_{k\in\mathbb{Z}_{>0}}$ does not converges to zero. We may then choose $C \in \mathbb{R}_{>0}$ and a subsequence $(\psi_n)_{n\in\mathbb{Z}_{>0}} \subseteq (\phi_k)_{k\in\mathbb{Z}_{>0}}$ such that $|\theta(\psi_n)| \geq C$ for all $n \in \mathbb{Z}_{>0}$, and such that $\|\psi_n^{(j)}\|_\infty < \frac{1}{4^n}$ for $j \in \{0, 1, \dots, n\}$. By defining $(\chi_j = 2^j\psi_j)_{j\in\mathbb{Z}_{>0}}$ we see that the sequence $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ and that the sequence $(|\theta(\chi_j)|)_{j\in\mathbb{Z}_{>0}}$ blows up to ∞ as $j \to \infty$.

A technical lemma is useful at this point.

1 Lemma *There exists a subsequence* $(\tilde{\theta}_k)_{k\in\mathbb{Z}_{>0}}$ *of* $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ *and a subsequence* $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ *of* $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ *such that*

$$\begin{aligned}|\tilde{\theta}_k(\tilde{\chi}_n)| &< \frac{1}{2^{n-k}}, \qquad k \in \{1,\dots,n-1\},\\ |\tilde{\theta}_n(\tilde{\chi}_n)| &> \sum_{k=1}^{n-1} |\tilde{\theta}_n(\tilde{\chi}_k)| + n, \qquad n \in \mathbb{Z}_{>0}.\end{aligned} \tag{10.3}$$

Furthermore, if $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ *is so chosen then*

$$\chi = \sum_{k=1}^{\infty} \tilde{\chi}_k \in \mathscr{D}(\mathbb{R};\mathbb{F}).$$

Proof Note that since $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, for fixed $k \in \mathbb{Z}_{>0}$ we have $\lim_{j\to\infty}|\theta_k(\chi_j)| = 0$. Therefore, for any $k_1,\dots,k_m \in \mathbb{Z}_{>0}$ we may choose a subsequence $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ of $(\chi_j)_{j\in\mathbb{Z}_{>0}}$ such that

$$|\tilde{\theta}_{k_a}(\tilde{\chi}_n)| < \frac{1}{2^{n-k_a}}, \qquad a \in \{1,\dots,m\}.$$

In particular, the first of equations (10.3) is satisfied. Since $\lim_{j\to\infty}|\theta(\chi_j)| = \infty$ we can refine the choice of subsequence $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ to further ensure that

$$|\theta(\tilde{\chi}_n)| > \sum_{k=1}^{n-1} |\theta(\tilde{\chi}_k)| + n, \qquad n \in \mathbb{Z}_{>0}.$$

What's more, since $\lim_{k\to\infty} \theta_k(\tilde{\chi}_j) = \theta(\tilde{\chi}_j)$, for all $j \in \mathbb{Z}_{>0}$ we may choose a subsequence $(\tilde{\theta}_k)_{k\in\mathbb{Z}_{>0}}$ of $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ such that the second of equations (10.3) holds.

For the second assertion we note that for $k \in \mathbb{Z}_{\geq 0}$ we have, for sufficiently large $n \in \mathbb{Z}_{>0}$,

$$\Big\|\sum_{j=n}^{\infty} \tilde{\chi}_j^{(k)}\Big\|_\infty \leq \sum_{j=n}^{\infty} \|\tilde{\chi}_j^{(k)}\|_\infty \leq \sum_{j=n}^{\infty} \|\chi_j^{(k)}\|_\infty \leq \sum_{j=n}^{\infty} \frac{1}{2^j} < \infty$$

by Example 2.4.2–**??**. This shows that all derivatives of the signals $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ converge to zero uniformly. From this it follows that χ is infinitely differentiable by Theorem 3.5.24. It further has compact support since $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, and so these signals have a common compact support. ▼

Using the subsequences $(\tilde{\theta}_k)_{k\in\mathbb{Z}_{>0}}$ and $(\tilde{\chi}_k)_{k\in\mathbb{Z}_{>0}}$ as specified in the lemma we have

$$\tilde{\theta}_k(\chi) = \sum_{j=1}^{k-1} \tilde{\theta}_k(\tilde{\chi}_j) + \tilde{\theta}_k(\tilde{\chi}_k) + \sum_{j=k+1}^{\infty} \tilde{\theta}_k(\tilde{\chi}_j).$$

The last term is bounded by the first of equations (10.3). By the second of equations (10.3) we have

$$\Big|\sum_{j=1}^{k-1} \tilde{\theta}_k(\tilde{\chi}_j) + \tilde{\theta}_k(\tilde{\chi}_k)\Big| \geq |\tilde{\theta}_k(\tilde{\chi}_k)| - \Big|\sum_{j=1}^{k-1} \tilde{\theta}_k(\tilde{\chi}_j)\Big| > n,$$

using Exercise 6.1.3. Thus $\lim_{k\to\infty}|\tilde{\theta}_k(\chi)| = \infty$, and so our initial assumption that θ is not continuous is false. ■

Let us consider the relationship between convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ and more usual types of convergence of sequences of signals.

10.2.23 Proposition (Convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ for signals) *For a sequence $(\mathsf{f}_j)_{j\in\mathbb{Z}_{>0}}$ of signals, the following statements hold:*

(i) if $(\mathsf{f}_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^1(\mathbb{R};\mathbb{F})$ then $(\theta_{\mathsf{f}_j})_{j\in\mathbb{Z}_{>0}}$ converges to θ_{f} in $\mathscr{D}'(\mathbb{R};\mathbb{F})$;

(ii) if $(\mathsf{f}_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^2(\mathbb{R};\mathbb{F})$ then $(\theta_{\mathsf{f}_j})_{j\in\mathbb{Z}_{>0}}$ converges to θ_{f} in $\mathscr{D}'(\mathbb{R};\mathbb{F})$;

(iii) if, for every compact continuous time-domain $\mathbb{T}\subseteq\mathbb{R}$ the sequence $(\mathsf{f}_j|\mathbb{T})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\mathsf{f}\in\mathsf{C}^0(\mathbb{T};\mathbb{F})$, then $(\theta_{\mathsf{f}_j})_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$;

(iv) if $(\mathsf{f}_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ for which

(a) $(\mathsf{f}_j(\mathsf{t}))_{j\in\mathbb{Z}_{>0}}$ converges to $\mathsf{f}(\mathsf{t})$ for almost every $\mathsf{t}\in\mathbb{R}$, and

(b) there exists $\mathsf{g}\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ such that $|\mathsf{f}_j(\mathsf{t})|\le\mathsf{g}(\mathsf{t})$ for almost every $\mathsf{t}\in\mathbb{R}$,

then $\mathsf{f}\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ and $(\theta_{\mathsf{f}_j})_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to θ_{f}.

Proof (i) First suppose that $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^1(\mathbb{R};\mathbb{F})$. Then for $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})$ we have

$$\begin{aligned}|\theta_{f_j}(\phi)-\theta_f(\phi)| &= \Big|\int_{\mathbb{R}}\big(f_j(t)-f(t)\big)\phi(t)\,\mathrm{d}t\Big|\\ &\le \int_{\mathbb{R}}|f_j(t)-f(t)||\phi(t)|\,\mathrm{d}t\\ &\le \|\phi\|_\infty\int_{\mathbb{R}}|f_j(t)-f(t)|\,\mathrm{d}t.\end{aligned}$$

Taking the limit as $j\to\infty$ then gives

$$\lim_{j\to\infty}|\theta_{f_j}(\phi)-\theta_f(\phi)|=0,$$

showing that $(\theta_{f_j})_{j\in\mathbb{Z}_{>0}}$ converges to θ_f in $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

(ii) Here we compute

$$\begin{aligned}|\theta_{f_j}(\phi)-\theta_f(\phi)| &= \Big|\int_{\mathbb{R}}\big(f_j(t)-f(t)\big)\phi(t)\,\mathrm{d}t\Big|\\ &\le \int_{\mathbb{R}}|f_j(t)-f(t)||\phi(t)|\,\mathrm{d}t\\ &\le \|f_j-f\|_2\|\phi\|_2,\end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Taking the limit as $j\to\infty$ gives the result.

(iii) Let $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})$ and for $\epsilon\in\mathbb{R}_{>0}$ let $N\in\mathbb{Z}_{>0}$ have the property that $|f_j(t)-f_k(t)|<\frac{\epsilon}{\|\phi\|_1}$, for $j,k\ge N$ and $t\in\operatorname{supp}(\phi)$, this being possible since $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly on $\operatorname{supp}(\phi)$. Now, for $j,k\ge N$ we compute

$$|\theta_{f_j}(\phi)-\theta_{f_k}(\phi)|\le\int_{\operatorname{supp}(\phi)}|f_j(t)-f_k(t)||\phi(t)|\,\mathrm{d}t\le\epsilon.$$

Thus $(\theta_{f_j}(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges.

(iv) That $f\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ is a consequence of the Dominated Convergence Theorem. Also by the Dominated Convergence Theorem,

$$\lim_{j\to\infty}\int_{\mathbb{R}}f_j(t)\phi(t)\,\mathrm{d}t=\int_{\mathbb{R}}f(t)\phi(t)\,\mathrm{d}t$$

for $\phi \in \mathscr{D}(\mathbb{R}; \mathbb{F})$. ■

10.2.24 Remark (The topology in $\mathscr{D}'(\mathbb{R}; \mathbb{F})$ is weaker than the signal space topologies) The converses of all of the assertions of Proposition 10.2.23 are false as the reader can show in ***missing stuff***. This means that convergence in spaces of signals is a more rigid notion than convergence in $\mathscr{D}'(\mathbb{R}; \mathbb{F})$, i.e., the topology in $\mathscr{D}'(\mathbb{R}; \mathbb{F})$ is weaker than the topologies we deal with for spaces of signals. This has its advantages and disadvantages. One advantage is that we can get useful convergence of sequences in cases that do not give convergence in the corresponding space of signals. One disadvantage is that by looking only at signals as elements of $\mathscr{D}'(\mathbb{R}; \mathbb{F})$ under the inclusion $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R}; \mathbb{F})$ we lose a lot of information about the signals. This is something to keep in mind, depending on what one is doing. •

Let us consider a nice example of convergence in $\mathscr{D}'(\mathbb{R}; \mathbb{F})$.

10.2.25 Example (The delta-signal is a limit of signals) Let us consider the sequence $(f_j)_{j \in \mathbb{Z}_{>0}}$ of signals defined by

$$f_j(t) = \begin{cases} j, & t \in [0, \frac{1}{j}], \\ 0, & \text{otherwise}. \end{cases}$$

In Figure 10.4 we show a few of the signals in this sequence. One can show (we

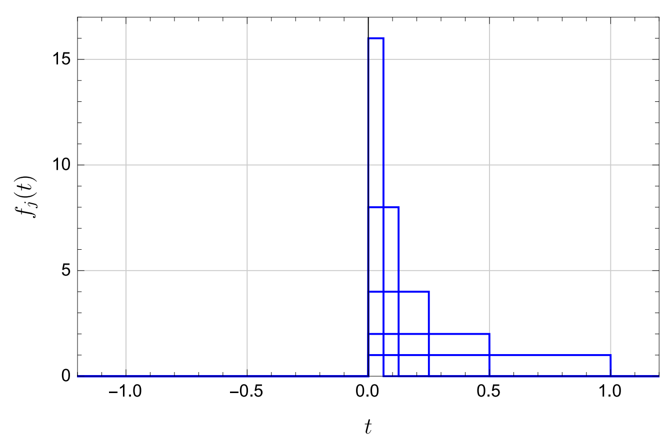


Figure 10.4 A sequence of signals converging to δ_0

will do this in Section 10.5.7) that $\delta_0 = \lim_{j \to \infty} f_j$, with the limit being taken in $\mathscr{D}'(\mathbb{R}; \mathbb{F})$. Note that this resolves the issue that came up in Section 10.1.1 regarding the limit of forces of increasing amplitude applied for decreasing time. •

10.2.6 Differentiation of distributions

Another item on our wish list of Section 10.1.3 was that we be able to differentiate generalised signals. Here we see that this can be done naturally.

Distributions have the remarkable property that they can *always* be differentiated. To define differentiation we use the following simple result.

10.2.26 Lemma (Differentiation of distributions through test signals) *For $\theta \in \mathscr{D}(\mathbb{R};\mathbb{F})$ define $\theta'\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by $\theta'(\phi) = -\theta(\phi')$. Then $\theta' \in \mathscr{D}(\mathbb{R};\mathbb{F})$.*

Proof It is clear that θ' is linear. Moreover, if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ then $(-\phi_j')_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Thus

$$\lim_{j\to\infty} \theta'(\phi_j) = \lim_{j\to\infty} \theta(-\phi_j) = 0,$$

giving continuity of θ', as desired. ∎

With the lemma the following definition makes sense.

10.2.27 Definition (Derivative of a distribution) If $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ the ***derivative*** of θ is the distribution θ' defined by $\theta'(\phi) = -\theta(\phi)$. •

10.2.28 Remark (Justification for signals in $\mathscr{D}(\mathbb{R};\mathbb{F})$ being infinitely differentiable) Our desire to differentiate our generalised signals accounts for the requirement that the test signals $\mathscr{D}(\mathbb{R};\mathbb{F})$ be infinitely differentiable. Indeed, were the test signals only continuous, then differentiation as we have defined it in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ would not be possible. One could, one supposes, consider test functions differentiable to some order less than infinity. However, this idea really arises naturally from the notion of the order of a distribution as we will discuss in Section 10.2.11. •

A consequence of the definition of derivative along with Proposition 10.2.12 is that every locally integrable signal can be differentiated! This is a strange fact on first encounter. Here is one place that one must really get used to the fact that distributions are not signals, but functions on the set of test functions. Thus the derivative of a non-differentiable signal is not a signal at all, but something possible rather different.

Also note that our definition immediately implies that distributions can be differentiated arbitrarily often. Indeed, a simple induction gives the following result.

10.2.29 Proposition (Higher-order derivatives of distributions) *If $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and if $\mathrm{k} \in \mathbb{Z}_{>0}$; denote by $\theta^{(\mathrm{k})} \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ the distribution obtained by differentiating θ k times. Then $\theta^{(\mathrm{k})}(\phi) = (-1)^{\mathrm{k}}\theta(\phi^{(\mathrm{k})})$.*

first time where we have used the fact that test signals are infinitely differentiable.

Let us consider some examples of derivatives of distributions.

10.2.30 Examples (Derivative of a distribution)

1. Let us begin with a simple general example that motivates the definition of the derivative of a distribution. Suppose that $f\colon \mathbb{R} \to \mathbb{F}$ is differentiable with derivative f'. Then, for $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$, an integration by parts gives

$$\theta_{f'}(\phi) = \int_{\mathbb{R}} f'(t)\phi(t)\,\mathrm{d}t = f(t)\phi(t)|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(t)\phi'(t)\,\mathrm{d}t = -\theta_f(\phi').$$

This shows that the "derivative of a distribution is the distribution of the derivative" when all terms are defined. We shall generalise this somewhat in Proposition 10.2.31.

2. Consider the ramp signal

$$\mathsf{R}(t) = \begin{cases} 0, & t \in \mathbb{R}_{\le 0}, \\ t, & t \in \mathbb{R}_{>0}. \end{cases}$$

We claim that $\theta'_{\mathsf{R}} = 1$. Indeed, if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ then

$$\begin{aligned} \theta'_{\mathsf{R}}(\phi) = &-\theta_{\mathsf{R}}(\phi) = -\int_{\mathbb{R}} \mathsf{R}(t)\phi'(t)\,\mathrm{d}t \\ = &-\int_0^\infty t\phi'(t)\,\mathrm{d}t = -t\phi(t)\Big|_0^\infty + \int_0^\infty \phi(t)\,\mathrm{d}t \\ = &\int_{\mathbb{R}} 1(t)\phi(t)\,\mathrm{d}t = \theta_1(\phi). \end{aligned}$$

3. Let us show that δ_0 is the derivative of the unit step signal 1. By definition of the derivative we have, for every test signal ϕ,

$$1'(\phi) = -1(\phi') = -\int_{\mathbb{R}} 1(t)\phi'(t)\,\mathrm{d}t = -\int_0^\infty \phi'(t) = -\phi(t)\Big|_0^\infty = \phi(0),$$

as desired. •

Since locally integrable signals give rise to distributions, it makes sense to ask, "For what class of signals is it true that $\theta'_f = \theta_{f'}$?" To address this, we recall from Section 5.9.6 the notion of a locally absolutely continuous signal.

10.2.31 Proposition (When is the derivative of distribution the distribution of a derivative?) *Let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and suppose that there exists a locally integrable signal $\mathsf{g}\colon \mathbb{R} \to \mathbb{F}$ for which $\theta' = \theta_{\mathsf{g}}$. Then there exists a locally absolutely continuous signal f such that $\theta = \theta_{\mathsf{f}}$ and $\mathsf{g} = \mathsf{f}'$ almost everywhere. Conversely, if $\theta = \theta_{\mathsf{f}}$ for a locally absolutely continuous signal f, then $\theta' = \theta_{\mathsf{f}'}$.*

Proof First suppose that $\theta' = \theta_g$ for $g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$. Then, for some $t_0 \in \mathbb{R}$, the signal

$$f_{t_0}(t) = \int_{t_0}^t g(\tau)\,\mathrm{d}\tau$$

satisfies $f'_{t_0}(t) = g(t)$ for almost every t, meaning that $\theta_{f'_{t_0}} = \theta_g = \theta'$. Thus θ_f and θ are primitives for θ' and so $\theta = \theta_{f_{t_0}} + \theta_h$ where h is a constant signal by Proposition 10.2.38. The first part of the result follows by taking $f = f_{t_0} + h$, and noting by Theorem 5.9.31 that f is locally absolutely continuous.

Next suppose that $\theta = \theta_f$ for a locally absolutely continuous signal f. Then

$$\begin{aligned}\theta'(\phi) &= -\theta(\phi') \\ &= -\int_{\mathbb{R}} f(t)\phi'(t)\,\mathrm{d}t \\ &= -f(t)\phi(t)\big|_{-\infty}^{\infty} + \int_{\mathbb{R}} f'(t)\phi(t)\,\mathrm{d}t \\ &= \theta_{f'}(\phi),\end{aligned}$$

as desired. ■

Conveniently, differentiation and limit can always be swapped for distributions.

10.2.32 Proposition (Limits and derivatives of distributions can be interchanged) *If $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence of distributions converging to $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, then the sequence $(\theta'_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to θ'.*

Proof We have

$$\left|\theta'(\phi) - \theta'_j(\phi)\right| = \left|\theta(\phi') - \theta_j(\phi')\right|.$$

Taking the limit as $j \to \infty$ gives the result. ■

For sums we then immediately have the following result.

10.2.33 Corollary (Infinite sums and derivatives of distributions commute) *Let $(\theta_j)_{n\in\mathbb{Z}_{>0}} \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$. If the sequence of partial sums for the series*

$$\sum_{j=1}^{\infty} \theta_j$$

converges to θ in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ then the sequence of partial sums for the series

$$\sum_{j=1}^{\infty} \theta'_j$$

converges to θ' in $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

One might think this is heaven indeed, since we seemingly no longer have to worry about swapping operations. However, the reader should bear in mind the caveat of Remark 10.2.24 and realise that sometimes one is loosing something when dealing with distributions.

In Section 10.5.7 we will examine conditions on a sequence of signals that ensure that these signals converge to δ_0. In that section we also give a few examples, and so these can be referred to to get more insight into convergence of sequences of regular distributions to a singular distribution. Here we exhibit one of these sequences—a sequence of infinitely differentiable functions—in order to illustrate the convergence of derivatives.

10.2.34 Example (Example 10.2.25 cont'd) In Example 10.5.25–3 we shall show that the sequence $(G_{\Omega,j})_{j\in\mathbb{Z}_{>0}}$ given by

$$G_{\Omega,j}(t) = j\frac{\exp(-\frac{(jt)^2}{4\Omega})}{\sqrt{4\pi\Omega}}$$

converges to δ_0 in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ for $\Omega \in \mathbb{R}_{>0}$. This sequence and its derivatives are shown in Figure 10.5 for $\Omega = \frac{1}{2}$. By Corollary 10.2.33 the sequence $(G'_{\Omega,j})_{j\in\mathbb{Z}_{>0}}$

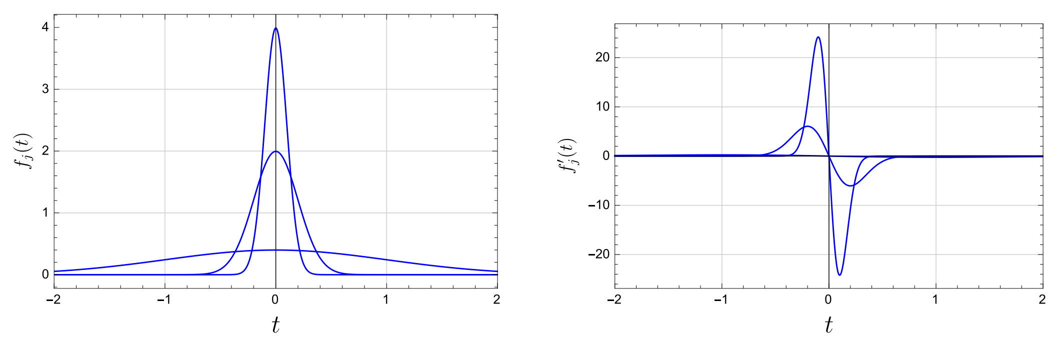


Figure 10.5 A sequence of signals converging to δ_0 (left) and δ'_0 (right)

converges to δ'_0 in $\mathscr{D}'(\mathbb{R};\mathbb{F})$. •

In Example 10.2.11–2 we showed that a signal can be multiplied by an infinitely differentiable signals and still be a distribution. Let us show that this resulting distribution obeys the product rule when differentiated.

10.2.35 Proposition (A product rule for functions and derivatives) *If $\phi_0 \colon \mathbb{R} \to \mathbb{F}$ is infinitely differentiable and if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then*

$$(\phi_0\theta)^{(1)} = \phi'_0\theta + \phi_0\theta'.$$

Proof We have

$$\begin{aligned}(\phi_0\theta)^{(1)}(\phi) = & -\phi_0\theta(\phi') = -\theta(\phi_0\phi') = \theta(\phi'_0\phi) - \theta((\phi_0\phi)^{(1)})\\ &= \phi'_0\theta(\phi) + \theta'(\phi_0\phi) = \phi'_0\theta(\phi) + \phi_0\theta'(\phi),\end{aligned}$$

as desired. ■

Let us consider a common situation where the use of the product rule is required.

10.2.36 Example (Differentiation of truncated signals) Let $f \in \mathsf{C}^\infty(\mathbb{R};\mathbb{F})$ and define $f_+ = 1 \cdot f$ to be the signal that truncates f to positive times. According to Proposition 10.2.35 we have

$$\begin{aligned}\theta'_{f_+}(\phi) &= (f'\theta_1)(\phi) + (f\theta'_1)(\phi)\\ &= \theta_1(f'\phi) + \theta'_1(f\phi)\\ &= \theta_{f'_+}(\phi) + f(0)\delta_0(\phi),\end{aligned}$$

where $f'_+ = 1 \cdot f'$. Provided one is prepared to take the time to understand properly the notation, the preceding equation can be written as

$$f'_+(t) = f'(t) \cdot 1(t) + f(0)\delta_0(t).$$

Note that this involves providing the delta-signal with "t" as argument. This should only be done after careful consideration!

Proceeding inductively as above, one may show that

$$\begin{aligned}\theta_{f_+} &= \theta_{1f}\\ \theta^{(1)}_{f_+} &= \theta_{f^{(1)}_+} + f(0)\delta_0\\ \theta^{(2)}_{f_+} &= \theta_{f^{(2)}} + f'(0)\delta_0 + f(0)\delta^{(1)}_0\\ &\vdots\\ \theta^{(n)}_{f_+} &= \theta_{f^{(j)}_+} + \sum_{j=1}^{n} g^{(n-j)}(0)\delta_0^{(j-1)},\end{aligned}$$

where $f^{(j)}_+ = 1 \cdot f^{(j)}$. This formula is useful when discussing the solution of differential equations using the Laplace transform. •

10.2.7 Integration of distributions

Recall from Section 3.4.6 that a primitive for a continuous-time signal $f\colon \mathbb{T} \to \mathbb{F}$ is a signal $g\colon \mathbb{T} \to \mathbb{F}$ with the property that $f = g'$. We wish to present a similar notion for distributions. One might expect that the idea here follows that for the derivative of a distribution.

10.2.37 Definition (Primitive of a distribution) A ***primitive*** for $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ is a distribution $\theta^{(-1)}$ that satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\theta^{(-1)} = \theta$. •

We let $\mathscr{D}^{(1)}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}(\mathbb{R};\mathbb{F})$ be those test functions that are derivatives of other test functions. That is,

$$\mathscr{D}^{(1)}(\mathbb{R};\mathbb{F}) = \{\phi' \mid \phi \in \mathscr{D}(\mathbb{R};\mathbb{F})\}.$$

Note that $\mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$ is a subspace of $\mathscr{D}(\mathbb{R};\mathbb{F})$ and that there are test signals that are not in $\mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$. To see that this is so, the reader might try to understand why $\wedge \notin \mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$.

With this notation we have the following result.

10.2.38 Proposition (Distributions have primitives) *Every distribution possesses a primitive. Furthermore, if $\theta^{(-1)}$ and $\tilde{\theta}^{(-1)}$ are primitives of $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then $\tilde{\theta}^{(-1)} - \theta^{(-1)} = \theta_{\mathrm{f}}$ where* f *is a constant signal.*

Proof Choose an arbitrary $\phi_0 \in \mathscr{D}(\mathbb{R};\mathbb{F})$ with the property that $\int_{\mathbb{R}} \phi_0(t)\,\mathrm{d}t = 1$. For $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ we can write

$$\phi = c_{\phi,\phi_0}\phi_0 + \psi_{\phi,\phi_0}$$

where

$$c_{\phi,\phi_0} = \int_{\mathbb{R}} \phi(t)\,\mathrm{d}t \tag{10.4}$$

and $\psi_{\phi,\phi_0} = \phi - c_{\phi,\phi_0}\phi_0$. We claim that $\psi_{\phi,\phi_0} \in \mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$. By Exercise 10.2.9 it suffices to check that $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ and that $\int_{\mathbb{R}} \psi_{\phi,\phi_0}(t)\,\mathrm{d}t = 0$. This, however, is a direct computation. This shows that every distribution ϕ admits a decomposition, unique in fact, of the type $\phi = c_{\phi,\phi_0}\phi_0 + \psi_{\phi,\phi_0}$ where ϕ_0 is as above and $\psi_{\phi,\phi_0} \in \mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$.

We claim that with any ϕ_0 as above,

$$\theta^{-1}(\phi) = c_{\phi,\phi_0}\theta^{(-1)}(\phi_0) - \theta(\psi^{(-1)}_{\phi,\phi_0})$$

defines a primitive of θ, where $\theta^{(-1)}(\phi_0)$ is an arbitrarily specified constant and where

$$\psi^{(-1)}_{\phi,\phi_0}(t) = \int_{-\infty}^{t} \psi_{\phi,\phi_0}(t)\,\mathrm{d}t.$$

Indeed, note that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\theta^{(-1)}(\phi) = -\theta^{(-1)}(\phi') = \theta(\phi)$$

for any $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ since $c_{\phi',\phi_0} = 0$. We still need to show that $\theta^{(-1)}$ is a distribution. To do this we must show that it is linear and continuous. Linearity is evident. To show continuity, let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, and write $\phi_j = c_{\phi_j,\phi_0}\phi_0 + \psi_{\phi_j,\phi_0}$ as above. We claim that both sequences $(c_{\phi_j,\phi_0})_{j\in\mathbb{Z}_{>0}}$ and $(\psi_{\phi_j,\phi_0})_{j\in\mathbb{Z}_{>0}}$ must tend to zero, the former in $\mathbb{F}$ and the latter in $\mathscr{D}(\mathbb{R};\mathbb{F})$. By (10.4) we then deduce the convergence to zero of $(c_{\phi_j,\phi_0})_{j\in\mathbb{Z}_{>0}}$. The convergence to zero of $(\psi_{\phi_j,\phi_0})_{j\in\mathbb{Z}_{>0}}$ immediately follows. We claim that $(\psi^{(-1)}_{\phi_j,\phi_0})_{j\in\mathbb{Z}_{>0}}$ also tends to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Indeed, if $\mathrm{supp}(\psi_{\phi_j,\phi_0}) \subseteq [-a,a]$, $j \in \mathbb{Z}_{>0}$, then it follows since $\psi_{\phi_j,\phi_0} \in \mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$ that $\mathrm{supp}(\psi_{\phi_j,\phi_0}) \subseteq [-a,a]$, $j \in \mathbb{Z}_{>0}$. We then have

$$\theta^{(-1)}(\phi_j) = c_{\phi_j,\phi_0}\theta^{(-1)}(\phi_0) - \theta(\psi^{(-1)}_{\phi_j,\phi_0}).$$

It then follows that $\lim_{j\to\infty}\theta^{(-1)}(\phi_j) = 0$, as desired.

To prove the last assertion of the result let $\theta^{(-1)}$ and $\tilde{\theta}^{(-1)}$ be two primitives for θ and let $\phi_0 \in \mathscr{D}(\mathbb{R};\mathbb{F})$ satisfy $\int_{\mathbb{R}} \phi_0(t)\,\mathrm{d}t = 1$. For $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ we have

$$\begin{aligned}\theta^{(-1)}(\phi) - \tilde{\theta}^{(-1)}(\phi) &= \theta^{(-1)}(c_{\phi,\phi_0}\phi_0 + \psi_{\phi,\phi_0}) - \tilde{\theta}^{(-1)}(c_{\phi,\phi_0}\phi_0 + \psi_{\phi,\phi_0})\\ &= c_{\phi,\phi_0}\theta^{(-1)}(\phi_0) - c_{\phi,\phi_0}\tilde{\theta}^{(-1)}(\phi_0) + \theta^{(-1)}(\psi_{\phi,\phi_0}) - \tilde{\theta}^{(-1)}(\psi_{\phi,\phi_0})\\ &= c_{\phi,\phi_0}\theta^{(-1)}(\phi_0) - c_{\phi,\phi_0}\tilde{\theta}^{(-1)}(\phi_0)\end{aligned}$$

since $\psi_{\phi,\phi_0} \in \mathscr{D}^{(1)}(\mathbb{R};\mathbb{F})$. The result now follows since

$$c_{\phi,\phi_0}\theta^{(-1)}(\phi_0) - c_{\phi,\phi_0}\tilde{\theta}^{(-1)}(\phi_0) = (\theta^{(-1)}(\phi_0) - \tilde{\theta}^{(-1)}(\phi_0))\int_{\mathbb{R}} \phi(t)\,\mathrm{d}t = \theta_f(\phi)$$

where $f(t) = \theta^{(-1)}(\phi_0) - \tilde{\theta}^{(-1)}(\phi_0)$. ■

There is also a version of integration by parts for distributions.

10.2.39 Proposition (Integration by parts for distributions) *If* $f \in C^\infty(\mathbb{R}; \mathbb{F})$ *and if* $\theta \in \mathscr{D}'(\mathbb{R}; \mathbb{F})$ *then*

$$(f^{(1)}\theta)^{(-1)} = f\theta - (f\theta^{(1)})^{(-1)} + \theta_g,$$

where g *is a constant signal.*

Proof By Proposition 10.2.35 we have

$$f^{(1)}\theta = (f\theta)^{(1)} - f\theta^{(1)}.$$

This means that both $(f^{(1)}\theta)^{(-1)}$ and $f\theta - (f\theta^{(1)})^{(-1)}$ are primitives for $f^{(1)}\theta$, and so differ by a constant signal by Proposition 10.2.38. ■

10.2.8 Distributions depending on parameters

Situations often arise where distributions are applied to classes of test signals that depend in some way on a parameter. Also, it can sometimes arise that distributions themselves depend on a parameter. In either of these cases, one would like to understand the dependence on parameter after hitting the test signal with a distribution (in the first case) and applying the distribution to a test signal (in the second case). One can consider the results in this section as being analogous to those like Theorem 5.9.16, where the dependence of integrals on parameters if discussed.

Let us first consider a test signal depending on a parameter. We let $I \subseteq \mathbb{R}$ be an interval and consider a function $\phi\colon I \times \mathbb{R} \to \mathbb{F}$. A typical point in $I \times \mathbb{R}$ we denote by (λ, t), thinking of λ as being a parameter and t as being the independent variable. For $(\lambda, t) \in I \times \mathbb{R}$ we define functions $\phi^\lambda\colon \mathbb{R} \to \mathbb{F}$ and $\phi_t\colon I \to \mathbb{F}$ by $\phi^\lambda(t) = \phi_t(\lambda) = \phi(\lambda, t)$. If, for each $\lambda \in I$, $\phi^\lambda \in \mathscr{D}(\mathbb{R}; \mathbb{F})$, then, given $\theta \in \mathscr{D}'(\mathbb{R}; \mathbb{F})$, we define $\Phi_{\theta,\phi}\colon I \to \mathbb{F}$ by

$$\Phi_{\theta,\phi}(\lambda) = \theta(\phi^\lambda).$$

Following the notation of Section **??**, for $r, s \in \mathbb{Z}_{\geq 0}$, we shall denote by $\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi(\lambda, t)$ the associated partial derivative of ϕ at $(\lambda, t) \in I \times \mathbb{R}$, in case this derivative exists. Note that one can think of these partial derivatives as simply taking values in $\mathbb{F}$ since they are partial derivatives with respect to a single variable, cf. Theorem **??**. For such partial derivatives, we adapt our notation from above and denote

$$(\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)^\lambda(t) = (\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)_t(\lambda) = \boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi(\lambda, t).$$

The following result indicates the character of the function $\Phi_{\theta,\phi}$.

10.2.40 Theorem (Distributions applied to test signals with parameter dependence) *Let* $I \subseteq \mathbb{R}$ *be an interval, let* $k \in \mathbb{Z}_{\geq 0}$, *and let* $\phi\colon I \times \mathbb{R} \to \mathbb{F}$ *have the following properties:*

(i) for each $\lambda \in I$, *the map* $t \mapsto \phi(\lambda, t)$ *is an element of* $\mathscr{D}(\mathbb{R}; \mathbb{F})$;

(ii) there exists a compact interval $K \subseteq \mathbb{R}$ *such that* $\operatorname{supp}(\phi^\lambda) \subseteq K$ *for each* $\lambda \in I$;

(iii) for each $\mathrm{r} \in \mathbb{Z}_{\geq 0}$, $\mathbf{D}_1^{\mathrm{k}}\mathbf{D}_2^{\mathrm{r}}\phi \colon \mathrm{I} \times \mathbb{R} \to \mathbb{F}$ *is continuous.*

Then, for any $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, $\Phi_{\theta,\phi}$ *is* k*-times continuously differentiable and, moreover,*

$$\Phi_{\theta,\phi}^{(\mathrm{k})}(\lambda) = \theta((\mathbf{D}_1^{\mathrm{k}}\phi)^{\lambda}).$$

Proof We first give the proof for $k = 0$. Let $\lambda \in I$ and let $(\epsilon_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to zero and such that $\lambda + \epsilon_j \in I$ for every $j \in \mathbb{Z}_{>0}$. Define $\psi_j^{\lambda} \in \mathscr{D}(\mathbb{R};\mathbb{F})$ by

$$\psi_j^{\lambda}(t) = \phi(\lambda + \epsilon_j, t).$$

The following lemma is then useful.

1 Lemma *The sequence* $(\psi_{\mathrm{j}}^{\lambda})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *converges to* ϕ^{λ} *in* $\mathscr{D}(\mathbb{R};\mathbb{F})$*.*

Proof First of all, by hypothesis,

$$\operatorname{supp}(\psi_j^{\lambda}) \subseteq K, \qquad j \in \mathbb{Z}_{>0}.$$

Thus the functions ψ_j^{λ}, $j \in \mathbb{Z}_{>0}$, have support contained in a common compact set. Let $r \in \mathbb{Z}_{\geq 0}$. Let $I' \subseteq I$ be the smallest compact interval for which $\lambda + \epsilon_j \in I'$ for every $j \in \mathbb{Z}_{>0}$. Since $\boldsymbol{D}_2^r\phi|I' \times K$ is continuous with compact support, by Theorem **??** it follows that it is uniformly continuous. This implies that, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$|\boldsymbol{D}^r\psi_j^{\lambda}(t) - \boldsymbol{D}^r\phi^{\lambda}(t)| = |\boldsymbol{D}_2^r\phi(\lambda + \epsilon_j, t) - \boldsymbol{D}_2^r\phi(\lambda, t)| < \epsilon, \qquad j \geq N,\ t \in K.$$

Since $r \in \mathbb{Z}_{\geq 0}$ is arbitrary, this implies that we have the desired convergence of $(\psi_j^{\lambda})_{j\in\mathbb{Z}_{>0}}$ to ϕ^{λ}. ▼

It then follows immediately from continuity of θ that

$$\lim_{j\to\infty} \Phi_{\theta,\phi}(\lambda + \epsilon_j) = \lim_{j\to\infty} \theta(\phi^{\lambda+\epsilon_j}) = \theta(\lim_{j\to\infty} \phi^{\lambda+\epsilon_j}) = \theta(\lim_{j\to\infty} \psi_j^{\lambda}) = \theta(\phi^{\lambda}) = \Phi_{\theta,\phi}(\lambda).$$

Continuity of $\Phi_{\theta,\phi}$ at λ then follows from Theorem 3.1.3.

Now we prove the theorem when $k = 1$. We first note that, by hypothesis, $(\boldsymbol{D}_1\phi) \in \mathscr{D}(\mathbb{R};\mathbb{F})$. We let (ϵ_j) be a sequence, none of whose terms are zero, converging to zero as above. Now we take

$$\psi_j^{\lambda}(t) = \frac{\phi(\lambda + \epsilon_j, t) - \phi(\lambda, t)}{\epsilon_j}.$$

The following lemma is then key.

2 Lemma *The sequence* $(\psi_{\mathrm{j}}^{\lambda})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *converges to* $(\mathbf{D}_1\phi)^{\lambda}$ *in* $\mathscr{D}(\mathbb{R};\mathbb{F})$*.*

Proof First of all, by hypothesis,

$$\operatorname{supp}(\psi_j^{\lambda}) \subseteq K, \qquad j \in \mathbb{Z}_{>0}.$$

Thus the functions ψ_j^{λ}, $j \in \mathbb{Z}_{>0}$, have support contained in a common compact set.

Let $r \in \mathbb{Z}_{\geq 0}$. Let $I' \subseteq I$ be the smallest compact interval for which $\lambda + \epsilon_j \in I'$ for every $j \in \mathbb{Z}_{>0}$. Now define $\psi_r \colon I' \times K \to \mathbb{F}$ by

$$\psi_r(\ell, t) = \begin{cases} \frac{\boldsymbol{D}_2^r\phi(\ell,t) - \boldsymbol{D}_2^r\phi(\lambda,t)}{\ell-\lambda}, & \ell \neq \lambda, \\ \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t), & \ell = \lambda. \end{cases}$$

It is clear from the hypotheses that ψ_r is continuous on

$$\{(\ell, t) \in I' \times K \mid \ell \neq \lambda\}.$$

Moreover, since the derivative $\boldsymbol{D}_1\boldsymbol{D}_2^r\phi$ exists and is continuous,

$$\lim_{\ell \to \lambda} \frac{\boldsymbol{D}_2^r\phi(\ell, t) - \boldsymbol{D}_2^r\phi(\lambda, t)}{\ell - \lambda} = \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t),$$

showing that ψ_r is continuous on $I' \times K$ by Theorem 3.1.3. Since ψ_r has compact support, it is uniformly continuous by Theorem **??**. Therefore, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$|\psi_r(\lambda + \epsilon_j, t) - \psi_r(\lambda, t)| < \epsilon, \qquad j \geq N, \ t \in K.$$

Using the definition of ψ_r, this implies that, for every $j \geq N$ and for every $t \in K$,

$$\left| \frac{\boldsymbol{D}_2^r\phi(\lambda + \epsilon_j, t) - \boldsymbol{D}_2^r\phi(\lambda, t)}{\epsilon_j} - \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t) \right| = |\boldsymbol{D}^r\psi_j^\lambda(t) - \boldsymbol{D}^r(\boldsymbol{D}_1\phi^\lambda)(t)| < \epsilon.$$

Since $r \in \mathbb{Z}_{\geq 0}$ is arbitrary, this gives convergence of $(\psi_j^\lambda)_{j \in \mathbb{Z}_{>0}}$ to $(\boldsymbol{D}_1\phi)^\lambda$. ▼

By continuity of θ we then have

$$\begin{aligned} \lim_{j \to \infty} \frac{\Phi_{\theta,\phi}(\lambda + \epsilon_j) - \Phi_{\theta,\phi}(\lambda)}{\epsilon_j} &= \lim_{j \to \infty} \frac{\theta(\phi^{\lambda + \epsilon_j}) - \theta(\phi^\lambda)}{\epsilon_j} \\ &= \theta(\lim_{j \to \epsilon} \psi_j^\lambda) = \theta((\boldsymbol{D}_1\phi)^\lambda), \end{aligned}$$

showing that $\Phi_{\theta,\phi}$ is differentiable with derivative as stated in the theorem for the case of $k = 1$.

Now suppose that the theorem is true for $j \in \{0, 1, \dots, m\}$ and suppose that the hypotheses of the theorem hold for $k = m + 1$. We let $\psi = \boldsymbol{D}_1^m\phi$ and verify that ψ satisfies the hypotheses of the theorem for $k = 1$. First note that, for each $\lambda \in I$, $t \mapsto \psi(\lambda, t)$ is the mth derivative of an element $\mathscr{D}(\mathbb{R}; \mathbb{F})$ and so is an element of $\mathscr{D}(\mathbb{R}; \mathbb{F})$. Since $\mathrm{supp}(\psi^\lambda) \subseteq \mathrm{supp}(\phi^\lambda)$, we also have the second hypothesis of the theorem. Finally, since

$$\boldsymbol{D}_1\boldsymbol{D}_2^r\psi = \boldsymbol{D}_1\boldsymbol{D}_2^r\boldsymbol{D}_1^m\phi = \boldsymbol{D}_1^{m+1}\boldsymbol{D}_2^r\phi$$

by Theorem **??**, the final hypothesis of the theorem also holds. Therefore, by the induction hypothesis, $\Phi_{\theta,\psi}$ is continuously differentiable. But, since

$$\Phi_{\theta,\psi}(\lambda) = \theta((\boldsymbol{D}_1^m\phi)^\lambda) = \Phi_{\theta,\phi}^{(m)}(\lambda),$$

this implies that $\Phi_{\theta,\phi}$ is $m + 1$-times continuously differentiable, and

$$\Phi_{\theta,\phi}^{(m+1)}(\lambda) = \theta((\boldsymbol{D}_1^{m+1}\phi)^\lambda)$$

as desired. ■

The following result is almost immediate from the theorem.

10.2.41 Corollary (Property of distributions applied to test functions of two variables) *Let $\phi\colon \mathbb{R}^2 \to \mathbb{F}$ be infinitely differentiable with compact support. Then we have $\Phi_{\theta,\phi} \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Moreover, for each $\mathrm{r} \in \mathbb{Z}_{>0}$,*

$$\Phi^{(\mathrm{r})}_{\theta,\phi}(\mathrm{s}) = \theta(\mathbf{D}^{\mathrm{r}}_1\phi^{\mathrm{s}}).$$

Proof Since Theorem 10.2.40 implies that $\Phi_{\theta,\phi}$ is infinitely differentiable, one only needs to show that this function has compact support. Since ϕ has compact support, there exists compact intervals $I, J \subseteq \mathbb{R}$ such that $\mathrm{supp}(\phi) \subseteq I \times J$. If $s \in \mathbb{R}\setminus I$ that $\phi^s(t) = 0$ for all $t \in \mathbb{R}$, and this immediately gives $\theta(\phi^s) = 0$ and so $\mathrm{supp}(\Phi_{\theta,\phi}) \subseteq I$. ■

The following result will be useful when we discuss convolution.

10.2.42 Corollary (Distributions applied to a special class of test signals) *Denote by $\tau\colon \mathbb{R}^2 \to \mathbb{R}$ the map given by $\tau(\mathrm{s},\mathrm{t}) = \mathrm{t}-\mathrm{s}$. Then, if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ and $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$, then $\Phi_{\theta,\tau^*\phi} \in \mathscr{E}(\mathbb{R};\mathbb{F})$.*

Proof In order to use Theorem 10.2.40 we need to show that, for any $s \in \mathbb{R}$, there exists a neighbourhood U of s and a compact set $K \subseteq \mathbb{R}$ such that $\mathrm{supp}((\tau^*\phi)^s) \subseteq K$ for every $s \in U$. We take $U = (s-1, s+1)$. Let $K' = [a,b]$ be a compact interval such that $\mathrm{supp}(\phi) \subseteq K'$ and define $K = (a-1, b+1)$. If $t \in \mathbb{R}\setminus K$ and $s \in U$ then $t-s \in \mathbb{R}\setminus K'$, as can be directly verified. Thus $\tau^*\phi(s,t) = \phi(t-s) = 0$. This shows that $\mathrm{supp}((\tau^*\phi)^s) \subseteq K$ for every $s \in U$. Thus $\Phi_{\theta,\tau^*\phi}|U$ is infinitely differentiable by Theorem 10.2.40. Since this is true for a neighbourhood of any $s \in \mathbb{R}$, we conclude that ϕ is infinitely differentiable. ■

Next we consider the situation where a distribution is allowed to depend on a parameter. We first consider a rather general setup. Let $(\Lambda, \mathscr{A}, \mu)$ be a measure space (where we will suppose the parameters live) and suppose that $\theta\colon \Lambda \to \mathscr{D}'(\mathbb{R};\mathbb{F})$ is an assignment of a distribution to each parameter in Λ. Then we can define $F_\theta\colon \Lambda \times \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by

$$F_\theta(\lambda,\phi) = \langle\theta(\lambda);\phi\rangle,$$

using the notation mentioned in Notation 10.2.9. Correspondingly, let us define $F_{\theta,\phi}\colon \Lambda \to \mathbb{F}$ by

$$F_{\theta,\phi}(\lambda) = F_\theta(\lambda,\phi),$$

and let us suppose that, for each $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$, $F_{\theta,\phi} \in \mathsf{L}^{(1)}((\Lambda,\mathscr{A};\mu);\mathbb{F})$. We can then define $\Theta_\theta\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by

$$\Theta_\theta(\phi) = \int_\Lambda F_{\theta,\phi}\,\mathrm{d}\mu.$$

The next result indicates when Θ_θ is a distribution.

10.2.43 Proposition (Distributions arising from integrating parameters) *Let $(\Lambda, \mathscr{A}, \mu)$ be a measure space, let $\theta\colon \Lambda \to \mathscr{D}'(\mathbb{R};\mathbb{F})$, and (with the notation above) suppose that $\mathrm{F}_{\theta,\phi} \in \mathsf{L}^{(1)}((\Lambda,\mathscr{A},\mu);\mathbb{F})$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. If, for every converging sequence $(\phi_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$, there exists $\mathrm{M} \in \mathbb{R}_{>0}$ such that the function*

$$\lambda \mapsto \sup\{|\mathrm{F}_{\theta,\phi_{\mathrm{j}}}(\lambda)| \mid \mathrm{j} \in \mathbb{Z}_{>0}\}$$

is μ-integrable, then $\Theta_\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$.

Proof Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero. For $\lambda\in\Lambda$ we have

$$\lim_{j\to\infty} F_{\theta,\phi_j}(\lambda) = \lim_{j\to\infty}\langle\theta(\lambda);\phi_j\rangle = 0,$$

using continuity of $\theta(\lambda)$. Define

$$\overline{F}_\theta(\lambda) = \sup\{|F_{\theta,\phi_j}(\lambda)| \mid j\in\mathbb{Z}_{>0}\},$$

and note that $\overline{F}_\theta$ is measurable by Propositions 5.6.11 and 5.6.18, and integrable by hypothesis. Moreover,

$$|F_{\theta,\phi_j}(\lambda)| \le \overline{F}_\theta(\lambda)$$

for every $\lambda\in\Lambda$ and $j\in\mathbb{Z}_{>0}$. Therefore, by the Dominated Convergence Theorem,

$$\lim_{j\to\infty}\Theta_\theta(\phi_j) = \lim_{j\to\infty}\int_\Lambda F_{\theta,\phi_j}\,\mathrm{d}\mu = \int_\Lambda \lim_{j\to\infty} F_{\theta,\phi_j}\,\mathrm{d}\mu = 0,$$

giving the desired continuity. ■

10.2.9 Fubini's Theorem for distributions

Let ϕ be an infinitely differentiable function with compact support in *two* variables. Then we can define $\phi^s(t) = \phi(s,t)$, so defining an element $\phi^s\in\mathscr{D}(\mathbb{R};\mathbb{F})$. By Corollary 10.2.41, the function $s\mapsto\theta_f(\phi^s)$ is in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Let us denote this function by $\Phi_{f,\phi}$. Now suppose that $g\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ has the property that $g\Phi_{f,\phi}$ is integrable. Thus the integral

$$\int_{\mathbb{R}} g(s)\Big(\int_{\mathbb{R}} f(t)\phi(s,t)\,\mathrm{d}t\Big)\,\mathrm{d}s$$

exists. One can now swap the rôles of s and t in the above and ask whether the integral

$$\int_{\mathbb{R}} f(t)\Big(\int_{\mathbb{R}} g(s)\phi(s,t)\,\mathrm{d}s\Big)\,\mathrm{d}t$$

exists, and whether the above two integrals agree. This is, of course, a place where one might be able to apply Fubini's Theorem for integration. Indeed, because ϕ is assumed to have compact support, one can show that the hypotheses of Fubini's Theorem apply. What we wish to do in this section is replace θ_f and θ_g with arbitrary distributions θ and ρ, and ask when one can swap the rôles of θ and ρ when we apply the distributions to a function of two variables.

We adopt the notation of Section 10.2.8, and for $f\colon\mathbb{R}^2\to\mathbb{F}$ define $f^s, f_t\colon\mathbb{R}\to\mathbb{F}$ by $f^s(t) = f_t(s) = f(s,t)$. Let $\phi\colon\mathbb{R}^2\to\mathbb{F}$ be infinitely differentiable with compact support. Note that ϕ^s and ϕ_t are elements of $\mathscr{D}(\mathbb{R};\mathbb{F})$ for every $s,t\in\mathbb{R}$ by Theorem **??**. Let $\theta\in\mathscr{D}'(\mathbb{R};\mathbb{F})$ and define $\Phi_{\theta,\phi},\Psi_{\theta,\phi}\colon\mathbb{R}\to\mathbb{F}$ by

$$\Phi_{\theta,\phi}(s) = \theta(\phi^s),\quad \Psi_{\theta,\phi}(t) = \theta(\phi_t).$$

By Corollary 10.2.41, if we have $\theta,\rho\in\mathscr{D}'(\mathbb{R};\mathbb{F})$, then the expressions $\theta(\Phi_{\rho,\phi})$ and $\rho(\Psi_{\theta,\phi})$ make sense. Fubini's Theorem for distributions says that they agree.

10.2.44 Theorem (Fubini's Theorem for distributions) *For an infinitely differentiable function $\phi\colon \mathbb{R}^2 \to \mathbb{F}$ with compact support and for $\theta, \rho \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, $\theta(\Phi_{\rho,\phi}) = \rho(\Psi_{\theta,\phi})$.*

Proof We begin with technical lemma. Let $\mathscr{D}(\mathbb{R}^2;\mathbb{F})$ be the set of infinitely differentiable functions from $\mathbb{R}^2$ to $\mathbb{F}$ with compact support. Note that the map $\iota\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \times \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathscr{D}(\mathbb{R}^2;\mathbb{F})$ given by

$$\iota(\psi \times \chi)(s,t) = \psi(s)\chi(t)$$

is an injection.

1 Lemma *If $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ then there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\operatorname{span}_{\mathbb{F}}(\operatorname{image}(\iota))$ with the following properties:*

(i) there exists a compact set $K \subseteq \mathbb{R}^2$ such that $\operatorname{supp}(\phi_j) \subseteq K$ for each $j \in \mathbb{Z}_{>0}$;

(ii) for each $k, l \in \mathbb{Z}_{>0}$, the sequence $(\boldsymbol{D}_1^k\boldsymbol{D}_2^l\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\boldsymbol{D}_1^k\boldsymbol{D}_2^l\phi$.

Proof ▼

Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\operatorname{image}(\iota)$ as in the lemma above. For each $j \in \mathbb{Z}_{>0}$ we write

$$\phi_j(s,t) = \sum_{k=1}^{m_j} \psi_{j,k}(s)\chi_{j,k}(t).$$

Note that linearity of θ allows us to write

$$\Phi_{\rho,\phi_j}(s) = \rho(\phi_j^s) = \rho\Big(\sum_{k=1}^{m_j} \psi_{j,k}(s)\chi_{j,k}\Big) = \sum_{k=1}^{m_j} \psi_{j,k}(s)\rho(\chi_{j,k}).$$

Similarly,

$$\Psi_{\theta,\phi_j}(t) = \sum_{k=1}^{m_j} \chi_{j,k}(t)\theta(\psi_{j,k}).$$

Now we use another lemma.

2 Lemma *If $\phi \in \mathscr{D}(\mathbb{R}^2;\mathbb{F})$ and if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{D}(\mathbb{R}^2;\mathbb{F})$ for which*

(i) there exists a compact set $K \subseteq \mathbb{R}^2$ such that $\operatorname{supp}(\phi_j) \subseteq K$ for each $j \in \mathbb{Z}_{>0}$ and

(ii) for each $k, l \in \mathbb{Z}_{>0}$, the sequence $(\boldsymbol{D}_1^k\boldsymbol{D}_2^l\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\boldsymbol{D}_1^k\boldsymbol{D}_2^l\phi$,

then

$$\lim_{j\to\infty} \theta(\Phi_{\rho,\phi_j}) = \theta(\Phi_{\rho,\phi}), \quad \lim_{j\to\infty} \rho(\Psi_{\theta,\phi_j}) = \rho(\Psi_{\theta,\phi}).$$

Proof Clearly it suffices to prove that $\lim_{j\to\infty} \rho(\Psi_{\theta,\phi_j}) = \rho(\Psi_{\theta,\phi})$ as the other conclusion follows in a similar manner.

We first claim that $(\Psi_{\theta,\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to $\Psi_{\theta,\phi}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Suppose otherwise so that, for some $r \in \mathbb{Z}_{\geq 0}$, the sequence $(\Psi^{(r)}_{\theta,\phi_j})_{j\in\mathbb{Z}_{>0}}$ does not converge uniformly to $\Psi^{(r)}_{\theta,\phi}$. Thus, possibly by replacing $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ with a subsequence, there exists $\alpha \in \mathbb{R}_{>0}$ such that, for each $j \in \mathbb{Z}_{>0}$, there exists $t_j \in \mathbb{R}$ such that

$$|\Psi^{(r)}_{\theta,\phi_j}(t_j) - \Psi^{(r)}_{\theta,\phi}(t_j)| \geq \alpha.$$

By Corollary 10.2.41 we have

$$\Psi^{(r)}_{\theta,\phi}(t) = \theta(\boldsymbol{D}_2^r\phi_t), \quad \Psi^{(r)}_{\theta,\phi_j}(t) = \theta(\boldsymbol{D}_2^r\phi_{j,t}), \qquad j \in \mathbb{Z}_{>0}.$$

Therefore, for each $j \in \mathbb{Z}_{>0}$, there exists $t_j \in \mathbb{R}$ such that

$$|\theta(\boldsymbol{D}_2^r\phi_{j,t_j}) - \theta(\boldsymbol{D}_2^r\phi_{t_j})| \geq \alpha.$$

Now let $\epsilon \in \mathbb{R}_{>0}$. By (ii) there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sup\{|\boldsymbol{D}_1^m\boldsymbol{D}_2^r(\phi_j - \phi)(s,t)| \mid s,t \in \mathbb{R}\} < \epsilon$$

for $j \geq N$. Thus

$$\sup\{|\boldsymbol{D}_1^m\boldsymbol{D}_2^r(\phi_j - \phi)(s,t_j)| \mid s \in \mathbb{R}\} < \epsilon$$

for all $j \geq N$. Thus $(\phi_{j,t_j})_{j\in\mathbb{Z}_{>0}}$ converges to ϕ_{t_j} in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Thus, by continuity of θ,

$$\lim_{j\to\infty}|\theta(\boldsymbol{D}_2^r\phi_{j,t_j}) - \theta(\boldsymbol{D}_2^r\phi_{t_j})| = 0.$$

This contradiction implies that, indeed, $(\Psi_{\theta,\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to $\Psi_{\theta,\phi}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$.

From this the lemma immediately follows from the continuity of ρ. ▼

Then

$$\theta(\Phi_{\rho,\phi_j}) = \sum_{k=1}^{m_j} \theta(\psi_{j,k})\rho(\chi_{j,k}) = \rho(\Psi_{\theta,\phi_j}),$$

as desired. ■

10.2.10 Some deeper properties of distributions

Thus far, it has pretty much been fun and games for distributions. However, there comes a time when one wants to understand what a distribution "really is." After all, all we have done thus far is to show that distributions have some useful properties and signals can be represented by distributions in a manner which seems reasonably apt. However, how do we know whether distributions are not just too good to be true? There is certainly some evidence for this in that (1) all distributions are differentiable and (2) differentiation can always be swapped with limits. To make distributions really respectable, we need to be able to say something useful about their structure. That is to say, we need to see if there is there a nice way to think of distributions that has some relationship with something we believe we understand. We address this in two ways: (1) by showing that distributions are limits of locally integrable signals; and (2) by showing that distributions are, in an appropriate sense, always derivatives of some order of locally integrable signals.

The first result we state, we do not prove since the most natural proof involves convolution which we consider in Chapter 11. However, now is a good time to at least state the result.

10.2.45 Theorem (Distributions are limits of locally integrable signals) *If $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then there exists a sequence $(\mathrm{f_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$ such that the sequence $(\theta_{\mathrm{f_j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to θ.*

Proof In Theorem 11.3.26 we will show something even stronger, namely that there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ such that the sequence $(\theta_{\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to θ in $\mathscr{D}'(\mathbb{R};\mathbb{F})$. ■

Next we show that a distribution is always a finite derivative of a locally integrable signal. To prove this result we first need a technical fact. The fact is a local one concerning the behaviour of distributions. To understand the result we introduce a subset of the set $\mathscr{D}(\mathbb{R};\mathbb{F})$ of test signals. Let $\mathbb{T} = [a,b]$ be a compact time-domain and let $\mathscr{D}(\mathbb{T};\mathbb{F})$ denote those test signals whose support is contained in $\mathbb{T}$. A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}} \subseteq \mathscr{D}(\mathbb{T};\mathbb{F})$ ***converges to zero*** in $\mathscr{D}(\mathbb{T};\mathbb{F})$ if it converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$.

10.2.46 Lemma (A local boundedness property for distributions) *Let $\mathbb{T}$ be a compact time-domain and let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$. Then there exists $\mathrm{M} \in \mathbb{R}_{>0}$ and $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ such that for each $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$ we have*

$$|\theta(\phi)| \leq \mathrm{M}\|\phi^{(\mathrm{k})}\|_\infty$$

Proof Let $\mathbb{T} = [a,b]$. First note that the sequences $((b-a)^m\|\phi_j^{(m)}\|_\infty)_{j\in\infty}$, $m \in \mathbb{Z}_{\geq 0}$, converge to zero if and only if the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Note that, since for $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$ we have

$$\phi^{(m)}(t) = \int_a^t \phi^{(m+1)}(\tau)\,\mathrm{d}\tau,$$

we have

$$\|\phi^{(m)}\|_\infty \leq (b-a)\|\phi^{(m+1)}\|_\infty \tag{10.5}$$

for every $m \in \mathbb{Z}_{\geq 0}$.

Now we proceed with the proof proper, using contradiction. Suppose that it is not possible to find such an M and k as asserted in the theorem statement. Then for each $j \in \mathbb{Z}_{>0}$ there exists a nonzero $\phi_j \in \mathscr{D}(\mathbb{T};\mathbb{F})$ such that

$$|\theta(\phi_j)| > j(b-a)^j\|\phi_j^{(j)}\|_\infty. \tag{10.6}$$

Then define

$$\psi_j = \frac{\phi_j}{j(b-a)^j\|\phi_j^{(j)}\|_\infty},$$

noting that $\psi_j \in \mathscr{D}(\mathbb{T};\mathbb{F})$. Then we have, for $m \leq j$,

$$(b-a)^m\|\psi_j^{(m)}\|_\infty = \frac{(b-a)^m\|\phi_j^{(m)}\|_\infty}{j(b-a)^j\|\phi_j^{(j)}\|_\infty} \leq \frac{1}{j}$$

since, by (10.5),

$$(b-a)^m\|\phi_j^{(m)}\|_\infty \leq (b-a)^j\|\phi_j^{(j)}\|_\infty, \qquad m < j.$$

Therefore, for each $m \in \mathbb{Z}_{>0}$ the sequence $((b-a)^m \|\psi_j^{(m)}\|_\infty)_{j\in\mathbb{Z}_{>0}}$ converges to zero, and so $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{T};\mathbb{F})$ according to the observation with which we began the proof. Therefore the sequence $(\theta(\psi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero. However, we also have

$$\theta(\psi_j) = \frac{\theta(\phi_j)}{j(b-a)^j \|\phi^{(j)}\|_\infty} > 1$$

by (10.6), thus arriving at a contradiction. ■

We now have the following rather non-obvious result.

10.2.47 Theorem (Distributions are locally finite-order derivatives of locally integrable signals) *Let $\mathbb{T}$ be a compact continuous time-domain. If $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then there exists $\mathrm{r} \in \mathbb{Z}_{\geq 0}$ and $\mathrm{f}_\theta \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ such that $\theta(\phi) = \theta^{(\mathrm{r})}_{\mathrm{f}_\theta}(\phi)$ for every $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$. Furthermore, we may take $\mathrm{r} = \mathrm{k}+1$ where $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ is as given by Lemma 10.2.46.*

Proof Let $M \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{>0}$ be chosen as in Lemma 10.2.46 so that for each $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$ we have $\theta(\phi) \leq M\|\phi^{(k)}\|_\infty$. Denote

$$\mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F}) = \{\phi^{(k+1)} \mid \phi \in \mathscr{D}(\mathbb{T};\mathbb{F})\},$$

noting that this is a subspace of $\mathscr{D}(\mathbb{T};\mathbb{F})$. On $\mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F})$ we consider the norm $\|\cdot\|_1$:

$$\|\phi^{(k+1)}\|_1 = \int_{\mathbb{T}} |\phi^{(k+1)}(t)|\,\mathrm{d}t.$$

Define a linear map $\alpha_\theta\colon \mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F}) \to \mathbb{F}$ by $\alpha_\theta(\phi^{(k+1)}) = \theta(\phi)$. We claim that α_θ is continuous using the norm $\|\cdot\|_1$. To see this, let $(\phi_j^{(k+1)})_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F})$ converging to zero in the norm $\|\cdot\|_1$. Thus, for any $\epsilon \in \mathbb{R}_{>0}$ we have $N \in \mathbb{Z}_{>0}$ such that

$$\int_{\mathbb{T}} |\phi_j^{(k+1)}(t)|\,\mathrm{d}t < \epsilon, \qquad j \geq N.$$

We then have

$$\begin{aligned} \phi_j^{(k)}(t) &= \int_{-\infty}^t \phi_j^{(k+1)}(\tau)\,\mathrm{d}\tau \\ \implies \quad |\phi_j^{(k)}(t)| &\leq \int_{-\infty}^t |\phi_j^{(k+1)}(\tau)|\,\mathrm{d}\tau \\ \implies \quad \|\phi_j^{(k)}\|_\infty &\leq \int_{\mathbb{T}} |\phi_j^{(k+1)}(t)|\,\mathrm{d}t. \end{aligned}$$

Therefore, for $\epsilon \in \mathbb{R}_{>0}$, if we choose $N \in \mathbb{Z}_{>0}$ sufficiently large that

$$\int_{\mathbb{T}} |\phi_j^{(k+1)}(t)|\,\mathrm{d}t \leq \frac{\epsilon}{M}, \qquad j \geq N,$$

then we have

$$|\alpha_\theta(\phi_j)| = |\theta(\phi_j)| \leq M\|\phi_j^{(k)}\|_\infty \leq M\int_{\mathbb{T}} |\phi_j^{(k+1)}(t)|\,\mathrm{d}t \leq \epsilon, \qquad j \geq N.$$

Thus the sequence $(\alpha_\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero, thus verifying our claim that α_θ is continuous at 0, and so continuous by virtue of Theorem 6.5.8.

Now think of $\mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F})$ as a subspace of $\mathsf{L}^{(1)}(\mathbb{T};\mathbb{F})$. By the Hahn–Banach Theorem, Theorem **??**, there exists a continuous linear map $\bar{\alpha}_\theta\colon \mathsf{L}^{(1)}(\mathbb{T};\mathbb{F})\to\mathbb{F}$ which agrees with α_θ on $\mathscr{D}^{(k+1)}(\mathbb{T};\mathbb{F})$. By Theorem **??** there exists $f_\theta\in\mathsf{L}^{(\infty)}(\mathbb{T};\mathbb{F})$ such that

$$\bar{\alpha}_\theta(\phi^{(k+1)}) = \int_{\mathbb{T}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t.$$

From this we immediately deduce

$$\theta(\phi) = \bar{\alpha}_\theta(\phi^{(k+1)}) = \theta_{g_\theta}(\phi^{(k+1)}) = (-1)^{k+1}\theta_{f_\theta}^{(k+1)}(\phi),$$

which is the result, since $\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})\subseteq\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$. ■

10.2.48 Remark (Distributions are finite-order derivatives of continuous signals) Note that since f in the statement of Theorem 10.2.47 is in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$, the signal

$$g(t) = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

is locally absolutely continuous. Thus we may deduce directly that $\theta(\phi) = \theta_g^{(r)}(\phi)$ for $\phi\in\mathscr{D}(\mathbb{T};\mathbb{F})$ where g is *continuous* (indeed, locally absolutely continuous). •

10.2.11 The order of a distribution

The local characterisation of distributions as derivatives leads one naturally to talk about the order of a distribution, and it is this we now do.

10.2.49 Definition (Order of a distribution) Let $\theta\in\mathscr{D}'(\mathbb{R};\mathbb{F})$.

(i) For a compact continuous time-domain $\mathbb{T}$ the $\mathbb{T}$-***order*** of θ is the smallest nonnegative integer k for which there exists a signal $f\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ satisfying

$$\theta(\phi) = \theta_f^{(k+1)}(\phi) \tag{10.7}$$

for all $\phi\in\mathscr{D}(\mathbb{T};\mathbb{F})$.

(ii) The ***order*** of θ is the smallest nonnegative integer k for which there exists a signal $f\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ satisfying (10.7) for each $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})$. If no such integer exists then θ has ***infinite order***. •

Some examples clarify the definitions.

10.2.50 Examples (Order of a distribution)

1. Note that $\delta_0 = \theta_1^{(1)}$. Thus δ_0 has $\mathbb{T}$-order zero for any $\mathbb{T}$, and also has order zero.
2. The distribution $\delta_0^{(m)}$ has $\mathbb{T}$-order zero if $0\notin\mathrm{int}(\mathbb{T})$. Indeed, if $0\notin\mathrm{supp}(\phi)$ then $\delta^{(m)}(\phi) = 0$. On the other hand, if $0\in\mathrm{int}(\mathbb{T})$ then the $\mathbb{T}$-order of $\delta_0^{(m)}$ is m. Indeed, in this case we have $\delta_0^{(m)}(\phi) = \theta_1^{(m+1)}(\phi) = (-1)^m\phi^{(m)}(0)$.

3. We consider the distribution

$$\theta = \sum_{n=1}^{\infty} n\delta_n^{(n)}.$$

First, we claim that the sum defines a distribution. To see this, let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero. Then there exists $N \in \mathbb{Z}_{>0}$ sufficiently large that $\mathrm{supp}(\phi) \subseteq [-N, N]$. Then we have

$$\theta(\phi_j) = \sum_{n=1}^{\infty} n(-1)^n \phi_j^{(n)}(n) = \sum_{n=1}^{N} n(-1)^n \phi_j^{(n)}(n).$$

The last sum is a finite sum of terms going to zero as $j \to \infty$ so we have $\lim_{j\to\infty} \theta(\phi_j) = 0$, thus $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$. If $\mathbb{T} \cap (1,\infty) = \emptyset$ then the $\mathbb{T}$-order of θ is zero since $\theta(\phi) = 0$ for any $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$. If $\max \mathbb{T} \cap \mathbb{Z} = N$ then the $\mathbb{T}$-order of θ is N, as is easily verified. Note that θ has infinite order since for any N one can find a $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ with $[-N, N] \in \mathrm{int}(\mathrm{supp}(\phi))$. •

If θ is a distribution of order k and if f is at least $k+1$-times continuously differentiable, then it is possible to define the product of θ with f to be the distribution $f\theta$ given by $(f\theta)(\phi) = \theta(f\phi)$. The following result indicates how this is done.

10.2.51 Proposition (Multiplication of distributions of finite order by functions that are finitely differentiable) *Let $\theta = \theta_g^{(k+1)}$ be a distribution of order k and let $f \in C^r(\mathbb{R};\mathbb{F})$ for $r \geq k+1$. Then the map*

$$\mathscr{D}(\mathbb{R};\mathbb{F}) \ni \phi \mapsto (-1)^{k+1}\theta_g((f\phi)^{(k+1)}) \in \mathbb{F}$$

defines a distribution which we denote $f\theta$.

Proof Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. Choose $T \in \mathbb{R}_{>0}$ such that $\mathrm{supp}(\phi_j) \subseteq [-T, T]$ for all $j \in \mathbb{Z}_{>0}$ and then note that

$$\begin{aligned}
|(f\theta)(\phi_j)| &= \left|(-1)^{k+1} \int_{\mathbb{R}} g(t)(f\phi)^{(k+1)}(t)\,\mathrm{d}t\right| \\
&\leq \left|\int_{\mathbb{R}} g(t) \sum_{m=0}^{k+1} f^{(m)}(t)\phi^{(k+1-m)}(t)\,\mathrm{d}t\right| \\
&= \|g\|_\infty \sum_{m=0}^{k+1} \|f^{(m)}\|_\infty \int_{-T}^{T} |\phi_j^{(k+1-m)}(t)|\,\mathrm{d}t,
\end{aligned}$$

where the ∞-norms are with respect to $[-T, T]$. Since the integrands go to zero uniformly in t, if we take the limit as $j \to \infty$ we can switch this with the integration by Theorem 3.5.23 and we get $\lim_{j\to\infty}|(f\theta)(\phi_j)| = 0$, as desired. ■

Finally, we discuss how simple distributions may be made to take not just test signals as argument, but signals that are merely differentiable to some extent. This sort of construction is often important in applications since one often wishes to give as an argument to a distribution something other than a test signal. The following result indicates why this is possible.

10.2.52 Theorem (Distributions of finite order only depend on finitely many derivatives) *Let $\mathbb{T}$ be a compact continuous time-domain, let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ have $\mathbb{T}$-order* k*, and let $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{T};\mathbb{F})$ satisfy*

$$\phi_1^{(j)}(t) = \phi_2^{(j)}(t), \qquad j \in \{0, 1, \dots, k+1\},\ t \in \mathbb{T} \cap \operatorname{supp}(\theta).$$

Then $\theta(\phi_1) = \theta(\phi_2)$.

Proof By Theorem 10.2.47 let $f_\theta \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ have the property that $\theta(\phi) = \theta^{(k+1)}_{f_\theta}(\phi)$ for all $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$. Note that since $\operatorname{supp}(\theta)$ is closed, $\mathbb{R} \setminus \operatorname{supp}(\theta)$ is open, and so is a finite or countable union of open intervals by Proposition 2.5.6. Since $\mathbb{T}$ is compact, only a finite number of these open intervals will intersect $\mathbb{T}$, and we denote these intervals by $(t_{1,m}, t_{2,m})$, $m \in \{1, \dots, n\}$ and we suppose that

$$t_{1,1} < \inf \mathbb{T} < t_{2,1} < t_{1,2} < t_{2,2} < \cdots < t_{1,n} < \sup \mathbb{T} < t_{2,n}.$$

It is convenient for our purposes to redefine $t_{1,1} = \inf \mathbb{T}$ and $t_{2,n} = \sup \mathbb{T}$. Then, for $\phi \in \mathscr{D}(\mathbb{T};\mathbb{F})$,

$$\begin{aligned}\theta(\phi) &= (-1)^{k+1} \int_{\mathbb{R}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t \\ &= \sum_{m=1}^{n} (-1)^{k+1} \int_{t_{1,m}}^{t_{2,m}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t + (-1)^{k+1} \int_{\mathbb{T}\cap\operatorname{supp}(\theta)} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t.\end{aligned}$$

The second term obviously only depends on $\phi^{(k+1)}(t)$ for $t \in \mathbb{T} \cap \operatorname{supp}(\theta)$. As for the first term, referring to our discussion of Section 10.2.4, we see that $f_\theta^{(k+1)}$ agrees with the zero distribution on each interval $(t_{1,m}, t_{2,m})$. We may then apply the integration by parts, Proposition 10.2.39, $k+1$ times to each term in

$$\sum_{m=1}^{n} (-1)^{k+1} \int_{t_{1,m}}^{t_{2,m}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t$$

to see that it depends only on $\phi^{(j)}(t_{1,m})$ and $\phi^{(j)}(t_{2,m})$ for $j \in \{0, 1, \dots, k+1\}$ and $m \in \{1, \dots, n\}$. Since $t_{1,m}, t_{2,m} \in \operatorname{supp}(\theta)$ for $m \in \{1, \dots, n\}$, the result follows. ■

Let us see how this works in an example.

10.2.53 Example (The delta-function evaluated on differentiable signals) Let us consider δ_0. Suppose that $f \in \mathsf{C}^1_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ is a signal with compact support containing 0 in its interior. Then we define

$$\delta_0(f) = (-1) \int_{\mathbb{R}} \mathsf{1}(t) f^{(1)}(t)\,\mathrm{d}t = -\int_0^\infty f^{(1)}\,\mathrm{d}t = -f(t)\Big|_0^\infty = f(0).$$

Thus the delta-signal acts on differentiable signals just as it does on test signals. Note that since $\delta_0(f)$ only depends on the value of f at $t = 0$, we ought to really be able to define δ_0 on signals in $\mathsf{C}^0(\mathbb{R};\mathbb{F})$. We shall see in Corollary 10.5.29 that it is indeed possible to do this. •

10.2.12 Measures as distributions

Exercises

10.2.1 Show that if $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R};\mathbb{F})$ then $\phi_1\phi_2 \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Thus $\mathscr{D}(\mathbb{R};\mathbb{F})$ is an algebra.

10.2.2 Which of the following signals is in $\mathscr{D}(\mathbb{R};\mathbb{F})$? For signals not in $\mathscr{D}(\mathbb{R};\mathbb{F})$, explain why they are not.

(a) $f(t) = \begin{cases} 1+\cos t, & t \in [-\pi,\pi], \\ 0, & \text{otherwise.} \end{cases}$

(b) $f(t) = \begin{cases} \wedge(t+1), & t \in [-2,-1], \\ \wedge(0), & t \in (-1,1), \\ \wedge(t-1), & t \in [1,2], \\ 0, & \text{otherwise.} \end{cases}$

(c) $f(t) = \arctan(t)$.

10.2.3 Which of the following sequences $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ of signals in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$? For sequences not converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, explain why they do not.

(a) $\phi_j(t) =$***missing stuff***

10.2.4 Let $\mathsf{L}\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ be defined by

$$\mathsf{L}(\phi) = \int_{\mathbb{R}} \phi(t)\,\mathrm{d}t.$$

Is L a distribution?

10.2.5 Recall from Example 8.1.6–2 the map $\sigma\colon \mathbb{R}\to\mathbb{R}$ defined by $\sigma(t) = -t$. For a signal f define $\sigma^*f(t) = f(-t)$, and for $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ define $\sigma^*\theta\colon \mathscr{D}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by $\sigma^*\theta(\phi) = \theta(\sigma^*\phi)$.

(a) Show that $\sigma^*\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$.

Recall that a signal f is ***even*** if $\sigma^*f = f$ and ***odd*** if $\sigma^*f = -f$. Say, then, that $\theta \in \mathscr{D}(\mathbb{R};\mathbb{F})$ is ***even*** if $\sigma^*\theta = \theta$ and ***odd*** if $\sigma^*\theta = -\theta$.

(b) Show that the following are equivalent:
1. θ is even;
2. $\theta(\phi) = 0$ for every odd $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$.

(c) Show that the following are equivalent:
1. θ is odd;
2. $\theta(\phi) = 0$ for every even $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$.

10.2.6 Consider the sequence of locally integrable signals $(f_j(t) = \sin(jt))_{j\in\mathbb{Z}_{>0}}$.

(a) Show that the sequence converges pointwise only at points $t = 2n\pi$, $n \in \mathbb{Z}$.

(b) Show that the sequence of distributions $(\theta_{f_j})_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to the zero distribution.

10.2.7 Let $\theta_1, \theta_2 \in \mathscr{D}'(\mathbb{R}; \mathbb{F})$ satisfy

1. $\operatorname{supp}(\theta_1) = \operatorname{supp}(\theta_2)$ and
2. $\theta_1(\phi) = \theta_2(\phi)$ for every $\phi \in \mathscr{D}(\mathbb{R}; \mathbb{F})$ for which $\operatorname{supp}(\phi) \subseteq \operatorname{supp}(\theta_1) = \operatorname{supp}(\theta_2)$.

Show that $\theta_1 = \theta_2$.

10.2.8 For the signal $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} \frac{1}{\sqrt{t}}, & t \in \mathbb{R}_{>0}, \\ 0, & t \in \mathbb{R}_{\leq 0}, \end{cases}$$

answer the following questions.

(a) Show that $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$, so that f defines a distribution θ_f.

(b) Show that the product of f with itself is not in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$, and so cannot be used to define a distribution in a direct manner.

10.2.9 Show that

$$\mathscr{D}^{(1)}(\mathbb{R}; \mathbb{F}) = \left\{\phi \in \mathscr{D}(\mathbb{R}; \mathbb{F}) \;\middle|\; \int_{\mathbb{R}} \phi(t)\,\mathrm{d}t = 0\right\}.$$

10.2.10 We did not define generalised discrete-time signals. The reason is that they are not necessary. Show how one may define the analogue of a delta-signal for discrete-time signal by asking that it have properties like those of the delta-signal. (Part of the question is that you should figure out what should be the adaptations to the discrete-time case of the properties in continuous-time case.)

10.2.11 Recall from Example 8.1.6–1 the map $\tau_a\colon \mathbb{R} \to \mathbb{R}$ defined by $\tau_a(t) = t - a$.

(a) Show that if $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{F})$ then $\tau_a^* f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{F})$ and show that $\tau_a^*\theta_f = \theta_{\tau_a^* f}$.

(b) Show that

$$\operatorname{supp}(\tau_a^*\theta) = \{t + a \mid t \in \operatorname{supp}(\theta)\}.$$

10.2.12 Let us revisit the mass/spring example of Section 10.1. The governing differential equation is

$$m\ddot{x}(t) + kx(t) = F(t). \tag{10.8}$$

For simplicity, take $m = k = 1$.

(a) Show by direct substitution that the solution to (10.8) is given by

$$x(t) = \int_0^t \sin(t - \tau)F(\tau)\,\mathrm{d}\tau$$

if the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
Hint: *First show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t, \tau)\,\mathrm{d}\tau = f(t, t) + \int_0^t \frac{\partial f(t, \tau)}{\partial t}\,\mathrm{d}\tau,$$

provided that all operations make sense.

For $\epsilon \in \mathbb{R}_{>0}$ define

$$F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & t \in [0, \epsilon], \\ 0, & \text{otherwise.} \end{cases}$$

(b) Compute the solution $x_\epsilon(t)$ to (10.8) when $F = F_\epsilon$ and with zero initial conditions.

(c) Plot x_ϵ for values of ϵ decreasing to zero, and comment on what the resulting solution seems to be converging to.

(d) Now consider the differential equation

$$\theta^{(2)} + \theta = \delta_0$$

for the distribution θ. Show that taking $\theta = \theta_x$ with $x(t) = 1(t)\sin t$ solves the differential equation. How does this compare to the limiting solution from part (c)? Does x satisfy the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$?

Section 10.3

Tempered distributions

The distributions $\mathscr{D}'(\mathbb{R};\mathbb{F})$ considered in the previous section are the most general sort of distribution we consider here. However, they can be, in some way, too general for some purposes. In particular, when we use distributions in the theory of Fourier transforms in Chapter 13 we will see that the setup with test signals with compact support leads to an asymmetry in the theory. Therefore, in this section we provide a different setup for distributions that utilises a larger class of test signals, giving rise to a correspondingly smaller class of distributions.

Do I need to read this section? Tempered distributions are rather important in the theory of the continuous-continuous Fourier transform which we present in Chapter 13. Thus readers interested in learning this part of the theory will need to know about tempered distributions. •

10.3.1 The Schwartz space of test signals

The set of test functions we consider in this section do not have compact support, but they do decay quickly at infinity. The following definition makes this precise.

10.3.1 Definition (Signal of rapid decay) A ***signal of rapid decay*** is a signal $f\colon \mathbb{R}\to\mathbb{F}$ having the property that for any $k\in\mathbb{Z}_{>0}$, $\lim_{|t|\to\infty} t^k f(t) = 0$. •

A useful characterisation of locally integrable signals of rapid decay is the following. The idea is that the signal can be multiplied by any polynomial and remain locally integrable.

10.3.2 Proposition (A property of locally integrable signals of rapid decay) *If* $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *is a signal of rapid decay then for each* $k\in\mathbb{Z}_{\geq 0}$ *the signal* $t\mapsto t^k f(t)$ *is in* $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$.

Proof Let $T\in\mathbb{R}_{>0}$ have the property that $|t^{k+2}f(t)|\leq 1$ for all $|t|\geq T$. This is possible since f is of rapid decay. Then we have

$$\begin{aligned}\int_{\mathbb{R}} |t^k f(t)|\,\mathrm{d}t &= \int_{-\infty}^{-T} |t^k f(t)|\,\mathrm{d}t + \int_{-T}^{T} |t^k f(t)|\,\mathrm{d}t + \int_{T}^{\infty} |t^k f(t)|\,\mathrm{d}t \\ &\leq 2\int_T^\infty \frac{1}{t^2}\,\mathrm{d}t + T^k \int_{-T}^{K} |f(t)|\,\mathrm{d}t < \infty,\end{aligned}$$

giving the result. ∎

Test signals of rapid decay generalise the test signals of Definition 10.2.1.

10.3.3 Definition (Schwartz signal) A ***test signal of rapid decay***, or a ***Schwartz signal***, is an infinitely differentiable map $\phi\colon \mathbb{R}\to\mathbb{F}$ with the property that for each $k\in\mathbb{Z}_{\geq 0}$ the signal $\phi^{(k)}$ is of rapid decay. The set of Schwartz signals is denoted $\mathscr{S}(\mathbb{R};\mathbb{F})$. •

10.3.4 Remark ($\mathscr{S}(\mathbb{R};\mathbb{F})$ is a vector space) One can easily verify that $\mathscr{S}(\mathbb{R};\mathbb{F})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{F}^{\mathbb{R}}$. •

Let us look at some examples of test signals of rapid decay.

10.3.5 Examples (Schwartz signals)

1. Note that every element of $\mathscr{D}(\mathbb{R};\mathbb{F})$ is also an element of $\mathscr{S}(\mathbb{R};\mathbb{F})$.
2. Consider a signal of the form
$$f(t) = \frac{1}{t^k + p_{k-1}t^{k-1} + \cdots + p_1 t + p_0}$$
where the polynomial
$$t^k + p_{k-1}t^{k-1} + \cdots + p_1 t + p_0$$
has no real roots. Then one can easily show using the quotient rule for derivatives that $\lim_{|t|\to\infty}|t^j f^{(r)}(t)| = \infty$ as long as $j \geq k + r$. Thus this signal is not in $\mathscr{S}(\mathbb{R};\mathbb{F})$.
3. The most often cited example of an test signal of rapid decay that is *not* in $\mathscr{D}(\mathbb{R};\mathbb{F})$ is the Gaussian $\phi(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}t^2}$. To see that this signal is indeed of rapid decay, note that $\phi^{(r)}(t) = P_r(t)\mathrm{e}^{-\frac{1}{2}t^2}$ for some polynomial P_r of degree r. Since the negative exponential goes to zero faster in the limit that any polynomial (this can be shown using l'Hôpital's Rule), it follows that the Gaussian is indeed a test signal of rapid decay.
4. If $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ then one checks that $P\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ for any polynomial $P \in \mathbb{F}[t]$ and that $\phi^{(k)} \in \mathscr{S}(\mathbb{R};\mathbb{F})$ for any $k \in \mathbb{Z}_{>0}$. (We shall prove this during the course of the proof of Proposition 10.3.18 below.) One can also check that the sum of two Schwartz signals is a Schwartz signal. Thus the Schwartz signals form an $\mathbb{F}$-vector space. •

As with test signals in $\mathscr{D}(\mathbb{R};\mathbb{F})$, the Schwartz functions come equipped with a natural notion of convergence.

10.3.6 Definition (Convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$) A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$ ***converges to zero*** if, for each $k, r \in \mathbb{Z}_{\geq 0}$, one has
$$\lim_{j\to\infty} \sup\left\{|t^k\phi_j^{(r)}(t)| \;\middle|\; t \in \mathbb{R}\right\} = 0.$$
A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$ ***converges*** to $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ if the sequence $(\phi_j-\phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

As with our notion of convergence in $\mathscr{D}(\mathbb{R};\mathbb{F})$, it is interesting to speculate whether convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ could have been prescribed by a norm. As with $\mathscr{D}(\mathbb{R};\mathbb{F})$, the answer for $\mathscr{S}(\mathbb{R};\mathbb{F})$ is, "No." However, the situation is somehow less dire for $\mathscr{S}(\mathbb{R};\mathbb{F})$ than it is for $\mathscr{D}(\mathbb{R};\mathbb{F})$. For example, it turns out that there is a metric on $\mathscr{S}(\mathbb{R};\mathbb{F})$ for which convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ is convergence with respect to the metric. This is not true for $\mathscr{D}(\mathbb{R};\mathbb{F})$. We leave these interesting matters for the motivated reader to explore in ***missing stuff***.

Note that unlike the situation for convergence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ we make no domain restrictions for sequences of test functions that converge. Also note that the definition of convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ implies, but is not implied by, the uniform convergence of derivatives of all orders. The following examples illustrate the notion of convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$.

10.3.7 Examples (Convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$)

1. We claim that a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Indeed, if let $T \in \mathbb{R}_{>0}$ have the property that $\operatorname{supp}(\phi_j) \subseteq [-T,T]$, $j \in \mathbb{Z}_{>0}$. Then for any $k,r \in \mathbb{Z}_{\geq 0}$ we have

$$\sup\left\{|t^k\phi^{(r)j}(t)| \;\middle|\; t\in\mathbb{R}\right\} \leq T^k \sup\left\{|\phi_j^{(r)}(t)| \;\middle|\; t\in\mathbb{R}\right\}.$$

The limit as $j\to\infty$ of the term on the right goes to zero since $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$.

2. Let ϕ be the Gaussian of Example 10.3.5–3 and consider the sequence $(\frac{1}{j}\phi)_{j\in\mathbb{Z}_{>0}}$. One can easily check that this sequence converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.
3. Again let ϕ be the Gaussian and now define $\phi_j(t) = \phi(t-j)$. One can then check that the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero pointwise, but not uniformly. Thus this sequence does not converge to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.
4. For $j\in\mathbb{Z}_{>0}$ define a signal ϕ_j as follows:

$$\phi_j(t) = \begin{cases} \frac{1}{j}\wedge(t+j^2), & t\in[-j^2-1,-j^2],\\ \frac{1}{j}\wedge(0), & t\in(-j^2,j^2),\\ \frac{1}{j}\wedge(t-j^2), & t\in[j^2,j^2+1],\\ 0, & \text{otherwise}.\end{cases}$$

One can check that these functions are all infinitely differentiable. Also, since ϕ_j, $j\in\mathbb{Z}_{>0}$, has compact support, it is in $\mathscr{D}(\mathbb{R};\mathbb{F})$, and therefore in $\mathscr{S}(\mathbb{R};\mathbb{F})$. One can also check that for each $r\in\mathbb{Z}_{\geq 0}$ the sequence of functions $(\phi_j^{(r)})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero. However, we claim that the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ does not converge to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Indeed, note that

$$\sup\{|t\phi_j(t)| \mid t\in\mathbb{R}\} = j\wedge(0).$$

Since this limit does not converge to zero as $j\to\infty$, our claim follows. In Figure 10.6 we show the first four signals in the sequence. The key feature of the sequence of signals is that the signals "spread out" faster as $j\to\infty$ than they decrease in magnitude. •

For the given notion of convergence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ there is a corresponding notion of continuity.

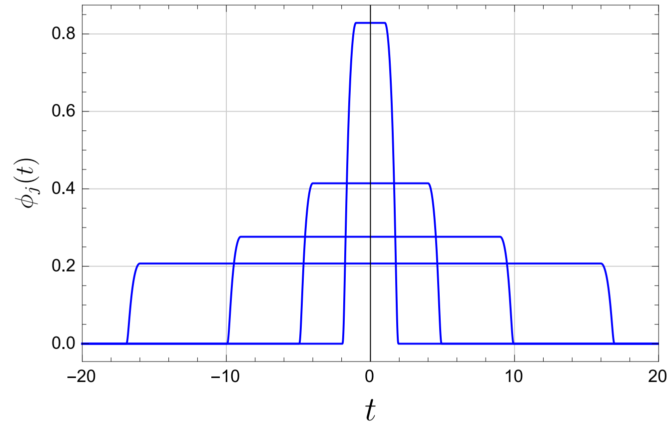


Figure 10.6 A sequence converging uniformly to zero, but not converging in $\mathscr{S}(\mathbb{R};\mathbb{F})$

10.3.8 Definition (Continuous linear maps on $\mathscr{S}(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon \mathscr{S}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ of numbers converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ that converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. •

10.3.2 Definition of tempered distributions

We now mirror for the Schwartz class of test signals the definition of a distribution.

10.3.9 Definition (Tempered distribution) A ***tempered distribution***, or a ***distribution of slow growth***, is a continuous linear map from $\mathscr{S}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of tempered distributions is denoted $\mathscr{S}'(\mathbb{R};\mathbb{F})$. •

10.3.10 Remark ($\mathscr{S}'(\mathbb{R};\mathbb{F})$ is a vector space) It is easy to check that $\mathscr{S}'(\mathbb{R};\mathbb{F})$ is a subspace of $\mathscr{D}'(\mathbb{R};\mathbb{F})$. The inclusion is proved below in Proposition 10.3.12, and the inheritance of the vector space structure is then readily verified.

Let us give some examples of tempered distributions.

10.3.11 Examples (Tempered distributions)

1. Any polynomial function $\mathrm{Ev}_{\mathbb{F}}(P)$, $P \in \mathbb{F}[\xi]$, defines a tempered distribution θ_P via
$$\theta_P(\phi) = \int_{\mathbb{R}} P(t)\phi(t)\,\mathrm{d}t.$$
2. We claim that if $f \in \mathsf{L}^{\mathrm{pow}}(\mathbb{R};\mathbb{F})$ then $\theta_f \in \mathscr{S}'(\mathbb{R};\mathbb{F})$. We first recall from Proposition 8.3.12 that f is a power signal then it is locally integrable. Thus

$\theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{F})$. If $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ then

$$\begin{aligned}\Big|\int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t\Big| &\le \lim_{T\to\infty}\int_{-T}^{T}|f(t)\phi(t)|\,\mathrm{d}t\\ &= \lim_{T\to\infty}\int_{-T}^{T}|T^{-1/2}f(t)T^{1/2}\phi(t)|\,\mathrm{d}t\\ &\le \lim_{T\to\infty}\Big(\frac{1}{T}\int_{-T}^{T}|f(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big(T\int_{-T}^{T}|\phi(t)|^2\,\mathrm{d}t\Big)^{1/2}\end{aligned}$$

It will follow that $\theta_f(\phi)$ is well-defined if we can show that the second term on the right is bounded. Choose $M \in \mathbb{R}_{>0}$ such that $|\phi(t)| \le \frac{M}{1+t^2}$ for all $t \in \mathbb{R}$. We then have

It is easy to show (and will be shown in Proposition 10.8.4) that $\mathscr{S}(\mathbb{R};\mathbb{F}) \subseteq \mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$. This shows that $\theta_f(\phi)$ is well-defined. The computations above also show that if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$, it follows that $(\theta_f(\phi_j))_{j\in\mathbb{Z}_{>0}}$ also converges to zero, using the fact that $(\|\phi_j\|_2)_{j\in\mathbb{Z}_{>0}}$ converges to zero. Thus θ_f is continuous, so giving an element of $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

3. The signal $f(t) = \mathrm{e}^t$ is not one of slow growth. One can also show that this signal does not define a tempered distribution by integration as is the case for signals of slow growth. For example, if one takes the signal in $\mathscr{S}(\mathbb{R};\mathbb{F})$ defined by

$$\phi(t) = \begin{cases}\frac{\wedge(t-1)}{\wedge(0)}\mathrm{e}^{-t}, & t \in [0,1],\\ \mathrm{e}^{-t}, & t < 1,\\ 0, & \text{otherwise}.\end{cases}$$

(see Figure 10.7), then one can see that the integral $\int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t$ diverges.

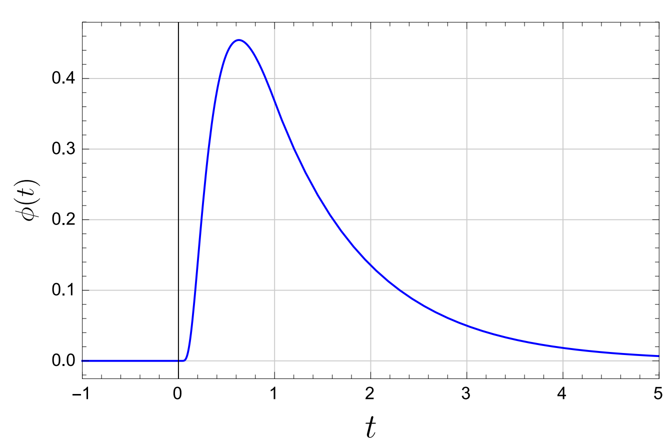


Figure 10.7 A test signal of slow growth on which e^t is undefined as a tempered distribution

4. Let $(c_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence with the property that there exists $M \in \mathbb{R}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$, and $N \in \mathbb{Z}_{>0}$ such that for $j \geq N$ we have $|c_j| \leq Mj^k$. We claim that for $\Delta \in \mathbb{R}_{>0}$,

$$\theta = \sum_{j=1}^{\infty} c_j \delta_{j\Delta}$$

is a tempered distribution. First of all, let us be sure we understand what θ really is. We are defining θ by

$$\theta(\phi) = \sum_{j=1}^{\infty} c_j \phi(j\Delta)$$

for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. Let us first check that this sum converges. Let $\tilde{N} \in \mathbb{Z}_{\geq 0}$ have the property that $|j^{k+2}\phi(j\Delta)| \leq 1$ for $j \geq \tilde{N}$. This is possible since $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. We then have

$$\sum_{j=1}^{\infty} |c_j\phi(j\Delta)| \leq \sum_{j=1}^{\max\{N,\tilde{N}\}-1} |c_j\phi(j\Delta)| + \sum_{j=\max\{N,\tilde{N}\}}^{\infty} \frac{M}{j^2} < \infty.$$

Now let $(\phi_\ell)_{\ell\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. We then have

$$\begin{aligned}
|\theta(\phi_\ell)| &= \Big|\sum_{j=1}^{\infty} c_j\phi_\ell(j\Delta)\Big| \leq \sum_{j=1}^{\infty} |c_j\phi_\ell(j\Delta)| \\
&\leq \sum_{j=1}^{N-1} |c_j\phi_\ell(j\Delta)| + \sum_{j=N}^{\infty} \frac{M}{\Delta^k} |(j\Delta)^k\phi_\ell(j\Delta)| \\
&\leq \sup\Big\{|\phi_\ell(t)| \sum_{j=1}^{N-1} |c_j| \;\Big|\; t \in \mathbb{R}\Big\} + \sup\Big\{|t^{k+2}\phi_\ell(t)| \frac{M}{\Delta^{k+2}} \sum_{j=N}^{\infty} \frac{1}{j^2} \;\Big|\; t \in \mathbb{R}\Big\},
\end{aligned}$$

the suprema existing since $(\phi_\ell)_{\ell\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Taking the limit as $\ell \to \infty$ shows that θ is indeed a tempered distribution. •

Let us now show that tempered distributions are distributions.

10.3.12 Proposition (Tempered distributions are distributions) *We have $\mathscr{S}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$. Moreover, tempered distributions $\theta_1, \theta_2 \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ agree if and only if they agree as distributions.*

Proof Clearly since $\mathscr{D}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{S}(\mathbb{R};\mathbb{F})$, if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ and $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ it makes sense to write $\theta(\phi)$. We need only check that if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$, then $(\theta(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathbb{F}$. However, this follows since such a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$ by Example 10.3.7–1.

Now suppose that $\theta_1 = \theta_2$ as tempered distributions. Thus $\theta_1(\phi) = \theta_2(\phi)$ for all $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. In particular, $\theta_1(\phi) = \theta_2(\phi)$ for all $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ and so $\theta_1 = \theta_2$ as distributions.

For the converse assertion, we refer ahead to Theorem 10.8.3(i) where it is shown that if $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ then there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{S}(\mathbb{R};\mathbb{F})$ which converges to ϕ in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Now suppose that $\theta_1 = \theta_2$ as distributions. Thus $\theta_1(\phi) = \theta_2(\phi)$ for all $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Now let $\psi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ and let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging in $\mathscr{S}(\mathbb{R};\mathbb{F})$ to ψ. Then, continuity of θ_1 and θ_2 gives

$$\theta_1(\psi) = \lim_{j\to\infty} \theta_1(\phi_j) = \lim_{j\to\infty} \theta_2(\phi_j) = \theta_2(\psi),$$

giving $\theta_1 = \theta_2$ as tempered distributions. ∎

The following result characterises a useful way of characterising those distributions that are tempered. Perhaps the most revealing way of interpreting the theorem is this. A distribution is tempered if it is continuous on $\mathscr{D}(\mathbb{R};\mathbb{F})$ if convergence to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ is defined using the notion of convergence to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.

10.3.13 Theorem (Alternative characterisation of tempered distributions) *If $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ then $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Conversely, if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and if $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ that converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$, then $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$.*

Proof Suppose that $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Continuity of θ ensures that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero.

Let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ have the property that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Also let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. To define $\theta(\phi)$ we let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ. This means that $(\phi - \phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. That this is possible is a consequence of Theorem 10.8.3(i) below. Let $j, k \in \mathbb{Z}_{>0}$ and note that

$$|\theta(\phi_j - \phi_k)| \le |\theta(\phi - \phi_j)| + |\theta(\phi - \phi_k)|.$$

By choosing j and k sufficiently large we can ensure that $|\theta(\phi_j - \phi_k)|$ is as small as desired, and this means that $(\theta(\phi - \phi_j))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence, and so converges in $\mathbb{F}$. This means that we can define $\theta(\phi) = \lim_{j\to\infty} \theta(\phi_j)$. To show that this definition does not depend on the choice of sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ, let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be another sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ again converging to ϕ in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Then

$$\begin{aligned}\Big|\lim_{j\to\infty} \theta(\phi_j) - \lim_{k\to\infty} \theta(\psi_k)\Big| &= \lim_{j,k\to\infty} |\theta(\phi_j - \psi_k)| \\ &\le \lim_{j,k\to\infty} |\theta(\phi - \phi_j)| + \lim_{j,k\to\infty} |\theta(\phi - \psi_k)|.\end{aligned}$$

Both of these last limits are zero and so the two limits are the same, and the notation $\theta(\phi)$ makes sense for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$.

We must still show that θ is linear and continuous. Linearity is simple. To show continuity let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero. Define $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ by

$$\psi(t) = \begin{cases} \mathrm{e}\exp\Big(-\frac{1}{1-t^2}\Big), & |t| < 1, \\ 0, & \text{otherwise}. \end{cases}$$

(The reader should figure out what the graph of this function looks like, since we will use properties of this graph in our arguments below.) Then define a sequence $(\psi_k)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ by $\psi_k(t) = \psi(\frac{t}{k})$. We make the following observations concerning this sequence.

1 Lemma *The following statements hold:*

(i) for each every compact set $\mathrm{K} \subseteq \mathbb{R}$*, the sequence* $(\psi_\mathrm{k}|\mathrm{K})_{\mathrm{k}\in\mathbb{Z}_{>0}}$ *converges uniformly to the function* $\mathrm{K} \ni \mathrm{t} \mapsto 1$*;*

(ii) for each $\mathrm{r} \in \mathbb{Z}_{>0}$*, the sequence* $(\psi_\mathrm{k}^{(\mathrm{r})})_{\mathrm{k}\in\mathbb{Z}_{>0}}$ *converges uniformly zero.*

Proof For the first assertion, let $K \subseteq \mathbb{R}$ be compact and let $T \in \mathbb{R}_{>0}$ be such that $K \subseteq [-T, T]$. For $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be sufficient large that $1 - \psi(\frac{T}{N}) < \epsilon$, this being possible since $\lim_{t\to 0}\psi(t) = 1$, the limit being an increasing one as t gets closer to zero. It then follows that for $t \in K \subseteq [-T, T]$ and for $k \geq N$ we have

$$|1 - \psi_k(t)| = |1 - \psi(\tfrac{t}{k})| \leq |1 - \psi(\tfrac{t}{N})| < \epsilon,$$

giving uniform convergence of $(\psi_k|K)_{k\in\mathbb{Z}_{>0}}$ to the function having the value 1 on K.

Now, for $r \in \mathbb{Z}_{>0}$, let

$$M_r = \sup\{|\psi^{(r)}(t)| \mid t \in \mathbb{R}\}.$$

For $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $N^{-r}M_r < \epsilon$. By the Chain Rule,

$$|\psi_k^{(r)}(t)| = |k^{-r}\psi^{(r)}(t)| < \epsilon$$

for $t \in \mathbb{R}$ and $k \geq N$. This gives the desired uniform convergence of $(\psi_k^{(r)})_{k\in\mathbb{Z}_{>0}}$ to zero. ▼

Next let us use this sequence to construct a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$.

2 Lemma *If* $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ *then the sequence* $(\phi\psi_\mathrm{j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *converges to* ϕ *in* $\mathscr{S}(\mathbb{R};\mathbb{F})$*.*

Proof Let $k, r \in \mathbb{Z}_{\geq 0}$ and let $\epsilon \in \mathbb{R}_{>0}$. Since $\lim_{|t|\to\infty} t^k\phi^{(r)}(t) = 0$, there exists $T \in \mathbb{R}_{>0}$ such that $|t^k\phi^{(r)}(t)| < \frac{\epsilon}{2}$ for all t such that $|t| \geq T$.

By the Leibniz Rule, Proposition 3.2.11, we have

$$(\phi\psi_j)^{(r)}(t) = \sum_{m=0}^{r}\binom{r}{m}\phi^{(r-m)}(t)\psi_j^{(m)}(t).$$

Thus

$$\phi^{(r)}(t) - (\phi\psi_j)^{(r)}(t) = \phi^{(r)}(t)(1 - \psi_j(t)) + \sum_{m=1}^{r}\binom{r}{m}\phi^{(r-m)}(t)\psi_j^{(m)}(t).$$

Let

$$B_r = \max\left\{\binom{r}{m} \;\middle|\; m \in \{0, 1, \dots, r\}\right\}.$$

For $m \in \{0, 1, \dots, r\}$ let

$$M_{m,k} = \sup\{|t^k\phi^{(m)}(t)| \mid t \in \mathbb{R}\}$$

and, using Lemma 1, let $N_1 \in \mathbb{Z}_{>0}$ be sufficiently large that

$$|1 - \psi_j(t)|M_{0,k} < \frac{\epsilon}{2}$$

for $t \in [-T, T]$ and $j \geq N_1$. Again using Lemma 1, let $N_2 \in \mathbb{Z}_{>0}$ be sufficiently large that

$$r|\psi_j^{(l)}(t)|B_r \max\{M_{1,k}, \dots, M_{r,k}\} < \frac{\epsilon}{2}, \qquad l \in \{1, \dots, r\},$$

for $t \in \mathbb{R}$ and $j \geq N_2$. Let $N = \max\{N_1, N_2\}$. Now, we consider two cases.

1. $|t| \leq T$: For $j \geq N$ we have

$$|t^k \phi^{(r)}(t)(1 - \psi_j(t))| \leq M_{0,k}|1 - \psi_j(t)| < \frac{\epsilon}{2}.$$

2. $|t| > T$: Since $1 - \psi_j(t) \in [0, 1]$ for every $t \in \mathbb{R}$, our definition of T immediately gives

$$|t^k \phi^{(r)}(t)(1 - \psi_j(t))| < \frac{\epsilon}{2}.$$

Thus, for every $t \in \mathbb{R}$ we have

$$|t^k \phi^{(r)}(t)(1 - \psi_j(t))| < \frac{\epsilon}{2}.$$

For $j \geq N$ and $m \in \{1, \dots, r\}$ we have

$$|t^k \phi^{(r-m)}(t)\psi_j^{(m)}(t)| \leq M_{r-m,k}|\psi_j^{(m)}(t)| \leq \max\{M_{1,k}, \dots, M_{r,k}\}|\psi_j^{(m)}(t)| < \frac{\epsilon}{2rB_r}$$

for every $t \in \mathbb{R}$. Thus, for $t \in \mathbb{R}$ and $j \geq N$ we then have

$$|t^k(\phi^{(r)}(t) - (\phi\psi_j)^{(r)}(t))| = \left|t^k\Big(\phi^{(r)}(t)(1 - \psi_j(t)) + \sum_{m=1}^{r} \binom{r}{m} \phi^{(r-m)}(t)\psi_j^{(m)}(t)\Big)\right| < \epsilon.$$

Since k and r are arbitrary, the sequence $(\phi - \phi\psi_j)_{j \in \mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R}; \mathbb{F})$ as desired. ▼

Continuing with the proof, for each $j \in \mathbb{Z}_{>0}$ note that the sequence $(\chi_{j,k} \triangleq \psi_k\phi_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R}; \mathbb{F})$ converges to ϕ_j in $\mathscr{S}(\mathbb{R}; \mathbb{F})$ by Lemma 2. Therefore, for each $j \in \mathbb{Z}_{>0}$, there exists $N_j \in \mathbb{Z}_{>0}$ sufficiently large that

$$|\theta(\phi_j - \psi_k\phi_j)| \leq \epsilon, \qquad k \geq N_j,$$

by our assumptions on θ. We claim that the sequence $(\psi_{N_j}\phi_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R}; \mathbb{F})$ converges to zero in $\mathscr{S}(\mathbb{R}; \mathbb{F})$. Indeed we have

$$\lim_{j \to \infty} \sup\left\{|t^m(\psi_{N_j}\phi_j)^{(r)}(t)| \;\middle|\; t \in \mathbb{R}\right\} = 0$$

by virtue of the Lemma 1, the fact that $(\phi_j)_{j \in \mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R}; \mathbb{F})$, and the formula

$$(\psi_{N_j}\phi_j)^{(r)} = \sum_{\ell=0}^{r} \binom{r}{\ell} \psi_{N_j}^{(\ell)}\phi_j^{(r-\ell)}.$$

This then gives

$$|\theta(\phi_j)| \leq |\theta(\phi_j - \psi_{N_j}\phi_j)| + |\theta(\psi_{N_j}\phi_j)|.$$

The two terms on the right go to zero as $j \to \infty$ by our hypotheses on θ, and so continuity of θ on $\mathscr{S}(\mathbb{R}; \mathbb{F})$ follows. ■

10.3.3 Properties of tempered distributions

In this section we record some of the basic facts about tempered distributions. Many of these follow, directly or with little effort, from their counterparts for distributions.

Since $\mathscr{S}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$ there is inherited from $\mathscr{D}'(\mathbb{R};\mathbb{F})$ the notion of convergence of a sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

10.3.14 Definition (Convergence in $\mathscr{S}'(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}'(\mathbb{R};\mathbb{F})$ is

(i) a ***Cauchy sequence*** if $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$, and

(ii) ***converges*** to a tempered distribution θ if, for every $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$, the sequence of numbers $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ converges to $\theta(\phi)$. •

What is not so clear is whether such a sequence converging in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ will converge to an element of $\mathscr{S}'(\mathbb{R};\mathbb{F})$. This is indeed the case.

10.3.15 Theorem (Cauchy sequences in $\mathscr{S}'(\mathbb{R};\mathbb{F})$ converge) *If $(\theta_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{S}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$ that is Cauchy, then it converges to some $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$.*

Proof The proof goes very much like that of Theorem 10.2.22. All one needs to do is choose the initial subsequence $(\psi_n)_{n\in\mathbb{Z}_{>0}}$ so as to have the additional property that

$$\sup\{|t^k\psi_n^{(j)}| \mid t \in \mathbb{R}\} < \frac{1}{4^n}, \qquad j,k \in \{0,1,\dots,n\}.$$

After replacing all occurrences of $\mathscr{D}(\mathbb{R};\mathbb{F})$ with $\mathscr{S}(\mathbb{R};\mathbb{F})$ and of $\mathscr{D}'(\mathbb{R};\mathbb{F})$ with $\mathscr{S}'(\mathbb{R};\mathbb{F})$, the same proof then gives the result in this case. ■

Let us give the analogue for tempered distributions of the fact that locally integrable signals are distributions. Note that Example 10.3.5–3 shows that there are locally integrable signals that are not to be regarded as tempered distributions.

10.3.16 Definition (Signal of slow growth) A measurable signal $f\colon \mathbb{R} \to \mathbb{F}$ is said to be ***of slow growth*** if there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$ such that

$$|f(t)| \le M(1+t^2)^N. \qquad \bullet$$

Since a signal of slow growth is bounded by a locally integrable function, such signals are themselves locally integrable. The following result gives the relationship between these signals and tempered distributions.

10.3.17 Proposition (Signals of slow growth are tempered distributions) *If $\mathrm{f}\colon \mathbb{R} \to \mathbb{F}$ is a signal of slow growth then $\theta_{\mathrm{f}} \in \mathscr{S}'(\mathbb{R};\mathbb{F})$. Moreover, if $\mathrm{f}_1, \mathrm{f}_2\colon \mathbb{R} \to \mathbb{F}$ are signals of slow growth for which $\theta_{\mathrm{f}_1} = \theta_{\mathrm{f}_2}$, then $\mathrm{f}_1(\mathrm{t}) = \mathrm{f}_2(\mathrm{t})$ for almost every $\mathrm{t} \in \mathbb{R}$.*

Proof First let us show that the integral

$$\int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t$$

exists for all $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. We have

$$\int_{\mathbb{R}} |f(t)\phi(t)|\,dt \le \int_{\mathbb{R}} M(1+t^2)^N|\phi(t)|\,dt.$$

By Proposition 10.3.2 the integral converges, showing that the map θ_f is well-defined. Now let us show that it defines a tempered distribution. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Then we compute

$$\begin{aligned}
|\theta_f(\phi_j)| &= \left|\int_{\mathbb{R}} f(t)\phi_j(t)\,dt\right| \le \int_{\mathbb{R}} M(1+t^2)^N|\phi_j(t)|\,dt\\
&= \int_{-\infty}^{-1} M(1+t^2)^N|\phi_j(t)|\,dt + \int_{-1}^{1} M(1+t^2)^N|\phi_j(t)|\,dt\\
&\quad + \int_{1}^{\infty} M(1+t^2)^N|\phi_j(t)|\,dt\\
&\le 2\sup\left\{|M(1+t^2)^N t^2\phi_j(t)| \int_{-\infty}^{2} \frac{1}{\tau^2}\,d\tau \;\middle|\; t\in\mathbb{R}\right\}\\
&\quad + \sup\left\{|M(1+t^2)^N\phi_j(t)| \int_{-1}^{1} M(1+t^2)^N|\phi_j(\tau)|\,d\tau \;\middle|\; t\in\mathbb{R}\right\},
\end{aligned}$$

the suprema existing since the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Taking the limit as $j\to\infty$ gives the desired conclusion, again since the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.

The last assertion follows the similar assertion in Proposition 10.2.12, along with Proposition 10.3.12. ■

Signals of slow growth also show up to give a natural class of signals which can be multiply tempered distributions.

10.3.18 Proposition (Tempered distributions can be multiplied by signals all of whose derivatives are of slow growth) *Let $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ and let $\phi_0\colon \mathbb{R}\to\mathbb{F}$ be an infinitely differentiable signal of slow growth, all of whose derivatives are also signals of slow growth. Then the map*

$$\mathscr{S}(\mathbb{R};\mathbb{F}) \ni \phi \mapsto \theta(\phi_0\phi) \in \mathbb{F}$$

defines an element of $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

Proof Linearity of the map is clear. To prove continuity, let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero. We claim that $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.

First we show that $\phi_0\phi_j \in \mathscr{S}(\mathbb{R};\mathbb{F})$ for each $j\in\mathbb{Z}_{>0}$. It is clear that $\phi_0\phi_j$ is infinitely differentiable. For each $r\in\mathbb{Z}_{\ge 0}$ let $M_r\in\mathbb{R}_{>0}$ and $N_r\in\mathbb{Z}_{>0}$ be such that

$$\phi_0^{(r)}(t) \le M_r(1+t^2)^{N_r}, \qquad t\in\mathbb{R}.$$

Then, for $k\in\mathbb{Z}_{>0}$,

$$\lim_{|t|\to\infty} |t^k(\phi_0(t)\phi_j)^{(r)}(t)| = 0$$

using Proposition 3.2.11 along with the fact that ϕ_j and all of its derivatives have slow growth.

Now we show that $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero. Let $k, r \in \mathbb{Z}_{\geq 0}$.

$$\lim_{j\to\infty} \sup\left\{|t^k(\phi_0\phi_j)^{(r)}(t)| \;\middle|\; t \in \mathbb{R}\right\}$$

again using Proposition 3.2.11 along with the fact that ϕ_j and all of its derivatives have slow growth.

Thus the result follows since

$$\lim_{j\to\infty} \theta(\phi_0\phi) = 0$$

for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. ■

The notions of regular, singular, support, and singular support are applied to $\mathscr{S}'(\mathbb{R};\mathbb{F})$ by restriction from $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

One can differentiate tempered distributions as they are distributions. It turns out that the derivative is again a tempered distribution.

10.3.19 Proposition (The derivative of a tempered distribution is a tempered distribution) *If $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ then $\theta' \in \mathscr{S}'(\mathbb{R};\mathbb{F})$.*

Proof This is easy to show. We let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero. Then $(-\phi_j')_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$, as is easily seen from the definition of convergence to zero. Therefore,

$$\lim_{j\to\infty} \theta'(\phi_j) = \lim_{j\to\infty} \theta(-\phi_j') = 0$$

as desired. ■

One can talk about tempered distributions of finite order, and tempered distributions are always locally of finite order by virtue of their being distributions. We shall see in Theorem 10.3.25 that even more is true for tempered distributions.

10.3.4 Tempered distributions depending on parameters

In this section we adapt our results from Section 10.2.8 to test signals from $\mathscr{S}(\mathbb{R};\mathbb{F})$ and distributions from $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

As in Section 10.2.8, we let $I \subseteq \mathbb{R}$ be an interval and consider a function $\phi\colon I\times\mathbb{R} \to \mathbb{F}$ and denote a typical point in $I\times\mathbb{R}$ by (λ, t). For $(\lambda, t) \in I\times\mathbb{R}$ we define functions $\phi^\lambda\colon \mathbb{R}\to\mathbb{F}$ and $\phi_t\colon I\to\mathbb{F}$ by $\phi^\lambda(t) = \phi_t(\lambda) = \phi(\lambda, t)$. If, for each $\lambda\in I$, $\phi^\lambda \in \mathscr{S}(\mathbb{R};\mathbb{F})$, then, given $\theta\in\mathscr{S}'(\mathbb{R};\mathbb{F})$, we define $\Phi_{\theta,\phi}\colon I\to\mathbb{F}$ by

$$\Phi_{\theta,\phi}(\lambda) = \theta(\phi^\lambda).$$

As in Section 10.2.8, we denote

$$(\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)^\lambda(t) = (\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)_t(\lambda) = \boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi(\lambda, t)$$

for $r, s \in \mathbb{Z}_{\geq 0}$.

The following result indicates the character of the function $\Phi_{\theta,\phi}$ in this case.

10.3.20 Theorem (Distributions applied to Schwartz signals with parameter dependence) *Let $\mathrm{I} \subseteq \mathbb{R}$ be an interval, let $\mathrm{k} \in \mathbb{Z}_{\geq 0}$, and let $\phi\colon \mathrm{I} \times \mathbb{R} \to \mathbb{F}$ have the following properties:*

(i) for each $\lambda \in \mathrm{I}$, the map $\mathrm{t} \mapsto \phi(\lambda, \mathrm{t})$ is an element of $\mathscr{S}(\mathbb{R}; \mathbb{F})$;

(ii) for each $\mathrm{r}, \mathrm{m} \in \mathbb{Z}_{\geq 0}$ there exists $\mathrm{C}_{\mathrm{k,r,m}} \in \mathbb{R}_{>0}$ such that

$$\sup\left\{|\mathrm{t}^{\mathrm{m}}\boldsymbol{D}_1^{\mathrm{k}+1}\boldsymbol{D}_2^{\mathrm{r}}\phi(\lambda, \mathrm{t})| \;\middle|\; \mathrm{t} \in \mathbb{R},\ \lambda \in \mathrm{I}\right\} < \mathrm{C}_{\mathrm{k,r,m}};$$

(iii) for each $\mathrm{r} \in \mathbb{Z}_{\geq 0}$, $\boldsymbol{D}_1^{\mathrm{k}+1}\boldsymbol{D}_2^{\mathrm{r}}\phi\colon \mathrm{I} \times \mathbb{R} \to \mathbb{F}$ is continuous.

Then, for any $\theta \in \mathscr{D}'(\mathbb{R}; \mathbb{F})$, $\Phi_{\theta,\phi}$ is k-times continuously differentiable and, moreover,

$$\Phi_{\theta,\phi}^{(\mathrm{k})}(\lambda) = \theta((\boldsymbol{D}_1^{\mathrm{k}}\phi)^{\lambda}).$$

Proof The proof follows closely that of Theorem 10.2.40, but we shall go through the details so as to understand clearly where the differences arise.

We first give the proof for $k = 0$. Let $\lambda \in I$ and let $(\epsilon_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to zero and such that $\lambda + \epsilon_j \in I$ for every $j \in \mathbb{Z}_{>0}$. Define $\psi_j^{\lambda} \in \mathscr{S}(\mathbb{R}; \mathbb{F})$ by

$$\psi_j^{\lambda}(t) = \phi(\lambda + \epsilon_j, t).$$

The following lemma is then useful.

1 Lemma *The sequence $(\psi_{\mathrm{j}}^{\lambda})_{\mathrm{j} \in \mathbb{Z}_{>0}}$ converges to ϕ^{λ} in $\mathscr{S}(\mathbb{R}; \mathbb{F})$.*

Proof Let $r, m \in \mathbb{Z}_{\geq 0}$. Let $I' \subseteq I$ be the smallest compact interval for which $\lambda + \epsilon_j \in I'$ for every $j \in \mathbb{Z}_{>0}$. Since

$$(\lambda, t) \mapsto t^m \boldsymbol{D}_2^r \phi(\lambda, t) \tag{10.9}$$

is continuous with bounded derivative and since $I \times \mathbb{R}$ is convex, by Proposition **??** it follows that the function (10.9) is uniformly continuous. This implies that, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$|t^m \boldsymbol{D}^r \psi_j^{\lambda}(t) - t^m \boldsymbol{D}^r \phi^{\lambda}(t)| = |t^m \boldsymbol{D}_2^r \phi(\lambda + \epsilon_j, t) - t^m \boldsymbol{D}_2^r \phi(\lambda, t)| < \epsilon, \qquad j \geq N,\ t \in \mathbb{R}.$$

Since $r, m \in \mathbb{Z}_{\geq 0}$ is arbitrary, this implies that we have the desired convergence of $(\psi_j^{\lambda})_{j \in \mathbb{Z}_{>0}}$ to ϕ^{λ}. ▼

It then follows immediately from continuity of θ that

$$\lim_{j \to \infty} \Phi_{\theta,\phi}(\lambda + \epsilon_j) = \lim_{j \to \infty} \theta(\phi^{\lambda + \epsilon_j}) = \theta(\lim_{j \to \infty} \phi^{\lambda + \epsilon_j}) = \theta(\lim_{j \to \infty} \psi_j^{\lambda}) = \theta(\phi^{\lambda}) = \Phi_{\theta,\phi}(\lambda).$$

Continuity of $\Phi_{\theta,\phi}$ at λ then follows from Theorem 3.1.3.

Now we prove the theorem when $k = 1$. We let (ϵ_j) be a sequence, none of whose terms are zero, converging to zero as above. Now we take

$$\psi_j^{\lambda}(t) = \frac{\phi(\lambda + \epsilon_j, t) - \phi(\lambda, t)}{\epsilon_j}.$$

The following lemma is then key.

2 Lemma *The sequence* $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ *converges to* $(\mathbf{D}_1\phi)^\lambda$ *in* $\mathscr{S}(\mathbb{R};\mathbb{F})$.

Proof Let $r,m\in\mathbb{Z}_{\geq 0}$. Define $\psi_{r,m}\colon I\times\mathbb{R}\to\mathbb{F}$ by

$$\psi_{r,m}(\ell,t)=\begin{cases}\frac{t^m\boldsymbol{D}_2^r\phi(\ell,t)-t^m\boldsymbol{D}_2^r\phi(\lambda,t)}{\ell-\lambda}, & \ell\neq\lambda,\\ t^m\boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda,t), & \ell=\lambda.\end{cases}$$

It is clear from the hypotheses that $\psi_{r,m}$ is continuous on

$$\{(\ell,t)\in I\times\mathbb{R}\mid\ \ell\neq\lambda\}.$$

Moreover, since the derivative $\boldsymbol{D}_1\boldsymbol{D}_2^r\phi$ exists and is continuous,

$$\lim_{\ell\to\lambda}\frac{t^m\boldsymbol{D}_2^r\phi(\ell,t)-t^m\boldsymbol{D}_2^r\phi(\lambda,t)}{\ell-\lambda}=t^m\boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda,t),\qquad t\in\mathbb{R},$$

showing that $\psi_{r,m}$ is continuous on $I\times\mathbb{R}$ by Theorem 3.1.3. Since $\psi_{r,m}$ is differentiable with bounded derivative and since $I\times\mathbb{R}$ is convex, it is uniformly continuous by Proposition **??**. Therefore, given $\epsilon\in\mathbb{R}_{>0}$, there exists $N\in\mathbb{Z}_{>0}$ such that

$$|\psi_{r,m}(\lambda+\epsilon_j,t)-\psi_{r,m}(\lambda,t)|<\epsilon,\qquad j\geq N,\ t\in\mathbb{R}.$$

Using the definition of $\psi_{r,m}$, this implies that, for every $j\geq N$ and for every $t\in\mathbb{R}$,

$$\left|\frac{t^m\boldsymbol{D}_2^r\phi(\lambda+\epsilon_j,t)-t^m\boldsymbol{D}_2^r\phi(\lambda,t)}{\epsilon_j}-t^m\boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda,t)\right|=|t^m\boldsymbol{D}^r\psi_j^\lambda(t)-t^m\boldsymbol{D}^r(\boldsymbol{D}_1\phi^\lambda)(t)|<\epsilon.$$

Since $r,m\in\mathbb{Z}_{\geq 0}$ are arbitrary, this gives convergence of $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ to $(\boldsymbol{D}_1\phi)^\lambda$. ▼

By continuity of θ we then have

$$\begin{aligned}\lim_{j\to\infty}\frac{\Phi_{\theta,\phi}(\lambda+\epsilon_j)-\Phi_{\theta,\phi}(\lambda)}{\epsilon_j}&=\lim_{j\to\infty}\frac{\theta(\phi^{\lambda+\epsilon_j})-\theta(\phi^\lambda)}{\epsilon_j}\\&=\theta(\lim_{j\to\epsilon}\psi_j^\lambda)=\theta((\boldsymbol{D}_1\phi)^\lambda),\end{aligned}$$

showing that $\Phi_{\theta,\phi}$ is differentiable with derivative as stated in the theorem for the case of $k=1$.

Now suppose that the theorem is true for $j\in\{0,1,\dots,m\}$ and suppose that the hypotheses of the theorem hold for $k=m+1$. We let $\psi=\boldsymbol{D}_1^m\phi$ and verify that ψ satisfies the hypotheses of the theorem for $k=1$. First note that, for each $\lambda\in I$, $t\mapsto\psi(\lambda,t)$ is the mth derivative of an element $\mathscr{S}(\mathbb{R};\mathbb{F})$ and so is an element of $\mathscr{S}(\mathbb{R};\mathbb{F})$. The second of the hypotheses of the theorem hold immediately. Finally, since

$$\boldsymbol{D}_2\boldsymbol{D}_2^r\psi=\boldsymbol{D}_2\boldsymbol{D}_2^r\boldsymbol{D}_1^m\phi=\boldsymbol{D}_1^{m+2}\boldsymbol{D}_2^r\phi$$

by Theorem **??**, the final hypothesis of the theorem also holds. Therefore, by the induction hypothesis, $\Phi_{\theta,\psi}$ is continuously differentiable. But, since

$$\Phi_{\theta,\psi}(\lambda)=\theta((\boldsymbol{D}_1^m\phi)^\lambda)=\Phi_{\theta,\phi}^{(m)}(\lambda),$$

this implies that $\Phi_{\theta,\phi}$ is $m+1$-times continuously differentiable, and

$$\Phi_{\theta,\phi}^{(m+1)}(\lambda)=\theta((\boldsymbol{D}_1^{m+1}\phi)^\lambda)$$

as desired. ■

The following corollary is what will be of primary importance for us.

10.3.21 Corollary (Property of tempered distributions applied to Schwartz functions of two variables) *Let* $\phi\colon \mathbb{R}^2 \to \mathbb{F}$ *be infinitely differentiable and such that, for each* $\mathrm{r}_1, \mathrm{r}_2, \mathrm{m} \in \mathbb{Z}_{\geq 0}$*, there exists* $\mathrm{C}_{\mathrm{r}_1,\mathrm{r}_2,\mathrm{m}} \in \mathbb{R}_{>0}$ *such that*

$$\sup\{(\mathrm{s}^2 + \mathrm{t}^2)^{\mathrm{m}/2}\mathbf{D}_1^{\mathrm{r}_1}\mathbf{D}_2^{\mathrm{r}_2}\phi(\mathrm{t},\mathrm{s}) \mid \mathrm{s},\mathrm{t} \in \mathbb{R}\} \leq \mathrm{C}_{\mathrm{r}_1,\mathrm{r}_2,\mathrm{m}}. \tag{10.10}$$

Then we have $\Phi_{\theta,\phi} \in \mathscr{S}(\mathbb{R};\mathbb{F})$*. Moreover, for each* $\mathrm{k} \in \mathbb{Z}_{>0}$*,*

$$\Phi_{\theta,\phi}^{(\mathrm{k})}(\mathrm{s}) = \theta((\mathbf{D}_1^{\mathrm{k}}\phi)^{\mathrm{s}}).$$

Proof In this case, the hypotheses of Theorem 10.3.20 are easily verified to hold for every $k \in \mathbb{Z}_{>0}$, and so $\Phi_{\theta,\phi}$ is infinitely differentiable. Now let $s \in \mathbb{R}$ and $r, m \in \mathbb{Z}_{\geq 0}$ and define $\psi_{r,m}^s \in \mathscr{S}(\mathbb{R};\mathbb{F})$ by

$$\psi_{r,m}^s(t) = s^m \boldsymbol{D}_1^r\phi(s,t).$$

Note that

$$s^m\Phi_{\theta,\phi}^{(r)}(s) = \psi_{r,m}^s(t).$$

Let $k \in \mathbb{Z}_{\geq 0}$. Since ϕ satisfies (10.10), let $C_{r,m,k}$ be such that

$$\sup\{|(1+t^2)^k(\psi_{r,m}^s)^{(k)}(t)| \mid s,t \in \mathbb{R}\} \leq C_{r,m,k}.$$

By Lemma 10.3.24 below, there exists $M \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|\theta(\psi)| \leq M\sup\left\{|(1+t^2)^k\psi^{(k)}(t)| \;\middle|\; t \in \mathbb{R}\right\}$$

for every $\psi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. Therefore, with M and k so chosen,

$$|s^m\Phi_{\theta,\phi}^{(r)}(s)| = |s^m\theta((\boldsymbol{D}_1^r\phi)^s)| = |\theta(s^m(\boldsymbol{D}_1^r\phi)^s)| = |\theta(\psi_{r,m}^s)| \leq MC_{r,m,k},$$

which shows that $\Phi_{\theta,\phi} \in \mathscr{S}(\mathbb{R};\mathbb{F})$, as desired. ■

The following result will be useful when we discuss convolution.

10.3.22 Corollary (Tempered distributions applied to a special class of test signals) *Denote by* $\tau\colon \mathbb{R}^2 \to \mathbb{R}$ *the map given by* $\tau(\mathrm{s},\mathrm{t}) = \mathrm{t} - \mathrm{s}$*. Then, if* $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ *and* $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$*, then* $\Phi_{\theta,\tau^*\phi} \in \mathscr{S}(\mathbb{R};\mathbb{F})$*.*

Proof We note that, for $r_1, r_2 \in \mathbb{Z}_{\geq 0}$,

$$\boldsymbol{D}_1^{r_1}\boldsymbol{D}_2^{r_2}\tau^*\phi(s,t) = (-1)^{r_1}\boldsymbol{D}^r\phi(t-s).$$

From this, and the fact that $\tau^*\phi$ is infinitely differentiable, we easily see that the hypotheses of Corollary 10.3.21 hold for every $k \in \mathbb{Z}_{\geq 0}$. ■

10.3.5 Fubini's Theorem for tempered distributions

Of course, since tempered distributions are distributions, Theorem 10.2.44 holds for tempered distributions. However, the results can be improved to account for the additional structure of $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

We let $\phi\colon \mathbb{R}^2 \to \mathbb{F}$ be infinitely differentiable and such that, for every $r_1, r_2, m \in \mathbb{Z}_{\geq 0}$, there exists $C_{r_1,r_2,m} \in \mathbb{R}_{>0}$ such that

$$\sup\{(s^2+t^2)^{m/2}\boldsymbol{D}_1^{r_1}\boldsymbol{D}_2^{r_2}\phi(t,s) \mid s,t \in \mathbb{R}\} \leq C_{r_1,r_2,m}.$$

As above, efine $\phi^s, \phi_t \in \mathscr{S}(\mathbb{R};\mathbb{F})$ by $\phi^s(t) = \phi_t(s) = \phi(s,t)$, noting that these functions obviously are elements of $\mathscr{S}(\mathbb{R};\mathbb{F})$. Let $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ and define $\Phi_{\theta,\phi}, \Psi_{\theta,\phi}\colon \mathbb{R} \to \mathbb{F}$ by

$$\Phi_{\theta,\phi}(s) = \theta(\phi^s), \quad \Psi_{\theta,\phi}(t) = \theta(\phi_t).$$

From Corollary 10.3.21 we have that both of these functions are elements of $\mathscr{S}(\mathbb{R};\mathbb{F})$.

Now we can state Fubini's Theorem for tempered distributions.

10.3.23 Theorem (Fubini's Theorem for tempered distributions) *For an infinitely differentiable function $\phi\colon \mathbb{R}^2 \to \mathbb{F}$ satisfying (10.10) and for $\theta, \rho \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, $\theta(\Phi_{\rho,\phi}) = \rho(\Psi_{\theta,\phi})$.*

Proof We begin with technical lemma. Let $\mathscr{S}(\mathbb{R}^2;\mathbb{F})$ be the set of infinitely differentiable functions from $\mathbb{R}^2$ to $\mathbb{F}$ satisfying (10.10). Note that the map $\iota\colon \mathscr{S}(\mathbb{R};\mathbb{F}) \times \mathscr{S}(\mathbb{R};\mathbb{F}) \to \mathscr{S}(\mathbb{R}^2;\mathbb{F})$ given by

$$\iota(\psi \times \chi)(s,t) = \psi(s)\chi(t)$$

is an injection.

1 Lemma *If $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ then there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\operatorname{span}_{\mathbb{F}}(\operatorname{image}(\iota))$ such that*

$$\lim_{j\to\infty} \sup\left\{|s^{k_1}t^{k_2}\mathbf{D}_1^{r_1}\mathbf{D}_2^{r_2}(\phi_j - \phi)(s,t)| \;\middle|\; s,t \in \mathbb{R}\right\} = 0$$

for every $k_1, k_2, r_1, r_2 \in \mathbb{Z}_{\geq 0}$.

Proof ▼

As in the proof of Theorem 10.2.44 we have

$$\Phi_{\rho,\phi_j}(s) = \sum_{k=1}^{m_j} \psi_{j,k}(s)\rho(\chi_{j,k}), \quad \Psi_{\theta,\phi_j}(t) = \sum_{k=1}^{m_j} \chi_{j,k}(t)\theta(\psi_{j,k}).$$

Now we use another lemma.

2 Lemma *If $\phi \in \mathscr{S}(\mathbb{R}^2;\mathbb{F})$ and if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{S}(\mathbb{R}^2;\mathbb{F})$ for which*

$$\lim_{j\to\infty} \sup\left\{|s^{k_1}t^{k_2}\mathbf{D}_1^{r_1}\mathbf{D}_2^{r_2}(\phi_j - \phi)(s,t)| \;\middle|\; s,t \in \mathbb{R}\right\} = 0$$

for every $k_1, k_2, r_1, r_2 \in \mathbb{Z}_{\geq 0}$, then

$$\lim_{j\to\infty} \theta(\Phi_{\rho,\phi_j}) = \theta(\Phi_{\rho,\phi}), \quad \lim_{j\to\infty} \rho(\Psi_{\theta,\phi_j}) = \rho(\Psi_{\theta,\phi}).$$

Proof Clearly it suffices to prove that $\lim_{j\to\infty} \rho(\Psi_{\theta,\phi_j}) = \rho(\Psi_{\theta,\phi})$ as the other conclusion follows in a similar manner.

We first claim that $(\Psi_{\theta,\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to $\Psi_{\theta,\phi}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Suppose otherwise so that, for some $k, r \in \mathbb{Z}_{\geq 0}$,

$$\lim_{j\to\infty} \sup\{|t^k(\Psi^{(r)}_{\theta,\phi_j}(t) - \Psi^{(r)}_{\theta,\phi}(t))| \mid t \in \mathbb{R}\} \neq 0.$$

Thus, possibly by replacing $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ with a subsequence, there exists $\alpha \in \mathbb{R}_{>0}$ such that, for each $j \in \mathbb{Z}_{>0}$, there exists $t_j \in \mathbb{R}$ such that

$$|t_j^k(\Psi^{(r)}_{\theta,\phi_j}(t_j) - \Psi^{(r)}_{\theta,\phi}(t_j))| \geq \alpha.$$

By Theorem 10.3.20 we have

$$\Psi^{(r)}_{\theta,\phi}(t) = \theta(\boldsymbol{D}_2^r\phi_t), \quad \Psi^{(r)}_{\theta,\phi_j}(t) = \theta(\boldsymbol{D}_2^r\phi_{j,t}), \qquad j \in \mathbb{Z}_{>0}.$$

Therefore, for each $j \in \mathbb{Z}_{>0}$, there exists $t_j \in \mathbb{R}$ such that

$$|t_j^k(\theta(\boldsymbol{D}_2^r\phi_{j,t_j}) - \theta(\boldsymbol{D}_2^r\phi_{t_j}))| \geq \alpha.$$

Let $\epsilon \in \mathbb{R}_{>0}$. By assumption there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sup\{s^l t^k \boldsymbol{D}_1^m \boldsymbol{D}_2^r(\phi_j - \phi)(s,t) \mid s, t \in \mathbb{R}\} < \epsilon,$$

for $j \geq N$. In particular,

$$\sup\{s^l t_j^k \boldsymbol{D}_1^m \boldsymbol{D}_2^r(\phi_j - \phi)(s,t) \mid s \in \mathbb{R}\} < \epsilon,$$

for $j \geq N$. Thus $(t_j^k\phi_{j,t_j})_{j\in\mathbb{Z}_{>0}}$ converges to $t_j^k\phi_{t_j}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Thus, by continuity of θ,

$$\lim_{j\to\infty} |\theta(t_j^k\boldsymbol{D}_2^r\phi_{j,t_j}) - \theta(t_j^k\boldsymbol{D}_2^r\phi_{t_j})| = \lim_{j\to\infty} |t_j^k(\boldsymbol{D}_2^r\phi_{j,t_j} - \theta(\boldsymbol{D}_2^r\phi_{t_j}))| = 0.$$

This contradiction implies that, indeed, $(\Psi_{\theta,\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to $\Psi_{\theta,\phi}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$.

From this the lemma immediately follows from the continuity of ρ. ▼

The last part of the proof is just like that for Theorem 10.2.44. ■

10.3.6 Some deeper properties of tempered distributions

Tempered distributions, being distributions, have the properties of Theorems 10.2.45 and 10.2.47. For tempered distributions one can say more. Indeed, we show that tempered distributions are always of finite order, not just locally of finite order. In order to prove this result, we have the following characterisation of tempered distributions, this providing an analogue of Lemma 10.2.46 for $\mathscr{S}'(\mathbb{R};\mathbb{F})$.

10.3.24 Lemma (A boundedness property for tempered distributions) *Let $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$. There then exists $\mathrm{M} \in \mathbb{R}_{>0}$ and $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ such that for each $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ we have*

$$|\theta(\phi)| \leq \mathrm{M}\sup\left\{|(1+\mathrm{t}^2)^{\mathrm{k}}\phi^{(\mathrm{k})}(\mathrm{t})| \;\middle|\; \mathrm{t} \in \mathbb{R}\right\}.$$

Proof To prove the result we indicate how one can reduce to the ideas used in the proof of Lemma 10.2.46. The principle idea of the proof of Lemma 10.2.46 is that a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ if and only if for each $m \in \mathbb{Z}_{\geq 0}$ the sequence $((b-a)^m\phi_j^{(m)})_{j\in\mathbb{Z}_{>0}}$ converges to zero where $\mathbb{T} = [a,b]$. We shall produce an equivalent characterisation for convergence to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. To do this we define

$$\|\phi\|_\infty^m = \left(\frac{\pi}{2}\right)^m \sup\left\{|(1+t^2)^m\phi^{(m)}(t)| \;\middle|\; t \in \mathbb{R}\right\},$$

for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. This allows us to state the following result.

1 **Sublemma** *A sequence $(\phi_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$ if and only if the sequence $(\|\phi_{\mathrm{j}}\|_\infty^{\mathrm{m}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges to zero for each $\mathrm{m} \in \mathbb{Z}_{\geq 0}$.*

Proof Define

$$\|\phi\|_\infty^{m,k} = \sup\left\{|(1+t^2)^m\phi^{(k)}(t)| \;\middle|\; t \in \mathbb{R}\right\}.$$

It is evident that

$$\|\phi\|_\infty^{m,k} \leq \|\phi\|_\infty^{\ell,k}, \qquad \ell \geq m. \tag{10.11}$$

For $t \in \mathbb{R}_{\geq 0}$ we have

$$\begin{aligned}|(1+t^2)^m\phi^{(k)}(t)| &= \left|(1+t^2)^m\int_t^\infty \phi^{(k+1)}(\tau)\,\mathrm{d}\tau\right| \\ &\leq (1+t^2)^m\int_t^\infty \frac{|(1+\tau^2)^{m+1}\phi^{(k+1)}(\tau)|}{(1+\tau^2)^{m+1}}\,\mathrm{d}\tau \\ &\leq \|\phi\|_\infty^{m+1,k+1}\int_t^\infty \frac{\mathrm{d}\tau}{1+\tau^2} \\ &\leq \frac{\pi}{2}\|\phi\|_\infty^{m+1,k+1},\end{aligned}$$

using the fact that $\int_0^\infty \frac{\mathrm{d}t}{1+t^2} = \frac{\pi}{2}$. In like manner we show that for $t \leq 0$ we have.

$$|(1+t^2)^m\phi^{(k)}(t)| = \left|(1+t^2)^m\int_{-\infty}^t \phi^{(k+1)}(\tau)\,\mathrm{d}\tau\right| \leq \frac{\pi}{2}\|\phi\|_\infty^{m+1,k+1}.$$

This shows then that

$$\|\phi\|_\infty^{m,k} \leq \frac{\pi}{2}\|\phi\|_\infty^{m+1,k+1}. \tag{10.12}$$

Next we compute

$$\begin{aligned}\phi^{(k)}(t) &= \int_{-\infty}^t \phi^{(k+1)}(\tau)\,\mathrm{d}\tau \\ &= (\tau-t)\phi^{(k+1)}(\tau)\Big|_{-\infty}^t - \int_{-\infty}^t (\tau-t)\phi^{(k+2)}(\tau)\,\mathrm{d}\tau \\ &= -\tfrac{1}{2}(\tau-t)^2\phi^{(k+2)}(\tau)\Big|_{-\infty}^t + \frac{1}{2}\int_{-\infty}^t (\tau-t)^2\phi^{(k+3)}(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_{-\infty}^t (\tau-t)^2\phi^{(k+3)}(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where we have twice integrated by parts. Therefore, for $t \leq 0$ we have

$$\begin{aligned}|(1+t^2)^m\phi^{(k)}(t)| &= \left|\tfrac{1}{2}(1+t^2)^m\int_{-\infty}^{t}(\tau-t)^2\phi^{(k+3)}(\tau)\,\mathrm{d}\tau\right| \\ &\leq \tfrac{1}{2}(1+t^2)^m\int_{-\infty}^{t}(\tau-t)^2\frac{|(1+\tau^2)^{m+2}\phi^{(k+3)}(\tau)|}{(1+\tau)^{m+2}}\,\mathrm{d}\tau \\ &\leq \frac{\|\phi\|_\infty^{m+2,k+3}}{2}\int_{-\infty}^{t}\frac{\tau^2}{(1+\tau^2)^2}\,\mathrm{d}\tau \leq \|\phi\|_\infty^{m+2,k+3},\end{aligned}$$

using the fact that $\int_{-\infty}^{0}\frac{t^2}{1+t^2}\,\mathrm{d}t = \frac{\pi}{4}$. A similar computation can be made for $t \in \mathbb{R}_{\geq 0}$ to conclude that

$$\|\phi\|_\infty^{m,k} \leq \|\phi\|_\infty^{m+2,k+3}. \tag{10.13}$$

Now we combine (10.11) and (10.12) to compute

$$\|\phi\|_\infty^{m,k} \leq \|\phi\|_\infty^{k,k} \leq \left(\frac{\pi}{2}\right)^n\|\phi\|_\infty^{n,n},$$

provided that $m \leq k$ and that $n \geq m, k$. From (10.13) we have

$$\|\phi\|_\infty^{m,k} \leq \|\phi\|_\infty^{m+2(m-k),k+3(m-k)} = \|\phi\|_\infty^{3m-2k,3m-2k},$$

provided that $m > k$. Choosing $n = \max\{k, 3m-2k\}$ we can then ensure that

$$\|\phi\|_\infty^{m,k} \leq \left(\frac{\pi}{2}\right)^n\|\phi\|_\infty^{n}.$$

Since convergence of $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$ means exactly that for any $m, k \in \mathbb{Z}_{>0}$ the sequence $(\|\phi\|_\infty^{m,k})_{j\in\mathbb{Z}_{>0}}$ converges to zero, the lemma now follows. ▼

We now state a simple lemma.

1 **Lemma** $(\frac{\pi}{2})^{\mathrm{m}}\|\phi\|_\infty^{\mathrm{m}} \leq (\frac{\pi}{2})^{\mathrm{m}+1}\|\phi\|_\infty^{\mathrm{m}+1}$, $\mathrm{m} \in \mathbb{Z}_{\geq 0}$.

Proof By (10.12) we have $\|\phi\|_\infty^{m} \leq \frac{\pi}{2}\|\phi\|_\infty^{m+1}$, and the result follows by multiplication by $(\frac{\pi}{2})^m$. ▼

The remainder of the theorem follows as the proof of Theorem 10.2.22, taking it up at the second paragraph. One needs only replace $(b-a)^m\|\phi^{(m)}\|_\infty$ with $(\frac{\pi}{2})^m\|\phi\|_\infty^m$, noting the inequalities of the second lemma above. ■

Using this nice property of tempered distributions, we can prove the following important and useful result. We note that in contrast to Theorem 10.2.47 for $\mathscr{D}'(\mathbb{R};\mathbb{F})$ which holds only locally, the following characterisation of $\mathscr{S}'(\mathbb{R};\mathbb{F})$ is global.

10.3.25 Theorem (Tempered distributions are finite-order derivatives of signals of slow growth) *If $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ then there exists $\mathrm{r} \in \mathbb{Z}_{\geq 0}$ and a signal $\mathrm{f}_\theta \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ of slow growth such that $\theta(\phi) = \theta^{(\mathrm{r})}_{\mathrm{f}_\theta}(\phi)$ for every $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. Furthermore, we may take $\mathrm{r} = \mathrm{k} + 1$ where $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ is as given by Lemma 10.3.24.*

Proof The result follows from Lemma 10.3.24 in much the same way that Theorem 10.2.47 follows from Lemma 10.2.46. We choose $M \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{>0}$ such that

$$|\theta(\phi)| \leq M \sup\left\{|(1+t^2)^k \phi^{(k)}(t)| \;\middle|\; t \in \mathbb{R}\right\}$$

for every $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. For $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ and $j \in \mathbb{Z}_{>0}$ define

$$\psi^j_\phi(t) = (1+t^2)^j \phi^{(j)}(t),$$

noting that $\psi^j_\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ for all $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$. Then define

$$\mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)} = \left\{\psi^{k+1}_\phi \;\middle|\; \phi \in \mathscr{S}(\mathbb{R};\mathbb{F})\right\},$$

and consider on $\mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)}$ the norm $\|\cdot\|_1$. Define a linear map $\alpha_\theta\colon \mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)} \to \mathbb{F}$ by $\alpha_\theta(\psi^{k+1}_\phi) = \theta(\phi)$. We claim that α_θ is continuous with respect to the norm $\|\cdot\|_1$. Let $(\psi^{k+1}_{\phi_j})_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)}$ relative to $\|\cdot\|_1$. For $t \in \mathbb{R}$ we have

$$\begin{aligned}
|(1+t^2)^k \phi_j^{(k)}(t)| &= \left|(1+t^2)^k \int_t^\infty \phi_j^{(k+1)}(\tau)\,\mathrm{d}\tau\right| \\
&\leq (1+t^2)^k \int_t^\infty \frac{|(1+\tau^2)^{k+1}\phi^{(k+1)}(t)|}{(1+\tau^2)^{k+1}}\,\mathrm{d}\tau \\
&\leq \int_t^\infty \frac{|(1+\tau^2)^{k+1}\phi^{(k+1)}(\tau)|}{1+\tau^2}\,\mathrm{d}\tau \\
&\leq \int_t^\infty |(1+\tau^2)^{k+1}\phi^{(k+1)}(\tau)|\,\mathrm{d}\tau \\
&\leq \int_{\mathbb{R}} |(1+\tau^2)^{k+1}\phi^{(k+1)}(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, if for $\epsilon \in \mathbb{R}_{>0}$ we choose $N \in \mathbb{Z}_{>0}$ such that

$$\int_{\mathbb{R}} \left|(1+t^2)^{k+1}\phi_j^{(k+1)}(t)\right|\mathrm{d}t < \frac{\epsilon}{M}, \qquad j \geq N,$$

this being possible since $(\psi^{k+1}_{\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges to zero relative to $\|\cdot\|_1$. Then

$$\begin{aligned}
|\alpha_\theta(\psi^{k+1}_{\phi_j})| = |\theta(\phi_j)| &\leq M \sup\left\{|(1+t^2)^k \phi_j^{(k)}| \;\middle|\; t \in \mathbb{R}\right\} \\
&\leq \int_{\mathbb{R}} \left|(1+t^2)^{k+1}\phi_j^{(k+1)}(t)\right|\mathrm{d}t < \epsilon, \qquad j \geq N.
\end{aligned}$$

This shows that α_θ is indeed continuous as claimed.

Note that $\mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)} \subseteq \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$. Then, by the Hahn–Banach theorem, Theorem **??**, there exists a continuous linear map $\bar{\alpha}_\theta\colon \mathsf{L}^1(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ such that $\bar{\alpha}_\theta|\mathscr{S}(\mathbb{R};\mathbb{F})^{(k+1)} = \alpha_\theta$. By Theorem **??** there exists $g_\theta \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ such that for each $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ we have

$$\bar{\alpha}_\theta(\psi^{k+1}_\phi) = \int_{\mathbb{R}} g_\theta(t)(1+t^2)^{k+1}\phi^{(k+1)}(t)\,\mathrm{d}t = \int_{\mathbb{R}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t,$$

where $f_\theta(t) = (1+t^2)^{k+1}g_\theta(t)$. Note that since g_θ is bounded, f_θ is a signal of slow growth (we may as well suppose that $g_\theta(t) \leq \|g_\theta\|_\infty$ for all $t \in \mathbb{R}$). Therefore,

$$\theta(\phi) = \bar{\alpha}_\theta(\psi_\phi^{k+1}) = \theta_{f_\theta}(\phi^{(k+1)}) = (-1)^{k+1}\theta_{f_\theta}^{(k+1)}(\phi),$$

as claimed. ■

Exercises

10.3.1 Show that if $\phi_1, \phi_2 \in \mathscr{S}(\mathbb{R};\mathbb{F})$ then $\phi_1\phi_2 \in \mathscr{S}(\mathbb{R};\mathbb{F})$. Thus $\mathscr{S}(\mathbb{R};\mathbb{F})$ is an algebra.

10.3.2 Which of the following signals is in $\mathscr{S}(\mathbb{R};\mathbb{F})$? For signals not in $\mathscr{S}(\mathbb{R};\mathbb{F})$, explain why they are not.

(a) $f(t) = \arctan(t)$.

(b) ***missing stuff***

10.3.3 Find a locally integrable signal f that is not of slow growth and for which $\theta_f \in \mathscr{S}'(\mathbb{R};\mathbb{F})$.

10.3.4 Which of the following sequences $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ of signals in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$? For sequences not converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$, explain why they do not.

(a) $\phi_j(t) =$***missing stuff***

Section 10.4

Integrable distributions

In this section we define a class of distributions that lies between the class $\mathscr{E}'(\mathbb{R};\mathbb{F})$ of distributions with compact support and the class $\mathscr{S}'(\mathbb{R};\mathbb{F})$ of tempered distributions. The class of distributions we describe here will be useful in Section **??** in our definition of convolution for distributions.

Do I need to read this section? The integrable distributions we consider in this section are not widely used. However, they will be used in our construction of convolution for distributions, and are indeed often used for constructions related to this. Therefore, this section is of secondary importance, and can be read at such time as one needs to really understand the details of the definition of convolution for distributions. •

10.4.1 Bounded test signals

We jump right to the definition since the pattern is by now well established, we hope. Our constructions rely on an understanding of the notions of integrability introduced in Section 6.7.7.

10.4.1 Definition (Bounded test signals) A ***bounded test signal*** is a signal $\phi\colon \mathbb{R} \to \mathbb{F}$ such that

(i) ϕ is infinitely differentiable and

(ii) $\lim_{|t|\to\infty} \phi^{(k)}(t) = 0$ for each $k \in \mathbb{Z}_{\geq 0}$.

The set of bounded test signals is denoted by $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. •

10.4.2 Remark ($\mathscr{B}_0(\mathbb{R};\mathbb{F})$ is a vector space) It is easy to verify that $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{R}}$. •

Let us consider some examples relating the test signals $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ to our other classes of test signals.

10.4.3 Examples (Bounded test signals)

1. It is clear that $\mathscr{S}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{B}_0(\mathbb{R};\mathbb{F})$.
2. An example of a signal in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ that is not in $\mathscr{S}(\mathbb{R};\mathbb{F})$ is $\phi(t) = \frac{1}{1+t^2}$, cf. Example 10.3.5–2.
3. It is clear that $\mathscr{B}_0(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$. However, the inclusion is equally as clearly strict; for example the signal $\phi(t) = t$ is in $\mathscr{E}(\mathbb{R};\mathbb{F})$ but not in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.
4. The function $\phi(t) = t\sin(t)$ is infinitely differentiable and decays to zero as $|t| \to \infty$. However, since $\phi'(t) = \sin(t) + t\cos(t)$, we see that ϕ' does not decay to zero as $|t| \to \infty$. Thus $\phi \notin \mathscr{B}_0(\mathbb{R};\mathbb{F})$. •

We can also define the notion of convergence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.

10.4.4 Definition (Convergence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$) A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ ***converges to zero*** if, for each $k\in\mathbb{Z}_{\geq 0}$, the sequence $(\phi_j^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero. A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ ***converges*** to $\phi\in\mathscr{B}_0(\mathbb{R};\mathbb{F})$ if the sequence $(\phi_j-\phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

Let us examine some characteristics of convergence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ via examples.

10.4.5 Examples (Convergence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$)

1. Note that a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Indeed, in Definition 10.3.3 one need only take $k=0$. It then follows from Example 10.3.7–1 that every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.
2. There are sequences of test signals in $\mathscr{S}(\mathbb{R};\mathbb{F})\subseteq\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, but not in $\mathscr{S}(\mathbb{R};\mathbb{F})$. Indeed, we saw one such sequence in Example 10.3.7–4.
3. The sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ defined by $\phi_j(t)=\frac{1}{j(1+t^2)}$ converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. More generally, if $\phi\in\mathscr{B}_0(\mathbb{R};\mathbb{F})$ then the sequence $(j^{-1}\phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{B}_)(\mathbb{R};\mathbb{F})$.
4. Let $\phi\in\mathscr{B}_0(\mathbb{R};\mathbb{F})$ and define $\phi_j(t)=j^{-1}\phi(j^{-1}t)$. Then one can verify that the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.
5. Let us define $f\colon\mathbb{R}\to\mathbb{F}$ by
$$f(t)=\begin{cases}\mathrm{e}\exp(-\frac{1}{1-t^2}), & |t|<1,\\ 0, & |t|\geq 1.\end{cases}$$
For $j\in\mathbb{Z}_{>0}$ define $\phi\in\mathscr{B}_0(\mathbb{R};\mathbb{F})$ by
$$\phi_j(t)=\begin{cases}j^{-1}, & t\in[-j,j],\\ j^{-1}f(j^2t+j^3), & t\in(-j-\frac{1}{j^2},-j),\\ j^{-1}f(j^2t-j^3), & t\in(j+\frac{1}{j^2},j),\\ 0 & |t|\geq j+\frac{1}{j^2}.\end{cases}$$
In Figure 10.8 we depict a few terms in this sequence. While this sequence converges uniformly to zero, one can show that the sequence $(\phi_j')_{j\in\mathbb{Z}_{>0}}$ does not converge uniformly to zero. •

Let us define the notion of continuity on $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.

10.4.6 Definition (Continuous linear maps on $\mathscr{B}_0(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon\mathscr{B}_0(\mathbb{R};\mathbb{F})\to\mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ of numbers converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ that converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. •

10.4.2 Definition of integrable distributions

As expected, we have the following definition for the class of distributions we are considering.

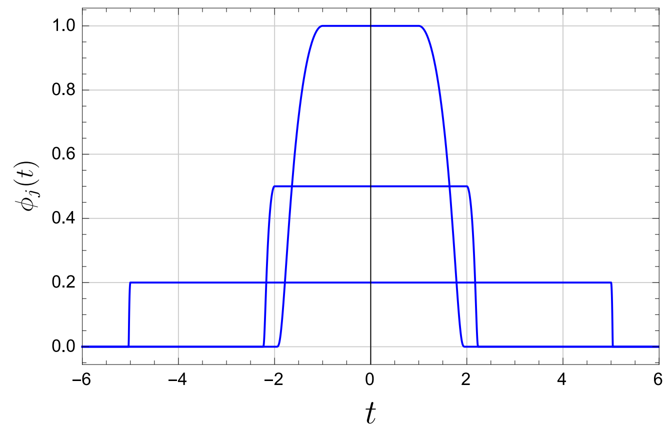


Figure 10.8 The 1st, 2nd, and 5th terms in a nonconverging sequence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$

10.4.7 Definition (Integrable distribution) An ***integrable distribution*** is a continuous linear map from $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of integrable distributions is denoted by $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$. •

10.4.8 Examples (Integrable distributions)

1. We claim that if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ then the map $\theta_f\colon \mathscr{B}_0(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ defined by

$$\theta_f(\phi) = \int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t.$$

First of all, since ϕ is bounded, the integral exists. Indeed, since $|f(t)\phi(t)| \leq \|\phi\|_\infty f(t)$, we have $\|f\phi\|_1 \leq \|\phi\|_\infty \|f\|_1 < \infty$. It is also obvious that θ_f is linear. It remains to show that θ_f is continuous. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero. This implies, in particular, that the sequence $(\|\phi_j\|)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathbb{R}$. For $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\|\phi_j\|_\infty \leq \frac{\epsilon}{\|f\|_1}$ for $j \geq N$. Then

$$|\theta_f(\phi_j)| \leq \int_{\mathbb{R}} |f(t)\phi_j(t)|\,\mathrm{d}t < \|\phi_j\|_\infty \|f\|_1 < \epsilon$$

when $j \geq N$. Thus $\lim_{j\to\infty} \theta_f(\phi_j) = 0$ as desired.

2. The map $\delta_0\colon \mathscr{B}_0(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ defined by $\delta_0(\phi) = \phi(0)$ is readily verified to be an integrable distribution. •

Let us prove some general results which clarify the relationship between integrable distributions and other classes of distributions.

10.4.9 Proposition (Integrable distributions are tempered distributions) *We have $\mathscr{B}_0'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{S}'(\mathbb{R};\mathbb{F})$. Moreover, integrable distributions $\theta_1, \theta_2 \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$ agree if and only if they agree as tempered distributions.*

Proof Firstly, if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$, then it follows immediately that the sequence also converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Therefore, if $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$, then $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};l\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. This shows that integrable distributions are tempered distributions.

Now let $\theta_1, \theta_2 \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ agree as integrable distributions. Then clearly θ_1 and θ_2 agree as tempered distributions since $\mathscr{S}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{B}_0(\mathbb{R};\mathbb{F})$.

Conversely, suppose that θ_1 and θ_2 agree as tempered distributions. By Proposition 10.3.12 it follows that θ_1 and θ_2 agree as distributions, i.e., that $\theta_1(\phi) = \theta_2(\phi)$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Let $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$. Then, by Theorem 10.8.3(iii), let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Then, continuity of θ_1 and θ_2 gives

$$\theta_1(\phi) = \lim_{j\to\infty} \theta_1(\phi_j) = \lim_{j\to\infty} \theta_2(\phi_j) = \theta_2(\phi),$$

showing that θ_1 and θ_2 agree as integrable distributions. ■

Next let us characterise membership in $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$ by using other classes of test functions. This is entirely analogous to Theorem 10.3.13 for tempered distributions.

10.4.10 Theorem (Alternative characterisation of integrable distributions) *If $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$ then $(\theta(\phi_\mathrm{j}))_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_\mathrm{j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Conversely, if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and if $(\theta(\phi_\mathrm{j}))_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_\mathrm{j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ that converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, then $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$.*

Proof Suppose that $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{S}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Continuity of θ ensures that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero.

Let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ have the property that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Also let $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$. To define $\theta(\phi)$ we let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ. This means that $(\phi - \phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. That this is possible is a consequence of Theorem 10.8.3(iii) below. Let $j, k \in \mathbb{Z}_{>0}$ and note that

$$|\theta(\phi_j - \phi_k)| \le |\theta(\phi - \phi_j)| + |\theta(\phi - \phi_k)|.$$

By choosing j and k sufficiently large we can ensure that $|\theta(\phi_j - \phi_k)|$ is as small as desired, and this means that $(\theta(\phi - \phi_j))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence, and so converges in $\mathbb{F}$. This means that we can define $\theta(\phi) = \lim_{j\to\infty} \theta(\phi_j)$. To show that this definition does not depend on the choice of sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ, let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be another sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ again converging to ϕ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Then

$$\begin{aligned}\left|\lim_{j\to\infty} \theta(\phi_j) - \lim_{k\to\infty} \theta(\psi_k)\right| &= \lim_{j,k\to\infty} |\theta(\phi_j - \psi_k)| \\ &\le \lim_{j,k\to\infty} |\theta(\phi - \phi_j)| + \lim_{j,k\to\infty} |\theta(\phi - \psi_k)|.\end{aligned}$$

Both of these last limits are zero and so the two limits are the same, and the notation $\theta(\phi)$ makes sense for $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$.

We must still show that θ is linear and continuous. Linearity is simple. To show continuity let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero. Let $(\psi_k)_{k\in\mathbb{Z}_{>0}}$ be the sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ characterised by Lemma 1 in the proof of Theorem 10.3.13.

1 Lemma *If $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ then the sequence $(\phi\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$.*

Proof This follows from taking the case of $k = 0$ in the proof of Lemma 2 used in proving Theorem 10.3.13. ▼

For each $j \in \mathbb{Z}_{>0}$ note that the sequence $(\chi_{j,k} \triangleq \psi_k\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converges to ϕ_j in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ by the lemma. Therefore, for each $j \in \mathbb{Z}_{>0}$, there exists $N_j \in \mathbb{Z}_{>0}$ sufficiently large that

$$|\theta(\phi_j - \psi_k\phi_j)| \le \epsilon, \qquad k \ge N_j,$$

by our assumptions on θ. We claim that the sequence $(\psi_{N_j}\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Indeed,

$$\lim_{j\to\infty} \sup\left\{|(\psi_{N_j}\phi_j)^{(r)}(t)| \;\middle|\; t \in \mathbb{R}\right\} = 0$$

by Lemma 1 from the proof of Theorem 10.3.13, the fact that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, and the formula

$$(\psi_{N_j}\phi_j)^{(r)} = \sum_{\ell=0}^{r} \binom{r}{\ell} \psi_{N_j}^{(\ell)}\phi_j^{(r-\ell)}.$$

This then gives

$$|\theta(\phi_j)| \le |\theta(\phi_j - \psi_{N_j}\phi_j)| + |\theta(\psi_{N_j}\phi_j)|.$$

The two terms on the right go to zero as $j \to \infty$ by our hypotheses on θ, and so continuity of θ on $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ follows. ■

10.4.3 Properties of integrable distributions

Let us define the notions of convergence of integrable distributions.

10.4.11 Definition (Convergence in $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$ is

(i) a ***Cauchy sequence*** if $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ and

(ii) ***converges*** to an integrable distribution θ if, for every $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$, the sequence $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ of numbers converges to $\theta(\phi)$. •

As one hopes, Cauchy sequences of integrable distributions converge.

10.4.12 Theorem (Cauchy sequences in $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$ converge) *If $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$ that is Cauchy, then it converges to some $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$.*

Proof The proof goes very much like that of Theorem 10.2.22. All one needs to do is choose the initial subsequence $(\psi_n)_{n\in\mathbb{Z}_{>0}}$ so as to have the additional property that

$$\|\psi_n^{(j)}\|_\infty < \frac{1}{4^n}, \qquad j,k \in \{0,1,\dots,n\}.$$

After replacing all occurrences of $\mathscr{D}(\mathbb{R};\mathbb{F})$ with $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ and of $\mathscr{D}'(\mathbb{R};\mathbb{F})$ with $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$, the same proof then gives the result in this case. ■

The class of signals that integrable distributions generalise are, unsurprisingly, the integrable signals.

10.4.13 Proposition (Integrable signals are integrable distributions) *If* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *then* $\theta_f \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$*. Moreover, if* $f_1, f_2 \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *for which* $\theta_{f_1} = \theta_{f_2}$ *then* $f_1(t) = f_2(t)$ *for almost every* $t \in \mathbb{R}$*.*

Proof The first statement of the proof is proved in Example 10.4.8–1.

The last assertion follows the similar assertion in Proposition 10.2.12, along with Propositions 10.3.12 and 10.4.9. ■

Let us characterise the functions that we can use to multiply integrable distributions.

10.4.14 Proposition (Integrable distributions can be multiplied by signals all of whose derivatives are bounded) *Let* $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$ *and let* $\phi_0\colon \mathbb{R} \to \mathbb{F}$ *be infinitely differentiable and such that* $\phi_0^{(k)} \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ *for each* $k \in \mathbb{Z}_{\geq 0}$*. Then the map*

$$\mathscr{B}_0(\mathbb{R};\mathbb{F}) \ni \phi \mapsto \theta(\phi_0\phi) \in \mathbb{F}$$

defines an element of $\mathscr{B}_0'(\mathbb{R};\mathbb{F})$*.*

Proof Linearity of the map is clear, and continuity follows from the computations in the second and third paragraphs of the proof of Proposition 10.3.18, taking $k = 0$. ■

The notions of regular, singular, support, and singular support are applied to $\mathscr{S}'(\mathbb{R};\mathbb{F})$ by restriction from $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

Of course, the derivative of an integrable distribution is an integrable distribution.

10.4.15 Proposition (The derivative of an integrable distribution is an integrable distribution) *If* $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$ *then* $\theta' \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$*.*

Proof This is easy to show. We let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero. Then $(-\phi_j')_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, as is easily seen from the definition of convergence to zero. Therefore,

$$\lim_{j\to\infty} \theta'(\phi_j) = \lim_{j\to\infty} \theta(-\phi_j') = 0$$

as desired. ■

10.4.4 Some deeper properties of integrable distributions

In this section we give some useful properties of integrable distributions. We begin by showing that integrable distributions have a boundedness property like as have seen for distributions and tempered distributions.

10.4.16 Lemma (A boundedness property for integrable distributions) *Let $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$. Then there exists $\mathrm{M} \in \mathbb{R}_{>0}$ and $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ such that, for each $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$, we have*

$$|\theta(\phi)| \leq \mathrm{M}\max\{\|\phi\|_\infty, \|\phi^{(1)}\|_\infty, \dots, \|\phi^{(\mathrm{k})}\|_\infty.$$

Proof For $m \in \mathbb{Z}_{\geq 0}$ define

$$\|\phi\|_\infty^m = \max\{\|\phi\|_\infty, \|\phi^{(1)}\|_\infty, \dots, \|\phi^{(m)}\|_\infty\}.$$

It is clear that a sequence $(\phi_j)_{j\in\infty}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converges to zero if and only if $(\|\phi_j^{(m)}\|_\infty)_{j\in\mathbb{Z}_{>0}}$ converges to zero for every $m \in \mathbb{Z}_{\geq 0}$. This, however, is easily seen to be equivalent to the convergence to zero of $(\|\phi_j\|_\infty^m)_{j\in\mathbb{Z}_{>0}}$ for each $m \in \mathbb{Z}_{\geq 0}$. One can now prove the lemma by picking up the proof of Lemma 10.2.46 in the second paragraph, replacing $(b-a)^m\|\phi^{(m)}\|_\infty$ with $\|\phi\|_\infty^m$, noting the obvious inequality $\|\phi\|_\infty^m \leq \|\phi\|_\infty^{m+1}$ for each $m \in \mathbb{Z}_{\geq 0}$. ■

The following notion will also be important for us.

10.4.17 Definition (Approximate unit and special approximate unit) A sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$

(i) is an ***approximate unit*** if

(a) for every compact set $K \subseteq \mathbb{R}$, $(\psi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to the function $K \ni t \mapsto 1$ and

(b) for each $r \in \mathbb{Z}_{\geq 0}$, there exists $M_r \in \mathbb{R}_{>0}$ such that $\|\phi_j^{(r)}\|_\infty \leq M_r$ for every $j \in \mathbb{Z}_{>0}$.

and

(ii) is a ***special approximate unit*** if

(a) for any compact set $K \subseteq \mathbb{R}$, there exists $N \in \mathbb{Z}_{>0}$ such that $\psi_j(t) = 1$ for every $t \in K$ and $j \geq N$ and

(b) for each $k \in \mathbb{Z}_{\geq 0}$, there exists $M_r \in \mathbb{R}_{>0}$ such that $\|\psi_j^{(r)}\|_\infty \leq M_r$ for every $j \in \mathbb{Z}_{>0}$. •

Such sequences of signals exist.

10.4.18 Examples (Approximate and special approximate units)

1. An example of an approximate unit is the sequence described in Lemma 1 in the proof of Theorem 10.3.13. Recall that if we define

$$\Psi(t) = \begin{cases} \mathrm{e}\exp\left(-\frac{1}{1-t^2}\right), & |t| < 1, \\ 0, & \text{otherwise}, \end{cases}$$

then the approximate unit is the sequence $(\Psi_j)_{j\in\mathbb{Z}_{>0}}$ where $\Psi_j(t) = \Psi(\frac{t}{j})$. In Figure 10.9 we show the graph of Ψ.

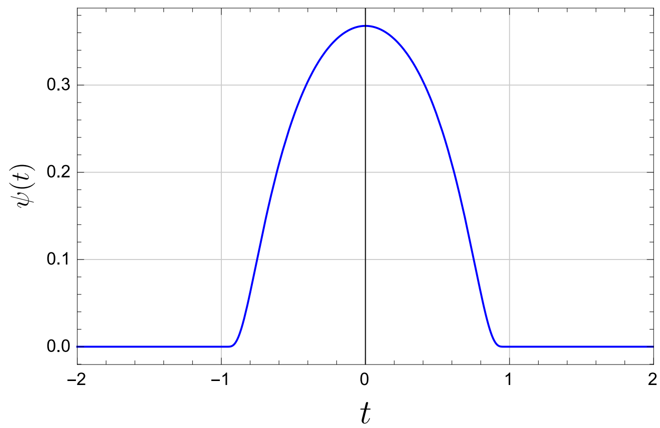


Figure 10.9 A signal used to construct an approximate unit

2. To give an example of a special approximate unit, let $\Psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ be defined by

$$\Psi(t) = \begin{cases} 0, & t \in (-\infty,-2], \\ \mathrm{e}\cdot\mathrm{e}^{-1/(1-(t+1)^2)}, & t \in (-2,-1), \\ 1, & t \in [-1,1], \\ \mathrm{e}\cdot\mathrm{e}^{-1/(1-(t-1)^2)}, & t \in (1,2), \\ 0, & t \in [2,\infty), \end{cases}$$

and depicted in Figure 10.10. As may be deduced from Example **??–??**, this

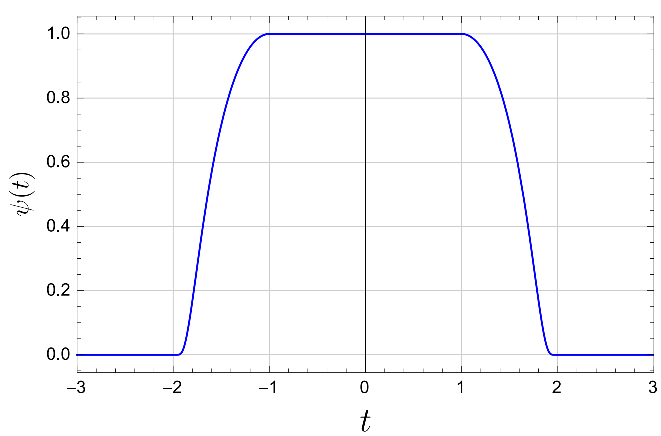


Figure 10.10 A signal used to construct a special approximate unit

signal is in $\mathscr{D}(\mathbb{R};\mathbb{F})$. The sequence $(\Psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ given by $\Psi_j(t) = \Psi(j^{-1}t)$, $j \in \mathbb{Z}_{>0}$, is then verified to be a special approximate unit in the sense of the above definition. •

Next we state a few equivalent characterisations of integrable distributions. As in the proof of Lemma 10.4.16, for $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$, denote

$$\|\phi\|_\infty^m = \max\{\|\phi\|_\infty, \|\phi^{(1)}\|_\infty, \dots, \|\phi^{(m)}\|_\infty\}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

With this notation we have the following result.

10.4.19 Theorem (Characterisation of integrable distributions) *For $\theta \in \mathscr{D}(\mathbb{R};\mathbb{F})$ the following statements are equivalent:*

(i) $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$;

(ii) there exists $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ such that, for every $\epsilon \in \mathbb{R}_{>0}$, there exists a compact set $\mathrm{K} \subseteq \mathbb{R}$ for which $|\theta(\phi)| < \epsilon\|\phi\|_\infty^{\mathrm{k}}$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ satisfying $\mathrm{supp}(\phi) \cap \mathrm{K} = \emptyset$;

(iii) for every approximate unit $(\Psi_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$, the sequence $(\theta(\Psi_{\mathrm{j}}))_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges;

(iv) for every special approximate unit $(\Psi_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$, the sequence $(\theta(\Psi_{\mathrm{j}}))_{\mathrm{j}\in\mathbb{Z}_{>0}}$ converges;

(v) there exists a compact set $\mathrm{K} \subseteq \mathbb{R}$, $\mathrm{M} \in \mathbb{R}_{>0}$, and $\mathrm{k} \in \mathbb{Z}_{\geq 0}$ such that $|\theta(\phi)| \leq \mathrm{M}\|\phi\|_\infty^{\mathrm{k}}$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ satisfying $\mathrm{supp}(\phi) \cap \mathrm{K} = \emptyset$.

Proof (i) $\implies$ (ii) Choose $k \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{R}_{>0}$ as in Lemma 10.4.16. Assume (ii) does not hold. Thus, assume that there exists $\epsilon \in \mathbb{R}_{>0}$ such that, for every compact set $K \subseteq \mathbb{R}$, there exists $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ satisfying $\mathrm{supp}(\phi) \cap K = \emptyset$ and such that $|\theta(\phi)| > \epsilon\|\phi\|_\infty^k$. Now we inductively construct a sequence $(K_j)_{j\in\mathbb{Z}_{>0}}$ of compact sets and $(\phi_j)_{j\in\mathbb{Z}_{>0}}$. Let $K_1 = [-1,1]$. By our assumption, there exists $\psi_1 \in \mathscr{D}(\mathbb{R};\mathbb{F})$ such that $\mathrm{supp}(\psi_1) \cap K_1 = \emptyset$ and such that $|\theta(\psi_1)| > \epsilon\|\phi_1\|_\infty^k$. We then take $\phi_1 = a_1\psi_1$ where $a_1 \in \mathbb{F}$ is chosen such that $\|\phi_1\|_\infty^k = 1$ and such that $\theta(\phi_1) \in \mathbb{R}_{>0}$. Then $\theta(\phi_1) > \epsilon$. Now suppose that $K_1, \dots, K_m$ and $\phi_1, \dots, \phi_m$ have been defined. Let $T_{m+1} \in \mathbb{R}_{>0}$ be such that $\mathrm{supp}(\phi_m) \subseteq (-T_{m+1}, T_{m+1})$ and such that

$$K_1 \cup \dots \cup K_m \subseteq (-T_{m+1}, T_{m+1}).$$

Take $K_{m+1} = [-T_{m+1}, T_{m+1}]$. Then there exists $\psi_{m+1} \in \mathscr{D}(\mathbb{R};\mathbb{F})$ such that $\mathrm{supp}(\psi_{m+1}) \cap K_{m+1} = \emptyset$. Then define $\phi_{m+1} = a_{m+1}\psi_{m+1}$ with $a_{m+1} \in \mathbb{F}$ chosen such that $\|\phi_{m+1}\|_\infty^k = 1$ and $\theta(\phi_{m+1}) \in \mathbb{R}_{>0}$. Then $\theta(\phi_{m+1}) > \epsilon$. Note that the supports of the functions ϕ_j, $j \in \mathbb{Z}_{>0}$, are pairwise disjoint. Thus we can define $\Phi_m \in \mathscr{D}(\mathbb{R};\mathbb{F})$, $m \in \mathbb{Z}_{>0}$, by

$$\Phi_m(x) = \sum_{j=1}^m \phi_j(m).$$

Moreover, we clearly have $\|\Phi_m\|_\infty^k = 1$, and so, by Lemma 10.4.16, $|\theta(\Phi_m)| \leq M$. However, we also have

$$\theta(\Phi_m) = \sum_{j=1}^m \theta(\phi_j) > m\epsilon.$$

Since this must hold for each $m \in \mathbb{Z}_{>0}$, we arrive at a contradiction.

(ii) $\implies$ (iii) Let $(\Psi_j)_{j\in\mathbb{Z}_{>0}}$ be the special approximate unit from Example 10.4.18–2 (although any other special approximate unit can be made to work). Note that for $j \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geq 0}$ we have $\phi_j^{(r)} = j^{-r}\Psi^{(r)}$ which gives

$$\|\Psi_j^{(r)}\|_\infty \leq \|\Psi^{(r)}\|_\infty, \qquad j \in \mathbb{Z}_{>0},\ r \in \mathbb{Z}_{\geq 0}. \tag{10.14}$$

For $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$, apply the higher-order Leibniz Rule, Proposition 3.2.11, to get

$$((1-\Psi_j)\phi)^{(r)} = \sum_{m=0}^{r} \binom{r}{m} (1-\Psi_j)^{(m)}\phi^{(r-m)}.$$

Now let

$$B_r = \max\left\{\binom{r}{m} \;\middle|\; m \in \{0,1,\dots,r\}\right\}.$$

Then

$$\|(1-\Psi_j)\phi\|_\infty^r \le rB_r\|1-\Psi\|_\infty^r\|\phi\|_\infty^r, \tag{10.15}$$

using (10.14). Now let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be an approximate unit and let $k \in \mathbb{Z}_{\ge 0}$ be chosen as in (ii). Then define

$$M_k = 4\sup\{\|\psi_j\|_\infty^k \mid j \in \mathbb{Z}_{>0}\}.$$

Let $\epsilon \in \mathbb{R}_{>0}$. By assumption, there exists a compact set K such that

$$|\theta(\phi)| < \frac{\epsilon}{kM_kB_k\|1-\Psi\|_\infty^k}\|\phi\|_\infty^k$$

for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ satisfying $\operatorname{supp}(\phi) \cap K = \emptyset$. Let $N_1 \in \mathbb{Z}_{>0}$ be sufficiently large that $\Psi_{N_1}(x) = 1$ for all x in a neighbourhood U of K. Since $1 - \Psi_{N_1}(x) = 0$ for all $x \in U$ we compute, for $l, m \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
|\theta((1-\Psi_{N_1})(\psi_l-\psi_m))| &< \frac{\epsilon}{kM_kB_k\|1-\Psi\|_\infty^k}\|(1-\Psi_{N_1})(\psi_l-\psi_m)\|_\infty^k \\
&\le \frac{\epsilon}{kM_kB_k\|1-\Psi\|_\infty^k}kB_k\|1-\Psi\|_\infty^k\|\psi_l-\psi_m\|_\infty^r \\
&\le \frac{\epsilon}{kM_kB_k\|1-\Psi\|_\infty^k}kB_k\|1-\Psi\|_\infty^k(\|\psi_l\|_\infty^r + \|\psi_m\|_\infty^r) \le \frac{\epsilon}{2},
\end{aligned}$$

using the triangle inequality and (10.15).

(For the next fifteen seconds we use some facts about distributions with compact support, as developed in Section 10.5.) Since the distribution $\Psi_{N_1}\theta$ has compact support and since the conditions for an approximate unit ensure that $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{E}(\mathbb{R};\mathbb{F})$, the sequence $(\theta(\Psi_{N_1}\psi_j))_{j\in\mathbb{Z}_{>0}}$ converges and so is Cauchy. Thus there exists $N_2 \in \mathbb{Z}_{>0}$ such that, if $l, m \ge N_2$,

$$|\theta(\Psi_{N_1}(\psi_l-\psi_m))| < \frac{\epsilon}{2}.$$

Thus, for $l, m \ge N_2$,

$$|\theta(\psi_l-\psi_m)| \le |\theta((1-\Psi_{N_1})(\psi_l-\psi_m))| + |\theta(\Psi_{N_1}(\psi_l-\psi_m))| \le \epsilon,$$

showing that the sequence $(\theta(\psi_j))_{j\in\mathbb{Z}_{>0}}$ is Cauchy, and so converges.

(iii) $\Longrightarrow$ (iv) This is clear.

(iv) $\Longrightarrow$ (v) Suppose that (v) does not hold. Thus, for every compact set $K \subseteq \mathbb{R}$, $M \in \mathbb{R}_{>0}$, and $k \in \mathbb{Z}_{\ge 0}$, there exists $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ such that $\operatorname{supp}(\phi) \cap K = \emptyset$ and $|\theta(\phi)| > M\|\phi\|_\infty^k$. For $j \in \mathbb{Z}_{>0}$ let $K_j = [-j, j]$. Then let $\phi_j \in \mathscr{D}(\mathbb{R};\mathbb{F})$ be such that $\operatorname{supp}(\phi_j) \cap K_j = \emptyset$ and such that $|\theta(\phi_j)| > j^2\|\phi\|_\infty^j$. Define

$$\Phi_j = \frac{\phi_j}{j\|\phi_j\|_\infty^j}$$

and note that $\operatorname{supp}(\Phi_j) \cap K_j = \emptyset$, $|\theta(\Phi_j)| > j$, and $\|\Phi_j\|_\infty^j < \frac{1}{j}$.

Now let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be a special approximate unit and note that $(\psi_j + \Phi_j)_{j\in\mathbb{Z}_{>0}}$ is also a special approximate unit. Moreover,

$$|\theta(\psi_j + \Phi_j) - \theta(\psi_j)| = |\theta(\Phi_j)| > j, \qquad j \in \mathbb{Z}_{>0},$$

from which we can infer that either $(\theta(\psi_j + \Phi_j))_{j\in\mathbb{Z}_{>0}}$ or $(\theta(\psi_j))_{j\in\mathbb{Z}_{>0}}$ diverges. Thus (iv) does not hold.

(v) $\implies$ (i) Let $\mathscr{B}(\mathbb{R};\mathbb{F})$ denote the set of infinitely differentiable functions on $\mathbb{R}$ such that the function and all of its derivatives are bounded. If $\Phi \in \mathscr{B}(\mathbb{R};\mathbb{F})$ and $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ then

$$(\Phi\phi)^{(r)} = \sum_{m=0}^{r} \binom{r}{m} \Phi^{(m)}\phi^{(r-m)}, \qquad r \in \mathbb{Z}_{\geq 0}.$$

As in (10.15) we have

$$\|\Phi\phi\|_\infty^r \leq rB_r\|\Phi\|_\infty^r\|\phi\|_\infty^r.$$

Let K, M, and k be chosen as in (v). Let $U \subseteq \mathbb{R}$ be open and such that $\operatorname{cl}(U)$ is compact and $K \subseteq U$. Let $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ be such that $\psi(x) = 1$ for all x in a neighbourhood of K and such that $\operatorname{supp}(\psi) \subseteq U$ (why does such a function ψ exist?). By Lemma 10.2.46 there exists $C \in \mathbb{R}_{>0}$ and $C \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$ such that, if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ is such that $\operatorname{supp}(\phi)$, then $|\theta(\phi)| \leq C\|\phi\|_\infty^m$. Without loss of generality we can assume that $m \geq k$. (This can be seen by understanding the proof of Lemma 10.2.46.) Then, for any $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$,

$$\begin{aligned}
|\theta(\phi)| &\leq |\theta((1-\psi)\phi)| + |\theta(\psi\phi)| \\
&\leq C\|(1-\psi)\phi\|_\infty^m + M\|\psi\phi\|_\infty^k \\
&\leq mCB_m\|(1-\psi)\|_\infty^m\|\phi\|_\infty^m + kMB_k\|\psi\|_\infty^k\|\phi\|_\infty^k \\
&\leq (mCB_m\|(1-\psi)\|_\infty^m + kMB_k\|\psi\|_\infty^k)\|\phi\|_\infty^m,
\end{aligned}$$

using the fact that $\|\phi\|_\infty^m \geq \|\phi\|_\infty^k$ since $m \geq k$.

Therefore, if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ which converges to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, then $\lim_{j\to\infty}\theta(\phi_j) = 0$, showing that θ is integrable by Theorems 10.3.13 and 10.4.10. ■

With this characterisation of integrable distributions at hand, we can give the following result that will be useful.

10.4.20 Corollary (Well-definedness of integrable distributions as limits) *If $\theta \in \mathscr{B}_0'(\mathbb{R};\mathbb{F})$ and if $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ and $(\psi_j')_{j\in\mathbb{Z}_{>0}}$ are approximate units, then*

$$\lim_{j\to\infty}\theta(\psi_j) = \lim_{j\to\infty}\theta(\psi_j').$$

Proof Let $(\Psi_j)_{j\in\mathbb{Z}_{>0}}$ be the special approximate unit of Example 10.4.18–2. We use the notation

$$\|\phi\|_\infty^m = \max\{\|\phi\|_\infty, \|\phi^{(1)}\|_\infty, \dots, \|\phi^{(m)}\|_\infty\}$$

from the proof of Lemma 10.4.16. Let $k \in \mathbb{Z}_{>0}$ be as in part (ii). Define

$$M_k = 4\sup(\{\|\psi_j\|_\infty^k \mid j \in \mathbb{Z}_{>0}\} \cup \{\|\psi_j'\|_\infty^k \mid j \in \mathbb{Z}_{>0}\})$$

Then let $K \subseteq \mathbb{R}$ be a compact set such that

$$|\theta(\phi)| \leq \frac{\epsilon}{kB_k M_k \|1 - \Psi\|_\infty^k} \|\phi\|_\infty^k$$

for every $\phi \in \mathscr{D}(\mathbb{R}; \mathbb{F})$ such that $\mathrm{supp}(\phi) \cap K = \emptyset$. Let $N_1 \in \mathbb{Z}_{>0}$ be such that $\Psi_{N_1}(t) = 1$ for all t in a neighbourhood of K. Then let $U \subseteq \mathbb{R}$ be a bounded open interval such that $\mathrm{supp}(\Psi_{N_1}) \subseteq U$. As per Lemma 10.2.46, let $C \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{>0}$ be such that

$$|\theta|(\phi) \leq C\|\phi\|_\infty^m$$

for all $\phi \in \mathscr{D}(\mathbb{R}; \mathbb{F})$ for which $\mathrm{supp}(\phi) \subseteq U$. Since, for any compact subset $L \subseteq \mathbb{R}$, each of the sequences $(\psi_j)_{j \in \mathbb{Z}_{>0}}$ and $(\psi_j')_{j \in \mathbb{Z}_{>0}}$ and all of their derivatives converge uniformly to the function equal to 1 on L, the sequence $(\psi_j - \psi_j')_{j \in \mathbb{Z}_{>0}}$ and all derivatives converge uniformly to the zero function. Thus there exists $N_2 \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned}
&\|(\psi_j - \psi_j')| \,\mathrm{supp}(\Psi_{N_1})\|_\infty^m \\
&\qquad \triangleq \sup\{|(\psi_j - \psi_j')^{(r)}(t)| \mid t \in \mathrm{supp}(\Psi_{N_1}),\ r \in \{0, 1, \dots, m\}\} < \frac{\epsilon}{2C\|\Psi\|_\infty^m}.
\end{aligned}$$

Then, for $j \geq N_2$ we have

$$\begin{aligned}
|\theta(\psi_j - \psi_j')| &\leq |\theta((1 - \Psi_{N_1})(\psi_j - \psi_j'))| + |\theta(\Psi_{N_1}(\psi_j - \psi_j'))| \\
&\leq \frac{\epsilon}{kB_k M_k \|1 - \Psi\|_\infty^k} \|(1 - \Psi_{N_1})(\psi_j - \psi_j')\|_\infty^k + C\|\Psi_{N_1}(\psi_j - \psi_j')\|_\infty^m \\
&\leq \frac{\epsilon}{kB_k M_k \|1 - \Psi\|_\infty^k} kB_k \|(1 - \Psi)\|_\infty^k \|\psi_j - \psi_j'\|_\infty^k + C\|\Psi\|_\infty^m \|\psi_j - \psi_j'\|_\infty^m \\
&\leq \frac{\epsilon}{M_k} (\|\psi_r\|_\infty^k + \|\psi_s'\|_\infty^k) + C\|\Psi\|_\infty^m \frac{\epsilon}{2C\|\Psi\|_\infty^m} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

using our estimates above, along with (10.14) and (10.15). Thus $\lim_{j \to \infty} \theta(\psi_j - \psi_j') = 0$. Since θ is integrable, the limits $\lim_{j \to \infty} \theta(\psi_j)$ and $\lim_{j \to \infty} \theta(\psi_j')$ exist by Theorem 10.4.19, and so must be equal. ■

10.4.5 Notes

Parts of Theorem 10.4.19 are from **[PD/JV:78]**.

Section 10.5

Distributions with compact support

In this section we specialise our set of distributions even further. That is, we increase the size of the test functions with a resulting decrease in the size of the distributions.

Do I need to read this section? This section can easily be skipped on a first reading. However, the results in Section 10.5.7 may be of general interest. •

10.5.1 The set of infinitely differentiable test signals

The test signals we consider form a rather large class.

10.5.1 Definition (Infinitely differentiable test signal) An ***infinitely differentiable test signal*** is an infinitely differentiable map $\phi\colon \mathbb{R} \to \mathbb{F}$. The set of infinitely differential test signals is denoted $\mathscr{E}(\mathbb{R};\mathbb{F})$. •

10.5.2 Remark ($\mathscr{E}(\mathbb{R};\mathbb{F})$ is a vector space) One can easily verify that $\mathscr{E}(\mathbb{R};\mathbb{F})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{F}^{\mathbb{R}}$. •

The set $\mathscr{E}(\mathbb{R};\mathbb{F})$ is a large set of signals, of course. It contains all polynomial functions, exponential functions, trigonometric functions, etc. It also contains the test signals in $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, and $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. As with $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, and $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, the important notion in $\mathscr{E}(\mathbb{R};\mathbb{F})$ is that of convergence.

10.5.3 Definition (Convergence in $\mathscr{E}(\mathbb{R};\mathbb{F})$) A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{E}(\mathbb{R};\mathbb{F})$ ***converges to zero*** if for each $r \in \mathbb{Z}_{\geq 0}$ and for each compact subset $K \subseteq \mathbb{R}$, the sequence $(\phi_j^{(r)}|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero. A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{E}(\mathbb{R};\mathbb{F})$ ***converges*** to $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$ if the sequence $(\phi_j - \phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

As with $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, and $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, we can ponder bemusedly the nature of convergence in $\mathscr{E}(\mathbb{R};\mathbb{F})$. It turns out that, as with $\mathscr{S}(\mathbb{R};\mathbb{F})$ and $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, there exists a metric on $\mathscr{E}(\mathbb{R};\mathbb{F})$ for which convergence is convergence in the metric. However, again as with $\mathscr{S}(\mathbb{R};\mathbb{F})$ and $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, there is no norm defining convergence in $\mathscr{E}(\mathbb{R};\mathbb{F})$.***missing stuff***

Let us explore the notion of convergence in $\mathscr{E}(\mathbb{R};\mathbb{F})$ through some examples.

10.5.4 Examples (Convergence in $\mathscr{E}(\mathbb{R};\mathbb{F})$)

1. Note that a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. It then follows from Example 10.4.5–1 that every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{S}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. And then it follows from Example 10.3.7–1 that every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ also converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$.

2. There are sequences of test signals in $\mathscr{B}_0(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, but not in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Let us give such a sequence. Let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be the sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ characterised in Lemma 1 from the proof of Theorem 10.3.13. Then define a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ by
$$\phi_j(t) = j^{-1}e^{t^2/5}\psi_j(t).$$
In Figure 10.11 we show a few terms in this sequence. While the sequence

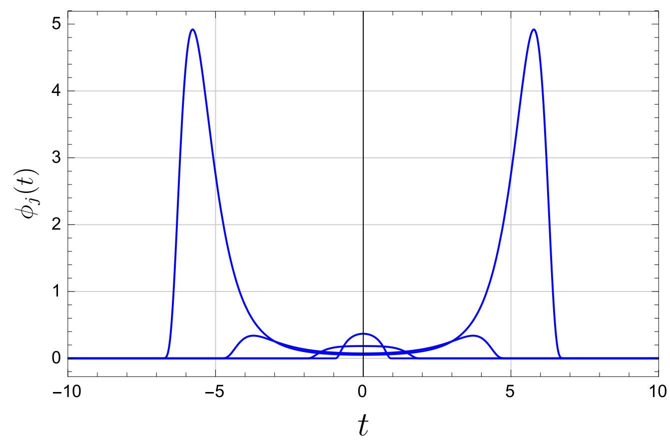


Figure 10.11 A few terms in a sequence converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$ but not in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$

of signals and all derivatives converges uniformly to zero on every compact interval (i.e., converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$), the sequence does not converge uniformly on $\mathbb{R}$. •

10.5.5 Definition (Continuous linear maps on $\mathscr{E}(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon \mathscr{E}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ that converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. •

10.5.2 Definition of distributions with compact support

The by now unsurprising definition is the following.

10.5.6 Definition (Distribution with compact support) A ***distribution with compact support*** is a continuous linear map from $\mathscr{E}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of distributions with compact support is denoted $\mathscr{E}'(\mathbb{R};\mathbb{F})$. •

10.5.7 Remark ($\mathscr{E}'(\mathbb{R};\mathbb{F})$ is a vector space) It is easy to check that $\mathscr{E}'(\mathbb{R};\mathbb{F})$ is a subspace of $\mathscr{B}'(\mathbb{R};\mathbb{F})$. The inclusion is proved below in Proposition 10.5.9, and the inheritance of the vector space structure is then readily verified. •

Do not at this point read anything literal into the words "with compact support" in the preceding definition. We will address this shortly.

Let us give some examples of distributions with compact support.

10.5.8 Examples (Distributions with compact support)

1. We claim that an integrable signal $f\colon \mathbb{R} \to \mathbb{F}$ with compact support defines an element θ_f of $\mathscr{E}'(\mathbb{R};\mathbb{F})$ by

$$\theta_f(\phi) = \int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t.$$

The integral clearly converges since, if $\operatorname{supp}(f) \subseteq [-T,T]$ we have

$$\int_{\mathbb{R}} |f(t)\phi(t)|\,\mathrm{d}t = \int_{-T}^{T} |f(t)\phi(t)|\,\mathrm{d}t \le \sup\left\{|\phi(t)|\int_{-T}^{T}|f(\tau)|\,\mathrm{d}\tau \;\middle|\; t \in [-T,T]\right\} < \infty.$$

We also claim that θ_f is continuous from $\mathscr{E}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. If $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$ we have

$$\begin{aligned}|\theta_f(\phi_j)| = \left|\int_{\mathbb{R}} f(t)\phi_j(t)\,\mathrm{d}t\right| &\le \int_{-T}^{T}|f(t)\phi_j(t)|\,\mathrm{d}t \\ &\le \sup\left\{|\phi_j(t)|\int_{-T}^{T}|f(\tau)|\,\mathrm{d}\tau \;\middle|\; t\in[-T,T]\right\}.\end{aligned}$$

Taking the limit as $j \to \infty$ shows that $\theta_f \in \mathscr{E}'(\mathbb{R};\mathbb{F})$, as claimed.

2. Let us show that $\delta_0^{(r)} \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ for each $r \in \mathbb{Z}_{\ge 0}$. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. We then have

$$\delta_0^{(r)}(\phi_j) = (-1)^r\phi^{(r)j}(0).$$

Since the sequence $(\phi_j^{(r)})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero on $[-T,T]$ for every $T \in \mathbb{R}_{>0}$, it then follows that $\lim_{j\to\infty}\delta_0^{(r)}(\phi_j) = 0$, so $\delta_0^{(r)} \in \mathscr{E}'(\mathbb{R};\mathbb{F})$.

3. While all derivatives of δ_0 are in $\mathscr{E}'(\mathbb{R};\mathbb{F})$, the "anti-derivative" of δ_0, the unit step 1, is not in $\mathscr{E}'(\mathbb{R};\mathbb{F})$. •

Let us show that distributions with compact support are tempered distributions.

10.5.9 Proposition (Distributions with compact support are integrable distributions) *We have $\mathscr{E}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{B}_0'(\mathbb{R};\mathbb{F})$. Moreover, distributions $\theta_1,\theta_2 \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ with compact support agree if and only if they agree as integrable distributions.*

Proof Let us first show that $\mathscr{E}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{B}_0'(\mathbb{R};\mathbb{F})$. Since $\mathscr{B}_0(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$, it makes sense to write $\theta(\phi)$ for $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ and $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$. We need to check that if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ then $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero if $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$. However, this follows since $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ implies convergence to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, as we saw in Example 10.5.4–1.

The final assertion follows as does the same part of Proposition 10.3.12, but now using Theorem 10.8.3(ii). ■

Recall that at this point the words "with compact support" in Definition 10.5.6 appear with no justification as concerns their relationship with elements in $\mathscr{D}(\mathbb{R};\mathbb{F})$ that have compact support. Therefore, we should establish this connection. First we note that since $\mathscr{D}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$, and since sequences converging to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ also converge to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, every element $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ defines a distribution in $\mathscr{D}'(\mathbb{R};\mathbb{F})$. The following result characterises those distributions in $\mathscr{E}'(\mathbb{R};\mathbb{F})$.

10.5.10 Proposition (A distribution with compact support is. . . a distribution with compact support) *A distribution $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ is in $\mathscr{E}'(\mathbb{R};\mathbb{F})$ if and only if $\operatorname{supp}(\theta)$ is a compact subset of $\mathbb{R}$.*

Proof First suppose that $\operatorname{supp}(\theta)$ is compact. Define $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ by asking that $\psi(t) = 1$ for all t in some open set containing $\operatorname{supp}(\theta)$. One can always do this by manipulating bump functions appropriately. By the definition of the support of a distribution, the value of θ on any element of $\mathscr{D}(\mathbb{R};\mathbb{F})$ is determined by its value on $\operatorname{supp}(\theta)$. In other words, if we define $\theta\colon \mathscr{E}(\mathbb{R};\mathbb{F}) \to \mathbb{F}$ by $\theta(\phi) = \theta(\psi\phi)$, then this map is well-defined. It is also linear and it is straightforward to check continuity. Thus this defines θ as an element of $\mathscr{E}'(\mathbb{R};\mathbb{F})$.

Now let $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ and think of it as an element of $\mathscr{D}'(\mathbb{R};\mathbb{F})$ by restriction to $\mathscr{D}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$. We claim that this element of $\mathscr{D}'(\mathbb{R};\mathbb{F})$ has compact support. Let $(K_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of compact subsets of $\mathbb{R}$ with the property that $K_j \subset K_k$ for $j < k$ and that $\mathbb{R} = \cup_{j\in\mathbb{Z}_{>0}} K_j$. Suppose that $\operatorname{supp}(\theta)$ is not compact. Then for each $j \in \mathbb{Z}_{>0}$ there exists $\phi_j \in \mathscr{D}(\mathbb{R};\mathbb{F})$ such that $\phi_j(t) = 0$ for all t in an open set containing K_j and such that $\theta(\phi_j) \neq 0$. Without loss of generality (by rescaling if necessary), suppose that $\theta(\phi_j) = 1$. We claim that the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Indeed, for any compact set $K \subseteq \mathbb{R}$ one can choose N sufficiently large that $\phi_j|K = 0$, $j \geq N$. Therefore, since θ is continuous we must have $\lim_{j\to\infty} \theta(\phi_j) = 0$, thus arriving at a contradiction. ■

As with Theorem 10.3.13 for tempered distributions, it is possible to characterise distributions with compact support using test functions from $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, or $\mathscr{B}_0(\mathbb{R};\mathbb{F})$, but with the notion of convergence inherited from $\mathscr{E}(\mathbb{R};\mathbb{F})$.

10.5.11 Theorem (Alternative characterisation of distributions with compact support) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ then $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Conversely, if $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and if $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ that converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, then $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$.*

Proof Suppose that $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$. Let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{B}_0(\mathbb{R};\mathbb{F}) \subseteq \mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Continuity of θ ensures that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero.

Let $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ have the property that $(\theta(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Also let $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$. To define $\theta(\phi)$ we let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ. This means that $(\phi - \phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. That this is possible is a consequence of Theorem 10.8.3(ii) below. Let $j, k \in \mathbb{Z}_{>0}$ and note that

$$|\theta(\phi_j - \phi_k)| \leq |\theta(\phi - \phi_j)| + |\theta(\phi - \phi_k)|.$$

By choosing j and k sufficiently large we can ensure that $|\theta(\phi_j - \phi_k)|$ is as small as desired, and this means that $(\theta(\phi - \phi_j))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence, and so converges in $\mathbb{F}$. This means that we can define $\theta(\phi) = \lim_{j\to\infty}\theta(\phi_j)$. To show that this definition does not depend on the choice of sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to ϕ, let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be another sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ again converging to ϕ in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Then

$$\begin{aligned}\left|\lim_{j\to\infty}\theta(\phi_j) - \lim_{k\to\infty}\theta(\psi_k)\right| &= \lim_{j,k\to\infty}|\theta(\phi_j - \psi_k)|\\ &\leq \lim_{j,k\to\infty}|\theta(\phi - \phi_j)| + \lim_{j,k\to\infty}|\theta(\phi - \psi_k)|.\end{aligned}$$

Both of these last limits are zero and so the two limits are the same, and the notation $\theta(\phi)$ makes sense for $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$.

We must still show that θ is linear and continuous. Linearity is simple. To show continuity let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero. Let $(\psi_k)_{k\in\mathbb{Z}_{>0}}$ be the sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ characterised by Lemma 1 in the proof of Theorem 10.3.13.

1 **Lemma** *If $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$ then the sequence $(\phi\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ in $\mathscr{E}(\mathbb{R};\mathbb{F})$.*

Proof Let $r \in \mathbb{Z}_{\geq 0}$, let $K \subseteq \mathbb{R}$ be compact, and let $\epsilon \in \mathbb{R}_{>0}$.

By the Leibniz Rule, Proposition 3.2.11, we have

$$(\phi\psi_j)^{(r)}(t) = \sum_{m=0}^{r}\binom{r}{m}\phi^{(r-m)}(t)\psi_j^{(m)}(t).$$

Thus

$$\phi^{(r)}(t) - (\phi\psi_j)^{(r)}(t) = \phi^{(r)}(t)(1 - \psi_j(t)) + \sum_{m=1}^{r}\binom{r}{m}\phi^{(r-m)}(t)\psi_j^{(m)}(t).$$

Let

$$B_r = \max\left\{\binom{r}{m}\;\middle|\; m \in \{0, 1, \dots, r\}\right\}.$$

For $m \in \{0, 1, \dots, r\}$ let

$$M_m = \sup\{|\phi^{(m)}(t)| \mid t \in K\}$$

and, using Lemma 1 from the proof of Theorem 10.3.13, let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$|1 - \psi_j(t)|M_0 < \frac{\epsilon}{2}$$

and

$$|\psi_j^{(l)}(t)|B_r\max\{M_1, \dots, M_r\} < \frac{\epsilon}{2}, \qquad l \in \{1, \dots, r\},$$

for $t \in K$ and $j \geq N$. Now, for $t \in K$ and $j \geq N$ we then have

$$|\phi^{(r)}(t) - (\phi\psi_j)^{(r)}(t)| = \left|\phi^{(r)}(t)(1 - \psi_k(t)) + \sum_{m=1}^{r}\phi^{(r-m)}(t)\psi_k^{(m)}(t)\right| < \epsilon.$$

Since K and r are arbitrary, the sequence $(\phi - \phi\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$ as desired. ▾

For each $j \in \mathbb{Z}_{>0}$ note that the sequence $(\chi_{j,k} \triangleq \psi_k \phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converges to ϕ_j in $\mathscr{E}(\mathbb{R};\mathbb{F})$ by the lemma. Therefore, for each $j \in \mathbb{Z}_{>0}$, there exists $N_j \in \mathbb{Z}_{>0}$ sufficiently large that

$$|\theta(\phi_j - \psi_k\phi_j)| \leq \epsilon, \qquad k \geq N_j,$$

by our assumptions on θ. We claim that the sequence $(\psi_{N_j}\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Indeed, for every compact subset $K \subseteq \mathbb{R}$ we have

$$\lim_{j\to\infty} \sup\left\{|(\psi_{N_j}\phi_j)^{(r)}(t)| \;\middle|\; t \in K\right\} = 0$$

by Lemma 1 from the proof of Theorem 10.3.13, the fact that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, and the formula

$$(\psi_{N_j}\phi_j)^{(r)} = \sum_{\ell=0}^{r} \binom{r}{\ell} \psi_{N_j}^{(\ell)} \phi_j^{(r-\ell)}.$$

This then gives

$$|\theta(\phi_j)| \leq |\theta(\phi_j - \psi_{N_j}\phi_j)| + |\theta(\psi_{N_j}\phi_j)|.$$

The two terms on the right go to zero as $j \to \infty$ by our hypotheses on θ, and so continuity of θ on $\mathscr{E}(\mathbb{R};\mathbb{F})$ follows. ■

10.5.3 Properties of distributions with compact support

In this section we record some of the basic facts about distributions with compact support. Many of these follow, directly or with little effort, from their counterparts for distributions.

Since $\mathscr{E}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$ there is inherited from $\mathscr{E}'(\mathbb{R};\mathbb{F})$ the notion of convergence of a sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{E}'(\mathbb{R};\mathbb{F})$.

10.5.12 Definition (Convergence in $\mathscr{E}'(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{E}'(\mathbb{R};\mathbb{F})$ is

(i) a ***Cauchy sequence*** if $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$, and

(ii) ***converges*** to a distribution θ with compact support if for every $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$, the sequence of numbers $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ converges to $\theta(\phi)$. •

As with tempered distributions, since distributions with compact support are distributions, Cauchy sequences have the property of converging in $\mathscr{D}'(\mathbb{R};\mathbb{F})$. It is also helpful if we have convergence in $\mathscr{E}'(\mathbb{R};\mathbb{F})$, and this is what the following result shows.

10.5.13 Theorem (Cauchy sequences in $\mathscr{E}'(\mathbb{R};\mathbb{F})$ converge) *If $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathscr{E}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$ the sequence converges to some $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$.*

Proof As with the proof of Theorem 10.3.15, the proof here can be made to mirror that of Theorem 10.2.22. To do this, we choose the initial subsequence $(\psi_n)_{n\in\mathbb{Z}_{>0}}$ such that it has the property that

$$\sup\left\{|\psi_n^{(j)}| \;\middle|\; t \in [-k,k]\right\} < \frac{1}{4^n}, \qquad j,k \in \{0,1,\dots,n\}.$$

Now the proof follows like that of Theorem 10.2.22, replacing $\mathscr{D}(\mathbb{R};\mathbb{F})$ with $\mathscr{E}(\mathbb{R};\mathbb{F})$ and $\mathscr{D}'(\mathbb{R};\mathbb{F})$ with $\mathscr{E}'(\mathbb{R};\mathbb{F})$. ■

Let us give the analogue for distributions with compact support of the fact that locally integrable signals are distributions. We recall from ***missing stuff*** the notion of the support of a measurable signal.

10.5.14 Proposition (Locally integrable signals with compact support are distributions with compact support) *If* $\mathsf{f}\colon \mathbb{R} \to \mathbb{F}$ *is a locally integrable signal with compact support then* $\theta_{\mathsf{f}} \in \mathscr{E}'(\mathbb{R};\mathbb{F})$*. Moreover, if* $\mathsf{f}_1, \mathsf{f}_2\colon \mathbb{R} \to \mathbb{F}$ *are locally integrable signals with compact support for which* $\theta_{\mathsf{f}_1} = \theta_{\mathsf{f}_2}$*, then* $\mathsf{f}_1(\mathsf{t}) = \mathsf{f}_2(\mathsf{t})$ *for almost every* $\mathsf{t} \in \mathbb{R}$*.*

Proof The first assertion is Example 10.5.8–1. The last assertion follows the similar assertion in Proposition 10.2.12, along with Propositions 10.3.12, 10.4.9, and 10.5.9. ■

Signals with compact support also show up to give a natural class of signals which can be multiply distributions with compact support.

10.5.15 Proposition (Distributions with compact support can be multiplied by smooth signals) *Let* $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ *and let* $\phi_0\colon \mathbb{R} \to \mathbb{F}$ *be an infinitely differentiable signal. Then the map*

$$\mathscr{E}(\mathbb{R};\mathbb{F}) \ni \phi \mapsto \theta(\phi_0\phi) \in \mathbb{F}$$

defines an element of $\mathscr{E}'(\mathbb{R};\mathbb{F})$*.*

Proof First of all, note that $\phi_0\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$. Now, linearity of the map is clear. To prove continuity, let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero. We claim that $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. It is clear that $\phi_0\phi_j \in \mathscr{E}(\mathbb{R};\mathbb{F})$ for each $j \in \mathbb{Z}_{>0}$, so we need only show that $(\phi_0\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. Let $K \subseteq \mathbb{R}$ be compact and let $r \in \mathbb{Z}_{\geq 0}$. By Proposition 3.2.11

$$(\phi_0\phi_j)^{(r)} = \sum_{k=1}^{r} \phi_0^{(k)}\phi_j^{(r-k)}.$$

If we let $\|\cdot\|_{K,\infty}$ be the infinity norm for functions restricted to K, then we have

$$\|(\phi_0\phi_j)^{(r)}\|_{K,\infty} \leq r\max\{\|\phi_0\|_{K,\infty}, \|\phi_0^{(1)}\|_{K,\infty}, \dots, \|\phi_0^{(r)}\|_{K,\infty}\} \\ \cdot \max\{\|\phi_j\|_{K,\infty}, \|\phi_j^{(1)}\|_{K,\infty}, \dots, \|\phi_j^{(r)}\|_{K,\infty}\}.$$

Letting $j \to \infty$, the second term on the right goes to zero, giving uniform convergence of $((\phi_0\phi_j)^{(r)})_{j\in\mathbb{Z}_{>0}}$ to zero on K.

Thus the result follows since

$$\lim_{j\to\infty} \theta(\phi_0\phi) = 0$$

for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$. ■

The notions of regular, singular, support, and singular support are applied to $\mathscr{E}'(\mathbb{R};\mathbb{F})$ by restriction from $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

One can differentiate distributions with compact support as they are distributions. It turns out that the derivative is again a distribution with compact support.

10.5.16 Proposition (The derivative of a distribution with compact support is a distribution with compact support) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ then $\theta' \in \mathscr{E}'(\mathbb{R};\mathbb{F})$.*

Proof We let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{E}(\mathbb{R};\mathbb{F})$ converging to zero. Then $(-\phi_j')_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{E}(\mathbb{R};\mathbb{F})$, as is easily seen from the definition of convergence to zero. Therefore,

$$\lim_{j\to\infty}\theta'(\phi_j) = \lim_{j\to\infty}\theta(-\phi_j') = 0$$

as desired. ■

One can talk about distributions with compact support of finite order, and distributions with compact support are always of finite order by virtue of their being tempered distributions. We shall see in Theorem 10.5.20 that even more is true for distributions with compact support.

10.5.4 Distributions with compact support depending on parameters

In this section we adapt our results from Sections 10.2.8 and 10.3.4 to test signals from $\mathscr{E}(\mathbb{R};\mathbb{F})$ and distributions from $\mathscr{E}'(\mathbb{R};\mathbb{F})$.

As previously, we let $I \subseteq \mathbb{R}$ be an interval and consider a function $\phi\colon I\times\mathbb{R}\to\mathbb{F}$ and denote a typical point in $I\times\mathbb{R}$ by (λ, t). For $(\lambda,t)\in I\times\mathbb{R}$ we define functions $\phi^\lambda\colon\mathbb{R}\to\mathbb{F}$ and $\phi_t\colon I\to\mathbb{F}$ by $\phi^\lambda(t)=\phi_t(\lambda)=\phi(\lambda,t)$. If, for each $\lambda\in I$, $\phi^\lambda\in\mathscr{E}(\mathbb{R};\mathbb{F})$, then, given $\theta\in\mathscr{E}'(\mathbb{R};\mathbb{F})$, we define $\Phi_{\theta,\phi}\colon I\to\mathbb{F}$ by

$$\Phi_{\theta,\phi}(\lambda) = \theta(\phi^\lambda).$$

As in Section 10.2.8, we denote

$$(\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)^\lambda(t) = (\boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi)_t(\lambda) = \boldsymbol{D}_1^s\boldsymbol{D}_2^r\phi(\lambda,t)$$

for $r,s\in\mathbb{Z}_{\geq 0}$.

The following result indicates the character of the function $\Phi_{\theta,\phi}$ in this case.

10.5.17 Theorem (Distributions with compact support applied to test signals with parameter dependence) *Let $\mathrm{I}\subseteq\mathbb{R}$ be an interval, let $\mathrm{k}\in\mathbb{Z}_{\geq 0}$, and let $\phi\colon \mathrm{I}\times\mathbb{R}\to\mathbb{F}$ have the following properties:*

(i) for each $\lambda\in\mathrm{I}$, the map $\mathrm{t}\mapsto\phi(\lambda,\mathrm{t})$ is an element of $\mathscr{E}(\mathbb{R};\mathbb{F})$;

(ii) for each $\mathrm{r}\in\mathbb{Z}_{\geq 0}$, $\mathbf{D}_1^{\mathrm{k}}\mathbf{D}_2^{\mathrm{r}}\phi\colon \mathrm{I}\times\mathbb{R}\to\mathbb{F}$ is continuous.

Then, for any $\theta\in\mathscr{E}'(\mathbb{R};\mathbb{F})$, $\Phi_{\theta,\phi}$ is k-times continuously differentiable and, moreover,

$$\Phi_{\theta,\phi}^{(\mathrm{k})}(\lambda) = \theta((\mathbf{D}_1^{\mathrm{k}}\phi)^\lambda).$$

Proof The proof follows closely that of Theorem 10.2.40, but we shall go through the details so as to understand clearly where the differences arise.

We first give the proof for $k=0$. Let $\lambda\in I$ and let $(\epsilon_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to zero and such that $\lambda+\epsilon_j\in I$ for every $j\in\mathbb{Z}_{>0}$. Define $\psi_j^\lambda\in\mathscr{E}(\mathbb{R};\mathbb{F})$ by

$$\psi_j^\lambda(t) = \phi(\lambda+\epsilon_j, t).$$

The following lemma is then useful.

1 Lemma *The sequence* $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ *converges to* ϕ^λ *in* $\mathscr{E}(\mathbb{R};\mathbb{F})$.

Proof Let $r \in \mathbb{Z}_{\geq 0}$ and let $K \subseteq \mathbb{R}$ be compact. Let $I' \subseteq I$ be the smallest compact interval for which $\lambda + \epsilon_j \in I'$ for every $j \in \mathbb{Z}_{>0}$. Since $\boldsymbol{D}_2^r\phi(\lambda, t)|I' \times K$ is continuous and since $I' \times K$ is compact, by Theorem **??** it follows that it is uniformly continuous. This implies that, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$|\boldsymbol{D}^r\psi_j^\lambda(t) - \boldsymbol{D}^r\phi^\lambda(t)| = |\boldsymbol{D}_2^r\phi(\lambda + \epsilon_j, t) - \boldsymbol{D}_2^r\phi(\lambda, t)| < \epsilon, \qquad j \geq N,\ t \in K.$$

Since $r \in \mathbb{Z}_{\geq 0}$ and K are arbitrary, this implies that we have the desired convergence of $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ to ϕ^λ. ▼

It then follows immediately from continuity of θ that

$$\lim_{j\to\infty} \Phi_{\theta,\phi}(\lambda + \epsilon_j) = \lim_{j\to\infty} \theta(\phi^{\lambda+\epsilon_j}) = \theta(\lim_{j\to\infty} \phi^{\lambda+\epsilon_j}) = \theta(\lim_{j\to\infty} \psi_j^\lambda) = \theta(\phi^\lambda) = \Phi_{\theta,\phi}(\lambda).$$

Continuity of $\Phi_{\theta,\phi}$ at λ then follows from Theorem 3.1.3.

Now we prove the theorem when $k = 1$. We let (ϵ_j) be a sequence, none of whose terms are zero, converging to zero as above. Now we take

$$\psi_j^\lambda(t) = \frac{\phi(\lambda + \epsilon_j, t) - \phi(\lambda, t)}{\epsilon_j}.$$

The following lemma is then key.

2 Lemma *The sequence* $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ *converges to* $(\mathbf{D}_1\phi)^\lambda$ *in* $\mathscr{E}(\mathbb{R};\mathbb{F})$.

Proof Let $r \in \mathbb{Z}_{\geq 0}$ and let $K \subseteq \mathbb{R}$ be compact. Let $I' \subseteq I$ be the smallest compact interval for which $\epsilon_j \in I'$ for every $j \in \mathbb{Z}_{>0}$. Define $\psi_r \colon I' \times K \to \mathbb{F}$ by

$$\psi_r(\ell, t) = \begin{cases} \frac{\boldsymbol{D}_2^r\phi(\ell,t) - \boldsymbol{D}_2^r\phi(\lambda,t)}{\ell-\lambda}, & \ell \neq \lambda, \\ \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t), & \ell = \lambda. \end{cases}$$

It is clear from the hypotheses that ψ_r is continuous on

$$\{(\ell, t) \in I' \times K \mid \ \ell \neq \lambda\}.$$

Moreover, since the derivative $\boldsymbol{D}_1\boldsymbol{D}_2^r\phi$ exists and is continuous,

$$\lim_{\ell\to\lambda} \frac{\boldsymbol{D}_2^r\phi(\ell, t) - \boldsymbol{D}_2^r\phi(\lambda, t)}{\ell - \lambda} = \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t), \qquad t \in K,$$

showing that ψ_r is continuous on $I \times \mathbb{R}$ by Theorem 3.1.3. Since ψ_r is continuous, it is uniformly continuous by Theorem **??**. Therefore, given $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$|\psi_r(\lambda + \epsilon_j, t) - \psi_r(\lambda, t)| < \epsilon, \qquad j \geq N,\ t \in K.$$

Using the definition of ψ_r, this implies that, for every $j \geq N$ and for every $t \in K$,

$$\left|\frac{\boldsymbol{D}_2^r\phi(\lambda + \epsilon_j, t) - \boldsymbol{D}_2^r\phi(\lambda, t)}{\epsilon_j} - \boldsymbol{D}_1\boldsymbol{D}_2^r\phi(\lambda, t)\right| = |\boldsymbol{D}^r\psi_j^\lambda(t) - \boldsymbol{D}^r(\boldsymbol{D}_1\phi^\lambda)(t)| < \epsilon.$$

Since $r \in \mathbb{Z}_{\geq 0}$ and K are arbitrary, this gives convergence of $(\psi_j^\lambda)_{j\in\mathbb{Z}_{>0}}$ to $(\boldsymbol{D}_1\phi)^\lambda$. ▼

By continuity of θ we then have

$$\lim_{j\to\infty}\frac{\Phi_{\theta,\phi}(\lambda+\epsilon_j)-\Phi_{\theta,\phi}(\lambda)}{\epsilon_j}=\lim_{j\to\infty}\frac{\theta(\phi^{\lambda+\epsilon_j})-\theta(\phi^{\lambda})}{\epsilon_j}$$
$$=\theta(\lim_{j\to\epsilon}\psi_j^{\lambda})=\theta((\boldsymbol{D}_1\phi)^{\lambda}),$$

showing that $\Phi_{\theta,\phi}$ is differentiable with derivative as stated in the theorem for the case of $k=1$.

Now suppose that the theorem is true for $j\in\{0,1,\dots,m\}$ and suppose that the hypotheses of the theorem hold for $k=m+1$. We let $\psi=\boldsymbol{D}_1^m\phi$ and verify that ψ satisfies the hypotheses of the theorem for $k=1$. First note that, for each $\lambda\in I$, $t\mapsto\psi(\lambda,t)$ is the mth derivative of an element $\mathscr{E}(\mathbb{R};\mathbb{F})$ and so is an element of $\mathscr{E}(\mathbb{R};\mathbb{F})$. The second of the hypotheses of the theorem hold immediately. Finally, since

$$\boldsymbol{D}_2\boldsymbol{D}_2^r\psi=\boldsymbol{D}_2\boldsymbol{D}_2^r\boldsymbol{D}_1^m\phi=\boldsymbol{D}_1^{m+2}\boldsymbol{D}_2^r\phi$$

by Theorem **??**, the final hypothesis of the theorem also holds. Therefore, by the induction hypothesis, $\Phi_{\theta,\psi}$ is continuously differentiable. But, since

$$\Phi_{\theta,\psi}(\lambda)=\theta((\boldsymbol{D}_1^m\phi)^{\lambda})=\Phi_{\theta,\phi}^{(m)}(\lambda),$$

this implies that $\Phi_{\theta,\phi}$ is $m+1$-times continuously differentiable, and

$$\Phi_{\theta,\phi}^{(m+1)}(\lambda)=\theta((\boldsymbol{D}_1^{m+1}\phi)^{\lambda})$$

as desired. ■

For the purposes of convolution, the following result will be important.

10.5.18 Corollary (Distributions with compact support applied to a special class of test signals) *Denote by $\tau\colon\mathbb{R}^2\to\mathbb{R}$ the map given by $\tau(\mathrm{s},\mathrm{t})=\mathrm{t}-\mathrm{s}$. Then, if $\phi\in\mathscr{E}(\mathbb{R};\mathbb{F})$ and $\theta\in\mathscr{E}'(\mathbb{R};\mathbb{F})$, then $\Phi_{\theta,\tau^*\phi}\in\mathscr{D}(\mathbb{R};\mathbb{F})$.*

Proof Since $\tau^*\phi$ is infinitely differentiable, by Theorem 10.5.17 we know that $\Phi_{\theta,\tau^*\phi}$ is infinitely differentiable. It remains to show that this function has compact support. ■

10.5.5 Fubini's Theorem for distributions with compact support

10.5.19 Lemma

*Proof***1 Lemma**

Proof Finally, we take the case where $\mathscr{T}=\mathscr{E}(\mathbb{R};\mathbb{F})$. Let $K\subseteq\mathbb{R}$ be compact and let $s\in K$. As in the first part of the proof, for each $j\in\mathbb{Z}_{>0}$ let $\rho_j\in(0,1)$ be such that

$$|\psi_{t,j}^{(r)}(s)-\phi^{(r+1)}(s+t)|\le|\epsilon_j||\phi^{(r+2)}(s+t+\rho_j\epsilon_j)|.$$

Then

$$|\psi_{t,j}^{(r)}(s)-\phi^{(r+1)}(s+t)|\le|\epsilon_j|\sup\{|\phi^{(r+2)}(x)|\mid x\in K\}.$$

Again, since the rightmost bound is independent of s, we get convergence of $(\psi_{t,j})_{j\in\mathbb{Z}_{>0}}$ to $\tau_t\phi'$ in $\mathscr{E}(\mathbb{R};\mathbb{F})$. ▼

■

10.5.6 Some deeper properties of distributions with compact support

Since $\mathscr{E}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$ it follows that Theorems 10.2.45 and 10.2.47 hold for distributions with compact support. The extra structure, however, allows us to provide a little more resolution in Theorem 10.2.47.

10.5.20 Theorem (Distributions with compact support are finite-order derivatives of signals with compact support) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ has support in the interior of a compact set $\mathrm{K} \subseteq \mathbb{R}$, then there exists signals $\mathrm{f}_{\theta,1},\dots,\mathrm{f}_{\theta,\mathrm{m}} \in \mathsf{C}^0(\mathbb{R};\mathbb{F})$ with support in K, and $\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{m} \in \mathbb{Z}_{\geq 0}$ such that*

$$\theta = \sum_{\mathrm{j}=1}^{\mathrm{m}} \theta^{(\mathrm{r}_\mathrm{j})}_{\mathrm{f}_{\theta,\mathrm{j}}}.$$

In particular, θ has finite order.

Proof Since K is compact we can write $\mathrm{int}(K)$ as a finite disjoint union of open intervals: $\mathrm{int}(K) = \cup_{j=1}^n (t_{1,j}, t_{2,j})$. We may as well also suppose that the closed intervals $[t_{1,j}, t_{2,j}]$, $j \in \{1,\dots,n\}$, are disjoint. Since $\mathrm{supp}(\theta) \subseteq \cup_{j=1}^n [t_{1,j}, t_{2,j}]$ it will suffice to prove the result for a distribution with support in a compact interval K, since then the general result will be obtained by simply (finitely) summing the expressions for each closed interval. In the case when K is a compact interval we may find an open $U \subseteq K$ for which $\mathrm{cl}(U)$ is compact and for which . By Theorem 10.2.47 there exists a continuous signal g and $r \in \mathbb{Z}_{\geq 0}$ for which $\theta(\psi) = \theta_g^{(r)}(\psi)$ for $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})_{\mathrm{cl}(U)}$. Now define $\chi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ so that it has support in U, and so that on some neighbourhood of K it takes the value 1. Such a function χ may be constructed using bump functions appropriately. For $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$ we have $\chi\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})_{\mathrm{cl}(U)}$, so giving, by Theorem 10.2.52,

$$\begin{aligned}\theta(\phi) = \theta(\chi\phi) &= (-1)^r \int_{\mathbb{R}} g(t)(\chi\phi)^{(r)}(t)\,\mathrm{d}t \\ &= (-1)^r \sum_{j=0}^{r} \binom{r}{j} \int_{\mathbb{R}} g(t)\chi^{(r-j)}(t)\phi^{(j)}(t)\,\mathrm{d}t.\end{aligned}$$

Letting

$$f_j = (-1)^{r+j}\binom{r}{j} g\chi^{(r-j)}$$

this gives

$$\theta(\phi) = \sum_{j=0}^{r} (-1)^j \int_{\mathbb{R}} f_j(t)\phi^{(j)}\,\mathrm{d}t = \sum_{j=0} \theta_{f_j}(\phi^{(j)}).$$

After a slight change of notation, the first part of the result now follows by using integration by parts, Proposition 10.2.39, and noting that $f_1,\dots,f_r$ have support in K.

To obtain the second assertion from the first, we let $r = \max\{r_1,\dots,r_m\}$ and define

$$f = \sum_{j=1}^{m} f_j^{(-(r-r_j))} \quad\Longrightarrow\quad f^{(r)} = \sum_{j=1}^{m} f^{(r_j)},$$

so giving the result. ■

Note that $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ having finite order does not mean that $\theta = \theta^{(r)}_{f_\theta}$ for f_θ with compact support. An example illustrates this caveat.

10.5.21 Example (The delta-signal as the derivative of signals) Let us consider $\delta_0 \in \mathscr{E}'(\mathbb{R};\mathbb{F})$. Note that if $f \in \mathsf{C}^0(\mathbb{R};\mathbb{F})$ has the property that $\delta_0 = \theta_f^{(k)}$ then it must be the case that $\theta_f = \delta_0^{(-k)}$. This means that $f(t) = 1(t)t^{k-1} + c$ for some constant c. Therefore, θ_f cannot have compact support. On the other hand one *can* write δ_0 as a finite linear combination of finite derivatives of continuous signals with compact support. This is guaranteed by Theorem 10.5.20, and can be realised concretely by defining

$$f_1(t) = \mathsf{R}(t)\frac{\wedge(t)}{\wedge(1)}, \quad f_2(t) = -2 1(t)\frac{\wedge^{(1)}(t)}{\wedge(1)} - \mathsf{R}(t)\frac{\wedge^{(2)}(t)}{\wedge(1)}.$$

A direct computation using Proposition 10.2.35 gives $\delta_0 = \theta_{f_1}^{(2)} + \theta_{f_2}^{(0)}$. In Figure 10.12 we plot f_1 and f_2, noting that they do have compact support. By rescaling the

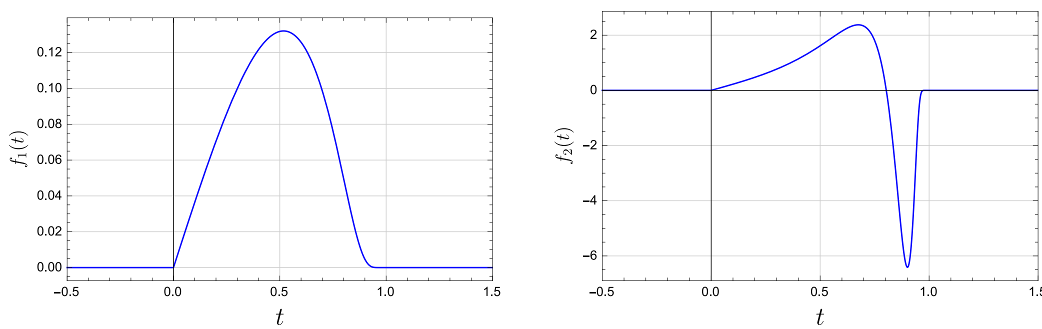


Figure 10.12 Signals f_1 and f_2 for which $\delta_0 = \theta_{f_1}^{(2)} + \theta_{f_2}^{(0)}$

argument of $\wedge$ one can make the support of these signals as small as desired. This is to be expected since δ_0 should be characterisable in terms of objects defined only in an arbitrarily small neighbourhood of $t = 0$. •

For distributions with compact support the conclusions of Theorem 10.2.52 can be sharpened somewhat.

10.5.22 Theorem (Distributions with compact support only depend on finitely many derivatives) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ has order k and if $\phi_1, \phi_2 \in \mathscr{E}(\mathbb{R};\mathbb{F})$ satisfy*

$$\phi_1^{(j)}(t) = \phi_2^{(j)}(t), \qquad j \in \{0, 1, \dots, k+1\},\ t \in \operatorname{supp}(\theta),$$

then $\theta(\phi_1) = \theta(\phi_2)$.

Proof The argument goes very much like that of Theorem 10.2.52. We let $f_\theta \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ have the property that $\theta(\psi) = \theta_{f_\theta}^{(k+1)}(\psi)$ for all $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Since $\operatorname{supp}(\theta)$ is compact we can write $\mathbb{R} \setminus \operatorname{supp}(\theta)$ as a finite collection of open intervals $\mathbb{T}_1, \dots, \mathbb{T}_n, \mathbb{T}_{n+1}, \mathbb{T}_{n+2}$, with these intervals being of the form $\mathbb{T}_j = (t_{1,j}, t_{2,j})$, $j \in \{1, \dots, n\}$, and $\mathbb{T}_{n+1} = (-\infty, t_L)$ and $\mathbb{T}_{n+2} = (t_R, \infty)$. We then have for $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$

$$\begin{aligned}\theta(\phi) &= (-1)^{k+1}\int_{\mathbb{R}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t\\ &= \sum_{m=1}^{n}(-1)^{k+1}\int_{t_{1,m}}^{t_{2,m}} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t + (-1)^{k+1}\int_{\mathbb{T}\cap\mathrm{supp}(\theta)} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t\\ &\quad + \int_{-\infty}^{t_L} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t + \int_{t_R}^{\infty} f_\theta(t)\phi^{(k+1)}(t)\,\mathrm{d}t.\end{aligned}$$

Just as in the proof of Theorem 10.2.52 the first three terms may be shown to depend only on $\phi^{(j)}(t)$ for $j \in \{0,1,\dots,k+1\}$ and $t \in \mathrm{supp}(\theta)$. As for the last term, note that on $(-\infty, t_L)$ and (t_R, ∞), θ agrees with the zero distribution. This means that on these intervals f_θ is a polynomial of degree at most k (its $(k+1)$st derivative must vanish). Now let $\psi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ have the property that it takes the value 1 on a neighbourhood of $[t_L, t_R]$. Then we can write $\phi = \psi\phi + (1-\psi)\phi$. Since $1-\psi$ vanishes on a neighbourhood of $\mathrm{supp}(\theta)$, from the definition of support we have $\theta((1-\psi)\phi) = 0$. We can then integrate by parts $k+1$ times the expression

$$\int_{-\infty}^{t_L} f_\theta(t)(\psi\phi)^{(k+1)}(t)\,\mathrm{d}t$$

to observe that it depends only on the value of $(\psi\phi)^{(j)}(t_L)$, $j \in \{1,\dots,k+1\}$. Similarly the expression

$$\int_{-\infty}^{t_L} f_\theta(t)(\psi\phi)^{(k+1)}(t)\,\mathrm{d}t$$

depends only on the value of $(\psi\phi)^{(j)}(t_R)$, $j \in \{1,\dots,k+1\}$. Since $(\psi\phi)^{(j)}(t_L) = \phi^{(j)}(t_L)$ and $(\psi\phi)^{(j)}(t_R) = \phi^{(j)}(t_R)$, the result follows. ■

A reading of the proof of the preceding theorem immediately gives the following corollary.

10.5.23 Corollary (A bound for the evaluation of distributions with compact support) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ and if $\phi \in \mathscr{E}(\mathbb{R};\mathbb{C})$, then there exists $\mathrm{M} \in \mathbb{R}_{>0}$, $\mathrm{k} \in \mathbb{Z}_{>0}$, and $\mathrm{K} \subseteq \mathbb{R}$ compact such that*

$$|\theta(\phi)| \leq \mathrm{M}\sup\{\mathrm{M}\phi^{(\mathrm{j})}(\mathrm{x}) \mid \mathrm{j} \in \{1,\dots,\mathrm{k}\},\ \mathrm{x} \in \mathrm{K}\}.$$

10.5.7 Some constructions with delta-signals

The distribution δ_0 and its derivatives are all examples of distributions with compact support. In this section we study some particular features of these signals. In Example 10.2.25 we considered a sequence of locally integrable signals that converge in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to δ_0. In general, a ***delta-sequence*** is a sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ of locally integrable signals having the property that $\lim_{j\to\infty}\theta_{f_j} = \delta_0$, with the limit being taken in $\mathscr{E}'(\mathbb{R};\mathbb{F})$. We shall encounter many examples of delta-sequences, and the following result will allow us to state that these indeed converge to δ_0.

10.5.24 Proposition (Characterisation of delta-sequences) *Let* $(f_j)_{j\in\mathbb{Z}_{>0}}$ *be a sequence in* $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *with the following properties:*

(i) there exists $M, T \in \mathbb{R}_{>0}$ *such that*

$$\int_{|t|\geq T} |f_j(t)|\,dt < M, \qquad j \in \mathbb{Z}_{>0};$$

(ii) for each $\delta \in (0,1)$ *the sequences* $(f_j|I_\delta)_{j\in\mathbb{Z}_{>0}}$ *and* $(f_j|I_{-\delta})_{j\in\mathbb{Z}_{>0}}$ *converge uniformly to zero, where* $I_\delta = [\delta, \delta^{-1}]$ *and* $I_{-\delta} = [-\delta, -\delta^{-1}]$*;*

(iii) for every $\delta \in \mathbb{R}_{>0}$ *the sequence*

$$\Big(\int_{|t|\leq\delta} f_j(t)\,dt\Big)_{j\in\mathbb{Z}_{>0}}$$

converges to 1*.*

Then $(f_j)_{j\in\mathbb{Z}_{>0}}$ *is a delta-sequence.*

Proof Let $\delta \in (0,1)$. For $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ we have

$$\begin{aligned}\theta_{f_j}(\phi) &= \int_{\mathbb{R}} f_j(t)\phi(t)\,dt \\ &= \int_{|t|\leq\delta} f_j(t)\phi(0)\,dt + \int_{|t|\leq\delta} f_j(t)(\phi(t)-\phi(0))\,dt + \int_{|t|\geq\delta} f_j(t)\phi(t)\,dt.\end{aligned}$$

Since ϕ has bounded derivatives there exists $C \in \mathbb{R}_{>0}$ such that $|\phi(t)-\phi(0)| \leq |t|C$ for $t \in \mathbb{R}$.***missing stuff*** Thus we have

$$\int_{|t|\leq\delta} |f_j(t)(\phi(t)-\phi(0))|\,dt \leq \delta M \int_{|t|<\delta} |f_j(t)|\,dt.$$

Therefore we can choose δ sufficiently small that

$$\int_{|t|\leq\delta} |f_j(t)(\phi(t)-\phi(0))|\,dt < \frac{\epsilon}{3}, \qquad j \in \mathbb{Z}_{>0},$$

by property (iii) of the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$. Since $(\phi f_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero on I_δ and $I_{-\delta}$ due to ϕ having compact support, we have

$$\lim_{j\to\infty} \int_{|t|\geq\delta} |f_j(t)\phi(t)|\,dt = \int_{|t|\geq\delta} \lim_{j\to\infty} |f_j(t)\phi(t)|\,dt = 0.$$

Thus we may choose $N_1 \in \mathbb{Z}_{>0}$ sufficiently large that

$$\int_{|t|\geq\delta} |f_j(t)\phi(t)|\,dt < \frac{\epsilon}{3}, \qquad j \geq N_1.$$

Now choose $N_2 \in \mathbb{Z}_{>0}$ sufficiently large that

$$\Big|\int_{\mathbb{R}} f_j(t)\,dt - 1\Big| < \frac{\epsilon}{3}.$$

Taking $N = \max\{N_1, N_2\}$ we see that

$$\Big|\int_{\mathbb{R}} f_j(t)\phi(t)\,dt - \phi(0)\Big| < \epsilon, \qquad j \geq N,$$

so giving the result. ■

Let us give a list a sequences of sequences of signals that may be verified to satisfy the hypotheses of this result, and such that they converge to δ_0 in $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

10.5.25 Examples (Delta-sequences)

1. A commonly used delta-sequence is given by $(f_j)_{j\in\mathbb{Z}_{>0}}$ where

$$f_j(t) = \begin{cases} j, & t \in [0, \frac{1}{j}], \\ 0, & \text{otherwise}. \end{cases}$$

This trivially verifies the hypotheses of Proposition 10.5.24. In Figure 10.13 (top left) we show some signals in this sequence. Note that we have made this

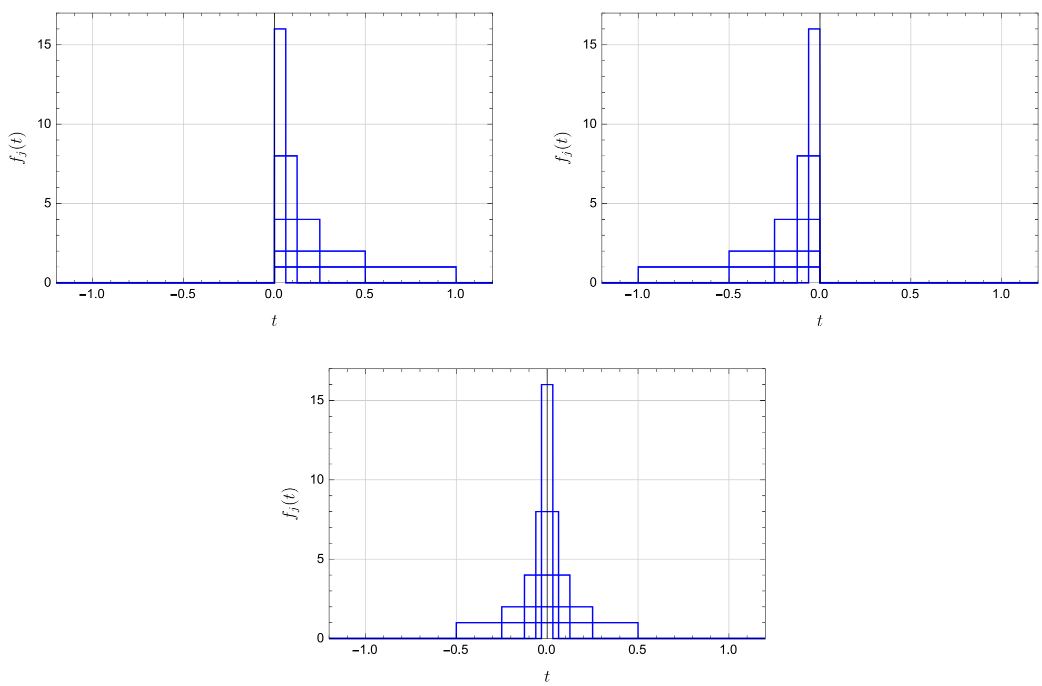


Figure 10.13 Three examples of delta-sequences

sequence one comprised signals that vanish for positive times. One could do the same with a sequence of signals that vanish for negative times by considering instead the sequence $(\sigma^* f_j)_{j\in\mathbb{Z}_{>0}}$ (Figure 10.13, top right). What's more, one could instead centre each of the signals at zero, and still maintain a delta-sequence (Figure 10.13, bottom).

The approximate identities from Example 11.3.7 below are all easily seen to define delta-sequences. We present these here, and the reader can easily verify that the hypotheses of Proposition 10.5.24 are satisfied by these signals, using the fact that these sequences are approximate identities.

2. Our next delta-sequence is denoted $(P_j)_{j\in\mathbb{Z}_{>0}}$ and defined by

$$P_j(t) = \frac{1}{\pi}\frac{j}{1 + j^2 t^2}.$$

This sequence is examined as an approximate identity in Example 11.3.7–1. A few terms of this sequence are shown in Figure 11.17 below, and there we can see the anticipated behaviour of concentration of the signal near 0.

Note that since this sequence is infinitely differentiable, it follows from Corollary 10.2.33 that the sequence $(P_j^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to $\delta_0^{(k)}$.

3. The sequence of signals $(G_{\Omega,j})_{j\in\mathbb{Z}_{>0}}$ given by

$$G_{\Omega,j}(t) = j\frac{\exp(-\frac{(jt)^2}{4\Omega})}{\sqrt{4\pi\Omega}}$$

converges to δ_0 in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ for every $\Omega\in\mathbb{R}_{>0}$. This sequences is examined as an approximate identity in Example 11.3.7–2. A few terms of this sequence are shown in Figure 11.18 below, and there we can see the anticipated behaviour of concentration of the signal near 0.

Note again that this sequence is infinitely differentiable, and so it follows from Corollary 10.2.33 that the sequence $(G_{\Omega,j}^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to $\delta_0^{(k)}$.

4. The sequence $(F_j)_{j\in\mathbb{Z}_{>0}}$ of signals defined by

$$F_j(t) = \begin{cases} \frac{\sin^2(\pi jt)}{\pi^2 jt^2}, & t\neq 0, \\ j, & t = 0\end{cases}$$

can easily be shown to satisfy thee hypotheses of Proposition 10.5.24, and so is a delta-sequence. This sequence is examined as an approximate identity in Example 11.3.7–3. In Figure 11.19 below we show a few terms in this sequence.

Again, this is a sequence of infinitely differentiable signals, so the sequences of its derivatives converge in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ to the corresponding derivatives of δ_0, as prescribed by Corollary 10.2.33. •

Note that delta-sequences may be thought of as sequences of signals that blow up at zero in just the right way, cf. Exercise 10.1.1. This idea leads to *negative* characterisations of delta-sequences. One such is the following which will have some consequences in Chapter 12.

10.5.26 Proposition (Characterisations of non-delta-sequences) *Let* $(f_j)_{j\in\mathbb{Z}_{>0}}$ *be a sequence in* $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$ *with the following properties:*

(i) there exists $M,T\in\mathbb{R}_{>0}$ *such that*

$$\int_{|t|\geq T}|f_j(t)|\,dt < M,\qquad j\in\mathbb{Z}_{>0};$$

(ii) for each $\delta\in(0,1)$ *the sequences* $(f_j|I_\delta)_{j\in\mathbb{Z}_{>0}}$ *and* $(f_j|I_{-\delta})_{j\in\mathbb{Z}_{>0}}$ *converge uniformly to zero, where* $I_\delta=[\delta,\delta^{-1}]$ *and* $I_{-\delta}=[-\delta,-\delta^{-1}]$*;*

(iii) $(\|f_j\|_2)_{j\in\mathbb{Z}_{>0}}$ *converges.*

Then $(f_j)_{j\in\mathbb{Z}_{>0}}$ *is not a delta-sequence.*

Proof As in the proof of Proposition 10.5.24 we have, for any $\delta \in (0,1)$ and $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$,

$$\lim_{j\to\infty} \int_{|t|\geq\delta} |f_j(t)\phi(t)|\,\mathrm{d}t = 0.$$

By the Cauchy–Schwarz–Bunyakovsky inequality we have

$$\Big|\int_{|t|\leq\delta} f_j(t)\phi(t)\,\mathrm{d}t\Big| \leq \Big(\int_{|t|\leq\delta} |f_j(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{|t|\leq\delta} |\phi(t)|^2\,\mathrm{d}t\Big)^{1/2},$$

this holding for all $\delta \in \mathbb{R}_{>0}$. Now let $(\delta_j)_{j\in\mathbb{Z}_{>0}}$ be a positive sequence converging to zero. Then

$$\begin{aligned}\Big|\int_{\mathbb{R}} f_j(t)\phi(t)\,\mathrm{d}t\Big| &= \Big|\int_{|t|\leq\delta_j} f_j(t)\phi(t)\,\mathrm{d}t + \int_{|t|\geq\delta_j} f_j(t)\phi(t)\,\mathrm{d}t\Big|\\ &\leq \Big|\int_{|t|\leq\delta_j} f_j(t)\phi(t)\,\mathrm{d}t\Big| + \int_{|t|\geq\delta_j} |f_j(t)\phi(t)|\,\mathrm{d}t\\ &\leq \Big(\int_{|t|\leq\delta} |f_j(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{|t|\leq\delta} |\phi(t)|^2\,\mathrm{d}t\Big)^{1/2} + \int_{|t|\geq\delta_j} |f_j(t)\phi(t)|\,\mathrm{d}t.\end{aligned}$$

Since $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathsf{L}^2(\mathbb{R};\mathbb{F})$ it follows that

$$\lim_{j\to\infty}\Big(\int_{|t|\leq\delta} |f_j(t)|^2\,\mathrm{d}t\Big)^{1/2} < \infty.$$

Since ϕ is continuous it follows that

$$\lim_{j\to\infty}\Big(\int_{|t|\leq\delta} |\phi(t)|^2\,\mathrm{d}t\Big)^{1/2} = 0.$$

Thus $\lim_{j\to\infty} \theta_{f_j}(\phi) = 0$, so showing that $\lim_{j\to\infty} \theta_{f_j}(\phi) = \delta_0(\phi)$ only if $\phi(0) = 0$. Thus $(f_j)_{j\in\mathbb{Z}_{>0}}$ is not a delta-sequence. ■

Let us give an example that illustrates the difference between a sequence that is a delta-sequence and one that is not.

10.5.27 Example (A non-delta-sequence) Define

$$f_j = \begin{cases}\sqrt{j}, & j \in [0, \frac{1}{j}],\\ 0, & \text{otherwise.}\end{cases}$$

We then compute $\|f_j\|_2 = 1$, so $(\|f_j\|_2)_{j\in\mathbb{Z}_{>0}}$ converges. Clearly the sequence satisfies all the conditions of Proposition 10.5.26, so is not a delta-sequence. In Figure 10.14 we show some terms in this sequence. The idea is that they do not blow up sufficiently fast relative to the rate at which their domain shrinks. A delta-sequence must maintain this "balance" in just the right way. The reader may explore this further in Exercise 10.5.5, also cf. Exercise 10.1.1. •

Now we shall show that the delta-signal and its derivatives are the only distributions which have a single point as their support.

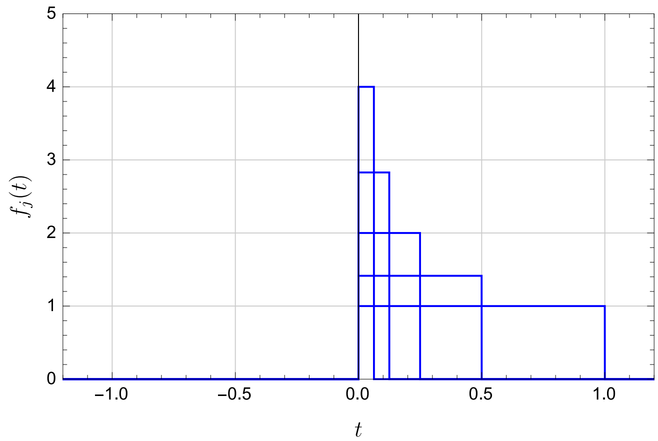


Figure 10.14 A sequence of signals that is not a delta-sequence

10.5.28 Proposition (Characterisation of distributions with point support) *If* $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ *has the property that* $\mathrm{supp}(\theta) = \{t_0\}$ *then there exists* $c_1,\dots,c_m \in \mathbb{F}$ *and* $r_1,\dots,r_m \in \mathbb{Z}_{\geq 0}$ *such that*

$$\theta = \sum_{j=1}^{m} c_j \delta_{t_0}^{(r_j)}.$$

Proof Since θ has compact support it has finite order $k \in \mathbb{Z}_{\geq 0}$. Then $\theta = \theta_{f_\theta}^{(k+1)}$ for $f_\theta \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$, or $\theta = \theta_{g_\theta}^{(k+2)}$ for $g_\theta \in \mathsf{C}^0(\mathbb{R};\mathbb{F})$. Since $\mathrm{supp}(\theta) = \{t_0\}$, θ agrees with the zero distribution on $(-\infty, t_0)$ and (t_0, ∞). This means that on $(-\infty, t_0)$ and (t_0, ∞) the continuous signal g_θ must be a polynomial of degree at most $k+1$ (its derivative of order $k+2$ must vanish). Thus we can write, using the argument in the proof of Theorem 10.5.22,

$$\theta(\phi) = \int_{-\infty}^{t_0} f_L(t)\phi^{(k+2)}(t)\,\mathrm{d}t + \int_{t_0}^{\infty} f_R(t)\phi^{(k+1)}(t)\,\mathrm{d}t,$$

where f_L and f_R are polynomials of degree at most $k+1$. Furthermore, since g_θ is continuous we must have $f_L(t_0) = f_R(t_0)$. Integrating by parts then gives $\theta(\phi)$ as a linear combination of $\phi(t_0), \phi^{(1)}(t_0), \dots, \phi^{(k)}(t_0)$, thus giving the result. ■

As a corollary to this we have the following (obvious) property of the delta-signal, showing that indeed one can give the delta-signal a continuous signal as an argument. This generalises Theorem 10.2.52 which says that the delta-signal can take *differentiable* signals as an argument.

10.5.29 Corollary (Order of derivatives for argument of derivatives of delta-signal) *Let* $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ *be a linear combination of* $\delta_{t_0}, \delta_{t_0}^{(1)}, \dots, \delta_{t_0}^{(k)}$. *If* $\phi_1, \phi_2 \in \mathscr{E}(\mathbb{R};\mathbb{F})$ *satisfy* $\phi_1^{(j)}(t_0) = \phi_2^{(j)}(t_0)$, $j \in \{0, 1, \dots, k\}$, *then* $\theta(\phi_1) = \theta(\phi_2)$.

Exercises

10.5.1 Show that if $\phi_1, \phi_2 \in \mathscr{E}(\mathbb{R};\mathbb{F})$ then $\phi_1\phi_2 \in \mathscr{E}(\mathbb{R};\mathbb{F})$. Thus $\mathscr{E}(\mathbb{R};\mathbb{F})$ is an algebra.

10.5.2 Which of the following locally integrable signals defines a distribution in $\mathscr{E}'(\mathbb{R};\mathbb{F})$?

(a) $f(t) = \arctan(t)$.

(b) ***missing stuff***

10.5.3 Which of the following signals is in $\mathscr{E}(\mathbb{R};\mathbb{F})$? For signals not in $\mathscr{E}(\mathbb{R};\mathbb{F})$, explain why they are not.

(a) $f(t) = \arctan(t)$.

(b) ***missing stuff***

10.5.4 Which of the following locally integrable signals defines a distribution in $\mathscr{E}'(\mathbb{R};\mathbb{F})$?

(a) $f(t) = \arctan(t)$.

(b) ***missing stuff***

10.5.5 Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ have the following properties:

1. there exists $M, T \in \mathbb{R}_{>0}$ such that
$$\int_{|t|\geq T} |f_j(t)|\,\mathrm{d}t < M, \qquad j \in \mathbb{Z}_{>0};$$
2. for each $\delta \in (0,1)$ the sequences $(f_j|I_\delta)_{j\in\mathbb{Z}_{>0}}$ and $(f_j|I_{-\delta})_{j\in\mathbb{Z}_{>0}}$ converge uniformly to zero, where $I_\delta = [\delta, \delta^{-1}]$ and $I_{-\delta} = [-\delta, -\delta^{-1}]$;
3. $(\|f_j\|_1)_{j\in\mathbb{Z}_{>0}}$ diverges.

Show that $(f_j)_{j\in\mathbb{Z}_{>0}}$ is not a delta-sequence.

Section 10.6

Ultradistributions

In this section we consider another class of distributions. This class is used in defining the Fourier transform of a distribution in Section **??**. The test functions we use in this section might appear particularly unmotivated. However, the appropriate motivation will appear in Section **??** when we show that the test functions considered here are the continuous-continuous Fourier transform of test functions from $\mathscr{D}(\mathbb{R};\mathbb{F})$.

Do I need to read this section? This section provides the prerequisite material for defining the CCFT of a distribution in **??**. Moreover, the connection between ultradistributions and the CCFT is very tight. Indeed, in this section we use some of the basic results from Section 13.1 regarding the properties of the CCFT. Therefore, to understand some of the proofs in this section will require reading Section 13.1. In terms of whether the present section is required reading, it should be read prior to, and maybe immediately prior to, one's reading of Section **??**. •

10.6.1 The space $\mathscr{Z}(\mathbb{R};\mathbb{F})$ of test signals

Let us begin by defining the collection of test signals for our new class of distributions. Let us give the definition for the moment, and then turn to discussing the various properties of these test signals.

10.6.1 Definition ($\mathscr{Z}(\mathbb{R};\mathbb{F})$) $\mathscr{Z}(\mathbb{R};\mathbb{F})$ denotes the set of signals ϕ for which there exists an entire function $a_\phi\colon \mathbb{C}\to\mathbb{C}$ such that

(i) $a_\phi(t+\mathrm{i}0)=\phi(t)$ for all $t\in\mathbb{R}$ and

(ii) there exists constants $a\in\mathbb{R}_{>0}$ and $C_k\in\mathbb{R}_{\geq 0}$, $k\in\mathbb{Z}_{\geq 0}$, such that, for each $k\in\mathbb{Z}_{\geq 0}$, we have $|z^k a_\phi(z)|\leq C_k \mathrm{e}^{a|\mathrm{Im}(z)|}$ for all $z\in\mathbb{C}$. •

In the usual circumstances, we would at this point list a collection of signals from $\mathscr{Z}(\mathbb{R};\mathbb{F})$, but this list will be absent here. There is a reason for this. As we shall see in Theorem **??**, to produce such signals requires computing $\mathscr{F}_{\mathrm{CC}}(\psi)$ for $\psi\in\mathscr{D}(\mathbb{R};\mathbb{C})$. Explicit computations of this type are not easily done, and would in any case produce signals that are not so recognisable. Thus the reader should content themselves with understanding $\mathscr{Z}(\mathbb{R};\mathbb{C})$ simply as a collection of test signals, some properties of which we enumerate above. Note that the same comments apply to our previous collections of test signals $\mathscr{D}(\mathbb{R};\mathbb{C})$, $\mathscr{S}(\mathbb{R};\mathbb{C})$, and $\mathscr{E}(\mathbb{R};\mathbb{C})$. While producing explicit examples of such signals helps us understand the properties of the collection, the examples are not strictly necessary to apply the theory of the associated distributions.

There is also a notion of convergence for test signals in $\mathscr{Z}(\mathbb{R};\mathbb{F})$, just as for all classes of test signals we encountered previously.

10.6.2 Definition (Convergence in $\mathscr{Z}(\mathbb{R};\mathbb{F})$) A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ ***converges to zero*** if

(i) there exists $a\in\mathbb{R}_{>0}$ and $C_k\in\mathbb{R}_{>0}$, $k\in\mathbb{Z}_{\geq 0}$, such that, for each $j\in\mathbb{Z}_{>0}$, the inequality
$$|z^k a_{\phi_j}(z)|\leq C_k e^{a|\mathrm{Im}(z)|},\qquad z\in\mathbb{C},$$
holds, and

(ii) the sequence $(a_{\phi_j})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero on any compact subset $K\subseteq\mathbb{C}$.

A sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ ***converges*** to $\phi\in\mathscr{Z}(\mathbb{R};\mathbb{F})$ if $(\phi_j-\phi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

Convergence in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ leads to the definition of continuity for maps.

10.6.3 Definition (Continuous linear maps on $\mathscr{Z}(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon\mathscr{D}(\mathbb{R};\mathbb{F})\to\mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ of numbers converges to zero for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ of test signals converging to zero. •

Let us understand the relationship between $\mathscr{Z}(\mathbb{R};\mathbb{F})$ and the other spaces of test signals. In the proof of the following result, we make use of Theorem **??** which characterises $\mathscr{Z}(\mathbb{R};\mathbb{F})$ in terms of the continuous-continuous Fourier transform.

10.6.4 Proposition (Relationship of $\mathscr{Z}(\mathbb{R};\mathbb{F})$ with $\mathscr{D}(\mathbb{R};\mathbb{F})$ and $\mathscr{S}(\mathbb{R};\mathbb{F})$) *The following statements hold:*

(i) $\mathscr{Z}(\mathbb{R};\mathbb{F})\cap\mathscr{D}(\mathbb{R};\mathbb{F})=\{0\}$;

(ii) $\mathscr{Z}(\mathbb{R};\mathbb{F})\subseteq\mathscr{S}(\mathbb{R};\mathbb{F})$.

Proof For the first assertion, let $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})\cap\mathscr{Z}(\mathbb{R};\mathbb{F})$. This means that there is an interval $[a,b]\subseteq\mathbb{R}$ for which $\phi(t)=0$ for all $t\in[a,b]$. However, since a_ϕ is entire, this implies that $a_\phi=0$ by analytic continuation.***missing stuff***

For the second assertion, since a_ϕ is entire, clearly ϕ is infinitely differentiable. Since $\mathscr{F}_{\mathrm{CC}}(\phi)\in\mathscr{D}(\mathbb{R};\mathbb{F})$ by Theorem **??**, the signal $t\mapsto t^k\mathscr{F}_{\mathrm{CC}}(\phi)(t)$ is also in $\mathscr{D}(\mathbb{R};\mathbb{F})$ for every $k\in\mathbb{Z}_{>0}$. From Proposition **??** we deduce that $\phi^{(k)}\in\mathscr{Z}(\mathbb{R};\mathbb{F})$. From property (ii) of Definition 10.6.1, taking $z\in\mathbb{R}$, we conclude that $\phi\in\mathscr{S}(\mathbb{R};\mathbb{F})$. ■

Let us record some facts that further elucidate the relationship between $\mathscr{Z}(\mathbb{R};\mathbb{C})$ and $\mathscr{S}(\mathbb{R};\mathbb{C})$. Here again we make use of the continuous-continuous Fourier transform to understand properties of $\mathscr{Z}(\mathbb{R};\mathbb{F})$.

10.6.5 Proposition (Further properties of $\mathscr{Z}(\mathbb{R};\mathbb{F})$ relative to $\mathscr{S}(\mathbb{R};\mathbb{F})$) *The following statements hold:*

(i) a sequence $(\phi_\mathrm{j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *converging to zero in* $\mathscr{Z}(\mathbb{R};\mathbb{F})$ *also converges to zero in* $\mathscr{S}(\mathbb{R};\mathbb{F})$;

(ii) $\mathscr{Z}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{S}(\mathbb{R};\mathbb{F})$.

Proof For the first assertion, note that by Theorem **??** that $(\mathscr{F}_{\mathrm{CC}}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$. This means that for each $k\in\mathbb{Z}_{\geq 0}$ the sequence $(\rho^k\mathscr{F}_{\mathrm{CC}}(\phi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero where $\rho(t)=t$. Following Proposition **??** we may then conclude that for each $k \in \mathbb{Z}_{\geq 0}$ the sequence $(\phi_j^{(k)})_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{Z}(\mathbb{R};\mathbb{F})$. Now note that convergence to zero in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ implies that for each $k, m \in \mathbb{Z}_{\geq 0}$ we have

$$\lim_{j\to\infty} \sup\left\{|t^m \phi_j^{(k)}(t)| \;\middle|\; t \in \mathbb{R}\right\} = 0,$$

so giving convergence to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$.

Let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ and by Theorem 10.8.3(i) choose a sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ converging to $\mathscr{F}_{\mathrm{CC}}(\phi)$. The sequence $(\overline{\mathscr{F}}_{\mathrm{CC}}(\psi_j))_{j\in\mathbb{Z}_{>0}}$ is then a sequence in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ that converges to ϕ by continuity of $\mathscr{F}_{\mathrm{CC}}$. ■

10.6.2 Definition of ultradistributions

Now we consider the set of distributions defined using $\mathscr{Z}(\mathbb{R};\mathbb{F})$ as test signals.

10.6.6 Definition (Ultradistribution) An ***ultradistribution*** is a continuous linear map from $\mathscr{Z}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of ultradistributions is denoted $\mathscr{Z}'(\mathbb{R};\mathbb{F})$. •

Ultradistributions have defined on them the operations usual for distributions. We list some of these.***missing stuff***

1. Ultradistributions can be added and multiplied by complex scalars to give them a $\mathbb{F}$-vector space structure.
2. If $\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ then one can define $\tau_a^*\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ and $\sigma^*\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ in the same manner as these are defined for distributions.***missing stuff***
3. The derivative of an ultradistribution θ is defined by $\theta'(\phi) = -\theta(\phi')$. To make sense of this, one must show that the set $\mathscr{Z}(\mathbb{R};\mathbb{F})$ is closed under differentiation, but this is easy to do.
4. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\tilde{\chi}\colon \mathbb{C} \to \mathbb{C}$ be an entire function satisfying

$$|\tilde{\chi}(z)| \leq C e^{a|\mathrm{Im}(z)|}(1 + |z|^k)$$

for some $C, a \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{\geq 0}$. Define $\chi(t) = \tilde{\chi}(t + \mathrm{i}0)$, supposing that χ is $\mathbb{F}$-valued. One can show easily that if $\phi \in \mathscr{Z}(\mathbb{R};\mathbb{F})$ then $\chi\phi \in \mathscr{Z}(\mathbb{R};\mathbb{F})$. Therefore, for such functions χ and for $\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ one can define $\chi\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ by $(\chi\theta)(\phi) = \theta(\chi\phi)$.

We now turn to recording some of the basic properties of ultradistributions.

10.6.7 Proposition (Tempered distributions are ultradistributions) $\mathscr{S}'(\mathbb{R};\mathbb{F}) \subseteq \mathscr{Z}'(\mathbb{R};\mathbb{F})$.

Proof Since $\mathscr{Z}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{S}(\mathbb{R};\mathbb{F})$ by Proposition 10.6.4 it follows that $\theta(\phi)$ is well-defined for $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ and $\phi \in \mathscr{Z}(\mathbb{R};\mathbb{F})$. Continuity of θ on $\mathscr{Z}(\mathbb{R};\mathbb{F})$ follows from Proposition 10.6.5. ■

This then gives a whole collection of ultradistributions. For instance, signals of slow growth define ultradistributions. One might hope that the set of distributions is contained in the set of ultradistributions. This is not the case, however, as the following counterexample shows.

10.6.8 Example (A distribution that is not an ultradistribution) Define $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ by $f(t) = \mathrm{e}^{t^2}$. We claim that $\theta_f \in \mathscr{D}'(\mathbb{R};\mathbb{R})$ but that $\theta_f \notin \mathscr{Z}'(\mathbb{R};\mathbb{R})$. That $f \in \mathscr{D}'(\mathbb{R};\mathbb{R})$ follows from Proposition 10.2.12.

10.6.3 Properties of ultradistributions

Ultradistributions have defined with them a notion of convergence in the usual manner.

10.6.9 Definition (Convergence in $\mathscr{Z}'(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}}$ in $\mathscr{Z}'(\mathbb{R};\mathbb{F})$

(i) is a ***Cauchy sequence*** if $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\phi \in \mathscr{Z}(\mathbb{R};\mathbb{F})$, and

(ii) ***converges*** to an ultradistribution θ if, for every $\phi \in \mathscr{Z}(\mathbb{R};\mathbb{F})$, $(\theta_j(\phi))_{j\in\mathbb{Z}_{>0}}$ converges to $\theta(\phi)$. •

One then has the hoped for relationship between Cauchy sequences and convergent sequences.

10.6.10 Theorem (Cauchy sequences in $\mathscr{Z}'(\mathbb{R};\mathbb{F})$ converge) *A sequence $(\theta_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathscr{Z}'(\mathbb{R};\mathbb{F})$ converges to some $\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ if and only if it is Cauchy.*

Proof This can be proved in the same manner as Theorem 10.2.22.***missing stuff*** ■

With this notion of convergence in mind, we state the following result, giving an analogue of Taylor's Theorem for ultradistributions.

10.6.11 Theorem (Taylor's Theorem for ultradistributions) *If $\theta \in \mathscr{Z}'(\mathbb{R};\mathbb{F})$ and* $\mathrm{a} \in \mathbb{F}$ *then*

$$\tau^*_{-\mathrm{a}}\theta = \sum_{\mathrm{j}=0}^{\infty} \frac{\mathrm{a}^{\mathrm{j}}}{\mathrm{j}!}\theta^{(\mathrm{j})}$$

Proof Let $\phi \in \mathscr{Z}$ and note that since a_ϕ is entire we may write

$$a_\phi(t-a) = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} a_\phi(t),$$

this being valid for all $a \in \mathbb{F}$. One has

$$\mathscr{F}_{\mathrm{CC}}\Big(\sum_{j=0}^{n} \frac{(-a)^j}{j!}\phi^{(j)}(t)\Big)(t) = \sum_{j=0}^{n} \frac{(-a)^j}{j!}(2\pi\mathrm{i}t)^j \mathscr{F}_{\mathrm{CC}}(\phi)(t),$$

using Proposition **??**. Since $\mathscr{F}_{\mathrm{CC}}(\phi)$ has compact support it follows that this series will converge in $\mathscr{D}(\mathbb{R};\mathbb{F})$ to $\mathrm{e}^{-2\pi\mathrm{i}ta}\mathscr{F}_{\mathrm{CC}}(\phi) = \mathscr{F}_{\mathrm{CC}}(\tau^*_1\phi)$ in the limit as $n \to \infty$. By Theorem **??** this means that

$$\sum_{j=0}^{n} \frac{(-a)^j}{j!}\phi^{(j)}(t)$$

converges in $\mathscr{Z}(\mathbb{R};\mathbb{F})$ as $n \to \infty$. The result now follows by continuity of $\mathscr{F}_{\mathrm{CC}}$ and the equality $\tau^*_{-a}\theta(\phi) = \theta(\tau^*_a\phi)$. ■

It might be helpful to think of "$\tau_{-a}\theta(t) = \theta(t + a)$," noting that this notation is something we are trying to avoid. This makes the relationship to Taylor's Theorem more transparent.

As with the test signals $\mathscr{Z}(\mathbb{R}; \mathbb{F})$ we did not invest much effort in enumerating explicit examples of ultradistributions. While it is true that tempered distributions are ultradistributions, there *are* ultradistributions that are not tempered. However, the best way to think of ultradistributions as being those objects which one gets after applying the continuous-continuous Fourier transform to distributions. This is what we do in Section **??**.

10.6.4 Some deeper properties of ultradistributions

Section 10.7

Periodic distributions

The next class of distributions we consider are those that are periodic. The development proceeds much as has been the case in the development of the sets $\mathscr{D}'(\mathbb{R};\mathbb{F})$, $\mathscr{S}'(\mathbb{R};\mathbb{F})$, and $\mathscr{E}'(\mathbb{R};\mathbb{F})$ of generalised signals.

Do I need to read this section? The material in this section is important in the development of the continuous-discrete Fourier transform of Chapter 12. In particular, if the reader is interested in understanding in a complete way the relationships between the four Fourier transforms we present, then the material in this section is important. •

10.7.1 Periodic test signals

Let us get straight to it.

10.7.1 Definition (Periodic test signal) A **T**-***periodic test signal*** is a signal $\psi\colon \mathbb{R}\to\mathbb{F}$ with the properties

(i) ψ is infinitely differentiable and

(ii) $\psi(t+T)=\psi(t)$ for all $t\in\mathbb{R}$.

The number T is the ***period*** for ψ, and the set of periodic test signals with period T is denoted $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. •

10.7.2 Remark ($\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is a vector space) One can easily verify that $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{F}^{\mathbb{R}}$. •

Let us consider some examples of periodic test signals.

10.7.3 Examples (Periodic test signals)

1. Harmonic signals, as discussed in Section 8.1.6, are certainly periodic test signals. Thus the signals $\sin(2\pi n\frac{t}{T})$, $\cos(2\pi n\frac{t}{T})$, $n\in\mathbb{Z}_{\geq 0}$, are $\mathbb{R}$-valued T-periodic test signals, and the signals $\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$, $n\in\mathbb{Z}$, are $\mathbb{C}$-valued T-periodic test signals.
2. If $\phi\in\mathscr{D}(\mathbb{R};\mathbb{F})$ then we may construct from ϕ a natural T-periodic test signal by
$$\phi_T(t)=\sum_{n\in\mathbb{Z}}\phi(t-nT). \tag{10.16}$$
Note that since ϕ has compact support, for each fixed t the sum in (10.16) is actually finite.
3. There is a particularly interesting collection of periodic test signals of the sort described above. A test signal $v\in\mathscr{D}(\mathbb{R};\mathbb{F})$ is **T**-***unitary*** if
$$\sum_{n\in\mathbb{Z}}v(t-nT)=1$$

for each $t \in \mathbb{R}$. The set of T-unitary signals is denoted $\mathscr{U}_T(\mathbb{R};\mathbb{F})$. This way of constructing periodic signals from aperiodic signals can be carried out in a general setting, and this is done in Section **??**.

The prototypical unitary test signal is

$$\upsilon_T(t) = \begin{cases} \frac{1}{c}\int_{|t|}^{T} \exp\left(-\frac{T^2}{\tau(T-\tau)}\right) \mathrm{d}\tau, & t \in (-T,T), \\ 0, & \text{otherwise}, \end{cases}$$

where

$$c = \int_0^T \exp\left(-\frac{T^2}{\tau(T-\tau)}\right) \mathrm{d}\tau.$$

In Figure 10.15 we plot υ_T. To check that this test signal is unitary we first note

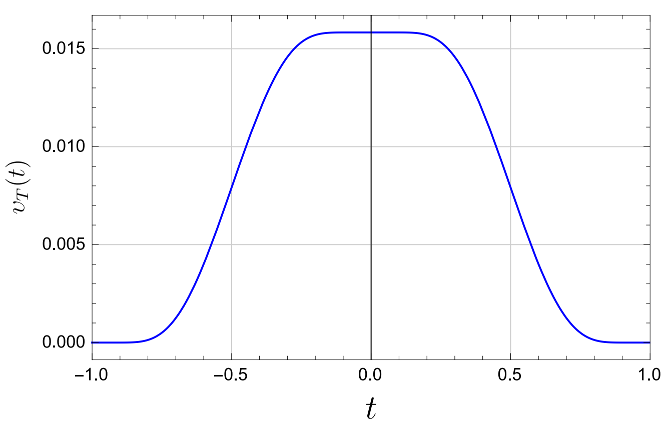


Figure 10.15 The unitary test signal υ_T for $T = 1$

that it is indeed infinitely differentiable. It is obviously differentiable away from $t = 0$ by virtue of the same arguments by which $\wedge$ is infinitely differentiable. At $t = 0$ one can check that the value of the signal is 1, and that all derivatives from the left and right are zero, cf. Example **??–??**. This shows that υ_T is infinitely differentiable. We also note that the sum

$$\sum_{n\in\mathbb{Z}} \upsilon_T(t - nT)$$

will, for a fixed t, be comprised of at most two summands. Indeed, if $t \in [jT,(j+1)T]$ then

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \upsilon_T(t-nT) &= \upsilon_T(t-jT) + \upsilon_T(t-(j+1)T) \\ &= \frac{1}{c}\Big(\int_{|t-jT|}^{T} \exp\left(-\frac{T^2}{\tau(T-\tau)}\right) \mathrm{d}\tau \\ &\quad + \int_{|t-(j+1)T|}^{T} \exp\left(-\frac{T^2}{\tau(T-\tau)}\right) \mathrm{d}\tau\Big). \end{aligned}$$

Note that the integrand $\exp\bigl(-\frac{T^2}{\tau(T-\tau)}\bigr)$ is positive and symmetric about $\frac{T}{2}$. Also note that the lower limits on the above two integrals are symmetric about $\frac{T}{2}$. Therefore

$$\int_{|t-(j+1)T|}^{T} \exp\Bigl(-\frac{T^2}{\tau(T-\tau)}\Bigr)\,\mathrm{d}\tau = \int_0^{|t-jT|} \exp\Bigl(-\frac{T^2}{\tau(T-\tau)}\Bigr)\,\mathrm{d}\tau.$$

This then gives

$$\sum_{n\in\mathbb{Z}} \upsilon_T(t-nT) = \frac{1}{c}\int_0^T \exp\Bigl(-\frac{T^2}{\tau(T-\tau)}\Bigr)\,\mathrm{d}\tau = 1,$$

as desired. •

As usual, it is important to specify a notion of convergence for the set of periodic test signals.

10.7.4 Definition (Convergence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) A sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ ***converges to zero*** if, for each $r\in\mathbb{Z}_{\geq 0}$, the sequence $(\psi_j^{(r)})_{j\in\mathbb{Z}_{>0}}$ converges to zero uniformly. A sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ ***converges*** to $\psi\in\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ if the sequence $(\psi_j-\psi)_{j\in\mathbb{Z}_{>0}}$ converges to zero. •

Note that there are none of the domain issues with convergence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ as arise with $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, and $\mathscr{E}(\mathbb{R};\mathbb{F})$. This is because, by virtue of periodicity, a convergent sequence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ can be understood by its behaviour on the compact set $[0,T]$.

10.7.5 Examples (Convergence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$)

1. For $k\in\mathbb{Z}_{>0}$ the sequence $(\frac{1}{n^k}\sin(2\pi n\frac{t}{T}))_{n\in\mathbb{Z}_{>0}}$ does not converge to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. The reason for this is that $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{1}{n^k}\sin(2\pi n\frac{t}{T})$ will be of the form $\pm(\frac{2\pi}{T})^k\sin(2\pi n\frac{t}{T})$ or $\pm(\frac{2\pi}{T})^k\cos(2\pi n\frac{t}{T})$ so that the sequence $(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{1}{n^k}\sin(2\pi n\frac{t}{T}))_{n\in\mathbb{Z}_{>0}}$ does not converge uniformly to zero.
2. The sequence $(\mathrm{e}^{-n}\sin(2\pi n\frac{t}{T}))_{n\in\mathbb{Z}_{>0}}$ does converge to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ since all derivatives will converge to zero uniformly in $[0,T]$. •

One also defines the notion of continuity of maps with domain $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

10.7.6 Definition (Continuous linear maps on $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) A linear map $\mathsf{L}\colon\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})\to\mathbb{F}$ is ***continuous*** if the sequence $(\mathsf{L}(\psi_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero for every sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ that converges to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. •

This seems rather like everything that has preceded it thus far. However, there is some useful additional structure for $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ that we have not yet revealed.

10.7.7 Proposition (Periodic test signals come from test signals) *If* $\psi \in \mathscr{D}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *then there exists* $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ *such that for each* $\mathrm{t} \in \mathbb{R}$

$$\psi(\mathrm{t}) = \sum_{\mathrm{n}\in\mathbb{Z}} \phi(\mathrm{t} - \mathrm{nT}).$$

Proof Let $\upsilon \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$. Note that $\psi\upsilon \in \mathscr{D}(\mathbb{R};\mathbb{F})$ since υ has compact support. We also have

$$\sum_{n\in\mathbb{Z}} \psi(t-nT)\upsilon(t-nT) = \psi(t)\sum_{n\in\mathbb{Z}} \upsilon(t-nT) = \psi(t),$$

by periodicity of ψ and since υ is unitary. The result therefore follows by taking $\phi = \psi\upsilon$. ■

10.7.2 Definition of periodic distributions

Let us begin with a preliminary construction. Recall from Example 8.1.6–1 the definition of $\tau_{t_0}\colon \mathbb{R} \to \mathbb{R}$ by $\tau_{t_0}(t) = t - t_0$ and from Example 8.1.13–1 the notation $\tau_{t_0}^* f(t) = f \circ \tau_{t_0}(t) = f(t - t_0)$ for $f\colon \mathbb{R} \to \mathbb{F}$. If $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ then we define $\tau_{t_0}^*\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ by $\tau_{t_0}^*\theta(\phi) = \theta(\tau_{-t_0}^*\phi)$. In particular, if $\theta = \theta_f$ for $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ then

$$\tau_{t_0}^*\theta_f(\phi) = \int_{\mathbb{R}} f(t)\phi(t+t_0)\,\mathrm{d}t = \int_{\mathbb{R}} f(t-t_0)\phi(t)\,\mathrm{d}t = \theta_{\tau_{t_0}^* f}(\phi).$$

With this as motivation, it makes sense to say that $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ is, by definition, T-periodic if $\tau_T^*\theta = \theta$. Note that there is no need for periodic test signals in this definition! Let us, therefore, make a definition using $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, and then show that it agrees with the natural definition we just gave in the absence of $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

Our definition is as follows.

10.7.8 Definition (Periodic distribution) A T-*periodic distribution* is a continuous linear map from $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ to $\mathbb{F}$. The set of T-periodic distributions is denoted $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. •

10.7.9 Remark ($\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ is a vector space) It is easy to check that $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is a subspace of $\mathscr{D}'(\mathbb{R};\mathbb{F})$. The inclusion is proved below in Theorem 10.7.10, and the inheritance of the vector space structure is then readily verified.

Now we can show that this definition of $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ agrees with our alternate characterisation above.

10.7.10 Theorem (Periodic distributions are... periodic distributions) *There exists a natural isomorphism from* $\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *to the subspace*

$$\mathscr{D}'_{\mathrm{T}}(\mathbb{R};\mathbb{F}) = \{\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F}) \mid \tau_{\mathrm{T}}^*\theta = \theta\}$$

of $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

Proof We shall construct the inverse of the stated isomorphism. Let $\upsilon \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$ and define a map $\iota_\upsilon\colon \mathscr{D}'_T(\mathbb{R};\mathbb{F}) \to \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ by

$$\iota_\upsilon(\theta)(\psi) = \theta(\upsilon\psi), \qquad \theta \in \mathscr{D}'_T(\mathbb{R};\mathbb{F}),\ \psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F}).$$

For ι_υ to make sense we must show that $\iota_\upsilon(\theta)$ is continuous. Let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. It is then clear that $(\upsilon\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}(\mathbb{R};\mathbb{F})$ since $\mathrm{supp}(\upsilon\psi_j) = \mathrm{supp}(\upsilon)$, $j \in \mathbb{Z}_{>0}$. We next claim that ι_υ is actually independent of υ. That is to say, if $\upsilon_1, \upsilon_2 \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$ then we have $\iota_{\upsilon_1} = \iota_{\upsilon_2}$. To see this, we first note that if $\theta \in \mathscr{D}'_T(\mathbb{R};\mathbb{F})$, $\upsilon \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$, and $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ then

$$\theta(\phi) = \theta\Big(\sum_{n\in\mathbb{Z}} \tau^*_{nT}\upsilon\phi\Big) = \sum_{n\in\mathbb{Z}} \theta(\tau^*_{nT}\upsilon\phi) = \Big(\sum_{n\in\mathbb{Z}} \tau^*_{nT}\upsilon\theta\Big)(\phi).$$

Therefore,

$$\begin{aligned}\theta(\upsilon_1\psi) &= \Big(\sum_{n\in\mathbb{Z}} \tau^*_{nT}\upsilon_2\theta\Big)(\upsilon_1\psi) = \Big(\sum_{n\in\mathbb{Z}} (\tau^*_{nT}\upsilon_2)\upsilon_1\theta\Big)(\psi)\\ &= \Big(\sum_{n\in\mathbb{Z}} \upsilon_2\tau^*_{-nT}(\upsilon_1\theta)\Big)(\psi) = \Big(\sum_{n\in\mathbb{Z}} \tau^*_{nT}\upsilon_1\theta\Big)(\upsilon_2\psi) = \theta(\upsilon_2\psi).\end{aligned}$$

This then establishes a natural linear map, which we denote simply by ι, from $\mathscr{D}'_T(\mathbb{R};\mathbb{F})$ to $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. We now show that ι is an isomorphism.

First let us show that ι is surjective. Let $\theta \in \mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. For $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ let ϕ_T denote the element of $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ defined by

$$\phi_T(t) = \sum_{n\in\mathbb{Z}} \phi(t - nT).$$

Then define $\tilde\theta \in \mathscr{D}(\mathbb{R};\mathbb{F})$ by $\tilde\theta(\phi) = \theta(\phi_T)$. We claim that $\tilde\theta \in \mathscr{D}'_T(\mathbb{R};\mathbb{F})$. This is clear since for $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ we have

$$\tau^*_T\tilde\theta(\phi) = \tilde\theta(\tau^*_T\phi) = \theta(\tau^*_T\phi_T) = \theta(\phi_T) = \tilde\theta(\phi).$$

We also claim that $\iota(\tilde\theta) = \theta$. Let $\upsilon \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$. From the proof of Proposition 10.7.7 note that for any $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we may write $\psi = \phi_T$ where $\phi = \upsilon\psi$. We then have

$$\iota(\tilde\theta)(\psi) = \tilde\theta(\upsilon\psi) = \tilde\theta(\phi) = \theta(\phi_T) = \theta(\psi),$$

as desired, and so showing that ι is surjective.

We lastly show that ι is injective. Suppose that $\iota(\theta) = 0$. Then $\iota(\theta)(\psi) = 0$ for every $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. In particular, $\iota(\theta)(\phi_T) = 0$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$, meaning that $\theta(\phi) = 0$ for every $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. Thus $\theta = 0$, so showing injectivity of ι. ■

Although the proof of the theorem is a little long-winded, it is elementary in that there are no difficult ideas to digest. But more importantly, it allows us to think of elements of $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ as distributions in the usual sense, and we shall subsequently do this without notice in the sequel. For this reason it is worth reproducing the following corollary that explicitly describes the isomorphism of Theorem 10.7.10.

10.7.11 Corollary (Explicit characterisation of periodic distributions) *For $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$ and $\mathrm{T} \in \mathbb{R}_{>0}$ the following are equivalent:*

(i) $\tau_{\mathrm{T}}^*\theta = \theta$;

(ii) *there exists a unique* $\tilde{\theta} \in \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *such that, for all* $\upsilon \in \mathscr{U}_{\mathrm{T}}(\mathbb{R};\mathbb{F})$, $\theta(\upsilon\psi) = \tilde{\theta}(\psi)$ *for each* $\psi \in \mathscr{D}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$;

(iii) *there exists a unique* $\tilde{\theta} \in \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *such that* $\theta(\phi) = \tilde{\theta}(\phi_{\mathrm{T}})$, *where* $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ *and where* ϕ_{T} *is defined as in (10.16).*

With these characterisations of T-periodic distributions, let us look at some examples.

10.7.12 Examples (Periodic distributions)

1. Let $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. Note that as an element in $\mathscr{D}(\mathbb{R};\mathbb{F})$ we have $\theta_f \in \mathscr{D}'_T(\mathbb{R};\mathbb{F})$. Therefore, we may think of $\theta_f \in \mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. Using Corollary 10.7.11 let us compute $\theta_f(\psi)$ for $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. For an arbitrary $\upsilon \in \mathscr{U}_T(\mathbb{R};\mathbb{F})$ and $a \in \mathbb{R}$, we have
$$\begin{aligned}\theta_f(\psi) &= \int_{\mathbb{R}} f(t)\upsilon(t)\psi(t)\,\mathrm{d}t\\ &= \sum_{n\in\mathbb{Z}} \int_{a+nT}^{a+(n+1)T} f(t)\upsilon(t)\psi(t)\,\mathrm{d}t\\ &= \sum_{n\in\mathbb{Z}} \int_{a}^{a+T} f(t+nT)\upsilon(t+nT)\psi(t+nT)\,\mathrm{d}t\\ &= \int_a^{a+T} f(t)\sum_{n\in\mathbb{Z}}\upsilon(t-nT)\psi(t)\,\mathrm{d}t\\ &= \int_a^{a+T} f(t)\psi(t)\,\mathrm{d}t.\end{aligned}$$
Thus we determine $\theta_f(\psi)$ is computed by integrating over any interval of length T in $\mathbb{R}$.

2. The ***delta-comb*** with period T is the T-periodic distribution defined by
$$⫛_T = \sum_{n\in\mathbb{Z}} \delta_{nT}.$$
Note that this is well-defined as an element of $\mathscr{D}'(\mathbb{R};\mathbb{F})$, and that it is clearly in $\mathscr{D}'_T(\mathbb{R};\mathbb{F})$. To compute how $⫛_T$ acts on an element of $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we again use Corollary 10.7.11 and compute, for $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$,
$$⫛_T(\psi) = ⫛_T(\upsilon\psi) = \sum_{n\in\mathbb{Z}}\delta_{nT}(\upsilon\psi) = \sum_{n\in\mathbb{Z}}\upsilon(nT)\psi(nT) = \psi(0)\sum_{n\in\mathbb{Z}}\upsilon(nT) = \psi(0).$$

This is analogous to the preceding example if one properly understands the symbols. That is to say, suppose that $a \in [(m-1)T, mT]$. Then we can write

$$⫛_T\,(\psi) = \int_a^{a+T} ⫛_T\,(t)\psi(t)\,\mathrm{d}t = \int_a^{a+T} \sum_{n\in\mathbb{Z}} \delta_{nT}(t)\psi(t)\,\mathrm{d}t = \psi(mT) = \psi(0).$$

These sorts of manipulations are perfectly acceptable, provided one understands what they actually mean! •

10.7.3 Properties of periodic distributions

In this section we record some of the basic facts about distributions with compact support. Many of these follow, directly or with little effort, from their counterparts for distributions.

Since $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}'(\mathbb{R};\mathbb{F})$, we have, in principle, a notion of convergence inherited from $\mathscr{D}(\mathbb{R};\mathbb{F})$. However, let us give a definition of convergence using $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

10.7.13 Definition (Convergence in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) A sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is

(i) a ***Cauchy sequence*** if $(\theta_j(\psi))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence for every $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, and

(ii) ***converges*** to a distribution θ with compact support if for every $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, the sequence of numbers $(\theta_j(\psi))_{j\in\mathbb{Z}_{>0}}$ converges to $\theta(\psi)$. •

Now we relate this notion of convergence to that inherited from $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

10.7.14 Theorem (Convergence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) *If $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{D}'_T(\mathbb{R};\mathbb{F}) \subseteq \mathscr{D}(\mathbb{R};\mathbb{F})$ that converges to $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{F})$, then $\theta \in \mathscr{D}'_T(\mathbb{R};\mathbb{F})$. Furthermore, such a sequence in $\mathscr{D}'_T(\mathbb{R};\mathbb{F})$ converges in $\mathscr{D}'(\mathbb{R};\mathbb{F})$ if and only if the corresponding sequence in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ determined by Theorem 10.7.10 converges in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.*

In particular, a sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is a Cauchy sequence if and only if it converges.

Proof For the first statement we need to show that $\tau_T^*\theta = \theta$. We have

$$\tau_T^*\theta(\phi) = \theta(\tau_{-T}^*\phi) = \lim_{j\to\infty}\theta_j(\tau_{-T}^*\phi) = \lim_{j\to\infty}\tau_T^*\theta_j(\phi) = \lim_{j\to\infty}\theta_j(\phi) = \theta(\phi),$$

this holding for any $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$.

For the second assertion, let $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, assuming that $\psi = \phi_T$ for $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$. We then have

$$\lim_{j\to\infty}\iota(\theta_j)(\psi) = \lim_{j\to\infty}\theta_j(\phi_T) = \theta(\phi_T) = \iota(\theta)(\psi),$$

thus giving convergence in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ from convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$. If $(\iota(\theta_j))_{j\in\mathbb{Z}_{>0}}$ converges in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ then we have

$$\lim_{j\to\infty}\theta_j(\phi) = \lim_{j\to\infty}\iota(\theta_j)(\phi_T) = \iota(\theta)(\phi) = \theta(\phi),$$

thus showing convergence in $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

The final assertion will follow if we can show that a sequence in $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is Cauchy if and only if the corresponding sequence in $\mathscr{D}'_T(\mathbb{R};\mathbb{F})$ is Cauchy. This follows from the same sort of arguments as used in the preceding part of the proof, and we leave the trivial working out of this to the reader. ■

Let us give the analogue for distributions with compact support of the fact that locally integrable signals are distributions. We recall from ***missing stuff*** the notion of the support of a measurable signal.

10.7.15 Proposition (Periodic integrable signals are periodic distributions) *If* $\mathrm{f}\colon \mathbb{R}\to\mathbb{F}$ *is a* T*-periodic locally integrable signal then* $\theta_{\mathrm{f}}\in\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$*. Moreover, if* $\mathrm{f}_1,\mathrm{f}_2\colon\mathbb{R}\to\mathbb{F}$ *are periodic locally integrable signals for which* $\theta_{\mathrm{f}_1}=\theta_{\mathrm{f}_2}$*, then* $\mathrm{f}_1(\mathrm{t})=\mathrm{f}_2(\mathrm{t})$ *for almost every* $\mathrm{t}\in\mathbb{R}$*.*

Proof From Proposition 10.2.12 we know that $\theta_f\in\mathscr{D}'(\mathbb{R};\mathbb{F})$. Thus we need only show that θ_f is T-periodic. This, however, is elementary. For $\psi\in\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we have

$$\tau_T^*\theta_f=\theta_{\tau_T^*f}=\theta_f,$$

using the computation preceding Definition 10.7.8.

The last assertion follows the similar assertion in Proposition 10.2.12. ■

Periodic signals also show up to give a natural class of signals which can be multiply periodic distributions.

10.7.16 Proposition (Periodic distributions can be multiplied by smooth periodic signals) *Let* $\theta\in\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *and let* $\psi_0\colon\mathbb{R}\to\mathbb{F}$ *be a* T*-periodic infinitely differentiable signal. Then the map*

$$\mathscr{D}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})\ni\psi\mapsto\theta(\psi_0\psi)\in\mathbb{F}$$

defines an element of $\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$*.*

Proof Linearity of the map is clear. To prove continuity, let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ converging to zero. We claim that $(\psi_0\psi_j)_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

It is clear that $\psi_0\psi_j\in\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ for each $j\in\mathbb{Z}_{>0}$, so we need only show that $(\psi_0\psi_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. By Proposition 3.2.11 the signal $(\psi_0\psi_j)^{(r)}$ is a sum of products formed by signals that are bounded on with signals that converge uniformly to zero on K. Thus $((\psi_0\psi_j)^{(r)})_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero, giving the desired conclusion.

Thus the result follows since

$$\lim_{j\to\infty}\theta(\psi_0\psi)=0$$

for every sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converging to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. ■

The notions of regular, singular, support, and singular support are applied to $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ by restriction from $\mathscr{D}'(\mathbb{R};\mathbb{F})$.

One can differentiate periodic distributions as they are distributions. It turns out that the derivative is again a periodic distribution.

10.7.17 Proposition (The derivative of a periodic distribution is a periodic distribution) *If* $\theta \in \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *then* $\theta' \in \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$*.*

Proof We let $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ converging to zero. Then $(-\psi'_j)_{j\in\mathbb{Z}_{>0}}$ is also a sequence converging to zero in $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, as is easily seen from the definition of convergence to zero. Therefore,

$$\lim_{j\to\infty}\theta'(\psi_j) = \lim_{j\to\infty}\theta(-\psi'_j) = 0$$

as desired. ■

One can talk about periodic distributions of finite order, and periodic distributions are always locally of finite order by virtue of their being distributions. We shall see in Theorem 10.7.19 that even more is true for distributions with compact support.

10.7.4 Some deeper properties of periodic distributions

We have already shown that there is a natural way the consider a periodic distribution as a regular distribution. Next we verify that periodic distributions are of slow growth.

10.7.18 Theorem (Periodic distributions are tempered distributions) $\mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$*, as a subspace of* $\mathscr{D}'(\mathbb{R};\mathbb{F})$*, is a subspace of* $\mathscr{S}'(\mathbb{R};\mathbb{F})$*.*

Proof Take $\theta \in \mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and regard this as an element of $\mathscr{D}'(\mathbb{R};\mathbb{F})$. Let $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ have the following properties:

1. $\mathrm{supp}(\phi) \subseteq (0,T)$;
2. there is a neighbourhood of $\frac{T}{2}$ on which ϕ takes the value 1;
3. $\phi(t) \in [0,1]$ for all $t \in \mathbb{R}$.

Then define $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ to be the T-periodic extension of ϕ. We can then write $\theta = \psi\theta + (1-\psi)\theta$, and we shall show that each summand is in $\mathscr{S}'(\mathbb{R};\mathbb{F})$. If $\theta_1 = \phi\theta$ and $\theta_2 = (1-\phi)\theta$ then $\theta_1, \theta_2 \in \mathscr{E}'(\mathbb{R};\mathbb{F})$. Furthermore,

$$\theta = \sum_{n\in\mathbb{Z}} \tau^*_{nT}\theta_1 + \sum_{n\in\mathbb{Z}} \tau^*_{nT}\theta_2.$$

The result will then follow if we can show that for any distribution β with compact support it follows that the sequence

$$\sum_{|n|\le N} \tau^*_{nT}\beta$$

of partial sums converges in $\mathscr{S}'(\mathbb{R};\mathbb{F})$. By Theorem 10.5.20, this will in turn follow if we can show that for $f \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ the sequence

$$\sum_{|n|\le N} \tau^*_{nT}\theta^{(r)}_f$$

of partial sums converges in $\mathscr{S}'(\mathbb{R};\mathbb{F})$ for $r \in \mathbb{Z}_{\geq 0}$. So let $\chi \in \mathscr{S}(\mathbb{R};\mathbb{F})$, $f \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$, and $r \in \mathbb{Z}_{\geq 0}$. For convenience, and without loss of generality, we suppose that $\operatorname{supp}(f) = [0,a]$ for some $a \in \mathbb{R}_{>0}$. Since $\chi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ we have

$$|\chi^{(r)}(t)| \leq \frac{M}{(1+t^2)^3}$$

for some $M \in \mathbb{R}_{>0}$. We may choose N sufficiently large that if $t \in \operatorname{supp}(\tau^*_{NT} f)$ then $(1+t^2)^{-1} \leq \epsilon$. In this case we have

$$|(\tau^*_{nT} f)(t)\phi^{(r)}(t)| \leq \frac{M\|f\|_\infty}{(1+t^2)^3} \leq \frac{M\|f\|_\infty \epsilon}{(1+t^2)^2} \leq \frac{M\|f\|_\infty \epsilon}{(1+(nT)^2)(1+t^2)}$$

for all $n \geq N$ and $t \in \mathbb{R}$. Therefore it follows that by taking N sufficiently large we have

$$\begin{aligned}\Big|\sum_{|n|\geq N} \tau^*_{nT}\theta^{(r)}_f(\chi)\Big| &= \Big|\sum_{|n|\geq N} (-1)^r \int_{\mathbb{R}} (\tau_{nT} f)(t)\chi^{(r)}(t)\,\mathrm{d}t\Big| \leq \sum_{|n|\geq N}\int_{\mathbb{R}} |\tau_{nT} f)(t)\chi^{(r)}(t)|\,\mathrm{d}t \\ &\leq \sum_{|n|\geq N}\int_{\mathbb{R}} \frac{M\|f\|_\infty \epsilon}{(1+(nT)^2)(1+t^2)}\,\mathrm{d}t = \frac{1}{2} M\|f\|_\infty \pi\epsilon \sum_{|n|\geq N}\frac{1}{1+(nT)^2}.\end{aligned}$$

Since the series in the preceding expression converges, this shows that by taking N sufficiently large we can make

$$\Big|\sum_{|n|\geq N} \tau^*_{nT}\theta^{(r)}_f(\chi)\Big|$$

as small as we like, which shows that $\sum_{n\in\mathbb{Z}} \tau^*_{nT}\theta^{(r)}_f$ converges in $\mathscr{S}'(\mathbb{R};\mathbb{F})$, so giving the result. ■

The following result now follows easily from Theorem 10.3.25, and provides the useful property of finite order for periodic generalised signals.

10.7.19 Theorem (Periodic distributions are finite-order derivatives of periodic signals) *If $\theta \in \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ then there exists $\mathrm{r} \in \mathbb{Z}_{\geq 0}$ and a T-periodic signal $\mathrm{f} \in \mathsf{C}^0(\mathbb{R};\mathbb{F})$ such that $\theta(\psi) = \theta^{(\mathrm{r})}_{\mathrm{f}}(\psi)$ for every $\psi \in \mathscr{D}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$.*

Proof From the proof of Theorem 10.7.18 we have

$$\theta = \sum_{n\in\mathbb{Z}} \tau^*_{nT}\theta_1 + \sum_{n\in\mathbb{Z}} \tau^*_{nT}\theta_2,$$

for distributions $\theta_1, \theta_2 \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ with support in $(0,T)$. Using Theorem 10.5.20 we may then write

$$\theta_j = \sum_{k_j=1}^{m_j} \theta^{(r_{j,k_j})}_{f_{j,k_j}}$$

for continuous signals f_{j,k_j}, $j \in \{1,2\}$, $k_j \in \{1,\dots,m_j\}$, with compact support. Note that

$$\tau^*_{nT}\theta^{(r_{j,k_j})}_{f_{j,k_j}} = \theta^{(r_{j,k_j})}_{\tau^*_{nT} f_{j,k_j}}, \qquad n \in \mathbb{Z},\ j \in \{1,2\},\ k_j \in \{1,\dots,m_j\}.$$

Thus

$$\theta = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{2} \sum_{k_j=1}^{m_j} \theta^{(r_{j,k_j})}_{\tau^*_{nT} f_{j,k_j}} = \sum_{j=1}^{m} \theta^{(r_j)}_{g_j}$$

where $m = m_1 + m_2$, $g_1, \dots, g_m$ are the T-periodic continuous signals given by

$$g_j = \begin{cases} \sum_{n\in\mathbb{Z}} \tau^*_{nT} f_{1,j}, & j \in \{1, \dots, m_1\}, \\ \sum_{n\in\mathbb{Z}} \tau^*_{nT} f_{2,m_1+j}, & j \in \{m_1+1, \dots, m_1+m_2\}, \end{cases}$$

and where

$$r_j = \begin{cases} r_{1,j}, & j \in \{1, \dots, m_1\}, \\ r_{2,m_1+j}, & j \in \{m_1+1, \dots, m_1+m_2\}. \end{cases}$$

Now we claim that for any T-periodic signal f and $r \in \mathbb{Z}_{>0}$ there exists a T-periodic solution g to the equation $g^{(r)} = f$. This can be shown inductively, and the essential idea is contained in the argument when $r = 1$. In this case we define

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau - \int_0^T f(\tau)\,\mathrm{d}\tau,$$

and we note that

$$g(t+T) = \int_0^{t+T} f(\tau)\,\mathrm{d}\tau = \int_0^t f(\tau)\,\mathrm{d}\tau + \int_t^{t+T} f(\tau)\,\mathrm{d}\tau = g(t),$$

so showing that g is T-periodic. We then take $r = \max\{r_1, \dots, r_{m_1+m_2}\}$ and then define h_j, $j \in \{1, \dots, m\}$, to be T-periodic signals satisfying $h_j^{(r)} = g_j^{(r_j)}$. This then gives $\theta = \theta_h^{(r)}$ where $h = \sum_{j=1}^m h_j$. ■

Exercises

10.7.1 Show that if $\phi_1, \phi_2 \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ then $\phi_1\phi_2 \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. Thus $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is an algebra.

Section 10.8

Inclusions between signals, test signals, and generalised signals

Having now presented all (okay, almost all; see Section 10.6.2) of our spaces of distributions, we shall now determine the inclusion relations between them.

Do I need to read this section? The material here is important, and at times extremely important. In particular, results concerning the density of spaces of test signals in spaces of distributions give important characterisations of distributions. •

In the preceding four sections we introduced the signal classes $\mathscr{D}(\mathbb{R};\mathbb{F})$, $\mathscr{S}(\mathbb{R};\mathbb{F})$, $\mathscr{E}(\mathbb{R};\mathbb{F})$, and $\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, and the generalised signal classes $\mathscr{D}'(\mathbb{R};\mathbb{F})$, $\mathscr{S}'(\mathbb{R};\mathbb{F})$, $\mathscr{E}'(\mathbb{R};\mathbb{F})$ and $\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. In the course of our presentation, we proved the following result.

10.8.1 Proposition (Inclusion relations between spaces of test signals and distributions) *The following inclusions hold:*

$$\begin{array}{ccccccc}
\mathscr{D}(\mathbb{R};\mathbb{F}) & \subseteq & \mathscr{S}(\mathbb{R};\mathbb{F}) & \subseteq & \mathscr{E}(\mathbb{R};\mathbb{F}) & \supseteq & \mathscr{D}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F}) \\
\cap & & \cap & & \cap & & \cap \\
\mathscr{E}'(\mathbb{R};\mathbb{F}) & \subseteq & \mathscr{S}'(\mathbb{R};\mathbb{F}) & \subseteq & \mathscr{D}'(\mathbb{R};\mathbb{F}) & \supseteq & \mathscr{D}'_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})
\end{array}$$

We wish to better understand some of these inclusions by providing some density relationships between the various sets of test signals and distributions. In order to do this we need to say what we mean by a dense subspace of the various spaces of test signals and associated distributions. The following definition achieves this.

10.8.2 Definition (Density between spaces of test signals and distributions) Let $\mathscr{T}_1,\mathscr{T}_2 \in \{\mathscr{D}(\mathbb{R};\mathbb{F}),\mathscr{S}(\mathbb{R};\mathbb{F}),\mathscr{E}(\mathbb{R};\mathbb{F}),\mathscr{B}_0(\mathbb{R};\mathbb{F}),\mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})\}$ and let $\mathscr{T}_1',\mathscr{T}_2' \in \{\mathscr{D}'(\mathbb{R};\mathbb{F}),\mathscr{S}'(\mathbb{R};\mathbb{F}),\mathscr{E}'(\mathbb{R};\mathbb{F}),\mathscr{D}'_{\mathrm{L}^1}(\mathbb{R};\mathbb{F}),\mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{F})\}$.

(i) If $\mathscr{T}_1 \subseteq \mathscr{T}_2$ then $\mathscr{T}_1$ is ***dense*** in $\mathscr{T}_2$ if, for each $\phi \in \mathscr{T}_2$, there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{T}_1$ such that $\lim_{j\to\infty}\phi_j = \phi$, the limit being taken in $\mathscr{T}_2$.

(ii) If $\mathscr{T}_1' \subseteq \mathscr{T}_2'$ then $\mathscr{T}_1'$ is ***dense*** in $\mathscr{T}_2'$ if, for each $\theta \in \mathscr{T}_2'$, there exists a sequence $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{T}_1'$ such that $\lim_{j\to\infty}\theta_j = \theta$, the limit being taken in $\mathscr{T}_2'$.

(iii) If $\mathscr{T}_1 \subseteq \mathscr{T}_2'$ then $\mathscr{T}_1$ is ***dense*** in $\mathscr{T}_2'$ if, for each $\theta \in \mathscr{T}_2'$, there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{T}_1$ such that $\lim_{j\to\infty}\theta_{\phi_j} = \theta$, the limit being taken in $\mathscr{T}_2'$. •

We then have the following important result.

10.8.3 Theorem (Density of spaces of test signals and distributions in one another) *The following statements hold:*

(i) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{S}(\mathbb{R};\mathbb{F})$*;*

(ii) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{E}(\mathbb{R};\mathbb{F})$*;*

(iii) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{B}_0(\mathbb{R};\mathbb{F})$*;*

(iv) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{D}'(\mathbb{R};\mathbb{F})$*;*

(v) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{S}'(\mathbb{R};\mathbb{F})$*.*

(vi) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathscr{E}'(\mathbb{R};\mathbb{F})$*.*

Proof (i) This is Lemma 2 in the proof of Theorem 10.3.13.

(ii) This is Lemma 1 in the proof of Theorem 10.5.11.

(iii) Let $\phi \in \mathscr{B}_0(\mathbb{R};\mathbb{F})$ and let $(\psi_k)_{k\in\mathbb{Z}_{>0}}$ be the sequence in $\mathscr{D}(\mathbb{R};\mathbb{F})$ characterised in Lemma 1 in the proof of Theorem 10.3.13. The arguments from the proof of Lemma 2 in the proof of Theorem 10.3.13 can be applied to show that the sequence $(\phi\psi_k)_{k\in\mathbb{Z}_{>0}}$ converges to ϕ in $\mathscr{B}_0(\mathbb{R};\mathbb{F})$. Indeed, a moments thought shows that the desired conclusion here follows directly from the computations in the proof of Lemma 2 in the proof of Theorem 10.3.13.

(iv) We will prove this as Theorem 11.3.26.

(v) We note that if $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{F})$ then there exists $f_\theta \in \mathsf{C}^0(\mathbb{R};\mathbb{F})$ and $r \in \mathbb{Z}_{\geq 0}$ of slow growth such that $\theta = \theta^{(r)}_{f_\theta}$. By Theorem 11.3.24 we may find a sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F}) = \mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ such that $\lim_{j\to\infty}\|f_\theta - \psi_j\|_\infty = 0$. Since $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ and since f_θ is of slow growth it follows that the sequence of signals $(\psi_j\phi)_{j\in\mathbb{Z}_{>0}}$ is uniformly bounded in j and t. We claim that this implies that the sequence $(\theta^{(r)}_{\psi_j})_{j\in\mathbb{Z}_{>0}}$ converges to $\theta^{(r)}_{f_\theta}$ in $\mathscr{S}'(\mathbb{R};\mathbb{F})$. Indeed, let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{F})$ and compute

$$\begin{aligned}
\lim_{j\to\infty} \theta^{(r)}_{\psi_j}(\phi) &= \lim_{j\to\infty}(-1)^r \int_{\mathbb{R}} \psi_j(t)\phi^{(r)}(t)\,\mathrm{d}t \\
&= (-1)^r \int_{\mathbb{T}} \lim_{j\to\infty} \psi_j(t)\phi^{(r)}(t)\,\mathrm{d}t \\
&= (-1)^r \int_{\mathbb{T}} g(t)\phi^{(r)}(t)\,\mathrm{d}t = \theta^{(r)}_{f_\theta}(\phi),
\end{aligned}$$

as desired, by the Dominated Convergence Theorem.

(vi) By Theorem 10.5.20, if $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{F})$ then there exists $f_1,\dots,f_m \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ and $r_1,\dots,r_m \in \mathbb{Z}_{\geq 0}$ such that

$$\theta = \sum_{j=1}^m \theta^{(r_j)}_{f_j}.$$

By Theorem 11.3.24 we may find sequences $(\psi_{j,k})_{k\in\mathbb{Z}_{>0}}$, $j \in \{1,\dots,m\}$, in $\mathscr{D}(\mathbb{R};\mathbb{F}) = \mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ so that $\lim_{k\to\infty}\|f_j - \psi_{j,k}\|_\infty = 0$. Furthermore, the support of the functions $\psi_{j,k}$, $k \in \mathbb{Z}_{>0}$, $j \in \{1,\dots,m\}$, is contained in some compact interval $\mathbb{T} \subseteq \mathbb{R}$. We claim that this implies that for each $j \in \{1,\dots,m\}$, the sequence $(\theta^{(r_j)}_{\psi_{j,k}})_{k\in\mathbb{Z}_{>0}}$ converges to $\theta^{(r_j)}_{f_j}$

in $\mathscr{E}'(\mathbb{R};\mathbb{F})$. Indeed, let $\phi \in \mathscr{E}(\mathbb{R};\mathbb{F})$ and compute

$$\begin{aligned}\lim_{k\to\infty}\theta^{(r_j)}_{\psi_{j,k}}(\phi) &= \lim_{k\to\infty}(-1)^{r_j}\int_{\mathbb{R}}\psi_{j,k}(t)\phi^{(r_j)}(t)\,\mathrm{d}t\\ &= (-1)^{r_j}\int_{\mathbb{T}}\lim_{k\to\infty}\psi_{j,k}(t)\phi^{(r_j)}(t)\,\mathrm{d}t\\ &= (-1)^{r_j}\int_{\mathbb{T}}f_j(t)\phi^{(r_j)}(t)\,\mathrm{d}t = \theta^{(r_j)}_{f_j}(\phi),\end{aligned}$$

as desired. Here we have used the fact that $\psi_{j,k}\phi^{(r_j)}$ is uniformly bounded in t and k, so making the interchange of the limit and the integral possible by the Dominated Convergence Theorem. We can then write

$$\theta(\phi) = \lim_{N\to\infty}\sum_{j=1}^{m}\sum_{k=1}^{N}\theta^{(r_j)}_{\psi_{j,k}}(\phi),$$

giving the result since $\theta^{(r_j)}_{\psi_{j,k}}$, $j\in\{1,\dots,m\}$, $k\in\mathbb{Z}_{>0}$, corresponds to an element in $\mathscr{D}(\mathbb{R};\mathbb{F})$ since $\mathscr{D}(\mathbb{R};\mathbb{F})$ is closed under differentiation. ■

Now let us consider the relationships between test signals and some of the signal spaces introduced in Chapter 8.***missing stuff***

10.8.4 Proposition (Inclusion relations between signal spaces and spaces of test signals and distributions) *The following statements hold:*

(i) $\mathscr{S}(\mathbb{R};\mathbb{F})\subseteq \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ *for* $p\in[1,\infty]$;

(ii) $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})\subseteq \mathscr{S}'(\mathbb{R};\mathbb{F})$ *for* $p\in[1,\infty]$;

(iii) $\mathscr{D}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathsf{L}^{p}(\mathbb{R};\mathbb{F})$ *for* $p\in[1,\infty)$;

(iv) $\mathscr{S}(\mathbb{R};\mathbb{F})$ *is a dense subspace of* $\mathsf{L}^{p}(\mathbb{R};\mathbb{F})$ *for* $p\in[1,\infty)$.

Proof (i) Let $\phi\in\mathscr{S}(\mathbb{R};\mathbb{F})$. By Proposition 10.3.2 there exists $T\in\mathbb{R}_{>0}$ such that $|\phi(t)|<\frac{1}{t^2}$. Since ϕ is infinitely differentiable it is also bounded on any compact subset of $\mathbb{R}$, and thus we deduce that $\mathscr{S}(\mathbb{R};\mathbb{F})\subseteq\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$. Now let $p\in[1,\infty)$. Choosing $T>1$ we have

$$\begin{aligned}\|\phi\|_p^p &= \int_{\mathbb{R}}|\phi(t)|^p\,\mathrm{d}t = \int_{|t\leq T|}|\phi(t)|^p\,\mathrm{d}t + \int_{|t|\geq T}|\phi(t)|^p\,\mathrm{d}t\\ &\leq \int_{|t\leq T|}|\phi(t)|^p\,\mathrm{d}t + 2\int_T^\infty t^{-2p}\,\mathrm{d}t \leq \int_{|t\leq T|}|\phi(t)|^p\,\mathrm{d}t + \frac{1}{2p-1}<\infty,\end{aligned}$$

so giving the result.

(ii) Let $f\in\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, $p\in[1,\infty]$, and let $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to zero in $\mathscr{S}(\mathbb{R};\mathbb{F})$. From part (i), $\phi_j\in\mathsf{L}^{(p')}(\mathbb{R};\mathbb{F})$, $j\in\mathbb{Z}_{>0}$, where $\frac{1}{p}+\frac{1}{p'}=1$. Furthermore,

$$\lim_{j\to\infty}\|\phi_j\|_{p'}^{p'} = \lim_{j\to\infty}\int_{\mathbb{R}}|\phi(t)|^{p'}\,\mathrm{d}t = \int_{\mathbb{R}}\lim_{j\to\infty}|\phi_j(t)|\,\mathrm{d}t = 0,$$

the last operation being valid since $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero on $\mathbb{R}$. We then have

$$\begin{aligned}|\theta_f(\phi_j)| &= \Big|\int_{\mathbb{R}} f(t)\phi_j(t)\,\mathrm{d}t\Big| \\ &\le \int_{\mathbb{R}} |f(t)\phi_j(t)|\,\mathrm{d}t \\ &\le \|f\|_p\|\phi_j\|_{p'},\end{aligned}$$

using Hölder's inequality, Lemma 6.7.51. Taking the limit as $j \to \infty$ gives the result.

(iii) This will be proved as Theorem 11.3.24.

(iv) This follows, using Exercise 6.6.2, from part (iii) and part (i) of Theorem 10.8.3.

■

Chapter 11

Convolution

The operation of convolution which we consider in this chapter is a remarkably useful one, and one which comes up in myriad ways. In this chapter itself we shall see how convolution can be used to generate nice approximations for general classes of signals. In ***missing stuff*** we shall see how convolutions arise in a natural way as representations for classes of systems. Convolution also arises in relation with the Fourier and Laplace transforms we consider in Chapters 12–14.***missing stuff*** This connection between convolution and transform theory is what makes transform theory so useful in the study of systems.

Despite this ubiquity of convolution, it is a subtle operation to understand. Indeed, perhaps *because* of the ubiquity of convolution, it is difficult to understand, as it is difficult to pinpoint *the* feature of convolution that makes it useful. Nonetheless, in this chapter we shall begin our study of convolution, giving some examples which, we hope might lead to come intuition about how convolution works. We shall also prove some of the basic results concerning convolution that will be useful in subsequent chapters.

Do I need to read this chapter? The basic definition of convolution should certainly be absorbed in reading these volumes. It is possible that the detailed results from Sections 11.2 and **??** can be read as needed in later chapters. Material from Section 11.3 provides a useful application of convolution, and for this reason it may be useful to read when one is making a first pass at this chapter. •

Contents

Section 11.1

Convolution of signals: Definitions, basic properties, and examples

This is an introductory section, defining the various sorts of convolution for signals that we will use, and giving some examples of when the operation of convolution is and is not defined. We postpone until Section 11.2 detailed results on when the operation of convolution of signals can be defined.

Do I need to read this section? If you are beginning to learn about convolution, this is where to begin. •

11.1.1 Convolution for aperiodic continuous-time signals

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We begin with signals defined on $\mathbb{R}$. For $a \in \mathbb{R}$ define the reparameterisation $\gamma_a\colon \mathbb{R} \to \mathbb{R}$ by $\gamma_a(t) = a - t$. Note that γ_a is the composition of a time reversal followed by a time shift by a. For $f\colon \mathbb{R} \to \mathbb{F}$ define $\gamma_a^* f$ by

$$\gamma_a^* f(t) = f \circ \gamma_a(t) = f(a - t).$$

With this notation we make the following result, recalling from Section 5.9.5 the notion of local integrability.

11.1.1 Definition (Convolution for aperiodic continuous-time signals) An ordered pair (f, g) of locally integrable $\mathbb{F}$-valued signals on $\mathbb{R}$ is ***convolvable*** if $(\gamma_t^* f)g \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{F})$ for almost every $t \in \mathbb{R}$. If (f, g) is convolvable then we denote

$$D(f, g) = \{t \in \mathbb{R} \mid (\gamma_t^* f)g \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{F})\}.$$

If (f, g) is convolvable then their ***convolution*** is the signal $f * g\colon \mathbb{R} \to \mathbb{F}$ defined by

$$f * g(t) = \int_{\mathbb{R}} (\gamma_t^* f) g \, \mathrm{d}\lambda$$

when $t \in D(f, g)$. If $t \notin D(f, g)$, we shall adopt the convention that $f * g(t) = 0$. •

11.1.2 Remark (Convolution only depends on "almost everywhere equal" equivalence class) By Proposition 5.7.11, if $f_1, f_2, g_1, g_2\colon \mathbb{R} \to \mathbb{F}$ are signals such that $f_1(t) = f_2(t)$ for almost every $t \in \mathbb{R}$ and $g_1(t) = g_2(t)$ for almost every $t \in \mathbb{R}$, then $(f_1, osig_1)$ is convolvable if and only if (f_2, g_2) is convolvable, and, if one of these pairs is convolvable, then $f_1 * g_1(t) = f_2 * g_2(t)$ for almost every $t \in \mathbb{R}$. For this reason, one can, and we very often will, think of convolution as mapping pairs of equivalence classes of signals to an equivalence class of signals using the equivalence relation where two signals are equivalent if they agree almost everywhere. Sometimes we will be careful to be explicit about when we are talking about equivalence classes, and sometimes we will not be so careful. •

Using the more familiar and penetrable Riemann integral notation for the Lebesgue integral, we have

$$f * g(t) = \begin{cases} \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s, & t \in D(f,g), \\ 0, & t \notin D(f,g). \end{cases}$$

We shall use this notation unless it is more convenient and/or clear to use the Lebesgue integral notation. Moreover, we shall often simply write

$$f * g(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s,$$

with the tacit understanding that this expression is to be applied only when $t \in D(f,g)$. In Exercise 11.1.1 the reader can show that there exists a convolvable pair (f,g) such that $D(f,g) \neq \mathbb{R}$.

In order to get some intuition about the operation of convolution, let us look at a simple concrete example.

11.1.3 Example (The mechanics of convolution) We consider two signals $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 1 + \frac{s}{2}, & s \in [-2,0], \\ 1 - s, & s \in (0,1], \\ 0, & \text{otherwise}, \end{cases} \qquad g(t) = \begin{cases} \frac{1}{2} + \frac{s}{2}, & s \in [-1,0], \\ \frac{1}{2} - \frac{s}{4}, & t \in (0,2], \\ 0, & \text{otherwise}. \end{cases}$$

We depict these signals in Figure 11.1 We note that $D(f,g) = \mathbb{R}$.

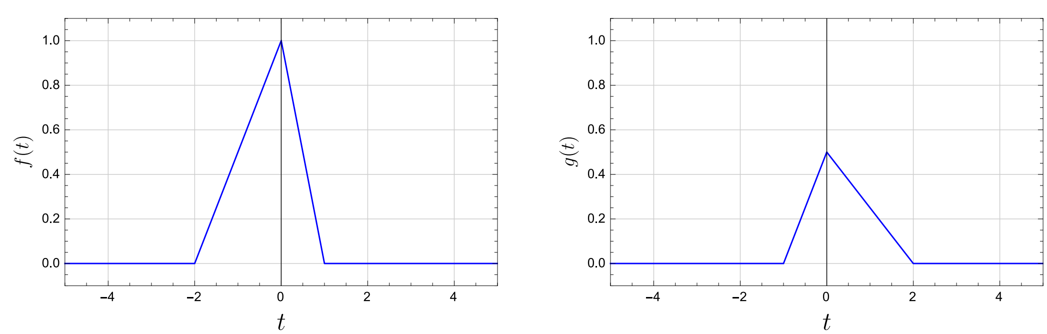


Figure 11.1 Two signals f (left) and g (right)

Let us first consider the character of the integrand of the convolution integral for various t. We show this in Figure 11.2. Note that the picture one should have in mind is of first time reversing the signal f and then "sliding it through g" starting at $-\infty$ and going to ∞. For times in the intersection of the supports of $\gamma_t f$ and g, the integrand at that time is the product of the two signals.

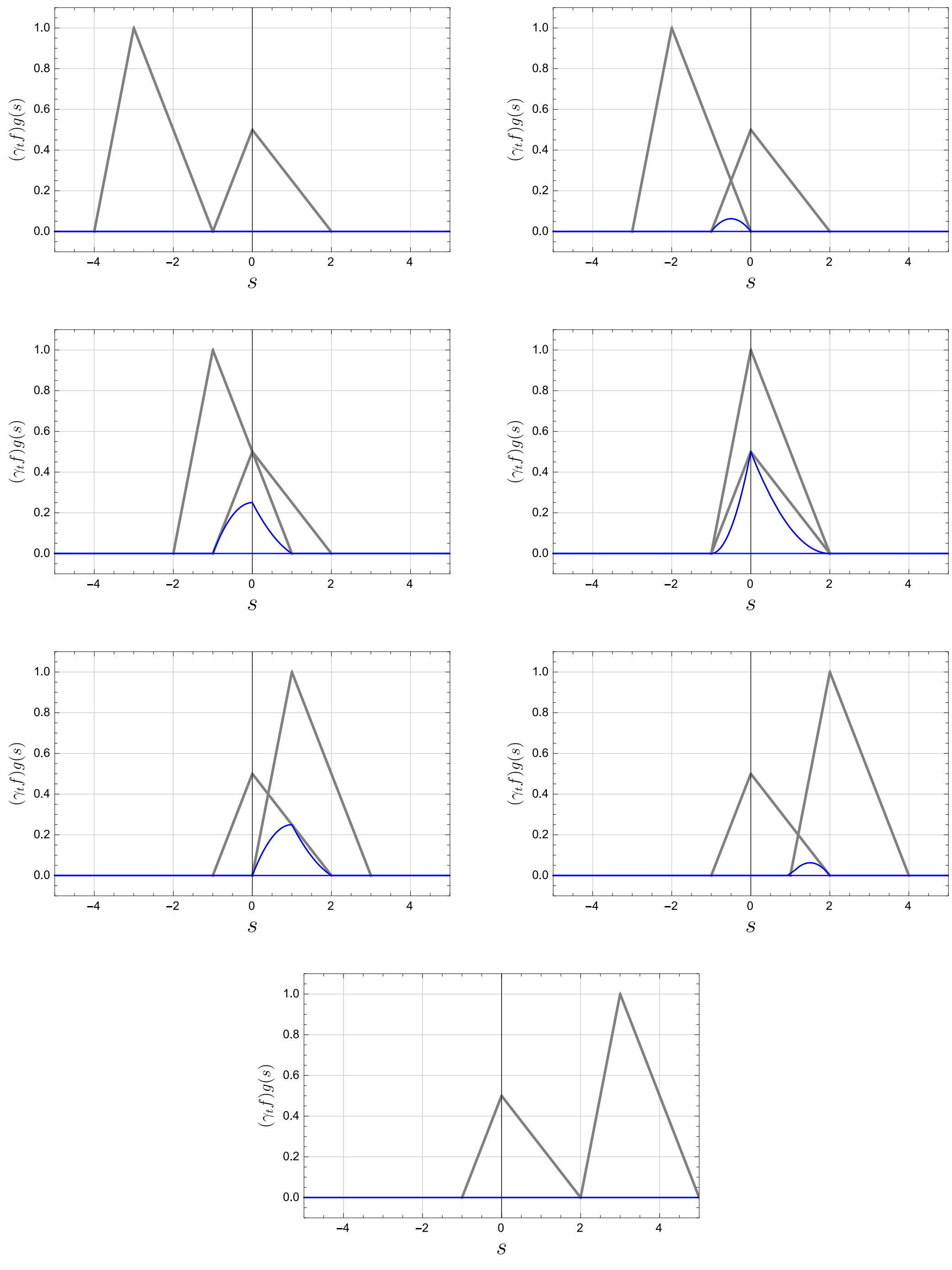


Figure 11.2 Integrand of convolution integral for signals from Figure 11.1 for $t \in \{-3, -2, -1, 0, 1, 2, 3\}$. In each plot, $\gamma_t f$ and g are shown in grey.

Next let us determine the convolution integral. The computation itself is merely tedious, and the result is

$$f * g(t) = \begin{cases} \frac{1}{24}(t+3)^3, & t \in [-3,-2], \\ \frac{1}{48}(-t^3+18t+30), & t \in (-2,-1], \\ \frac{1}{48}(-7t^3-18t^2+24), & t \in (-1,0], \\ \frac{1}{48}(7t^3-18t^2+24), & t \in (0,1], \\ \frac{1}{48}(t^3-18t+30), & t \in (1,2], \\ -\frac{1}{24}(t-3)^3, & t \in (2,3]. \end{cases}$$

We depict the convolution in Figure 11.3. A few comments are in order about the

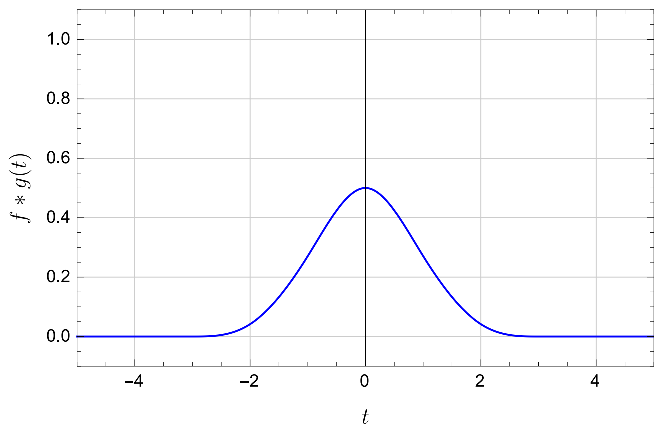


Figure 11.3 The convolution of the signals f and g from Figure 11.1

convolution here, and these reflect some truths about convolution in general.

1. The supports of f and g are "smeared" by convolution. That is, the support of $f * g$ is larger than either $\mathrm{supp}(f)$ or $\mathrm{supp}(g)$.
2. Each of the signals f and g is continuous, but not differentiable. However, one can verify that $f * g$ *is* differentiable. This reflects the fact that convolution has a "smoothing" property. •

Now that we have an example illustrating how convolution of signals, let us explore some basic properties of this operation. The following result is sometimes useful.

11.1.4 Proposition (Property of D(f, g)) *We have that* (f, g) *is convolvable if and only if* $(|\mathrm{f}|, |\mathrm{g}|)$ *is convolvable. Moreover, if* (f, g) *is convolvable, then* $\mathrm{D}(\mathrm{f}, \mathrm{g}) = \mathrm{D}(|\mathrm{f}|, |\mathrm{g}|)$.

Proof This is an immediate consequence, by Proposition 5.7.21, of the fact that $s \mapsto f(t-s)g(s)$ is integrable if and only if $s \mapsto |f(t-s)||g(s)|$ is integrable. ■

It is not easy to give a complete characterisation of all convolvable pairs. We dedicate Section 11.2 to describing some interesting subsets of convolvable pairs of signals. Also, it is not easy to describe in generality the character of a signal which is obtained by convolving two convolvable signals. However, it is useful to have the following property of a convolvable pair.

11.1.5 Theorem (Convolutions are locally integrable) *If* (f, g) *is a convolvable pair of signals, then* $\mathsf{f} * \mathsf{g}$ *is locally integrable.*

Proof We first begin with an observation, one which will be expanded upon and generalised in Section **??**. Let us consider, in the language of Proposition 5.7.65, the measures $\mu = f \cdot \lambda$ and $\nu = g \cdot \lambda$. These measures are, by Example **??–??**, absolutely continuous with respect to λ. By $\mu \times \nu$ denote the product measure on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Now consider the map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\Phi(\sigma, \tau) = \sigma + \tau$. On $\mathbb{R}$ we may consider the signed (if $\mathbb{F} = \mathbb{R}$) or complex (if $\mathbb{F} = \mathbb{C}$) measure $(\mu \times \nu)\Phi^{-1}$ which is the image of $\mu \times \nu$ under Φ (see Section 5.7.6). This measure on $\mathbb{R}$ is then an element of the topological dual of the continuous functions with compact support equipped with the ∞-norm (see ***missing stuff***). Moreover, if $h\colon \mathbb{R} \to \mathbb{F}$ is a continuous function with compact support,

$$\langle (\mu \times \nu)\Phi^{-1}; h\rangle = \langle \mu \times \nu; \Phi^* h\rangle.$$

Using the definition of the product measure, we can directly verify that

$$\langle (\mu \times \nu)\Phi^{-1}; h\rangle = \int_{\mathbb{R}^2} F_{f,g,h}\, \mathrm{d}\lambda_2,$$

where $F_{f,g,h}(\sigma, \tau) = h(\sigma + \tau) f(\tau) g(\sigma)$. Moreover, if $A \in \mathscr{L}(\mathbb{R})$ then

$$(\mu \times \nu)\Phi^{-1}(A) = \int_{\mathbb{R}^2} F_{f,g,A}\, \mathrm{d}\lambda_2,$$

where $F_{f,g,A}(\sigma, \tau) = \chi_A(\sigma + \tau) f(\tau) g(\sigma)$. We shall employ these relationships and the attendant constructions in the proof of the theorem and the corollary following.

By definition, both f and g are locally integrable. For $t \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ let us denote $t + S = \{t + x \mid x \in S\}$. Let us define $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi(s, t) = (s, t - s)$. If $A \in \mathscr{L}(\mathbb{R})$ then, recalling the notation $F_{f,g,A}$ from above, we have

$$F_{f,g,A} \circ \phi(s, t) = \chi_A(t) f(t - s) g(s).$$

Now let $N \in \mathscr{L}(\mathbb{R})$ have Lebesgue measure zero. Note that $s \mapsto F_{f,g,N} \circ \phi(s, t)$ is integrable if $t \in (\mathbb{R} \setminus N) \cup (D(f, g) \cap N)$, and so integrable for almost every $t \in \mathbb{R}$ since (f, g) is convolvable. Also, the map $t \mapsto F_{f,g,N} \circ \phi(s, t)$ is almost everywhere zero for every $s \in \mathbb{R}$, and so integrable for almost every $s \in \mathbb{R}$. If we define

$$h_{g,N}(\tau) = \int_{\mathbb{R}} \chi_{-\tau+N}\, g\, \mathrm{d}\lambda,$$

then we note that absolute continuity of the measure $g \cdot \lambda$ (see Example **??–??**) implies that $h_{g,N}(\tau) = 0$ for every $\tau \in \mathbb{R}$. Therefore,

$$\int_{\mathbb{R}} h_{g,N}\, \mathrm{d}\lambda = 0$$

for every $N \in \mathscr{L}(\mathbb{R})$ having zero Lebesgue measure. By Fubini's Theorem, whose hypotheses we have just verified, and the remarks at the beginning of the proof,

$$(\mu \times \nu)\Phi^{-1}(N) = \int_{\mathbb{R}^2} F_{f,g,N}\,\mathrm{d}\lambda_2 = \int_{\mathbb{R}} h_{g,N}\,\mathrm{d}\lambda = 0$$

for every set N of Lebesgue measure zero. Thus the measure $(\mu \times \nu)\Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure.

Now let $h\colon \mathbb{R} \to \mathbb{F}$ be continuous with compact support. Note that

$$F_{f,g,h} \circ \phi(s,t) = h(t) f(t-s) g(s).$$

By our remarks at the beginning of the proof we have

$$\langle (\mu \times \nu)\Phi^{-1}; h\rangle = \int_{\mathbb{R}^2} F_{f,g,h}\,\mathrm{d}\lambda_2 \in \mathbb{R}.$$

Since (f,g) is convolvable, $s \mapsto F_{h,f,g} \circ \phi(s,t)$ is integrable when

$$t \in D(f,g) \cup (\mathbb{R} \setminus \mathrm{supp}(h)).$$

In particular, this function is integrable for almost every $t \in \mathbb{R}$. Now consider the function $t \mapsto F_{h,f,g} \circ \phi(s,t)$. Since g is locally integrable, $g(s)$ is finite for almost every $s \in \mathbb{R}$. Therefore, $t \mapsto F_{h,f,g} \circ \phi(s,t)$ is integrable for almost every $s \in \mathbb{R}$ by Proposition 5.9.21. By the change of variable theorem, Theorem **??**, and Fubini's Theorem, whose hypotheses we just verified, we have

$$\int_{\mathbb{R}^2} F_{f,g,h}\,\mathrm{d}\lambda_2 = \int_{\mathbb{R}^2} F_{f,g,h} \circ \phi\,\mathrm{d}\lambda_2 = \int_{\mathbb{R}} h(f * g)\,\mathrm{d}\lambda,$$

which shows that

$$\langle (\mu \times \nu)\Phi^{-1}; h\rangle = \int_{\mathbb{R}} h(f * g)\,\mathrm{d}\lambda.$$

Thus $f * g$ is the Radon–Nikodym derivative of the absolutely continuous measure $(\mu \times \nu)\Phi^{-1}$ with respect to λ. By ***missing stuff***, $f * g$ is locally integrable. ■

11.1.6 Remark (Local integrability and convolution) The reader will observe that the proof of the preceding theorem, somewhat surprisingly, on some nontrivial measure theoretic developments. This is perhaps because convolution of signals is really a special case of convolution of measures, and some of the basic properties for convolutions of signals are most directly, and perhaps only, seen through the connection to convolution of measures. We shall examine the convolution of measures in Sections **??** and **??**. •

The following characterisation of convolution is useful for determining some of the properties of convolution.

11.1.7 Corollary (Characterisation of convolution) *For* $\mathrm{f}, \mathrm{g} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}; \mathbb{F})$ *the following statements are equivalent:*

(i) (f, g) *is convolvable;*

(ii) for every continuous signal $\mathrm{h}\colon \mathbb{R} \to \mathbb{F}$ *with compact support, it holds that*

$$\int_{\mathbb{R}^2} \mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}\,\mathrm{d}\lambda_2 \in \mathbb{R},$$

where $\mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}(\sigma, \tau) = \mathrm{h}(\sigma + \tau)\mathrm{f}(\tau)\mathrm{g}(\sigma)$.

Moreover, if (f, g) *is convolvable and if* $\mathrm{h}\colon \mathbb{R} \to \mathbb{F}$ *is continuous with compact support, then*

$$\int_{\mathbb{R}} \mathrm{h}(\mathrm{f} * \mathrm{g})\,\mathrm{d}\lambda = \int_{\mathbb{R}^2} \mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}\,\mathrm{d}\lambda_2.$$

Proof We continue using the notation from the proof of the theorem.

Suppose that (f, g) is convolvable and let $h\colon \mathbb{R} \to \mathbb{F}$ be continuous with compact support. Since (f, g) is convolvable, $s \mapsto F_{h,f,g} \circ \phi(s, t)$ is integrable when

$$t \in D(f, g) \cup (\mathbb{R} \setminus \mathrm{supp}(h)).$$

In particular, this function is integrable for almost every $t \in \mathbb{R}$. Now consider the function $t \mapsto F_{h,f,g} \circ \phi(s, t)$. The signals f and g are locally integrable. Then, as we saw in the proof of the preceding theorem, by Fubini's Theorem we have

$$\int_{\mathbb{R}} h(f * g)\,\mathrm{d}\lambda = \int_{\mathbb{R}^2} F_{f,g,h}\,\mathrm{d}\lambda_2.$$

Since $f * g$ is locally integrable by the preceding theorem, the integral on the left is finite by ***missing stuff***. This gives this part of the result.

Next suppose that $F_{f,g,h}$ is integrable with respect to λ_2 for every continuous signal h with compact support. With ϕ as above, it follows from the change of variable formula, Theorem **??**, that $|F_{f,g,h}| \circ \phi$ is also integrable with respect to λ_2 for every such h. Now, for a continuous compactly supported signal h, let

$$A_h = \{t \in \mathbb{R} \mid h(t) \neq 0\}.$$

By Fubini's Theorem, there exists a set $Z_h \subseteq A_h$ of measure zero such that, if $t \in A_h \setminus Z_h$, the function $s \mapsto |F_{f,g,h} \circ \phi(t, s)|$ is integrable. Now, for $j \in \mathbb{Z}$, define

$$h_j(t) = \begin{cases} 1, & t \in [j, j+1], \\ t - (j-1), & t \in [j-1, j), \\ -t + (j+2), & t \in (j+1, j+2], \\ 0, & \text{otherwise}. \end{cases}$$

Note that for each $t \in \mathbb{R}$ there exists $j_t \in \mathbb{Z}$ (not necessarily unique, but no matter) such that $h_{j_t}(t) = 1$. Moreover, for each $j \in \mathbb{Z}$, h_j is continuous with compact support. Note that the set $Z = \cup_{j \in \mathbb{Z}} Z_{h_j}$ has measure zero, being the countable union of sets of measure zero. Moreover, if $t \in \mathbb{R} \setminus Z$ then we have that $s \mapsto |F_{f,g,h_{j_t}} \circ \phi(t, s)|$ is integrable. However,

$$|F_{f,g,h_{j_t}} \circ \phi(t, s)| = |f(t-s)g(s)|,$$

showing that if $t \in \mathbb{R} \setminus Z$ then $t \in D(f, g)$. Thus (f, g) is convolvable. ■

In more familiar notation, the preceding result says that (f, g) is convolvable if and only if, for every compactly supported continuous signal h it holds that

$$\int_{\mathbb{R}} h(t) f * g(t)\,\mathrm{d}t = \iint_{\mathbb{R}^2} h(\sigma + \tau) f(\tau) g(\sigma)\,\mathrm{d}\sigma \mathrm{d}\tau.$$

Let us prove a result which makes somewhat precise the "smearing" of supports resulting from convolution that we observed in Example 11.1.3. In this result we make use of the Definition 5.9.4 which gives the support of a measurable signal.

11.1.8 Proposition (Support of convolution) *If* (f, g) *is a pair of convolvable* $\mathbb{F}$*-valued signals on* $\mathbb{R}$ *then*

$$\mathrm{supp}(\mathrm{f} * \mathrm{g}) \subseteq \mathrm{cl}(\mathrm{supp}(\mathrm{f}) + \mathrm{supp}(\mathrm{g})),$$

where $\mathrm{supp}(\mathrm{f}) + \mathrm{supp}(\mathrm{g}) = \{\mathrm{s} + \mathrm{t} \mid \mathrm{s} \in \mathrm{supp}(\mathrm{f}),\ \mathrm{t} \in \mathrm{supp}(\mathrm{g})\}$*. Moreover, the above inclusion is equality of sets if* $\mathrm{f}(\mathrm{t})$ *and* $\mathrm{g}(\mathrm{t})$ *are nonnegative for almost every* $\mathrm{t} \in \mathbb{R}$*.*

Proof If $\mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g)) = \mathbb{R}$ the first assertion holds trivially. So we suppose this not to be the case. Let $t \in \mathbb{R} \setminus \mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$ and let U be a neighbourhood of t contained in $\mathbb{R} \setminus \mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$, such a neighbourhood existing since $\mathbb{R} \setminus \mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g))$ is open. Let $h\colon \mathbb{R} \to \mathbb{F}$ be a continuous function with $\mathrm{supp}(h) \subseteq U$. (Such a function exists since U necessarily contains an open interval, and it is easy to explicitly define a continuous function with support contained in a prescribed interval; think of a function whose graph is triangular.) Then we have that, borrowing the notation of Corollary 11.1.7,

$$\int_{\mathbb{R}} h(f * g)\,\mathrm{d}\lambda = \int_{\mathbb{R}^2} F_{f,g,h}\,\mathrm{d}\lambda_2 = \int_{\mathrm{supp}(f)\times\mathrm{supp}(g)} F_{f,g,h}\,\mathrm{d}\lambda_2, \tag{11.1}$$

using the definition $F_{f,g,h}(\sigma, \tau) = h(\sigma+\tau) f(\tau) g(\sigma)$. However, if $(\sigma, \tau) \in \mathrm{supp}(f)\times\mathrm{supp}(g)$ it follows by assumption that $h(\sigma + \tau) = 0$, and so $F_{f,g,h}(\sigma, \tau) = 0$ as well. Thus the integrals from (11.1) vanish for every continuous function h with support in U. It follows from ***missing stuff*** that $U \subseteq \mathbb{R} \setminus \mathrm{supp}(f * g)$. Thus every open subset of $\mathbb{R} \setminus \mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$ is contained in $\mathbb{R} \setminus \mathrm{supp}(f * g)$. Equivalently, every closed subset of $\mathrm{supp}(f * g)$ is contained in $\mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$, which gives the first part of the result.

For the second assertion, note that if f and g are almost everywhere nonnegative, then so is $f * g$, being defined as the integral of two almost everywhere nonnegative signals. Let $U \subseteq \mathbb{R}$ be open and with the property that $f * g(t) = 0$ for almost every $t \in U$. If the only such open set is the empty set then $f * g$ is almost everywhere nonzero, and so almost everywhere positive. In this case the second assertion holds trivially. Thus we suppose that there exists a nonempty open set U such that $f * g(t) = 0$ for almost every $t \in U$. Then we let $K \subseteq U$ be a nonempty compact set and let $L \subseteq U$ be a compact set such that $K \subset L$. By Urysohn's Lemma, Theorem **??**, let $h\colon \mathbb{R} \to [0, 1]$ have compact support and have the property that $h(t) = 1$ for $t \in K$ and $h(t) = 0$ for $t \in \mathbb{R} \setminus L$. It follows that $h(t) = 0$ for $t \in \mathbb{R} \setminus U$. We, therefore, have

$$\int_{\mathbb{R}} h(f * g)\,\mathrm{d}\lambda = 0.$$

Let $H\colon \mathbb{R}^2 \to [0,1]$ be defined by $H(\sigma,\tau) = h(\sigma+\tau)$. By (11.1) it follows that the open set $H^{-1}((\frac{1}{2},\infty))$ (open since H is continuous) does not intersect $\mathrm{supp}(f)\times\mathrm{supp}(g)$. We claim that this implies that $K\cap\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g)) = \emptyset$. Indeed, if $t \in K\cap\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g))$ then $t = \sigma + \tau$ for $\sigma \in \mathrm{supp}(f)$ and $\tau \in \mathrm{supp}(g)$ and $h(t) = H(\sigma + \tau) = 1$. But then $(\sigma,\tau) \in \mathrm{supp}(f)\times\mathrm{supp}(g) \cap H^{-1}((\frac{1}{2},\infty))$, giving a contradiction. Thus we conclude by ***missing stuff*** that if $K \subseteq \mathbb{R}\setminus\mathrm{supp}(f*g)$ is compact then $K \subseteq \mathbb{R}\setminus\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g))$. This shows that $\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g)) \subseteq \mathrm{supp}(f*g)$ in this case, as is desired. ■

Let us first verify that convolution is commutative and distributive.

11.1.9 Proposition (Algebraic properties of convolution) *If* $\mathsf{f},\mathsf{g},\mathsf{h}\colon \mathbb{R}\to\mathbb{F}$ *are locally integrable then the following statements hold:*

(i) if (f,g) *is convolvable, then* (g,f) *is convolvable and* $\mathsf{f}*\mathsf{g} = \mathsf{g}*\mathsf{f}$*;*

(ii) if (f,g) *and* (f,h) *are convolvable, then* $(\mathsf{f},\mathsf{g}+\mathsf{h})$ *is convolvable and* $\mathsf{f}*(\mathsf{g}+\mathsf{h}) = \mathsf{f}*\mathsf{g}+\mathsf{f}*\mathsf{h}$.

Proof (i) Let $t \in D(f,g)$. Note that $\gamma_t\colon \mathbb{R}\to\mathbb{R}$ is a differentiable bijection that is monotonically decreasing. Moreover, $\gamma_t'(s) = -1$. Therefore, by Theorem 5.9.36 it follows that $((\gamma_t^*f)g)\circ\gamma_t$ is integrable. Moreover,

$$((\gamma_t^*f)g)\circ\gamma_t(s) = (\gamma_t^*f)(t-s)g(t-s) = f(s)g(t-s) = ((\gamma_t^*g)f)(s).$$

Thus $t \in D(g,f)$. Reversing the argument shows that if $t \in D(g,f)$ then $t \in D(f,g)$. Thus $D(f,g) = D(g,f)$. By Theorem 5.9.36 we also have

$$\int_{\mathbb{R}} (\gamma_t^*f)g\,\mathrm{d}\lambda = \int_{\mathbb{R}} (\gamma_t^*g)f\,\mathrm{d}\lambda,$$

which is the result.

(ii) This is a direct consequence of linearity of the integral, Proposition 5.7.17. ■

Thus, for a convolvable pair (f,g) we have (g,f) also convolvable and, moreover,

$$f*g(t) = g*f(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s = \int_{\mathbb{R}} f(s)g(t-s)\,\mathrm{d}s.$$

This is a formula that we shall employ without mention in our future uses of convolution.

If one is looking for the binary operation of convolution to have the properties of an algebra, one might observe that associativity is missing from the list of properties in the preceding result. This is because associativity does not always hold.

11.1.10 Example (Convolution is not generally associative) Let $w\colon \mathbb{R}\to\mathbb{R}$ be given by

$$w(t) = \begin{cases} 1-\cos t, & t \in [0,2\pi], \\ 0, & \text{otherwise}, \end{cases}$$

and define $f,g,h\colon \mathbb{R}\to\mathbb{R}$ by

$$f(t) = 1, \qquad g(t) = \begin{cases} \sin t, & t\in[0,2\pi], \\ 0, & \text{otherwise}, \end{cases} \qquad h(t) = \int_{-\infty}^{t} w(\tau)\,\mathrm{d}\tau.$$

Note that w is differentiable everywhere except at 0 and 2π, and that its derivative at all points of differentiability is $w'(t) = g(t)$. Thus we write $w' = g$ with the understanding that this holds except at 0 and 2π. As we will be computing integrals of these signals, this will not make a difference.

We then compute

$$f * g(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s = \int_0^{2\pi} \sin s\,\mathrm{d}s = 0$$

and

$$\begin{aligned} g * h(t) &= \int_{\mathbb{R}} g(t-s)h(s)\,\mathrm{d}s = \int_{\mathbb{R}} w'(t-s)\Big(\int_0^s w(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} w(t-s)w(s)\,\mathrm{d}s = w * w(t), \end{aligned}$$

using integration by parts. Note that w is strictly positive on $(0, 2\pi)$ and zero elsewhere. Therefore,

$$w * w(t) = \int_{\mathbb{R}} w(t-s)w(s)\,\mathrm{d}s = \int_0^{2\pi} w(t-s)w(s)\,\mathrm{d}s$$

is strictly positive whenever the set $\{t - s \mid s \in (0, 2\pi)\}$ intersects $(0, 2\pi)$, i.e., whenever $t \in (0, 4\pi)$. Thus $w * w$ is strictly positive on $(0, 4\pi)$ and zero elsewhere. Thus we have $(f * g) * h = 0$ and

$$f * (g * h)(t) = \int_{\mathbb{R}} f(t-s)w * w(s)\,\mathrm{d}s = \int_{\mathbb{R}} w * w(s)\,\mathrm{d}s,$$

giving $f*(g*h)$ as being a nonzero constant signal. In particular, $(f*g)*h \neq f*(g*h)$. •

Despite the preceding result, we shall see that there are many classes of signals for which, when the convolution operation is restricted to them, it is associative. We shall see instances of this in Section 11.2.

Let us close this section by considering an important property of convolution: that of continuity of the convolved signal. We shall see in Section 11.2 that, in many cases, the convolution of signals is a continuous signal. However, this is not *always* the case, as the following example shows.

11.1.11 Example (A convolvable pair whose convolution is discontinuous) We let $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t^{-1/2}, & t \in (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

One easily verifies that (f, f) is convolvable, that $D(f, f) = \mathbb{R}$, and that

$$f * f(t) = \begin{cases} \pi, & t \in (0, 1], \\ 2(\csc^{-1}(\sqrt{t}) - \tan^{-1}(\sqrt{t-1})), & t \in (1, 2), \\ 0, & \text{otherwise,} \end{cases}$$

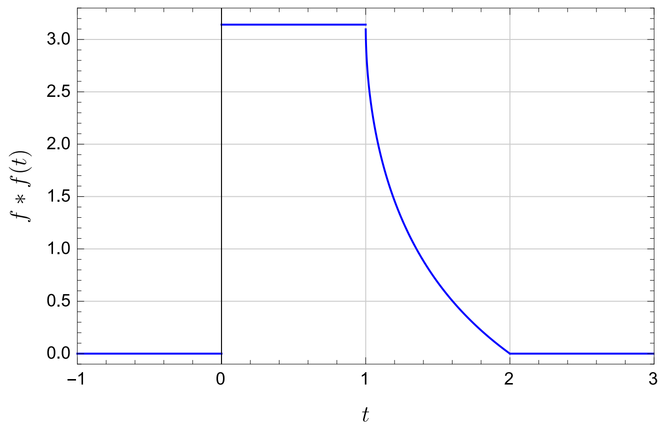


Figure 11.4 A convolution $f * f$ of unbounded signals that is discontinuous

recalling from Section 3.6.4 the definitions of $\csc^{-1}$ and $\tan^{-1}$. In Figure 11.4 we depict the convolution $f * f$, and we see that it is discontinuous at $t = 0$. •

The preceding example may make one think that at times where a signal becomes unbounded, this will lead to the convolution being discontinuous. The next example shows that the truth is more subtle than this.

11.1.12 Example (An unbounded signal with a continuous convolution with itself) Here we take $f\colon \mathbb{R} \to \mathbb{R}$ to be defined by

$$g(t) = \begin{cases} t^{-1/4}, & t \in (0,1], \\ 0, & \text{otherwise}. \end{cases}$$

The computation of the convolution integral here involves special functions that are not quite elementary, including but not limited to, the Γ-function from Exercise **??**. However, once one swallows this, one sees that (f, f) is convolvable with $D(f, f) = \mathbb{R}$. In Figure 11.5 we show depict the convolution $g * g$, and we note that it is continuous, despite f being unbounded. •

As we shall see in Section 11.2, in particular Theorem 11.2.8, the difference between the preceding two examples is that in Example 11.1.11 the signal is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$, whereas the signal from Example 11.1.12 is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{F})$.

This is all we shall say in general about convolution for signals defined on $\mathbb{R}$. In Section 11.2 we shall give many more important results on convolution in this setting, taking into account particular collections of signals.

11.1.2 Convolution for periodic continuous-time signals

Again we let $\mathbb{F} \in \{\mathbb{R};\mathbb{C}\}$. And again we consider signals defined on $\mathbb{R}$, but now we ask that the signals we consider be T-periodic for a fixed $T \in \mathbb{R}_{>0}$. It will be

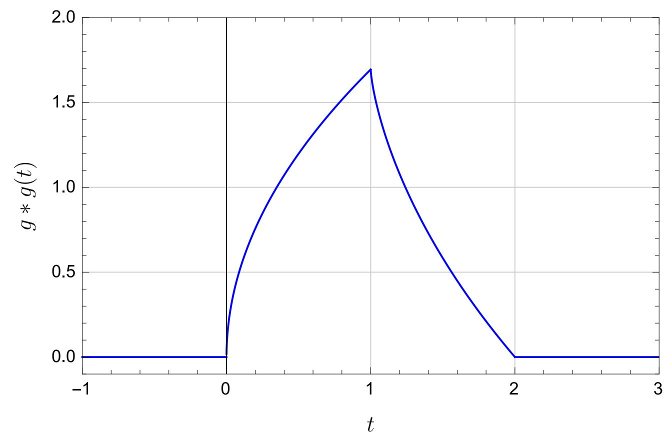


Figure 11.5 A convolution $g * g$ of unbounded signals that is continuous

convenient in this section to have at hand the notion of a T-periodic set $S \subseteq \mathbb{R}$, by which we mean that $\{T + t \mid t \in S\} = S$.

We still denote $\gamma_a : \mathbb{R} \to \mathbb{R}$ by $\gamma_a(t) = a - t$ and note that if $f : \mathbb{R} \to \mathbb{F}$ is T-periodic then $\gamma_a^* f$ is also T-periodic. Thus we can make the following definition.

11.1.13 Definition (Convolution for T-periodic continuous-time signals) An ordered pair (f, g) of signals from $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ is ***periodically convolvable*** if $(\gamma_t^* f)g|[0, T] \in \mathsf{L}^{(1)}([0, T]; \mathbb{F})$ for almost every $t \in \mathbb{R}$. If (f, g) is convolvable then we denote

$$D(f, g) = \{t \in [0, T] \mid (\gamma_t^* f)g|[0, T] \in \mathsf{L}^{(1)}([0, T]; \mathbb{F})\}.$$

If (f, g) is periodically convolvable then their ***periodic convolution*** is the signal $f * g : \mathbb{R} \to \mathbb{F}$ defined by

$$f * g(t) = \int_{[0,T]} (\gamma_t^* f)g \, \mathrm{d}\lambda_{[0,T]}$$

when $t \in D(f, g)$. If $t \notin D(f, g)$, we shall adopt the convention that $f * g(t) = 0$. •

Of course, Remark 11.1.2 applies equally well here, and so periodic convolution can be thought of as mapping pairs of equivalence classes of signals to an equivalence class of signals under the equivalence of almost everywhere equality.

Using Riemann integral notation, the periodic convolution of periodic signals can be written as

$$f * g(t) = \begin{cases} \int_0^T f(t - s)g(s) \, \mathrm{d}s, & t \in D(f, g), \\ 0, & t \notin D(f, g). \end{cases}$$

Moreover, we shall often simply write

$$f * g(t) = \int_0^T f(t - s)g(s) \, \mathrm{d}s,$$

with the understanding that this holds only almost everywhere.

11.1.14 Remarks (On periodic convolution of periodic signals)

1. By Lemma 8.3.5 we can consider the periodic convolution of T-periodic signals or of signals whose value at $t+T$ is equal to their value at t for *almost every* $t \in \mathbb{R}$. We shall frequently make use of this lack of distinction without mention.
2. For aperiodic convolution we required signals to be locally integrable. For periodic convolution, local integrability demands that signals be integrable over each period. Thus the domain of convolution in this case is clearly defined, and it is $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.
3. Note that there can be no essential ambiguity between which convolution is meant for periodic signals since the notion of convolution from Section 11.1.1 only exists for T-periodic signals when one of the signals is zero; see Exercise 11.1.3.
4. In our definitions above, the integration is performed over the integral $[0,T]$. The definition, however, is independent of particular the interval of length T over which integration is performed. We shall sometimes use this fact to change the interval of integration, frequently to $[-\frac{T}{2},\frac{T}{2}]$. •

Let us give an example of a periodic convolution to see how it works.

11.1.15 Example (The mechanics of periodic convolution) We consider two 1-periodic signals $f, g_N \colon \mathbb{R} \to \mathbb{R}$ defined on $(-\frac{1}{2},\frac{1}{2}]$ by

$$f(t) = \begin{cases} 0, & t \in (-\frac{1}{2},-\frac{1}{4}], \\ 1, & t \in (-\frac{1}{4},\frac{1}{4}], \\ 0, & t \in (\frac{1}{4},\frac{1}{2}], \end{cases} \qquad g_N(t) = \begin{cases} \frac{\sin(2\pi N t)}{\pi t}, & t \neq 0, \\ 2N, & t = 0. \end{cases}$$

Here we think of N as a parameter in $\mathbb{Z}_{>0}$. We plot the graphs of f and g_N for various N in Figure 11.6. One can verify that $f, g_N)$ is periodically convolvable with $D(f,g) = \mathbb{R}$. In Figures 11.7, 11.8, and 11.9 we show the integrands for various t's and N's, in order to try to get some feeling for what is happening with the convolution integral. The periodic convolution itself is shown in Figure 11.10. The closed-form expression for the convolution in this case is only given in terms of special functions we have not introduced; thus we do not provide these expressions, only plotting the results.

Let us make some comments about these periodic convolutions.

1. Let us first make some comments about signal g_N for various N. As N increases these signals become more "focused" around $t = 0$. That is, the values of the signal around $t = 0$ grow large compared to the values away from $t = 0$ as $N \to \infty$. As $N \to \infty$, the signals g_N exhibit some oscillatory behaviour whose frequency becomes larger.
2. Now let us compare f with $f * g_N$ as N varies. In some sense, as $N \to \infty$, $f * g_N$ approximates f. Let us make some observations about the nature of this approximation.

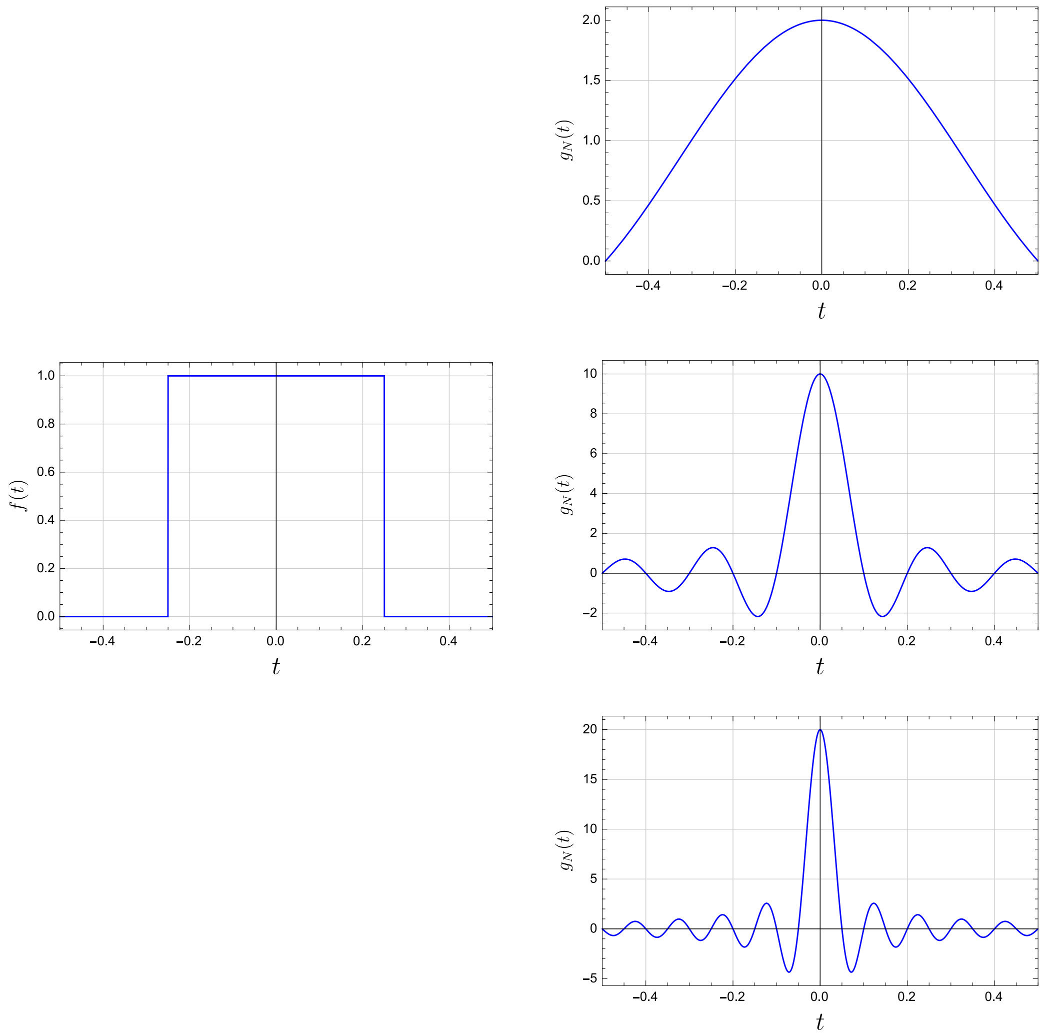


Figure 11.6 Two 1-periodic signals f (left) and g_N (right top, middle, and bottom), the latter for $N \in \{1, 5, 10\}$

(a) The approximation of f by $f * g_N$ is by infinitely differentiable signals for each N, despite the act that f is itself discontinuous.

(b) Away from the points of discontinuity for f, the approximation by $f * g_N$ appear to get better as $N \to \infty$.

(c) Around points of discontinuity of f, the approximation is quite rough. Looking at the integrands from Figures 11.7, 11.8, and 11.9, we can get some hints as to why this might be. There we see that as the point of discontinuity of $\gamma_t^* f$ passes through $t = 0$ the convolution picks up the oscillatory behaviour of g_N. This effect is often called "ringing." It is a little difficult to be precise about this, but after awhile one develops some intuition.

The sequence of signals $(g_N)_{N \in \mathbb{Z}_{>0}}$ that we see here will be very important to us in

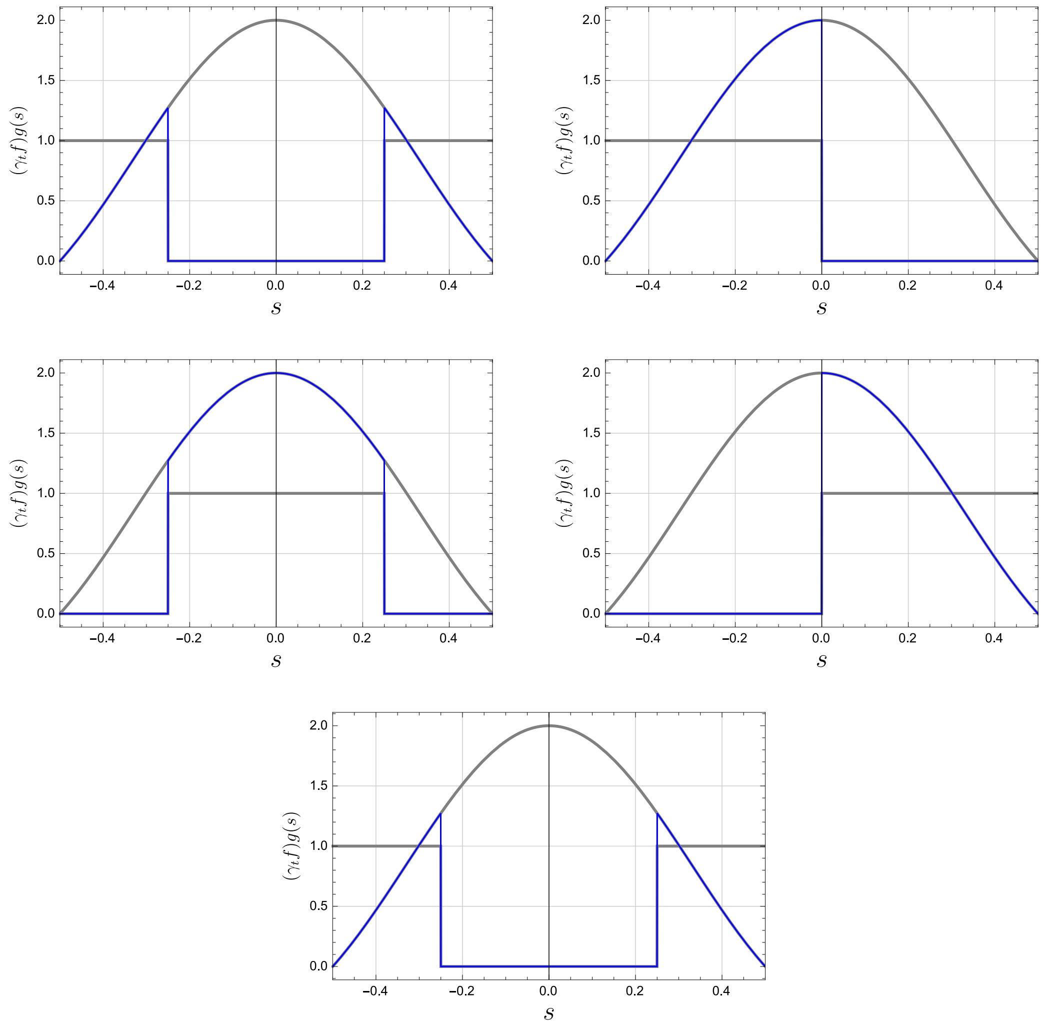


Figure 11.7 Integrand of periodic convolution integral for signals from Figure 11.6 for $N = 1$ and $t \in \{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}\}$. In each plot $\gamma_t f$ and g are shown in grey.

Chapter 12, and we shall see there why the periodic convolution integrals above are useful. •

Now let us explore the basic properties of periodic convolution. The pattern here follows that for the aperiodic convolution from the preceding section. Thus we skip or sketch proofs that mirror their counterparts we have already seen.

As for aperiodic signals, we have the following result.

11.1.16 Proposition (Property of D(f, g)) *We have that* (f, g) *is periodically convolvable if and only if* $(|\mathsf{f}|, |\mathsf{g}|)$ *is periodically convolvable. Moreover, if* (f, g) *is periodically convolvable, then* $\mathrm{D}(\mathsf{f}, \mathsf{g}) = \mathrm{D}(|\mathsf{f}|, |\mathsf{g}|)$.

The periodic convolution of two periodically convolvable signals is particularly

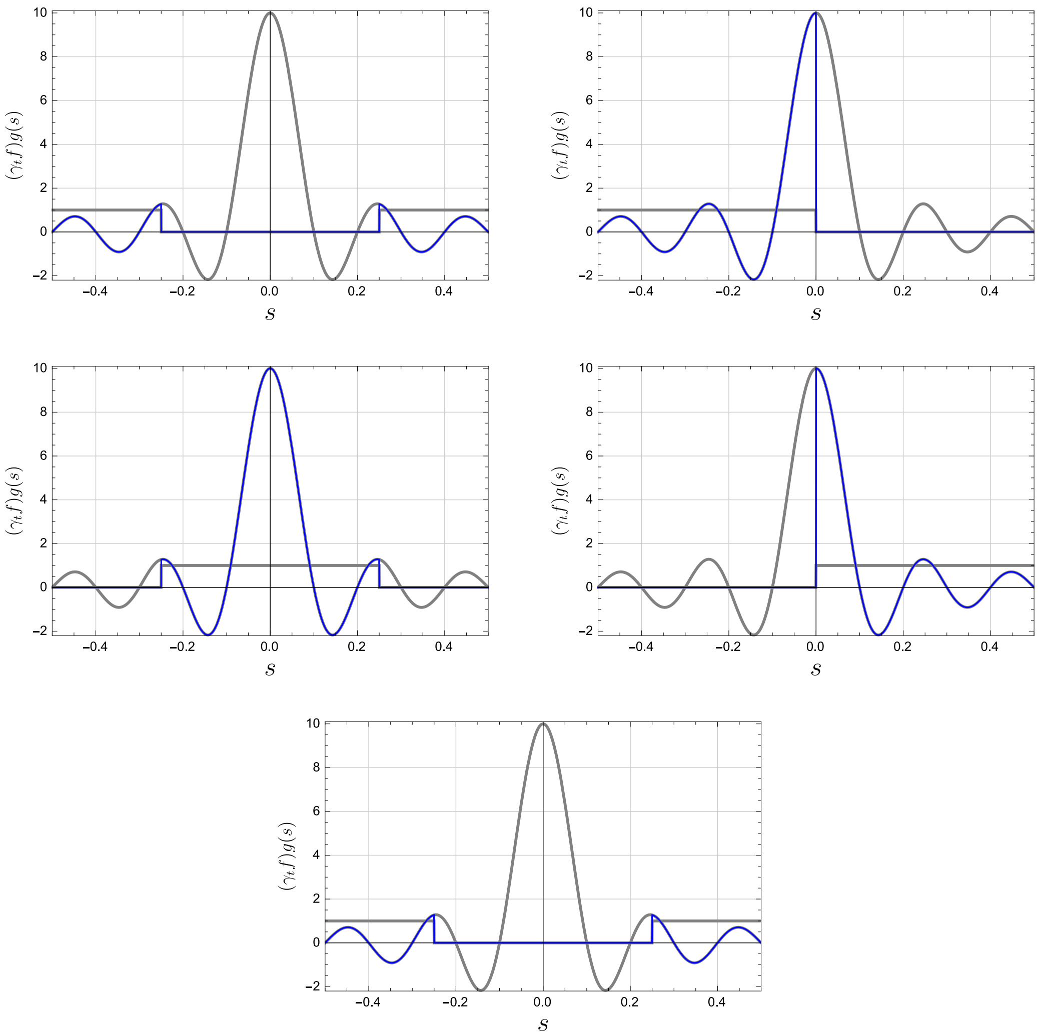


Figure 11.8 Integrand of periodic convolution integral for signals from Figure 11.6 for $N = 5$ and $t \in \{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}\}$. In each plot $\gamma_t f$ and g are shown in grey.

nice. We also show that the notion of periodic convolvability is vacuous when the signals being convolved are in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, as we assume.

11.1.17 Theorem (Periodic convolutions are periodic and integrable) *If* $\mathrm{f}, \mathrm{g} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *then* (f, g) *is convolvable and* $\mathrm{f} * \mathrm{g} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$.

Proof First we show that $f*g(t)$ is defined for almost every $t \in \mathbb{R}$ and is integrable over any period. Let us define $F_{f,g}\colon \mathbb{R}^2 \to \mathbb{F}$ by $F_{f,g}(s,t) = f(\tau)g(\sigma)$. If we take $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be defined by $\phi(s,t) = (s, t-s)$ then $F_{f,g} \circ \phi(s,t) = f(t-s)g(s)$. Since f and g are locally integrable, by Corollary 5.8.8 $F_{f,g}$ is also locally integrable. By the change of variable

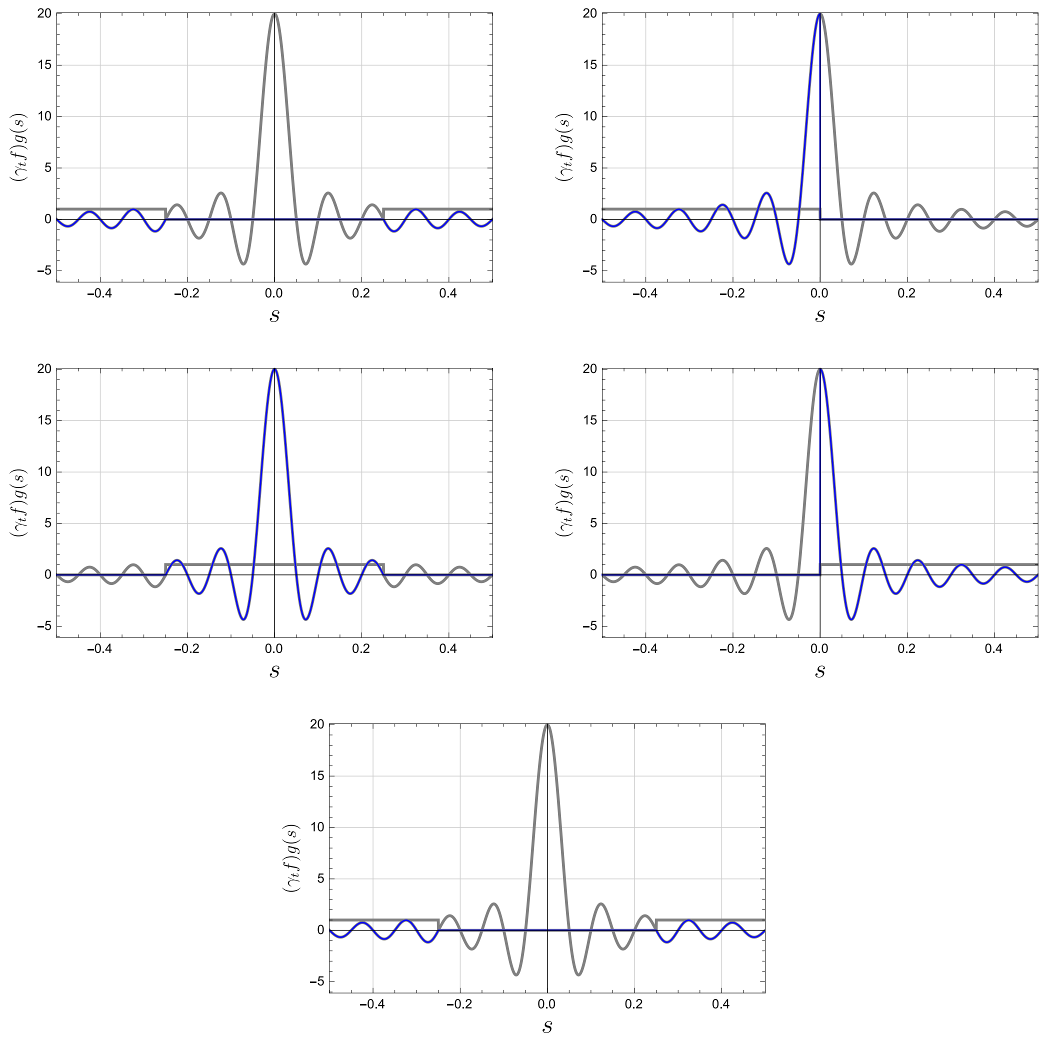


Figure 11.9 Integrand of periodic convolution integral for signals from Figure 11.6 for $N = 10$ and $t \in \{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}\}$. In each plot $\gamma_t f$ and g are shown in grey.

formula, Theorem **??**, $F_{f,g} \circ \phi$ is also locally integrable. Therefore,

$$\int_{[0,T]\times[0,T]} |F_{f,g}| \, d\lambda_2 \in \mathbb{R}.$$

By Fubini's Theorem we then have that $s \mapsto f(t-s)g(s)$ in integrable over $[0,T]$ for almost every $t \in [0,T]$ and that $f * g|[0,T] \in \mathsf{L}^{(1)}([0,T];\mathbb{F})$.

We next claim that $D(f,g)$ is T-periodic. Indeed, if $s \mapsto f(t-s)g(s)$ is integrable, then, since $f(t+T-s) = f(t-s)$ (i.e., $\gamma_{t+T}f = \gamma_t f$), $s \mapsto f(t+T-s)g(s)$ is integrable.

Finally we show that $f * g$ is T periodic. Let $t \in \mathbb{R}$. First suppose that $t \in D(f,g)$. Then,

$$f * g(t+T) = \int_{[0,T]} (\gamma^*_{t+T} f) g \, d\lambda = \int_{[0,T]} (\gamma^*_t f) f \, d\lambda = f * g(t).$$

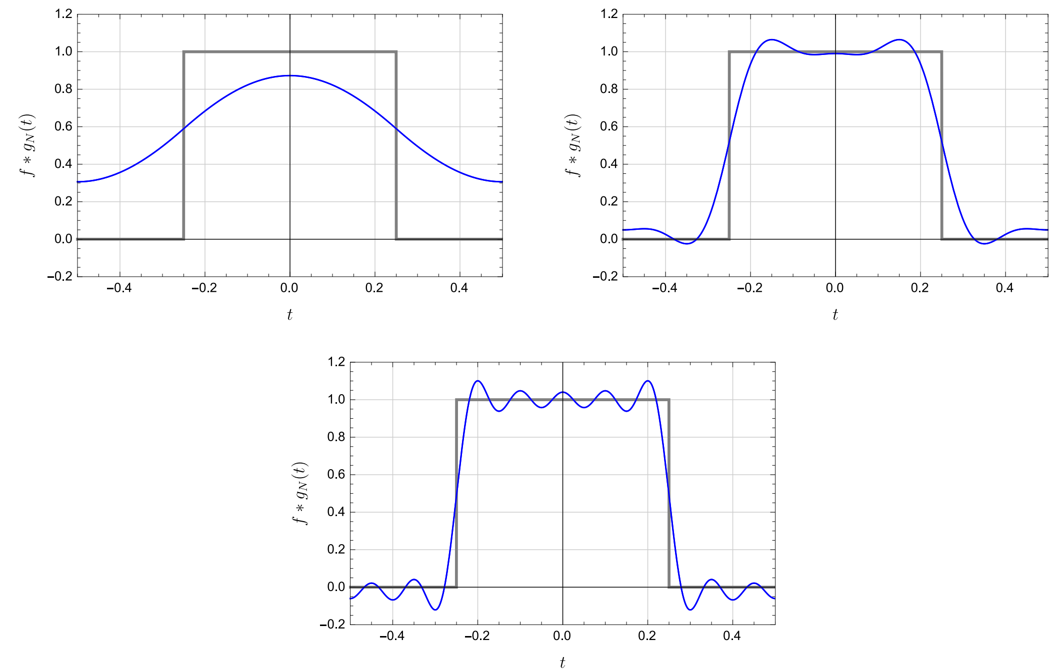


Figure 11.10 The periodic convolution of the signals f and g_N from Figure 11.6 for $N \in \{1, 5, 10\}$. The signal f is shown in grey.

Also, if $t \notin D(f, g)$ then $t + T \notin D(f, g)$ and so

$$f * g(t + T) = 0 = f * g(t),$$

giving the result. ■

The theorem has the following useful corollary.

11.1.18 Corollary (Characterisation of periodic convolution) *For* $\mathsf{f}, \mathsf{g} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R}; \mathbb{F})$ *we have*

$$\int_{[0,\mathrm{T}]} \mathrm{h}(\mathrm{f} * \mathrm{g})\, \mathrm{d}\lambda = \int_{[0,\mathrm{T}]\times[0,\mathrm{T}]} \mathrm{F}_{\mathrm{f,g,h}}\, \mathrm{d}\lambda_2$$

for every T*-periodic continuous signal* $\mathrm{h}\colon \mathbb{R} \to \mathbb{F}$*, where* $\mathrm{F}_{\mathrm{f,g,h}}(\sigma, \tau) = \mathrm{h}(\sigma + \tau)\mathrm{f}(\tau)\mathrm{g}(\sigma)$*.*

Proof If $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\phi(s, t) = (s, t - s)$ then $F_{f,g,h} \circ \phi(s, t) = h(t) f(t - s) g(s)$. Define

$$P = \{(s, t) \in \mathbb{R}^2 \mid \phi(s, t) \in [0, T] \times [0, T]\};$$

see Figure 11.11. By the change of variables theorem we have

$$\int_{[0,T]\times[0,T]} F_{f,g,h}\, \mathrm{d}\lambda_2 = \int_P F_{f,g,h} \circ \phi\, \mathrm{d}\lambda_2.$$

By Fubini's Theorem we have

$$\int_P F_{f,g,h} \circ \phi\, \mathrm{d}\lambda_2 = \int_{[0,T]} g\Big(\int_{s+[0,T]} G_{f,h}\, \mathrm{d}\lambda\Big)\, \mathrm{d}\lambda,$$

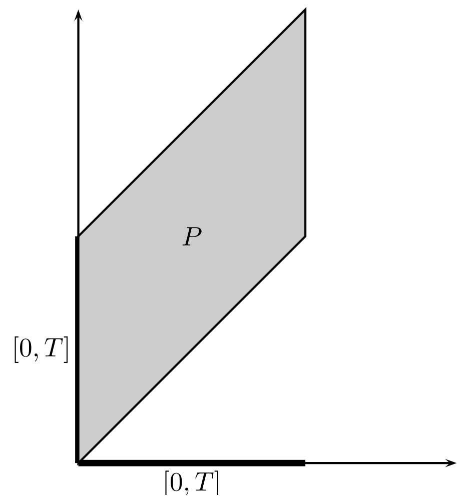


Figure 11.11 The change of variable domain for the proof of Corollary 11.1.18

where $G_{f,h}(s,t) = h(t)f(t-s)$. Periodicity of f and h then ensures that

$$\int_{s+[0,T]} G_{f,h}\,\mathrm{d}\lambda = \int_{[0,T]} G_{f,h}\,\mathrm{d}\lambda,$$

so giving

$$\int_P F_{f,g,h} \circ \phi\,\mathrm{d}\lambda_2 = \int_{[0,T]} g\Big(\int_{[0,T]} G_{f,h}\,\mathrm{d}\lambda\Big)\,\mathrm{d}\lambda.$$

Another application of Fubini's Theorem gives

$$\int_{[0,T]} g\Big(\int_{[0,T]} G_{f,h}\,\mathrm{d}\lambda\Big)\,\mathrm{d}\lambda = \int_{[0,T]} h(f * g)\,\mathrm{d}\lambda,$$

which gives the result. ■

As with the convolution for aperiodic signals from Section 11.1.1, we can say something about the support of the convolution of two T-periodic signals. In order to do this, it is convenient to define the map $\phi_T\colon \mathbb{R} \to [0,T)$ by noting that if $t \in \mathbb{R}$ then $t - kT \in [0,T)$ for some unique $k \in \mathbb{Z}$. We then define $\phi_T(t) = t - kT$.

11.1.19 Proposition (Support of periodic convolution) *If* (f, g) *is a pair of* T*-periodic* $\mathbb{F}$*-valued periodically convolvable signals, then*

$$(\mathrm{supp}(\mathrm{f} * \mathrm{g}) \cap [0,\mathrm{T})) \subseteq \phi_{\mathrm{T}}(\mathrm{cl}(\mathrm{supp}(\mathrm{f}) \cap (-\mathrm{T}, 2\mathrm{T}) + \mathrm{supp}(\mathrm{g})(-\mathrm{T}, 2\mathrm{T}))).$$

Moreover, the above inclusion is equality of sets if $\mathrm{f}(\mathrm{t})$ *and* $\mathrm{g}(\mathrm{t})$ *are nonnegative for almost every* $\mathrm{t} \in [0,\mathrm{T}]$.

Proof Note that since $f * g$ is T-periodic, $\mathrm{supp}(f * g)$ is invariant under translations by T:

$$\{t + T \mid t \in \mathrm{supp}(f * g)\} = \mathrm{supp}(f * g).$$

Similar statements hold for the sets $\operatorname{supp}(f)$, $\operatorname{supp}(g)$, and $\operatorname{supp}(f)+\operatorname{supp}(g)$. Moreover, note that

$$\{s \in \mathbb{R} \mid s+t \in [0,T),\ t \in [0,T)\} \cup \{s \in \mathbb{R} \mid s-t \in [0,T),\ t \in [0,T)\} \\ \cup \{s \in \mathbb{R} \mid t-s \in [0,T),\ t \in [0,T)\} = (-T, 2T).$$

Therefore, taking this all into account, a moments thought shows that the result is equivalent to the assertion that

$$\operatorname{supp}(f * g) \subseteq \operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g)),$$

with equality occurring when f and g are almost everywhere nonnegative.

If $\operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g)) = \mathbb{R}$ the first assertion holds trivially. So we suppose this not to be the case. Let $t \in \mathbb{R} \setminus \operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g))$, noting that $t + kT \in \mathbb{R} \setminus \operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g))$ for every $k \in \mathbb{Z}$. Now let U be a neighbourhood of t contained in $\mathbb{R} \setminus \operatorname{cl}(\operatorname{supp}(f)+\operatorname{supp}(g))$, such a neighbourhood existing since $\mathbb{R} \setminus \operatorname{cl}(\operatorname{supp}(f)+\operatorname{supp}(g))$ is open. Let us also assume that U can be contained in an interval J of length T, this being possible without loss of generality. The neighbourhood U can then be made T-periodic by translating it by kT, $k \in \mathbb{Z}$. We then have an open set U containing the points $t + kT$, $k \in \mathbb{Z}$. Let $h \colon \mathbb{R} \to \mathbb{F}$ be a continuous T-periodic function with $\operatorname{supp}(h) \subseteq U$. Then we have that, borrowing the notation of Corollary 11.1.18,

$$\int_J h(f * g)\, d\lambda = \int_{J \times J} F_{f,g,h}\, d\lambda_2 = \int_{(\operatorname{supp}(f)\cap J)\times(\operatorname{supp}(g)\cap J)} F_{f,g,h}\, d\lambda_2, \tag{11.2}$$

using the definition $F_{f,g,h}(\sigma, \tau) = h(\sigma+\tau)f(\sigma)g(\tau)$. However, if $(\sigma, \tau) \in \operatorname{supp}(f)\times\operatorname{supp}(g)$ it follows by assumption that $h(\sigma + \tau) = 0$, and so $F_{f,g,h}(\sigma, \tau) = 0$ as well. Thus the integrals from (11.2) vanish for every continuous T-periodic function h with support in U. It follows from ***missing stuff*** that $U \subseteq \mathbb{R} \setminus \operatorname{supp}(f * g)$. Thus every open subset of $\mathbb{R} \setminus \operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g))$ is contained in $\mathbb{R} \setminus \operatorname{supp}(f * g)$. Equivalently, every closed subset of $\operatorname{supp}(f * g)$ is contained in $\operatorname{cl}(\operatorname{supp}(f) + \operatorname{supp}(g))$, which gives the first part of the result.

For the second assertion, note that if f and g are almost everywhere nonnegative, then so is $f * g$, being defined as the integral of two almost everywhere nonnegative signals. Let $U \subseteq \mathbb{R}$ be open and with the property that $f * g(t) = 0$ for almost every $t \in U$. If the only such open set is the empty set then $f * g$ is almost everywhere nonzero, and so almost everywhere positive. In this case the second assertion holds trivially. Thus we suppose that there exists a nonempty open set U such that $f * g(t) = 0$ for almost every $t \in U$. Without loss of generality we suppose that U is strictly contained in an interval J of length T. Then we let $K \subseteq U$ be a nonempty compact set and let $L \subseteq U$ be a compact set such that $K \subset L$. By Urysohn's Lemma, Theorem **??**, let $h \colon J \to [0, 1]$ have compact support and have the property that $h(t) = 1$ for $t \in K$ and $h(t) = 0$ for $t \in J \setminus L$. It follows that $h(t) = 0$ for $t \in J \setminus U$. Next, T-periodically extend h to be defined on all of $\mathbb{R}$, still denoting the periodically extended signal by h. Similarly, translate K by kT, $k \in \mathbb{Z}$, to get a T-periodic set which is a union of compact sets, the translations of K. Still denote this set by K. We then have

$$\int_J h(f * g)\, d\lambda = 0.$$

Let $H\colon \mathbb{R}^2 \to [0,1]$ be defined by $H(\sigma,\tau) = h(\sigma+\tau)$. By (11.2) it follows that the open set $H^{-1}((\frac{1}{2},\infty))$ (open since H is continuous) does not intersect $\mathrm{supp}(f)\times\mathrm{supp}(g)$. We claim that this implies that $K\cap\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g)) = \emptyset$. Indeed, if $t \in K\cap\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g))$ then $t = \sigma+\tau$ for $\sigma \in \mathrm{supp}(f)$ and $\tau \in \mathrm{supp}(g)$ and $h(t) = H(\sigma+\tau) = 1$. But then $(\sigma,\tau) \in \mathrm{supp}(f)\times\mathrm{supp}(g)\cap H^{-1}((\frac{1}{2},\infty))$, giving a contradiction. Thus we conclude by ***missing stuff*** that if $K \subseteq \mathbb{R}\setminus\mathrm{supp}(f*g)$ is constructed as above, then $K \subseteq \mathbb{R}\setminus\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g))$. This shows that $\mathrm{cl}(\mathrm{supp}(f)+\mathrm{supp}(g)) \subseteq \mathrm{supp}(f*g)$ in this case, as is desired. ■

Now let us record the algebraic structure of periodic convolution. Here, periodicity allows us to make stronger assertions that we were able to make in the aperiodic case, principally since periodic convolutions are integrable over their period by Theorem 11.1.17.

11.1.20 Theorem ($\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is an associative, commutative algebra without unit, when equipped with convolution as product) *If $f,g,h\colon \mathbb{R}\to\mathbb{F}$ are such that their restrictions to $[0,T]$ are in $\mathsf{L}^{(1)}([0,T];\mathbb{F})$, then the following statements hold:*

(i) $\|f*g\|_1 \le \|f\|_1\|g\|_1$;

(ii) $f*g = g*f$;

(iii) *if* $(f*g)*h = f*(g*h)$;

(iv) $f*(g+h) = f*g+f*h$;

(v) *(recalling Remark 11.1.2) there is no equivalence class of signals $[u] \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ such that $[u*f] = [f]$ for every $[f] \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.*

Proof (i) This is a straightforward estimate using Fubini's Theorem, the change of variable theorem, periodicity of f, and Proposition 5.7.21:

$$\begin{aligned}\|f*g\|_1 &= \int_0^T |f*g(t)|\,\mathrm{d}t = \int_0^T \Big|\int_0^T f(t-s)g(s)\,\mathrm{d}s\Big|\,\mathrm{d}t\\ &\le \int_0^T \Big(\int_0^T |f(t-s)g(s)|\,\mathrm{d}s\Big)\,\mathrm{d}t = \int_0^T |g(s)|\Big(\int_{-s}^{T-s} |f(\tau)|\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ &\le \|f\|_1 \int_0^T |g(s)|\,\mathrm{d}s = \|f\|_1\|g\|_1.\end{aligned}$$

(ii) The proof that $D(f,g) = D(g,f)$ follows as in the proof of Proposition 11.1.9. Also, as in the proof of Proposition 11.1.9, the change of variable theorem gives $f*g = g*f$.

(iii) Here we use Fubini's Theorem, the change of variable theorem, and periodicity of f and g to compute

$$\begin{aligned}(f*g)*h(t) &= \int_0^T f*g(t-s)h(s)\,\mathrm{d}s\\ &= \int_0^T\Big(\int_0^T f(t-s-r)g(r)\,\mathrm{d}r\Big)h(s)\,\mathrm{d}s\\ &= \int_0^T\Big(\int_s^{s+T} f(t-\tau)g(\tau-s)\,\mathrm{d}\tau\Big)h(s)\,\mathrm{d}s\\ &= \int_0^T\Big(\int_0^T f(t-\tau)g(\tau-s)\,\mathrm{d}\tau\Big)h(s)\,\mathrm{d}s\\ &= \int_0^T f(t-\tau)\Big(\int_0^T g(\tau-s)h(s)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\\ &= \int_0^T f(t-\tau)g*h(\tau)\,\mathrm{d}\tau = f*(g*h)(t).\end{aligned}$$

(iv) This follows directly from linearity of the integral, Proposition 5.7.17.
(v) We use a lemma.

1 Lemma *If* $\mathrm{u}\in\mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ *then there exists* $\mathrm{r}\in\mathbb{R}_{>0}$ *such that*

$$\Big|\int_{\mathrm{t-r'}}^{\mathrm{t+r'}}\mathrm{u(s)\,ds}\Big|<1,\qquad \mathrm{t}\in\mathbb{R},\ \mathrm{r'}\in(0,\mathrm{r}].$$

Proof Let $t\in\mathbb{R}$. By Proposition 5.9.24 and Theorem 5.9.31, the function

$$r\mapsto\int_{t-r}^{t+r}u(s)\,\mathrm{d}s$$

is continuous since u is locally integrable. Therefore, since the value of this function is zero at $r=0$, there exists $r_t\in\mathbb{R}_{>0}$ such that

$$\Big|\int_{t-r}^{t+r}u(s)\,\mathrm{d}s\Big|<\frac{1}{2}$$

for every $r\in(0,r_t)$. Note that $((-r_t,r_t))_{t\in[0,T]}$ is an open cover of $[0,T]$. By compactness of $[0,T]$, we can apply Theorem 2.5.30 to assert the existence of $r\in\mathbb{R}_{>0}$ such that, for each $t\in[0,T]$, there exists $s_t\in[0,T]$ such that

$$(t-r,t+r)\cap[0,T]\subseteq(s_t-r_{s_t},s_t+r_{s_t}).\tag{11.3}$$

Now let $t\in[0,T]$ and let $s_t\in[0,T]$ be such that (11.3) holds, this being possible by definition of r. Let $r'\in(0,r]$. Then the preceding inclusion and the definition of r_{s_t} immediately gives

$$\Big|\int_{t-r'}^{t+r'}u(s)\,\mathrm{d}s\Big|=\Big|\int_{t-r'}^{s_t}u(s)\,\mathrm{d}s+\int_{s_t}^{t+r'}u(s)\,\mathrm{d}s\Big|\le\Big|\int_{t-r'}^{s_t}u(s)\,\mathrm{d}s\Big|+\Big|\int_{s_t}^{t+r'}u(s)\,\mathrm{d}s\Big|<1,$$

using the usual convention that

$$\int_a^b\mathrm{d}s=-\int_b^a\mathrm{d}s$$

when $a>b$. The lemma follows from periodicity of u. ▼

Now let $f = \chi_{[-r,r]}$ be the characteristic function of the interval $[-r, r]$. By assumption, there exists $Z \subseteq \mathbb{R}$ of zero measure such that $u * f(t) = f(t)$ for every $t \in \mathbb{R} \setminus Z$. For $t \in [-r, r] \cap (\mathbb{R} \setminus Z)$ we have

$$1 = f(t) = u * f(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t-s)f(s)\,\mathrm{d}s = \int_{-r}^{-r} u(t-s)\,\mathrm{d}s = \int_{t-r}^{t+r} u(s)\,\mathrm{d}s < 1,$$

using the change of variables theorem. This gives a contradiction. ■

We comment here that Examples 11.1.11 and 11.1.12, while presented in the context of aperiodic signals on the time-domain $\mathbb{R}$, are equally valid for periodic signals by simply appropriately periodically extending the signals in the aperiodic case. In particular, there is a signal $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ for which $f * f$ is discontinuous and there is an unbounded signal $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$ for which $f * f$ is continuous.

11.1.3 Convolution for aperiodic discrete-time signals

The next class of signals for which we consider convolution is the class of signals defined on a discrete time-domain of the form $\mathbb{Z}(\Delta)$. The situation for discrete-time signals is somewhat simpler than that for continuous-time signals since we do not have to deal with the subtleties of integration, but instead can just deal with summation.

11.1.21 Definition (Convolution for aperiodic discrete-time signals) Let $\Delta \in \mathbb{R}_{>0}$. An ordered pair (f, g) of $\mathbb{F}$-valued signals on $\mathbb{Z}(\Delta)$ is ***convolvable*** if the signal

$$\mathbb{Z}(\Delta) \ni j\Delta \mapsto f(k\Delta - j\Delta)g(j\Delta) \in \mathbb{F}$$

is in $\ell^1(\mathbb{Z}(\Delta); \mathbb{F})$ for every $k\Delta \in \mathbb{Z}(\Delta)$. If (f, g) is convolvable then their ***convolution*** is the signal $f * g \colon \mathbb{Z}(\Delta) \to \mathbb{F}$ defined by

$$f * g(k\Delta) = \Delta \sum_{j \in \mathbb{Z}} f(k\Delta - j\Delta)g(j\Delta). \qquad \bullet$$

Let us consider a simple example in order to understand how discrete convolution works.

11.1.22 Example (The mechanics of discrete convolution) On the time-domain $\mathbb{Z}(1) = \mathbb{Z}$ and for $N \in \mathbb{Z}_{>0}$ let us define $f, g_N \colon \mathbb{Z} \to \mathbb{R}$ by

$$f(k) = \begin{cases} 1, & k \in \{-5, \dots, 5\}, \\ 0, & \text{otherwise}, \end{cases} \qquad g_N(t) = \begin{cases} \dfrac{\sin(2\pi\Omega N k)}{\pi k}, & k \neq 0, \\ 2\Omega N, & \text{otherwise}. \end{cases}$$

We take $\Omega = \frac{1}{20\pi}$ and in Figure 11.12 we plot f and g_N for various N. In Figure 11.13 we show the convolution $f * g_N$ for various N. The only thing we will point out here is that as N gets large, the convolution $f * g_N$ approaches f. This is rather similar to what we saw in Example 11.1.15. However, it turns out that there are some issues here with the signals being discrete. We shall consider these in ***missing stuff***. •

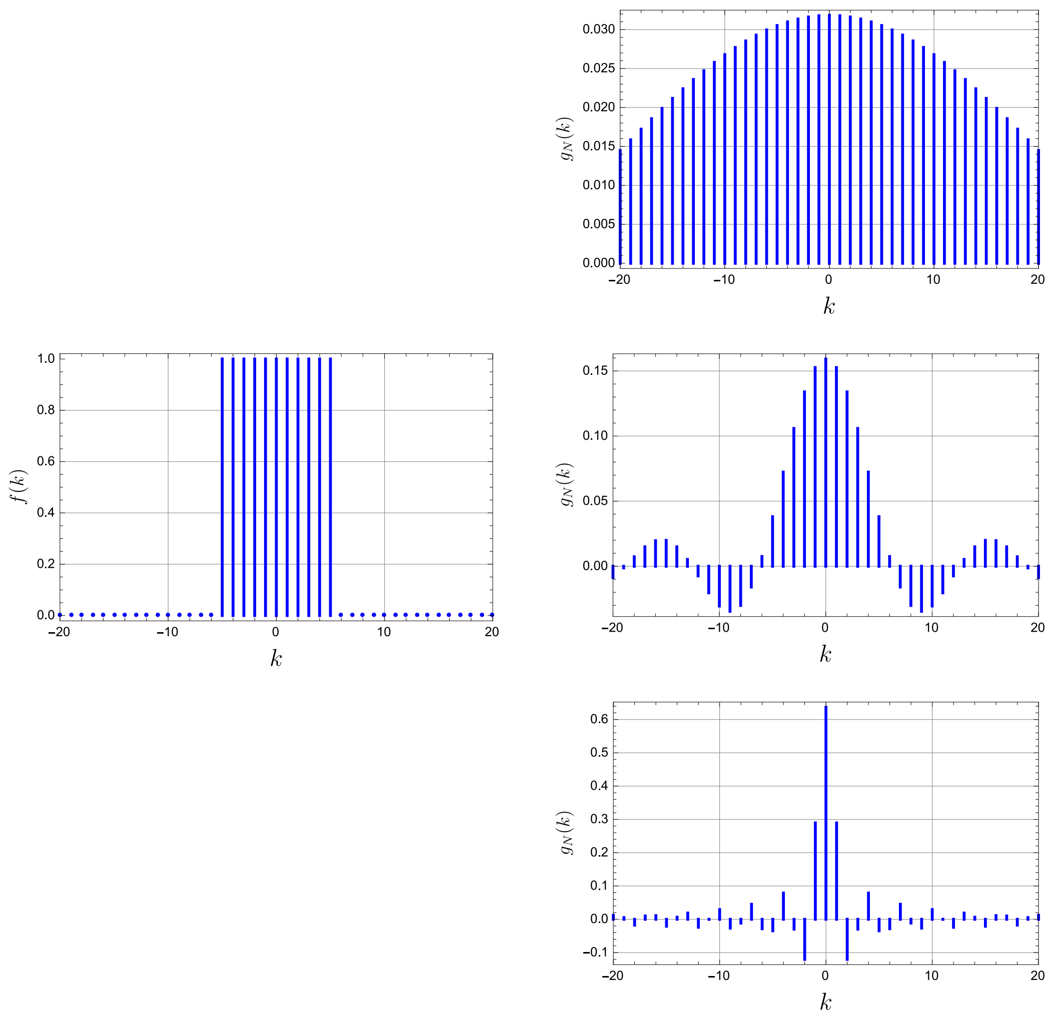


Figure 11.12 Two signals f (left) and g_N (right top, middle, and bottom), the latter for $N \in \{1, 5, 20\}$

As we saw in Theorem 11.1.20 for periodic signals, and as we shall see in Theorem 11.2.1 for aperiodic signals, in the continuous case there is no signal which serves as a unit for the binary operation of convolution. For discrete-time signals, however, there is a unit.

11.1.23 Example (Convolution with the unit pulse) Let $\Delta \in \mathbb{R}_{>0}$ and let $f \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ be an arbitrary signal. Recall from Example 8.1.9–4 the unit pulse $\mathsf{P}\colon \mathbb{Z}(\Delta) \to \mathbb{R}$ defined by

$$\mathsf{P}(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

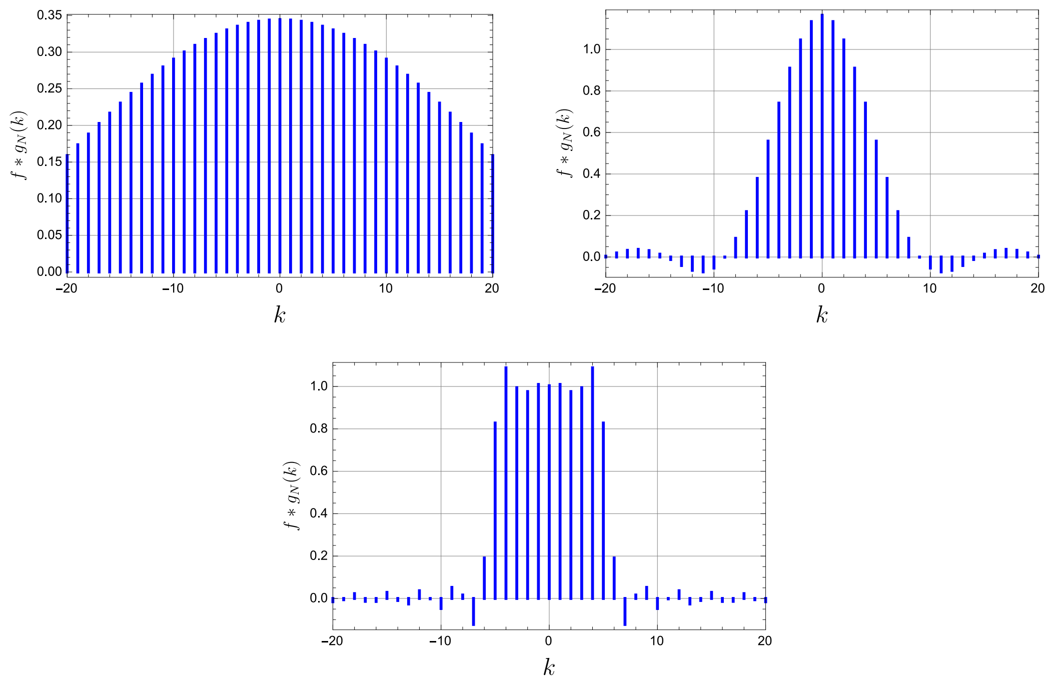


Figure 11.13 The convolution of the signals f and g_N from Figure 11.12 for $N \in \{1, 5, 20\}$.

Let us correspondingly define $\mathsf{P}_N \colon \mathbb{Z}(\Delta) \to \mathbb{R}$ by

$$\mathsf{P}_N(t) = \begin{cases} 1, & t = N\Delta \\ 0, & \text{otherwise} \end{cases}$$

for $N \in \mathbb{Z}$. We then directly compute

$$f * \mathsf{P}_N(k\Delta) = \Delta \sum_{j \in \mathbb{Z}_{>0}} f(k\Delta - j\Delta)\mathsf{P}_N(j\Delta) = \Delta f(k\Delta - N\Delta).$$

In particular, $f * (\Delta^{-1}\mathsf{P}) = f$, and so we see that discrete-time signals always possess a unit under the binary operation of convolution. •

Although the characterisation is not as deep as in the continuous-time case, we provide the following characterisation of convolvable discrete-time signals.

11.1.24 Theorem (Characterisation of discrete convolution) *Let $\Delta \in \mathbb{R}_{>0}$. For $\mathrm{f}, \mathrm{g} \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ the following statements are equivalent:*

(i) (f, g) *is convolvable;*

(ii) for every signal $\mathrm{h} \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ with finite support (i.e., $\mathrm{h}(\mathrm{t}) \neq 0$ for finitely many $\mathrm{t} \in \mathbb{Z}(\Delta)$), it holds that

$$\sum_{(\mathrm{j},\mathrm{k}) \in \mathbb{Z}^2} \mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}(\mathrm{j}, \mathrm{k}) \in \mathbb{R},$$

where $\mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}(\mathrm{j}, \mathrm{k}) = \mathrm{h}(\mathrm{j}\Delta + \mathrm{k}\Delta)\mathrm{f}(\mathrm{k}\Delta)\mathrm{g}(\mathrm{j}\Delta)$.

Moreover, if (f, g) *is convolvable and if* $\mathrm{h} \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ *has finite support, then*

$$\Delta \sum_{\mathrm{j} \in \mathbb{Z}} \mathrm{h}(\mathrm{j}\Delta)\mathrm{f} * \mathrm{g}(\mathrm{j}\Delta) = \Delta^2 \sum_{(\mathrm{j},\mathrm{k}) \in \mathbb{Z}^2} \mathrm{F}_{\mathrm{f},\mathrm{g},\mathrm{h}}(\mathrm{j}, \mathrm{k}).$$

Proof Suppose that (f, g) is convolvable and let $h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ have finite support. Define $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ by $\phi(j, k) = (j, k - j)$ and note that

$$F_{f,g,h} \circ \phi(j, k) = h(k\Delta) f(k\Delta - j\Delta) g(j\Delta).$$

Since (f, g) is convolvable, it follows that $j \mapsto F_{f,g,h} \circ \phi(j, k)$ is in $\ell^1(\mathbb{Z}, \mathbb{F})$ for every $k \in \mathbb{Z}$. Since h has finite support, the signal $k \mapsto F_{f,g,h} \circ \phi(j, k)$ is in $\ell^1(\mathbb{Z}; \mathbb{F})$ for every $j \in \mathbb{Z}$. Then we have

$$\sum_{k \in \mathbb{Z}} h(k\Delta) f * g(k) = \Delta \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} F_{f,g,h} \circ \phi(j, k).$$

Since (f, g) is convolvable and since h has finite support, the sum on the left converges absolutely. Thus the sum on the right converges absolutely and so, since ϕ is a bijection, we have

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} F_{f,g,h} \circ \phi(j, k) = \sum_{j,k \in \mathbb{Z}^2} F_{f,g,h}(j, k),$$

which gives this part of the theorem following the constructions of Section 2.4.7.

Now suppose that, for every signal $h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ with finite support, it holds that

$$\sum_{(j,k) \in \mathbb{Z}^2} |F_{f,g,h}(j, k)| \in \mathbb{R}.$$

By the constructions of Section 2.4.7 it follows that

$$\sum_{(j,k) \in \mathbb{Z}^2} |F_{f,g,h} \circ \phi(j, k)| \in \mathbb{R},$$

with ϕ as above. By Fubini's Theorem the function $j \mapsto |F_{f,g,h} \circ \phi(j, k)|$ is in $\ell^1(\mathbb{Z}; \mathbb{F})$ for every $k \in \mathbb{Z}$ and for every finitely supported $h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$. As above,

$$F_{f,g,h} \circ \phi(j, k) = h(k\Delta) f(k\Delta - j\Delta) g(j\Delta),$$

and so choosing

$$h(k\Delta) = \begin{cases} 1, & k = m, \\ 0, & \text{otherwise}, \end{cases}$$

we see that $j \mapsto |f(m\Delta - j\Delta) g(j\Delta)|$ is in $\ell^1(\mathbb{Z}; \mathbb{F})$ for every $m \in \mathbb{Z}$. Thus (f, g) is convolvable. ■

As with the convolution of continuous-time signals, we can characterise the support of the convolution of discrete-time signals.

11.1.25 Proposition (Support of discrete convolution) *Let $\Delta \in \mathbb{R}_{>0}$. If (f, g) is a pair of convolvable $\mathbb{F}$-valued signals on $\mathbb{Z}(\Delta)$, then*

$$\mathrm{supp}(\mathrm{f} * \mathrm{g}) \subseteq \mathrm{supp}(\mathrm{f}) + \mathrm{supp}(\mathrm{g}),$$

where $\mathrm{supp}(\mathrm{f}) + \mathrm{supp}(\mathrm{g}) = \{\mathrm{s} + \mathrm{t} \mid \mathrm{s} \in \mathrm{supp}(\mathrm{f}),\ \mathrm{t} \in \mathrm{supp}(\mathrm{g})\}$*. Moreover, the above inclusion is equality of sets if* $\mathrm{f}(\mathrm{t})$ *and* $\mathrm{g}(\mathrm{t})$ *are nonnegative for every* $\mathrm{t} \in \mathbb{Z}(\Delta)$*.*

Proof If $\mathrm{supp}(f) + \mathrm{supp}(g) = \mathbb{Z}(\Delta)$ the first assertion holds trivially. So we suppose this not to be the case. Let $t \in \mathbb{Z}(\Delta) \setminus (\mathrm{supp}(f) + \mathrm{supp}(g))$ and let $h\colon \mathbb{Z}(\Delta) \to \mathbb{F}$ be a signal with $\mathrm{supp}(h) = \{t\}$. Then we have that, borrowing the notation of Theorem 11.1.24,

$$\Delta \sum_{j \in \mathbb{Z}} h(j\Delta) f * g(j\Delta) = \Delta^2 \sum_{(j,k) \in \mathbb{Z}^2} F_{f,g,h}(j,k) = \Delta^2 \sum_{(j,k) \in \mathrm{supp}(f) \times \mathrm{supp}(g)} F_{f,g,h}(j,k), \qquad (11.4)$$

using the definition $F_{f,g,h}(j,k) = h(j\Delta + k\Delta) f(k\Delta) g(j\Delta)$. However, if $(j\Delta, k\Delta) \in \mathrm{supp}(f) \times \mathrm{supp}(g)$ it follows by assumption that $h(j\Delta + k\Delta) = 0$, and so $F_{f,g,h}(j,k) = 0$ as well. Thus the sums from (11.4) vanish for every signal h with $\mathrm{supp}(h) = \{t\}$. It follows that $\{t\} \subseteq \mathbb{R} \setminus \mathrm{supp}(f * g)$, which gives the first part of the result.

For the second assertion, note that if f and g are nonnegative, then so is $f * g$, being defined as the sum of two nonnegative signals. If there is no $t \in \mathbb{Z}(\Delta)$ where $f * g(t) = 0$ then $f * g$ is everywhere nonzero, and so everywhere positive. In this case the second assertion holds trivially. So let $t \in \mathbb{Z}(\Delta)$ be such that $f * g(t) = 0$ and let $h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ be such that $\mathrm{supp}(h) = \{t\}$. We, therefore, have

$$\sum_{j \in \mathbb{Z}} h(j\Delta) f * g(j\Delta) = 0.$$

According to (11.4) we have

$$\sum_{(j,k) \in \mathrm{supp}(f) \times \mathrm{supp}(g)} F_{f,g,h}(j,k) = 0.$$

Since the sum is a sum of strictly positive or strictly negative terms, it follows that all terms in the sum are zero. By the definitions of h and $F_{f,g,h}$, this means that, for $(j\Delta, k\Delta) \in \mathrm{supp}(f) \times \mathrm{supp}(g)$, $f(j\Delta) g(k\Delta) = 0$ whenever $j\Delta + k\Delta = t$. Thus $t \notin \mathrm{supp}(f) + \mathrm{supp}(g)$, giving the second assertion of the proposition. ■

Next we give a few algebraic properties of discrete convolution.

11.1.26 Proposition (Algebraic properties of discrete convolution) *If* $\mathrm{f}, \mathrm{g}, \mathrm{h}\colon \mathbb{Z}(\Delta) \to \mathbb{F}$ *then the following statements hold:*

(i) if (f, g) *is convolvable, then* (g, f) *is convolvable and* $\mathrm{f} * \mathrm{g} = \mathrm{g} * \mathrm{f}$*;*

(ii) if (f, g) *and* (f, h) *are convolvable, then* $(\mathrm{f}, \mathrm{g} + \mathrm{h})$ *is convolvable and* $\mathrm{f} * (\mathrm{g} + \mathrm{h}) = \mathrm{f} * \mathrm{g} + \mathrm{f} * \mathrm{h}$.

Proof (i) Let $k \in \mathbb{Z}$ and consider the bijection $\gamma_k\colon \mathbb{Z} \to \mathbb{Z}$ given by $\gamma_k(j) = k - j$. Since (f, g) is convolvable, the signal $\gamma_k^* f g$ is in $\ell^1(\mathbb{Z}; \mathbb{F})$. By the constructions of Section 2.4.7 it follows that $(\gamma_k^* f g) \circ \gamma_k$ is in $\ell^1(\mathbb{Z}; \mathbb{F})$. Since

$$(\gamma_k^* f g) \circ \gamma_k(j) = (\gamma_k^* f g)(k - j) = f(j) g(k - j),$$

it follows that $j \mapsto f(j\Delta)g(k\Delta - j\Delta)$ is in $\ell^1(\mathbb{Z};\mathbb{F})$, showing that (g, f) is convolvable. The constructions of Section 2.4.7 further give

$$\sum_{j\in\mathbb{Z}} f(j)g(k-j) = \sum_{j\in\mathbb{Z}} (\gamma_k^* f g)\circ\gamma_k(j) = \sum_{j\in\mathbb{Z}} (\gamma_k^* f g)(j) = \sum_{j\in\mathbb{Z}} f(k-j)g(j).$$

(ii) This is a direct consequence of linearity of convergent sums, Proposition 2.4.30. It is also a consequence of linearity of the integral since sums are actually integrals with respect to a suitable measure by Example 5.7.10. ■

11.1.27 Example (Discrete convolution is not generally associative) Let us take $\Delta = 1$ and define $f, g, h\colon \mathbb{Z} \to \mathbb{R}$ by

$$f(j) = 1, \qquad g(j) = \begin{cases} 1, & j = 1, \\ -1, & j = 2, \\ 0, & \text{otherwise}, \end{cases} \qquad h(j) = \begin{cases} 0, & j \le 0, \\ 1, & j = 1, \\ 2, & g \ge 2. \end{cases}$$

We have

$$f * g(k) = \sum_{j\in\mathbb{Z}} f(k-j)g(j) = \sum_{j\in\mathbb{Z}} g(j) = 0$$

for every $k \in \mathbb{Z}$. We also directly compute

$$g * h(j) = \begin{cases} 1, & j \in \{2, 3\}, \\ 0, & \text{otherwise} \end{cases}$$

and consequently also directly we compute $f * (g * h)(j) = 2$ for every $j \in \mathbb{Z}$. This shows, in particular, that $f * (g * h) \neq (f * g) * h$, as desired. •

11.1.4 Convolution for periodic discrete-time signals

The next class of convolutions we consider is that for periodic discrete-time signals. For periodic discrete convolution, there is no danger of any pairs of signals not being convolvable since, as we shall shortly see, this convolution involves finite sums. Thus we make the following definition.

11.1.28 Definition (Convolution for periodic discrete-time signals) Let $\Delta \in \mathbb{R}_{>0}$ and let $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$. The ***convolution*** of the pair $(f, g) \in \ell_{\mathrm{per},T}(\mathbb{Z}(\Delta);\mathbb{F})$ is the signal $f * g\colon \mathbb{Z}(\Delta) \to \mathbb{F}$ defined by

$$f * g(k\Delta) = \Delta \sum_{j=0}^{N-1} f(k\Delta - j\Delta)g(j\Delta).$$

Let us make some comments about discrete periodic convolution.

11.1.29 Remarks (On periodic convolution of discrete-time periodic signals)

1. For a pair (f, g) of T-periodic discrete-time signals defined on $\mathbb{Z}(\Delta)$, when we write $f * g$ there can be no ambiguity about whether we mean "convolution" or "periodic convolution." Indeed, if the pair (f, g) is convolvable in the sense of Definition 11.1.21, then one of f and g must be zero. The reader can prove this in Exercise 11.1.5.
2. While the sum in Definition 11.1.28 is from 0 to $N-1$, by periodicity of f and g the sum can be performed over any collection of length N of consecutive integers. •

Let us give an example of discrete periodic convolution. We shall be a little brief here, having devoted much effort in the previous three versions of convolution to understanding what convolution "is."

11.1.30 Example (The mechanics of discrete periodic convolution) We let $\Delta = 1$ and take $T = 10$. We define two T-periodic signals f and g on $\mathbb{Z}$ by defining them on $\{-5, -4, \dots, 4, 5\}$ to be

$$f(k) = \begin{cases} 1, & k \in \{-2, -1, 0, 1, 2\}, \\ 0, & \text{otherwise}, \end{cases} \qquad g(k) = \begin{cases} 1, & k = 0, \\ \frac{1}{2}, & k \in \{-1, 1\}, \\ 0, & \text{otherwise}. \end{cases}$$

In Figure 11.14 we depict these signals on one period. Their convolution is shown

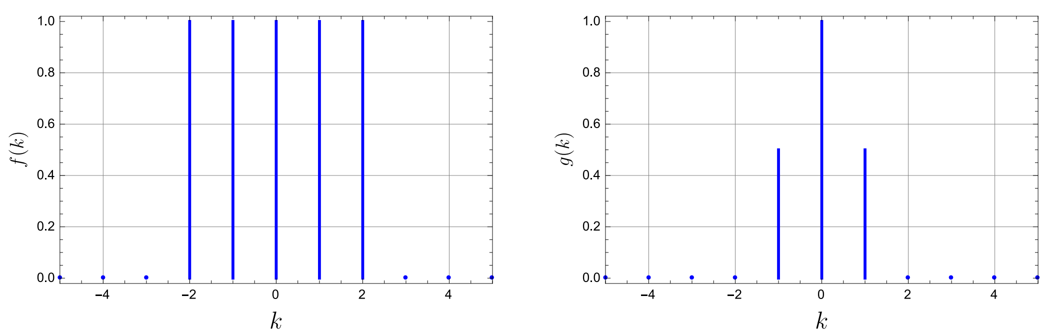


Figure 11.14 Two periodic discrete-time signals

in Figure 11.15. •

As with aperiodic discrete convolution, periodic discrete convolution possesses a unit.

11.1.31 Example (Convolution with the periodic unit pulse) Let $\Delta \in \mathbb{R}_{>0}$ and let $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$. Let $f \in \ell_{\mathrm{per},T}(\mathbb{Z}(\Delta); \mathbb{F})$ be an arbitrary T-periodic discrete-time signal. In Example 8.1.24–1 we defined the T-periodic unit pulse $\mathsf{P}_{\mathrm{per},T} \in \ell_{\mathrm{per},T}(\mathbb{Z}(\Delta); \mathbb{F})$ by

$$\mathsf{P}_{\mathrm{per},T}(t) = \begin{cases} 1, & t = kT \text{ for some } k \in \mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases}$$

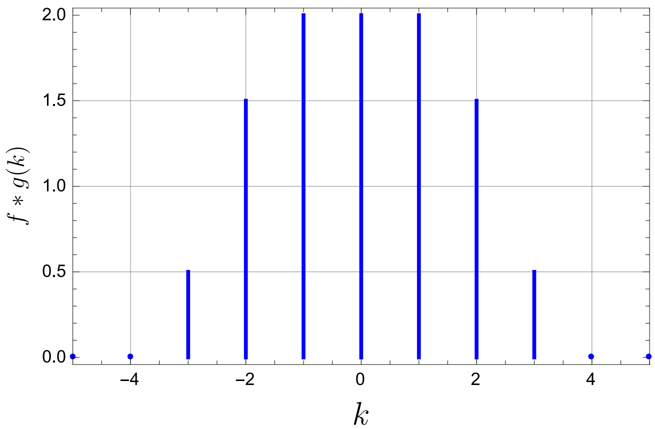


Figure 11.15 The periodic convolution of the signals from Figure 11.14

We then compute

$$f * \mathsf{P}_{\mathrm{per},T}(k\Delta) = \Delta \sum_{j=0}^{N-1} f(k\Delta - j\Delta)\mathsf{P}_{\mathrm{per},T}(j\Delta) = \Delta f(k\Delta).$$

Thus $f * \mathsf{P}_{\mathrm{per},T} = \Delta f$, showing that $\Delta^{-1}\mathsf{P}_{\mathrm{per},T}$ serves as a multiplicative identity if the product is given by convolution. •

We should verify that discrete periodic convolution gives rise to periodic signals.

11.1.32 Proposition (Discrete periodic convolutions are periodic) *Let* $\Delta \in \mathbb{R}_{>0}$ *and let* $\mathrm{T} = \mathrm{N}\Delta$ *for some* $\mathrm{N} \in \mathbb{Z}_{>0}$. *If* $\mathrm{f}, \mathrm{g} \in \ell_{\mathrm{per},\mathrm{T}}(\mathbb{Z}(\Delta); \mathbb{F})$ *then* $\mathrm{f} * \mathrm{g} \in \ell_{\mathrm{per},\mathrm{T}}(\mathbb{Z}(\Delta); \mathbb{F})$.

Proof This is a simple computation:

$$f * g(k\Delta + N\Delta) = \Delta \sum_{j=0}^{N-1} f(k\Delta + N\Delta - j\Delta)g(j\Delta) = \Delta \sum_{j=0}^{N-1} f(k\Delta - j\Delta)g(j\Delta) = f * g(k\Delta). \quad \blacksquare$$

Let us now characterise the support for discrete periodic convolutions. We recall from the paragraph preceding Proposition 11.1.19 the definition of $\phi_T \colon \mathbb{R} \to [0, T)$.

11.1.33 Proposition (Support of discrete periodic convolution) *Let* $\Delta \in \mathbb{R}_{>0}$ *and let* $\mathrm{T} = \mathrm{N}\Delta$ *for some* $\mathrm{N} \in \mathbb{Z}_{>0}$. *If* (f, g) *is a pair of convolvable* T*-periodic* $\mathbb{F}$*-valued signals on* $\mathbb{Z}(\Delta)$*, then*

$$(\mathrm{supp}(\mathrm{f} * \mathrm{g}) \cap [0, \mathrm{T})) \subseteq \phi_{\mathrm{T}}(\mathrm{supp}(\mathrm{f}) \cap (-\mathrm{T}, 2\mathrm{T}) + \mathrm{supp}(\mathrm{g})(-\mathrm{T}, 2\mathrm{T})).$$

Moreover, the above inclusion is equality of sets if $\mathrm{f}(\mathrm{t})$ *and* $\mathrm{g}(\mathrm{t})$ *are nonnegative for every* $\mathrm{t} \in \mathbb{Z}(\Delta)$.

Proof Note that since $f * g$ is T-periodic, $\mathrm{supp}(f * g)$ is invariant under translations by T:

$$\{t + T \mid t \in \mathrm{supp}(f * g)\} = \mathrm{supp}(f * g).$$

Similar statements hold for the sets $\mathrm{supp}(f)$, $\mathrm{supp}(g)$, and $\mathrm{supp}(f)+\mathrm{supp}(g)$. Moreover, note that

$$\{s \in \mathbb{Z}(\Delta) \mid s + t \in [0, T),\ t \in [0, T)\} \cup \{s \in \mathbb{Z}(\Delta) \mid s - t \in [0, T),\ t \in [0, T)\} \\ \cup \{s \in \mathbb{Z}(\Delta) \mid t - s \in [0, T),\ t \in [0, T)\} = \mathbb{Z}(\Delta) \cap (-T, 2T).$$

Therefore, taking this all into account, a moments thought shows that the result is equivalent to the assertion that

$$\mathrm{supp}(f * g) \subseteq (\mathrm{supp}(f) + \mathrm{supp}(g)),$$

with equality occurring when f and g are almost everywhere nonnegative.

If $\mathrm{supp}(f) + \mathrm{supp}(g) = \mathbb{Z}(\Delta)$ the first assertion holds trivially. So we suppose this not to be the case. Let $t \in \mathbb{Z}(\Delta) \cap [0, T) \setminus (\mathrm{supp}(f) + \mathrm{supp}(g))$, noting that $t + kT \in \mathbb{Z}(\Delta) \setminus (\mathrm{supp}(f) + \mathrm{supp}(g))$ for every $k \in \mathbb{Z}$. Let $h\colon \mathbb{Z}(\Delta) \to \mathbb{F}$ be a T-periodic signal with

$$\mathrm{supp}(h) = \{t + kT \mid k \in \mathbb{Z}\}.$$

Then we have that, borrowing the notation of Theorem 11.1.24,

$$\sum_{j=0}^{N-1} h(j\Delta) f * g(j\Delta) = \Delta \sum_{(j,k)\in\{0,1,\dots,N-1\}^2} F_{f,g,h}(j, k) = \Delta \sum_{(j,k)\in\mathrm{supp}(f)\times\mathrm{supp}(g)\cap[0,T)^2} F_{f,g,h}(j, k), \tag{11.5}$$

using the definition $F_{f,g,h}(j, k) = h(j + k) f(j) g(k)$. However, if $(j\Delta, k\Delta) \in \mathrm{supp}(f) \times \mathrm{supp}(g) \cap [0, T)^2$ it follows by assumption that $h(j\Delta + k\Delta) = 0$, and so $F_{f,g,h}(j, k) = 0$ as well. Thus the sums from (11.5) vanish for every T-periodic signal h with the support as described. It follows that

$$\{t + kT \mid k \in \mathbb{Z}\} \subseteq \mathbb{R} \setminus \mathrm{supp}(f * g),$$

which gives the first part of the result.

For the second assertion, note that if f and g are nonnegative, then so is $f * g$, being defined as the sum of two nonnegative signals. If there is no $t \in \mathbb{Z}(\Delta)$ where $f * g(t) = 0$ then $f * g$ is everywhere nonzero, and so everywhere positive. In this case the second assertion holds trivially. So let $t \in \mathbb{Z}(\Delta) \cap [0, T)$ be such that $f * g(t) = 0$ and note that $f * g(t + kT) = 0$ for every $k \in \mathbb{Z}$. Let $h\colon \mathbb{Z}(\Delta) \to \mathbb{F}$ be a T-periodic signal with

$$\mathrm{supp}(h) = \{t + kT \mid k \in \mathbb{Z}\}.$$

We, therefore, have

$$\sum_{j=1}^{N-1} h(j\Delta) f * g(j\Delta) = 0.$$

According to (11.5) we have

$$\sum_{(j,k)\in\mathrm{supp}(f)\times\mathrm{supp}(g)\cap[0,T)^2} F_{f,g,h}(j, k) = 0.$$

Since the sum is a sum of strictly positive or strictly negative terms, it follows that all terms in the sum are zero. By the definitions of h and $F_{f,g,h}$, this means that, for $(j\Delta, k\Delta) \in \operatorname{supp}(f) \times \operatorname{supp}(g) \cap [0,T)^2$, $f(j\Delta)g(k\Delta) = 0$ whenever $j\Delta + k\Delta = t$. Thus $t \notin \operatorname{supp}(f) + \operatorname{supp}(g)$, giving the second assertion of the proposition. ■

11.1.34 Theorem ($\ell_{\mathrm{per,T}}(\mathbb{Z}(\Delta);\mathbb{F})$ is an associative, commutative algebra with unit when equipped with convolution as product) *Let* Δ *and let* $\mathrm{T} = \mathrm{N}\Delta$ *for some* $\mathrm{N} \in \mathbb{Z}_{>0}$. *If* $\mathrm{f}, \mathrm{g}, \mathrm{h}\ell_{\mathrm{per,T}}(\mathbb{Z}(\Delta);\mathbb{F})$, *then the following statements hold:*

(i) $\|\mathrm{f} * \mathrm{g}\|_1 \le \|\mathrm{f}\|_1\|\mathrm{g}\|_1$;

(ii) $\mathrm{f} * \mathrm{g} = \mathrm{g} * \mathrm{f}$;

(iii) *if* $(\mathrm{f} * \mathrm{g}) * \mathrm{h} = \mathrm{f} * (\mathrm{g} * \mathrm{h})$;

(iv) $\mathrm{f} * (\mathrm{g} + \mathrm{h}) = \mathrm{f} * \mathrm{g} + \mathrm{f} * \mathrm{h}$;

(v) *there exists* $\mathrm{u} \in \ell_{\mathrm{per,T}}(\mathbb{Z}(\Delta);\mathbb{F})$ *such that* $\mathrm{u} * \mathrm{f} = \mathrm{f}$ *for every* $\mathrm{f} \in \ell_{\mathrm{per,T}}(\mathbb{Z}(\Delta);\mathbb{F})$.

Proof (i) We compute

$$\begin{aligned}\|f * g\|_1 &= \Delta \sum_{k=0}^{N-1} |f * g(k)| \le \Delta^2 \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} |f(k\Delta - j\Delta)g(j\Delta)| \\ &= \Delta^2 \sum_{j=0}^{N-1} |g(j\Delta)| \sum_{m=-j\Delta}^{N-1-j\Delta} |f(m\Delta)| = \Delta\|f\|_1 \sum_{j=0}^{N-1} |g(j\Delta)| = \|f\|_1\|g\|_1.\end{aligned}$$

(ii) We compute

$$\begin{aligned}f * g(k\Delta) &= \Delta \sum_{j=0}^{N-1} f(k\Delta - j\Delta)g(j\Delta) = \Delta \sum_{m=k-(N-1)}^{k} f(m\Delta)g(k\Delta - m\Delta) \\ &= \Delta \sum_{l=0}^{N-1} f(l\Delta)g(k\Delta - l\Delta) = g * f(k\Delta).\end{aligned}$$

(iii) We compute

$$\begin{aligned}f * (g * h)(l\Delta) &= \Delta \sum_{k=0}^{N-1} f(l\Delta - k\Delta) g * h(k\Delta) = \Delta^2 \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} f(l\Delta - k\Delta)g(k\Delta - j\Delta)h(j\Delta) \\ &= \Delta^2 \sum_{m=-j}^{N-1-j}\sum_{j=0}^{N-1} f(l\Delta - m\Delta - j\Delta)g(m\Delta)h(j\Delta) \\ &= \Delta^2 \sum_{j=0}^{N-1}\sum_{m=0}^{N-1} f(l\Delta - j\Delta - m\Delta)g(m\Delta)h(j\Delta) = \Delta \sum_{j=0}^{N-1} f * g(l\Delta - j\Delta)h(j\Delta) \\ &= (f * g) * h(l\Delta).\end{aligned}$$

(iv) This follows by linearity of finite summation.

(v) This follows from Example 11.1.31. ■

11.1.5 Notes

The reader should be aware that the basic treatment of convolution in many texts is a little sloppy. For example, it is very often stated that convolution is associative. Our Example 11.1.10, from **[EBH/GLW:90]**, shows that this is not generally true, although in Theorem 11.2.1 we show that this is true for $\mathsf{L}^1(\mathbb{R};\mathbb{F})$. In Theorem 11.1.20 we show that periodic convolution *is* always associative, however. Also, convolution for causal signals is associative. Thus one has to exercise care when utilising the "natural" algebraic properties of convolution.

Various definitions of convolution are possible, and comparisons of these are made, for example, in [**PD/JV:78**, **AK:82**]

Exercises

11.1.1 Consider the signals $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t^{-1/2}, & t \in (0,1], \\ 0, & \text{otherwise} \end{cases}$$

and $g(t) = f(-t)$.

(a) Show that (f, g) is convolvable and that $D(f, g) = \mathbb{R} \setminus \{0\}$.

(b) Compute $f * g$.

11.1.2 Show that if either f or g has compact support, then the pair (f, g) is convolvable.

11.1.3 Let (f, g) be T-periodic $\mathbb{F}$-valued signals whose convolution over $\mathbb{R}$,

$$f * g(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s,$$

exists for almost every $t \in \mathbb{R}$. Show that one of f or g must be zero, i.e., zero almost everywhere.

11.1.4 Let $\Delta \in \mathbb{R}_{>0}$ and let $f \in \mathbb{F}^{\mathbb{Z}(\Delta)}$. Show that there exist signals $g, h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ such that $g * h = f$, i.e., show that discrete convolution is "surjective."

11.1.5 Let (f, g) be discrete-time T-periodic $\mathbb{F}$-valued signals defined on $\mathbb{Z}(\Delta)$ whose convolution,

$$f * g(k\Delta) = \Delta \sum_{j \in \mathbb{Z}} f(k\Delta - j\Delta)g(j\Delta),$$

exists for every $k \in \mathbb{Z}$. Show that one of f or g must be zero.

11.1.6 Let $\Delta \in \mathbb{R}_{>0}$, let $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$, and let $f \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ be T-periodic. Show that there exist T-periodic signals $g, h \in \mathbb{F}^{\mathbb{Z}(\Delta)}$ such that $g * h = f$, i.e., show that periodic discrete convolution is "surjective."

Section 11.2

Convolvable pairs of signals

In this section we consider the convolution between signals from various spaces. It is really not possible to stage the most general result for when the convolution of two signals is defined, nor is it necessarily interesting to do so. Thus, in this section we shall give those results which will be of interest to us in our subsequent uses of convolution.

Do I need to read this section? If one wishes to understand the basic results about when the operation of convolution is defined between two signals, then this is the section to read. •

11.2.1 Convolution in $\mathsf{L}^1(\mathbb{R};\mathbb{F})$

The first case where convolution makes sense is when it is applied to integrable signals. The following result gives the space of integrable signals some rather useful algebraic structure.

11.2.1 Theorem ($\mathsf{L}^1(\mathbb{R};\mathbb{F})$ is an associative, commutative algebra without unit, when equipped with convolution as product) *If* $f, g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *then* (f, g) *is convolvable and* $f * g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$*. Furthermore, for* $f, g, h \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$*, the following statements hold:*

(i) $\|f * g\|_1 \le \|f\|_1\|g\|_1$*;*

(ii) $f * g = g * f$*;*

(iii) $(f * g) * h = f * (g * h)$*;*

(iv) $f * (g + h) = f * g + f * h$*;*

(v) *(recalling Remark 11.1.2) there is no equivalence class of signals* $[u] \in \mathsf{L}^1(\mathbb{R};\mathbb{F})$ *such that* $[u * f] = [f]$ *for every* $[f] \in \mathsf{L}^1(\mathbb{R};\mathbb{F})$*.*

Proof Define $F_{f,g}\colon \mathbb{R}^2 \to \mathbb{F}$ by $F_{f,g}(\sigma, \tau) = f(\sigma)g(\tau)$. By Corollary 5.8.8 $F_{f,g} \in \mathsf{L}^{(1)}(\mathbb{R}^2;\mathbb{F})$. Now consider the change of variable $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(s, t) = (t - s, s)$, so that

$$F_{f,g} \circ \phi(s, t) = f(t - s)g(s).$$

By the change of variable theorem, Theorem **??**, $F_{f,g} \circ \phi \in \mathsf{L}^{(1)}(\mathbb{R}^2;\mathbb{F})$, and so by Fubini's Theorem, the function $s \mapsto f(t - s)g(s)$ is integrable for almost every $t \in \mathbb{R}$. Thus (f, g) is convolvable.

(i) Moreover, using the change of variable theorem and Fubini's Theorem again,

$$\int_{\mathbb{R}}\Big|\int_{\mathbb{R}} f(t - s)g(s)\,\mathrm{d}s\Big|\,\mathrm{d}t \le \int_{\mathbb{R}^2} |F_{f,g} \circ \phi|\,\mathrm{d}\lambda_2 = \int_{\mathbb{R}^2} |F_{f,g}|\,\mathrm{d}\lambda_2 = \|f\|_1\|g\|_1,$$

as desired.

(ii) This is Proposition 11.1.9(i).

(iii) We have

$$\begin{aligned}(f * g) * h(t) &= \int_{\mathbb{R}} f * g(t-s)h(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t-s-r)g(r)\,\mathrm{d}r\Big)h(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t-\tau)g(\tau-s)\,\mathrm{d}\tau\Big)h(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} f(t-\tau)\Big(\int_{\mathbb{R}} g(\tau-s)h(s)\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}} f(t-\tau)g * h(\tau)\,\mathrm{d}\tau = f * (g * h)(t),\end{aligned}$$

using the change of variable theorem and Fubini's Theorem.

(iv) This is simply linearity of the integral, Proposition 5.7.17.

(v) Let $u \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ be such that, for every $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$, $u * f(t) = f(t)$ for almost every $t \in \mathbb{R}$. We first use a lemma that is an adaptation of Lemma 1 from the proof of Theorem 11.1.20.

1 Lemma *If* $\mathrm{u} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *then there exists* $\mathrm{r} \in \mathbb{R}_{>0}$ *such that*

$$\Big|\int_{\mathrm{t}-\mathrm{r}'}^{\mathrm{t}+\mathrm{r}'} \mathrm{u}(\mathrm{s})\,\mathrm{ds}\Big| < 1, \qquad \mathrm{t} \in \mathbb{R},\ \mathrm{r}' \in (0,\mathrm{r}].$$

Proof Let $t \in \mathbb{R}$. By Proposition 5.9.24 and Theorem 5.9.31, the function

$$r \mapsto \int_{t-r}^{t+r} u(s)\,\mathrm{d}s$$

is continuous since u is locally integrable. Therefore, since the value of this function is zero at $r = 0$, there exists $r_t \in \mathbb{R}_{>0}$ such that

$$\Big|\int_{t-r}^{t+r} u(s)\,\mathrm{d}s\Big| < \frac{1}{2}$$

for every $r \in (0, r_t)$. Now let $T \in \mathbb{R}_{>0}$ be sufficiently large that

$$\int_{\mathbb{R}} |u(s)|\,\mathrm{d}s - \int_{-T}^{T} |u(s)|\,\mathrm{d}s < 1,$$

this being possible since

$$\lim_{T\to\infty} \int_{-T}^{T} |u(s)|\,\mathrm{d}s < \infty.$$

Note that $((-r_t, r_t))_{t\in[-2T,2T]}$ is an open cover of $[-2T, 2T]$. By compactness of $[-2T, 2T]$, we can apply Theorem 2.5.30 to assert the existence of $\rho \in \mathbb{R}_{>0}$ such that, for each $t \in [-2T, 2T]$, there exists $s_t \in [-2T, 2T]$ such that

$$(t-\rho, t+\rho) \cap [-2T, 2T] \subseteq (s_t - r_{s_t}, s_t + r_{s_t}).$$

Let $r = \min\{\rho, \frac{T}{2}\}$.

Now let $t \in \mathbb{R}$ and let $r' \in (0, r]$. If $t \in [-2T, 2T]$ then let $s_t \in [-2T, 2T]$ be such that

$$(t - r, t + r) \cap [-2T, 2T] \subseteq (s_t - r_{s_t}, s_t + r_{s_t}),$$

this being possible by definition of r. Then the preceding inclusion and the definition of r_{s_t} immediately gives

$$\Big|\int_{t-r'}^{t+r'} u(s)\,\mathrm{d}s\Big| = \Big|\int_{t-r'}^{s_t} u(s)\,\mathrm{d}s + \int_{s_t}^{t+r'} u(s)\,\mathrm{d}s\Big| \le \Big|\int_{t-r'}^{s_t} u(s)\,\mathrm{d}s\Big| + \Big|\int_{s_t}^{t+r'} u(s)\,\mathrm{d}s\Big| < 1$$

using the usual convention that

$$\int_a^b \mathrm{d}s = -\int_b^a \mathrm{d}s$$

when $a > b$. If $t \in (-\infty, -2T)$ then, by the definition of T, we have

$$\Big|\int_{t-r'}^{t+r'} u(s)\,\mathrm{d}s\Big| \le \int_{t-r}^{t+r} |u(s)|\,\mathrm{d}s \le \int_{-\infty}^{-2T} |u(s)|\,\mathrm{d}s < 1.$$

Similarly, if $t \in (2T, \infty)$ then

$$\Big|\int_{t-r'}^{t+r'} u(s)\,\mathrm{d}s\Big| < 1,$$

and the lemma follows. ▼

Now let $f = \chi_{[-r,r]}$ be the characteristic function of the interval $[-r, r]$. By assumption, there exists $Z \subseteq \mathbb{R}$ of zero measure such that $u * f(t) = f(t)$ for every $t \in \mathbb{R} \setminus Z$. For $t \in [-r, r] \cap (\mathbb{R} \setminus Z)$ we have

$$1 = f(t) = u * f(t) = \int_{\mathbb{R}} u(t - s) f(s)\,\mathrm{d}s = \int_{-r}^{-r} u(t - s)\,\mathrm{d}s = \int_{t-r}^{t+r} u(s)\,\mathrm{d}s < 1,$$

using the lemma and the change of variables theorem. This gives a contradiction. ■

The last four assertions of the preceding theorem exactly say that $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$, equipped with convolution as product, forms an associative, commutative algebra without unit. The first assertion says that, equipped with the 1-norm, the resulting algebra is what is known as a ***Banach algebra***. This property has the following corollary.

11.2.2 Corollary (Continuity of L^1-convolution) *The map* $(\mathrm{f}, \mathrm{g}) \mapsto \mathrm{f} * \mathrm{g}$ *from* $\mathsf{L}^1(\mathbb{R}; \mathbb{F}) \times \mathsf{L}^1(\mathbb{R}; \mathbb{F})$ *to* $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

Proof We first state a lemma that will be of use later as well.

1 Lemma *Let* $(\mathsf{U}, \|\cdot\|_{\mathsf{U}})$, $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$, *and* $(\mathsf{W}, \|\cdot\|_{\mathsf{W}})$ *be (semi)normed* $\mathbb{F}$*-vector spaces. A bilinear map* $\mathsf{B}\colon \mathsf{U} \times \mathsf{V} \to \mathsf{W}$ *is continuous, where* $\mathsf{U} \times \mathsf{V}$ *is equipped with the product topology, if and only there exists* $\mathrm{M} \in \mathbb{R}_{>0}$ *such that*

$$\|\mathsf{B}(\mathrm{u}, \mathrm{v})\|_{\mathsf{W}} \le \mathrm{M}\|\mathrm{u}\|_{\mathsf{U}}\|\mathrm{v}\|_{\mathsf{V}} \qquad (11.6)$$

for every $(\mathrm{u}, \mathrm{v}) \in \mathsf{U} \times \mathsf{V}$.

Proof First suppose that there exists $M \in \mathbb{R}_{>0}$ such that (11.6) holds. Let $(u_0, v_0) \in \mathsf{U}\times\mathsf{V}$ and let $\epsilon \in \mathbb{R}_{>0}$. Let

$$\delta = \min\left\{\sqrt{\frac{\epsilon}{3M}}, \frac{\epsilon}{3M\|u_0\|_{\mathsf{U}}}, \frac{\epsilon}{3M\|v_0\|_{\mathsf{V}}}\right\},$$

allowing that the last two terms might be infinite if either u_0 or v_0 are zero. Suppose that $(u, v) \in \mathsf{U}\times\mathsf{V}$ are such that $\|u - u_0\|_{\mathsf{U}}, \|v - v_0\|_{\mathsf{V}} < \delta$. We then compute

$$\begin{aligned}\|\mathsf{B}(u,v) - \mathsf{B}(u_0,v_0)\|_{\mathsf{W}} &\le \|\mathsf{B}(u-u_0, v-v_0)\|_{\mathsf{W}} + \|\mathsf{B}(u-u_0,v_0)\|_{\mathsf{W}} + \|\mathsf{B}(u_0, v-v_0)\|_{\mathsf{W}} \\ &\le M\|u-u_0\|_{\mathsf{U}}\|v-v_0\|_{\mathsf{V}} + M\|u-u_0\|_{\mathsf{U}}\|v_0\|_{\mathsf{V}} + M\|u_0\|_{\mathsf{U}}\|v-v_0\|_{\mathsf{V}} < \epsilon,\end{aligned}$$

giving continuity of B at (u_0, v_0).

Now suppose that B is continuous. Thus B is continuous at $(0, 0)$. Given this, let $M \in \mathbb{R}_{>0}$ be such that

$$\|u\|_{\mathsf{U}}, \|v\|_{\mathsf{V}} < \frac{2}{\sqrt{M}} \quad\Longrightarrow\quad \|\mathsf{B}(u,v)\|_{\mathsf{W}} < 1.$$

Then, for $(u, v) \in \mathsf{U}\times\mathsf{V}$,

$$\begin{aligned}\Big\|\frac{u}{\sqrt{M}\|u\|_{\mathsf{U}}}\Big\|_{\mathsf{U}}, \Big\|\frac{v}{\sqrt{M}\|v\|_{\mathsf{V}}}\Big\|_{\mathsf{V}} < \frac{2}{\sqrt{M}} \quad&\Longrightarrow\quad \Big\|\mathsf{B}\Big(\frac{u}{\sqrt{M}\|u\|_{\mathsf{U}}}, \frac{v}{\sqrt{M}\|v\|_{\mathsf{V}}}\Big)\Big\|_{\mathsf{W}} < 1 \\ &\Longrightarrow\quad \|\mathsf{B}(u,v)\|_{\mathsf{W}} < M,\end{aligned}$$

as claimed. ▼

The corollary follows immediately from the lemma. ■

The next result indicates an additional property of the algebra $\mathsf{L}^1(\mathbb{R};\mathbb{F})$, noting that the product of convolution makes this set into a ring.

11.2.3 Proposition ($\mathsf{L}^1(\mathbb{R};\mathbb{F})$ is not an integral domain) *There exists* $f, g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *with the following properties:*

(i) f *and* g *are each bounded and nowhere zero;*

(ii) $f * g(t) = 0$ *for every* $t \in \mathbb{R}$.

Proof Let $\boldsymbol{\alpha} = (\alpha_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{F})$ and define $L_{\boldsymbol{\alpha}} \colon \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F}) \to \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ by

$$L_{\boldsymbol{\alpha}}(f)(t) = \sum_{j\in\mathbb{Z}} \alpha_j f(t-j).$$

Let us verify that this map is well-defined. Certainly, for each $t \in \mathbb{R}$, the sum defining the number $L_{\boldsymbol{\alpha}}(f)(t)$ converges since $\boldsymbol{\alpha} \in \ell^1(\mathbb{Z};\mathbb{F})$. Moreover, by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}}\Big|\sum_{j\in\mathbb{Z}} f(t-j)\Big|\,\mathrm{d}t \le \int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}|\alpha_j||f(t-j)|\,\mathrm{d}t = \sum_{j\in\mathbb{Z}}|\alpha_j|\int_{\mathbb{R}}|f(t-j)|\,\mathrm{d}t = \|f\|_1\sum_{j\in\mathbb{Z}}|\alpha_j| < \infty,$$

showing that $L_{\boldsymbol{\alpha}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$.

Now let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \ell^1(\mathbb{Z};\mathbb{F})$ and let $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$. Then

$$L_{\boldsymbol{\alpha}} \circ L_{\boldsymbol{\beta}}(f)(t) = L_{\boldsymbol{\alpha}}\Big(\sum_{j\in\mathbb{Z}} \beta_j f(t-j)\Big) = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \alpha_k\beta_j f(t-j-k) = \sum_{m\in\mathbb{Z}} \gamma_m f(t-m),$$

where

$$\gamma_m = \sum_{\substack{j,k\in\mathbb{Z}\\ j+k=m}} \alpha_j\beta_k, \tag{11.7}$$

using Proposition 2.4.30. Also, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \ell^1(\mathbb{Z};\mathbb{F})$ and for $f, g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ we compute

$$\begin{aligned}
L_{\boldsymbol{\alpha}}(f) * L_{\boldsymbol{\beta}}(g)(t) &= \int_{\mathbb{R}} \Big(\sum_{j\in\mathbb{Z}} \alpha_j f(t-s-j)\Big)\Big(\sum_{k\in\mathbb{Z}} g(s-k)\Big)\,\mathrm{d}s \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha_j\beta_k \int_{\mathbb{R}} f(t-s-j)g(s-k)\,\mathrm{d}s \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha_j\beta_k \int_{\mathbb{R}} f(t-\tau-j-k)g(\tau)\,\mathrm{d}\tau \\
&= \sum_{m\in\mathbb{Z}} \gamma_m f * g(t-m) = L_{\boldsymbol{\alpha}} \circ L_{\boldsymbol{\beta}}(f * g),
\end{aligned}$$

where γ_m, $m \in \mathbb{Z}$, is as given in (11.7), where we swap the sums and the integral using Fubini's Theorem and where we use the change of variables formula.

Now define $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \ell^1(\mathbb{Z};\mathbb{F})$ by

$$\alpha_j = \begin{cases} \frac{1}{\pi}\frac{(-1)^{j/2}}{1-j^2}, & j \text{ even}, \\ \frac{1}{4}, & j \in \{-1,1\}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta_j = \begin{cases} \frac{1}{\pi}\frac{(-1)^{j/2}}{1-j^2}, & j \text{ even}, \\ -\frac{1}{4}, & j \in \{-1,1\}, \\ 0, & \text{otherwise}. \end{cases}$$

If we define the 2π-periodic signals

$$F(t) = \tfrac{1}{2}(|\cos(t)| + \cos(t)), \quad G(t) = \tfrac{1}{2}(|\cos(t)| - \cos(t)),$$

one verifies by direct computation (here we use the notion of the continuous-discrete Fourier transform discussed in Chapter 12) that

$$\mathscr{F}_{\mathrm{CD}}(F)(2\pi j) = 2\pi\alpha_j, \quad \mathscr{F}_{\mathrm{CD}}(G)(2\pi j) = 2\pi\beta_j.$$

Thus

$$F(t) = \sum_{j\in\mathbb{Z}} \alpha_j \mathrm{e}^{\mathrm{i}jt}, \quad G(t) = \sum_{j\in\mathbb{Z}} \beta_j \mathrm{e}^{\mathrm{i}jt}.$$

Note that $F(t)G(t) = 0$ for every $t \in \mathbb{R}$. Therefore,

$$0 = F(t)G(t) = \Big(\sum_{j\in\mathbb{Z}} \alpha_j \mathrm{e}^{\mathrm{i}jt}\Big)\Big(\sum_{k\in\mathbb{Z}} \beta_k \mathrm{e}^{\mathrm{i}kt}\Big) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha_j\beta_k \mathrm{e}^{\mathrm{i}(j+k)t} = \sum_{m\in\mathbb{Z}} \gamma_m \mathrm{e}^{\mathrm{i}mt},$$

where γ_m, $m \in \mathbb{Z}$, is as given by (11.7), and where we use Proposition 2.4.30 which is valid since the sums are absolutely convergent. Injectivity of the CDFT (see Theorem 12.2.1) implies that $\gamma_m = 0$ for each $m \in \mathbb{Z}$. Therefore, it follows from our computations above for the composition $L_\alpha \circ L_\beta$ and for the convolution $L_\alpha(f) * L_\beta(g)$ that, for any $f_1, g_1 \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ we have

$$L_\alpha(f_1) * L_\beta(g_1)(t) = L_\alpha \circ L_\beta(f_1 * g_1)(t) = 0$$

for all $t \in \mathbb{R}$. If we take $f_1 = g_1 = \chi_{(-1,1]}$ we see that $L_\alpha(f_1)(t)$ and $L_\beta(g_1)(t)$ are nonzero for every $t \in \mathbb{R}$. Thus the result follows taking $f = L_\alpha(f_1)$ and $g = L_\beta(g_1)$. ■

Another interesting result that holds for convolution in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ is that every signal is a convolution of two other signals in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$. The second part of the result makes mention of the CCFT which we will study in detail in Chapter 13.

11.2.4 Theorem (Convolution in $\mathsf{L}^1(\mathbb{R};\mathbb{F})$ is "surjective") *If* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *then there exists* $\mathrm{g}, \mathrm{h} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *such that* $\mathrm{f}(\mathrm{t}) = \mathrm{g} * \mathrm{h}(\mathrm{t})$ *for almost every* $\mathrm{t} \in \mathbb{R}$*. Moreover,* g *and* h *can be chosen such that* g *is an element of the closure (using the* L^1*-norm) of the ideal generated by* f *and such that* h *and* $\mathscr{F}_{\mathrm{CC}}(\mathrm{h})$ *are even positive signals.*

Proof In our proof of this result we freely make use of some results we have not yet proved. In particular, we use facts regarding the continuous-continuous Fourier transform presented in Chapter 13.

The proof begins with the construction of a function with certain properties. Our construction is based on the following basic interpolation result.

1 Lemma *Let* $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{R}_{>0}$ *satisfy* $\mathrm{a} > \mathrm{b} > \mathrm{c}$ *and let* $\mathrm{y}_1, \mathrm{y}_2 \in \mathbb{R}_{>0}$ *satisfy* $\mathrm{y}_2 > \mathrm{y}_1$ *and* $\mathrm{b} > (\mathrm{y}_2 - \mathrm{y}_1)^{-1}$*. Then, for each* $\sigma_1 \in (\mathrm{b}, \mathrm{a})$ *and* $\sigma_2 \in (\mathrm{c}, \mathrm{b})$ *for which the lines*

$$\mathrm{s} \mapsto \mathrm{y}_1 + \sigma_1 \mathrm{s}, \quad \mathrm{s} \mapsto \mathrm{y}_2 + \sigma_2(\mathrm{s} - 1)$$

intersect at a point $(\bar{\mathrm{s}}, \bar{\alpha})$ *for which* $\bar{\mathrm{s}} \in \mathbb{Q}$*, there exists* $\psi\colon [0,1] \to \mathbb{R}$ *with the following properties:*

(i) $\psi(0) = \mathrm{y}_1$*;*

(ii) $\psi(1) = \mathrm{y}_2$*;*

(iii) $\psi'(0) = \sigma_1$*;*

(iv) $\psi'(1) = \sigma_2$*;*

(v) $\psi''(0) = \psi''(1) = 0$*;*

(vi) $\psi''(\mathrm{s}) < 0$ *for* $\mathrm{s} \in (0,1)$*.*

Proof Let us write $\bar{s} = \frac{n}{d}$ and let $k \in \mathbb{Z}_{>0}$ be such that $kn \geq 2$ and $k(d-n) \geq 2$. Let $L\colon [0,1] \to \mathbb{R}$ be the function

$$L(s) = \begin{cases} y_1 + \sigma_1 s, & s \in [0, \bar{s}], \\ y_2 + \sigma_2(s-1), & s \in (\bar{s}, 1], \end{cases}$$

noting that L is continuous with a graph consisting of two line segments with positive slope, the slope for the leftmost being larger than that of the rightmost. Let ψ be the Bernstein polynomial of degree kd for L:

$$\psi(s) = \sum_{j=0}^{kd} L\Big(\frac{j}{kd}\Big)\binom{kd}{j} s^j (1-s)^{kd-j}$$

(see Section 3.5.6 for our discussion of Bernstein polynomials). It immediately follows that

$$\psi(0) = L(0) = y_1, \quad \psi(1) = L(1) = y_2.$$

By Lemma 3.5.20(**??**) we have

$$\psi^{(r)}(s) = \frac{(kd)!}{(kd-r)!} \sum_{j=0}^{kd-r} \Delta_h^r L\Big(\frac{j}{kd}\Big)\binom{kd-r}{j} s^j (1-s)^{kd-r-j} \tag{11.8}$$

for $r \in \{1, \dots, kd\}$ and where $h = \frac{1}{kd}$. Thus, in particular,

$$\psi'(0) = \frac{(kd)!}{(kd-1)!}\Delta_h^1 L(0), \quad \psi'(1) = \frac{(kd)!}{(kd-1)!}\Delta_h^1 L\Big(\frac{kd-1}{kd}\Big)$$

and

$$\psi''(0) = \frac{(kd)!}{(kd-2)!}\Delta_h^2 L(0), \quad \psi''(1) = \frac{(kd)!}{(kd-2)!}\Delta_h^2 L\Big(\frac{kd-2}{kd}\Big).$$

Since

$$\Delta_h^1 L(0) = \frac{L(h) - L(0)}{h}, \quad \Delta_h^1 L\Big(\frac{kd-1}{kd}\Big) = \frac{L(1) - L(1-h)}{h}$$

and

$$\Delta_h^2 L(0) = \frac{L(2h) - 2L(h) + L(0)}{h}, \quad \nabla_h^2 L \Delta_h^2 L\Big(\frac{kd-2}{kd}\Big) = \frac{L(1) - 2L(1-h) + L(1-2h)}{h}$$

and since

$$2h = \frac{2}{kd} \leq \frac{kn}{kd} = \bar{s}$$

and

$$1 - 2h = \frac{kd-2}{kd} \geq \frac{kn}{kd} = \bar{s} \tag{11.9}$$

by definition of k, we have

$$\psi'(0) = \sigma_1, \quad \psi'(1) = \sigma_2, \quad \psi''(0) = \psi''(1) = 0.$$

The final assertion, that $\psi''(s) < 0$ for $s \in (0,1)$ will follow from Lemma 3.5.20(**??**) if we can show that $\Delta_h^2 L(s) < 0$ for every $s \in (0, 1-2h)$.

Now let us prove that $\psi''(s) < 0$ for $s \in (0,1)$. Let $s \in [0, 1-2h]$. If $s + 2h \leq \bar{s}$ or if $s \geq \bar{s}$, it is immediate that

$$\Delta_h^2 L(s) = \frac{L(s+2h) - 2L(s+h) + L(s)}{h} = 0.$$

If $\bar{s} \in (s, s+2h)$ note that the point $(s+h, \frac{1}{2}(L(s)+L(s+2h))$ in the plane is the midpoint on the line connecting the points $(s, L(s))$ and $(s+2h, L(s+2h))$. This proves that $\Delta_h^2 L(s) \leq 0$ for every $s \in [0, 1-2h]$. Moreover, if $\bar{s} \in (s, s+2h)$ then $\Delta_h^2 L(s) < 0$. Now note that, by (11.9),

$$\frac{kn-1}{kd} = \bar{s} - h < \bar{s} \leq \frac{kd-2}{kd}.$$

Therefore, for $s \in (0,1)$ we note that, in the case of $r = 2$, the sum in (11.8) is one of nonpositive terms, and has the term

$$\Delta_h^r L(\bar{s} - h)\binom{kd-r}{j} s^j (1-s)^{kd-r-j}$$

as one if its summands. This term is negative as we showed above. Thus $\psi''(s) < 0$, as claimed. ▼

In Figure 11.16 we illustrate how one can think of the function ψ from the lemma.

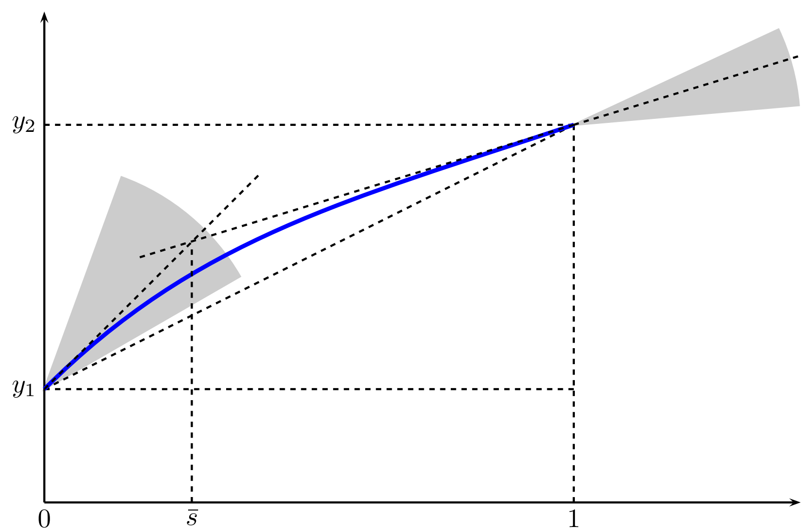


Figure 11.16 The function from Lemma 1 in the proof of Theorem 11.2.4; the shaded regions are the admissible slopes at the endpoints

Using the previous lemma, the following lemma provides the function we need.

2 Lemma *Let $(s_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $s_1 = 0$ and $s_{j+1} > 2s_j$ for $j \in \mathbb{Z}_{>0}$. Then there exists a function $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ with the following properties:*

(i) ϕ is twice continuously differentiable;

(ii) $\phi(s_j) = j$, $j \in \mathbb{Z}_{>0}$;

(iii) $s_j\phi'(s_j) < 2$, $j \geq 2$;

(iv) $\phi''(s) \leq 0$ for $s \in \mathbb{R}_{\geq 0}$.

Proof We construct ϕ by defining it on each of the intervals $[s_{j-1}, s_j]$ in such a way that when the definitions on each of these intervals are combined to give a function defined on $\mathbb{R}_{\geq 0}$ with the desired properties.

Let us give a few preliminary constructions. Define $b_1 = s_2^{-1}$, $a_1 = \frac{3}{2}b_1$, and $c_1 = (s_3 - s_2)^{-1}$. Then, for $j \geq 2$, recursively define $a_j = b_{j-1}$, $b_j = c_{j-1}$, and $c_j = (s_{j+2} - s_{j+1})^{-1}$. In our constructions, the interval (b_j, a_j) will be the valid set of slopes for graph of our function ϕ as it passes through (s_j, j). Note that $c_j < b_j < a_j$ for each $j \in \mathbb{Z}_{>0}$.

Let us now select which slopes we choose. Let $j \in \mathbb{Z}_{>0}$, let $\sigma_j \in (b_j, a_j)$ and $\sigma_{j+1} \in (b_{j+1}, a_{j+1})$, and let

$$\lambda_j(s) = j + \sigma_j(s - s_j), \quad \mu_j(s) = j + 1 + \sigma_{j+1}(s - s_{j+1})$$

be the lines through (s_j, j) and $(s_{j+1}, j+1)$ with slopes σ_j and σ_{j+1}, respectively. These lines intersect at a point in the plane whose s-coordinate is

$$s_{\sigma_j,\sigma_{j+1}} = \frac{1 + \sigma_j s_j - \sigma_{j+1} s_{j+1}}{\sigma_j - \sigma_{j+1}}.$$

One can verify that, as σ_j varies in the interval (b_j, a_j) and σ_{j+1} varies in the interval (b_{j+1}, a_{j+1}), $s_{\sigma_j,\sigma_{j+1}}$ varies throughout the interval

$$\Big(\frac{1 + a_j s_j - b_{j+1} s_{j+1}}{a_j - b_{j+1}}, \frac{1 + b_j s_j - a_{j+1} s_{j+1}}{b_j - a_{j+1}}\Big).$$

Moreover, if $\sigma_j \in (b_j, a_j)$ is fixed, $s_{\sigma_j,\sigma_{j+1}}$ varies throughout the interval

$$\Big(\frac{1 + \sigma_j s_j - b_{j+1} s_{j+1}}{\sigma_j - b_{j+1}}, \frac{1 + \sigma_j s_j - a_{j+1} s_{j+1}}{\sigma_j - a_{j+1}}\Big).$$

We then select the slopes σ_1 and σ_2 so that

$$\alpha_1 \triangleq \frac{s_{\sigma_1,\sigma_2} - s_1}{s_2 - s_1} \in \mathbb{Q}.$$

For $j \geq 3$ we then recursively define σ_j by asking that

$$\alpha_{j-1} \triangleq \frac{s_{\sigma_{j-1},\sigma_j} - s_{j-1}}{s_j - s_{j-1}} \in \mathbb{Q}.$$

These constructions are possible by Proposition 2.2.15.

Let $j \geq 2$. By Lemma 1 let $\psi_j \colon [0, 1] \to \mathbb{R}$ satisfy

1. $\psi_j(0) = j - 1$;
2. $\psi_j(1) = j$;
3. $\psi_j'(0) = \sigma_{j-1}(s_j - s_{j-1})$;
4. $\psi_j'(1) = \sigma_j(s_j - s_{j-1})$;
5. $\psi_1''(0) = \psi_1''(1) = 0$;
6. $\psi_1''(s) < 0$ for $s \in (0, 1)$.

Then define $\phi_j \colon [s_{j-1}, s_j] \to \mathbb{R}$ by

$$\phi_j(s) = \psi_j\Big(\frac{s - s_{j-1}}{s_j - s_{j-1}}\Big)$$

and note that

1. $\phi_j(s_{j-1}) = j - 1$,
2. $\phi_j(s_j) = j$,
3. $\phi_j'(s_{j-1}) = \sigma_{j-1}$,
4. $\phi_j'(s_j) = \sigma_j$,
5. $\phi_j''(s_{j-1}) = \phi_j''(s_j) = 0$, and
6. $\phi''(s) < 0$ for $s \in (s_{j-1}, s_j)$.

Then, if we define $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by asking that $\phi|[s_{j-1}, s_j] = \phi_j$, we see that ϕ has the first two properties asserted in the statement of the lemma.

For the final assertion of the lemma, note that by the Mean Value Theorem we have

$$\phi(s_j) - \phi(s_{j-1}) = \phi'(\bar{s}_j)(s_j - s_{j-1})$$

for some $\bar{s}_j \in (s_{j-1}, s_j)$. Since ϕ'' is negative on (s_{j-1}, s_j) it follows that $\phi'(s_j) < \phi'(\bar{s}_j)$. Since $\phi(s_j) = j$ and $\phi(s_{j-1}) = j - 1$ this gives

$$\phi'(s_j)(s_j - s_{j-1}) < 1.$$

Since $s_{j-1} < \frac{1}{2}s_j$ we have the final property asserted for ϕ. ▼

With the preceding technical constructions in place, we proceed with the proof proper. For $s \in \mathbb{R}_{>0}$ define $F_s\colon \mathbb{R} \to \mathbb{R}$ by asking that

$$\mathscr{F}_{\mathrm{CC}}(F_s)(\nu) = \begin{cases} 1 - \frac{|\nu|}{s}, & |\nu| \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

One can verify that

$$F_s(t) = \frac{\sin^2(\pi s t)}{s\pi^2 t^2},$$

cf. Example 13.1.3–4. Define

$$E(s) = \|F_s * f - f\|_1.$$

By Theorem 13.2.36, $\lim_{s\to\infty} E(s) = 0$.

Now choose a sequence $(s_j)_{j\in\mathbb{Z}_{>0}}$ by letting $s_j = 0$ and by taking $s_j > 2s_{j-1}$ such that $E(s) < j^{-1}$ if $s > s_j$. Then, by Lemma 2, let ϕ be such that

1. ϕ is twice continuously differentiable,
2. $\phi(s_j) = j$, $j \in \mathbb{Z}_{>0}$,
3. $s_j\phi'(s_j) < 2$, $j \geq 2$, and
4. $\phi''(s) \leq 0$ for $s \in \mathbb{R}_{\geq 0}$.

Then, for $j \geq 2$, since ϕ'' is nonpositive and using integration by parts,

$$\begin{aligned}
\int_{s_j}^{s_{j+1}} sE(s)|\phi''(s)|\,ds &< \; -j^{-2}\int_{s_j}^{s_{j+1}} s\phi''(s)\,ds \\
&= \; -j^{-2}\Big(s\phi'(s)|_{s_j}^{s_{j+1}} - \int_{s_j}^{s_{j+1}} \phi'(s)\,ds\Big) \\
&= j^{-2}(s_j\phi'(s_j) - s_{j+1}\phi'(s_{j+1}) + \phi(s_{j_1}) - \phi(s_j)) \leq 3j^{-2}.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}_{\geq 0}} sE(s)|\phi''(s)|\,ds < \int_{s_1}^{s_2} sE(s)|\phi''(s)|\,ds + 3\sum_{j=2}^{\infty}\frac{1}{j^2} < \infty$$

using Example 2.4.2–**??**. Since

$$E(s) = \int_{\mathbb{R}} |F_s * f(t) - f(t)|\,dt,$$

it follows from Fubini's Theorem that

$$(s,t) \mapsto (F_s * f(t) - f(t))s\phi''(s)$$

is integrable on $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Therefore, again by Fubini's Theorem, the function

$$s \mapsto (F_s * f(t) - f(t))s\phi''(s)$$

is integrable for almost every $t \in \mathbb{R}$ and that the function g defined by

$$g(t) = f(t) + \int_{\mathbb{R}_{\geq 0}} (F_s * f(t) - f(t))s\phi''(s)\,\mathrm{d}s$$

is integrable.

By Proposition 13.1.18 we have

$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \mathscr{F}_{\mathrm{CC}}(f)(\nu) + \mathscr{F}_{\mathrm{CC}}(f)(\nu) \int_{\mathbb{R}_{\geq 0}} (\mathscr{F}_{\mathrm{CC}}(F_s)(\nu) - 1)s\phi''(s)\,\mathrm{d}s.$$

For $\nu \in \mathbb{R}_{>0}$ we use the form of $\mathscr{F}_{\mathrm{CC}}(F_s)$ to compute

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 0}} (\mathscr{F}_{\mathrm{CC}}(F_s)(\nu) - 1)s\phi''(s)\,\mathrm{d}s &= -\int_0^\nu s\phi''(s)\,\mathrm{d}s - \nu\int_\nu^\infty \phi''(s)\,\mathrm{d}s \\
&= -s\phi'(s)|_0^\nu + \int_0^\nu \phi'(s)\,\mathrm{d}s - \nu\phi'(s)|_\nu^\infty \\
&= \phi(\nu) - \phi(0) = \phi(\nu) - 1,
\end{aligned}$$

using integration by parts. Now extend ϕ to be defined on $\mathbb{R}$ by asking that $\phi(-s) = \phi(s)$ for $s \in \mathbb{R}_{<0}$. Then, since $\mathscr{F}_{\mathrm{CC}}(F_s)(-\nu) = \mathscr{F}_{\mathrm{CC}}(F_s)(\nu)$, we have

$$\int_{\mathbb{R}_{\geq 0}} (\mathscr{F}_{\mathrm{CC}}(F_s)(\nu) - 1)s\phi''(s)\,\mathrm{d}s = \phi(\nu) - 1$$

for all $\nu \in \mathbb{R}$. Therefore,

$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \mathscr{F}_{\mathrm{CC}}(f)(\nu)\phi(\nu), \qquad \nu \in \mathbb{R}. \tag{11.10}$$

Next let $\psi(s) = \frac{1}{\phi(s)}$, this making sense since $\phi(s) \in \mathbb{R}_{>0}$ for every $s \in \mathbb{R}$. We have

$$\psi'(s) = -\frac{\phi'(s)}{\phi(s)^2}, \qquad \psi''(s) = -\frac{\phi''(s)}{\phi(s)^2} + 2\frac{\phi'(s)^2}{\phi(s)^3}.$$

This implies that ψ' is negative and ψ'' is positive in $\mathbb{R}_{\geq 0}$. Moreover, $\lim_{s\to\infty} \psi(s) = 0$ and $\lim_{s\to\infty} \psi'(s) = 0$. By the Mean Value Theorem we have

$$\psi(t_2) - \psi(t_1) = \psi'(\bar{t})(t_2 - t_1)$$

for every $t_1, t_2 \in \mathbb{R}_{\geq 0}$ satisfying $t_1 < t_2$ and for some $\bar{t} \in (t_1, t_2)$. Since ψ' is increasing we have $\psi'(t_2) > \psi'(\bar{t})$ and so

$$\psi'(t_2)t_2 > \psi(t_2) - \psi(t_1) + \psi'(t_2)t_1$$

for all $t_1, t_2 \in \mathbb{R}_{\geq 0}$ satisfying $t_1 < t_2$. Let $\epsilon \in \mathbb{R}_{>0}$ and let $t_1 \in \mathbb{R}_{\geq 0}$ be sufficiently large that $\psi(t) < \epsilon$ for $t \geq t_1$. Then

$$\psi'(t_2)t_2 > \psi(t_2) - \epsilon + \psi'(t_2)t_1,$$

and taking the limit as $t_2 \to \infty$ we have

$$\lim_{t_2 \to \infty} \psi'(t_2)t_2 > -\epsilon,$$

which, since ϵ is arbitrary, gives

$$\lim_{t_2 \to \infty} \psi'(t_2)t_2 = 0 \tag{11.11}$$

Now, by integration by parts and (11.11), we have

$$\int_{\mathbb{R}_{\geq 0}} s\psi''(s)\,\mathrm{d}s = s\psi'(s)\big|_0^\infty - \int_{\mathbb{R}_{\geq 0}} \psi'(s)\,\mathrm{d}s = \psi(0) < \infty.$$

Therefore, since $s \mapsto F_s(t)$ is bounded by 1 for every $t \in \mathbb{R}$, we can define h by

$$h(t) = \int_{\mathbb{R}_{\geq 0}} F_s(t)s\psi''(s)\,\mathrm{d}s.$$

One can easily see that

$$(s, t) \mapsto F_s(t)s\psi''(s)$$

is integrable on $\mathbb{R}_{\geq 0} \times \mathbb{R}$. By Fubini's Theorem it follows that h is integrable.

We now have

$$\mathscr{F}_{\mathrm{CC}}(h)(\nu) = \int_{\mathbb{R}_{\geq 0}} \mathscr{F}_{\mathrm{CC}}(F_s)(\nu)s\psi''(s)\,\mathrm{d}s.$$

For $\nu \in \mathbb{R}_{>0}$ we compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CC}}(h)(\nu) &= \int_\nu^\infty (s - \nu)\psi''(s)\,\mathrm{d}s \\ &= s\psi'(s)\big|_\nu^\infty - \int_\nu^\infty \psi'(s)\,\mathrm{d}s - \nu\psi'(s)\big|_\nu^\infty = \psi(\nu),\end{aligned}$$

using integration by parts and (11.11). As above, since $\mathscr{F}_{\mathrm{CC}}(F_s)$ and ψ are even, we have

$$\mathscr{F}_{\mathrm{CC}}(h)(\nu) = \psi(\nu), \qquad \nu \in \mathbb{R}. \tag{11.12}$$

Thus, combining (11.10) and (11.12), we have $\mathscr{F}_{\mathrm{CC}}(f) = \mathscr{F}_{\mathrm{CC}}(g)\mathscr{F}_{\mathrm{CC}}(h)$, and the first assertion of the proposition follows from Proposition 13.1.18.

It is clear from the definitions that h and $\mathscr{F}_{\mathrm{CC}}(h)$ are positive and even. Let $s \in \mathbb{R}_{>0}$ and define

$$\Phi_s(\nu) = \begin{cases} 0, & \nu \in (-\infty, -s], \\ (F_s)'(\nu)(1 + \frac{\nu}{s}) + F_s(\nu), & \nu \in (-s, 0], \\ (F_s)'(\nu)(1 - \frac{\nu}{s}) - F_s(\nu), & s \in (0, s], \\ 0, & s \in (s, \infty), \end{cases}$$

and note that Φ_s is the derivative of ϕF_s at those points where the latter is differentiable. Note that Φ_s is piecewise continuous with compact support. Therefore, as we showed in Corollary 13.2.28, the function

$$\Psi_s(t) = \int_{\mathbb{R}} \phi(\nu) F_s(\nu) e^{2\pi i \nu t} \, d\nu$$

is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ and $\mathscr{F}_{\mathrm{CC}}(\Psi_s) = \phi F_s$. By (11.10) we have

$$\mathscr{F}_{\mathrm{CC}}(\Psi_s)\mathscr{F}_{\mathrm{CC}}(f) = \mathscr{F}_{\mathrm{CC}}(F_s)\mathscr{F}_{\mathrm{CC}}(g).$$

By Proposition 13.1.18 it follows that $F_s * g = \Psi_s * f$ and so $F_s * g$ is in the ideal generated by f for every $s \in \mathbb{R}_{>0}$. By Theorem 13.2.36, $\lim_{s\to\infty} F_s * g = g$, the limit being taken with respect to the L^1-norm. It follows, therefore, that g is in the closure of the ideal generated by f, as claimed. ■

11.2.5 Remark (The character of factorisation in $\mathsf{L}^1(\mathbb{R};\mathbb{F})$) The proof of Theorem 11.2.20 below is easily adapted to prove the following, which is an alternative version of Theorem 11.2.4.

Let $\epsilon \in \mathbb{R}_{>0}$. If $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ then there exists $g, h \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ such that

(i) $f(t) = g * h(t)$ for almost every $t \in \mathbb{R}$,

(ii) g is in the closed ideal generated by f, and

(iii) $\|f - g\|_1 < \epsilon$.

In fact, the preceding result is somewhat easier to prove than Theorem 11.2.20 since the topology on $\mathsf{L}^1(\mathbb{R};\mathbb{F})$ is a norm topology, and is not defined by a family of seminorms, as is the topology on $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$. •

Note that one of the factors, namely h, in the convolution $f = g*h$ in the statement of the preceding theorem is a positive-valued signal. It makes sense to ask whether every nonnegative-valued signal is the convolution of two nonnegative-valued signals. This is not the case, as the following example shows.

11.2.6 Example (A nonnegative signal that is not the convolution of two nonnegative signals) Let $C_{1/2}$ be the Cantor set from Example 2.5.42 with $\epsilon = \frac{1}{2}$. We recall the following:

1. $C_{1/2}$ is compact;
2. $\mathrm{int}(A) = \emptyset$;
3. every point of $C_{1/2}$ is an accumulation point.

We will show that $\chi_{C_{1/2}}$, the characteristic function of $C_{1/2}$, is not the convolution of two nonnegative signals. We do this with a few lemmata.

1 Lemma *If* $f, g \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{R}_{\geq 0})$ *then* $f * g$ *is lower semicontinuous.*

Proof By Proposition 5.6.39 let $(f_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence of simple functions satisfying

1. $f_{j+1}(t) \geq f_j(t)$, $j \in \mathbb{Z}_{>0}$, and
2. $f(t) = \lim_{j \to \infty} f_j(t)$.

By Corollary 11.2.10 below, $f_j * g$ is continuous and bounded for each $j \in \mathbb{Z}_{>0}$. By the Monotone Convergence Theorem,

$$\lim_{j \to \infty} f_j * g(t) = f * g(t)$$

for every $t \in \mathbb{R}$. By Proposition **??** it follows that $f * g$ is lower semicontinuous. ▼

Thus it follows that if $\chi_{C_{1/2}}$ is the convolution of two nonnegative signals, then $\chi_{C_{1/2}}$ must be almost everywhere equal to a signal that is lower semicontinuous. This is not the case, as the following lemma shows.

2 Lemma *The signal* $\chi_{C_{1/2}}$ *is not almost everywhere equal to a lower semicontinuous signal.*

Proof Suppose that $\chi_{C_{1/2}}$ is almost everywhere equal to the function f. Let Z be the set of measure zero where f and $\chi_{C_{1/2}}$ differ. Let $t_0 \in \mathbb{R}$. If $t_0 \in \mathbb{R} \setminus C_{1/2}$ then, by closedness of $C_{1/2}$, there exists a neighbourhood U of t_0 such that $U \subseteq \mathbb{R} \setminus C_{1/2}$. Choose $r \in \mathbb{R}_{>0}$ such that $\mathsf{B}(r, t_0) \subseteq U$. For $j \in \mathbb{Z}_{>0}$ note that $\mathsf{B}(\frac{r}{j}, t_0) - Z \neq \emptyset$ since Z has measure zero. Choose $t_j \in \mathsf{B}(\frac{r}{j}, t_0) - Z$ so that the sequence $(t_j)_{j \in \mathbb{Z}}$ converges to t_0. Note that

$$\limsup_{j \to \infty} f(t_j) = \lim_{j \to \infty} \chi_{C_{1/2}}(t_j) = 0.$$

Next suppose that $t_0 \in C_{1/2}$. Note that $C_{1/2} \cap \mathsf{B}(r, t_0)$ has positive measure for every $r \in \mathbb{R}_{>0}$ (why?). Thus, for each $j \in \mathbb{Z}_{>0}$, we can take $t_j \in (C_{1/2} \cap \mathsf{B}(\frac{1}{j}, t_0)) - Z$. The sequence $(t_j)_{j \in \mathbb{Z}_{>0}}$ then converges to t_0 and satisfies $f(t_1) = 1$. Thus

$$\limsup_{j \to \infty} f(t_j) = \lim_{j \to \infty} \chi_{C_{1/2}}(t_j) = 0.$$

Thus the preceding holds, for some sequence $(t_j)_{j \in \mathbb{Z}_{>0}}$, for every $t_0 \in \mathbb{R}$, show that f cannot by lower semicontinuous by Proposition **??**. ▼

The previous lemma gives us an example of a nonnegative signal that is not the convolution of two nonnegative signals. •

Let us summarise the algebraic structure of $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$.

11.2.7 Theorem (The algebraic structure of $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$) *The algebra* $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$ *with the product defined by convolution has the following properties:*

(i) the multiplicative structure is commutative and associative;

(ii) it has no multiplicative unit;

(iii) the ring associated with the multiplicative structure has no primes;

(iv) the ring associated with the multiplicative structure is not an integral domain.

11.2.2 Convolution between $\mathsf{L}^{\mathrm{p}}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{\mathrm{q}}(\mathbb{R};\mathbb{F})$

In this section we consider the convolution between signals living in various L^p-spaces. There are various flavours of such results, but many of them are consequences of the following result, sometimes known as ***Young's inequality***.

11.2.8 Theorem (Convolution between $\mathsf{L}^{\mathrm{p}}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{\mathrm{q}}(\mathbb{R};\mathbb{F})$) *Let* $\mathrm{p},\mathrm{q},\mathrm{r}\in[1,\infty]$ *satisfy* $\frac{1}{\mathrm{r}}=\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}-1$*. If* $\mathrm{f}\in\mathsf{L}^{(\mathrm{p})}(\mathbb{R};\mathbb{F})$ *and* $\mathrm{g}\in\mathsf{L}^{(\mathrm{q})}(\mathbb{R};\mathbb{F})$ *then* (f,g) *is convolvable,* $\mathrm{f}*\mathrm{g}\in\mathsf{L}^{(\mathrm{r})}(\mathbb{R};\mathbb{F})$*, and* $\|\mathrm{f}*\mathrm{g}\|_{\mathrm{r}}\le\|\mathrm{f}\|_{\mathrm{p}}\|\mathrm{g}\|_{\mathrm{q}}$.

Proof The proof is conducted by considering various cases. Note that because convolution is commutative by Proposition 11.1.9(i), we do not have to consider all p's and q's as in the statement of the theorem.

$q=p'$: Let us first take the case of $r=\infty$ so that $\frac{1}{p}+\frac{1}{q}=1$, i.e., $q=p'$, the conjugate index of p. Let $t\in\mathbb{R}$. Then $s\mapsto f(t-s)$ is in $\mathsf{L}^{(q)}(\mathbb{R};\mathbb{F})$ and $s\mapsto g(s)$ is in $\mathsf{L}^{(p')}(\mathbb{R};\mathbb{F})$. By Hölder's inequality, Lemma 6.7.51 and Exercise 6.7.8, the signal $s\mapsto f(t-s)g(s)$ is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ and the 1-norm of this signal is bounded by $\|f\|_p\|g\|_{p'}$, keeping in mind that $\|\gamma_t^*f\|_p=\|f\|_p$ by translation invariance of the Lebesgue integral. Thus, in this case, $D(f,g)=\mathbb{R}$ and, moreover,

$$|h(t)|=\int_{\mathbb{R}}|f(t-s)g(s)|\,\mathrm{d}s\le\|f\|_p\|g\|_{p'},$$

giving $h\in\mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$, as desired in this case.

The remaining cases are subcases of the case $r\in[1,\infty)$, noting that $p,q\le r$ in this case. Define a, b, and c by

$$a=p\Big(1-\frac{1}{q}\Big),\quad b=\frac{r}{q},\quad c=\frac{q}{q-1}.$$

One easily verifies that $a\in[0,1)$, $b\in[1,\infty)$, and $c\in(1,\infty]$. Note that $c=q'$, the conjugate index of q. Define

$$h(t)=\int_{\mathbb{R}}|f(t-s)g(s)|\,\mathrm{d}s,$$

noting that this integral is always defined, although it may be infinite. We now consider various cases.

$q=1$, $p=r<\infty$: If $a=0$ then $q=1$ and $p=r$. In this case we can suppose that $p,r<\infty$ since the case of $p=r=\infty$ is covered in the first part of the proof. Thus we take $p\in[1,\infty)$ and let p' be the conjugate index. We compute

$$\begin{aligned}|h(t)|=\int_{\mathbb{R}}|f(t-s)g(s)|\,\mathrm{d}s&=\int_{\mathbb{R}}|f(t-s)||g(s)|^{1/p'}|g(s)|^{1/p}\Big)\,\mathrm{d}s\\&\le\|g\|_1^{1/p'}\Big(\int_{\mathbb{R}}|f(t-s)|^p|g(s)|\,\mathrm{d}s\Big)^{1/p},\end{aligned}$$

using Hölder's inequality in the form of Lemma 6.7.51. Now we compute

$$\begin{aligned}\int_{\mathbb{R}}|h(t)|^p\,\mathrm{d}t&\le\|g\|_1^{p/p'}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|f(t-s)|^p|g(s)|\,\mathrm{d}s\Big)\,\mathrm{d}t=\|g\|_1^{p/p'}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|f(t-s)|^p\,\mathrm{d}t\Big)|g(s)|\mathrm{d}s\\&\le\|g\|_1^{p/p'+1}\|f\|_p^p=\|g\|_1^p\|f\|_p^p,\end{aligned}$$

using Fubini's Theorem. This gives the result in this case.

$q \in (1, \infty)$: Here we take $a \neq 0$ so that $q \in (1, \infty)$. Note that

$$h(t) = \int_{\mathbb{R}} |f(t-s)|^{1-a} |f(t-s)|^{a} |g(s)|\, \mathrm{d}s.$$

An application of Hölder's inequality in the form of Lemma 6.7.51 gives

$$h(t) \le \Big(\int_{\mathbb{R}} |f(t-s)|^{q(1-a)} |g(s)|^{q}\, \mathrm{d}s\Big)^{1/q} \Big(\int_{\mathbb{R}} |f(t-s)|^{q' a}\, \mathrm{d}s\Big)^{1/q'}.$$

Thus, since $q'a = p$,

$$h(t) \le \|f\|_p^a \Big(\int_{\mathbb{R}} |f(t-s)|^{q(1-s)} |g(s)|^{q}\, \mathrm{d}s\Big)^{1/q},$$

giving

$$h^q(t) \le \|f\|_p^{qa} \int_{\mathbb{R}} |f(t-s)|^{q(1-s)} |g(s)|^{q}\, \mathrm{d}s,$$

Thus, since $qb = r$,

$$\begin{aligned} \|h\|_r^b = \|h\|_{qb}^b = \Big(\int_{\mathbb{R}} h^{qb}(t)\, \mathrm{d}t\Big)^{1/b} &= \|h^q\|_b \\ &\le \|f\|_p^{qa} \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(t-s)|^{q(1-s)} |g(s)|^{q}\, \mathrm{d}s\Big)^{b} \mathrm{d}t\Big)^{1/b} \\ &\le \|f\|_p^{qa} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(t-s)|^{qb(1-s)} |g(s)|^{qb}\, \mathrm{d}t\Big)^{1/b} \mathrm{d}s \end{aligned}$$

using the integral version of the Minkowski inequality, Lemma 6.7.53. Evaluating the integrals and noting that $qb(1-a) = p$ gives

$$\|h\|_r^b \le \|f\|_p^{qa} \|f\|_p^{q(1-a)} \|g\|_q^q = \|f\|_p^q \|g\|_q^q,$$

which gives $\|h\|_r^b < \infty$. ■

The preceding theorem, applied in various cases, gives a few useful corollaries.

11.2.9 Corollary (Convolution between $\mathsf{L}^1(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^p(\mathbb{R};\mathbb{F})$) *If* $p \in [1, \infty]$, *if* $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, *and if* $g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$, *then* (f, g) *is convolvable,* $f * g \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, *and* $\|f * g\|_p \le \|f\|_p \|g\|_1$.

Proof This follows from the "$a = 0$" case in the proof of Theorem 11.2.8. ■

11.2.10 Corollary (Convolution between $\mathsf{L}^p(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{p'}(\mathbb{R};\mathbb{F})$) *Let* $p \in [1, \infty]$ *and let* $p' \in [1, \infty]$ *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ *and* $g \in \mathsf{L}^{(p')}(\mathbb{R};\mathbb{F})$ *then* (f, g) *is convolvable,* $D(f, g) = \mathbb{R}$, *and* $f * g \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{F})$.

Proof That (f, g) is convolvable and that $f * g \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ follows from the "$r = \infty$" case (i.e., the $q = p'$ case) of the proof of Theorem 11.2.8. Moreover, that part of the proof shows that $D(f, g) = \mathbb{R}$. It remains to show that $f * g$ is continuous. Key to this is the following lemma, recalling that, if $a \in \mathbb{R}$, then $\tau_a^* f(t) = f(t - a)$.

1 Lemma *If* $p \in [1,\infty)$ *and if* $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, *then* $\lim_{a\to 0}\|f - \tau_a^* f\|_p = 0$.

Proof Let $\epsilon \in \mathbb{R}_{>0}$. Choose $g \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{C})$ so that $\|f-g\|_p < \frac{\epsilon}{3}$ by part (ii) of Theorem 8.3.11. Suppose that $\mathrm{supp}(g) \subseteq [\alpha,\beta]$. By uniform continuity of g (cf. Theorem 3.1.24), choose $\delta \in (0,1)$ so that $|g(t-a)-g(t)| < \frac{\epsilon}{3(\beta-\alpha+2)^{1/p}}$ when $|a| < \delta$. Then

$$\|\tau_a g - f\|_p = \Big(\int_{\alpha-1}^{\beta+1} |g(t-a)-g(t)|^p \,\mathrm{d}t\Big)^{1/p} < \Big(\int_{\alpha-1}^{\beta+1} \frac{\epsilon^p}{3^p(\beta-\alpha+2)}\,\mathrm{d}t\Big)^{1/p} = \frac{\epsilon}{3}$$

for $|a| < \delta$. We then have

$$\begin{aligned}\|\tau_a^* f - f\|_p &\le \|\tau_a^* f - \tau_a^* g\|_p + \|\tau_a^* g - f\|_p + \|f-g\|_p \\ &= 2\|f-g\|_p + \|\tau_a^* g - f\|_p < \epsilon,\end{aligned}$$

as claimed. ▼

By commutativity of convolution, Proposition 11.1.9(i), we need only consider $p \in [1,\infty)$. Recall the notation $\sigma^* f(t) = f(-t)$ and note that, for any $t \in \mathbb{R}$, $\tau_t^*\sigma^* f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ if $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, as a consequence of the change of variable theorem, Theorem 5.9.36. Now let $\epsilon \in \mathbb{R}_{>0}$ and choose $\delta \in \mathbb{R}_{>0}$ such that

$$\|\tau_a^*\tau_t^*\sigma^* f - \tau_t^*\sigma^* f\|_p < \frac{\epsilon}{\|g\|_{p'}}$$

for $|a| < \delta$, this being possible by the lemma. Then, using Hölder's inequality in the form of either Lemma 6.7.51 and Exercise 6.7.8, we have

$$\begin{aligned}|f * g(t+a) - f * g(t)| &\le \int_{\mathbb{R}} |f(t+a-s) - f(t-s)||g(s)|\,\mathrm{d}s \\ &\le \|\tau_a^*\tau_t^*\sigma^* f - \tau_t^*\sigma^* f\|_p \|g\|_{p'} < \epsilon\end{aligned}$$

for $|a| < \delta$. This gives the desired continuity of $f * g$. ■

11.2.11 Remark (Continuity of translation) In Lemma 1 in the proof of the preceding corollary we show, essentially, that translation is continuous in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ when $p \in [1,\infty)$. This conclusion is false when $p = \infty$. Indeed, if $f \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$, then $\lim_{a\to\infty}\|f - \tau_a^* f\|_\infty = 0$ if and only if f is almost everywhere equal to a uniformly continuous signal. •

The following result records the continuity of convolution in the case we are considering.

11.2.12 Corollary (Continuity of L^p-convolution) *Let* $p,q,r \in [1,\infty]$ *satisfy* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *The map* $(f,g) \mapsto f * g$ *from* $\mathsf{L}^p(\mathbb{R};\mathbb{F}) \times \mathsf{L}^q(\mathbb{R};\mathbb{F})$ *to* $\mathsf{L}^r(\mathbb{R};\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

Proof This follows from Lemma 1 from the proof of Corollary 11.2.2. ■

11.2.3 Convolution of signals with restrictions on their support

Thus far, we have considered convolvability of signals based upon their integrability properties. In this section we consider conditions for convolvability that are based on the support of one or both of the signals.

Our first result is the following.

11.2.13 Proposition (Convolvability when one of the signals has compact support) *If* $f \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{F})$ *has compact support and if* $g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$*, then* (f,g) *is convolvable.*

Proof For $t \in \mathbb{R}$ we have

$$\Big|\int_{\mathbb{R}} g(t-s)f(s)\,\mathrm{d}s\Big| \leq \|f\|_\infty \int_{\mathrm{supp}(f)} |g(t-s)|\,\mathrm{d}s < \infty$$

since the signal $s \mapsto g(t-s)$ is locally integrable. ■

Next we consider the convolvability of signals that are causal. It goes without saying that the entire discussion can be adapted to acausal signals, but the most natural and important applications are to causal signals. For a causal signal $f\colon \mathbb{R} \to \mathbb{F}$ we denote $\sigma(f) = \inf \mathrm{supp}(f)$. Thus $f(t) = 0$ for almost every $t \in (-\infty, \sigma(f)]$.

The following result is the basic one concerning the convolution of causal signals.

11.2.14 Proposition (Convolution of causal signals) *If* $f, g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *are causal then* (f,g) *is convolvable and*

$$f * g(t) = \begin{cases} \int_{\sigma(g)}^{t-\sigma(f)} f(t-s)g(s)\,\mathrm{d}s, & t \in [\sigma(f)+\sigma(g),\infty) \cap D(f,g), \\ 0, & \textit{otherwise}. \end{cases}$$

Proof Let $t \in \mathbb{R}$. Let $s > t - \sigma(f)$. Then $t - s < \sigma(f)$. Therefore, for almost every $s \in \mathbb{R} \setminus [\sigma(g), t-\sigma(f)]$ it holds that $f(t-s)g(s) = 0$. In particular, if $t < \sigma(f) + \sigma(g)$ then $f(t-s)g(s) = 0$ for almost every $s \in \mathbb{R}$, giving $f * g(t) = 0$ for every $t < \sigma(f)+\sigma(g)$. If $t \geq \sigma(f) + \sigma(g)$ then it also holds that

$$\int_{\mathbb{R}} f(t-s)g(s) = \int_{\sigma(g)}^{t-\sigma(f)} f(t-s)g(s)\,\mathrm{d}s$$

in this case, and this gives the result. ■

In Example 11.2.1 the reader can provide the analogous statement for acausal signals.

Of particular interest is the case where signals have their support contained in $\mathbb{R}_{\geq 0}$. In this case we have the following corollary of the above result.

11.2.15 Corollary (Convolution for signals with support in $\mathbb{R}_{\geq 0}$) *If* $f, g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *satisfy* $\mathrm{supp}(f), \mathrm{supp}(g) \subseteq \mathbb{R}_{\geq 0}$*, then*

(i) (f,g) *is convolvable,*

(ii) $\mathrm{supp}(f * g) \subseteq \mathbb{R}_{\geq 0}$*, and*

(iii) $f * g(t) = \displaystyle\int_0^t f(t-s)g(s)\,\mathrm{d}s,\ t \in \mathbb{R}_{\geq 0}$.

This shows that signals in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ with support in $\mathbb{R}_{\geq 0}$ are closed under the product of convolution. With this in mind, for $f, g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ we define $f \circledast g \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ by

$$f \circledast g(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}_{\geq 0},$$

so that $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is a *bona fide* algebra. With the convolutions between the various L^p-spaces, the natural topologies to consider on the various signal spaces were prescribed by the appropriate norms. However, for the signal spaces $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$, $p \in [1,\infty)$, there is no useful norm topology. We shall provide a locally convex topology for $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$, using tools from Chapter **??**. Thus on $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ we consider the family of seminorms $\|\cdot\|_{p,T}$, $T \in \mathbb{R}_{>0}$, defined by

$$\|f\|_{p,T} = \Big(\int_0^T |f(t)|^p \,\mathrm{d}t\Big)^{1/p},$$

and we note from ***missing stuff*** that the resulting locally convex topology on $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is Fréchet. In this section, we focus on $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$, this being the most important case since, for $p \in [1,\infty)$, $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F}) \subseteq \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ by Exercise 8.3.8. We refer the reader to Exercise 11.2.2 for a result on convolution in $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$.

With this notation, we have the following result that provides the basic structure of the algebra $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$.

11.2.16 Theorem ($\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is an associative, commutative algebra without unit, when equipped with convolution as a product) *For* $f, g, h \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$, *the following statements hold:*

(i) $\|f \circledast g\|_{1,T} \le \|f\|_{1,T}\|g\|_{1,T}$ *for every* $T \in \mathbb{R}_{>0}$*;*

(ii) $f \circledast g = g \circledast f$*;*

(iii) $(f \circledast g) \circledast h = f \circledast (g \circledast h)$*;*

(iv) $f \circledast (g + h) = f \circledast g + f \circledast h$*;*

(v) (recalling Remark 11.1.2) there is no equivalence class of signals $[u] \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ *such that* $[u \circledast f] = [f]$ *for every* $[f] \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$.

Proof Only parts (iii) and (v) do not follow from already proved facts.

(iii) We have

$$\begin{aligned}(f \circledast g) \circledast h(t) &= \int_0^t f * g(t-s)h(s)\,\mathrm{d}s \\ &= \int_0^t \Big(\int_0^{t-s} f(t-s-r)g(r)\,\mathrm{d}r\Big)h(s)\,\mathrm{d}s \\ &= \int_0^t \Big(\int_s^t f(t-\tau)g(\tau-s)\,\mathrm{d}\tau\Big)h(s)\,\mathrm{d}s \\ &= \int_0^t f(t-\tau)\Big(\int_0^\tau g(\tau-s)h(s)\,\mathrm{d}s\Big)\mathrm{d}\tau \\ &= \int_0^t f(t-\tau)g * h(\tau)\,\mathrm{d}\tau = f * (g * h)(t),\end{aligned}$$

using the change of variable theorem and Fubini's Theorem.

(v) The only assertion we have not yet proved is that there is no multiplicative unit. Let $u \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ be such that $u \circledast f(t) = f(t)$ for every $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ and for almost every $t \in \mathbb{R}_{\ge 0}$. This implies that $u \circledast f(t) = f(t)$ for every $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ and for almost every $t \in [0,1]$. Let $\hat{u} \in \mathsf{L}^1(\mathbb{R};\mathbb{F})$ be defined such that $\hat{u} = u(t)$ for $t \in [0,1]$ and such that it is zero off $[0,1]$. By Lemma 1 from the proof of Theorem 11.2.1, there exists $r \in \mathbb{R}_{>0}$ such that

$$\left|\int_{t-r'}^{t+r'} u(s)\,\mathrm{d}s\right| < 1, \qquad t \in \mathbb{R},\ r' \in (0,r].$$

Let $\rho = \min\{r,1\}$ and take $f = \chi_{[0,\rho]}$. By hypothesis, there exists a set $Z \subseteq [0,1]$ such that $u \circledast f(t) = f(t)$ for every $t \in [0,\rho] \setminus Z$. Thus there exists $t \in [0,1] \setminus Z$ such that $u \circledast f(t + \frac{\rho}{2}) = f(t + \frac{\rho}{2})$. Then

$$\begin{aligned} 1 = f(t+\tfrac{\rho}{2}) = u \circledast f(t+\tfrac{\rho}{2}) = u * f(t+\tfrac{\rho}{2}) &= \int_{\mathbb{R}} u(t+\tfrac{\rho}{2}-s)f(s)\,\mathrm{d}s \\ = \int_0^{\rho} u(t+\tfrac{\rho}{2}-s)\,\mathrm{d}s &= \int_{t-\rho/2}^{t+\rho/2} u(\tau)\,\mathrm{d}\tau < 1, \end{aligned}$$

the contradiction giving us the desired result. ■

We can also prove that convolution is continuous in $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ using the appropriate topology.

11.2.17 Corollary (Continuity of $\mathsf{L}^1_{\mathrm{loc}}$-convolution) *The map* $(\mathrm{f},\mathrm{g}) \mapsto \mathrm{f}\circledast\mathrm{g}$ *from* $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F}) \times \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ *to* $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

Proof Let $f, g \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ and let U be a neighbourhood of $f \circledast g$. Thus there exists $T, \epsilon \in \mathbb{R}_{>0}$ such that

$$U(T,\epsilon,f\circledast g) \triangleq \{h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F}) \mid \|h - f\circledast g\|_{1,T} < \epsilon\} \subseteq U.$$

By Lemma 1 from the proof of Corollary 11.2.2, the map $(f',g') \mapsto f'\circledast g'$ is continuous in the topology defined by the seminorm $\|\cdot\|_{1,T}$. Therefore, there exists $\delta \in \mathbb{R}_{>0}$ such that, if $f', g' \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ satisfy

$$\|f - f'\|_{1,T}, \|g - g'\|_{1,T} < \delta,$$

then

$$\|f\circledast g - f'\circledast g'\|_{1,T} < \epsilon.$$

That is, the set

$$\{(f'g') \mid \|f - f'\|_{1,T}, \|g - g'\|_{1,T} < \delta\}$$

is mapped to U by convolution, and this gives the desired continuity. ■

The following famous theorem gives some properties of this algebra (which are summarised in the corollary following the theorem).

11.2.18 Theorem (Titchmarsh[1] Convolution Theorem) *If* $\mathsf{f}, \mathsf{g} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ *are such that* $\sigma(\mathsf{f}), \sigma(\mathsf{g}) > -\infty$, *then* $\sigma(\mathsf{f} * \mathsf{g}) = \sigma(\mathsf{f}) + \sigma(\mathsf{g})$.

Proof The proof is an indirect one that relies on establishing a few facts about the so-called ***Volterra operator***. This is the linear map $V\colon \mathsf{L}^{(1)}([0,1];\mathbb{F}) \to \mathsf{L}^{(1)}([0,1];\mathbb{F})$ defined by

$$V(f)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

Let us first examine the form of V and its iterates. To do so it is convenient to denote by $\hat{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ the signal corresponding to $f \in \mathsf{L}^{(1)}([0,1];\mathbb{F})$ according to

$$\hat{f}(t) = \begin{cases} f(t), & t \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{Z}_{>0}$ let us also define $h_k\colon \mathbb{R} \to \mathbb{F}$ by

$$h_k(t) = \begin{cases} \frac{t^{k-1}}{(k-1)!}, & t \in \mathbb{R}_{\geq 0}, \\ 0, & t \in \mathbb{R}_{<0}. \end{cases}$$

With this notation we have the following lemma.

1 Lemma *If* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *and if* $\mathrm{k} \in \mathbb{Z}_{>0}$ *then* $\mathrm{V}^{\mathrm{k}}(\mathrm{f})(\mathrm{t}) = \mathrm{h}_{\mathrm{k}} * \hat{\mathrm{f}}(\mathrm{t})$ *for every* $\mathrm{t} \in [0,1]$.

Proof Let $f \in \mathsf{L}^{(1)}([0,1];\mathbb{F})$ and compute

$$V(f)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau = \int_0^t f(\tau)h_1(t-\tau)\,\mathrm{d}\tau = h_1 * f(t)$$

for all $t \in [0,1]$. This establishes the lemma for $k = 1$. Now suppose that the lemma holds for $k = r$ and compute

$$V^{r+1}(f)(t) = V(V^r(f))(t) = h_1 * (h_r * f)(t) = (h_1 * h_r) * f(t)$$

for $t \in [0,1]$, using the induction hypothesis and associativity of convolution. The result now follows since one easily verifies that $h_1 * h_r(t) = h_{r+1}$. ▼

The following result provides the invariant subspaces of the Volterra operator.

2 Lemma *For a closed subspace* $\mathsf{S} \subseteq \mathsf{L}^{(1)}([0,1];\mathbb{F})$, *the following two statements are equivalent:*

(i) *there exists* $\mathrm{b} \in [0,1]$ *such that*

$$\mathsf{S} = \{\mathrm{f} \mid \mathrm{f}(\mathrm{t}) = 0 \textit{ for almost every } \mathrm{t} \in [0,\mathrm{a}]\};$$

(ii) $\mathrm{V}(\mathsf{S}) \subseteq \mathsf{S}$.

Proof (i) $\Longrightarrow$ (ii) Supposing that S is as hypothesised for some $b \in [0,1]$, let $f \in \mathsf{S}$ and let $t \in [0,b]$. Then

$$V(f)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau = 0,$$

[1]Edward Charles Titchmarsh (1899-1963) was an English mathematician, all of whose work was in the area of analysis, including complex function theory and Fourier analysis.

and so $V(f) \in \mathsf{S}$.

(ii) $\Longrightarrow$ (i) First suppose that $\mathsf{S} \subseteq \mathsf{C}^0([0,1];\mathbb{F})$ is such that $V(\mathsf{S}) \subseteq \mathsf{S}$. For $b \in [0,1]$ denote

$$\mathsf{S}_b = \{f \in \mathsf{C}^0([0,1];\mathbb{F}) \mid f(t) = 0 \text{ for all } t \in [0,b]\}.$$

We shall prove that there exists $b \in [0,1]$ such that, if $f \in \mathsf{S}$, then $f \in \mathsf{S}_b$. In our proof of this fact, we shall use the fact that the dual of $\mathsf{C}^0([0,1];\mathbb{F})$ is $\overline{\mathsf{BV}}([0,1];\mathbb{F})$—the vector space of normalised functions of bounded variation***missing stuff*** (see Definition **??**)—and that the natural pairing between $\varphi \in \mathsf{BV}([0,1];\mathbb{F})$ and $f \in \mathsf{C}^0([0,1];\mathbb{F})$ is

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}\varphi(t);$$

see Theorem **??**. We shall denote by $\mu_\varphi\colon \mathscr{B}([0,1]) \to \mathbb{R}$***missing stuff*** the signed or complex measure defined on $[0,1]$ by

$$\mu_\varphi(A) = \int_0^1 \chi_A(t)\,\mathrm{d}\varphi(t)$$

for $A \in \mathscr{B}([0,1])$. We shall implicitly think of functions defined on a subinterval of $\mathbb{R}$ as being extended to be defined on all of $\mathbb{R}$ by taking them to be zero off the subinterval on which they are defined. In particular, we shall not make use of the "$\hat{\cdot}$" notation from Lemma 1.

Let $\mathsf{S} \subseteq \mathsf{C}^0([0,1];\mathbb{F})$ be a subspace invariant under V as above. Let $f \in \mathsf{S}$ and let S_f be the smallest subspace of S containing f and invariant under V. Let $\varphi \in \overline{\mathsf{BV}}([0,1];\mathbb{F})$ be such that $\varphi(V^k(f)) = 0$ for every $k \in \mathbb{Z}_{>0}$, i.e., φ annihilates the subspace S_f, cf. Theorem **??**. By Lemma 1, $\varphi(h_k * f) = 0$ for every $k \in \mathbb{Z}_{>0}$. Let $\sigma^*\varphi \in \mathsf{BV}([-1,0];\mathbb{F})$ be defined by $\sigma^*\phi(t) = \varphi(-t)$. By***missing stuff*** we have

$$\varphi(h_k * f) = \int_0^1 h_k * f(t)\,\mathrm{d}\varphi(t) = \int_0^1 h_k * f(t)\,\mathrm{d}(\sigma^*\varphi)(-t) = ((h_k * f) * (\sigma^*\varphi))(0) = 0$$

for every $k \in \mathbb{Z}_{>0}$. By associativity of convolution***missing stuff*** we then have

$$(h_k * (f * (\sigma^*\varphi)))(0) = 0, \qquad k \in \mathbb{Z}_{>0}.$$

Thus

$$\int_{\mathbb{R}} h_k(-t)\,\mathrm{d}(f * (\sigma^*\varphi))(t) = 0, \qquad k \in \mathbb{Z}_{>0}. \tag{11.13}$$

Let $\sigma^*\mu_\varphi$ be the signed or complex measure on $[-1,0]$ associated with $\sigma^*\varphi$.

We claim that $\operatorname{supp}(f * (\sigma^*\mu_\varphi)) \subseteq \mathbb{R}_{\geq 0}$.***missing stuff*** Indeed, suppose that $t_0 \in \mathbb{R}_{<0}$ lies in $\operatorname{supp}(\mu_\varphi)$. Then, by***missing stuff***, there exists a continuous function $g \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ such that $\operatorname{supp}(g) \subseteq [t_0-\epsilon, t_0+\epsilon]$ and such that $f*(\sigma^*\mu_\varphi)(g) \neq 0$. By the Weierstrass Approximation Theorem, let $(g_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of polynomial functions converging uniformly to g on $[t_0-\epsilon, 0]$. By (11.13) we have $f * (\sigma^*\mu_\varphi)(g_j) = 0$ for every $j \in \mathbb{Z}_{>0}$. Continuity of $f * (\sigma^*\mu_\varphi)$ on the normed vector space $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}), \|\cdot\|_\infty)$***missing stuff*** then ensures that

$$f * (\sigma^*\mu_\varphi)(g) = \lim_{j\to\infty} f * (\sigma^*\mu_\varphi)(g_j) = 0,$$

and the resulting contradiction implies that $\mathrm{supp}(f * (\sigma^*\mu_\varphi)) \subseteq \mathbb{R}_{\geq 0}$, as claimed.

Let $[\alpha, \delta]$ be the smallest compact interval such that $\mathrm{supp}(f) \subseteq [\alpha, \beta]$. We claim that $\mathrm{ann}(\mathsf{S}_f) = \mathrm{ann}(\mathsf{S}_\alpha)$. Since $f \in \mathsf{S}_\alpha$ and since S_α is invariant under V from the first part of the proof, $\mathsf{S}_f \subseteq \mathsf{S}_\alpha$. Therefore, $\mathrm{ann}(\mathsf{S}_\alpha) \subseteq \mathrm{ann}(\mathsf{S}_f)$ by Proposition **??**. Conversely, suppose that $\varphi \in \mathrm{ann}(\mathsf{S}_f)$. Let $[\delta, \gamma]$ be the smallest compact interval such that $\mathrm{supp}(\mu_\varphi) \subseteq [\alpha, \beta]$. By***missing stuff*** and***missing stuff*** we have $\mathrm{supp}(f * (\sigma^*\mu_\varphi)) = [\alpha - \gamma, \beta - \delta]$. Thus $\alpha - \gamma \geq 0$ and so $\gamma \leq \alpha$. Thus $\varphi \in \mathrm{ann}(\mathsf{S}_\alpha)$, as claimed.

Now note that S is the closed span of the union of the subspaces S_f for $f \in \mathsf{S}$. Our arguments above show that S is the closed span of the union of subspaces of the form $\mathsf{S}_{\sigma(f)}$ for $f \in \mathsf{S}$. We claim that this implies that there exists $b \in [0, 1]$ so that $\mathsf{S} = \mathsf{S}_b$. Indeed, take

$$b = \inf\{\sigma(f) \mid f \in \mathsf{S}\}.$$

First, if g is in the span of the union of the subspaces $\mathsf{S}_{\sigma(f)}$ for $f \in \mathsf{S}$ then

$$g = c_1 g_1 + \cdots + c_k g_k$$

for some $c_j \in \mathbb{F}$ and $g_j \in \mathsf{S}_{\sigma(f_j)}$, $j \in \{1, \dots, k\}$, where $f_j \in \mathsf{S}$. It immediately follows that $g(t) = 0$ for

$$t \in \min\{\sigma(f_1), \dots, \sigma(f_k)\} \geq b.$$

Thus $g \in \mathsf{S}_b$. Next let g be in the closed linear span of the union of subspaces of the form $\mathsf{S}_{\sigma(f)}$ for $f \in \mathsf{S}$. Then there exists a sequence $(g_j)_{j \in \mathbb{Z}_{>0}}$ in S_b converging uniformly, and so pointwise, to g. It follows immediately that $g \in \mathsf{S}_b$. Thus the closed linear span of the subspaces S_f for $f \in \mathsf{S}$ is S_b, as claimed.

The above arguments prove the second half of the lemma for continuous functions. Let us now prove this half of the lemma for integrable signals. Thus we let $\mathsf{S} \subseteq \mathsf{L}^{(1)}([0, 1]; \mathbb{F})$ be invariant under V. Then $V(\mathsf{S})$ is invariant under V and and, by Theorem 5.9.31, is comprised of functions that are absolutely continuous, and so continuous. By our arguments above, if

$$b = \inf\{\sigma(Vf) \mid f \in \mathsf{S}\},$$

then $V(\mathsf{S}) = \mathsf{S}_b$. Thus, if $f \in \mathsf{S}$,

$$\int_0^t f(\tau)\, \mathrm{d}\tau = 0, \qquad t \in [0, b].$$

Thus, by Lemma 5.9.30, $f(t) = 0$ for almost every $t \in [0, b]$ and so $\mathsf{S} = \mathsf{S}_b$, completing the proof. ▼

Using the preceding result on the invariant subspaces of the Volterra operator, we can prove the following result. Here we adopt the convention that functions defined on an interval are extended to $\mathbb{R}$ by taking them to be zero off the interval of definition, making no special notation for the extended function.

3 Lemma *Let* $\mathrm{b} \in \mathbb{R}_{>0}$*. If* $\mathrm{f}, \mathrm{g} \in \mathsf{L}^2([0, \mathrm{b}]; \mathbb{F})$ *satisfy* $\sigma(\mathrm{f}) = 0$ *and* $\sigma(\mathrm{f} * \mathrm{g}) \geq \mathrm{b}$*, then* $\mathrm{g}(\mathrm{t}) = 0$ *for almost every* $\mathrm{t} \in [0, \mathrm{b}]$*.*

Proof For $k \in \mathbb{Z}_{>0}$ let h_k be as defined in the proof of Lemma 2 and note that $\sigma(h_k) = 0$. By Proposition 11.1.8 we then have

$$\sigma(h_k * f * g) \geq \sigma(h_k) + \sigma(f * g) = \sigma(f * g) \geq b.$$

By Corollary 11.2.10, $h_k * f * g$ is continuous and so $h_k * f * g(t) = 0$ for all $t \in [0, b]$. Therefore, in particular,

$$0 = h_k * f * g(1) = \int_0^1 h_k * f(t) g(1-t)\,\mathrm{d}t.$$

Thus the signal $t \mapsto \bar{g}(1-t)$ is orthogonal in $\mathsf{L}^1([0,1];\mathbb{F})$ to $h_k * f = V^k(f)$, $k \in \mathbb{Z}_{>0}$, using Lemma 1. Thus, by Lemma 2, $t \mapsto \bar{g}(1-t)$ is orthogonal in $\mathsf{L}^2([0,b];\mathbb{F})$ to a subspace of the form $\mathsf{L}^2([a,b];\mathbb{F})$ for some $a \in [0,b]$. Moreover, as we saw in the proof of Lemma 2, $a = 0$ since $\sigma(f) = 0$. Therefore, g is almost everywhere zero, as claimed. ▼

Finally, we use the last lemma to prove the theorem. By translating the signals and by swapping them, we can assume without loss of generality that $\sigma(f) = 0$ and $\sigma(g) \geq 0$. By Proposition 11.1.8 it follows that $\sigma(f * g) \geq 0$. Let $M \in \mathbb{R}_{>0}$. Define

$$b = \begin{cases} \sigma(f * g), & \sigma(f * g) < \infty, \\ M, & \sigma(f * g) = \infty. \end{cases}$$

Since $\sigma(f * g) \geq \sigma(g)$ it suffices to show that $\sigma(f) \geq b$.

Now define $f_1 = h_1 * f$ and $g_1 = h_1 * g$. Since

$$f_1(t) = \int_0^t f(\tau)\,\mathrm{d}\tau, \quad g_1(t) = \int_0^t g(\tau)\,\mathrm{d}\tau$$

it follows from Theorem 5.9.31 that f_1 and g_1 are locally absolutely continuous and that their derivatives are almost everywhere equal to f and g, respectively. Therefore, $\sigma(f_1) = \sigma(f) = 0$ and $\sigma(g_1) = \sigma(g)$. Now, by Proposition 11.1.8, associativity of convolution, and the easily verified fact that $h_1 * h_1 = h_2$,

$$\sigma(f_1 * g_1) = \sigma(h_1 * h_1 * f * g) = \sigma(h_2) + \sigma(f * g) \geq b.$$

Now let $f_2, g_2 \colon \mathbb{R} \to \mathbb{F}$ be defined so that they agree with f_1 and g_1 restricted to $[0, b]$ and are zero elsewhere. Note that, for $t \in [0, b]$,

$$f_1 * g_1(t) = \int_0^t f_1(t-s) g_1(s)\,\mathrm{d}s = \int_0^t f_2(t-s) g_2(s)\,\mathrm{d}s = f_2 * g_2(t).$$

Since f_1 and g_1 are continuous, f_2 and g_2 are bounded and so in $\mathsf{L}^2([0,b];\mathbb{F})$. Since $\sigma(f_2) = \sigma(f_1) = 0$ and $\sigma(f_2 * g_2) \geq b$, by Lemma 3 it follows that g_2 is almost everywhere zero. Thus $\sigma(g_1) \geq b$ and so, since g_1 is continuous, for every $t \in [0, b]$,

$$0 = g_1(t) = \int_0^t g(\tau)\,\mathrm{d}\tau.$$

This gives $g(t) = 0$ for almost every $t \in [0, b]$ by Lemma 5.9.30. Thus the theorem follows. ■

The following consequence of the Titchmarsh Convolution Theorem as stated above is one that often carries the name of the theorem.

11.2.19 Corollary ($\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is an integral domain) *When equipped with the product* $\circledast$*,* $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ *is an integral domain.*

Proof If $f \circledast g(t) = 0$ for almost every $t \in \mathbb{R}_{\ge 0}$ then $\sigma(f \circledast g) = \infty$. It follows from the Titchmarsh Convolution Theorem that at least one of $\sigma(f)$ or $\sigma(g)$ must be infinite, which gives the result. ■

Next let us give the analogue of Theorem 11.2.4 for the algebra $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$.

11.2.20 Theorem (Convolution in $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is "surjective") *If* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ *then there exists* $\mathrm{g}, \mathrm{h} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ *such that* $\mathrm{f}(\mathrm{t}) = \mathrm{g} \circledast \mathrm{h}(\mathrm{t})$ *for almost every* $\mathrm{t} \in \mathbb{R}_{\ge 0}$*. Moreover, given* $\epsilon \in \mathbb{R}_{>0}$ *and* $\mathrm{T} \in \mathbb{R}_{>0}$*,* g *can be chosen such that*

(i) g *is in the closed ideal generated by* f *and*

(ii) $\|\mathrm{f} - \mathrm{g}\|_{1,\mathrm{T}}\,\mathrm{dt} < \epsilon$*.*

Proof First of all, let us denote $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F}) = \mathbb{F} \oplus \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ and define a product in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ by

$$(\alpha, f) \cdot (\beta, g) = (\alpha\beta, \alpha g + \beta f + f \circledast g).$$

Note that $(1, 0)$ is then an identity in this algebra. Moreover, if for $T \in \mathbb{R}_{>0}$ we define a seminorm on $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$, denoted by $\|\cdot\|_{1,T}$ (accepting a mild abuse of notation), by

$$\|(\alpha, f)\|_{1,T} = |\alpha| + \int_0^T |f(t)|\,\mathrm{d}t.$$

Note that $\|(\alpha, f) \cdot (\beta, g)\|_{1,T} \le \|(\alpha, f)\|_{1,T}\|(\beta, g)\|_{1,T}$, as may be directly verified. Let us agree to write $I = (1, 0)$ so that $(\alpha, f) = \alpha I + (0, f)$, which we simply write as $\alpha I + f$.

We now prove a few lemmata.

1 Lemma *If* $\mathrm{f} \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ *has the property that* $\|\mathrm{f}\|_{1,\mathrm{T}} < 1$ *for every* $\mathrm{T} \in \mathbb{R}_{>0}$*, then* $\mathrm{I} - \mathrm{f}$ *is invertible and*

$$(\mathrm{I} - \mathrm{f})^{-1} = \mathrm{I} + \sum_{\mathrm{n}=1}^{\infty} \mathrm{f}^{\mathrm{n}},$$

where f^{n} *denotes the* n*-fold product of* f *with itself, using the product* $\circledast$*.*

Proof First we claim that multiplication in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ is continuous. That is, we show that the map $(u, v) \mapsto u \cdot v$ is continuous, where the domain is equipped with the product topology. Let $u_0, v_0 \in \overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F})$ and let N be a neighbourhood of (u_0, v_0). Let U be a neighbourhood of $u_0 \cdot v_0$. By ***missing stuff*** there exists $T \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$ such that

$$U(T, \epsilon) \triangleq \{u \in \overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{F}) \mid \|u - u_0\|_{1,T} < \epsilon\} \subseteq U.$$

First suppose that $u_0 \ne 0$. Let $\delta \in \mathbb{R}_{>0}$ be such that

$$\delta\Big(\|v_0\|_{1,T} + \frac{\epsilon}{2\|u_0\|_{1,T}}\Big) < \frac{\epsilon}{2}$$

and let $u \in U(T, \delta)$ and $v \in U(T, \frac{\epsilon}{2\|u\|_{1,T}})$. Then

$$\|v\|_{1,T} \le \|v - v_0\|_{1,T} + \|v_0\|_{1,T} < \|v_0\|_{1,T} + \frac{\epsilon}{2\|u\|_{1,T}}$$

and so

$$\|u \cdot v - u_0 \cdot v_0\|_{1,T} \leq \|(u - u_0) \cdot v\|_{1,T} + \|u_0 \cdot (v - v_0)\|_{1,T} < \epsilon,$$

giving the desired continuity of multiplication in case $u_0 \neq 0$ by ***missing stuff***. If $u_0 = 0$ than, taking $u, v \in U(T, \sqrt{\epsilon})$, we have

$$\|u \cdot v - u_0 \cdot v_0\|_{1,T} < \epsilon,$$

completing the proof of our claim that multiplication is continuous.

We next claim that the sum $\sum_{n=1}^{\infty} f^n$ converges in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$. Let $T \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$. Then

$$\sum_{n=1}^{N} \|f^n\|_{1,T} \leq \sum_{n=1}^{N} \|f\|_{1,T}^n \leq \frac{\|f\|_{1,T}}{1 - \|f\|_{1,T}},$$

using Example 2.4.2–**??**. Thus

$$\sum_{n=1}^{\infty} \|f^n\|_{1,T} \leq \frac{\|f\|_{1,T}}{1 - \|f\|_{1,T}},$$

giving convergence of the sum on the left. Thus, if $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{n=N}^{\infty} \|f^n\|_{1,T} \leq \epsilon.$$

Therefore, if $j, k \geq N$,

$$\Big\| \sum_{n=j+1}^{k} f^n \Big\|_T \leq \sum_{n=j+1}^{k} \|f\|_{1,T}^n \leq \epsilon,$$

showing that the sequence of partial sums is Cauchy with respect to the seminorm $\|\cdot\|_{1,T}$, and so is convergent with respect to that seminorm. This gives convergence of the sum in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$ by ***missing stuff***.

Now, using continuity of multiplication to swap the multiplication and the sum, we have

$$(I - f)\Big(I + \sum_{n=1}^{\infty} f^n\Big) = I + \sum_{n=1}^{\infty} f^n - f - \sum_{n=2}^{\infty} f^n = I,$$

giving the lemma. ▼

2 Lemma *Let* $\mathrm{T} \in \mathbb{R}_{>0}$. *If* $\mathrm{G}(\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F}))$ *denotes the set of invertible (with respect to multiplication) elements of* $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$, *then the map* $\mathrm{G}(\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})) \ni \alpha\mathrm{I} + \mathrm{f} \mapsto (\alpha\mathrm{I} + \mathrm{f})^{-1}$ *is continuous in the topology defined by the seminorm* $\|\cdot\|_{1,\mathrm{T}}$.

Proof Let $v \in \overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$ and let $\epsilon \in \mathbb{R}_{>0}$. Let $\delta \in \mathbb{R}_{>0}$ be sufficiently small that

$$\frac{\|v^{-1}\|_{1,T}^2 \delta'}{1 - \|v^{-1}\|_{1,T}\delta'} < \epsilon$$

for all $\delta' \in (0, \delta)$. Then, if $\|u - v\|_{1,T} < \delta$ we have

$$\begin{aligned}\|u^{-1} - v^{-1}\|_{1,T} &= \|((I - v^{-1} \cdot (v-u))^{-1} - I) \cdot v^{-1}\|_{1,T} \\ &\le \Big\|\sum_{j=1}^{\infty} (v^{-1} \cdot (v-u))^j\Big\|_{1,T} \|v^{-1}\|_{1,T} \le \Big(\sum_{j=1}^{\infty} \|v^{-1} \cdot (v-u)\|_{1,T}^j\Big)\|v^{-1}\|_{1,T} \\ &\le \|v^{-1}\|_{1,T} \frac{\|v^{-1} \cdot (v-u)\|_{1,T}}{1 - \|v^{-1} \cdot (v-u)\|_{1,T}} \le \frac{\|v^{-1}\|_{1,T}^2 \|u - v\|_{1,T}}{1 - \|v^{-1}\|_{1,T}\|u - v\|_{1,T}} < \epsilon,\end{aligned}$$

using Lemma 1 and Proposition 6.4.2. This gives the result. ▼

3 Lemma *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{F})$*, let* $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}_{\ge 0})$ *satisfy*

$$\int_{\mathbb{R}_{\ge 0}} u(t)\, dt = 1,$$

and let $T \in \mathbb{R}_{>0}$*. Let* $\beta \in (\frac{1}{2}, 1)$ *and define*

$$v = (\beta I + (1-\beta)u)^{-1}.$$

Then

$$\|v \cdot f - f\|_{1,T} \le \frac{1-\beta}{2\beta - 1} \|u \circledast f - f\|_{1,T}.$$

Proof First note that

$$\beta I + (1-\beta)u = \beta\Big(I + \frac{\beta - 1}{\beta} u\Big)$$

and that

$$\Big\|\frac{\beta-1}{\beta} u\Big\|_{1,T} < \|u\|_{1,T} \le 1$$

for every $T \in \mathbb{R}_{>0}$. It follows from Lemma 1 that $\beta I + (1-\beta)u$ is invertible and that

$$(\beta I + (1-\beta)u)^{-1} = \beta^{-1}\Big(I + \sum_{n=1}^{\infty} \frac{\beta-1}{\beta} u^n\Big).$$

We have

$$\begin{aligned} v \cdot f - f &= ((\beta I + (1-\beta)u)^{-1} - I) \cdot f \\ &= ((\beta I + (1-\beta)u)^{-1} - (\beta I + (1-\beta)u)^{-1} \cdot (\beta I + (1-\beta)u)) \cdot f \\ &= \frac{1-\beta}{\beta}(f - u \circledast f) \cdot \Big(I + \frac{1-\beta}{\beta} u\Big)^{-1} \\ &= \frac{1-\beta}{\beta}(f - u \circledast f)\Big(I + \sum_{n=1}^{\infty}\Big(\frac{\beta-1}{\beta}\Big)^n u^n\Big).\end{aligned}$$

Therefore, by our hypothesis on u,

$$\begin{aligned}\|v \cdot f - f\|_{1,T} &\le \frac{1-\beta}{\beta} \|u \circledast f - f\|_{1,T} \sum_{n=0}^{\infty} \Big(\frac{1-\beta}{\beta}\Big)^n \\ &= \frac{1-\beta}{\beta}\|u \circledast f - f\|_{1,T} \frac{\beta}{2\beta - 1} = \frac{1-\beta}{2\beta-1}\|u \circledast f - f\|_{1,T},\end{aligned}$$

as claimed. ▼

Using the preceding lemmata, the key to proving the theorem is then the following inductive lemma.

4 Lemma *Let* $\beta \in (\frac{1}{2}, 1)$. *There exists sequences* $(\mathrm{u_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *in* $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ *and* $(\mathrm{h_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *in* $\overline{\mathsf{L}}^{-1}_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ *with the following properties for each* $\mathrm{k} \in \mathbb{Z}_{>0}$*:*

(i) $\displaystyle\int_{\mathbb{R}_{\geq 0}} \mathrm{u_k(t)\,dt} = 1$;

(ii) $\mathrm{h_k} = \beta^{\mathrm{k}}\mathrm{I} + (1-\beta)\displaystyle\sum_{\mathrm{j}=1}^{\mathrm{k}} \beta^{\mathrm{j}-1}\mathrm{u_j}$;

(iii) $\|\mathrm{h_k^{-1}\cdot f - h_{k-1}^{-1}\cdot f}\|_{1,\mathrm{kT}} < \frac{\epsilon}{2^{\mathrm{k}}}$, *where* $\mathrm{h_0 = I}$.

Proof First of all, note that if u_k satisfies (i) and if h_k satisfies (i), then h_k is invertible with respect to multiplication. To see that, note that

$$h_k = \beta^k\Big(I - \frac{\beta-1}{\beta^k}\sum_{j=1}^{k}\beta^{j-1}u_j\Big)$$

and that, for any $T \in \mathbb{R}_{>0}$,

$$\Big\|\frac{\beta-1}{\beta^k}\sum_{j=1}^{k}\beta^{j-1}u_j\Big\|_{1,T} \leq \frac{1-\beta}{\beta^k}\sum_{j=1}^{\infty}\beta^{j-1} = \frac{1}{\beta^k} < 1,$$

using Example 2.4.2–**??**. From Lemma 1 it follows that h_k is invertible as claimed.

Let $u \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ satisfy

$$\int_{\mathbb{R}_{\geq 0}} u(t)\,\mathrm{d}t = 1$$

and, for $\sigma \in \mathbb{R}_{>0}$, define $u_\sigma(t) = \sigma u(\sigma t)$. Then, by Theorem 11.3.9, let σ be sufficiently large that

$$\|u_\sigma \circledast f - f\|_{1,T} < \frac{\epsilon(2\beta-1)}{2(1-\beta)}.$$

Take $u_1 = u_\sigma$ and take $h_1 = \beta + (1-\beta)u_1$. Note that, as we showed at the beginning of the proof, h_1 is invertible. Moreover, by Lemma 3, the estimate (iii) holds. Thus we have the conditions of the lemma for $k = 1$.

Now suppose that $u_1, \dots, u_k$ and $h_1, \dots, h_k$ satisfy the four conditions of the lemma. Let u_σ, $\sigma \in \mathbb{R}_{>0}$, be the family of functions defined above. Then define

$$\begin{aligned}
w_\sigma &= (\beta I + (1-\beta)u_\sigma)^{-1},\\
u'_{\sigma,j} &= w_\sigma \cdot u_j, \qquad j \in \{1,\dots,k\},\\
h'_{\sigma,k} &= \beta^k I + (1-\beta)\sum_{j=1}^{k}\beta^{j-1}u'_{\sigma,j},\\
h_{\sigma,k+1} &= h'_{\sigma,k}\cdot w_\sigma^{-1}.
\end{aligned}$$

Note that, by Lemma 1 and Example 2.4.2–??,

$$\|w_\sigma\|_{1,(k+1)T} = \beta^{-1}\Big\|\Big(I - \frac{\beta-1}{\beta}u_\sigma\Big)^{-1}\Big\|_{1,(k+1)T} = \beta^{-1}\Big\|I + \sum_{j=1}^{\infty}\Big(\frac{\beta-1}{\beta}\Big)^j u_\sigma^j\Big\|_{1,(k+1)T}$$
$$\leq \beta^{-1}\frac{1}{1-\frac{1-\beta}{\beta}} = \frac{1}{2\beta-1}. \tag{11.14}$$

Then

$$\|h_k - h'_{\sigma,k}\|_{1,(k+1)T} = \Big\|\beta^k I + (1-\beta)\sum_{j=1}^{k}\beta^{j-1}u_j - \beta^k I - (1-\beta)\sum_{j=1}^{k}\beta^{j-1}u'_{\sigma,j}\Big\|_{1,(k+1)T}$$
$$\leq (1-\beta)\sum_{j=1}^{k}\beta^{j-1}\|u_j - u'_{\sigma,j}\|_{1,(k+1)T}$$
$$\leq (1-\beta)\Big(\sum_{j=1}^{\infty}\beta^{j-1}\Big)\max\{\|u_j - u'_{\sigma,j}\|_{1,(k+1)T} \mid j \in \{1,\dots,k\}\}$$
$$\leq \max\{\|u_j - w_\sigma \cdot u_j\|_{1,(k+1)T} \mid j \in \{1,\dots,k\}\},$$

using Example 2.4.2–??. By Lemma 3 it then follows that

$$\|h_k - h'_{\sigma,k}\|_{1,(k+1)T} \leq \frac{1-\beta}{2\beta-1}\max\{\|u_j - u_\sigma \circledast u_j\|_{1,(k+1)T} \mid j \in \{1,\dots,k\}\}. \tag{11.15}$$

Note that

$$\begin{aligned}\|h^{-1}_{\sigma,k+1}\cdot f - h_k^{-1}\cdot f\|_{1,(k+1)T} &= \|(h'_{\sigma,k})^{-1}\cdot w_\sigma \cdot f - h_k^{-1}\cdot f\|_{1,(k+1)T}\\ &\leq \|(h'_{\sigma,k})^{-1}\cdot w_\sigma\cdot f - h_k^{-1}\cdot w_\sigma\cdot f\|_{1,(k+1)T} + \|h_k^{-1}\cdot w_\sigma\cdot f - h_k^{-1}\cdot f\|_{1,(k+1)T}\\ &\leq \|(h'_{\sigma,k})^{-1} - h_k^{-1}\|_{1,(k+1)T}\|w_\sigma\cdot f\|_{1,(k+1)T} + \|h_k^{-1}\|_{1,(k+1)T}\|w_\sigma\cdot f - f\|_{1,(k+1)T}\\ &\leq \frac{1}{2\beta-1}\|(h'_{\sigma,k})^{-1} - h_k^{-1}\|_{1,(k+1)T}\|f\|_{1,(k+1)T} + \frac{1-\beta}{2\beta-1}\|h_k^{-1}\|_{1,(k+1)T}\|u_\sigma\circledast f - f\|_{1,(k+1)T},\end{aligned} \tag{11.16}$$

where we have used (11.14) and Lemma 3.

By Theorem 11.3.9 let σ be sufficiently large that

$$\frac{1-\beta}{2\beta-1}\|h_k^{-1}\|_{1,(k+1)T}\|u_\sigma\circledast f - f\|_{1,(k+1)T} < \frac{\epsilon}{2^{k+1}}. \tag{11.17}$$

By Lemma 2, let $\delta \in \mathbb{R}_{>0}$ be sufficiently small that, if $w \in G(\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F}))$ satisfies $\|w\|_{1,(k+1)T} \leq \delta$, then

$$\frac{1}{2\beta-1}\|f\|_{1,(k+1)T}\|w^{-1}\|_{1,(k+1)T} \leq \frac{\epsilon}{2^{k+2}}$$

Arguing as we did to obtain (11.17), we can take σ sufficiently large that

$$\frac{1-\beta}{2\beta-1}\max\{\|u_j - u_\sigma\circledast u_j\|_{1,(k+1)T} \mid j\in\{1,\dots,k\}\} \leq \delta.$$

By (11.15) and the definition of δ it follows that

$$\frac{1}{2\beta-1}\|(h'_{\sigma,k})^{-1}-h_k^{-1}\|_{1,(k+1)T}\|f\|_{1,(k+1)T}<\frac{\epsilon}{2^{k+1}} \tag{11.18}$$

for σ sufficiently large.

Now let σ be sufficiently large that both (11.17) and (11.18) hold and define $u_{k+1}=u_\sigma$ and $h_{k+1}=h_{\sigma,k+1}$. By (11.16) it follows that

$$\|h_{k+1}^{-1}\cdot f-h_k^{-1}\cdot f\|_{1,(k+1)T}<\frac{\epsilon}{2^{k+1}}.$$

Moreover, by definition of $h_{\sigma,k+1}$, we have

$$h_{k+1}=\Big(\beta^k I+(1-\beta)\sum_{j=1}^{k}\beta^{j-1}w_\sigma\cdot u_j\Big)(\beta I+(1-\beta)u_{k+1})=\beta^{k+1}I+(1-\beta)\sum_{j=1}^{k+1}\beta^{j-1}u_j,$$

and this completes the proof of the lemma. ▼

Now we complete the proof of the theorem. We define $g_k=h_k^{-1}\cdot f$ so that $f=g_k\cdot h_k$ for each $k\in\mathbb{Z}_{>0}$. We claim that the sequence $(h_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ to

$$h\triangleq(1-\beta)\sum_{j=1}^{\infty}\beta^{j-1}u_j. \tag{11.19}$$

First of all, note that the sum in the preceding expression converges in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$. To see this, note that for each $T\in\mathbb{R}_{>0}$

$$\sum_{j=1}^{\infty}\beta^{j-1}\|u_j\|_{1,T}\leq\sum_{j=1}^{\infty}\beta^{j-1}=\frac{1}{1-\beta},$$

using Example 2.4.2–**??**. This gives convergence of the sum on the left. Thus, if $\epsilon\in\mathbb{R}_{>0}$, there exists $N\in\mathbb{Z}_{>0}$ such that

$$\sum_{j=N}^{\infty}\beta^{j-1}\|u_j\|_{1,T}\leq\epsilon.$$

Therefore, if $j,k\geq N$,

$$\Big\|\sum_{m=j+1}^{k}\beta^{m-1}u_k\Big\|_{1,T}\leq\sum_{m=j+1}^{k}\beta^{m-1}\|u_m\|_{1,T}\leq\epsilon,$$

showing that the sequence of partial sums for the sum in (11.19) is Cauchy with respect to the seminorm $\|\cdot\|_{1,T}$, and so the sum is convergent with respect to that seminorm. This gives convergence of the sum in (11.19) in $\overline{\mathsf{L}}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ by ***missing stuff***. To show that the sequence $(h_j)_{j\in\mathbb{Z}_{>0}}$ converges to h, for each $T\in\mathbb{R}_{>0}$ we compute

$$\|h_k-h\|_{1,T}=\|\beta^k I\|_{1,T}=\beta^k.$$

Clearly then, $\lim_{k\to\infty}\|h_k - h\|_{1,T} = 0$, giving the desired convergence. By Lemma 2 and continuity of multiplication (proved during the course of the proof of Lemma 1), we can define

$$g = \lim_{k\to\infty} g_k = \lim_{k\to\infty} h_k^{-1} \cdot f.$$

Let us now show that g is in the closed ideal generated by f. This will follow if we can show that g_k is in the closed ideal of $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ generated by f for each $k \in \mathbb{Z}_{>0}$. To see that this is true, note that, by Lemma 1, h_k^{-1} has the form

$$h_k^{-1} = \alpha I + \sum_{j=1}^{\infty} v_j,$$

where $\alpha \in \mathbb{F}$, where $v_j \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$, and where the series on the right converges. Therefore,

$$g_k = \Big(\alpha I + \sum_{j=1}^{\infty} v_j\Big) \cdot f = \alpha f + \sum_{j=1}^{\infty} v_j \circledast f.$$

by continuity of multiplication. The sum is clearly in the closed ideal generated by f. Thus, to show that g_k is in the closed ideal generated by f, it suffices to show that f is in the closed ideal generated by f, cf. Theorem **??**. However, if u_σ, $\sigma \in \mathbb{R}_{>0}$, is the family of functions used above, then, by Theorem 11.3.9, $\lim_{\sigma\to\infty}\|f \circledast u_\sigma - f\|_{1,T} = 0$ for every $T \in \mathbb{R}_{>0}$. This shows that, indeed, f is in the closed ideal generated by itself by virtue of ***missing stuff***.

Now let us prove that the bound (ii) holds. For each $k \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
\|f - g_k\|_{1,T} &= \|f - f \cdot h_k^{-1}\|_{1,T} \\
&\leq \|f - f \cdot g_1^{-1}\|_{1,T} + \|f \cdot g_1^{-1} - f \cdot g_2^{-1}\|_{1,2T} + \cdots + \|f \cdot g_{k-1}^{-1} - f \cdot g_k^{-1}\|_{1,kT} \\
&\leq \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.
\end{aligned}$$

Therefore,

$$\|f - g\|_{1,T} = \lim_{k\to\infty} \|f - g_k\|_{1,T} < \epsilon,$$

as desired. ■

With the preceding results, we can summarise the algebraic character of $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$.

11.2.21 Theorem (The algebraic structure of $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$) *The algebra $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ with the product defined by convolution has the following properties:*

(i) the multiplicative structure is commutative and associative;

(ii) it has no multiplicative unit;

(iii) the ring associated with the multiplicative structure has no primes;

(iv) the ring associated with the multiplicative structure is an integral domain.

11.2.4 Convolution in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$

Much of the work for periodic convolution has been done in Section 11.1.2. In particular, in Theorem 11.1.20 we stated the fundamental theorem regarding convolution in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. From this result and from Lemma 1 from the proof of Corollary 11.2.2, we have the following result.

11.2.22 Corollary (Continuity of $\mathsf{L}^1_{\mathrm{per},T}$-convolution) *The map* $(f,g)\mapsto f*g$ *from* $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})\times\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *to* $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

We can explore the algebra $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ further by providing a few results regarding the algebraic properties of the algebra.

11.2.23 Theorem ($\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is not an integral domain) *There exists* $f,g\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *with the following properties:*

(i) f *and* g *are each bounded, continuous, and nonzero;*

(ii) $f*g(t)=0$ *for every* $t\in\mathbb{R}$.

Proof Consider the two signals

$$f(t)=\sum_{n=1}^{\infty}\frac{\cos(2\pi(2n)\frac{t}{T})}{(2n)^2},\quad g(t)=\sum_{j=1}^{\infty}\frac{\cos(2\pi(2n-1)\frac{t}{T})}{(2n-1)^2},$$

and make the following observations, making reference to the CDFT discussed in Chapter 12.

1. Since the series $\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$ and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ converge absolutely, the series defining f and g converge uniformly to a continuous function by the Weierstrass M-test.
2. We have $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})=0$ for n odd and $\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1})=0$ for n even.

As a result, $\mathscr{F}_{\mathrm{CD}}(f)\mathscr{F}_{\mathrm{CD}}(g)$ is the zero signal. However, by Proposition 12.1.19,

$$\mathscr{F}_{\mathrm{CD}}(f*g)=\mathscr{F}_{\mathrm{CD}}(f)\mathscr{F}_{\mathrm{CD}}(g),$$

giving $\mathscr{F}_{\mathrm{CD}}(f*g)=0$. Thus $f*g$ is the zero signal by Lemma 1 from the proof of Theorem 12.2.1, noting that $f*g$ is continuous by Corollary 11.2.29. ■

We can prove a surjectivity result for periodic convolution. The result here makes reference to the continuous-discrete Fourier transform which we discuss in detail in Chapter 12.

11.2.24 Theorem (Convolution in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is “surjective”) *If* $f\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *then there exists* $g,h\in\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *such that* $f(t)=g*h(t)$ *for almost every* $t\in\mathbb{R}$. *Moreover,* g *and* h *can be chosen such that* g *is an element of the closure (using the* L^1*-norm) of the ideal generated by* f *and such that* h *and* $\mathscr{F}_{\mathrm{CD}}(h)$ *are even positive signals.*

Proof Let $\hat{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ be defined by

$$\hat{f}(t) = \begin{cases} f(t), & t \in [0,T], \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 11.2.4 there exists $\hat{g}, \hat{h} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ such that

1. $\hat{f}(t) = \hat{g} * \hat{h}(t)$ for almost every $t \in \mathbb{R}$,
2. $\hat{g}$ is in the closed ideal generated by $\hat{f}$, and
3. $\hat{h}$ and $\mathscr{F}_{\mathrm{CC}}(\hat{h})$ are even and positive.

As we shall show in the proof of Proposition **??** below, we can define $g \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ by

$$g(t) = \sum_{j\in\mathbb{Z}} \hat{g}(t + jT),$$

with this sum being defined for almost every $t \in \mathbb{R}$. Similarly we define

$$h(t) = \sum_{j\in\mathbb{Z}} \hat{h}(t + jT)$$

and, moreover, note that

$$f(t) = \sum_{j\in\mathbb{Z}} \hat{f}(t + jT).$$

We claim that $f(t) = g * h(t)$ for almost every $t \in \mathbb{R}$. Indeed, for any t for which the summations are defined, we have

$$\begin{aligned}
\int_0^T g(t-s)h(s)\,\mathrm{d}s &= \int_0^T \Big(\sum_{j\in\mathbb{Z}} \hat{g}(t-s+jT)\Big)\Big(\sum_{k\in\mathbb{Z}} \hat{h}(s+kT)\Big)\,\mathrm{d}s \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_0^T \hat{g}(t-s+jT)\hat{h}(s+kT)\,\mathrm{d}s \\
&= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \int_{jT}^{(j+1)T} \hat{g}(t-s+lT)\hat{h}(s)\,\mathrm{d}s \\
&= \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \hat{g}(t+lT-s)\hat{h}(s)\,\mathrm{d}s = \sum_{l\in\mathbb{Z}} \hat{f}(t+lT) = f(t),
\end{aligned}$$

using Fubini's Theorem to swap the sum and integral and also using the change of variable formula.

Finally, we claim that h is even. Indeed, for $t \in \mathbb{R}$,

$$h(-t) = \sum_{j\in\mathbb{Z}} \hat{h}(-t+jT) = \sum_{j\in\mathbb{Z}} \hat{h}(t-jT) = \sum_{j\in\mathbb{Z}} \hat{h}(t+jT),$$

using the fact that $\hat{h}$ is even. Then, from Proposition 12.1.6(iii), we also have that $\mathscr{F}_{\mathrm{CD}}(h)$ is even, concluding the proof. ■

11.2.25 Remark (The character of factorisation in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) The proof of Theorem 11.2.20 above is easily adapted to prove the following, which is an alternative version of Theorem 11.2.4.

Let $\epsilon \in \mathbb{R}_{>0}$. If $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ then there exists $g, h \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ such that

(i) $f(t) = g * h(t)$ for almost every $t \in \mathbb{R}$,

(ii) g is in the closed ideal generated by f, and

(iii) $\|f - g\|_1 < \epsilon$.

In fact, the preceding result is somewhat easier to prove than Theorem 11.2.20 since the topology on $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ is a norm topology, and is not defined by a family of seminorms, as is the topology on $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$. •

missing stuff

We can now summarise the algebraic structure of $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

11.2.26 Theorem (The algebraic structure of $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R}_{\geq 0};\mathbb{F})$) *The algebra $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R}_{\geq 0};\mathbb{F})$ with the product defined by convolution has the following properties:*

(i) the multiplicative structure is commutative and associative;

(ii) it has no multiplicative unit;

(iii) the ring associated with the multiplicative structure has no primes;

(iv) the ring associated with the multiplicative structure is not an integral domain.

11.2.5 Convolution in $\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$

In this section we give the analogous results for periodic signals to the results from Section 11.2.2 for aperiodic signals.

11.2.27 Theorem (Convolution between $\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^q_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) *Let $\mathrm{p}, \mathrm{q}, \mathrm{r} \in [1, \infty]$ satisfy $\frac{1}{\mathrm{r}} = \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} - 1$. If $\mathrm{f} \in \mathsf{L}^{(\mathrm{p})}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$ and $\mathrm{g} \in \mathsf{L}^{(\mathrm{q})}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$ then (f, g) is convolvable, $\mathrm{f} * \mathrm{g} \in \mathsf{L}^{(\mathrm{r})}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$, and $\|\mathrm{f} * \mathrm{g}\|_{\mathrm{r}} \leq \|\mathrm{f}\|_{\mathrm{p}}\|\mathrm{g}\|_{\mathrm{q}}$.*

Proof This is proved exactly as is Theorem 11.2.8, but replacing integrals over $\mathbb{R}$ with integrals over $[0, T]$. The details of the translation can be easily performed by any exceptionally bored reader. ■

The following corollaries single out the most interesting cases of the preceding theorem. They follow from Theorem 11.2.27 in the same manner as the corresponding corollaries to Theorem 11.2.8.

11.2.28 Corollary (Convolution between $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$) *If $\mathrm{p} \in [1, \infty]$, if $\mathrm{f} \in \mathsf{L}^{(\mathrm{p})}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$, and if $\mathrm{g} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$, then (f, g) is convolvable, $\mathrm{f} * \mathrm{g} \in \mathsf{L}^{(\mathrm{p})}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$, and $\|\mathrm{f} * \mathrm{g}\|_{\mathrm{p}} \leq \|\mathrm{f}\|_{\mathrm{p}}\|\mathrm{g}\|_1$.*

11.2.29 Corollary (Convolution between $\mathsf{L}^{\mathsf{p}}_{\mathsf{per,T}}(\mathbb{R};\mathbb{F})$ and $\mathsf{L}^{\mathsf{p}'}_{\mathsf{per,T}}(\mathbb{R};\mathbb{F})$) *Let* $p \in [1,\infty]$ *and let* $p' \in [1,\infty]$ *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* $f \in \mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *and* $g \in \mathsf{L}^{(p')}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *then* (f,g) *is convolvable,* $D(f,g) = \mathbb{R}$, *and* $f * g \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$.

Proof The following lemma is key.

1 Lemma *If* $p \in [1,\infty)$ *and if* $f \in \mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, *then* $\lim_{a\to 0}\|f - \tau_a^* f\|_p = 0$.

Proof Let $\epsilon \in \mathbb{R}_{>0}$. Choose $g \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ so that $\|f - g\|_p < \frac{\epsilon}{3}$ by part (i) of Theorem 12.2.42. By uniform continuity of g (cf. Theorem 3.1.24), choose $\delta \in (0,1)$ so that $|g(t-a) - g(t)| < \frac{\epsilon}{3T^{1/p}}$ when $|a| < \delta$. Then

$$\|\tau_a g - f\|_p = \Big(\int_0^T |g(t-a) - g(t)|^p \,\mathrm{d}t\Big)^{1/p} < \Big(\int_0^T \frac{\epsilon^p}{3^p T}\,\mathrm{d}t\Big)^{1/p} = \frac{\epsilon}{3}$$

for $|a| < \delta$. We then have

$$\begin{aligned}\|\tau_a^* f - f\|_p &\le \|\tau_a^* f - \tau_a^* g\|_p + \|\tau_a^* g - f\|_p + \|f - g\|_p\\ &= 2\|f - g\|_p + \|\tau_a^* g - f\|_p < \epsilon,\end{aligned}$$

as claimed. ▼

The corollary now follows from the lemma in the same manner as Corollary 11.2.10 follows from Lemma 1. ■

11.2.30 Corollary (Continuity of $\mathsf{L}^{\mathsf{p}}_{\mathsf{per,T}}$-convolution) *Let* $p, q, r \in [1,\infty]$ *satisfy* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *The map* $(f,g) \mapsto f * g$ *from* $\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) \times \mathsf{L}^{q}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *to* $\mathsf{L}^{r}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

11.2.6 Convolution in $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$

Now we turn to convolutions of discrete-time signals. First we consider the case of absolutely summable, aperiodic signals, i.e., signals in $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$. The following result records the basic algebraic structure of this space of signals with the convolution as product.

11.2.31 Theorem ($\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ is an associative, commutative algebra with unit, when equipped with the convolution as product) *If* $f, g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ *then* (f,g) *is convolvable and* $f * g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$. *Furthermore, for* $f, g, h \in \ell^1(\mathbb{R};\mathbb{F})$, *the following statements hold:*

(i) $\|f * g\|_1 \le \|f\|_1\|g\|_1$;

(ii) $f * g = g * f$;

(iii) $(f * g) * h = f * (g * h)$;

(iv) $f * (g + h) = f * g + f * h$;

(v) *there exists* $u \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ *such that* $u * f = f$ *for every* $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$.

Proof Define $F_{f,g}\colon \mathbb{Z}(\Delta^2 \to \mathbb{F}$ by $F_{f,g}(\sigma,\tau) = f(\sigma)g(\tau)$. By Corollary 5.8.8

$$\sum_{(j,k)\in\mathbb{Z}^2} |F_{f,g}(j\Delta, k\Delta)| < \infty.$$

Now consider the change of variable $\phi\colon \mathbb{Z}(\Delta)^2 \to \mathbb{Z}(\Delta)^2$ given by $\phi(j\Delta,k\Delta) = ((k-j)\Delta, j\Delta)$, so that

$$F_{f,g}\circ\phi(j\Delta,k\Delta) = f((k-j)\Delta)g(j\Delta).$$

By Theorem 2.4.5 (why is this the right theorem to use?),

$$\sum_{(j,k)\in\mathbb{Z}^2} |f((k-j)\Delta)g(j\Delta)| < \infty.$$

By Fubini's Theorem, the function $j\Delta \mapsto f((k-j)\Delta)g(j\Delta)$ is integrable for every $k\in\mathbb{Z}$. Thus (f,g) is convolvable.

(i) Moreover, using a change of index and Fubini's Theorem again,

$$\begin{aligned}\Delta^2\sum_{k\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}} f((k-j)\Delta)g(j\Delta)\Big| &\le \Delta^2\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|f((k-j)\Delta)g(j\Delta)|\\ &= \Delta^2\sum_{(j,k)\in\mathbb{Z}^2}|f(k\Delta)||g(j\Delta)| = \|f\|_1\|g\|_1,\end{aligned}$$

as desired.

(ii) This is Proposition 11.1.26(i).

(iii) We have

$$\begin{aligned}(f*g)*h(k\Delta) &= \Delta\sum_{j\in\mathbb{Z}} f*g((k-j)\Delta)h(j\Delta)\\ &= \Delta^2\sum_{j\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}} f(k-j-l)g(l)\Big)h(j\Delta)\\ &= \Delta^2\sum_{j\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}} f((k-l)\Delta)g((l-j)\Delta)\Big)h(j\Delta)\\ &= \Delta^2\sum_{l\in\mathbb{Z}} f((k-l)\Delta)\Big(\sum_{j\in\mathbb{Z}} g((l-j)\Delta)h(j\Delta)\Big)\\ &=\Delta\sum_{l\in\mathbb{Z}} f((k-l)\Delta)g*h(l\Delta) = f*(g*h)(k\Delta),\end{aligned}$$

using a change of index and Fubini's Theorem.

(iv) This is simply linearity of the integral, Proposition 5.7.17.

(v) This was shown in Example 11.1.23. ■

We can show that the convolution we are considering in this section is continuous.

11.2.32 Corollary (Continuity of ℓ^1-convolution) *The map* $(\mathrm{f},\mathrm{g}) \mapsto \mathrm{f}*\mathrm{g}$ *from* $\ell^1(\mathbb{Z}(\Delta);\mathbb{F}) \times \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ *to* $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

Proof This follows from Lemma 1 from the proof of Corollary 11.2.2. ■

The following result gives some additional algebraic structure for $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$.

11.2.33 Proposition ($\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ is not an integral domain) *There exists* $f, g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ *with the following properties:*

(i) f *and* g *are not everywhere zero;*

(ii) $f * g(k\Delta) = 0$ *for every* $k \in \mathbb{Z}$.

Proof We shall assume some things about the CDFT and the DCFT which we discuss ion detail in Chapter 12 and in Section 14.1, respectively. Define $F, G \in \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{F})$ by asking that for $\nu \in [0, \Delta^{-1}]$ we have

$$F(\nu) = \begin{cases} \nu, & \nu \in [0, \frac{1}{4}\Delta^{-1}], \\ \frac{1}{4}\Delta^{-1}(1-\nu), & \nu \in (\frac{1}{4}\Delta^{-1}, \frac{1}{2}\Delta^{-1}], \\ 0, & \nu \in (\frac{1}{2}\Delta^{-1}, \Delta^{-1}] \end{cases}$$

and

$$G(\nu) = \begin{cases} 0, & \nu \in [0, \frac{1}{2}\Delta^{-1}], \\ \nu - \frac{1}{2}\Delta^{-1}, & \nu \in (\frac{1}{2}\Delta^{-1}, \frac{3}{4}\Delta^{-1}], \\ \frac{3}{4}\Delta^{-1} - \nu, & \nu \in (\frac{3}{4}\Delta^{-1}, \Delta^{-1}]. \end{cases}$$

Clearly we have $FG(\nu) = 0$ for every $\nu \in \mathbb{R}$. Note that F and G satisfy the hypotheses of Corollary 12.2.35. Thus, as we showed in the proof of that corollary,

$$\mathscr{F}_{\mathrm{CD}}(F), \mathscr{F}_{\mathrm{CD}}(G) \in \ell^1(\mathbb{Z}(\Delta), \mathbb{F}).$$

Moreover, injectivity of the CDFT proved in Theorem 12.2.1 gives that $\mathscr{F}_{\mathrm{CD}}(F)$ and $\mathscr{F}_{\mathrm{CD}}(G)$ are nonzero. By Proposition 14.1.12 we have

$$\mathscr{F}_{\mathrm{DC}}(\mathscr{F}_{\mathrm{CD}}(F) * \mathscr{F}_{\mathrm{CD}}(G))(\nu) = F(\nu)G(\nu) = 0$$

for every $\nu \in \mathbb{R}$. Injectivity of the DCFT proved in Theorem 14.1.14 gives $\mathscr{F}_{\mathrm{CD}}(F)(k\Delta) * \mathscr{F}_{\mathrm{CD}}(G)(k\Delta) = 0$ for every $k \in \mathbb{Z}$. ■

Note that in $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ the matter of factorisation, such as we developed in Theorems 11.2.4, 11.2.20, and 11.2.24 for various classes of continuous-time signals, is not as interesting for $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ since we always have $f = u * f$ where u is the unit alluded to in Theorem 11.2.31.

11.2.7 Convolution in $\ell^p(\mathbb{Z}(\Delta);\mathbb{F})$

In this section we give the analogous results for discrete-time signals to the results from Section 11.2.2 for continuous-time signals.

11.2.34 Theorem (Convolution between $\ell^p(\mathbb{Z}(\Delta);\mathbb{F})$ and $\ell^q(\mathbb{Z}(\Delta);\mathbb{F})$) *Let* $p, q, r \in [1, \infty]$ *satisfy* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *If* $f \in \ell^p(\mathbb{Z}(\Delta);\mathbb{F})$ *and* $g \in \ell^q(\mathbb{Z}(\Delta);\mathbb{F})$ *then* (f, g) *is convolvable,* $f * g \in \ell^r(\mathbb{Z}(\Delta);\mathbb{F})$*, and* $\|f * g\|_r \le \|f\|_p\|g\|_q$.

Proof This is proved exactly as is Theorem 11.2.8, but replacing integrals over $\mathbb{R}$ with sums over $\mathbb{Z}$, replacing the use of Lemma 6.7.51 and Exercise 6.7.8 with Lemma 6.7.16 and Exercise 6.7.2, respectively, and replacing the use of Lemma 6.7.53 with Lemma 6.7.18. ■

The following corollaries single out the most interesting cases of the preceding theorem. They follow from Theorem 11.2.34 in the same manner as the corresponding corollaries to Theorem 11.2.8.

11.2.35 Corollary (Convolution between $\ell^1(\mathbb{Z}(\Delta);\mathbb{F})$ and $\ell^p(\mathbb{Z}(\Delta);\mathbb{F})$) *If* $p \in [1,\infty]$, *if* $f \in \ell^p(\mathbb{Z}(\Delta);\mathbb{F})$, *and if* $g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{F})$, *then* (f,g) *is convolvable,* $f * g \in \ell^p(\mathbb{Z}(\Delta);\mathbb{F})$, *and* $\|f * g\|_p \le \|f\|_p\|g\|_1$.

11.2.36 Corollary (Convolution between $\ell^p(\mathbb{Z}(\Delta);\mathbb{F})$ and $\ell^{p'}(\mathbb{Z}(\Delta);\mathbb{F})$) *Let* $p \in [1,\infty]$ *and let* $p' \in [1,\infty]$ *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* $f \in \ell^p(\mathbb{Z}(\Delta);\mathbb{F})$ *and* $g \in \ell^{p'}(\mathbb{Z}(\Delta);\mathbb{F})$ *then* (f,g) *is convolvable.*

Note that the continuous-time versions of the preceding corollary, Corollaries 11.2.10 and 11.2.29, have the additional conclusion that the resulting convolution is continuous. Such conclusions do not have significance in the discrete-time case.

11.2.37 Corollary (Continuity of ℓ^p-convolution) *Let* $p, q, r \in [1,\infty]$ *satisfy* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *The map* $(f,g) \mapsto f * g$ *from* $\ell^p(\mathbb{Z}(\Delta);\mathbb{F}) \times \ell^q(\mathbb{Z}(\Delta);\mathbb{F})$ *to* $\ell^r(\mathbb{Z}(\Delta);\mathbb{F})$ *is continuous, where the domain is equipped with the product topology.*

11.2.8 Convolution and regularity

In this section we indicate how the regularity properties of one of the signals in a convolvable pair are transferred to the convolution of the signals. The basic result is the following.

11.2.38 Theorem (Differentiability and convolution for aperiodic signals) *Let* $f, g \in \mathsf{L}^{(0)}(\mathbb{R};\mathbb{F})$ *have the following properties:*

(i) f *is locally absolutely continuous;*

(ii) *for each compact set* $K \subseteq \mathbb{R}$, *the functions*

$$(s,t) \mapsto f(t-s)g(s), \quad (s,t) \mapsto f'(t-s)g(s),$$

when restricted to $\mathbb{R} \times K$, *are integrable.*

Then (f,g) *and* (f',g) *are convolvable,* $f * g$ *is locally absolutely continuous, and*

$$(f * g)'(t) = f' * g(t)$$

for almost every $t \in \mathbb{R}$.

Proof For each $k \in \mathbb{Z}_{>0}$ the hypotheses of the theorem ensure that

$$(s,t) \mapsto f(t-s)g(s), \quad (s,t) \mapsto f'(t-s)g(s)$$

are integrable when restricted to $\mathbb{R} \times [-k,k]$. By Fubini's Theorem it follows that $s \mapsto f(t-s)g(s)$ is integrable for almost every $t \in [-k,k]$. Since this is true for every $k \in \mathbb{Z}_{>0}$ it follows that (f,g) and (f',g) is convolvable.

Define $F\colon \mathbb{R}^2 \to \mathbb{F}$ by $F(t,s) = f(t-s)g(s)$. The hypotheses ensure that

1. $t \mapsto F(t,s)$ is locally absolutely continuous for every $s \in \mathbb{R}$ and,
2. for every compact set $K \subseteq \mathbb{R}$, the functions

$$(t,s) \mapsto F(t,s)g(s), \quad (t,s) \mapsto \boldsymbol{D}_1F(t,s),$$

when restricted to $K \times \mathbb{R}$, are integrable.

The result now follows immediately from Theorem 5.9.17. ■

By inductively applying the preceding result, we have the following.

11.2.39 Corollary (Higher derivatives and convolution for aperiodic signals) *Let* $g \in L^{(1)}(\mathbb{R};\mathbb{F})$ *and let* $f \in C^k(\mathbb{R};\mathbb{F})$ *have the property that* $f^{(r)}$ *is bounded for* $r \in \{0,1,\dots,k\}$. *Then* $(f^{(r)}, g)$, $r \in \{0,1,\dots,k\}$, *is convolvable,* $f * g \in C^k_{bdd}(\mathbb{R};\mathbb{F})$, *and* $(f * g)^{(r)} = f^{(r)} * g$ *for each* $r \in \{0,1,\dots,k\}$.

Proof By Corollary 11.2.10 it follows that $(f^{(r)}, g)$ is convolvable and that $f^{(r)} * g \in C^0_{bdd}(\mathbb{R};\mathbb{F})$ for each $r \in \{0,1,\dots,k\}$. Moreover, for each $r \in \{0,1,\dots,k-1\}$, the hypotheses of Theorem 11.2.38 applied to the pair $(f^{(r)}, g)$ gives that $f^{(r)} * g$ is locally absolutely continuous. As we have already shown, its derivative is continuous. Therefore, by Theorem 3.4.30 we conclude that $f^{(r)} * g$ is continuously differentiable. By Theorem 11.2.38 we have the formula $(f^{(r)} * g)' = f^{(r+1)} * g$. An elementary induction then gives the desired formula $(f * g)^{(r+1)} = f^{(r+1)} * g$. ■

One can also give differentiability results for convolutions of other sorts of signals. For example, for signals with support in $\mathbb{R}_{\geq 0}$ we have the following result.

11.2.40 Theorem (Differentiability and convolution for signals with support in $\mathbb{R}_{\geq 0}$) *Let* $f, g \in L^{(1)}_{loc}(\mathbb{R}_{\geq 0};\mathbb{F})$ *have the following properties:*

(i) f *is locally absolutely continuous;*

(ii) for each compact set $K \subseteq \mathbb{R}$, *the functions*

$$(s,t) \mapsto f(t-s)g(s), \quad (s,t) \mapsto f'(t-s)g(s),$$

when restricted to $\mathbb{R} \times K$, *are integrable.*

Then (f, g) *and* (f', g) *are convolvable,* $f \circledast g$ *is locally absolutely continuous, and*

$$(f \circledast g)'(t) = f(0)g(t) + f' \circledast g(t)$$

for almost every $t \in \mathbb{R}_{\geq 0}$.

Proof This can be proved in a slick way using distributions. However, we give a direct distribution-free proof.

We shall think of f and g as being defined on $\mathbb{R}$ by asking that they be zero on $\mathbb{R}_{<0}$. Let $\epsilon \in \mathbb{R}_{>0}$ and define $f_\epsilon\colon \mathbb{R} \to \mathbb{F}$ by

$$f_\epsilon(t) = \begin{cases} f(t), & t \in \mathbb{R}_{\geq 0}, \\ f(0)(1 + \frac{t}{\epsilon}), & t \in [-\epsilon, 0). \end{cases}$$

Note that f_ϵ is locally absolutely continuous. Now let $K \subseteq \mathbb{R}$ and note that the set

$$\{(s,t) \mid t \in K,\ s \in [0,t]\}$$

is compact and, moreover, contains the support of the functions

$$(s,t) \mapsto f_\epsilon(t-s)g(s), \quad (s,t) \mapsto f'_\epsilon(t-s)g(s)$$

when restricted to $K \times \mathbb{R}$. Since both f_ϵ and g are locally integrable, Fubini's Theorem allows us to conclude that these restricted functions are, in fact, integrable. Thus the pair (f_ϵ, g) satisfies the hypotheses of Theorem 11.2.38. Therefore, $f_\epsilon \circledast g$ is locally absolutely continuous and $(f_\epsilon \circledast g)'(t) = f'_\epsilon \circledast g(t)$ for almost every $t \in \mathbb{R}_{\geq 0}$.

Now we use a lemma.

1 Lemma *Let* $\mathrm{F}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{F}$ *have the following properties:*

(i) $\mathrm{t}\mapsto \mathrm{F}(\mathrm{s},\mathrm{t})$ *is locally absolutely continuous for almost every* $\mathrm{s}\in\mathbb{R}$*;*

(ii) the functions

$$(\mathrm{s},\mathrm{t})\mapsto \mathrm{F}(\mathrm{s},\mathrm{t}),\quad (\mathrm{s},\mathrm{t})\mapsto \mathbf{D}_2\mathrm{F}(\mathrm{s},\mathrm{t})$$

are locally integrable.

Let $\mathrm{a},\mathrm{b}\colon\mathbb{R}\to\mathbb{R}$ *be locally absolutely continuous and have the property that* $\mathrm{a}(\mathrm{t})<\mathrm{b}(\mathrm{t})$ *for every* $\mathrm{t}\in\mathbb{R}$*. Then*

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(\int_{\mathrm{a(t)}}^{\mathrm{b(t)}}\mathrm{F}(\mathrm{s},\mathrm{t})\,\mathrm{ds}\Big)=\mathrm{F}(\mathrm{a}(\mathrm{t}),\mathrm{t})-\mathrm{F}(\mathrm{b}(\mathrm{t}),\mathrm{t})+\int_{\mathrm{a(t)}}^{\mathrm{b(t)}}\mathbf{D}_2\mathrm{F}(\mathrm{s},\mathrm{t})\,\mathrm{ds}.$$

Proof Define $G\colon\mathbb{R}^3\to\mathbb{F}$ by

$$G(u,v,w)=\int_{a(u)}^{b(v)}F(s,w)\,\mathrm{d}s$$

and define $\boldsymbol{d}\colon\mathbb{R}\to\mathbb{R}^3$ by $\boldsymbol{d}(t)=(t,t,t)$. Note that

$$G\circ\boldsymbol{d}(t)=\int_{a(t)}^{b(t)}F(s,t)\,\mathrm{d}s.$$

For almost every $t\in\mathbb{R}$, by Theorem **??**, and Theorems 5.9.17 and 5.9.31, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)}F(s,t)\,\mathrm{d}s&=\boldsymbol{D}_1G(\boldsymbol{d}(t))\circ\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(t)+\boldsymbol{D}_2G(\boldsymbol{d}(t))\circ\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(t)+\boldsymbol{D}_3G(\boldsymbol{d}(t))\circ\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(t)\\&=F(a(t),t)-F(b(t),t)+\int_{a(t)}^{b(t)}\boldsymbol{D}_2F(s,t)\,\mathrm{d}s,\end{aligned}$$

as desired. ▼

Note that for $t\in\mathbb{R}_{\geq 0}$ we have

$$f\circledast g(t)=\int_0^t f(t-s)g(s)\,\mathrm{d}s=\int_0^t f_\epsilon(t-s)g(s)\,\mathrm{d}s,$$

so that, for almost every $t\in\mathbb{R}_{\geq 0}$, by the lemma,

$$\begin{aligned}(f\circledast g)'(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f_\epsilon(t-s)g(s)\,\mathrm{d}s=f_\epsilon(0)g(t)+\int_0^t f_\epsilon'(t-s)g(s)\,\mathrm{d}s\\&=f(0)g(t)+\int_0^t f'(t-s)g(s)\,\mathrm{d}s,\end{aligned}$$

as stated. ■

For higher-order derivatives we have the following result.

11.2.41 Corollary (Higher derivatives and convolution for signals with support in $\mathbb{R}_{\geq 0}$) *Let* $\mathrm{g}\in\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ *and let* $\mathrm{f}\in\mathsf{C}^{\mathrm{k}}(\mathbb{R}_{\geq 0};\mathbb{F})$*. Then* $(\mathrm{f}^{(\mathrm{r})},\mathrm{g})$, $\mathrm{r}\in\{0,1,\dots,\mathrm{k}\}$, *is convolvable,* $\mathrm{f}*\mathrm{g}\in\mathsf{C}^{\mathrm{k}}(\mathbb{R};\mathbb{F})$, *and* $(\mathrm{f}*\mathrm{g})^{(\mathrm{r})}=\mathrm{f}^{(\mathrm{r})}*\mathrm{g}$ *for each* $\mathrm{r}\in\{0,1,\dots,\mathrm{k}\}$.

Proof This follows immediately from induction using Theorem 11.2.40. ■

For periodic signals, the result is the following.

11.2.42 Theorem (Differentiability and convolution for periodic signals) *Let* $f, g \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *be such that* f *is locally absolutely continuous with* $f' \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$*. Then* (f, g) *and* (f', g) *are convolvable,* $f * g$ *is locally absolutely continuous, and*

$$(f * g)'(t) = f' * g(t)$$

for almost every $t \in \mathbb{R}$.

Proof Let $K \subseteq \mathbb{R}$ be compact. Since integrable periodic signals are locally integrable, from Fubini's Theorem we have that the signals

$$(s, t) \mapsto f(t - s)g(s), \quad (s, t) \mapsto f'(t - s)g(s)$$

are integrable when restricted to $[0, T] \times K$. The result then follows immediately from Theorem 5.9.17. ■

For higher-order derivatives we have the following result.

11.2.43 Corollary (Higher derivatives and convolution for periodic signals) *Let* $g \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R}_{\geq 0};\mathbb{F})$ *and let* $f \in \mathsf{C}^{k}_{\mathrm{per},T}(\mathbb{R}_{\geq 0};\mathbb{F})$*. Then* $(f^{(r)}, g)$*,* $r \in \{0, 1, \dots, k\}$*, is convolvable,* $f * g \in \mathsf{C}^{k}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$*, and* $(f * g)^{(r)} = f^{(r)} * g$ *for each* $r \in \{0, 1, \dots, k\}$.

Proof This follows immediately from induction using Theorem 11.2.42. ■

11.2.9 Notes

Proposition 11.2.3 is from [**EBH/GLW:90**].

Theorem 11.2.4 is from [**WR:57**], and its adaptation to prove Theorem 11.2.24 is from [**WR:58**]. Various versions of this result have been stated or shown to not be true. For example, **PK:73** shows that it is not true that every compactly supported signal is the convolution of two compactly supported signals. A generalisation of Theorem 11.2.4 to locally compact groups is given by **PJC:59**.

The existence in Lemma 1 of the concave function used in the proof of Theorem 11.2.4 follows the technique of **EP/JAR:77**.

Theorem 11.2.18 was first proved by **ECT:26**. This theorem seems to defy direct proof. The original proof of **ECT:26** relies on methods from the theory of analytic functions. Proofs in a similar style are also given by **MMC:41** and **JD:47**, the latter proof also using the Laplace transform. A proof using methods of real function theory can be found spread out in the papers [**JGM:51**, **JGM/CR-N:52**, **JGM:52**]. A more or less elementary (but still not direct) proof is given by **RD:88a**. Proofs of Titchmarsh's Convolution Theorem involving functional analysis methods are given by **GKK:57** and **MSB:57**. Our proof is based on this sort of proof in that we use the characterisation of invariant subspaces of the Volterra operator. Our proof of the character of these invariant subspaces follows the measure theoretic arguments of **WFDJr:57**.

Exercises

11.2.1 State and prove the version of Proposition 11.2.14 that is valid for acausal signals.

11.2.2 Let $p, q, r \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and let $f \in \mathsf{L}^{(p)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$ and $g \in \mathsf{L}^{(q)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$. Show that (f, g) is convolvable, that $f \circledast g \in \mathsf{L}^{(r)}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$, and that $\|f \circledast g\|_{r,T} \leq \|f\|_{p,T}\|g\|_{q,T}$ for every $T \in \mathbb{R}_{>0}$.

Section 11.3

Approximation and regularisation

One of the most useful applications of convolution is in the construction of approximations of general signals by signals with desired properties. For instance, the regularity results of Section 11.2.8 indicate that convolution often inherits the smoothness of the smoother of the signals being convolved. We shall also see that convolution allows us a means of approximating signals by signals with restrictions on their support. One way to regard this procedure of approximation is this. Note that in Theorems 11.2.1, 11.2.16, and 11.1.20 we showed that our spaces of continuous-time signals did not have a unit for the convolution product. However, we shall see that these spaces have approximate units, by which we mean a sequence of signals which, when convolved with a signal from the space, produce a sequence of signals converging to the original signal in an appropriate sense. This is the notion of an "approximate identity," and these will play an important rôle in our study of various Fourier transforms in Chapters 12 and 13, and Section 14.1.

Do I need to read this section? The material in this section is important to understand since the ideas we consider feature prominently in our discussion of Fourier inversion in Chapters 12 and 13, and Section 14.1. Moreover, this section gives an important application of the convolution product, and so is an essential part of coming to grips with convolution in general. •

11.3.1 Approximate identities on $\mathbb{R}$

We shall encounter the notion of an approximate identity for various classes of signals. In this section we study approximate identities for aperiodic signals defined on $\mathbb{R}$.

There are many variations on the definition of an approximate identity. An example of one version is what we characterised in Section 10.5.7 as a "delta-sequence," i.e., a sequence in $\mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{F})$ converging to δ_0 in $\mathscr{E}'(\mathbb{R};\mathbb{F})$. Here we choose a slightly different definition that is not equivalent to our notion of a delta-sequence, but serves our purposes here. The reader may encounter other definitions, some of which may be equivalent, some of which may not be. Just which definition one uses depends on the sort of approximations one wishes to make.

11.3.1 Definition (Approximate identity for aperiodic signals defined on $\mathbb{R}$) An ***approximate identity*** on $\mathbb{R}$ is a sequence $(u_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ with the following properties:

(i) $\displaystyle\int_{\mathbb{R}} u_j(t)\,\mathrm{d}t = 1,\ j\in\mathbb{Z}_{>0}$;

(ii) there exists $M\in\mathbb{R}_{>0}$ such that $\|u_j\|_1 \le M$ for each $j\in\mathbb{Z}_{>0}$;

(iii) for each $\alpha \in \mathbb{R}_{>0}$,

$$\lim_{j\to\infty} \int_{\mathbb{R}\setminus[-\alpha,\alpha]} |u_j(t)|\,dt = 0. \qquad \bullet$$

Before we give some examples of approximate identities, let us show why they are useful. We do this by way of two approximation theorems.

11.3.2 Theorem (Approximation in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$ using approximate identities) *Let $p \in [1,\infty)$. If $(u_j)_{j\in\mathbb{Z}_{>0}}$ is an approximate identity on $\mathbb{R}$ and if $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, then the sequence $(f * u_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$.*

Proof Note that

$$f(t) - f * u_j(t) = \int_{\mathbb{R}} (f(t) - f(t-\tau))u_j(\tau)\,d\tau,$$

using the fact that

$$\int_{\mathbb{R}} u_j(\tau)\,d\tau = 1.$$

Recalling the integral version of Minkowski's inequality, Lemma 6.7.53, we have

$$\begin{aligned}
\|f - f * u_j\|_p &= \Big(\int_{\mathbb{R}} \Big|\int_{\mathbb{R}} (f(t) - f(t-s))u_j(\tau)\,d\tau\Big|^p\,dt\Big)^{1/p} \\
&\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |(f(t) - f(t-\tau))u_j(\tau)|^p\,dt\Big)^{1/p}\,d\tau \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(t) - f(t-\tau)|^p\,dt\Big)^{1/p} |u_j(\tau)|\,d\tau \\
&\le \int_{\mathbb{R}} \|f - \tau_\tau^* f\|_p |u_j(\tau)|\,d\tau,
\end{aligned}$$

recalling the notation $\tau_\tau^* f(t) = f(t-\tau)$. Let $\epsilon \in \mathbb{R}_{>0}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \le M$ for each $j \in \mathbb{Z}_{>0}$. By Lemma 1 from the proof of Corollary 11.2.10 let $\delta \in \mathbb{R}_{>0}$ be sufficiently small that $\|f - \tau_a^* f\| < \frac{\epsilon}{2M}$ for $a \in (0,\delta]$. Then

$$\int_{-\delta}^{\delta} \|f - \tau_\tau^* f\|_p |u_j(\tau)|\,d\tau \le \frac{\epsilon}{2M} \int_{\mathbb{R}} |u_j(\tau)|\,d\tau \le \frac{\epsilon}{2}. \tag{11.20}$$

Also let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)|\,d\tau < \frac{\epsilon}{4\|f\|_p}, \qquad j \ge N.$$

Then, noting that $\|f - \tau_\tau^* f\|_p \le 2\|f\|_p$ by the triangle inequality and invariance of the norm under translation, we compute

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} \|f - \tau_\tau^* f\|_p |u_j(\tau)|\,ds \le 2\|f\|_p \int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)|\,d\tau < \frac{\epsilon}{2}. \tag{11.21}$$

Combining (11.20) and (11.21) we see that, for $j \ge N$,

$$\|f - f * u_j\|_p < \epsilon,$$

giving the result. ■

For continuous signals we also have an approximation result using approximate identities.

11.3.3 Theorem (Approximation in $C^0_{bdd}(\mathbb{R};\mathbb{F})$ using approximate identities) *If* $(u_j)_{j\in\mathbb{Z}_{>0}}$ *is an approximate identity on* $\mathbb{R}$ *and if* $f \in C^0_{bdd}(\mathbb{R};\mathbb{F})$*, then, for each compact set* $K \subseteq \mathbb{R}$*, the sequence* $(f * u_j|K)_{j\in\mathbb{Z}_{>0}}$ *converges uniformly to* $f|K$.

Proof Let $K \subseteq \mathbb{R}$ be compact and let $\epsilon \in \mathbb{R}_{>0}$. Choose $T \in \mathbb{R}_{>0}$ sufficiently large that $K \subseteq [-T, T]$. As in the proof of Theorem 11.3.2, noting that

$$\int_{\mathbb{R}} u_j(\tau)\,d\tau = 1$$

we have

$$f(t) - f * u_j(t) = \int_{\mathbb{R}} (f(t) - f(t-\tau))u_j(\tau)\,d\tau \tag{11.22}$$

for every $t \in \mathbb{R}$ and $j \in \mathbb{Z}_{>0}$. Note that since f is bounded this integral makes sense for all $t \in \mathbb{R}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \leq M$ for each $j \in \mathbb{Z}_{>0}$. Note that f is uniformly continuous on $[-T, T]$. Thus there exists $\delta \in \mathbb{R}_{>0}$ such that

$$|f(t) - f(t-\tau)| < \frac{\epsilon}{2M}$$

when $t \in [-T, T]$ and $|\tau| < \delta$. Then

$$\int_{-\delta}^{\delta} |f(t) - f(t-\tau)||u_j(\tau)|\,d\tau \leq \frac{\epsilon}{2M}\int_{\mathbb{R}} |u_j(\tau)|\,d\tau < \frac{\epsilon}{2}. \tag{11.23}$$

Now let $C = \|f\|_\infty$ and note that, for every $t_1, t_2 \in \mathbb{R}$, $|f(t_1) - f(t_2)| \leq 2C$ using the triangle inequality. Now there exists $N \in \mathbb{Z}_{>0}$ such that

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)|\,d\tau < \frac{\epsilon}{4C}$$

for $j \geq N$. Therefore, if $j \geq N$ we have

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |f(t) - f(t-\tau)||u_j(\tau)|\,d\tau < \frac{\epsilon}{2}. \tag{11.24}$$

Putting (11.22), (11.23), and (11.24) together we have

$$|f(t) - f * u_j(t)| < \epsilon, \qquad j \geq N,\ t \in K,$$

giving the result. ■

Our next approximation result also deals with continuous signals. Here we get the stronger result of uniform convergence on $\mathbb{R}$, but by adding the hypothesis of uniform continuity.

11.3.4 Theorem (Approximation in $C^0_{unif,bdd}(\mathbb{R};\mathbb{F})$ using approximate identities) *If* $(u_j)_{j\in\mathbb{Z}_{>0}}$ *is an approximate identity on* $\mathbb{R}$ *and if* $f \in C^0_{unif,bdd}(\mathbb{R};\mathbb{F})$*, then the sequence* $(f * u_j|K)_{j\in\mathbb{Z}_{>0}}$ *converges uniformly to* f.

Proof Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \leq M$ for every $j \in \mathbb{Z}_{>0}$. Choose $\delta \in \mathbb{R}_{>0}$ so that $|f(t-s) - f(t)| < \frac{\epsilon}{2M}$ for $|s| < \delta$, this being possible by uniform continuity of f. Let $N \in \mathbb{Z}_{>0}$ be such that

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(s)|\,\mathrm{d}s \leq \frac{\epsilon}{2\|f\|_\infty}.$$

For each $j \in \mathbb{Z}$ we have

$$\int_{\mathbb{R}} u_j(s)\,\mathrm{d}s = 1$$

and so, for each $t \in \mathbb{R}$ and $j \geq N$,

$$\begin{aligned}|f * u_j(t) - f(t)| &= \Big|\int_{\mathbb{R}} u_j(s)(f(t-s) - f(t))\,\mathrm{d}s\Big| \\ &\leq \int_{-\delta}^{\delta} |u_j(s)||f(t-s) - f(t)|\,\mathrm{d}s + \int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(s)||f(t-s) - f(t)|\,\mathrm{d}s \\ &\leq \|u_j\|_1 \frac{\epsilon}{2M} + 2\|f\|_\infty \int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(s)|\,\mathrm{d}s < \epsilon,\end{aligned}$$

giving the result. ■

Our final result is a pointwise convergence result.

11.3.5 Theorem (Pointwise approximations using even approximate identities) *Let $(u_j)_{j\in\mathbb{Z}_{>0}}$ be an approximate identity such that $u_j(-t) = u_j(t)$ for each $j \in \mathbb{Z}_{>0}$ and $t \in \mathbb{R}$. If $f \in L^{(\infty)}(\mathbb{R};\mathbb{F})$ and if, for $t_0 \in \mathbb{R}$, the limits $f(t_0-)$ and $f(t_0+)$ exist, then $(f * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ converges to $\frac{1}{2}(f(t_0-) + f(t_0+))$.*

Proof We may obviously assume that f is not almost everywhere zero. First suppose that f is continuous at t_0. Let $\epsilon \in \mathbb{R}_{>0}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \leq M$ for every $j \in \mathbb{Z}_{>0}$. Let $\delta \in \mathbb{R}_{>0}$ be such that, if $|\tau| < \delta$, then $|f(t_0 - \tau) - f(t_0)| \leq \frac{\epsilon}{2M}$. Then

$$\int_{-\delta}^{\delta} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,\mathrm{d}\tau \leq \frac{\epsilon}{2M}\int_{\mathbb{R}} |u_j(\tau)|\,\mathrm{d}\tau < \frac{\epsilon}{2}.$$

Note that $|f(t_0 - \tau) - f(t_0)| \leq 2\|f\|_\infty$. Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)|\,\mathrm{d}\tau < \frac{\epsilon}{4\|f\|_\infty}$$

for $j \geq N$. Then

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,\mathrm{d}\tau < \frac{\epsilon}{2}$$

for $j \geq N$.

Now, as in the proofs of Theorems 11.3.2 and 11.3.3, we have

$$f(t_0) - f * u_j(t_0) = \int_{\mathbb{R}} (f(t_0) - f(t_0-\tau))u_j(\tau)\,\mathrm{d}\tau$$

and so

$$|f(t_0) - f * u_j(t_0)| \leq \int_{-\delta}^{\delta} |u_j(\tau)||f(t_0 - \tau) - f(t_0)| \,\mathrm{d}\tau \\ + \int_{\mathbb{R}\setminus[-\delta,\delta]} |u_j(\tau)||f(t_0 - \tau) - f(t_0)| \,\mathrm{d}\tau < \epsilon$$

for $j \geq N$.

Thus our result holds if f is continuous at t_0. If the limits $f(t_0-)$ and $f(t_0+)$ both exist but are not necessarily equal, then one applies the argument for signals continuous at $t = 0$ to the signal $g(t) = \frac{1}{2}(f(t_0 - t) + f(t_0 + t))$. Convergence of $(g * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ to $g(0)$ then implies convergence of $(f * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ to $\frac{1}{2}(f(t_0-) + f(t_0+))$ using the fact that

$$\int_{\mathbb{R}} f(t_0 - s)u_j(\tau) \,\mathrm{d}\tau = \frac{1}{2} \int_{\mathbb{R}} (f(t_0 - \tau) + f(t_0 + \tau))u_j(\tau) \,\mathrm{d}\tau$$

by evenness of u_j, $j \in \mathbb{Z}_{>0}$. ■

Pointwise convergence at Lebesgue points in `convolution.pdf.gz`

There is a large class of approximate identities that arise in the following way.

11.3.6 Proposition (Approximate identities on $\mathbb{R}$ generated by a single signal) *If* $\mathrm{u} \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{F})$ *satisfies*

$$\int_{\mathbb{R}} \mathrm{u(t)} \,\mathrm{dt} = 1,$$

then the sequence $(\mathrm{u_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *defined by* $\mathrm{u_j(t) = ju(jt)}$ *is an approximate identity on* $\mathbb{R}$*.*

Proof By the change of variables formula we have

$$\int_{\mathbb{R}} u_j(t) \,\mathrm{d}t = \int_{\mathbb{R}} u(t) \,\mathrm{d}t$$

and $\|u_j\|_1 = \|u\|_1$ for every $j \in \mathbb{Z}_{>0}$, immediately giving the first two properties of an approximate identity. For the third property, let $\alpha \in \mathbb{R}_{>0}$. Note that since $u \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{F})$ the limit

$$\int_{-R}^{R} |u(t)| \,\mathrm{d}t$$

exists and so

$$\lim_{R\to 0} \int_{\mathbb{R}\setminus[-R,R]} |u(t)| \,\mathrm{d}t = 0.$$

Therefore, using the change of variables theorem,

$$\lim_{j\to\infty} \int_{\mathbb{R}\setminus[-\alpha,\alpha]} |u_j(t)| \,\mathrm{d}t = \lim_{j\to\infty} \int_{\mathbb{R}\setminus[-j\alpha,j\alpha]} |u(\tau)| \,\mathrm{d}\tau = 0,$$

as desired. ■

Let us give some examples of approximate identities.

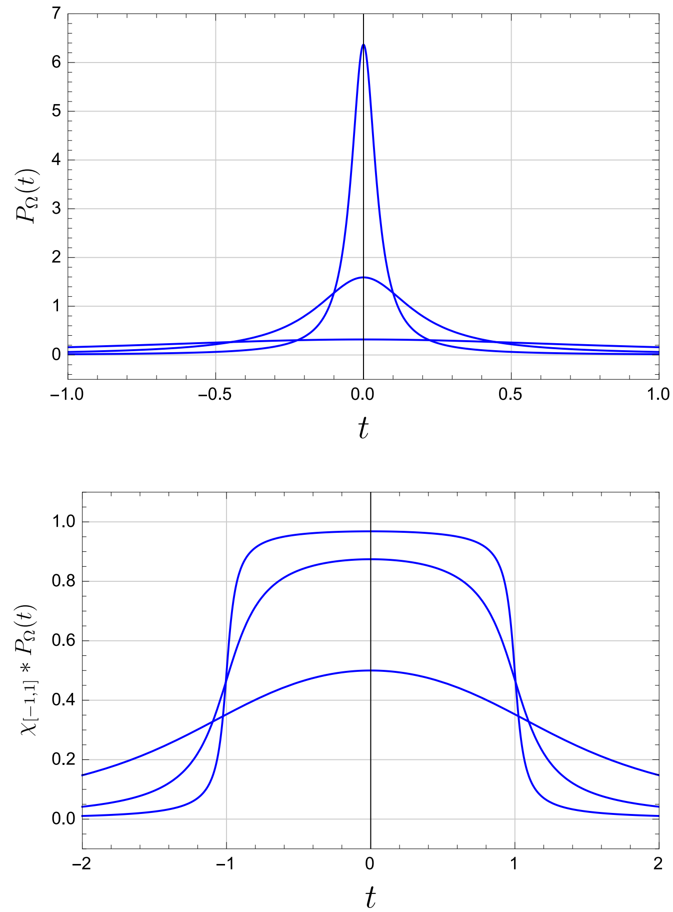


Figure 11.17 The Poisson kernel P_Ω for $\Omega \in \{1, 5, 20\}$ (top) and the corresponding approximations by convolution of the characteristic function of $[-1, 1]$ (bottom)

11.3.7 Examples (Approximate identities on $\mathbb{R}$)

1. Let us define ***Poisson kernel*** on $\mathbb{R}$ for $\Omega \in \mathbb{R}_{>0}$ by

$$P_\Omega(t) = \frac{1}{\pi} \frac{\Omega}{1 + \Omega^2 t^2}, \qquad t \in \mathbb{R}.$$

In Figure 11.17 we plot it for a few values of Ω.
It is clear that $P_\Omega \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{R})$ (see Exercise 8.3.9). Also, by the change of variable theorem we compute

$$\frac{\Omega}{\pi} \int_{\mathbb{R}} \frac{1}{1 + \Omega^2 t^2} \, \mathrm{d}t = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1 + \tau^2} \, \mathrm{d}\tau = \frac{1}{\pi} \tan^{-1}(\tau)|_{-\infty}^{\infty} = 1,$$

recalling that, as we showed in the proof of Theorem 3.6.18,

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}(x) = \frac{1}{1 + x^2}$$

(also see Example **??–??**). Note that

$$jP_\Omega(jt) = \frac{1}{\pi}\frac{j\Omega}{1+(j\Omega)^2t^2} = P_{j\Omega}.$$

Therefore, by Proposition 11.3.6, $(P_{j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for every $\Omega \in \mathbb{R}_{>0}$. Moreover, this also shows that the limit as $j \to \infty$ in Theorems 11.3.2, 11.3.3, 11.3.4, and 11.3.5 can be replaced with the limit as $\Omega \to \infty$. Said precisely, if $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$, then $\lim_{\Omega\to\infty}\|f - f * P_\Omega\|_p = 0$ and, if $f \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{F})$ then the family of signals $(f * P_\Omega)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to f on every compact set.***missing stuff*** In Figure 11.17 we show a few approximations of the signal $\chi_{[-1,1]}$ by convolution with the Poisson kernel.

There is an alternative representation of the Poisson kernel that will be useful to us in ***missing stuff***. To make this representation, we think of the Poisson kernel as being a function of t and Ω and make the change of variable

$$\mathbb{R}_{>0} \times \mathbb{R}(\Omega, t) \mapsto (\tfrac{1}{\Omega}, t) \in \mathbb{R}_{>0} \times \mathbb{R},$$

calling the new variables (x, y). In these variables, the Poisson kernel is expressed as

$$P(x, y) = \frac{1}{\pi}\frac{x}{x^2 + y^2}.$$

We can think of P as being defined on the plane, and indeed the complex plane. In doing this, the limit as $\Omega \to \infty$ becomes the limit as $x \to 0$ from the right, i.e., approaching the imaginary axis in the complex plane.

2. Here, for $\Omega \in \mathbb{R}_{>0}$, we define the ***Gauss–Weierstrass kernel*** on $\mathbb{R}$ by

$$G_\Omega(t) = \frac{\exp(-\frac{t^2}{4\Omega})}{\sqrt{4\pi\Omega}}$$

for $t \in \mathbb{R}$. In Figure 11.18 Note that $G_\Omega \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ by Exercise 8.3.9. By Lemma 1 from Example 5.3.32–**??** and a change of variable, we easily determine that $\|G_\Omega\|_1 = 1$ for every $\Omega \in \mathbb{R}_{>0}$. Thus, if we define

$$G_{\Omega,j}(t) = jG_\Omega(jt),$$

we see from Proposition 11.3.6 that the sequence $(G_{\Omega,j})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity. In Figure 11.18 we show a few approximations by convolution with the Gauss–Weierstrass kernel of the characteristic function of $[-1, 1]$.

3. The ***Fejér kernel***[2] on $\mathbb{R}$ is defined for $\Omega \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}$ by

$$F_\Omega(t) = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \neq 0, \\ \Omega, & t = 0. \end{cases}$$

In Figure 11.19 we show the Fejér kernel for a few values of Ω.

[2]Lipót Fejér (1880-1959) was a Hungarian mathematician whose main area of mathematical activity was harmonic analysis.

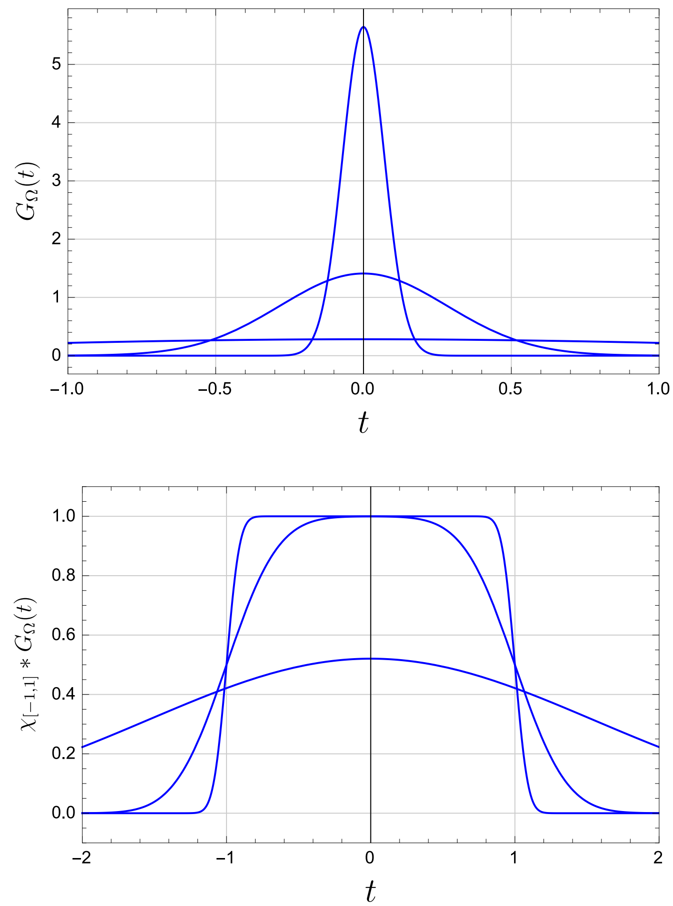


Figure 11.18 The Gauss–Weierstrass kernel G_Ω for $\Omega \in \{1, 5, 20\}$ (top) and the corresponding approximations by convolution of the characteristic function of $[-1, 1]$ (bottom)

Let us show that $F_\Omega \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{R})$ and that $\|F_\Omega\|_1 = 1$. First we prove a couple of lemmata. First we define $\operatorname{sinc}\colon \mathbb{R} \to \mathbb{R}$ by

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

With this function defined, we have the following lemma.

1 Lemma $\lim_{\mathrm{T}\to\infty} \int_{-\mathrm{T}}^{\mathrm{T}} \operatorname{sinc}(\mathrm{t})\,\mathrm{dt} = \pi.$

Proof Define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2\mathrm{i}z}, & z \neq 0, \\ 1, & z = 0 \end{cases}$$

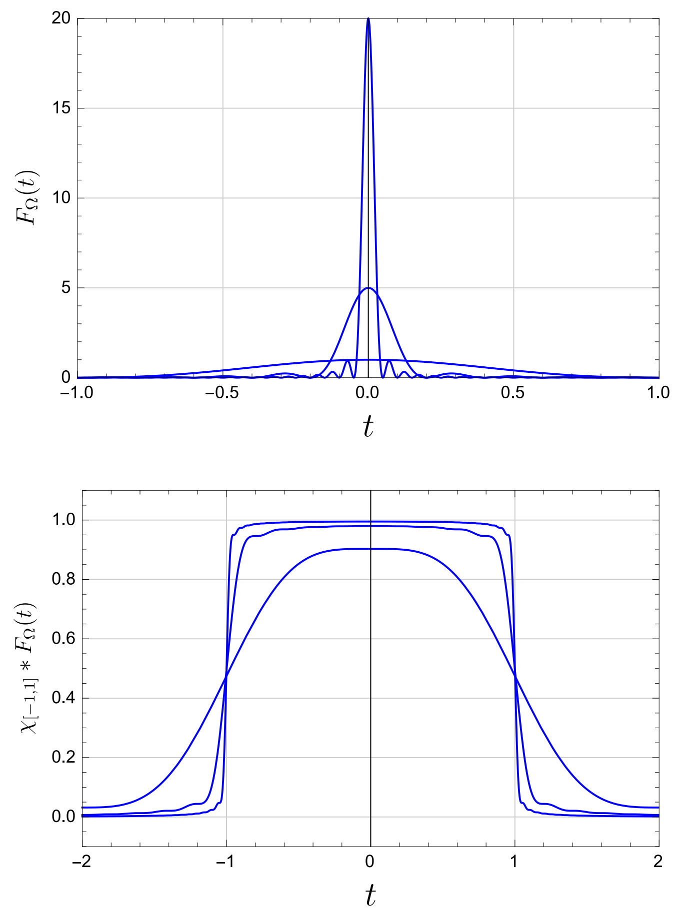


Figure 11.19 The Fejér kernel F_Ω for $\Omega \in \{1, 5, 20\}$ (top) and the corresponding approximations by convolution of the characteristic function of $[-1, 1]$ (bottom)

and note that

$$F(\mathrm{i}y) = \frac{\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{-\mathrm{i}y}}{2\mathrm{i}(\mathrm{i}y)} = \frac{\operatorname{sinc}(y)}{\mathrm{i}}.$$

It is clear that F is analytic on $\mathbb{C} \setminus \{0\}$. Since $\lim_{z\to 0} F(z) = 1$ it follows from ***missing stuff*** that F is, in fact, analytic. Note that if we define

$$F_+(z) = \frac{\mathrm{e}^z}{2\mathrm{i}z}, \quad F_-(z) = \frac{\mathrm{e}^{-z}}{2\mathrm{i}z}$$

then $F(z) = F_+(z) + F_-(z)$ for $z \neq 0$, but that at $z = 0$ both F_+ and F_- have singularities.

Now we define some contours in $\mathbb{C}$:

$$\begin{aligned}
\gamma_T &= \{0 + \mathrm{i}y \in \mathbb{C} \mid y \in [-T, T]\},\\
\gamma'_T &= \{0 + \mathrm{i}y \mid y \in [-T, 1]\} \cup \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [-\tfrac{1}{2}, \tfrac{1}{2}]\} \cup \{0 + \mathrm{i}y \mid y \in [1, T]\}, \quad T > 1,\\
C_{+,T} &= \{T\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [-\pi, \pi]\},\\
C_{-,T} &= \{-T\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [-\pi, \pi]\}.
\end{aligned}$$

We depict these contours in Figure 11.20 with their positive orientations. Let

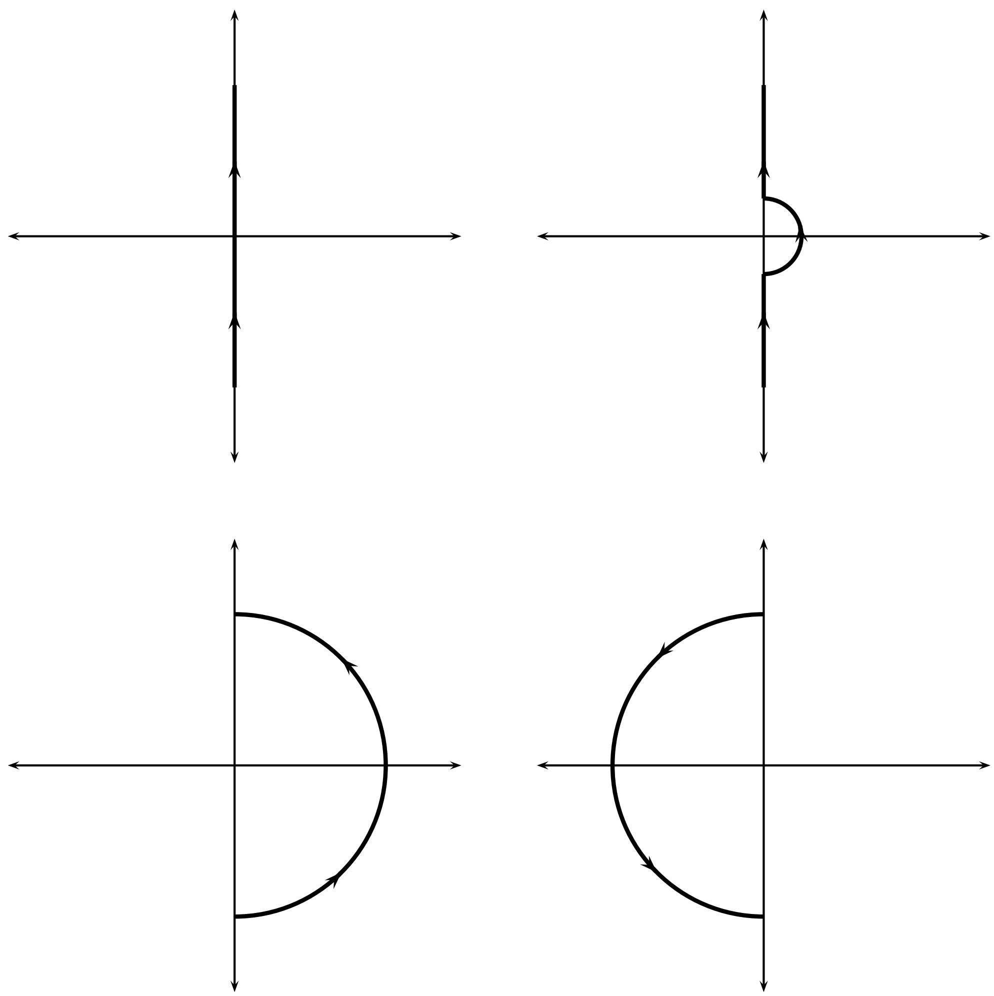


Figure 11.20 The contours γ_T (topleft), γ'_T (top right), $C_{+,T}$ (bottom left), and $C_{-,T}$ (bottom right)

us also denote

$$\Gamma_{+,T} = \gamma'_T \cup C_{+,T}, \quad \Gamma_{-,T} = \gamma'_T \cup C_{-,T},$$

taking the counterclockwise orientation about these contours as being positive, as usual.

Note that, by direct computation, we have

$$\int_{-T}^{T} \operatorname{sinc}(t)\,\mathrm{d}t = \int_{\gamma_T} F(z)\,\mathrm{d}z.$$

Note that since F is entire we have

$$\int_{-T}^{T} \operatorname{sinc}(t)\,\mathrm{d}t = \int_{\gamma'_T} F(z)\,\mathrm{d}z$$

by virtue of ***missing stuff***. Therefore,

$$\int_{-T}^{T} \operatorname{sinc}(t)\,\mathrm{d}t = \int_{\gamma'_T} F_+(z)\,\mathrm{d}z - \int_{\gamma'_T} F_-(z)\,\mathrm{d}z. \tag{11.25}$$

By the Residue Theorem, since $F_+(z)$ has a simple pole at $z = 0$,

$$\int_{\Gamma_{-,T}} F_+(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname{Res}(F_+, 0) = 2\pi\mathrm{i} \lim_{z\to 0} zF_+(z) = \pi.$$

Since F_- is analytic on and within $\Gamma_{+,T}$ we have

$$\int_{\Gamma_{+,T}} F_-(z)\,\mathrm{d}z = 0.$$

Note that, by Jordan's Lemma (***missing stuff***), we have

$$\lim_{T\to\infty} \int_{C_{-,T}} F_+(z)\,\mathrm{d}z = \lim_{T\to\infty} \int_{C_{+,T}} F_-(z)\,\mathrm{d}z = 0.$$

Therefore, since

$$\int_{\Gamma_{-,T}} F_+(z)\,\mathrm{d}z = \int_{\gamma'_T} F_+(z)\,\mathrm{d}z + \int_{C_{-,T}} F_+(z)\,\mathrm{d}z$$

and

$$\int_{\Gamma_{+,T}} F_-(z)\,\mathrm{d}z = -\int_{\gamma'_T} F_-(z)\,\mathrm{d}z + \int_{C_{+,T}} F_-(z)\,\mathrm{d}z$$

(keeping orientations of contours in mind), we have

$$\lim_{T\to\infty} \int_{\gamma'_T} F_+(z)\,\mathrm{d}z = \pi$$

and

$$\lim_{T\to\infty} \int_{\gamma'_T} F_-(z)\,\mathrm{d}z = 0,$$

giving the lemma by virtue of (11.25). ▼

2 Lemma $\int_0^{\mathrm{T}} \mathrm{sinc(t)^2\,dt} = \int_0^{2\mathrm{T}} \mathrm{sinc(t)\,dt - T\,sinc(T)^2}.$

Proof We differentiate both sides of the proposed equation with respect to T:

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_0^T \operatorname{sinc}(t)^2\,\mathrm{d}t = \operatorname{sinc}(T)^2$$

and (noting the definition of sinc)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}\Big(\int_0^{2T}\operatorname{sinc}(t)\,\mathrm{d}t - \frac{\sin(T)^2}{T}\Big) &= \operatorname{sinc}(2T) + \frac{\sin(T)(\sin(T) - 2T\cos(T))}{T^2}\\ &= \frac{\sin(T)^2}{T^2} = \operatorname{sinc}(T)^2,\end{aligned}$$

noting that $2\cos(T)\sin(T) = \sin(2T)$. Thus both sides of the proposed equality have the same derivatives. Moreover, since they both have the value 0 at $x = 0$ and since they are differentiable, it follows that the two sides of the proposed equation must indeed be equal. ▼

3 Lemma *We have* $\operatorname{sinc} \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ *and* $\operatorname{sinc}^2 \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{R})$*.*

Proof That $\operatorname{sinc} \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ is shown in Example 3.4.20. Since sinc is continuous, so is sinc^2. Thus sinc^2 is bounded on $[-1, 1]$. Thus, noting that sinc^2 is even,

$$\begin{aligned}\int_{\mathbb{R}} |\operatorname{sinc}^2(t)|\,\mathrm{d}t &= \int_{-1}^{1} |\operatorname{sinc}^2(t)|\,\mathrm{d}t + 2\int_1^\infty \Big|\frac{\sin(t)^2}{t^2}\Big|\,\mathrm{d}x\\ &\le \int_{-1}^{1} |\operatorname{sinc}^2(t)|\,\mathrm{d}t + \int_1^\infty \frac{1}{t^2}\,\mathrm{d}t < \infty,\end{aligned}$$

giving $\operatorname{sinc}^2 \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$. One similarly shows that $\operatorname{sinc}^2 \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{R})$. ▼

From the second lemma and then the first lemma (noting that sinc is an even function) we have

$$\lim_{T\to\infty}\int_0^T \operatorname{sinc}(t)^2\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T \operatorname{sinc}(t)\,\mathrm{d}t = \frac{\pi}{2}.$$

Thus

$$\int_{\mathbb{R}} \operatorname{sinc}(t)^2\,\mathrm{d}t = \pi,$$

using evenness of sinc^2. By the third lemma, $F_\Omega \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$. Moreover, by the change of variable theorem,

$$\int_{\mathbb{R}} F_\Omega(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{\mathbb{R}} \operatorname{sinc}(\tau)^2\,\mathrm{d}\tau = 1,$$

showing that F_Ω a candidate for defining an approximate identity according to Proposition 11.3.6. Note that

$$jF_\Omega(jt) = \frac{\sin^2(\pi j\Omega t)}{\pi^2 j\Omega t^2} = F_{j\Omega}(t).$$

Thus, just as for the Poisson kernel, $(F_{j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for every $\Omega \in \mathbb{R}_{>0}$. And, also just as for the Poisson kernel, we can replace the limit as $j \to \infty$ in Theorems 11.3.2, 11.3.3, 11.3.4, and 11.3.5 with the limit as $\Omega \to \infty$.

While we are talking about the Fejér kernel, it is a good moment to prove a formula that will be useful to us in Section 13.2.

4 Lemma $F_\Omega(t) = \dfrac{1}{\Omega}\displaystyle\int_0^\Omega \Big(\int_{-a}^{a} e^{2\pi i\nu t}\,d\nu\Big)\,da.$

Proof An easy calculation (which will be performed in Example 13.1.3–3) gives

$$\int_{-a}^{a} e^{2\pi i\nu t}\,d\nu = \begin{cases} \frac{\sin(2\pi a t)}{\pi t}, & t \neq 0, \\ 2a, & t = 0. \end{cases}$$

The result now follows by elementary integration. ▼

4. The next approximate identity we consider is the ***de la Vallée Poussin kernel*** on $\mathbb{R}$ which is defined for $\Omega \in \mathbb{R}_{>0}$ by

$$V_\Omega(t) = 2F_{2\Omega}(t) - F_\Omega(t).$$

In Figure 11.21 we show the de la Vallée Poussin kernel for a few values of Ω. From the properties of the Fejér kernel we immediately have that $\|V_\Omega\|_1 = 1$. Moreover, we have

$$jV_\Omega(jt) = 2jF_{2\Omega}(jt) - jF_\Omega(jt) = 2F_{2j\Omega}(t) - F_{j\Omega}(t) = V_{j\Omega}(t).$$

Thus we deduce that $(V_\Omega)_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for every $\Omega \in \mathbb{R}_{>0}$. As we have seen above for the Poisson and Fejér kernels, we can replace the limit as $j \to \infty$ in Theorems 11.3.2, 11.3.3, 11.3.4, and 11.3.5 with the limit as $\Omega \to \infty$.

5. The ***Dirichlet kernel*** on $\mathbb{R}$ is defined for $\Omega \in \mathbb{R}_{>0}$ by

$$D_\Omega(t) = \begin{cases} \frac{\sin(2\pi\Omega t)}{\pi t}, & t \neq 0, \\ 2\Omega, & t = 0 \end{cases}$$

for $t \in \mathbb{R}$. In Figure 11.22 we show the Dirichlet kernel for a few values of Ω. Note that,

$$jD_\Omega(jt) = \frac{\sin(2\pi j\Omega t)}{\pi t} = D_{j\Omega}(t),$$

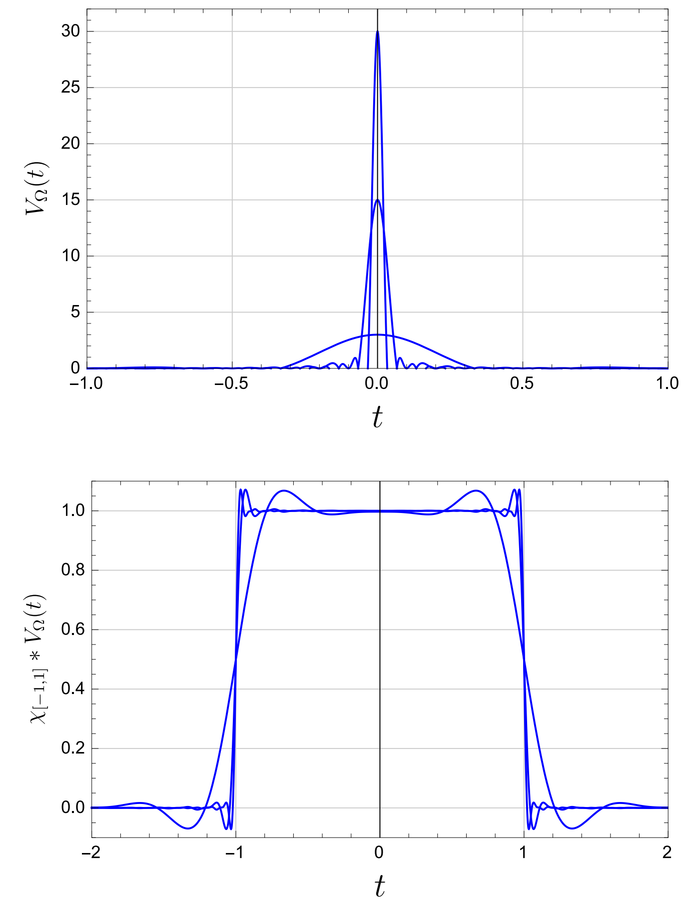


Figure 11.21 The de la Vallée Poussin kernel V_Ω for $\Omega \in \{1, 5, 10\}$ (top) and the corresponding approximations by convolution of the characteristic function of $[-1, 1]$ (bottom)

similarly to what we have seen for the Poisson and Fejér kernels. However, it is not the case that $(\Omega^{-1} D_{j\Omega})_{j \in \mathbb{Z}_{>0}}$ is an approximate identity. For example, by the change of variable theorem and Lemma 1 above, we have

$$\lim_{T \to \infty} \int_{-T}^{T} D_\Omega(t)\, dt = \lim_{T \to \infty} \frac{\Omega}{\pi} \int_{-T}^{T} \operatorname{sinc}(t)\, dt = \Omega,$$

and so the Dirichlet kernel does not have unit integral. Also as can be seen from Lemma 3, $D_\Omega \notin \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{R})$. So the Dirichlet kernel fails to define an approximate identity in the way that the Fejér kernel does. However, it could still be the case that the sequence $(D_{j\Omega})_{j \in \mathbb{Z}_{>0}}$ has the approximating properties of an approximate identity. It turns out that this is true, sort of. It is not true strictly. For example, there exists $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{F})$ such that the sequence $(f * D_{j\Omega})_{j \in \mathbb{Z}_{>0}}$ does not converge to f in $\mathsf{L}^1(\mathbb{R}; \mathbb{F})$.***missing stuff*** But in Figure 11.22 we show a few approximations of the characteristic function $\chi_{[-1,1]}$, and we see that, indeed,

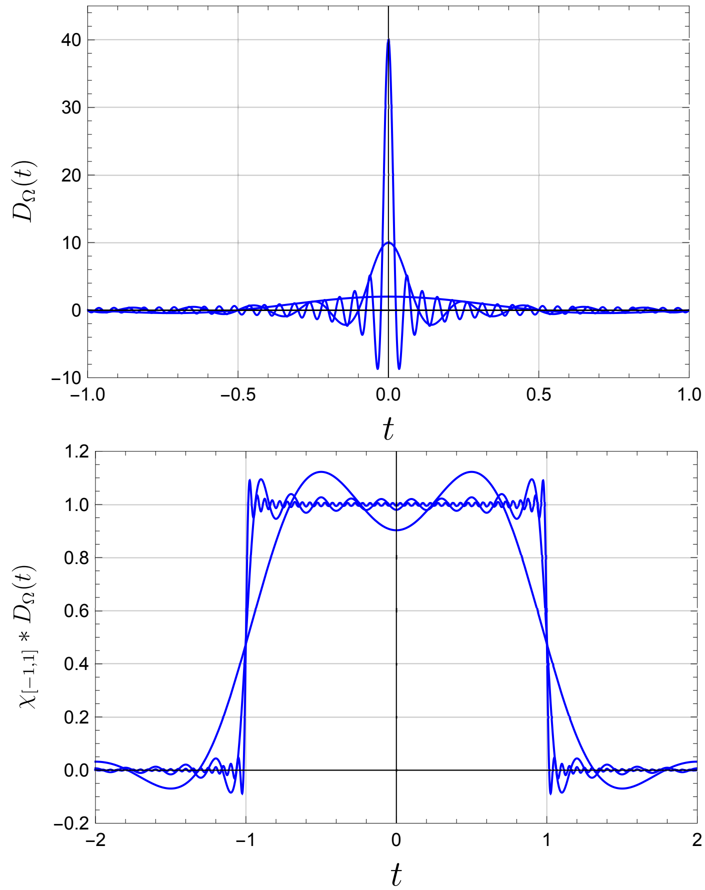


Figure 11.22 The Dirichlet kernel D_Ω for $\Omega \in \{1, 5, 20\}$ (top) and the corresponding approximations by convolution of the characteristic function of $[-1, 1]$ (bottom)

some sort of approximation seems to be taking place. We shall discuss these matters in some detail when we talk about Fourier integrals in Section 13.2. •

11.3.2 Approximate identities on $\mathbb{R}_{\geq 0}$

In this section we consider approximate identities for signals defined on the continuous time-domain $\mathbb{R}_{\geq 0}$. Let $p \in [1, \infty)$. In Section 11.2.3 we considered the structure of convolution in $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$, focusing on the case of $p = 1$. Here we shall use the locally convex topological structure of $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{F})$ discussed in Section 11.2.3.

We begin with the definition of an approximate identity on $\mathbb{R}_{\geq 0}$. Again, the reader should be aware of possible variations, not all equivalent, to the definition we give.

11.3.8 Definition (Approximate identity for signals defined on $\mathbb{R}_{\geq 0}$) An *approximate identity* on $\mathbb{R}_{\geq 0}$ is a sequence $(u_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathsf{L}^{(1)}(\mathbb{R}_{\geq 0}; \mathbb{F})$ with the following properties:

(i) $\displaystyle\int_{\mathbb{R}_{\geq 0}} u_j(t)\,\mathrm{d}t = 1$, $j \in \mathbb{Z}_{>0}$;

(ii) there exists $M \in \mathbb{R}_{>0}$ such that $\|u_j\|_1 \leq M$ for each $j \in \mathbb{Z}_{>0}$;

(iii) for each $\alpha \in \mathbb{R}_{>0}$,

$$\lim_{j\to\infty} \int_\alpha^\infty |u_j(t)|\,\mathrm{d}t = 0. \qquad \bullet$$

The following result adapts Theorem 11.3.2 to the present case.

11.3.9 Theorem (Approximation in $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$ using approximate identities) *If $(u_j)_{j\in\mathbb{Z}_{>0}}$ is an approximate identity on $\mathbb{R}_{\geq 0}$ and if $f \in \mathsf{L}^{(p)}(\mathbb{R}_{\geq 0};\mathbb{F})$, then the sequence $(f \circledast u_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in the topology on $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{F})$.*

Proof Let $T \in \mathbb{R}_{>0}$ and let $f_T = f\chi_{[0,T]}$ so that $f_T \in \mathsf{L}^p(\mathbb{R}_{\geq 0};\mathbb{F})$. Let us think of the signals u_j, $j \in \mathbb{Z}_{>0}$, and f_T as being defined on $\mathbb{R}$ by asking that they take the value 0 for negative times. Then $(u_j)_{j\in\mathbb{Z}_{>0}}$ is obviously an approximate identity on $\mathbb{R}$, as discussed in Section 11.3.1. By Theorem 11.3.2 it follows that $(f_T * u_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$. Now let $t \in [0,T]$. Then, for each $j \in \mathbb{Z}_{>0}$,

$$f \circledast u_j(t) = \int_0^t f(t-s)u_j(s)\,\mathrm{d}s = \int_0^t f_T(t-s)u_j(s)\,\mathrm{d}s = f_T \circledast u_j(t) = f_T * u_j(t).$$

That is to say, $f \circledast u_j|[0,T]$ only depends on $f|[0,T]$. Therefore,

$$\lim_{j\to\infty}\|f \circledast u_j - f\|_{p,T} = \lim_{j\to\infty}\|f_T * u_j - f_T\|_p = 0,$$

giving the result by ***missing stuff***. ■

The result can also be adapted to continuous signals. Thus we consider $\mathsf{C}^0(\mathbb{R}_{\geq 0};\mathbb{F})$ and on this space of signals we use the locally convex topology defined by the family of seminorms $\|\cdot\|_{\infty,T}$, $T \in \mathbb{R}_{>0}$, defined by

$$\|f\|_{\infty,T} = \sup\{|f(t)| \mid t \in [0,T]\}.$$

As in ***missing stuff***, this topology is Fréchet. In this case we have the following result.

11.3.10 Theorem (Approximation in $\mathsf{C}^0(\mathbb{R}_{\geq 0};\mathbb{F})$ using approximate identities) *If $(u_j)_{j\in\mathbb{Z}_{>0}}$ is an approximate identity on $\mathbb{R}_{\geq 0}$ and if $f \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R}_{\geq 0};\mathbb{F})$, then the sequence $(f \circledast u_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in the topology on $\mathsf{C}^0(\mathbb{R}_{\geq 0};\mathbb{F})$.*

Proof Let $T \in \mathbb{R}_{>0}$ and define $f_T \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R}_{\geq 0};\mathbb{F})$ by

$$f_T(t) = \begin{cases} f(0), & t \in \mathbb{R}_{<0}, \\ f(t), & t \in [0,T], \\ f(T), & t \in (T,\infty). \end{cases}$$

Let u_j, $j \in \mathbb{Z}_{>0}$, be extended to be defined on $\mathbb{R}$ by asking that it take the value 0 for negative times. By Theorem 11.3.3 the sequence $(f_T * u_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to f_T on $[0,T]$. As in the proof of Theorem 11.3.9, $f \circledast u_j(t) = f * u_j(t)$ for $t \in [0,T]$. Thus

$$\lim_{j\to\infty}\|f \circledast u_j - f\|_{\infty,T} = \lim_{j\to\infty} \sup\{|f_T * u_j(t) - f(t)| \mid t \in [0,T]\} = 0,$$

giving the result by ***missing stuff***. ■

The special class of approximate identities on $\mathbb{R}$ characterised in Proposition 11.3.6 are easily adapted to the case of approximate identities on $\mathbb{R}_{\geq 0}$. The following result is easily proved along the same lines as Proposition 11.3.6.

11.3.11 Proposition (Approximate identities on $\mathbb{R}_{\geq 0}$ generated by a single signal) *If* $\mathsf{u} \in \mathsf{L}^{(1)}(\mathbb{R}_{\geq 0}; \mathbb{F})$ *satisfies*

$$\int_{\mathbb{R}_{\geq 0}} \mathsf{u}(\mathsf{t})\,\mathsf{dt} = 1,$$

then the sequence $(\mathsf{u}_\mathsf{j})_{\mathsf{j}\in\mathbb{Z}_{>0}}$ *defined by* $\mathsf{u}_\mathsf{j}(\mathsf{t}) = \mathsf{j}\mathsf{u}(\mathsf{j}\mathsf{t})$ *is an approximate identity on* $\mathbb{R}_{\geq 0}$*.*

It is also straightforward to adapt the examples of approximate identities given in Example 11.3.7. We invite the reader to compute some approximations by convolution with these approximate identities to see how they work. Unsurprisingly, the reader will find that they work rather like those in Example 11.3.7.

11.3.12 Examples (Approximate identities on $\mathbb{R}_{\geq 0}$)

1. For $t \in \mathbb{R}_{\geq 0}$ and for $\Omega \in \mathbb{R}_{>0}$ define the ***Poisson kernel*** on $\mathbb{R}_{\geq 0}$ by

$$P^+_\Omega(t) = \frac{2}{\pi}\frac{\Omega}{1+\Omega^2 t^2}.$$

   As in Example 11.3.7–1, we have $P^+_\Omega \in \mathsf{L}^{(1)}(\mathbb{R}_{\geq 0}; \mathbb{F})$ and $\|P^+_\Omega\|_1 = 1$ for every $\Omega \in \mathbb{R}_{>0}$. Moreover, we also have $jP^+_(jt) = P^+_{j\Omega}(t)$ and so $(P^+_{j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for every $\Omega \in \mathbb{R}_{>0}$.

2. As in Example 11.3.7–2 we define the ***Gauss–Weierstrass kernel*** on $\mathbb{R}_{\geq 0}$ for $\Omega \in \mathbb{R}_{>0}$ by

$$G^+_\Omega(t) = \frac{\exp(-\frac{t^2}{4\Omega})}{\sqrt{\pi\Omega}}.$$

   One verifies from Example 11.3.7–2 that $G^+_\Omega \in \mathsf{L}^{(1)}(\mathbb{R}_{\geq 0}; \mathbb{F})$ and that $\|G^+_\Omega\|_1$. Thus, if we define $G^+_{\Omega,j}(t) = jG^+_\Omega(jt)$, we have that $(G^+_{\Omega,j})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity.

3. The ***Fejér kernel*** on $\mathbb{R}_{\geq 0}$ is defined for $\Omega \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{\geq 0}$ by

$$F^+_\Omega(t) = \begin{cases} 2\frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \in \mathbb{R}_{>0}, \\ 2\Omega, & t = 0. \end{cases}$$

   Following Example 11.3.7–3 we note that $F^+_\Omega \in \mathsf{L}^{(1)}(\mathbb{R}_{\geq 0}; \mathbb{F})$, $\|F^+_\Omega\|_1 = 1$, and $jF^+_\Omega(jt) = F^+_{j\Omega}(t)$. Thus $(F^+_{j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity.

4. The ***de la Vallée Poussin kernel*** on $\mathbb{R}_{\geq 0}$ is defined for $\Omega \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{\geq 0}$ by

$$V^+_\Omega(t) = 2F^+_{2\Omega}(t) - F^+_\Omega(t).$$

   From our computations for the Fejér kernel we have $\|V^+_\Omega\|_1 = 1$ and $jV^+_\Omega(jt) = V^+_{j\Omega}(t)$. Thus $(V^+_{j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity on $\mathbb{R}_{\geq 0}$. •

11.3.3 Periodic approximate identities

In this section we shall discuss approximate identities for periodic signals. The first part of the discussion, that giving the definitions and the basic approximation theorems, follows along the same lines as the preceding two sections. However, we then turn to some rather deep connections between approximate identities on $\mathbb{R}$ as discussed in Section 11.3.1 and periodic approximate identities. This discussion relies heavily on material from Chapters 12 and 13.

Let us begin with the more or less familiar parts of the discussion.

11.3.13 Definition (Approximate identity for periodic signals) A T-*periodic approximate identity* on $\mathbb{R}$ is a sequence $(u_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ with the following properties:

(i) $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} u_j(t)\,\mathrm{d}t = 1$, $j\in\mathbb{Z}_{>0}$;

(ii) there exists $M\in\mathbb{R}_{>0}$ such that $\|u_j\|_1 \le M$ for each $j\in\mathbb{Z}_{>0}$;

(iii) for each $\alpha\in(0,\frac{T}{2}]$,
$$\lim_{j\to\infty}\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |u_j(t)|\,\mathrm{d}t = 0. \qquad \bullet$$

We can now state approximation theorems that are analogous to those in Section 11.3.1.

11.3.14 Theorem (Approximation in $\mathsf{L}^{\mathrm{p}}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$ using approximate identities) *Let* $\mathrm{p}\in[1,\infty)$. *If* $(\mathrm{u_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *is a* T*-periodic approximate identity on* $\mathbb{R}$ *and if* $\mathrm{f}\in\mathsf{L}^{(\mathrm{p})}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$*, then the sequence* $(\mathrm{f}*\mathrm{u_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *converges to* f *in* $\mathsf{L}^{\mathrm{p}}_{\mathrm{per,T}}(\mathbb{R};\mathbb{F})$.

Proof Note that
$$f(t) - f*u_j(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t)-f(t-\tau))u_j(\tau)\,\mathrm{d}\tau,$$
using the fact that
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} u_j(\tau)\,\mathrm{d}\tau = 1.$$
Recalling the integral version of Minkowski's inequality, Lemma 6.7.53, we have
$$\begin{aligned}
\|f-f*u_j\|_p &= \Big(\int_{-\frac{T}{2}}^{\frac{T}{2}}\Big|\int_{-\frac{T}{2}}^{\frac{T}{2}}(f(t)-f(t-s))u_j(\tau)\,\mathrm{d}\tau\Big|^p\,\mathrm{d}t\Big)^{1/p}\\
&\le \int_{-\frac{T}{2}}^{\frac{T}{2}}\Big(\int_{-\frac{T}{2}}^{\frac{T}{2}}|(f(t)-f(t-\tau))u_j(\tau)|^p\,\mathrm{d}t\Big)^{1/p}\,\mathrm{d}\tau\\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}}\Big(\int_{-\frac{T}{2}}^{\frac{T}{2}}|f(t)-f(t-\tau)|^p\,\mathrm{d}t\Big)^{1/p}|u_j(\tau)|\,\mathrm{d}\tau\\
&\le \int_{-\frac{T}{2}}^{\frac{T}{2}}\|f-\tau_\tau^* f\|_p|u_j(\tau)|\,\mathrm{d}\tau,
\end{aligned}$$

recalling the notation $\tau_\tau^* f(t) = f(t-\tau)$. Let $\epsilon \in \mathbb{R}_{>0}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \le M$ for each $j \in \mathbb{Z}_{>0}$. By Lemma 1 from the proof of Corollary 11.2.29 let $\delta \in (0, \frac{T}{2}]$ be sufficiently small that $\|f - \tau_a^* f\| < \frac{\epsilon}{2M}$ for $a \in (0, \delta]$. Then

$$\int_{-\delta}^{\delta} \|f - \tau_\tau^* f\|_p |u_j(\tau)| \, d\tau \le \frac{\epsilon}{2M} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_j(\tau)| \, d\tau \le \frac{\epsilon}{2}. \tag{11.26}$$

Also let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)| \, d\tau < \frac{\epsilon}{4\|f\|_p}, \qquad j \ge N.$$

Then, noting that $\|f - \tau_\tau^* f\|_p \le 2\|f\|_p$ by the triangle inequality and invariance of the norm under translation, we compute

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} \|f - \tau_\tau^* f\|_p |u_j(\tau)| \, ds \le 2\|f\|_p \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)| \, d\tau < \frac{\epsilon}{2}. \tag{11.27}$$

Combining (11.26) and (11.27) we see that, for $j \ge N$,

$$\|f - f * u_j\|_p < \epsilon,$$

giving the result. ■

For continuous signals we also have an approximation result using approximate identities.

11.3.15 Theorem (Approximation in $\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ using approximate identities) *If $(u_j)_{j\in\mathbb{Z}_{>0}}$ is a T-periodic approximate identity on $\mathbb{R}$ and if $f \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, then the sequence $(f * u_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to f.*

Proof Let $\epsilon \in \mathbb{R}_{>0}$. As in the proof of Theorem 11.3.14, noting that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} u_j(\tau) \, d\tau = 1$$

we have

$$f(t) - f * u_j(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t) - f(t-\tau)) u_j(\tau) \, d\tau \tag{11.28}$$

for every $t \in \mathbb{R}$ and $j \in \mathbb{Z}_{>0}$. Note that since f is bounded this integral makes sense for all $t \in \mathbb{R}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \le M$ for each $j \in \mathbb{Z}_{>0}$. Note that f is uniformly continuous on each interval of length T, and so is uniformly continuous. Thus there exists $\delta \in (0, \frac{T}{2}]$ such that

$$|f(t) - f(t-\tau)| < \frac{\epsilon}{2M}$$

for $t \in \mathbb{R}$ and $|\tau| < \delta$. Then, for every $j \in \mathbb{Z}_{>0}$,

$$\int_{-\delta}^{\delta} |f(t) - f(t-\tau)| |u_j(\tau)| \, d\tau \le \frac{\epsilon}{2M} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_j(\tau)| \, d\tau < \frac{\epsilon}{2}. \tag{11.29}$$

Now let $C = \|f\|_\infty$ and note that, for every $t_1, t_2 \in \mathbb{R}$, $|f(t_1) - f(t_2)| \leq 2C$ using the triangle inequality. Now there exists $N \in \mathbb{Z}_{>0}$ such that

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)|\,d\tau < \frac{\epsilon}{4C}$$

for $j \geq N$. Therefore, if $j \geq N$ we have

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |f(t) - f(t-\tau)||u_j(\tau)|\,d\tau < \frac{\epsilon}{2}. \tag{11.30}$$

Putting (11.28), (11.29), and (11.30) together we have

$$|f(t) - f * u_j(t)| < \epsilon, \qquad j \geq N,\ t \in \mathbb{R},$$

giving the result. ■

We also have a result concerning pointwise convergence of approximations.

11.3.16 Theorem (Pointwise approximations using even approximate identities) *Let* $(u_j)_{j\in\mathbb{Z}_{>0}}$ *be a* T*-periodic approximate identity such that* $u_j(-t) = u_j(t)$ *for each* $j \in \mathbb{Z}_{>0}$ *and* $t \in (-\frac{T}{2}, \frac{T}{2})$*. If* $f \in L^{(\infty)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ *and if, for* $t_0 \in \mathbb{R}$*, the limits* $f(t_0-)$ *and* $f(t_0+)$ *exist, then* $(f * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ *converges to* $\frac{1}{2}(f(t_0-) + f(t_0+))$.

Proof We may obviously assume that f is not almost everywhere zero. First suppose that f is continuous at t_0. Let $\epsilon \in \mathbb{R}_{>0}$. Let $M \in \mathbb{R}_{>0}$ be such that $\|u_j\|_1 \leq M$ for every $j \in \mathbb{Z}_{>0}$. Let $\delta \in (0, \frac{T}{2}]$ be such that, if $|\tau| < \delta$, then $|f(t_0 - \tau) - f(t_0)| \leq \frac{\epsilon}{2M}$. Then

$$\int_{-\delta}^{\delta} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,d\tau \leq \frac{\epsilon}{2M}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u_j(\tau)|\,d\tau < \frac{\epsilon}{2}.$$

Note that $|f(t_0 - \tau) - f(t_0)| \leq 2\|f\|_\infty$. Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)|\,d\tau < \frac{\epsilon}{4\|f\|_\infty}$$

for $j \geq N$. Then

$$\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,d\tau < \frac{\epsilon}{2}$$

for $j \geq N$.

Now, as in the proofs of Theorems 11.3.14 and 11.3.15, we have

$$f(t_0) - f * u_j(t_0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t_0) - f(t_0-\tau))u_j(\tau)\,d\tau$$

and so

$$|f(t_0) - f * u_j(t_0)| \leq \int_{-\delta}^{\delta} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,d\tau + \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |u_j(\tau)||f(t_0-\tau) - f(t_0)|\,d\tau < \epsilon$$

for $j \geq N$.

Thus our result holds if f is continuous at t_0. If the limits $f(t_0-)$ and $f(t_0+)$ both exist but are not necessarily equal, then one applies the argument for signals continuous at $t = 0$ to the signal $g(t) = \frac{1}{2}(f(t_0 - t) + f(t_0 + t))$. Convergence of $(g * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ to $g(0)$ then implies convergence of $(f * u_j(t_0))_{j\in\mathbb{Z}_{>0}}$ to $\frac{1}{2}(f(t_0-) + f(t_0+))$ using the fact that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_0 - s)u_j(\tau)\,\mathrm{d}\tau = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t_0 - \tau) + f(t_0 + \tau))u_j(\tau)\,\mathrm{d}\tau$$

by evenness of u_j, $j \in \mathbb{Z}_{>0}$. ■

Now that we understand some of the approximation properties of periodic approximate identities, we can see some of the ways in which they can be produced. A basic result here shows how approximate identities on $\mathbb{R}$ give rise to periodic approximate identities. To state the result, we look ahead to Section **??** for the notion of the periodisation of a signal.

11.3.17 Theorem (Periodic approximate identities from aperiodic ones) *Let* $(\mathrm{u}_j)_{j\in\mathbb{Z}_{>0}}$ *be an approximate identity on* $\mathbb{R}$ *and let* $\mathrm{T} \in \mathbb{R}_{>0}$. *Then* $(\mathrm{per}_\mathrm{T}(\mathrm{u}_j))_{j\in\mathbb{Z}_{>0}}$ *is a* T*-periodic approximate identity.*

Proof From Proposition **??** the first two properties of a periodic approximate identity are immediately verified. For the third, let $\alpha \in (0, \frac{T}{2}]$. Then

$$\begin{aligned}
\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |\mathrm{per}_T(u_j)(t)|\,\mathrm{d}t &\leq \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |u_j(t)|\,\mathrm{d}t + \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} \Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} |u_j(t + kT)|\Big)\,\mathrm{d}t \\
&\leq \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |u_j(t)|\,\mathrm{d}t + \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k\in\mathbb{Z}\setminus\{0\}} |u_j(t + kT)|\,\mathrm{d}t \\
&= \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |u_j(t)|\,\mathrm{d}t + \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_j(t + kT)|\,\mathrm{d}t \\
&= \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} |u_j(t)|\,\mathrm{d}t + \sum_{m\in\mathbb{Z}\setminus\{0\}} \int_{(m+\frac{1}{2})T}^{(m-\frac{1}{2})T} |u_j(\tau)|\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}\setminus[-\alpha,\alpha]} |u_j(\tau)|\,\mathrm{d}\tau,
\end{aligned}$$

using Fubini's Theorem to swap the sum and integral and using the change of variables theorem. Taking the limit as $j \to \infty$ gives the result since $(u_j)_{j\in\mathbb{Z}_{>0}}$ is an approximate identity. ■

Combining the preceding theorem with Proposition 11.3.6 immediately gives the following corollary.

11.3.18 Corollary (A special class of periodic approximate identities) *Let* $\mathrm{u} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{F})$ *satisfy*

$$\int_{\mathbb{R}} \mathrm{u}(\mathrm{t})\,\mathrm{d}\mathrm{t}$$

and define $u_j \in L^{(1)}(\mathbb{R};\mathbb{F})$ *by* $u_j(t) = ju(jt)$*. Then, for any* $T \in \mathbb{R}_{>0}$*,* $(\mathrm{per}_T(u_j))_{j\in\mathbb{Z}_{>0}}$ *is a periodic approximate identity.*

The matter of computing $\mathrm{per}_T(f)$ is often facilitated by the so-called Poisson Summation Formula, presented in Section **??**, which relies on the theory of Fourier transforms presented in Chapters 12 and 13. In Section **??** we use the Poisson Summation Formula to compute periodisations of the approximate identities on $\mathbb{R}$ given in Example 11.3.7. Here we shall simply refer ahead to these computations and give the resulting periodic approximate identities, as well as give the approximations for a concrete example.

11.3.19 Examples (Periodic approximate identities)

1. In Example **??–??** the periodisation of the Poisson kernel

$$P_\Omega(t) = \frac{1}{\pi}\frac{\Omega}{1+\Omega^2 t^2},$$

is computed, and this periodisation is determined to be $\mathrm{per}_T(P_\Omega) = \frac{1}{T}P^{\mathrm{per}}_{T,\Omega}$, where

$$P^{\mathrm{per}}_{T,\Omega}(t) = \frac{1-(\mathrm{e}^{-\frac{2\pi}{\Omega T}})^2}{1-2\mathrm{e}^{-\frac{2\pi}{\Omega T}}\cos(2\pi\frac{t}{T}) + (\mathrm{e}^{-\frac{2\pi}{\Omega T}})^2}.$$

We call $P^{\mathrm{per}}_{T,\Omega}$ the **T*-periodic Poisson kernel***. Note that, according to Theorem 11.3.17 and our computations of Example 11.3.7–1, the sequence $(\frac{1}{T}P^{\mathrm{per}}_{T,j\Omega})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for every $\Omega \in \mathbb{R}_{>0}$. In Figure 11.23 we plot the periodic Poisson kernel for $T = 2$ and a few values of Ω. We also show the corresponding approximations to the periodic extension of the characteristic function of $[-\frac{1}{2},\frac{1}{2}]$.

There is another representation of the periodic Poisson kernel that we shall use. To understand this representation, we think of the Poisson kernel as a function of the two variables (Ω, t). We then make a change of variable

$$\mathbb{R}_{>0}\times[0,T] \ni (\Omega,t) \mapsto (\mathrm{e}^{-2\pi/\Omega}, 2\pi\tfrac{t}{T}) \in (0,1)\times[0,2\pi],$$

calling the new variables (r,θ). The periodic Poisson kernel expressed in these coordinates is then

$$P^{\mathrm{per}}(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$

which we think of as being a function on the unit disk in the plane, possibly the complex plane. Note that $r \to 1$ as $\Omega \to \infty$. Thus limits for large Ω correspond to approaching the boundary of the disk. This interpretation is explored ***missing stuff***.

2. In Example **??–??** we determined the periodisation of the Gauss–Weierstrass kernel

$$G_\Omega(t) = \frac{\exp(-\frac{t^2}{4\Omega})}{\sqrt{4\pi\Omega}}$$

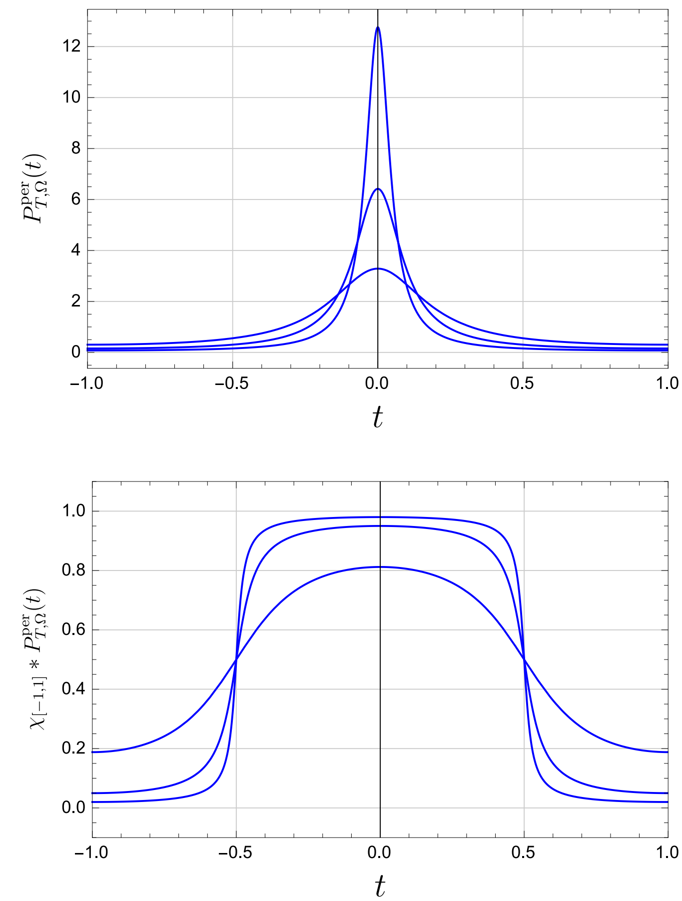


Figure 11.23 The 2-periodic Poisson kernel $P^{\mathrm{per}}_{\Omega}$ for $\Omega \in \{5, 10, 20\}$ (top) and the corresponding approximations by convolution of the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$ (bottom)

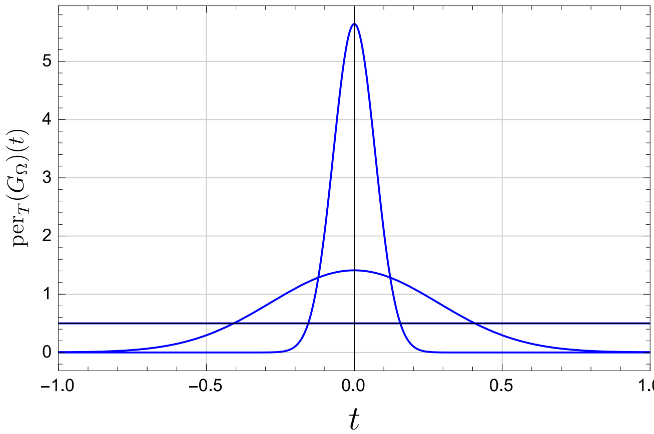
to be given by the infinite series

$$\mathrm{per}_T(G_\Omega)(t) = \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{4\pi^2\Omega n^2}{T^2}\Big)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

This infinite series converges uniformly by the Weierstrass M-test. If we define

$$G_{\Omega,j}(t) = jG_\Omega(jt),$$

then $(\mathrm{per}_T(G_{\Omega,j}))_{j\in\mathbb{Z}_{>0}}$ is an approximate identity. In Figure 11.24 we plot the periodic Gauss–Weierstrass kernel for $T = 2$ and a few values of Ω. We also show the corresponding approximations to the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$.



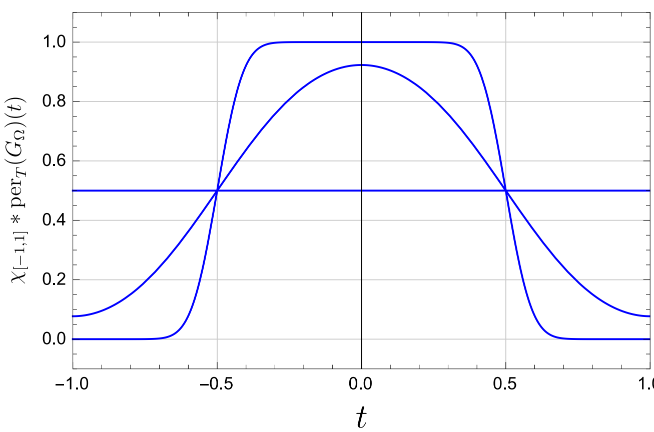


Figure 11.24 The 2-periodic Gauss–Weierstrass kernel $\mathrm{per}_T(G_\Omega)$ for $\Omega \in \{1, 5, 20\}$ (top) and the corresponding approximations by convolution of the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$ (bottom)

3. In Example **??–??** we determine the periodisation of the Fejér kernel

$$F_\Omega(t) = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \neq 0, \\ \Omega, & t = 0. \end{cases}$$

In order to express this, for $N \in \mathbb{Z}_{>0}$ and $T \in \mathbb{R}_{>0}$, we define the ***periodic Fejér kernel*** and the ***periodic Dirichlet kernel*** by

$$F^{\mathrm{per}}_{T,N}(t) = \begin{cases} \dfrac{1}{N}\dfrac{\sin^2(N\pi\frac{t}{T})}{\sin^2(\pi\frac{t}{T})}, & t \notin \mathbb{Z}(T), \\ N, & t \in \mathbb{Z}(T) \end{cases}$$

and

$$D^{\mathrm{per}}_{T,N}(t) = \begin{cases} \frac{\sin((2N+1)\pi\frac{t}{T})}{\sin(\pi\frac{t}{T})}, & t \notin \mathbb{Z}(T), \\ 2N+1, & t \in \mathbb{Z}(T), \end{cases}$$

respectively. With these signals defined, we can then write the general form of the periodisation of F_Ω:

$$\mathrm{per}_T(F_\Omega)(t) = \tfrac{1}{T}\Big(\tfrac{N}{T\Omega}F^{\mathrm{per}}_{T,N}(t) + (1 - \tfrac{N}{T\Omega})D^{\mathrm{per}}_{T,N-1}(t)\Big),$$

where $N \in \mathbb{Z}_{>0}$ is the smallest integer such that $N \geq T\Omega$. Note that, according to Theorem 11.3.17 and our computations of Example 11.3.7–3, the sequence $(\mathrm{per}_T(F_{j\Omega}))_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for each $\Omega \in \mathbb{R}_{>0}$. In particular, when $T\Omega \in \mathbb{Z}$, this shows that $(\frac{1}{T}F^{\mathrm{per}}_{T,N})_{N\in\mathbb{Z}_{>0}}$ is an approximate identity. In Figure 11.25 we plot the periodic Fejér kernel for $T = 2$ and a few values of

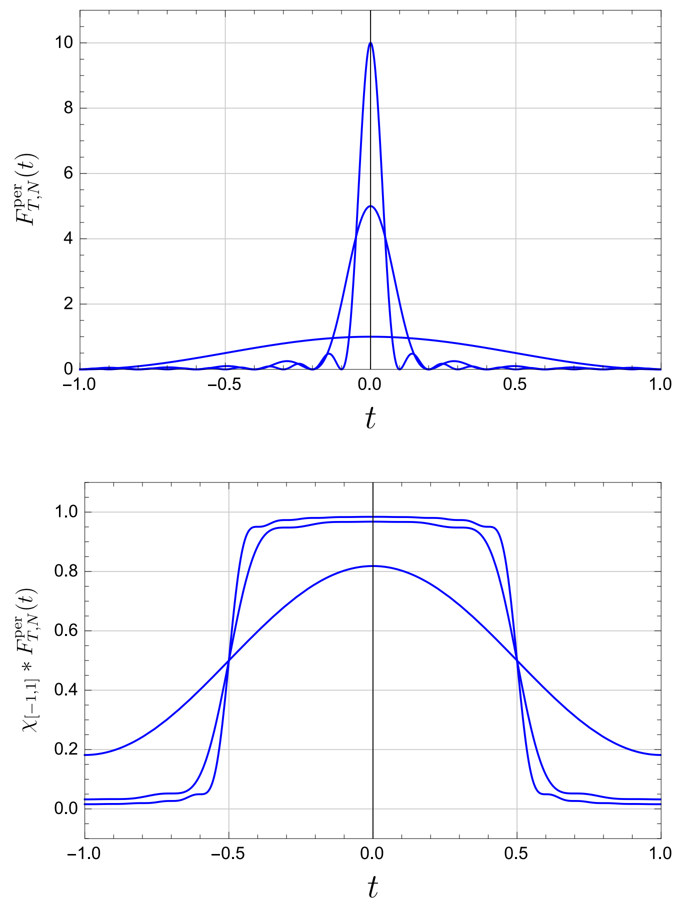


Figure 11.25 The 2-periodic Fejér kernel $F^{\mathrm{per}}_{T,N}$ for $\Omega \in \{1, 5, 10\}$ (top) and the corresponding approximations by convolution of the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$ (bottom)

Ω. We also show the corresponding approximations to the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. Note that in the plot given, $T\Omega$ is an integer, and so $\mathrm{per}_T(F_\Omega) = \frac{1}{T}F^{\mathrm{per}}_{T,N}$ in this case.

4. In Example **??–??** we considered the periodisation of the de la Vallée Poussin kernel
$$V_\Omega(t) = 2F_{2\Omega}(t) - F_\Omega(t).$$
In the case when the period T of the periodisation satisfies $T\Omega \in \mathbb{Z}$ then we saw that
$$\mathrm{per}_T(V_\Omega)(t) = \tfrac{1}{T} V^{\mathrm{per}}_{T,N}(t),$$
where $N = T\Omega$ and where
$$V^{\mathrm{per}}_{T,N}(t) \triangleq 2F^{\mathrm{per}}_{T,2N}(t) - F^{\mathrm{per}}_{T,N}(t).$$
By Theorem 11.3.17 and our computations of Example 11.3.7–4, the sequence $(\mathrm{per}_T(V_{j\Omega}))_{j\in\mathbb{Z}_{>0}}$ is an approximate identity for each $\Omega \in \mathbb{R}_{>0}$. In Figure 11.26 we plot the periodic de la Vallée Poussin kernel for $T = 2$ and a few values of

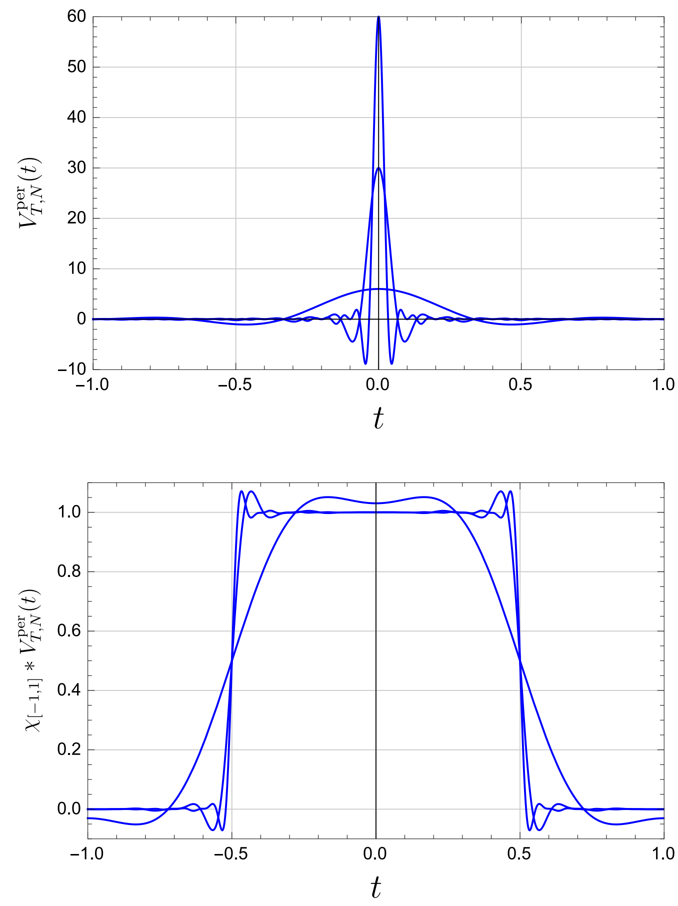


Figure 11.26 The 2-periodic de la Vallée Poussin kernel $V^{\mathrm{per}}_{T,N}$ for $\Omega \in \{1, 5, 10\}$ (top) and the corresponding approximations by convolution of the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$ (bottom)

Ω. We also show the corresponding approximations to the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. Note that in the plot given, $T\Omega$ is an integer, and so $\mathrm{per}_T(V_\Omega) = \frac{1}{T} V^{\mathrm{per}}_{T,N}$ in this case.

5. The last periodic approximate identity we consider is *not* actually a periodic approximate identity, but it is so important that we include it in our list of examples. The Dirichlet kernel

$$D_\Omega(t) = \begin{cases} \frac{\sin(2\pi\Omega t)}{\pi t}, & t \neq 0, \\ 2\Omega, & t = 0 \end{cases}$$

introduced in Example 11.3.7–5 is not integrable, and so it cannot be used as in Theorem 11.3.17 to construct an approximate identity. Moreover, its periodisation cannot be computed using the Poisson Summation Formula as was the case for the Poisson, Gauss–Weierstrass, and Fejér kernels. Nonetheless, in Example **??** we computed the periodisation of D_Ω to be

$$\mathrm{per}_T(D_\Omega)(t) = \frac{1}{T} \begin{cases} D^{\mathrm{per}}_{N,T}(t), & \Omega \notin \mathbb{Z}(T^{-1}), \\ D^{\mathrm{per}}_{N,T}(t) + \cos(2\pi(N+1)\frac{t}{T}), & \Omega \in \mathbb{Z}(T^{-1}), \end{cases}$$

where N is largest integer such that $N < T\Omega$. While we cannot use Theorem 11.3.17 to deduce that $\mathrm{per}_T(D_\Omega)$ gives rise to an approximate identity, it still might be the case that $\mathrm{per}_T(D_\Omega)$ *does* give rise to an approximate identity. For example, note that $\mathrm{per}_T(D_\Omega) \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and so $\mathrm{per}_T(D_\Omega)$ is bounded since it is periodic. Therefore, $\mathrm{per}_T(D_\Omega) \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$, and so this requirement of a periodic approximate identity is met. Moreover, we compute

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} D^{\mathrm{per}}_{N,T}(t)\,\mathrm{d}t = \sum_{j=-N}^{N} \int_0^T \mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}}\,\mathrm{d}t = T,$$

noting that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}}\,\mathrm{d}t = 0$$

for $j \neq 0$. From this we conclude that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{per}_T(D_\Omega)(t)\,\mathrm{d}t = 1$$

for every $T, \Omega \in \mathbb{R}_{>0}$. However, the following lemma shows that $\mathrm{per}_T(D_\Omega)$ cannot be used to define a periodic approximate identity.

1 Lemma *For* $\mathrm{T} \in \mathbb{R}_{>0}$, $\lim_{\mathrm{N}\to\infty} \|\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\|_1 = \infty$.

Proof Since $\mathrm{per}_T(D_\Omega)$ is even, it suffices to consider its restriction to $[0, \frac{T}{2}]$. Note that

$$0 < \sin(x) < x,\ x \in (0, \tfrac{\pi}{2}] \quad \Longrightarrow \quad 0 < \sin(\pi\tfrac{t}{T}) < \pi\tfrac{t}{T},\ t \in (0, \tfrac{T}{2}].$$

We then have

$$\begin{aligned}\int_0^{\frac{T}{2}} |D^{\mathrm{per}}_{T,N}(t)|\,\mathrm{d}t &\geq \sum_{k=1}^{N-1}\int_{\frac{k}{2N+1}T}^{\frac{k+1}{2N+1}T} |D^{\mathrm{per}}_{T,N}(t)|\,\mathrm{d}t\\ &\geq \sum_{k=1}^{N-1}\int_{\frac{k}{2N+1}T}^{\frac{k+1}{2N+1}T}\Big|\frac{\sin((2N+1)\pi\frac{t}{T})}{\pi\frac{t}{T}}\Big|\,\mathrm{d}t\\ &\geq \sum_{k=1}^{N-1}\Big(\pi\frac{k+1}{2N+1}\Big)^{-1}\int_{\frac{k}{2N+1}T}^{\frac{k+1}{2N+1}T}|\sin((2N+1)\pi\tfrac{t}{T})|\,\mathrm{d}t\\ &= \sum_{k=1}^{N-1}\frac{2N+1}{\pi(k+1)}\frac{2T}{(2N+1)\pi} = \frac{2T}{\pi^2}\sum_{k=1}^{N-1}\frac{1}{k+1}.\end{aligned}$$

By Example 2.4.2–**??**, this gives

$$\lim_{N\to\infty}\int_0^{\frac{T}{2}} |D^{\mathrm{per}}_{T,N}(t)|\,\mathrm{d}t = \infty,$$

which gives the lemma. ▼

Thus condition (ii) of Definition 11.3.13 cannot be satisfied for the sequence $(\mathrm{per}_T(D_{j\Omega}))_{j\in\mathbb{Z}_{>0}}$. Nonetheless, as we did in Example 11.3.19–5, we can ponder on whether $(\mathrm{per}_T(D_{j\Omega}))_{j\in\mathbb{Z}_{>0}}$ can be used to approximate signals. As was the case in Example 11.3.19–5, for $(D_{j\Omega})_{j\in\mathbb{Z}_{>0}}$, there exists $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ such that $(f * \mathrm{per}_T(D_{j\Omega}))_{j\in\mathbb{Z}_{>0}}$ does not converge in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$. But, for some classes of signals, these approximations by convolution do indeed give meaningful results. To this end, in Figure 11.27 we plot the periodisation $\mathrm{per}_T(D_\Omega)$ of the Dirichlet kernel for $T = 2$ and a few values of Ω. We also show the corresponding approximations to the periodic extension of the characteristic function of $[-\frac{1}{2},\frac{1}{2}]$. Note that in the plot given, $T\Omega$ is always an integer, and so $\mathrm{per}_T(D_\Omega) \neq \frac{1}{T}D^{\mathrm{per}}_{T,N}$ in this case. •

11.3.4 Regularisation of signals on $\mathbb{R}$

Next we consider approximating signals by signals that are infinitely differentiable. Key to this is the following idea.

11.3.20 Definition A sequence $(\rho_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{F})$ is ***regularising*** if

(i) $\rho_j(t) \geq 0$ for all $t \in \mathbb{R}$;

(ii) $\int_{\mathbb{R}} \rho_j(t)\,\mathrm{d}t = 1$ for $j \in \mathbb{Z}_{>0}$;

(iii) $\mathrm{supp}(\phi_j) = [-\delta_j,\delta_j]$ and $\lim_{j\to\infty}\delta_j = 0$.

If $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ and if $(\rho_j)_{j\in\mathbb{Z}_{>0}}$ is a regularising sequence, then the sequence $(f * \rho_j)_{j\in\mathbb{Z}_{>0}}$ is a ***regularisation*** of f. •

Note that it is clear by Proposition 10.5.24 that a regularising sequence is a delta-sequence. However, it is a rather special delta-sequence since it is comprised of test signals. The following example shows that regularising sequences exist.

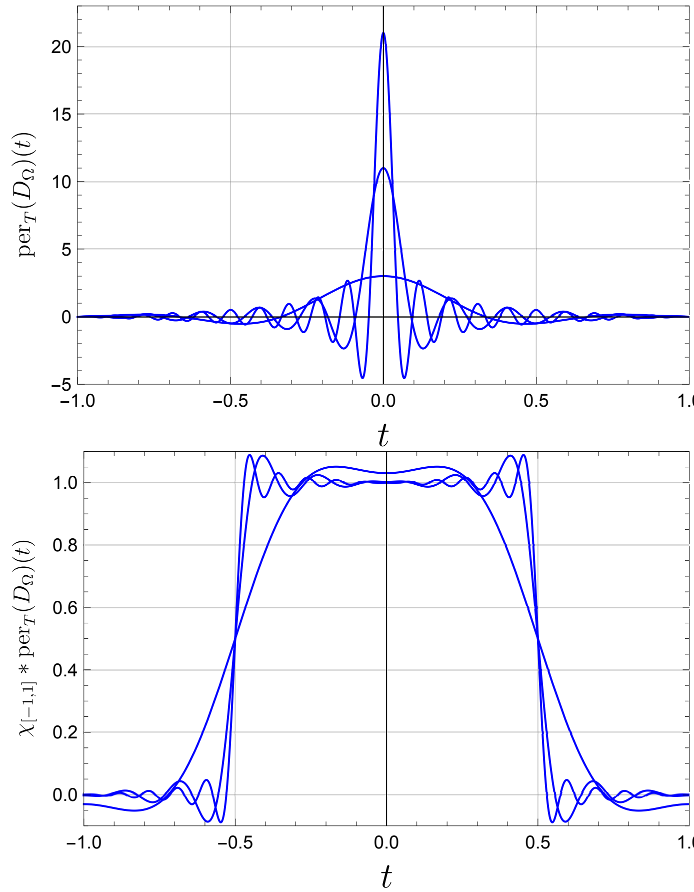


Figure 11.27 The 2-periodisation of D_Ω for $\Omega \in \{1, 5, 10\}$ (top) and the corresponding approximations by convolution of the periodic extension of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$ (bottom)

11.3.21 Example Define $\rho_j(t) = j \wedge (jt)$. One readily checks that the sequence $(\rho_j)_{j \in \mathbb{Z}_{>0}}$ is a regularising sequence. In Figure 11.28 we show a few terms in this sequence. •

The following theorem shows that the sequence of regularisations of a signal converges to the signal in a suitable sense.

11.3.22 Theorem *If* $f \in \mathsf{L}^{(p)}(\mathbb{R}; \mathbb{F})$, $p \in [1, \infty)$, *and* $(\rho_j)_{j \in \mathbb{Z}_{>0}}$ *is a regularising sequence then* $\lim_{j \to \infty} \|f - f * \rho_j\|_p = 0$.

Proof We shall first prove the result for a compactly supported continuous signal g. Let $(\rho_j)_{j \in \mathbb{Z}_{>0}}$ be a regularising sequence. Let us consider approximating g by $g * \rho_j$. We have

$$g(t) - g * \rho_j(t) = \int_{\mathbb{R}} (g(t) - g(t - s))\rho_j(s)\, \mathrm{d}s,$$

using the fact that $\int_{\mathbb{R}} \rho_j(t)\, \mathrm{d}t = 1$. Recalling the integral version of Minkowski's in-

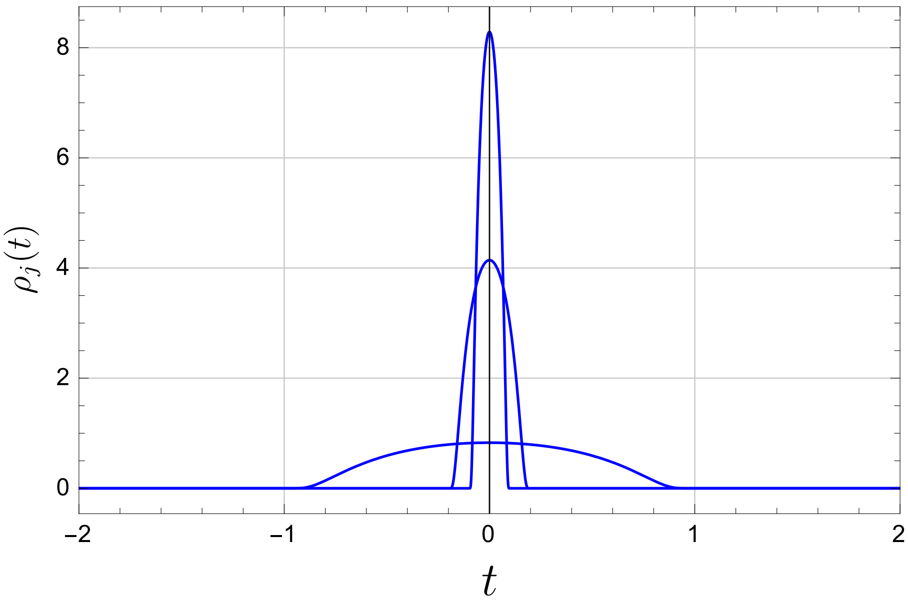


Figure 11.28 A regularising sequence

equality, Lemma 6.7.53, we have

$$\begin{aligned}\|g - g * \rho_j\|_p &= \Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}(g(t) - g(t-\tau))\rho_j(\tau)\,\mathrm{d}\tau\Big|^p\,\mathrm{d}t\Big)^{1/p}\\ &\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|(g(t) - g(t-\tau))\rho_j(\tau)|^p\,\mathrm{d}t\Big)^{1/p}\,\mathrm{d}\tau\\ &\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|g(t) - g(t-\tau)|^p\,\mathrm{d}t\Big)^{1/p}|\rho_j(\tau)|\,\mathrm{d}\tau\\ &\le \int_{\mathbb{R}}\|g - g_\tau\|_p|\rho_j(\tau)|\,\mathrm{d}\tau,\end{aligned}$$

where $g_\tau(t) = g(t-\tau)$. Note that the integral above is really over $[-\delta_j, \delta_j]$ since ρ_j has its support in this interval. We claim that $\lim_{\tau\to 0}\|g - g_\tau\|_p = 0$. Indeed, since $g \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ by choosing $\delta > 0$ sufficiently small we can ensure that $|g(t) - g(t-\tau)|$ is as small as we like for $|\tau| < \delta$. In this case, it is clear that $\|g - g_\tau\|_p$ can also be made as small as we like by taking τ sufficiently close to zero. Therefore, by taking j sufficiently large we can ensure that

$$\|g - g * \rho_j\|_p < \frac{\epsilon}{2},$$

by virtue of δ_j being sufficiently small. This gives the result for g. By Theorem 8.3.11(ii) there exists $g \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ so that $\|f - g\|_p < \frac{\epsilon}{2}$. Thus the result follows for general $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$.

For the actual assertion of the theorem, consider the signal $(f - g) * \rho_j$. By Corollary 11.2.9 we have

$$\|(f - g) * \rho_j\|_p \le \|f - g\|_p\|\rho_j\|_1 = \|f - g\|_p.$$

This then gives

$$\begin{aligned}\|f - f * \rho_j\|_p &= \|f - g + g - g * \rho_j + g * \rho_j - f * \rho_j\|_p \\ &\leq \|f - g\|_p + \|g - g * \rho_j\|_p + \|(f - g) * \rho_j\|_p \\ &\leq 2\|f - g\| + \|g - g * \rho_j\|_p < \epsilon,\end{aligned}$$

provided $g \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ is chosen sufficiently close to f in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$ and that j is sufficiently large, so giving the result. ■

The theorem shows that a signal in $\mathsf{L}^{(p)}(\mathbb{R};\mathbb{F})$ can be well approximated by an infinitely differentiable signal. This can be shown via an example.

11.3.23 Example We consider $f(t) = \frac{1}{1+t^2}$. •

Let us now establish the density of $\mathscr{D}(\mathbb{R};\mathbb{F})$ in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$ for $p \in [1,\infty)$. This result is one that we have used time and again in the text to this point, and so must be considered one of some importance. We actually proved this result during the course of the proof of Theorem 11.3.22. However, it is useful to have the proof we give below since it gives the construction that was useful in the proof of Theorem **??**.

11.3.24 Theorem *$\mathscr{D}(\mathbb{R};\mathbb{F})$ is dense in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$ for $p \in [1,\infty)$.*

Proof We let $\phi \in \mathscr{D}(\mathbb{R};\mathbb{F})$ have the property that $\int_{\mathbb{R}} \phi(t)\,\mathrm{d}t = 1$ and that there exists a neighbourhood of 0 on which ϕ takes the value 1. We leave to the reader the exercise of showing that such a test signals exists. We define $f_j(t) = f(t)\phi(\frac{t}{j})$ and we note that as $j \to \infty$ the signals f and f_j agree on a sequence of intervals that covers $\mathbb{R}$ in the limit. Thus we have $\lim_{j\to\infty}\|f - f_j\|_p = 0$. Now take $\rho_k = k\phi(kt)$. As in the proof of Theorem 11.3.22 we have, for each $j \in \mathbb{Z}_{>0}$, $\lim_{k\to\infty}\|f_j - f_j * \rho_k\|_p = 0$. We also have $\lim_{j\to\infty}\|f - f * \rho_j\|_p = 0$. Therefore

$$\lim_{k\to\infty}\|f - f * \rho_k - (f_j - f_j * \rho_k)\|_p = 0$$

$$\begin{aligned}\|f - f_j * \rho_j\|_p &= \|f - f_j * \rho_k + f_j * \rho_k - f_j * \rho_j\|_p \\ &\leq \|f - f * \rho_k\|_p +\end{aligned}$$

■

11.3.25 Corollary *$\mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ is dense in $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ for $p \in [1,\infty)$.*

Proof In Theorem 11.3.24 we proved that $\mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ is dense in $\mathsf{L}^p(\mathbb{R};\mathbb{F})$. Furthermore, if one investigates the proof, one can see that if $\mathrm{supp}(f)$ is compact then one can choose $\phi \in \mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ so that $\lambda(\mathrm{supp}(\phi) \setminus \mathrm{supp}(f))$ is as small as one likes. Now one proceeds as follows, assuming that $f \in \mathsf{L}^p(\mathbb{T};\mathbb{F})$ for some open time-domain $\mathbb{T}$. Let $\tilde{\mathbb{T}} \subseteq \mathbb{T}$ be a compact time-domain sufficiently large that $|f - f\chi_{\tilde{\mathbb{T}}}|_p < \frac{\epsilon}{2}$. Then choose $\phi \in \mathsf{C}^\infty_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ so that $\|f\chi_{\tilde{\mathbb{T}}} - \phi\|_p < \frac{\epsilon}{2}$ and that $\mathrm{supp}(\phi) \subseteq \mathbb{T}$. It now follows that $\|f - \phi\|_p < \epsilon$, which gives the result. ■

11.3.5 Regularisation of periodic signals

11.3.6 Regularisation of generalised signals

11.3.26 Theorem

Proof ■

Exercises

11.3.1

Chapter 12

The continuous-discrete Fourier transform

In this chapter we begin our discussion of Fourier transform theory. As we shall see, there are four natural sorts of transforms, depending on the character of the signal involved. The order of presentation of the four transforms is not quite uniquely determinable from the point of view of combining pedagogy and logic. Our choice here is to present first the CDFT on the basis that it is perhaps the easiest to motivate. Based on the dream of Fourier, our version of which is presented in Section 9.6.1, our hope is that the harmonic signals $\{e^{2\pi i n \frac{t}{T}}\}_{n\in\mathbb{Z}}$ have the property that a suitable large class of periodic signals can be written as infinite linear combinations of them. In our way of presenting things, this dream is not immediately apparent. Indeed, our presentation begins with a transform point of view, the idea being that we wish to produce a frequency-domain representation of a time-domain signal. However, the dream is realised in two ways.

1. In Section 12.2 we present the dream as being precisely the idea that the transform from time-domain to frequency-domain is invertible. We give suitable properties of signals which ensure that they can indeed be recovered from their frequency-domain representations.
2. In Section 12.3 we consider the harmonic signals $\{e^{2\pi i n \frac{t}{T}}\}_{n\in\mathbb{Z}}$ as an orthogonal set in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. We show in Theorem 12.3.3 that this set (more precisely, the orthonormal set associated to it) is a Hilbert basis for $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Using Theorem 7.3.25 we then conclude that every signal in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ can be written as an infinite sum of harmonic signals, with the sum converging in the 2-norm.

Thus, with a little patience, the reader shall see that our way of thinking about things as transforms provides a coherent picture of the subject of Fourier analysis since many of the questions in Fourier analysis can be thought of in terms of the transform and its inverse.

Do I need to read this chapter? If you are learning Fourier transform theory, this is where your journey begins. •

Contents

Section 12.1

The L^1-CDFT

In this section we define the CDFT and give some of its properties. The point of view in this section is simply that, given a signal $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, we assign to it another signal $\mathscr{F}_{\mathrm{CD}}(f)$. We think of f as being a time-domain signal and of $\mathscr{F}_{\mathrm{CD}}(f)$ as being its frequency-domain representation. We do this because this provides a nice motivation for what we are doing. However, from the point of view of the theory, there is nothing in particular to be gained from thinking of f as being a function of time or of $\mathscr{F}_{\mathrm{CD}}(f)$ as being a function of frequency.

Do I need to read this section? If you are reading this chapter then you are reading this section. •

12.1.1 Definitions and computations

We assume the reader has read the motivational ideas from Section 9.6.1 and from the preamble to this chapter and to this section. Therefore, we merely give the definition.

12.1.1 Definition (CDFT) The ***continuous-discrete Fourier transform*** or ***CDFT*** assigns to $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{F}_{\mathrm{CD}}(f)\colon \mathbb{Z}(T^{-1}) \to \mathbb{C}$ by

$$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t, \qquad n \in \mathbb{Z}. \qquad \bullet$$

12.1.2 Remarks (Comments on the definition of the CDFT)

1. Note that the expression for $\mathscr{F}_{\mathrm{CD}}(f)$ makes sense if and only if $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, so the CDFT is most naturally defined on such signals.
2. Note that if $f_1, f_2 \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ have the property that $f_1(t) = f_2(t)$ for almost every $t \in \mathbb{R}$, then we have $\mathscr{F}_{\mathrm{CD}}(f_1) = \mathscr{F}_{\mathrm{CD}}(f_2)$ by Proposition 5.7.11. Therefore, $\mathscr{F}_{\mathrm{CD}}$ is well-defined as a map from equivalence classes in $\mathsf{L}^{1}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Frequently we shall be interested in this equivalence class version of the CDFT, and we shall explicitly indicate that we are working with $\mathsf{L}^{1}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ rather than $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ in such cases. However, we shall adhere to our convention of denoting equivalence classes of signals in $\mathsf{L}^{1}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ by f rather than with some more cumbersome notation.
3. It is convenient for the purposes of general discussion to always think of $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{R})$ as being a subspace of $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, and so always use complex exponentials for the Fourier series rather than the real trigonometric functions.
4. It is not uncommon to see $\mathscr{F}_{\mathrm{CD}}(f)$ defined as having domain $\mathbb{Z}$ rather than $\mathbb{Z}(T^{-1})$. The reason for our choice of $\mathbb{Z}(T^{-1})$ for the domain of $\mathscr{F}_{\mathrm{CD}}(f)$ is not

perfectly clear at this time, except that the points nT^{-1}, $n \in \mathbb{Z}$, are the frequencies of the harmonics in the Fourier series for f.

5. Another very common alternative convention for the CDFT is to define it as we have done, but scaled by $\frac{1}{T}$. There are good reasons to do this, and there are good reasons to do as we have done. So the reader needs to simply be aware of what conventions are in effect. •

Let us compute the CDFT for some simple examples.

12.1.3 Examples (Computing the CDFT)

1. We let $f \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{R})$ be the 1-periodic extension of the signal $\tilde{f}(t) = t$; the signal is depicted in Figure 12.1. We directly compute

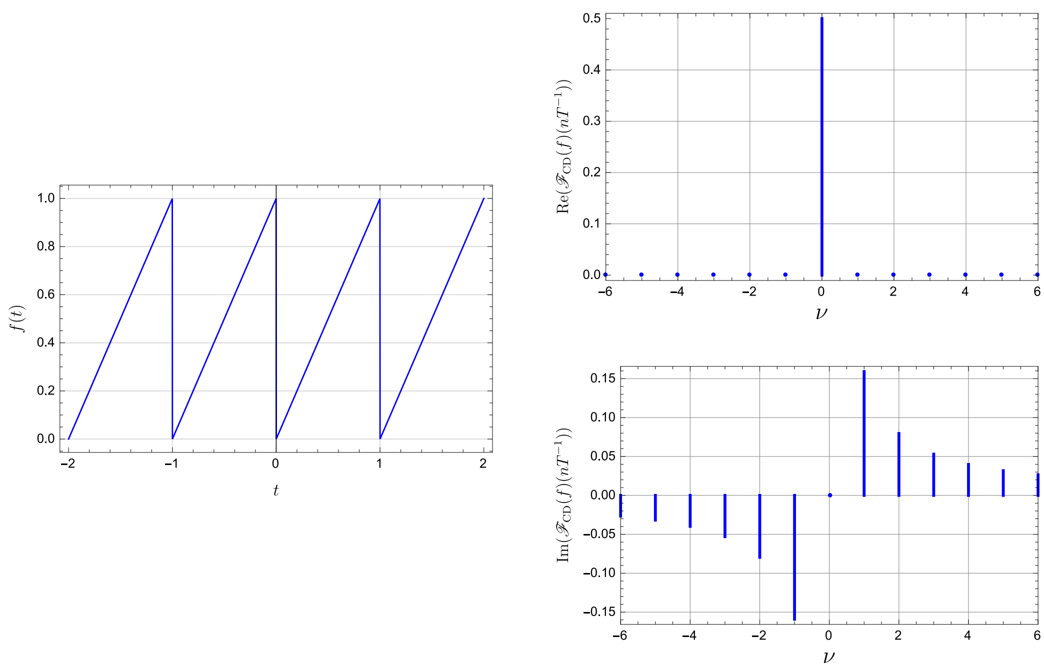


Figure 12.1 The 1-periodic extension of $t \mapsto t$ on $[0,1]$ (left) and the real (top right) and imaginary (bottom right) parts of its CDFT

$$\mathscr{F}_{\mathrm{CD}}(f)(0) = \int_0^1 t\,\mathrm{d}t = \frac{1}{2}$$

and, using integration by parts,

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f)(n) &= \int_0^1 t\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t\\ &= -t\frac{1}{2\pi\mathrm{i}n}\mathrm{e}^{-2\pi\mathrm{i}nt}\Big|_0^1 + \frac{1}{2\pi\mathrm{i}n}\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t\\ &= -\frac{1}{2\pi\mathrm{i}n} + \frac{1}{2\pi\mathrm{i}n}\frac{1}{2\pi\mathrm{i}n}\mathrm{e}^{2\pi\mathrm{i}nt}\Big|_0^1\\ &= -\frac{1}{2\pi\mathrm{i}n} = \frac{\mathrm{i}}{2n\pi},\end{aligned}$$

provided that $n \neq 0$. Therefore,

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \begin{cases}\frac{1}{2}, & n = 0,\\ \frac{\mathrm{i}}{2n\pi}, & \text{otherwise.}\end{cases}$$

2. We consider the signal $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \square_{2,1,0}(t) - 1$ and depicted in Figure 12.2. Thus f is the 1-periodic extension of the signal defined on $[0, 1]$ by

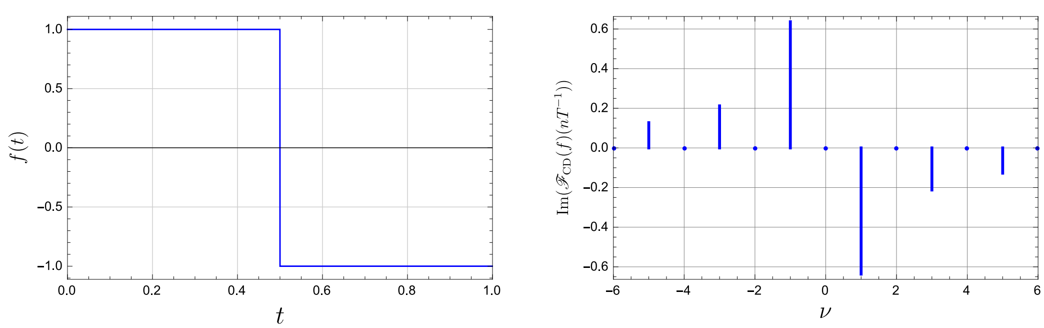


Figure 12.2 The signal $\square_{2,1,0}(t) - 1$ (left) and its CDFT (right)

$$(f|[0,1])(t) = \begin{cases}1, & t \in [0, \frac{1}{2}],\\ -1, & t \in (\frac{1}{2}, 1].\end{cases}$$

We compute

$$\mathscr{F}_{\mathrm{CD}}(f)(0) = \int_0^1 f(t)\,\mathrm{d}t = 0.$$

For $n \neq 0$ we have

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f)(n) &= \int_0^1 f(t)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t - \int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\ &= -\frac{\mathrm{e}^{-2\pi\mathrm{i}nt}}{2\pi\mathrm{i}n}\Big|_0^{\frac{1}{2}} + \frac{\mathrm{e}^{-2\pi\mathrm{i}nt}}{2\pi\mathrm{i}n}\Big|_{\frac{1}{2}}^1 = \frac{1-\mathrm{e}^{\mathrm{i}n\pi}}{2\pi\mathrm{i}n} - \frac{\mathrm{e}^{\mathrm{i}n\pi}-1}{2\pi\mathrm{i}n} \\ &= \mathrm{i}\frac{(-1)^n - 1}{n\pi},\end{aligned}$$

using the identity $\mathrm{e}^{\mathrm{i}n\pi} = (-1)^n$ for $n \in \mathbb{Z}$. Thus we have

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \begin{cases} 0, & n = 0, \\ \mathrm{i}\frac{(-1)^n-1}{n\pi} & \text{otherwise.}\end{cases}$$

We plot this in Figure 12.2.

3. Next we consider the signal $g = \triangle_{\frac{1}{2},1,0}$ depicted in Figure 12.3. Thus g is the

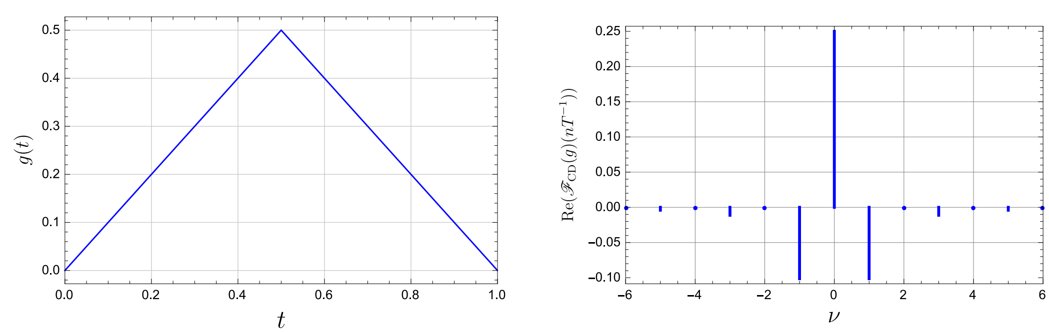


Figure 12.3 The signal $\triangle_{\frac{1}{2},1,0}$ (left) and its CDFT (right)

1-periodic extension of the signal

$$(g|[0,1])(t) = \begin{cases} t, & t \in [0,\frac{1}{2}], \\ 1-t, & t \in (\frac{1}{2},1].\end{cases}$$

We then compute

$$\mathscr{F}_{\mathrm{CD}}(g)(0) = \int_0^1 g(t)\,\mathrm{d}t = \int_0^{\frac{1}{2}} t\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)\,\mathrm{d}t = \frac{t^2}{2}\Big|_0^{\frac{1}{2}} + \Big(t - \frac{t^2}{2}\Big)\Big|_{\frac{1}{2}}^1 = \frac{1}{4},$$

and for $n \neq 0$,

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(g)(n) &= \int_0^1 g(t)\mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t \\
&= \int_0^{\frac{1}{2}} t\mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)\mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t \\
&= -\frac{t\mathrm{e}^{-2\pi \mathrm{i} nt}}{2\pi \mathrm{i} n}\Big|_0^{\frac{1}{2}} + \frac{1}{2\pi \mathrm{i} n}\int_0^{\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t + \frac{t\mathrm{e}^{-2\pi \mathrm{i} nt}}{2\pi \mathrm{i} n}\Big|_{\frac{1}{2}}^1 - \frac{1}{2\pi \mathrm{i} n}\int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t \\
&= -\frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4\mathrm{i} n\pi} + \frac{\mathrm{e}^{-2\pi \mathrm{i} nt}}{4n^2\pi^2}\Big|_0^{\frac{1}{2}} - \frac{\mathrm{e}^{-2\pi \mathrm{i} nt}}{2\pi \mathrm{i} n}\Big|_{\frac{1}{2}}^1 + \frac{1}{2\pi \mathrm{i} n} - \frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4\mathrm{i} n\pi} - \frac{\mathrm{e}^{-2\pi \mathrm{i} nt}}{4n^2\pi^2}\Big|_{\frac{1}{2}}^1 \\
&= -\frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4\mathrm{i} n\pi} + \frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4n^2\pi^2} - \frac{1}{4n^2\pi^2} - \frac{1}{2\pi \mathrm{i} n} + \frac{\mathrm{e}^{-\mathrm{i} n\pi}}{2\pi \mathrm{i} n} + \frac{1}{2\pi \mathrm{i} n} \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4\mathrm{i} n\pi} - \frac{1}{4n^2\pi^2} + \frac{\mathrm{e}^{-\mathrm{i} n\pi}}{4n^2\pi^2} \\
&= \frac{\mathrm{e}^{-\mathrm{i} n\pi} - 1}{2n^2\pi^2} = \frac{(-1)^n - 1}{2n^2\pi^2}.
\end{aligned}$$

We have used the fact that $\mathrm{e}^{-\mathrm{i} n\pi} = (-1)^n$ for $n \in \mathbb{Z}$. Thus we have

$$\mathscr{F}_{\mathrm{CD}}(g)(n) = \begin{cases} \frac{1}{4}, & n = 0, \\ \frac{(-1)^n - 1}{2n^2\pi^2}, & \text{otherwise.} \end{cases} \tag{12.1}$$

This CDFT is plotted in Figure 12.3.

4. The T-periodic signal we consider next we define not be specifying it on $[0, T]$, but on $[-\frac{T}{2}, \frac{T}{2}]$. We take $a \in (0, \frac{T}{2}]$ and define $f_a\colon \mathbb{R} \to \mathbb{R}$ by defining it on $[-\frac{T}{2}, \frac{T}{2}]$ by

$$f_a(t) = \begin{cases} 1, & t \in [-a, a], \\ 0, & \text{otherwise.} \end{cases}$$

We plot this signal in Figure 12.4. We compute the CDFT of this signal, noting that T-periodicity gives

$$\mathscr{F}_{\mathrm{CD}}(f_a)(nT^{-1}) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f_a(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t.$$

For $n \neq 0$ we have

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} f_a(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \int_{-a}^{a} \mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = -\frac{T\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}}{2\pi \mathrm{i} n}\Big|_{-a}^{a} \\
&= \frac{1}{\pi\frac{n}{T}}\frac{1}{2\mathrm{i}}(\mathrm{e}^{2\pi \mathrm{i} n\frac{a}{T}} - \mathrm{e}^{-2\pi \mathrm{i} n\frac{a}{T}}) = \frac{\sin(2\pi a\frac{n}{T})}{\pi\frac{n}{T}}.
\end{aligned}$$

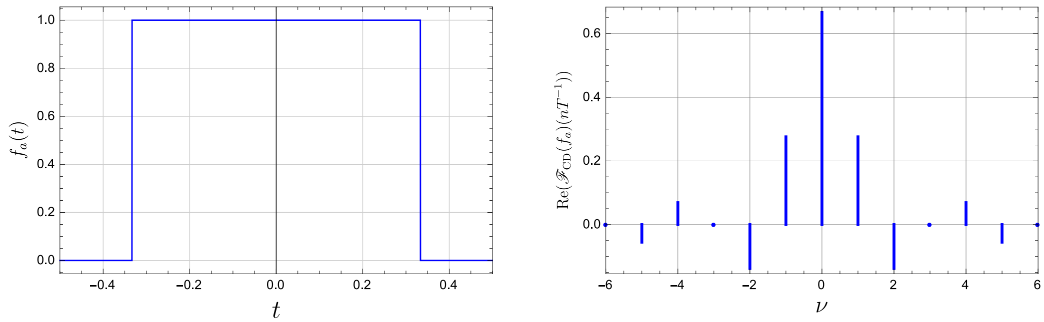


Figure 12.4 The signal f_a for $T = 1$ and $a = \frac{1}{3}$ (left) and its CDFT (right)

We also directly compute $\mathscr{F}_{\mathrm{CD}}(f_a)(0) = 2a$. In summary,

$$\mathscr{F}_{\mathrm{CD}}(f_a)(nT^{-1}) = \begin{cases} 2a, & n = 0, \\ \frac{\sin(2\pi a \frac{n}{T})}{\pi \frac{n}{T}}, & n \neq 0. \end{cases}$$

We plot the CDFT of f_a in Figure 12.4.

5. As with the previous example, we define a T-periodic signal by prescribing it on $[-\frac{T}{2}, \frac{T}{2}]$. The signal we denote by g_a for $a \in (0, \frac{T}{2}]$:

$$g_a(t) = \begin{cases} 1 + \frac{t}{a}, & t \in [-a, 0], \\ 1 - \frac{t}{a}, & t \in (0, a], \\ 0, & \text{otherwise.} \end{cases}$$

We compute the CDFT of g_a, first for $n \neq 0$:

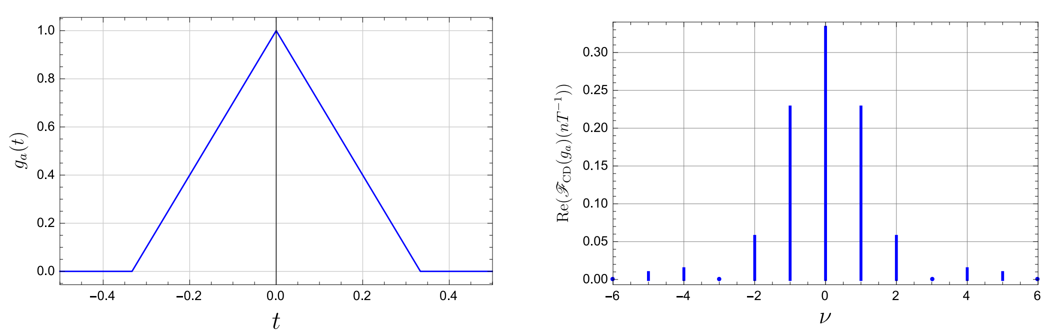


Figure 12.5 The signal g_a for $T = 1$ and $a = \frac{1}{3}$ (left) and its CDFT (right)

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(g_a)(nT^{-1}) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} g_a(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\ &= \int_{-a}^{0}(1+\tfrac{t}{a})\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t + \int_{0}^{a}(1-\tfrac{t}{a})\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\ &= 2\int_0^a (1-\tfrac{t}{a})\cos(2\pi n\tfrac{t}{T})\,\mathrm{d}t,\end{aligned}$$

using a change of variable and the identity $\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})$. One may now use a messy integration by parts to compute

$$\mathscr{F}_{\mathrm{CD}}(g_a)(nT^{-1}) = \frac{\sin(\pi a\frac{n}{T})^2}{\pi^2 a(\frac{n}{T})^2}$$

for $n \neq 0$. For $n = 0$ we compute

$$\mathscr{F}_{\mathrm{CD}}(g_a)(0) = \int_{-a}^{0}(1+\tfrac{t}{a})\,\mathrm{d}t + \int_0^a (1-\tfrac{t}{a})\,\mathrm{d}t = a.$$

Thus we have

$$\mathscr{F}_{\mathrm{CD}}(g_a)(nT^{-1}) = \begin{cases} a, & n = 0, \\ \frac{\sin(\pi a\frac{n}{T})^2}{\pi^2 a(\frac{n}{T})^2}, & n \neq 0. \end{cases}$$

This CDFT is plotted in Figure 12.5. •

Sometimes one also considers somewhat different versions of Fourier transforms, defined using real, rather than complex, harmonic functions. These are easy to define.

12.1.4 Definition (CDCT and CDST)

(i) The ***continuous-discrete cosine transform*** or ***CDCT*** assigns to $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{C}_{\mathrm{CD}}\colon \mathbb{Z}_{\geq 0}(T^{-1}) \to \mathbb{C}$ by

$$\mathscr{C}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\cos(2\pi n\tfrac{t}{T})\,\mathrm{d}t, \qquad n \in \mathbb{Z}_{\geq 0}.$$

(ii) The ***continuous-discrete sine transform*** or ***CDST*** assigns to $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{S}_{\mathrm{CD}}\colon \mathbb{Z}_{>0}(T^{-1}) \to \mathbb{C}$ by

$$\mathscr{S}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\sin(2\pi n\tfrac{t}{T})\,\mathrm{d}t, \qquad n \in \mathbb{Z}_{>0}.$$ •

Let us give the relationship between the CDFT, and the CDCT and the CDST.

12.1.5 Proposition (The CDFT, and the CDCT and the CDST) *For* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(0) = \mathscr{C}_{\mathrm{CD}}(\mathrm{f})(0)$;

(ii) $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) = \mathscr{C}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) - \mathrm{i}\mathscr{S}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})$ *and* $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(-\mathrm{nT}^{-1}) = \mathscr{C}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) + \mathrm{i}\mathscr{S}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})$ *for every* $\mathrm{n} \in \mathbb{Z}_{>0}$;

(iii) $\mathscr{C}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) = \frac{1}{2}(\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) + \mathscr{F}_{\mathrm{CD}}(\mathrm{f})(-\mathrm{nT}^{-1}))$ *for every* $\mathrm{n} \in \mathbb{Z}_{\geq 0}$;

(iv) $\mathscr{S}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}) = \frac{1}{2\mathrm{i}}(\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(-\mathrm{nT}^{-1}) - \mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1}))$ *for every* $\mathrm{n} \in \mathbb{Z}_{>0}$.

Proof This follows by direct computation using Euler's formula

$$\mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}} = \cos(2\pi n \tfrac{t}{T}) + \mathrm{i}\sin(2\pi n \tfrac{t}{T}). \qquad \blacksquare$$

Sometimes it is easier to compute the CDFT using the CDCT and the CDST, along with the relations from the preceding result. However, for dealing with generalities the CDFT is by far the more preferable, so we will deal exclusively with it for this purpose.

12.1.2 Properties of the CDFT

Before we go any further, let us provide some elementary properties of the CDFT. Recall from Example 8.1.6–2 the time-domain reparameterisation $\sigma\colon \mathbb{R} \to \mathbb{R}$ defined by $\sigma(t) = -t$, and from Example 8.1.13–2 the domain transformation σ^* defined by $\sigma^* f(t) = f(-t)$. Clearly, if $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $\sigma^* f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Also, if $a \in \mathbb{R}$ then in Example 8.1.6–1 we defined the reparameterisation $\tau_a\colon \mathbb{R} \to \mathbb{R}$ by $\tau_a(t) = t - a$ and in Example 8.1.13–1 the corresponding domain transformation τ_a^* by $\tau_a^* f(t) = f(t-a)$. Again, if $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $\tau_a^* f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. In like manner, if $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $\bar{f} \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ denotes the signal defined by $\bar{f}(t) = \overline{f(t)}$. For $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ let us also define the signal $\overline{\mathscr{F}}_{\mathrm{CD}}(f)\colon \mathbb{Z}(T^{-1}) \to \mathbb{C}$ by

$$\overline{\mathscr{F}}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}}\,\mathrm{d}t.$$

At this point it is not clear why we should care about $\overline{\mathscr{F}}_{\mathrm{CD}}(f)$, but it will come up in Section 14.1. With the preceding notation we have the following result.

12.1.6 Proposition (Elementary properties of the CDFT) *If* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *and if* $\mathrm{a} \in \mathbb{R}$, *then the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{CD}}(\mathrm{f})} = \overline{\mathscr{F}}_{\mathrm{CD}}(\bar{\mathrm{f}})$;

(ii) $\mathscr{F}_{\mathrm{CD}}(\sigma^*\mathrm{f}) = \sigma^*(\mathscr{F}_{\mathrm{CD}}(\mathrm{f})) = \overline{\mathscr{F}}_{\mathrm{CD}}(\mathrm{f})$;

(iii) *if* f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})$ *is even (resp. odd);*

(iv) *if* f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})$ *is real and even (resp. imaginary and odd);*

(v) $\mathscr{F}_{\mathrm{CD}}(\tau_{\mathrm{a}}^*\mathrm{f})(\mathrm{nT}^{-1}) = \mathrm{e}^{-2\pi \mathrm{i} n \frac{\mathrm{a}}{\mathrm{T}}}\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})$.***missing stuff***

Proof (i) This follows directly from the definition of $\mathscr{F}_{\mathrm{CD}}$ and $\overline{\mathscr{F}}_{\mathrm{CD}}$.

(ii) We compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(\sigma^* f)(nT^{-1}) &= \int_0^T f(-t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \int_0^T f(t)\mathrm{e}^{2\pi\mathrm{i}(-n)\frac{t}{T}}\,\mathrm{d}t\\ &= \sigma^*(\mathscr{F}_{\mathrm{CD}}(f)) = \overline{\mathscr{F}}_{\mathrm{CD}}(f).\end{aligned}$$

(iii) This follows immediately from (ii).

(iv) If f is real and even then $\bar{f} = f = \sigma^* f$. Evenness of $\mathscr{F}_{\mathrm{CD}}(f)$ follows from (ii) and realness of $\mathscr{F}_{\mathrm{CD}} f$ follows since

$$\overline{\mathscr{F}_{\mathrm{CD}}(f)} = \overline{\mathscr{F}}_{\mathrm{CD}}(\bar{f}) = \overline{\mathscr{F}}_{\mathrm{CD}}(f) = \sigma^*(\mathscr{F}_{\mathrm{CD}}(f)) = \mathscr{F}_{\mathrm{CD}}(f),$$

where we have used (i), (ii), and (iii). In like manner, if f is real and odd then we have $\bar{f} = f$ and $\sigma^* f = -f$. The same computations then show that $\overline{\mathscr{F}_{\mathrm{CD}}(f)} = -\mathscr{F}_{\mathrm{CD}}(f)$, meaning that $\mathscr{F}_{\mathrm{CD}}(f)$ is odd and imaginary.

(v) We compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(\tau_a^* f)(nT^{-1}) &= \int_0^T \tau_a^* f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \int_0^T f(t-a)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\\ &= \int_{-a}^{T-a} f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t+a}{T}}\,\mathrm{d}t = \frac{\mathrm{e}^{-2\pi\mathrm{i}n\frac{a}{T}}}{T}\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\\ &= \mathrm{e}^{-2\pi\mathrm{i}n\frac{a}{T}}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}),\end{aligned}$$

as desired. ■

12.1.7 Examples (Elementary properties of the CDFT)

1. We consider the signal $f(t) = \square_{2,1,0}(t) - 1$ introduced in Example 12.1.3–2. This signal is real and odd, and we see that $\mathscr{F}_{\mathrm{CD}}(f)$ is imaginary and odd, as predicted by part (iv) of Proposition 12.1.6.
2. We consider the signal $g(t) = \triangle_{\frac{1}{2},1,0}(t) - 1$ introduced in Example 12.1.3–3. This signal is real and even, and we see that $\mathscr{F}_{\mathrm{CD}}(g)$ is real and even, again as in part (iv) of Proposition 12.1.6.
3. We consider the example initiated in Example 8.1.24. We consider the signal $\tilde{f}$ defined on $[\frac{1}{3}, \frac{4}{3}]$ by $\tilde{f}(t) = t$. Note that we have changed the "f" in Example 8.1.24 to "$\tilde{f}$" here in order to match the notation of Proposition 12.1.6(v). In Figure 12.6 we show the periodic extension of $\tilde{f}$ as well as the periodic extension of the signal f defined by $f(t) = \tilde{f}(t + \frac{1}{3})$. Note that f_{per} is continuous on $[0, 1]$ but that $\tilde{f}_{\mathrm{per}}$ is not, but is continuous on the shifted interval $[\frac{1}{3}, \frac{4}{3}]$. Let us determine the CDFT for each signal. For f we compute

$$\mathscr{F}_{\mathrm{CD}}(f_{\mathrm{per}})(0) = \int_0^1 (t + \tfrac{1}{3})\,\mathrm{d}t = \frac{5}{6},$$

and for $n \neq 0$,

$$\mathscr{F}_{\mathrm{CD}}(f_{\mathrm{per}})(n) = \int_0^1 (t + \tfrac{1}{3})\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t = \int_0^1 t\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t = \frac{\mathrm{i}}{2n\pi}$$

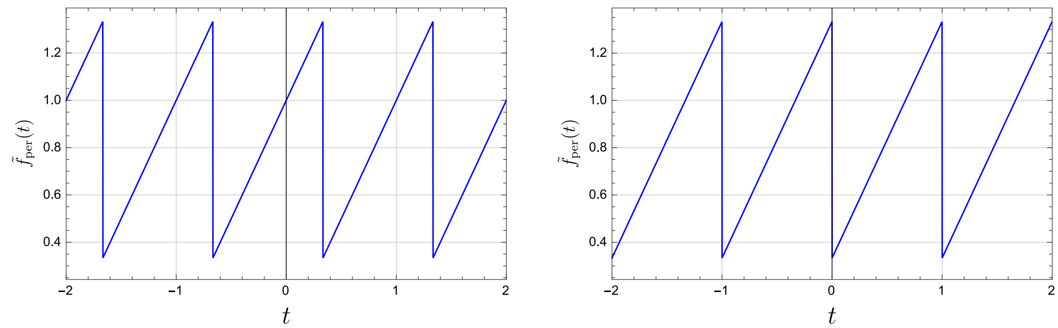


Figure 12.6 A signal $\tilde{f}$ (top) and its shift to the origin f (bottom), both periodically extended

(this is the same computation as we performed in Example 12.1.3–1). For $\tilde{f}_{\mathrm{per}}$ we note that for $t \in [0,1]$ we have

$$\tilde{f}_{\mathrm{per}} = \begin{cases} t+1, & t \in [0, \frac{1}{3}], \\ t, & t \in (\frac{1}{3}, 1]. \end{cases}$$

Thus we compute

$$\begin{aligned} \mathscr{F}_{\mathrm{CD}}(\tilde{f}_{\mathrm{per}})(0) &= \int_0^{\frac{1}{3}} (t+1)\,\mathrm{d}t + \int_{\frac{1}{3}}^1 t\,\mathrm{d}t \\ &= (\tfrac{t^2}{2} + t)\Big|_0^{\frac{1}{3}} + \tfrac{t^2}{2}\Big|_{\frac{1}{3}}^1 = \frac{7}{18} + \frac{4}{9} = \frac{5}{6}, \end{aligned}$$

and for $n \neq 0$ we compute

$$\begin{aligned} \mathscr{F}_{\mathrm{CD}}(\tilde{f}_{\mathrm{per}})(n) &= \int_0^{\frac{1}{3}} (t+1)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t + \int_{\frac{1}{3}}^1 t\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\ &= \int_0^1 t\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t + \int_0^{\frac{1}{3}} \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\ &= \frac{\mathrm{i}}{2n\pi} + \frac{\mathrm{e}^{-2\pi\mathrm{i}n/3} - 1}{-2\pi\mathrm{i}n} = \mathrm{e}^{-2\pi\mathrm{i}n/3}\frac{\mathrm{i}}{2n\pi}. \end{aligned}$$

Note that this is exactly as predicted by Proposition 12.1.6(v), and we could have saved ourselves some computation by noting this, of course. •

Let us now show that when $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, the sequence $(\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}))_{n\in\mathbb{Z}_{>0}}$ is reasonably well behaved.

12.1.8 Theorem (Riemann–Lebesgue Lemma) *If* $f \in \mathsf{L}^{(1)}([a,b];\mathbb{C})$ *then*

$$\lim_{|n|\to\infty} \int_a^b f(t)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = 0.$$

In particular, if $(\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{n}\mathrm{T}^{-1}))_{\mathrm{n}\in\mathbb{Z}_{>0}}$ *are the values of the CDFT of* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *then* $\lim_{|\mathrm{n}|\to\infty}|\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{n}\mathrm{T}^{-1})| = 0$.

Proof We first establish the result when f is continuously differentiable on $[a,b]$. In this case, integration by parts yields

$$\int_a^b f(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \frac{T}{2\pi \mathrm{i} n} f(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\Big|_a^b - \frac{T}{2\pi \mathrm{i} n}\int_a^b f'(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t.$$

Taking the modulus of both sides of this equation gives

$$\Big|\int_a^b f(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| \le \frac{T}{2|n|\pi}\Big(|f(b)| + |f(a)| + \int_a^b |f'(t)|\,\mathrm{d}t\Big), \tag{12.2}$$

where we have used the triangle inequality, along with the inequality

$$\Big|\int_a^b g(t)\,\mathrm{d}t\Big| \le \int_a^b |g(t)|\,\mathrm{d}t. \tag{12.3}$$

Taking the limit as $n \to \infty$ in (12.2) gives the result when f is continuously differentiable.

By Corollary 11.3.25 the continuously differentiable signals are dense in $\mathsf{L}^1([a,b];\mathbb{C})$. Therefore, if $f \in \mathsf{L}^{(1)}([0,T];\mathbb{C})$, then for any $\epsilon \in \mathbb{R}_{>0}$ there exists a continuously differentiable $g_\epsilon\colon [0,T] \to \mathbb{C}$ with the property that

$$\int_a^b |f(t) - g_\epsilon(t)|\,\mathrm{d}t < \frac{\epsilon}{2}.$$

Thus g_ϵ is close to f in L^1. The triangle inequality and (12.3) now give

$$\begin{aligned}
\Big|\int_a^b f(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| &= \Big|\int_a^b (f(t) - g_\epsilon(t) + g_\epsilon(t))\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| \\
&\le \Big|\int_a^b (f(t) - g_\epsilon(t))\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| + \Big|\int_a^b g_\epsilon(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| \\
&\le \int_a^b \Big|f(t) - g_\epsilon(t)\Big|\,\mathrm{d}t + \Big|\int_a^b g_\epsilon(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big|.
\end{aligned}$$

As the lemma is true for g_ϵ, there exists $N \in \mathbb{Z}_{>0}$ so that, provided that $|n| \ge N$, we have

$$\Big|\int_a^b g_\epsilon(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| < \frac{\epsilon}{2}.$$

Thus, for $|n| \ge N$ we have

$$\Big|\int_a^b f(t)\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\Big| < \epsilon,$$

giving the result. ■

12.1.9 Remark (What the Riemann–Lebesgue Lemma does not say) Note that the decaying of the CDFT to zero at infinity says nothing about the summability properties of the sequence $(\mathscr{F}_{CD}(f)(nT^{-1}))_{n\in\mathbb{Z}}$. Indeed, the coefficients may decay quite slowly indeed, cf. Theorem 12.1.18. This is a crucial matter when it comes to attempting to invert the CDFT in Section 12.2. •

The following result gives an interpretation of Theorem 12.1.8 in terms of the ideas of Section 6.5. To make sense of the statement recall from Section 8.2.2 that $\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{F})$ denotes the set of signals in $\ell(\mathbb{Z}(T^{-1});\mathbb{F})$ that decay to zero at infinity.

12.1.10 Corollary (The CDFT is continuous) *$\mathscr{F}_{CD}$ is a continuous linear mapping from $(\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}),\|\cdot\|_1)$ to $(\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}),\|\cdot\|_\infty)$.*

Proof Linearity of $\mathscr{F}_{CD}$ follows directly from linearity of the integral. The Riemann–Lebesgue Lemma tells us that the domain of the CDFT is indeed $\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$. Note that for $n\in\mathbb{Z}$ we have

$$|\mathscr{F}_{CD}(f)(nT^{-1})| = \left|\int_0^T f(t)e^{-2\pi i n\frac{t}{T}}\,dt\right| \le \int_0^T |f(t)|\,dt = \|f\|_1.$$

Since this holds for every $n\in\mathbb{Z}$, this shows that

$$\|\mathscr{F}_{CD}(f)\|_\infty \le \|f\|_1.$$

Thus $\mathscr{F}_{CD}$ is bounded and so continuous by Theorem 6.5.8. ■

The following formula is sometimes a helpful one.

12.1.11 Proposition (Fourier Reciprocity Relation for the CDFT) *If $f,g\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ are such that $\mathscr{F}_{CD}(f)\in\ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ then*

$$\int_0^T f(t)g(t)\,dt = \frac{1}{T}\sum_{n\in\mathbb{Z}}\mathscr{F}_{CD}(f)(nT^{-1})\mathscr{F}_{CD}(g)(-nT^{-1}).$$

In particular,

$$\int_0^T |f(t)|^2\,dt = \frac{1}{T}\sum_{n\in\mathbb{Z}}|\mathscr{F}_{CD}(f)(nT^{-1})|^2.$$

Proof As we shall see in Theorem 12.2.33, and as follows more or less immediately from the Weierstrass M-test, the series

$$\frac{1}{T}\sum_{n\in\mathbb{Z}}\mathscr{F}_{CD}(f)(nT^{-1})e^{2\pi i n\frac{t}{T}}$$

converges uniformly to a (necessarily continuous) signal that is equal to f almost everywhere. Thus we can assume, without loss of generality, that f is continuous so that

$$f(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\mathscr{F}_{CD}(f)(nT^{-1})e^{2\pi i n\frac{t}{T}}, \qquad t\in\mathbb{R}.$$

Thus f is bounded since it is periodic and so $t \mapsto f(t)g(t)$ is bounded by the integrable signal $\|f\|_\infty g$, and so is integrable. By the Dominated Convergence Theorem and Proposition 12.1.6 we have

$$\begin{aligned}\int_0^T f(t)g(t)\,\mathrm{d}t &= \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) \int_0^T g(t)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\ &= \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\overline{\mathscr{F}}_{\mathrm{CD}}(g)(nT^{-1}) \\ &= \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathscr{F}_{\mathrm{CD}}(g)(-nT^{-1}),\end{aligned}$$

as desired.

The last relation in the statement of the proposition is proved by taking $g = \bar{f}$, and using Proposition 12.1.6. ■

The final assertion of the preceding result is a special case of ***Parseval's equality*** which we shall explore more generally in Section 12.3; see Corollary 12.3.4.

12.1.3 Differentiation, integration, and the CDFT

We next consider the relationships between the CDFT of a signal and the CDFT of its derivative, in those case when the signal is in some sense differentiable. These relationships are important to understand since they give some hint that the CDFT of a signal might actually say something about the signal itself.

12.1.12 Proposition (The CDFT and differentiation) *Suppose that* $\mathrm{f} \in \mathrm{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *and suppose that there exists a piecewise continuous signal* $\mathrm{f}'\colon [0,\mathrm{T}] \to \mathbb{C}$ *with the property that*

$$\mathrm{f}(\mathrm{t}) = \mathrm{f}(0) + \int_0^{\mathrm{t}} \mathrm{f}'(\tau)\,\mathrm{d}\tau, \qquad \mathrm{t} \in [0,\mathrm{T}].$$

Then

$$\mathscr{F}_{\mathrm{CD}}(\mathrm{f}'_{\mathrm{per}})(\mathrm{n}\mathrm{T}^{-1}) = \frac{2\pi\mathrm{i}\mathrm{n}}{\mathrm{T}}\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{n}\mathrm{T}^{-1}), \qquad \mathrm{n} \in \mathbb{Z}.$$

Proof Let $(t_0, t_1, \dots, t_k)$ be the endpoints of a partition having the property that f' is continuous on each subinterval (t_j, t_{j-1}), $j = 1, \dots, k$. Integration by parts of the expression for $\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})$ on (t_j, t_{j-1}) gives

$$\int_{t_{j-1}}^{t_j} f'(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\Big|_{t_{j-1}}^{t_j} + \frac{2\pi\mathrm{i}n}{T}\int_{t_{j-1}}^{t_j} f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t.$$

Over the entire interval $[0, T]$ we then have

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f'_{\mathrm{per}})(nT^{-1}) &= \int_0^T f'(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\ &= \sum_{j=1}^k \int_{t_{j-1}}^{t_j} f'(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\ &= \frac{2\pi\mathrm{i}n}{T}\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \frac{2\pi\mathrm{i}n}{T}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}),\end{aligned}$$

using the fact that f is continuous and that $f(0) = f(T)$. ■

12.1.13 Example (The CDFT and differentiation) We consider the signals f and g introduced in parts 2 and 3 of Example 12.1.3. Note that g satisfies the conditions of Proposition 12.1.12 and that

$$g(t) = \int_0^t f(\tau)\,d\tau, \qquad t \in [0,1].$$

Consistent with Proposition 12.1.12, we have

$$\mathscr{F}_{CD}(f)(n) = 2\pi i n \mathscr{F}_{CD}(g)(n). \qquad \bullet$$

The preceding result may be applied iteratively if a periodic signal is more than once differentiable. Upon doing this, we obtain the following characterisation of the CDFT.

12.1.14 Corollary (The CDFT and higher-order derivatives) *Suppose that* $f \in C^{r-1}_{per,T}(\mathbb{R};\mathbb{C})$ *for* $r \in \mathbb{Z}_{>0}$ *and suppose that there exists a piecewise continuous signal* $f^{(r)}\colon [0,T] \to \mathbb{C}$ *with the property that*

$$f^{(r-1)}(t) = f^{(r-1)}(0) + \int_0^t f^{(r)}(\tau)\,d\tau.$$

Then

$$\mathscr{F}_{CD}(f^{(r)}_{per})(nT^{-1}) = \Big(\frac{2\pi i n}{T}\Big)^r \mathscr{F}_{CD}(f)(nT^{-1}).$$

The next result records the manner in which integration acts relative to the CDFT.

12.1.15 Proposition (The CDFT and integration) *If* $f \in L^{(1)}_{per,T}(\mathbb{R};\mathbb{C})$, *if* $\int_0^T f(t)\,dt = 0$, *and if we define* $g\colon [0,T] \to \mathbb{C}$ *by*

$$g(t) = \int_0^t f(\tau)\,d\tau,$$

then

$$\mathscr{F}_{CD}(g_{per})(nT^{-1}) = \begin{cases} T\displaystyle\int_0^T f(t)\,dt - \int_0^T tf(t)\,dt, & n = 0,\\ \dfrac{T}{2\pi i n}\mathscr{F}_{CD}(f)(nT^{-1}), & n \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

Proof Using integration by parts we compute for $n \neq 0$

$$\begin{aligned}\mathscr{F}_{CD}(g_{per})(nT^{-1}) &= \int_0^T g(t)e^{-2\pi i n\frac{t}{T}}\,dt\\ &= -\frac{T}{2\pi i n}e^{-2\pi i n\frac{t}{T}}g(t)\Big|_0^T + \frac{T}{2\pi i n}\int_0^T f(t)e^{-2\pi i n\frac{t}{T}}\,dt\\ &= \frac{T}{2\pi i n}\mathscr{F}_{CD}(f),\end{aligned}$$

as desired. For $n = 0$ we have, again by integration by parts,

$$\mathscr{F}_{\mathrm{CD}}(g)(0) = \int_0^T g(t)\,\mathrm{d}t = tg(t)\Big|_0^T - \int_0^T tf(t)\,\mathrm{d}t = T\int_0^T f(t)\,\mathrm{d}t - \int_0^T tf(t)\,\mathrm{d}t,$$

as stated. ■

12.1.16 Example (The CDFT and integration) Consider again the signals f and g introduced in parts 2 and 3 of Example 12.1.3. We have $g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$, and we note that indeed the CDFT's of f and g are related as in Proposition 12.1.15. •

12.1.4 Decay of the CDFT

In this section we examine how properties of a signal, particularly its "smoothness," are reflected in its CDFT.

Given our results in the preceding section relating differentiability of a signal to its CDFT, we immediately have the following summary of the behaviour of the CDFT as the smoothness of a signal improves.

1. If $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the Fourier coefficients satisfy
$$\lim_{|n|\to\infty} |\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = 0.$$
This is the Riemann–Lebesgue Lemma, Theorem 12.1.8.
2. If $f \in \mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$. This is simply Parseval's equality and this will be discussed further in Section 12.3, also, cf. Proposition 12.1.11.
3. If f satisfies the conditions of Proposition 12.1.12 then $(\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ as we shall show in Corollary 12.2.35. Note that by Theorem 8.2.7 this is a stronger condition on the coefficients than one gets from a signal simply being in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.
4. If $f \in \mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the CDFT of f satisfies
$$\lim_{|n|\to\infty} |n^r\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = 0.$$
This follows from Corollary 12.1.14.
5. A converse of the preceding implication also holds. Precisely, if $\lim_{|n|\to\infty} n^{r+1+\epsilon}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = 0$ for some $\epsilon \in \mathbb{R}_{>0}$, then $f \in \mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. This follows exactly as in Corollary 14.1.11 proved below for the DCFT.
6. If $f \in \mathsf{C}^\infty_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is infinitely differentiable then the Fourier coefficients satisfy
$$\lim_{|n|\to\infty} |n^k\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = 0$$
for any $k \in \mathbb{Z}_{\geq 0}$. This follows from an inductive application of Corollary 12.1.14.

Let us give a result which relates the CDFT to real analyticity as defined in Definition **??**. Note that, in the context here, if $I \subseteq \mathbb{R}$ is an interval, then $f\colon I \to \mathbb{C}$ is ***real analytic*** if its real and imaginary parts are real analytic, i.e., don't let the fact that the signal is $\mathbb{C}$-values lure you into thoughts of holomorphicity.

12.1.17 Theorem (The CDFT for real analytic signals) *A signal* $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *is almost everywhere equal to a real analytic signal if and only if there exists* $C, \alpha \in \mathbb{R}_{>0}$ *such that* $|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le Ce^{-\alpha|n|}$ *for every* $n \in \mathbb{Z}$.

Proof First suppose that $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is almost everywhere equal to a real analytic signal. We assume without loss of generality that f is itself real analytic. By Theorem **??**, for each $t_0 \in [0,T]$ there exists a neighbourhood U of t_0 and $M, r \in \mathbb{R}_{>0}$ such that

$$|f^{(m)}(t)| \le M m! r^{-m}$$

for each $t \in U$ and $m \in \mathbb{Z}_{\ge 0}$. Since $[0,T]$ is compact, we can cover $[0,T]$ with a finite number of intervals for which the above estimate holds. Therefore, we can assume the estimate holds for every $t \in [0,T]$, and so for every $t \in \mathbb{R}$ by periodicity of f.

Then, since f is infinitely differentiable, by Corollary 12.1.14, we have

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le \left|\frac{T}{2\pi n}\right|^m |\mathscr{F}_{\mathrm{CD}}(f^{(m)})(nT^{-1})| \le \left|\frac{T}{2\pi n}\right|^m TMm!r^{-m} = \frac{Am!}{(nr)^m},$$

where $A = MT(\frac{T}{2\pi})^m$, this being valid for $n \neq 0$ and for all $m \in \mathbb{Z}_{\ge 0}$. Let m_n be the largest integer less than $|n|r$. Then we have

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le \frac{Am_n!}{m_n^{m_n}}.$$

By Stirling's formula, ***missing stuff***,

$$\lim_{n\to\infty} \frac{m_n!}{\sqrt{2\pi m_n}(\frac{m_n}{e})^{m_n}} = 1.$$

Thus there exists $N_1 \in \mathbb{Z}_{>0}$ sufficiently large that

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le 2A\sqrt{2\pi m_n}e^{-m_n}, \qquad |n| \ge N_1.$$

Since $\lim_{n\to\infty} \sqrt{m_n}e^{-m_n/2} = 0$, there exists $N_2 \in \mathbb{Z}_{>0}$ such that $2A\sqrt{2\pi m_n}e^{-m_n/2} \le 1$ for $|n| \ge N_2$. Thus, if $N = \max\{N_1, N_2\}$ we have

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le e^{-m_n/2}, \qquad |n| \ge N.$$

Since $m_n + 1 \ge |n|r$ this gives

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le \sqrt{e}e^{-r|n|/2}, \qquad |n| \ge N.$$

Let us take $\alpha = \frac{r}{2}$ and define

$$C = \max\{\sqrt{e}, |\mathscr{F}_{\mathrm{CD}}(f)(-NT^{-1})|e^{\alpha|N|}, \dots, |\mathscr{F}_{\mathrm{CD}}(f)(NT^{-1})|e^{\alpha|N|}\}.$$

Then we have

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le Ce^{\alpha|n|}$$

for all $n \in \mathbb{Z}$, as desired.

Conversely, suppose that there exists $C, \alpha \in \mathbb{R}_{>0}$ such that $|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \le Ce^{-\alpha|n|}$ for all $n \in \mathbb{Z}$. By the Weierstrass M-test, cf. Theorem 12.2.33 below, the series

$$\frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})e^{2\pi i n\frac{t}{T}}$$

converges to a continuous function that is almost everywhere to f. Let us suppose, without loss of generality, that f is defined by this series. By repeated application of Theorem 3.5.24 and the Weierstrass M-test we have

$$f^{(m)}(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\Big(\frac{2\pi \mathrm{i} n}{T}\Big)^m \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}}$$

for every $m \in \mathbb{Z}_{\geq 0}$, and this limit is continuous. If we take

$$C' = \frac{A}{T}, \quad r' = \frac{2\pi}{T}$$

we have

$$|f^{(m)}(t)| \leq C' \sum_{n\in\mathbb{Z}} (nr')^m \mathrm{e}^{-\alpha|n|} = C' + 2C' \sum_{n=1}^{\infty} (nr')^m \mathrm{e}^{-\alpha|n|}.$$

For $n \in \mathbb{Z}_{>0}$ and $x \in [n, n+1]$ we have

$$(nr')^m \mathrm{e}^{-\alpha|n|} \leq \mathrm{e}^{\alpha}(r'x)^m \mathrm{e}^{-\alpha x}$$

and so

$$|f^{(m)}(t)| \leq C' + 2C'\mathrm{e}^{\alpha}(r')^m \sum_{n\in\mathbb{Z}} \int_n^{n+1} x^m \mathrm{e}^{-\alpha x}\,\mathrm{d}x \leq C' + 2C'\mathrm{e}^{\alpha}(r')^m + \int_0^{\infty} x^m \mathrm{e}^{-\alpha x}\,\mathrm{d}x.$$

By a repeated application of integration by parts we have

$$\int_0^{\infty} x^m \mathrm{e}^{-\alpha x}\,\mathrm{d}x = \frac{1}{\alpha}\frac{m!}{\alpha^m}.$$

Thus

$$|f^{(m)}(t)| \leq C' + \frac{2C'\mathrm{e}^{\alpha}}{\alpha}\Big(\frac{r'}{\alpha}\Big)^m m!$$

for each $m \in \mathbb{Z}_{\geq 0}$. Since

$$\lim_{m\to\infty}\Big(C' + \frac{2C'\mathrm{e}^{\alpha}}{\alpha}\Big(\frac{r'}{\alpha}\Big)^m m!\Big)\Big(\frac{\alpha}{r'}\Big)^m \frac{1}{m!} = \frac{2C'\mathrm{e}^{\alpha}}{\alpha},$$

there exists $N \in \mathbb{Z}_{>0}$ sufficiently large that

$$C' + \frac{2C'\mathrm{e}^{\alpha}}{\alpha}\Big(\frac{r'}{\alpha}\Big)^m m! \leq \frac{4C'\mathrm{e}^{\alpha}}{\alpha}\Big(\frac{r'}{\alpha}\Big)^m m!$$

for every $m \geq N$. Now take $r = \frac{\alpha}{r'}$ and

$$C = \max\Big\{\frac{4C'\mathrm{e}^{\alpha}}{\alpha}, \|f\|_{\infty}, \dots, \frac{r^N}{N!}\|f^{(N)}\|_{\infty}\Big\}.$$

Then we have $|f^{(m)}(t)| \leq Cm!r^{-m}$ for every $m \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}$, giving analyticity of f by Theorem **??**. ■

Finally, let us show that any general estimate for the rate of decay of the values for the CDFT is not possible.

12.1.18 Theorem (The CDFT decays arbitrarily slowly generally) *If* $G \in c_0(\mathbb{Z}(T^{-1}); \mathbb{R}_{\geq 0})$ *then there exists* $f \in L^{(1)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{C})$ *such that*

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \geq G(nT^{-1}), \qquad n \in \mathbb{Z}.$$

Proof We first state a couple of lemmata having to do with the construction of sequences with desirable properties.

1 Lemma *If* $(\alpha_j)_{j\in\mathbb{Z}_{\geq 0}}$ *is a sequence in* $\mathbb{R}_{\geq 0}$ *satisfying* $\lim_{j\to\infty} \alpha_j = 0$*, then there exists a sequence* $(\beta_j)_{j\in\mathbb{Z}_{\geq 0}}$ *satisfying*

(i) $\beta_j \geq \alpha_j$, $j \in \mathbb{Z}_{\geq 0}$,

(ii) $\beta_{j+2} + \beta_j \geq 2\beta_{j+1}$, $j \in \mathbb{Z}_{\geq 0}$, *and*

(iii) $\lim_{j\to\infty} \beta_j = 0$.

Proof Let $M \in \mathbb{R}_{>0}$ be such that $\beta_j \leq M$ for each $j \in \mathbb{Z}_{\geq 0}$. Let $N_0 = 0$ and let $N_1 > N_0$ be such that $\alpha_j \leq \frac{M}{2}$ for $j \geq N_1$. Now suppose that we have defined $N_0, N_1, \dots, N_n \in \mathbb{Z}_{>0}$ such that

$$N_0 < N_1 < \cdots < N_n$$

and $\alpha_j \leq M2^{-m}$ whenever $j \geq N_m$. Then choose $N_{n+1} > N_n + \frac{1}{2}(N_n - N_{n-1})$ such that $\alpha_j \leq M2^{-(n+1)}$ whenever $j \geq N_{n+1}$. This inductively defines a sequence $(N_m)_{m\in\mathbb{Z}_{>0}}$. Define $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by asking that $h(N_m) = M2^{m-1}$ and that h be linear between N_m and N_{m+1} for each $m \in \mathbb{Z}_{\geq 0}$. The resulting function is easily seen to be strictly convex by construction. We define $\beta_j = h(j)$ for each $j \in \mathbb{Z}_{\geq 0}$. Since h is convex we have the second conclusion. Since $\lim_{m\to\infty} \beta_{N_m} = 0$ the third conclusion follows. Finally, the construction of the sequence $(N_m)_{m\in\mathbb{Z}_{\geq 0}}$ ensures that the first conclusion holds. ▼

2 Lemma *If* $(\beta_j)_{j\in\mathbb{Z}_{\geq 0}}$ *is a sequence satisfying the first two conclusions of the preceding lemma, then*

$$\sum_{j=0}^{\infty} (j+1)(\beta_{j+k+2} + \beta_{j+k} - 2\beta_{j+k+1}) = \beta_k$$

for each $k \in \mathbb{Z}_{\geq 0}$.

Proof An elementary induction on N gives

$$\sum_{j=0}^{N} (j+1)(\beta_{j+k+2} + \beta_{j+k} - 2\beta_{j+k+1}) = \beta_k - (N+1)(\beta_{k+N+1} - \beta_{k+N+2}) - \beta_{k+N+1}$$

for each $N \in \mathbb{Z}_{\geq 0}$. The lemma will follow if we can show that

$$\lim_{N\to\infty} (N+1)(\beta_{k+N+1} - \beta_{k+N+2}) = 0.$$

Let N' be the largest integer less than $\frac{N}{2}$. We have

$$\beta_{k+N'+1} - \beta_{k+N+2} = (\beta_{k+N'+1} - \beta_{k+N'+2}) + (\beta_{k+N'+2} - \beta_{k+N'+3}) + \cdots + (\beta_{k+N+1} - \beta_{k+N+2}).$$

By the second property from the preceding lemma,

$$\beta_{k+N'+j} - \beta_{k+N'+j+1} \geq \beta_{k+N'+j+1} - \beta_{k+N'+j+2}$$

for each $j \in \{1, \dots, N - N'\}$. This gives

$$\beta_{k+N'+1} - \beta_{k+N+2} \geq (N'+1)(\beta_{k+N+1} - \beta_{k+N+2}) \geq \frac{N+1}{2}(\beta_{k+N+1} - \beta_{k+N+2}) \geq 0.$$

Since

$$\lim_{N\to\infty} \beta_{k+N'+1} - \beta_{k+N+2} = 0$$

(noting that N' is determined by N), the lemma follows. ▼

For $G \in \mathsf{c}_0(\mathbb{Z}(T^{-1}); \mathbb{R}_{\geq 0})$ define

$$\alpha_j = T(G(jT) + G(-jT)), \qquad j \in \mathbb{Z}_{\geq 0}.$$

By Lemma 1 let $(\beta_j)_{j\in\mathbb{Z}_{\geq 0}}$ be a sequence satisfying the conclusions of that lemma for the associated sequence $(\alpha_j)_{j\in\mathbb{Z}_{\geq 0}}$. Define $(\beta_n)_{n\in\mathbb{Z}}$ by asking that $\beta_n = \beta_{-n}$ for $n \in \mathbb{Z}_{<0}$. Define

$$f(t) = \sum_{j=0}^{\infty} (j+1)(\beta_{j+2} + \beta_j - 2\beta_{j+1}) F^{\mathrm{per}}_{T,j}(t),$$

where $F^{\mathrm{per}}_{T,j}$, $j \in \mathbb{Z}_{\geq 0}$, is the Fejér kernel of Example 11.3.19–3. By the properties of the sequence $(\beta_j)_{j\in\mathbb{Z}_{\geq 0}}$ and the positivity of the Fejér kernel, it follows that $f(t) \in \overline{\mathbb{R}}_{\geq 0}$ for each $t \in \mathbb{R}$. Applying Lemma 2 with $k = 0$ gives

$$\|f\|_1 = \sum_{j=0}^{\infty} (j+1)(\beta_{j+1} + \beta_j - 2\beta_{j+1}) \|F^{\mathrm{per}}_{T,j}\|_1 = \beta_0 < \infty,$$

using the fact that $\|F^{\mathrm{per}}_{T,j}\|_1 = 1$ by a direct computation using Lemma **??** from Example **??**–**??**.***missing stuff*** This shows that $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

Now we compute the CDFT of f. For $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) &= \sum_{j=0}^{\infty} (j+1)(\beta_{j+2} + \beta_j - 2\beta_{j+1}) \mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,j})(nT^{-1}) \\
&= \frac{1}{T} \sum_{j=|n|}^{\infty} (j+1)(\beta_{j+2} + \beta_j - 2\beta_{j+1}) \Big(1 - \frac{|n|}{j+1}\Big) \\
&= \frac{1}{T} \sum_{j=|n|}^{\infty} (j+1)(\beta_{j+2} + \beta_j - 2\beta_{j+1}) - \frac{|n|}{T} \sum_{j=|n|}^{\infty} (\beta_{j+2} + \beta_j - 2\beta_{j+1}) \\
&= \frac{1}{T} \sum_{j=0}^{\infty} (j+1+|n|)(\beta_{j+|n|+2} + \beta_{j+|n|} - 2\beta_{j+|n|+1}) \\
&\qquad - \frac{|n|}{T} \sum_{j=0}^{\infty} (\beta_{j+|n|+2} + \beta_{j+|n|} - 2\beta_{j+|n|+1}) = \frac{\beta_{|n|}}{T},
\end{aligned}$$

using Lemma 2 and Lemma **??** from Example **??**–**??**. Finally, note that

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = \frac{\beta_{|n|}}{T} \geq \frac{\alpha_{|n|}}{T} \geq G(nT^{-1}),$$

as desired. ■

12.1.5 Convolution, multiplication, and the L^1-CDFT

An important rôle is played in Fourier transform theory by convolution. In this section we investigate this for periodic signals and the CDFT.

As we saw in Theorem 11.1.20, the product of convolution makes $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ into an algebra with some particular properties. It makes sense to ask how this algebra structure appears after the CDFT is applied. It turns out that the answer is very simple: Convolution in the time-domain becomes multiplication in the frequency-domain. This is the content of the following theorem.

12.1.19 Proposition (The CDFT of a convolution is the product of the CDFT's) *If* $\mathrm{f}, \mathrm{g} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{C})$ *then*

$$\mathscr{F}_{\mathrm{CD}}(\mathrm{f} * \mathrm{g})(\mathrm{n}\mathrm{T}^{-1}) = \mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{n}\mathrm{T}^{-1})\mathscr{F}_{\mathrm{CD}}(\mathrm{g})(\mathrm{n}\mathrm{T}^{-1})$$

for every $\mathrm{n} \in \mathbb{Z}$.

Proof This is a fairly straightforward application of Fubini's Theorem, the change of variables theorem, and periodicity of f:

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(f * g)(nT^{-1}) &= \int_0^T f * g(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \int_0^T \Big(\int_0^T f(t-s)g(s)\,\mathrm{d}s\Big)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\
&= \int_0^T g(s)\Big(\int_0^T f(t-s)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&= \int_0^T g(\sigma)\Big(\int_{-\sigma}^{T-\sigma} f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{\sigma+\tau}{T}}\,\mathrm{d}\tau\Big)\,\mathrm{d}\sigma \\
&= \Big(\int_0^T g(\sigma)\mathrm{e}^{-2\pi\mathrm{i}n\frac{\sigma}{T}}\,\mathrm{d}\sigma\Big)\Big(\int_0^T f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{\tau}{T}}\,\mathrm{d}\tau\Big) \\
&= \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1}).
\end{aligned}$$

(The reader may wish to compare this computation to that performed at some length in the proof of Theorem 11.1.17.) ■

Mostly, the previous result is of theoretical importance since, as we shall see in ***missing stuff***, convolution arises in an essential way when one talks about linear systems. However, sometimes the result can be used to easily compute the CDFT of a signal knowing that it is a convolution.

12.1.20 Example (The CDFT of a convolution) Let us define $f, g \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ by asking that for $t \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$f(t) = \begin{cases} 1, & t \in [-\frac{1}{4}, \frac{1}{4}], \\ 0, & \text{otherwise} \end{cases}$$

and

$$g(t) = \begin{cases} \frac{1}{2} + t, & t \in [-\frac{1}{2}, 0], \\ \frac{1}{2} - t, & t \in (0, \frac{1}{2}]. \end{cases}$$

In Examples 12.1.3–4 and 5 we computed a=1/4 T=1

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \begin{cases} \frac{1}{2}, & n = 0, \\ \frac{\sin(\frac{n\pi}{2})}{n\pi}, & n \neq 0, \end{cases} \qquad \mathscr{F}_{\mathrm{CD}}(g)(n) = \begin{cases} \frac{1}{4}, & n = 0, \\ \frac{\sin(\frac{n\pi}{2})^2}{\pi^2 n^2}, & n \neq 0. \end{cases}$$

One can verify that $g = f * f$ and so, by Proposition 12.1.19, we have $\mathscr{F}_{\mathrm{CD}}(g) = \mathscr{F}_{\mathrm{CD}}(f)\mathscr{F}_{\mathrm{CD}}(f)$. This is true. •

It is possible to swap the rôles of convolution and multiplication in the above result, recalling from Section 11.1.3 the definition of convolution for aperiodic discrete-time (in this case, discrete-frequency) signals.

12.1.21 Proposition (The CDFT of a product is the convolution of the CDFT's) *If* $\mathsf{f}, \mathsf{g} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathsf{T}}(\mathbb{R};\mathbb{C})$ *and if* $\mathscr{F}_{\mathrm{CD}}(\mathsf{f}), \mathscr{F}_{\mathrm{CD}}(\mathsf{g}) \in \ell^1(\mathbb{Z}(\mathsf{T}^{-1});\mathbb{C})$*, then*

$$\mathscr{F}_{\mathrm{CD}}(\mathsf{fg})(\mathsf{nT}^{-1}) = \mathscr{F}_{\mathrm{CD}}(\mathsf{f}) * \mathscr{F}_{\mathrm{CD}}(\mathsf{g})(\mathsf{nT}^{-1})$$

for every $\mathsf{n} \in \mathbb{Z}$*.*

Proof Our proof relies on some facts about the inverse of the CDFT presented in Section 12.2 and some facts about the DCFT presented in Section 14.1.

By Theorem 12.2.33 it follows that f and g are almost everywhere equal to continuous signals. Let us without loss of generality assume that f and g are continuous.

Note that

$$\mathscr{F}_{\mathrm{CD}}(f) * \mathscr{F}_{\mathrm{CD}}(g) \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$$

by Theorem 11.2.31. By Proposition 12.1.6(ii), Theorem 14.1.16, Proposition 14.1.12, and the fact that $\overline{\mathscr{F}}_{\mathrm{CD}} = \mathscr{F}_{\mathrm{DC}}^{-1}$, we have

$$\mathscr{F}_{\mathrm{DC}}(\mathscr{F}_{\mathrm{CD}}(f) * \mathscr{F}_{\mathrm{CD}}(g)) = \mathscr{F}_{\mathrm{DC}}(\mathscr{F}_{\mathrm{DC}}^{-1}(\sigma^* f) * \mathscr{F}_{\mathrm{DC}}^{-1}(\sigma^* g)) = \sigma^* f \sigma^* g.$$

By the definition of the DCFT this gives

$$\frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f) * \mathscr{F}_{\mathrm{CD}}(g)(nT^{-1})\mathrm{e}^{-2\pi\mathrm{i}n\frac{\tau}{T}} = f(-\tau)g(-\tau), \qquad \tau \in \mathbb{R},$$

the sum on the left converging uniformly by the Weierstrass M-test. Making the change of variable $t = -\tau$ we have

$$\frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f) * \mathscr{F}_{\mathrm{CD}}(g)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} = f(t)g(t), \qquad t \in \mathbb{R},$$

Taking the CDFT of the expression on the left, swapping the integral and sum by virtue of Theorem 3.5.23 and using Lemma 12.3.2 below, we get

$$\mathscr{F}_{\mathrm{CD}}(f) * \mathscr{F}_{\mathrm{CD}}(g)(nT^{-1}) = \mathscr{F}_{\mathrm{CD}}(fg)(nT^{-1}),$$

as desired. ■

Exercises

12.1.1 Suppose that f is the $2T$-periodic extension of $g \in L^{(1)}([-T,T];\mathbb{C})$.

(a) Argue that the natural harmonic signals to use to define the CDFT for f are $(\tilde{e}_n = e^{i\pi n \frac{t}{T}})_{n\in\mathbb{Z}}$.

(b) Show that

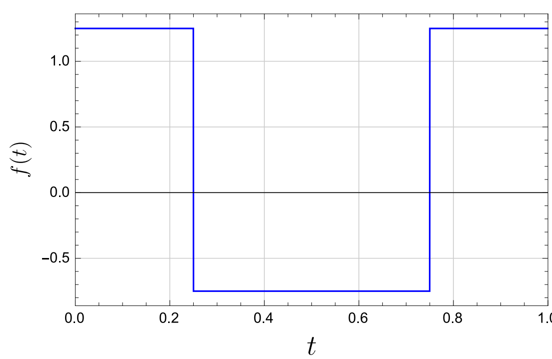
$$\mathscr{F}_{CD}(f)(\tfrac{n}{2T}) = \int_{-T}^{T} g(t) e^{-i\pi n \frac{t}{T}}\, dt.$$

(c) Give the formulae for the CDCT and the CDST in this case.

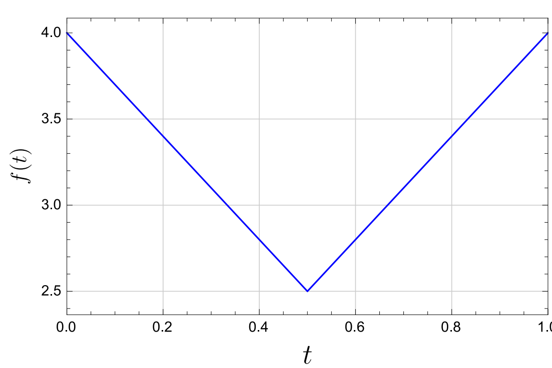
12.1.2 In this exercise you will be given the graphs of T-periodic signals for $T = 1$. For each signal, without just grinding away at the computations, determine the CDFT.

Hint: *Use Example 12.1.3.*

(a) The graph is



(b) The graph is



12.1.3 Let $f \in L^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

(a) For $a \in \mathbb{R}$, show that $|\mathscr{F}_{CD}(\tau_a^* f)(nT^{-1})| = |\mathscr{F}_{CD}(f)(nT^{-1})|$ for each $n \in \mathbb{Z}$.

(b) For which values of $a \in \mathbb{R}$ is it true that $\arg(\mathscr{F}_{CD}(\tau_a^* f)(nT^{-1})) = \arg(\mathscr{F}_{CD}(f)(nT^{-1}))$ for every $n \in \mathbb{Z}$? Does your conclusion depend on f?

(c) Find a nontrivial codomain transformation $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $\arg(\mathscr{F}_{CD}(\phi \circ f)(nT^{-1})) = \arg(\mathscr{F}_{CD}(f)(nT^{-1}))$ for every $n \in \mathbb{Z}$.

12.1.4 Prove the following result.

Proposition *Let* $f \in L^{(1)}([0,T];\mathbb{R})$ *and denote by* $f_{even} \in L^{(1)}_{per,2T}(\mathbb{R};\mathbb{R})$ *and* $f_{odd} \in L^{(1)}_{per,2T}(\mathbb{R};\mathbb{R})$ *the even and odd extensions. Then*

$$\begin{aligned}
\mathscr{C}_{CD}(f_{even})(n(2T)^{-1}) &= 2\int_0^T f_{even}(t)\cos(\pi n \tfrac{t}{T})\,dt, \qquad n \in \mathbb{Z}_{\geq 0},\\
\mathscr{S}_{CD}(f_{even})(n(2T)^{-1}) &= 0, \qquad n \in \mathbb{Z}_{>0},\\
\mathscr{C}_{CD}(f_{odd})(n(2T)^{-1}) &= 0, \qquad n \in \mathbb{Z}_{\geq 0},\\
\mathscr{S}_{CD}(f_{odd})(n(2T)^{-1}) &= 2\int_0^T f_{odd}(t)\sin(\pi n \tfrac{t}{T})\,dt, \qquad n \in \mathbb{Z}_{>0}.
\end{aligned}$$

12.1.5 Let $f, g \in L^{(1)}_{per,T}(\mathbb{R};\mathbb{C})$ and suppose that $\mathscr{F}_{CD}(f) \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$. Show that

$$\int_0^T f(t)g(t)\,dt = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{CD}(f)(nT^{-1})\mathscr{F}_{CD}(g)(-nT^{-1}),$$

making sure to understand why all integrals and sums exist.

12.1.6 In Table 12.1 you are given the graphs of four functions, all defined on $[0, 2\pi]$, along with four Fourier series. You are not told which graph goes with which Fourier series. You are to match the graph with the appropriate Fourier series, providing justification for your choice.

12.1.7 In each of the following problems, you will be asked to provide a 1-periodic signal (i.e., a periodic signal with period 1) $f\colon \mathbb{R} \to \mathbb{C}$ with prescribed properties. In all cases, you are not allowed to use the explicit form of the CDFT of f, i.e., all explanations must be given in terms of f.

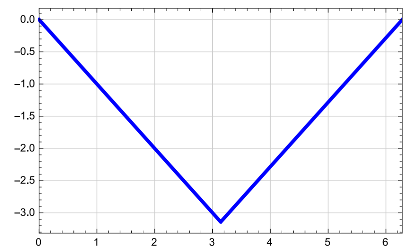
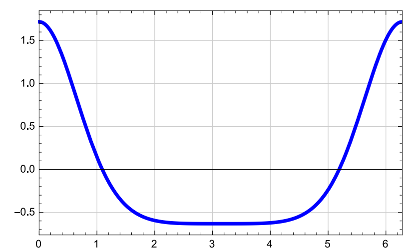
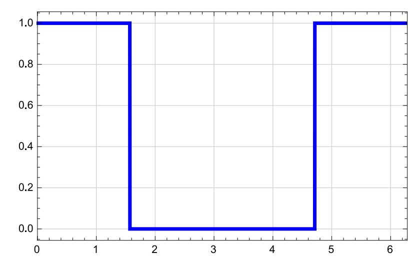
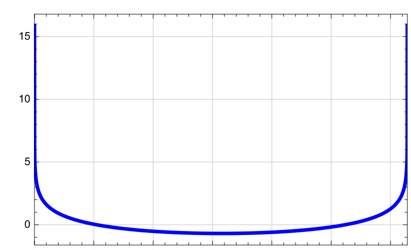
(a) Properties:
 1. $\lim_{|n|\to\infty} \mathscr{F}_{CD}(f)(n) = 0$ and
 2. $\lim_{|n|\to\infty} n\mathscr{F}_{CD}(f)(n) \neq 0$.

(b) Properties:
 1. $\lim_{|n|\to\infty} n\mathscr{F}_{CD}(f)(n) = 0$ and
 2. $\lim_{|n|\to\infty} n^2\mathscr{F}_{CD}(f)(n) \neq 0$.

(c) Properties:
 1. $\lim_{|n|\to\infty} n^r\mathscr{F}_{CD}(f)(n) = 0$ for every $r \in \mathbb{Z}_{\geq 0}$.

(d) Properties:
 1. $\mathscr{F}_{CD}(f) \in \ell^2(\mathbb{Z};\mathbb{C})$ and
 2. $\mathscr{F}_{CD}(f) \notin \ell^1(\mathbb{Z};\mathbb{C})$.

Table 12.1 Table of Fourier series and graphs of functions

Fourier series	Graphs
1. $\pi + 2\sum_{n=1}^{\infty} \frac{(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2})}{n}\cos(nt)$	**1.** 
2. $2\pi\sum_{n=1}^{\infty} \frac{1}{n}\cos(nt)$	**2.** 
3. $-\pi^2 + 2\sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2}\cos(nt)$	**3.** 
4. $2\pi\sum_{n=1}^{\infty} \frac{1}{n!}\cos(nt)$	**4.** 

Section 12.2

Inversion of the CDFT

Now that we have defined the CDFT and given some of its more basic properties, let us turn to the question, "Does the CDFT of a signal faithfully represent the signal." If one thinks about the situation illustrated in Figure 9.7 where we show time- and frequency-domain representations of two music clips, if the frequency-domain representation is to have any value then we ought to be able to recover from it the time-domain representation. In this section we investigate this "inversion" process.

Do I need to read this section? The topic of transform inversion is one of the most important in Fourier analysis. So this section is an important one. •

12.2.1 Preparatory work

The CDFT takes $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and returns $\mathscr{F}_{\mathrm{CD}}(f) \in \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$. Our objective is to ascertain whether there is a way of retrieving f if we are given $\mathscr{F}_{\mathrm{CD}}(f)$. First let us show that the inverse of the CDFT exists, at least in a set theoretic sense.

12.2.1 Theorem (The CDFT is injective) *The map* $\mathscr{F}_{\mathrm{CD}}\colon \mathsf{L}^1_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(\mathrm{T}^{-1});\mathbb{C})$ *is injective.*

Proof Let us recall from Example 11.3.19–3 the definition of the periodic Fejér kernel:

$$F^{\mathrm{per}}_{T,N}(t) = \begin{cases} \dfrac{1}{N}\dfrac{\sin^2(N\pi\frac{t}{T})}{\sin^2(\pi\frac{t}{T})}, & t \notin \mathbb{Z}(T),\\ N, & t \in \mathbb{Z}(T).\end{cases}$$

According to Lemma **??** from Example **??**–**??** below, $F^{\mathrm{per}}_{T,N}$ is a finite linear combination of the harmonic signals $t \mapsto \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$, $n \in \mathbb{Z}$. Also recall that in Example 11.3.19–3 we verified that $(\frac{1}{T}F^{\mathrm{per}}_{T,N})_{N\in\mathbb{Z}_{>0}}$ is a periodic approximate identity. We use these facts in the following lemma.

1 **Lemma** *Let* $\mathrm{f}_1, \mathrm{f}_2 \in \mathsf{C}^0_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{F})$ *and suppose that* $\mathscr{F}_{\mathrm{CD}}(\mathrm{f}_1)(\mathrm{n}\mathrm{T}^{-1}) = \mathscr{F}_{\mathrm{CD}}(\mathrm{f}_2)(\mathrm{n}\mathrm{T}^{-1})$ *for all* $\mathrm{n} \in \mathbb{Z}$. *Then* $\mathrm{f}_1 = \mathrm{f}_2$.

Proof By linearity, the theorem amounts to showing that if $f \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ and if $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = 0$ for $n \in \mathbb{Z}$, then $f = 0$. Also by linearity of the integral, we may as well suppose that $\mathbb{F} = \mathbb{R}$, as if this is not so, we may apply the theorem separately to the real and imaginary parts of f.

We suppose that $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = 0$ for $n \in \mathbb{Z}$ and that $f \neq 0$. By translation (cf. Proposition 12.1.6) and multiplication by -1 if necessary, we may suppose that $f(0) \in \mathbb{R}_{>0}$. Note that the relation

$$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = 0$$

implies, by periodicity of f and of $e^{2\pi i n\frac{t}{T}}$, $n \in \mathbb{Z}$, that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-2\pi i n\frac{t}{T}}\,dt = 0, \quad n \in \mathbb{Z}.$$

By linearity of the integral this means that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g(t)\,dt = 0 \tag{12.4}$$

where g is any finite linear combination of the harmonic signals $(e^{2\pi i n\frac{t}{T}})_{n\in\mathbb{Z}}$.

Now we use the properties of $F^{\mathrm{per}}_{T,N}$ to proceed with the proof. As f is continuous and $f(0) \neq 0$, we can choose $\alpha \in \mathbb{R}_{>0}$ so that $f(t) \geq \frac{1}{2}f(0)$ for all $t \in [-\alpha, \alpha]$. We then write

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)F^{\mathrm{per}}_{T,N}\,dt = \int_{-\alpha}^{\alpha} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt + \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt.$$

Let

$$M_\alpha = \sup\{|f(t)| \mid \alpha \leq |t| \leq \tfrac{T}{2}\},$$

noting that $M_\alpha < \infty$ as f is continuous. Thus we have

$$\left|\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt\right| \leq M_\alpha \int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} F^{\mathrm{per}}_{T,N}(t)\,dt,$$

and so

$$\lim_{N\to\infty}\left|\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt\right| = 0$$

since $(\frac{1}{T}F^{\mathrm{per}}_{T,N})_{N\in\mathbb{Z}_{>0}}$ is a periodic approximate identity. We also have

$$\int_{-\alpha}^{\alpha} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt \geq \frac{1}{2}f(0)\int_{-\alpha}^{\alpha} F^{\mathrm{per}}_{T,N}(t)\,dt.$$

Thus, again since $(\frac{1}{T}F^{\mathrm{per}}_{T,N})_{N\in\mathbb{Z}_{>0}}$ is a periodic approximate identity,

$$\lim_{N\to\infty}\int_{-\alpha}^{\alpha} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt \geq \frac{1}{2}f(0)T.$$

Now choose $N_0 \in \mathbb{Z}_{>0}$ sufficiently large that, for all $N \geq N_0$,

$$\left|\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\alpha,\alpha]} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt\right| < \frac{1}{8}f(0)T$$

and

$$\int_{-\alpha}^{\alpha} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt \geq \frac{1}{4}f(0)T.$$

This gives

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)F^{\mathrm{per}}_{T,N}(t)\,dt \geq \frac{1}{8}f(0)T \in \mathbb{R}_{>0},$$

so contradicting (12.4). Thus, if f is continuous and nonzero, it must hold that $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) \neq 0$ for some $n \in \mathbb{Z}$, so giving the result. ▼

Now let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. It is sufficient to show that if $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = 0$ for every $n \in \mathbb{Z}$ then $f(t) = 0$ for almost every $t \in \mathbb{R}$. Define $F\colon [0,T] \to \mathbb{C}$ by

$$F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

We claim that F_{per} is continuous. By Proposition 5.9.24 and Theorem 5.9.31 it follows that F is continuous. Moreover,

$$F(T) = \int_0^T f(t)\,\mathrm{d}t = \mathscr{F}_{\mathrm{CD}}(f)(0) = 0 = F(0),$$

and so we conclude that F_{per} is continuous. For $n \neq 0$, using Fubini's Theorem we compute

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(nT^{-1}) &= \int_0^T F(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \int_0^T \Big(\int_0^t f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}\tau\Big)\mathrm{d}t \\
&= \int_0^T f(\tau)\Big(\int_\tau^T \mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\Big)\,\mathrm{d}\tau \\
&= \frac{T}{2\pi\mathrm{i}n}\Big(\int_0^T f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{\tau}{T}}\,\mathrm{d}\tau - \mathrm{e}^{-2\pi\mathrm{i}n}\int_0^T f(\tau)\,\mathrm{d}\tau\Big) \\
&= \frac{T}{2\pi\mathrm{i}n}(\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) - \mathscr{F}_{\mathrm{CD}}(f)(0)) = 0,
\end{aligned}$$

since $\mathscr{F}_{\mathrm{CD}}(f) = 0$. Now consider the signal $G \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ defined by $G(t) = F_{\mathrm{per}}(t) - \mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(0)$. We have

$$\mathscr{F}_{\mathrm{CD}}(G)(nT^{-1}) = \mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(nT^{-1}) - \mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(0)\int_0^T \mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = 0, \qquad n \in \mathbb{Z}.$$

By Lemma 1 it follows that the signal G is zero since it is continuous. Thus

$$f(t) = F'_{\mathrm{per}}(t) = G'(t) = 0$$

for almost every $t \in \mathbb{R}$ using Lemma 5.9.30. ■

Note that the proof of Theorem 12.2.22 is quite detailed and involved, using special properties of harmonic signals. This is to be expected since we are proving something nontrivial, namely that $\mathscr{F}_{\mathrm{CD}}$ possesses an inverse of some sort. In the course of the proof we made use of the periodic Fejér kernel defined in Example 11.3.19–3. As we saw when we defined the periodic Fejér kernel, it defined a periodic approximate identity. Moreover, in the proof of Theorem 12.2.22 we used precisely the properties of an approximate identity in the proof. We shall again and again see this theme of approximate identities playing a crucial rôle in transform inversion.

Theorem 12.2.1 raises the question about the surjectivity of the CDFT. It turns out that it is not surjective as the following result indicates.

12.2.2 Proposition (The CDFT is not onto $\mathsf{c}_0(\mathbb{Z}(\mathrm{T}^{-1});\mathbb{C})$) *The map* $\mathscr{F}_{\mathrm{CD}}\colon \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(\mathrm{T}^{-1});\mathbb{C})$ *is not surjective.*

Proof We argue abstractly. As a map of Banach spaces, $\mathscr{F}_{\mathrm{CD}}$ is continuous. If it is an linear isomorphism then it is a homeomorphism by Theorem 6.5.24. Let $\mathscr{F}^{-1}_{\mathrm{CD}}\colon \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ be the inverse of $\mathscr{F}_{\mathrm{CD}}$. Thus, by Theorem 6.5.8 it follows that there exists $M \in \mathbb{R}_{>0}$ such that $\|\mathscr{F}^{-1}_{\mathrm{CD}}(F)\|_1 \leq M\|F\|_\infty$ for every $F \in \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$. In particular, it follows that $\|f\|_1 \leq M\|\mathscr{F}_{\mathrm{CD}}(f)\|_\infty$ for every $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. We recall from Example 11.3.19–5 the sequence $(D^{\mathrm{per}}_{T,N})_{N\in\mathbb{Z}_{>0}}$ defined by the periodic Dirichlet kernel. By Lemma **??** from Example **??** below we have

$$D^{\mathrm{per}}_{T,N}(t) = \sum_{|n|\leq N} \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

From this we conclude that $\|\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{T,N})\|_\infty = 1$ for every $N \in \mathbb{Z}_{>0}$. By Lemma 1 from Example 11.3.19–5 we have $\lim_{N\to\infty}\|D^{\mathrm{per}}_{T,N}\|_1 = \infty$. This, however, contradicts the conclusion, arrived at by assuming that $\mathscr{F}_{\mathrm{CD}}$ is surjective, that $\|D^{\mathrm{per}}_{T,N}\|_1 \leq M\|\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{T,N})\|_\infty$ for every $N \in \mathbb{Z}_{>0}$. Thus $\mathscr{F}_{\mathrm{CD}}$ cannot be surjective. ■

12.2.3 Remarks (On inversion of the CDFT)

1. As the reader knows from Proposition 1.3.9, Theorem 12.2.1 implies the existence of a left-inverse for the CDFT. That is to say, we are ensured the existence of a map $\mathscr{I}_{\mathrm{CD}}\colon \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ with the property that $\mathscr{I}_{\mathrm{CD}} \circ \mathscr{F}_{\mathrm{CD}}(f) = f$ for every $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Since $\mathscr{F}_{\mathrm{CD}}$ is not surjective, this inverse is not unique. Therefore, from the multitude of possible left-inverse, one would want one with useful properties. This is one way to view the results in this section.
2. Another approach would be to *propose* a possible left-inverse, and then consider classes of functions in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ for which the proposed left-inverse actually does act as a left-inverse should; namely, one recovers the original function after an application of the CDFT and the proposed left-inverse. This approach is, in fact, the one we adopt, and it is the one predominantly adopted when one considers inversion of the CDFT. As we shall see, there are various possible left-inverses, each with its own advantages and disadvantages. •

12.2.2 Fourier series

Apropos to Remark 12.2.3–2, and motivated by the ramblings in Section 9.6.1 and other places, a seemingly good candidate for the inverse of the CDFT is the map $\mathscr{F}^{-1}_{\mathrm{CD}}\colon \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ defined by

$$\mathscr{F}^{-1}_{\mathrm{CD}}(F)(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} F(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

(The reason for the factor $\frac{1}{T}$ is not so awfully important.) Well, this is unlikely to literally work since $\mathscr{F}^{-1}_{\mathrm{CD}}(F)$ will not be defined for frequency-domain signals in

$\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$ that decay slowly at infinity. However, all may not be lost because all we are interested in is computing $\mathscr{F}_{\mathrm{CD}}^{-1}$ on image($\mathscr{F}_{\mathrm{CD}}$). Thus, maybe it holds that

$$f(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

However, this fails too, and actually fails badly in some sense; we discuss this in Section 12.2.3. Thus the "obvious" inverse for $\mathscr{F}_{\mathrm{CD}}$ does not work on signals in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. However, maybe it works on some subset of $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ that is large enough to contain most signals of interest. We shall see that this is sort of true in that signals for which the naïve inverse of $\mathscr{F}_{\mathrm{CD}}$ does not work tend to be a little pathological.

With the above as setup, we make the following definition.

12.2.4 Definition (Fourier series) For $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the ***Fourier series*** of f is the series

$$\mathrm{FS}[f](t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}},$$

disregarding the convergence of this series. The ***real Fourier series*** of f is

$$\mathrm{FS}[f](t) = \frac{1}{2T}\mathscr{C}_{\mathrm{CD}}(0) + \frac{1}{T}\sum_{n=1}^{\infty}\Big(\mathscr{C}_{\mathrm{CD}}(f)(nT^{-1})\cos(2\pi n\tfrac{t}{T}) + \mathscr{S}_{\mathrm{CD}}(f)(nT^{-1})\sin(2\pi n\tfrac{t}{T})\Big),$$

again disregarding convergence of the series. •

As we shall see in a moment, we are justified in using the same symbol for the Fourier series and the real Fourier series.

12.2.5 Remark (The meaning of "disregarding the convergence of this series") The disregarding of the convergence of the series defining $\mathrm{FS}[f]$ is perhaps unsettling. What we mean by this is that we will be considering various ways in which this series converges. However, it is possible to assign a precise meaning to $\mathrm{FS}[f]$ in any case, so let us describe this. We may define $\mathrm{FS}[f] \in \mathscr{D}'_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ by

$$\mathrm{FS}[f](\psi) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathscr{F}_{\mathrm{CD}}(\psi)(-nT^{-1}), \tag{12.5}$$

for $\psi \in \mathscr{D}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Note that this sum makes sense for all $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ since $\mathscr{F}_{\mathrm{CD}}(f) \in \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$ (and so is bounded, in particular), since $\lim_{n\to\infty}|n|^k|\mathscr{F}_{\mathrm{CD}}(\psi)(nT^{-1})| = 0$ since ψ is infinitely differentiable, and using Exercise 8.2.3. The rationale for this formula comes from the Fourier Reciprocity Relation, Proposition 12.1.11. Indeed, if $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ then this result gives

$$\int_0^T \mathrm{FS}[f](t)\psi(t)\,\mathrm{d}t = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathscr{F}_{\mathrm{CD}}(\psi)(-nT^{-1}),$$

which is the relation (12.5) in this case. However, the formula (12.5) is valid for all $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Despite this interpretation being accurate and valid, we shall not use it. •

12.2.6 Remark (The usual rôle of Fourier series) In most texts that deal with Fourier series, the chapter dealing with the topics we are now discussing would be titled "Fourier series." For us Fourier series arise in terms of inverting the CDFT, and so are not the principal objects of study, but just something that comes up along the way. Thus the emphasis and arrangement of ideas is a little different for us than for many other treatments. For example, in the usual treatment, one does not often think of the CDFT as a map, but more or less thinks of the coefficients

$$c_n(f) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t$$

in the complex case or

$$a_n(f) = 2\int_0^T f(t)\cos(2\pi n\tfrac{t}{T})\,\mathrm{d}t, \quad b_n(f) = 2\int_0^T f(t)\sin(2\pi n\tfrac{t}{T})\,\mathrm{d}t$$

as being computed, and then retrieving f using the Fourier series. The coefficients are called the ***Fourier coefficients***. The transform approach we give here more easily connects with the CCFT and the other Fourier transforms that we will encounter later. But the reader should still be aware that the usual treatment of the material in this chapter has a different slant than we give here, although the content of the presentation is broadly equivalent. •

The question we spend some time answering in this section is, "For what signals does the Fourier series converge?" In studying convergence of Fourier series we shall consider the Nth partial sum which is simply

$$f_N(t) = \frac{1}{T}\sum_{n=-N}^{N} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$$

for the Fourier series and

$$f_N(t) = \frac{1}{2T}\mathscr{C}_{\mathrm{CD}}(0) + \frac{1}{T}\sum_{n=1}^{N}\Big(\mathscr{C}_{\mathrm{CD}}(f)(nT^{-1})\cos(2\pi n\tfrac{t}{T}) + \mathscr{S}_{\mathrm{CD}}(f)(nT^{-1})\sin(2\pi n\tfrac{t}{T})\Big)$$

for the real Fourier series. The use of f_N for both partial sums is acceptable because, as the reader can show in Exercise 12.2.1, they are actually the same.

It is useful to have a formula for the Nth partial sum. Quite miraculously, there is a very nice explicit expression, and involving the periodic Dirichlet kernel from Examples 11.3.19–3 and 11.3.19–5:

$$D^{\mathrm{per}}_{T,N}(t) = \begin{cases} \frac{\sin((2N+1)\pi\frac{t}{T})}{\sin(\pi\frac{t}{T})}, & \theta \notin \mathbb{Z}, \\ 2N+1, & \theta \in \mathbb{Z}. \end{cases}$$

Note that like its sister, the periodic Fejér kernel $F^{\mathrm{per}}_{T,N}$ which we saw in the proof of Theorem 12.2.1, the periodic Dirichlet kernel is "concentrated" around $t = 0$ for large N, even though it is not a periodic approximate identity. The two kernels $F^{\mathrm{per}}_{T,N}$ and $D^{\mathrm{per}}_{T,N}$ form an integral part of the analysis of Fourier series, with each being the appropriate object at different stages of the game.

For us, the essential part played by the periodic Dirichlet kernel is given in the following lemma.

12.2.7 Lemma (Partial sums and the periodic Dirichlet kernel) *For* $\mathsf{f} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathsf{T}}(\mathbb{R};\mathbb{C})$ *we have*

$$\mathrm{f}_{\mathrm{N}}(\mathrm{t}) = \frac{1}{\mathrm{T}}\int_{-\frac{\mathrm{T}}{2}}^{\frac{\mathrm{T}}{2}} \mathrm{f}(\mathrm{t}-\tau)\mathrm{D}^{\mathrm{per}}_{\mathrm{T},\mathrm{N}}(\tau)\,\mathrm{d}\tau$$

for every $\mathrm{N} \in \mathbb{Z}_{>0}$.

Proof We compute

$$\begin{aligned} f_N(t) &= \frac{1}{T}\sum_{|n|\le N} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} = \frac{1}{T}\sum_{|n|\le N}\Big(\int_0^T f(s)\mathrm{e}^{-2\pi\mathrm{i}n\frac{s}{T}}\,\mathrm{d}s\Big)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} \\ &= \frac{1}{T}\sum_{|n|\le N}\Big(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(s)\mathrm{e}^{-2\pi\mathrm{i}n\frac{s}{T}}\,\mathrm{d}s\Big)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\Big(\sum_{|n|\le N}\mathrm{e}^{2\pi\mathrm{i}n\frac{(t-s)}{T}}\Big)f(s)\,\mathrm{d}s. \end{aligned} \tag{12.6}$$

Applying Lemma **??** from Example **??** below in the case when $\theta = 2\pi\frac{t-s}{T}$ gives

$$\sum_{|n|\le N}\mathrm{e}^{2\pi\mathrm{i}n\frac{(t-s)}{T}} = \frac{\sin((2N+1)\pi\frac{t-s}{T})}{\sin(\pi\frac{t-s}{T})} = D^{\mathrm{per}}_{T,N}(t-s).$$

This equality, after substitution into (12.6) with this followed by a change of variable $\tau = t - s$, gives

$$f_N(t) = \frac{1}{T}\int_{-\frac{T}{2}+t}^{\frac{T}{2}+t} f(t-\tau)D^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau.$$

Since both f and $D^{\mathrm{per}}_{T,N}$ are T-periodic we have

$$f_N(t) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)D^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau,$$

as desired. ■

Note that this gives the Nth partial sum as the T-periodic convolution of f with $D^{\mathrm{per}}_{T,N}$, recalling from Section 11.1.2 the definition of convolution for periodic signals.

12.2.8 Notation ($\mathbf{D^{per}_{T,N}f}$) Motivated by the above, for $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and for $N \in \mathbb{Z}_{>0}$ we shall from now on denote the Nth partial sum by

$$D^{\mathrm{per}}_{T,N}f(t) = \frac{1}{T}\sum_{|n|\le N} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}.$$

The notation is intended to be suggestive of convolution, and also serves to make clear the essential rôle of the periodic Dirichlet kernel in Fourier series. •

Also, rather than speak of convergence of the Fourier series to f we speak of convergence of the sequence $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ to f. At various times, the convergence will be pointwise, uniform, or bounded, as required by the situation. The reader may wish to revisit Section 3.5 to recall these various notions of convergence since we will proceed as if they are known.

Before we get to the specific results on convergence of Fourier series, it is perhaps useful to have an example to preview what we might expect.

12.2.9 Example (A sample Fourier series) We let $f \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{R})$ be the 1-periodic extension of the signal $\tilde{f}(t) = t$. In Example 12.1.3–1 we computed

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \begin{cases} \frac{1}{2}, & n = 0, \\ \frac{\mathrm{i}}{2n\pi}, & \text{otherwise.} \end{cases}$$

Equivalently we may compute the CDCT and CDST to be

$$\mathscr{C}_{\mathrm{CD}}(f)(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} \qquad \mathscr{S}_{\mathrm{CD}}(f)(n) = -\frac{1}{n\pi}, \qquad n \in \mathbb{Z}_{>0}.$$

Thus we have

$$\mathrm{FS}[f](t) = \frac{1}{2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{i}\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}}{2n\pi} = \sum_{n=1}^{\infty} -\frac{1}{n\pi}\sin(2n\pi t).$$

In Figure 12.7 are shown a few of the partial sums. Let us make a few observations.

1. The partial sums seem to be converging nicely at all points where the signal is continuous. One might be led to speculate that continuity is related to convergence of Fourier series. This is not true. Indeed, for the signal we are considering here, at points of continuity, the signal is not just continuous, but infinitely differentiable. We will see in Corollary 12.2.25 that, in fact, differentiability at a point implies convergence of the Fourier series at this point.
2. At the points of discontinuity the partial sums behave peculiarly. Indeed, as we take more terms in the partial sums, the region for which the approximation is good gets larger, but the approximation near the point of discontinuity gets no better. We shall see in Section 12.2.6 that this is a somewhat general phenomenon. •

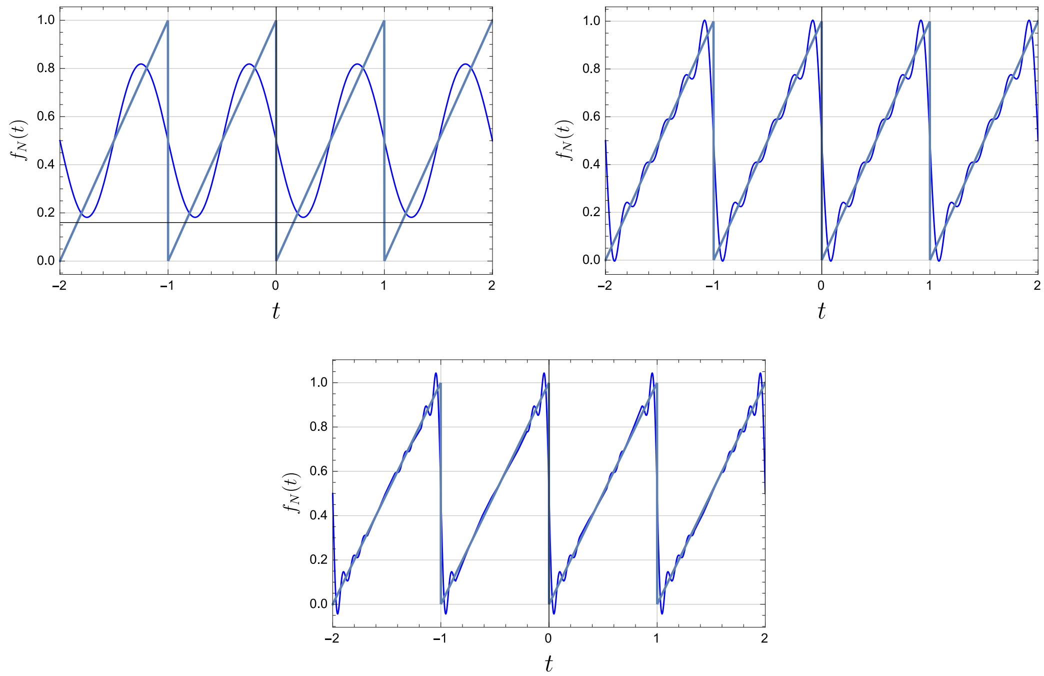


Figure 12.7 The 1st (top left), 5th (top right), and 10th (bottom) partial sums for the Fourier series for $f(t) = t$

12.2.3 Divergence of Fourier series

Before we get to results regarding *convergence* of Fourier series, it is first useful, to properly frame our discussion, to say a few words about *divergence* of Fourier series. There are various modes of convergence one may discuss, including pointwise convergence, uniform convergence, and convergence in the L^1-norm. In this section we shall see that the Fourier series interacts well with none of these modes of convergence in any general way. Since our discussion is a little involved at points, let us point out that the essential ideas are expressed in Example 12.2.10, and in the statements of Theorems 12.2.18, 12.2.20, and 12.2.21.

Let us begin with pointwise convergence. First of all, note that pointwise convergence of Fourier series to signals in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is sort of meaningless since signals with the same CDFT, and so the same Fourier series, will generally only agree almost everywhere. However, for continuous signals, the matter of pointwise convergence has some meaning since continuous signals agreeing almost everywhere are necessarily equal by Exercise 5.9.8. With this as background, our first divergent Fourier series is for a continuous signal with a Fourier series diverging at a single point.

12.2.10 Example (A continuous signal whose Fourier series diverges at a point) We define a continuous function 2π-periodic signal whose Fourier series diverges at $t = 0$. The construction of the continuous signal on is as follows. For $k \in \mathbb{Z}_{>0}$ define

$\alpha_k = \frac{1}{k^2}$ and $n_k = 3^{k^4}$. Note that for each $t \in (0, \pi]$ there exists a unique $k \in \mathbb{Z}_{>0}$ for which $t \in [\frac{\pi}{n_k}, \frac{\pi}{n_{k-1}}]$. With this in mind, for $t \in [0, \pi]$ define

$$f(t) = \begin{cases} \alpha_k \sin(n_k t), & t \in [\frac{\pi}{n_k}, \frac{\pi}{n_{k-1}}], \\ 0, & t = 0. \end{cases}$$

We then take f to be the even extension of its restriction to $[0, \pi]$. In Figure 12.8 we plot f between $-\pi$ and π. The idea is that on the intervals $[\frac{\pi}{n_k}, \frac{\pi}{n_{k-1}}]$, $k \in \mathbb{Z}_{>0}$,

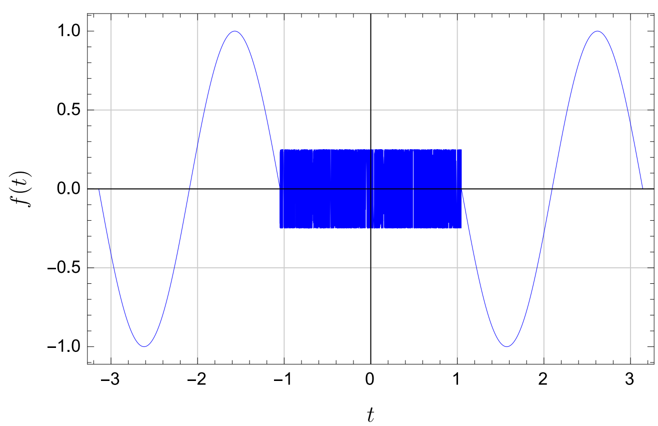


Figure 12.8 A continuous signal with a Fourier series divergent at 0

the signal is sinusoidal with an amplitude that decreases with k. These intervals accumulate at $t = 0$. They are actually compressed rather tightly around $t = 0$, and in Figure 12.8 we can effectively only see two of the intervals. Note that at the endpoints of each of these intervals the signal is zero, so the signal is continuous at all points away from $t = 0$. The signal is also continuous at $t = 0$ as we now show. Let $\epsilon \in \mathbb{R}_{>0}$ and let k_ϵ be the smallest natural number for which $\frac{1}{k^2} < \epsilon$. If we take $\delta = \frac{\pi}{n_{k_\epsilon}}$ then it follows that if $|t| < \delta$ then $|f(t)| < \epsilon$, thereby showing continuity of f at $t = 0$. We may also show that f is of bounded variation on any closed interval not containing 0. Therefore, the convergence of $(D^{\mathrm{per}}_{T,N} f)_{N \in \mathbb{Z}_{>0}}$ to f is uniform on any closed subset of $[-\pi, \pi]$ not containing $t = 0$.

We show that despite this the Fourier series diverges at $t = 0$. To show this we shall show that

$$\lim_{N \to \infty} \int_0^\pi f(t) \frac{\sin(n_N t)}{t} \, \mathrm{d}t = \infty,$$

which suffices by Theorem 12.2.22 and by evenness of f. We have

$$I_N \triangleq \int_0^\pi f(t) \frac{\sin(n_N t)}{t} \, \mathrm{d}t = \sum_{k=1}^\infty \alpha_k \int_{\pi/n_k}^{\pi/n_{k-1}} \sin(n_k t) \frac{\sin(n_N t)}{t} \, \mathrm{d}t.$$

Let us define

$$i_{N,k} = \int_{\pi/n_k}^{\pi/n_{k-1}} \sin(n_k t)\frac{\sin(n_N t)}{t}\,\mathrm{d}t$$

so that $I_N = \sum_{k=1}^{\infty} \alpha_k i_{N,k}$. For $N \neq k$ we have

$$\begin{aligned}
i_{N,k} &= \frac{1}{2}\int_{\pi/n_k}^{\pi/n_{k-1}} \frac{\cos|n_k - n_N|t - \cos(n_k + n_N)t}{t}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{1}^{\pi|n_k-n_N|/n_{k-1}} \frac{\cos\tau}{\tau}\mathrm{d}\tau - \frac{1}{2}\int_{1}^{\pi|n_k-n_N|/n_k} \frac{\cos\tau}{\tau}\mathrm{d}\tau \\
&\quad + \frac{1}{2}\int_{1}^{\pi(n_k-n_N)/n_{k-1}} \frac{\cos\tau}{\tau}\mathrm{d}\tau - \frac{1}{2}\int_{1}^{\pi(n_k-n_N)/n_k} \frac{\cos\tau}{\tau}\mathrm{d}\tau.
\end{aligned}$$

Note that none of the upper limits is less than $\frac{2\pi}{3}$ and since

$$\int_1^{\infty} \frac{\cos\tau}{\tau}\,\mathrm{d}\tau < \infty$$

it follows that $\lim_{N\to\infty}|i_{N,k}| < \infty$ provided that $N \neq k$.

We also have

$$\begin{aligned}
i_{N,N} &= \frac{1}{2}\int_{\pi/n_N}^{\pi/n_{N-1}} \frac{1-\cos(2n_N t)}{t}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{2\pi}^{2\pi n_N/n_{N-1}} \frac{1-\cos\tau}{\tau}\,\mathrm{d}\tau \\
&= \frac{1}{2}\log(n_N/n_{N-1}) - \frac{1}{2}\int_{2\pi}^{2\pi n_N/n_{N-1}} \frac{\cos\tau}{\tau}\,\mathrm{d}\tau.
\end{aligned}$$

As $N \to \infty$ the second term is bounded. Therefore

$$\lim_{N\to\infty} I_N = \lim_{N\to\infty} \frac{\alpha_N}{2}\log(n_N/n_{N-1}) + \beta$$

where $|\beta| < \infty$. Note that

$$\frac{\alpha_N}{2}\log(n_N/n_{N-1}) = \frac{(N^4-(N-1)^4)\log 3}{2N^2},$$

thus showing that $\lim_{N\to\infty}|I_N| = \infty$, thus showing divergence of $(D^{\mathrm{per}}_{T,N} f(0))_{N\in\mathbb{Z}_{>0}}$ as desired. •

Now we turn to divergence of Fourier series on more general sets and for classes of signals. Our discussion here will be a little general in the beginning since it is convenient to make some of the constructions in an abstract setting first. We begin by considering a useful sort of class of periodic signals.

12.2.11 Definition (Homogeneous Banach space of periodic signals) A ***homogeneous Banach space of*** $\mathbf{T}$***-periodic signals*** is a Banach space $(\mathsf{V}, \|\cdot\|)$ with the following properties:

(i) V is a subspace of $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$;

(ii) $\|f\|_1 \le \|f\|$ for every $f \in \mathsf{V}$;

(iii) $\|\tau_a^* f\| = \|f\|$ for every $f \in \mathsf{V}$ and $a \in \mathbb{R}$;

(iv) the map $a \mapsto \tau_a^* f$ is continuous for every $f \in \mathsf{V}$.

A homogeneous Banach space of T-periodic signals $(\mathsf{V}, \|\cdot\|)$ is ***regular***

(v) if $\mathsf{E}_{2\pi i n T^{-1}} f \in \mathsf{V}$ and

(vi) if $\|\mathsf{E}_{2\pi i n T^{-1}} f\| = \|f\|$

for every $f \in \mathsf{V}$ and $n \in \mathbb{Z}$, where $\mathsf{E}_{2\pi i n T^{-1}}$ denotes the harmonic signal $t \mapsto \mathrm{e}^{2\pi i n \frac{t}{T}}$. •

There are two homogeneous Banach spaces of periodic signals in which we will be interested.

12.2.12 Examples (Homogeneous Banach spaces of periodic signals)

1. We claim that $(\mathsf{L}^1_{\mathrm{per},T}, \|\cdot\|_1)$ is a homogeneous Banach space of T-periodic signals. The only nonobvious property to verify is the last one. This, however, is proved as Lemma 1 in the proof of Corollary 11.2.29.
2. We claim that $(\mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C}), T\|\cdot\|_\infty)$ is a homogeneous Banach space of T-periodic signals. The second of the properties is verified thusly:

$$\|f\|_1 = \int_0^T |f(t)|\,\mathrm{d}t \le \|f\|_\infty \int_0^T \mathrm{d}t = T\|f\|_\infty.$$

   The third property is obvious and the fourth property is exactly the statement that continuous periodic signals are uniformly continuous, cf. Theorem 3.1.24.

•

One of the useful features of homogeneous Banach spaces is that they admit the following useful approximation result. This result really belongs to Section 12.2.7, but we state it here since this is the only place we shall use this general result. We shall find the shorthand

$$F^{\mathrm{per}}_{T,N} f(t) = \frac{1}{T}\int_0^T f(t-\tau) F^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau$$

useful, where $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

12.2.13 Lemma (Approximations in homogeneous Banach spaces) *If* $(\mathsf{V}, \|\cdot\|)$ *is a homogeneous Banach space of* T*-periodic signals and if* $\mathrm{f} \in \mathsf{V}$*, then the sequence* $(\mathrm{F}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges to* f *in* V *in the topology induced by the norm* $\|\cdot\|$*.*

Proof We begin with a couple of sublemmata that have to do with integrals of functions taking values in a Banach space. We consider a general situation before we specialise to the situation of the lemma. To this end, for an $\mathbb{F}$-Banach space $(\mathsf{V}, \|\cdot\|)$, let us denote by $\ell^\infty([a,b];\mathsf{V})$ the subspace of $\mathbb{F}^{[a,b]}$ as follows:

$$\ell^\infty([a,b];\mathsf{V}) = \{\gamma\colon [a,b] \to \mathsf{V} \mid \ \sup\{\|\gamma(t)\| \mid \ t \in [a,b]\} < \infty\}.$$

This has the $\mathbb{F}$-vector space structure given by pointwise operations of vector addition and scalar multiplication:

$$(\gamma_1 + \gamma_2)(t) = \gamma_1(t) + \gamma_2(t), \quad (a\gamma)(t) = a(\gamma(t)).$$

We also define a norm $\|\cdot\|_\infty$ on $\ell^\infty([a,b];\mathsf{V})$ by

$$\|\gamma\|_\infty = \sup\{\|\gamma(t)\| \mid \ t \in [a,b]\}.$$

This is readily verified to be a norm. Also denote by $\mathsf{C}^0([a,b];\mathsf{V})$ the set of continuous maps from $[a,b]$ to V. We can now state our first lemma.

1 Sublemma *If $(\mathsf{V}, \|\cdot\|)$ is a $\mathbb{F}$-Banach space, if $\gamma \in \mathsf{C}^0([a,b];\mathsf{V})$, and if $k \in \mathbb{Z}_{>0}$, define $\gamma_k\colon [a,b] \to \mathsf{V}$ by asking that*

$$\gamma_k(t) = \begin{cases} \gamma(a + \frac{j(b-a)}{k}), & t \in [a + \frac{j(b-a)}{k}, a + \frac{(j+1)(b-a)}{k}),\ j \in \{0,1,\dots,k-1\}, \\ \gamma(a + \frac{(k-1)(b-a)}{k}), & t = b. \end{cases}$$

Then the sequence $(\gamma_k)_{k\in\mathbb{Z}_{>0}}$ converges to γ in the norm $\|\cdot\|_\infty$.

Proof Note that γ is uniformly continuous since $[a,b]$ is compact. This is proved exactly as the standard Heine–Borel Theorem is proved for $\mathbb{R}$-valued functions, but using the norm $\|\cdot\|$ in place of the absolute value $|\cdot|$. Let $\epsilon \in \mathbb{R}_{>0}$ and, by uniform continuity of γ, let $\delta \in \mathbb{R}_{>0}$ be sufficiently small that if $t_1, t_2 \in [a,b]$ satisfy $|t_1 - t_2| < \delta$ then $\|\gamma(t_1) - \gamma(t_2)\| < \epsilon$. Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\frac{b-a}{N} < \frac{\delta}{2}$. For $k \geq N$ let $t \in [a,b)$ and let $j \in \{0,1,\dots,k-1\}$ be such that $t \in [a + \frac{j(b-a)}{k}, a + \frac{(j+1)(b-a)}{k})$. Then

$$\|\gamma(t) - \gamma_k(t)\| = \|\gamma(t) - \gamma(a + \tfrac{j(b-a)}{k})\| < \epsilon.$$

Also,

$$\|\gamma(b) - \gamma_k(b)\| = \|\gamma(b) - \gamma(a + \tfrac{(k-1)(b-a)}{k})\| < \epsilon.$$

Thus $\|\gamma - \gamma_k\|_\infty < \epsilon$ for $k \geq N$, giving the desired result. ▼

2 Sublemma *Let $(\mathsf{V}, \|\cdot\|)$ be a $\mathbb{F}$-Banach space, let $\gamma \in \mathsf{C}^0([a,b];\mathsf{V})$, and for $k \in \mathbb{Z}_{>0}$ define*

$$S_k = \frac{(b-a)}{k} \sum_{j=1}^{k} \gamma(a + \tfrac{j(b-a)}{k}).$$

Then the sequence $(S_k)_{k\in\mathbb{Z}_{>0}}$ converges in V.

Proof Let $\mathscr{S}$ be the set of all piecewise constant maps from $[a,b]$ to V. Thus $\sigma\colon [a,b] \to \mathsf{V}$ is an element of $\mathscr{S}$ if there exists a partition $(I_1,\dots,I_k)$ of $[a,b]$ such that $\sigma|I_j$ is constant for each $j \in \{1,\dots,k\}$. Note that $\mathscr{S}$ is a $\mathbb{C}$-vector space with pointwise operations of addition and scalar multiplication. Define a linear map $I\colon \mathscr{S} \to \mathsf{V}$ by

$$I(\sigma) = \sum_{j=1}^{k} \lambda(I_j)\sigma_j,$$

where $P = (I_1,\dots,I_k)$ is a partition such that $\sigma|I_j$ takes the value $\sigma_j \in \mathsf{V}$. Let $\mathrm{EP}(P) = \{t_0,t_1,\dots,t_k\}$. Note that

$$\|I(\sigma)\| \le \sum_{j=1}^{k}(t_j - t_{j-1})\|\sigma_j\| \le T\|\sigma\|_\infty.$$

Thus I is a bounded linear map. Thus, by Proposition 6.5.11, the map I extends to a bounded linear map $\bar{I}$ from the completion of $\mathscr{S}$ into V. By Sublemma 1 it follows that $\gamma \in \mathrm{cl}(\mathscr{S})$, since in that sublemma we provided a sequence $(\gamma_k)_{k\in\mathbb{Z}_{>0}}$ in $\mathscr{S}$ converging to γ. Therefore,

$$\bar{I}(\gamma) = \lim_{k\to\infty} I(\gamma_k),$$

the limit existing by virtue of Proposition 6.5.11. ▼

3 Sublemma *If* $(\mathsf{V},\|\cdot\|)$ *is a homogeneous Banach space of* T*-periodic signals, if* $\mathrm{f} \in \mathsf{V}$*, and if* $\mathrm{g} \in \mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$*, then the limit*

$$\lim_{\mathrm{k}\to\infty} \frac{\mathrm{T}}{\mathrm{k}} \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{g}\big(\tfrac{\mathrm{jT}}{\mathrm{k}}\big)\tau^*_{\mathrm{jT/k}}\mathrm{f}$$

exists in V *and is almost everywhere equal to* $\mathrm{g} * \mathrm{f}$.

Proof Let us first suppose that $\mathsf{V} = \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, that $\|\cdot\| = \|\cdot\|_1$, and that f is continuous. Since f and g are continuous, the signal $s \mapsto g(s)f(t-s)$ is continuous. Therefore, by Theorems 3.4.9 and 3.4.11 it follows that

$$\int_0^T g(s)f(t-s)\,\mathrm{d}s = \lim_{k\to\infty} \frac{T}{k}\sum_{j=1}^{k} g\big(\tfrac{jT}{k}\big)\tau^*_{jT/k}f(t)$$

for every $t \in \mathbb{R}$. Note that for each $k \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned}\frac{T}{k}\int_0^T\Big|\sum_{j=1}^{k} g\Big(\frac{jT}{k}\Big)\tau^*_{jT/k}f(t)\Big|\,\mathrm{d}t &\le \frac{T}{k}\sum_{j=1}^{k}\Big|g\Big(\frac{jT}{k}\Big)\Big|\int_0^T |\tau^*_{jT/k}f(t)|\,\mathrm{d}t\\ &\le \|g\|_\infty\|f\|_1.\end{aligned}$$

Thus we can apply the Dominated Convergence Theorem to get the result in this case.

Next suppose that $\mathsf{V} = \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, that $\|\cdot\| = \|\cdot\|_1$, and that f is a general signal. Let $\epsilon \in \mathbb{R}_{>0}$. By Theorem 6.7.56 let $h \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ be such that

$$\|f - h\|_1 < \max\Big\{\frac{\epsilon}{3\|g\|_1}, \frac{\epsilon}{3\|g\|_\infty}\Big\}.$$

As we showed in the preceding paragraph,

$$\lim_{k\to\infty}\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}h = g * h.$$

Thus let $N \in \mathbb{Z}_{>0}$ be such that

$$\left\|\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}h - g * h\right\| < \frac{\epsilon}{3}$$

for $k \geq N$. Then, for $k \geq N$ we have

$$\begin{aligned}\left\|\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}f - g * f\right\| &\leq \left\|\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}(f-h) - g * (f-h)\right\| \\ &\quad + \left\|\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}h - g * h\right\| \\ &\leq \|g\|_\infty\|f-h\|_1 + \|g\|_1\|f-h\|_1 + \frac{\epsilon}{3} < \epsilon,\end{aligned}$$

using Theorem 11.2.1. This gives the result in this case.

Finally, we can prove the result in the general case. Thus we let $(\mathsf{V}, \|\cdot\|)$ be a homogeneous Banach space of T-periodic signals and we let $f \in \mathsf{V}$. By Proposition 6.5.4 and the definition of a homogeneous Banach space of T-periodic signals, the map

$$[0, T] \ni s \mapsto u(s)\tau^*_s f \in \mathsf{V}$$

is continuous in the norm topology induced by $\|\cdot\|$.***missing stuff*** With this in mind, let us for notational convenience define $F \in \mathsf{C}^0([0, T]; \mathsf{V})$ by

$$F(s) = u(s)\tau^*_s f.$$

By Sublemma 2 the limit

$$\lim_{k\to\infty}\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}f$$

exists in V with respect to the norm $\|\cdot\|$. Let us denote this limit by $\overline{F}$. Since $\|\cdot\|_1 \leq \|\cdot\|$, one readily verifies that F being continuous in the topology of the norm $\|\cdot\|$ implies its being continuous in the topology of the norm $\|\cdot\|_1$. Therefore, as we proved above, the limit

$$\lim_{k\to\infty}\frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}f,$$

exists in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and is almost everywhere equal to $g * f$. Denote

$$S_k = \frac{T}{k}\sum_{j=1}^{k} g\left(\tfrac{jT}{k}\right)\tau^*_{jT/k}f.$$

For $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be such that $\|\overline{F} - S_k\| < \frac{\epsilon}{2}$ and $\|g * f - S_k\|_1 < \frac{\epsilon}{2}$ for $k \geq N$. Then we have

$$\|\overline{F} - g * f\|_1 \leq \|\overline{F} - S_k\|_1 + \|g * f - S_k\|_1 \leq \|\overline{F} - S_k\| + \|g * f - S_k\|_1 < \epsilon$$

giving $\|\overline{F} - g * f\|_1 = 0$ and so $\overline{F}$ almost everywhere equal to $g * f$, as desired. ▼

We first prove a technical estimate from which the result of the lemma follows easily.

4 **Sublemma** *If* $(\mathsf{V}, \|\cdot\|)$ *is a homogeneous Banach space of* T*-periodic signals, if* $\mathrm{f} \in \mathsf{V}$*, and if* $\mathrm{u} \in \mathrm{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *satisfies* $\int_0^{\mathrm{T}} \mathrm{u(s)\,ds} = 1$*, then*

$$\|\mathrm{f} - \mathrm{f} * \mathrm{u}\| \leq \int_0^{\mathrm{T}} \|\mathrm{f} - \tau_{\mathrm{s}}^* \mathrm{f}\| |\mathrm{u(s)}| \,\mathrm{ds}.$$

Proof We have

$$f(t) - f * u(t) = \int_0^T (f(t) - f(t-s))u(s)\,\mathrm{d}s.$$

By the preceding sublemma and by Theorems 3.4.9 and 3.4.11, we have

$$f - f * u = \lim_{k\to\infty} \frac{T}{k} \sum_{j=1}^{k} u\Big(\tfrac{jT}{k}\Big)(f - \tau_{jT/k}^* f),$$

the limit being taken in V with respect to the norm $\|\cdot\|$. By continuity of the norm we have

$$\begin{aligned}\|f - f * u\| &= \Big\|\lim_{k\to\infty} \frac{T}{k} \sum_{j=1}^{k} u\Big(\tfrac{jT}{k}\Big)(f - \tau_{jT/k}^* f)\Big\| \\ &\leq \lim_{k\to\infty} \frac{T}{k} \sum_{j=1}^{k} \Big|u\Big(\tfrac{jT}{k}\Big)\Big| \|f - \tau_{jT/k}^* f\| = \int_0^T |u(s)| \|f - \tau_s f\| \,\mathrm{d}s,\end{aligned}$$

the last equality holding by Theorems 3.4.9 and 3.4.11, along with the fact that u and the map $s \mapsto \|f - \tau_s^* f\|$ are continuous. ▼

Let $\epsilon \in \mathbb{R}_{>0}$. By the sublemma,

$$\Big\|f - F_{T,j}^{\mathrm{per}} f\Big\| \leq \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \|f - \tau_s^* f\| |F_{T,j}^{\mathrm{per}}(s)| \,\mathrm{d}s. \tag{12.7}$$

Let $M \in \mathbb{R}_{>0}$ be such that $\|F_{T,j}^{\mathrm{per}}\|_1 \leq M$ for each $j \in \mathbb{Z}_{>0}$. By the properties of homogeneous Banach spaces of T-periodic signals, there exists $\delta \in (0, \frac{T}{2}]$ such that

$$\|f - \tau_s^* f\| < \frac{T\epsilon}{2M}$$

for $t \in \mathbb{R}$ and $|s| < \delta$. Then, for every $j \in \mathbb{Z}_{>0}$,

$$\frac{1}{T} \int_{-\delta}^{\delta} \|f - \tau_s^* f\| |F_{T,j}^{\mathrm{per}}(s)| \,\mathrm{d}s \leq \frac{\epsilon}{2M} \int_{-\frac{T}{2}}^{\frac{T}{2}} |F_{T,j}^{\mathrm{per}}(s)| \,\mathrm{d}s < \frac{\epsilon}{2}. \tag{12.8}$$

Now let $C = \|f\|$ and note that $\|f - \tau_s^* f\| \leq 2C$ using the triangle inequality and invariance of $\|\cdot\|$ under translation. Now, since $(\frac{1}{T}F^{\mathrm{per}}_{T,j})_{j\in\mathbb{Z}_{>0}}$ is an approximate identity by Example 11.3.19–3, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\frac{1}{T}\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} |F^{\mathrm{per}}_{T,j}(s)|\,\mathrm{d}s < \frac{\epsilon}{4C}$$

for $j \geq N$. Therefore, if $j \geq N$ we have

$$\frac{1}{T}\int_{[-\frac{T}{2},\frac{T}{2}]\setminus[-\delta,\delta]} \|f - \tau_s^* f\| |F^{\mathrm{per}}_{T,j}(s)|\,\mathrm{d}s < \frac{\epsilon}{2}. \tag{12.9}$$

Putting (12.7), (12.8), and (12.9) together we have

$$|f(t) - f * u_j(t)| < \epsilon, \qquad j \geq N,\ t \in \mathbb{R},$$

giving the result. ∎

Now let us consider a general definition that will be useful in characterising pointwise convergence of Fourier series.

12.2.14 Definition (Set of divergence) A subset $A \subseteq [0, T)$ is a ***set of divergence*** for a homogeneous Banach space of T-periodic signals $(\mathsf{V}, \|\cdot\|)$ if there exists $f \in \mathsf{V}$ such that $(D^{\mathrm{per}}_{T,N} f(t))_{N\in\mathbb{Z}_{>0}}$ diverges for every $t \in A$. •

For $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and $t \in \mathbb{R}$, let us introduce the following notation:

$$\overline{D}^{\mathrm{per}}_{T} f(t) = \sup\{\sup\{|D^{\mathrm{per}}_{T,m} f(t)| \mid m \in \{1,\dots,n\}\} \mid n \in \mathbb{Z}_{>0}\}.$$

The following lemma indicates the value of these definitions.

12.2.15 Lemma (Condition to be in a set of divergence) *If $(\mathsf{V}, \|\cdot\|)$ is a homogeneous Banach space of T-periodic signals, then $\mathrm{A} \subseteq [0, \mathrm{T})$ is a set of divergence for V if and only if there exists $\mathrm{f} \in \mathsf{V}$ such that $\overline{\mathrm{D}}^{\mathrm{per}}_{\mathrm{T}} \mathrm{f}(\mathrm{t}) = \infty$ for every $\mathrm{t} \in \mathrm{A}$.*

Proof It is clear that, if there exists $f \in \mathsf{V}$ such that $\overline{D}^{\mathrm{per}}_{T} f(t) = \infty$ for every $t \in A$, then $(D^{\mathrm{per}}_{T,N} f(t))_{N\in\mathbb{Z}_{>0}}$ does not converge for $t \in A$. Thus A is a set of divergence for V.

To prove the converse, we first prove a technical result.

1 Sublemma *Let $(\mathsf{V}, \|\cdot\|)$ be a homogeneous Banach space of T-periodic signals and let $\mathrm{g} \in \mathsf{V}$. Then there exists $\mathrm{f} \in \mathsf{V}$ and a sequence $(\alpha_{\mathrm{j}})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ in $\mathbb{R}_{>0}$ such that*

(i) $\alpha_{\mathrm{j}} > \alpha_{\mathrm{j}+1}$, $\mathrm{j} \in \mathbb{Z}_{>0}$,

(ii) $\lim_{\mathrm{j}\to\infty} \alpha_{\mathrm{j}} = \infty$, *and*

(iii) $\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{n}\mathrm{T}^{-1}) = \alpha_{|\mathrm{n}|}\mathscr{F}_{\mathrm{CD}}(\mathrm{g})(\mathrm{n}\mathrm{T}^{-1})$, $\mathrm{n} \in \mathbb{Z}$.

Proof By Lemma 12.2.13, for $n \in \mathbb{Z}_{>0}$ let $N_n \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\left\|g - F^{\mathrm{per}}_{T,N_n} g\right\| < 2^{-n}.$$

Define

$$f = g + \sum_{n=1}^{\infty} (g - F^{\mathrm{per}}_{T,N_n} g).$$

Since

$$\sum_{n=1}^{\infty}\|g - F^{\mathrm{per}}_{T,N_n}g\| < 1$$

by Example 2.4.2–**??**, it follows that the series in the definition for f converges in V by Theorem 6.4.6. Thus $f \in \mathsf{V}$. Moreover, since

$$\sum_{n=1}^{\infty}\|g - F^{\mathrm{per}}_{T,N_n}g\|_1 \le \sum_{n=1}^{\infty}\|g - F^{\mathrm{per}}_{T,N_n}g\|,$$

it follows that the sum defining f also converges in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Therefore, by continuity of $\mathscr{F}_{\mathrm{CD}}$ and by Proposition 12.1.19, it follows that

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f)(kT^{-1}) &= \mathscr{F}_{\mathrm{CD}}(g)(kT^{-1}) + \sum_{n=1}^{\infty}(\mathscr{F}_{\mathrm{CD}}(g)(kT^{-1}) - \mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}g)(kT^{-1}))\\ &= \mathscr{F}_{\mathrm{CD}}(g)(kT^{-1})\Big(1 + \sum_{n=1}^{\infty}\Big(1 - \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}(kT^{-1}))\Big)\Big).\end{aligned}$$

Let us define

$$\alpha_k = 1 + \sum_{n=1}^{\infty}\Big(1 - \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}(kT^{-1}))\Big)$$

From Example 12.2.45–4 we have

$$\frac{1}{T}\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}(kT^{-1})) = \begin{cases}1 - \frac{|k|}{N_n}, & |k| \in \{0, 1, \dots, (N_n - 1)\},\\ 0, & \text{otherwise.}\end{cases}$$

It follows that the sequence $(\alpha_k)_{k\in\mathbb{Z}_{>0}}$ is strictly monotonically increasing. Moreover, by the Monotone Convergence Theorem,

$$\lim_{k\to\infty}\alpha_k = 1 + \sum_{n=1}^{\infty}\Big(1 - \frac{1}{T}\lim_{k\to\infty}\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}(kT^{-1}))\Big) = \infty,$$

since $\lim_{k\to\infty}\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N_n}(kT^{-1})) = 0$ for each $n \in \mathbb{Z}_{>0}$. This gives the result since $\alpha_{-k} = \alpha_k$ for all $k \in \mathbb{Z}_{<0}$. ▼

Now suppose that A is a set of divergence for V and let $g \in \mathsf{V}$ have the property that $(D^{\mathrm{per}}_{T,N}g(t))_{N\in\mathbb{Z}_{>0}}$ diverges for each $t \in A$. By the sublemma, let $f \in \mathsf{V}$ and the sequence $(\alpha_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}_{>0}$ satisfy

1. $\alpha_j > \alpha_{j+1}$, $j \in \mathbb{Z}_{>0}$,
2. $\lim_{j\to\infty}\alpha_j = \infty$, and
3. $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \alpha_{|n|}\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1})$, $n \in \mathbb{Z}$.

Then we compute, for $n, m \in \mathbb{Z}_{>0}$ with $n > m$ and $t \in \mathbb{R}$,

$$\begin{aligned}
D^{\mathrm{per}}_{T,n}g(t) - D^{\mathrm{per}}_{T,m}g(t) &= \frac{1}{T}\sum_{j=m+1}^{n} \mathscr{F}_{\mathrm{CD}}(g)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} + \frac{1}{T}\sum_{j=-n}^{-m-1} \mathscr{F}_{\mathrm{CD}}(g)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} \\
&= \frac{1}{T}\sum_{j=m+1}^{n} \alpha_{|j|}^{-1}\mathscr{F}_{\mathrm{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} + \frac{1}{T}\sum_{j=-n}^{-m-1} \alpha_{|j|}^{-1}\mathscr{F}_{\mathrm{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} \\
&= \sum_{j=m+1}^{n} \alpha_j^{-1}(D^{\mathrm{per}}_{T,j}f(t) - D^{\mathrm{per}}_{T,j-1}f(t)) \\
&= \alpha_n^{-1}D^{\mathrm{per}}_{T,n}f(t) - \alpha_{m+1}^{-1}D^{\mathrm{per}}_{T,m}f(t) + \sum_{j=m+1}^{n-1}(\alpha_j^{-1} - \alpha_{j+1}^{-1})D^{\mathrm{per}}_{T,j}f(t).
\end{aligned}$$

Hence

$$|D^{\mathrm{per}}_{T,n}g(t) - D^{\mathrm{per}}_{T,m}g(t)| \le \alpha_{m+1}^{-1}|D^{\mathrm{per}}_{T,m}f(t)| \le \alpha_{m+1}^{-1}\overline{D}^{\mathrm{per}}_{T}f(t). \tag{12.10}$$

We claim that $\overline{D}^{\mathrm{per}}_{T}f(t) = \infty$ for each $t \in A$. Indeed, suppose that $\overline{D}^{\mathrm{per}}_{T}f(t) < \infty$ for some $t \in A$. We claim that $(D^{\mathrm{per}}_{T,n}g(t))_{n\in\mathbb{Z}_{>0}}$ is Cauchy, and so converges. Indeed, if $\epsilon \in \mathbb{R}_{>0}$ let $N \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\frac{\overline{D}^{\mathrm{per}}_{T}f(t)}{\alpha_N} < \epsilon.$$

By (12.10) we then have

$$|D^{\mathrm{per}}_{T,n}g(t) - D^{\mathrm{per}}_{T,m}g(t)| < \epsilon,$$

showing that $(D^{\mathrm{per}}_{T,n}g(t))_{n\in\mathbb{Z}_{>0}}$ indeed converges. This contradiction gives the result. ■

The following characterisation of sets of divergence is useful.

12.2.16 Theorem (Character of sets of divergence) *If* $(\mathsf{V}, \|\cdot\|)$ *is a regular homogeneous Banach algebra of* T*-periodic signals, then* $\mathrm{A} \subseteq [0, \mathrm{T})$ *is a set of divergence for* V *if and only if there exists a sequence* $(\mathrm{P}_j)_{j\in\mathbb{Z}_{>0}}$ *of trigonometric polynomials in* V *such that*

(i) $\sum_{j=1}^{\infty}\|\mathrm{P}_j\| < \infty$ *and*

(ii) $\sup\{\overline{\mathrm{D}}^{\mathrm{per}}_{\mathrm{T}}\mathrm{P}_j(\mathrm{t}) \mid j \in \mathbb{Z}_{>0}\} = \infty$ *for every* $\mathrm{t} \in \mathrm{A}$.

Proof To simplify notation, let us take $T = 1$, without loss of generality.

Let $(P_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of trigonometric polynomials satisfying conditions (i) and (ii). Let d_j be the degree of P_j and let $(m_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of integers for which

$$m_j > m_{j-1} + d_j + d_{j-1}. \tag{12.11}$$

With these definitions take

$$f(t) = \sum_{j=1}^{\infty} \mathrm{e}^{2\pi\mathrm{i}m_j t}P_j(t),$$

noting that the series converges uniformly by assumption (i) and the Weierstrass M-test. Note that since P_j is a trigonometric polynomial we have

$$P_j(t) = \sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(P_j)(n)\mathrm{e}^{2\pi\mathrm{i}nt} \quad\implies\quad \mathrm{e}^{2\pi\mathrm{i}m_j t}P_j(t) = \sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(P_j)(n - m_j)\mathrm{e}^{2\pi\mathrm{i}nt}$$

using Lemma 12.3.2, with the sums being finite. Therefore,

$$\mathscr{F}_{\mathrm{CD}}(\mathsf{E}_{2\pi\mathrm{i}m_j}P_j)(n) = \mathscr{F}_{\mathrm{CD}}(P_j)(n - m_j)$$

and so

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \sum_{j=1}^{\infty} \mathscr{F}_{\mathrm{CD}}(P_j)(n - m_j), \qquad n \in \mathbb{Z}.$$

Let $j \in \mathbb{Z}_{>0}$, let $n \in \{0, 1, \dots, d_j\}$, and compute

$$\begin{aligned} D^{\mathrm{per}}_{1,m_j+n}f(t) - D^{\mathrm{per}}_{1,m_j-n-1}f(t) &= \sum_{k=-m_j-n}^{m_j+n} \mathscr{F}_{\mathrm{CD}}(f)(k)\mathrm{e}^{2\pi\mathrm{i}kt} - \sum_{k=-m_j+n+1}^{m_j-n-1} \mathscr{F}_{\mathrm{CD}}(f)(k)\mathrm{e}^{2\pi\mathrm{i}kt} \\ &= \sum_{k=-m_j-n}^{m_j+n} \sum_{r=1}^{\infty} \mathscr{F}_{\mathrm{CD}}(P_r)(k - m_r)\mathrm{e}^{2\pi\mathrm{i}kt} - \sum_{k=-m_j+n+1}^{m_j-n-1} \sum_{r=1}^{\infty} \mathscr{F}_{\mathrm{CD}}(P_r)(k - m_r)\mathrm{e}^{2\pi\mathrm{i}kt}. \end{aligned} \tag{12.12}$$

We shall examine the preceding sums when the inner sum is fixed at $r \in \mathbb{Z}_{>0}$. In this case we make a change of index $l = k - m_r$ and make some manipulations to get that the rth term in (12.12) is

$$\sum_{l=-m_r-m_j-n}^{-m_r-m_j+n} \mathscr{F}_{\mathrm{CD}}(P_r)(l)\mathrm{e}^{2\pi\mathrm{i}(m_r+l)t} + \sum_{l=m_j-m_r-n}^{m_j-m_r+n} \mathscr{F}_{\mathrm{CD}}(P_r)(l)\mathrm{e}^{2\pi\mathrm{i}(m_r+l)t}. \tag{12.13}$$

Now we make a few observations.

1. Note that, by definition of d_r,

$$\mathscr{F}_{\mathrm{CD}}(P_r)(l) = 0, \qquad l > d_r \ \text{ or } \ l < -d_r.$$

2. By (12.11) we have

$$m_j - m_{j+1} < -d_j - d_{j+1}, \quad m_{j+1} - m_{j+2} < -d_{j+1} - d_{j+2}.$$

   Adding these inequalities we have

$$m_j - m_{j+2} < d_j - 2d_{j+1} - d_{j+2} < -d_j - d_{j+2}.$$

   Carrying on, we have that $m_j - m_r < -d_j - d_r$ whenever $r > j$. Therefore, by this inequality and the definition of n,

$$m_j - m_r + n \leq m_j - m_r + d_j < -d_r.$$

   Therefore, whenever $r > j$, we have that the rth term in (12.12) is zero since all terms in the sum (12.13) are zero.

3. Let $r < j$. By (12.11) we have

$$\begin{aligned} m_r + d_r < m_{r+1} - d_{r+1} < m_{r+1} + d_{r+1} < m_{r+2} - d_{r+2} < m_{r+2} + d_{r+2} < \\ \cdots < m_{j-1} - d_{j-1} < m_{j-1} + d_{j-1} < m_j - d_j. \end{aligned}$$

Thus

$$-m_j - m_r + n < m_r - m_j < -d_j - d_r < -d_r.$$

Also,

$$m_j - m_r - n \geq m_j - m_r - d_j > d_r.$$

These preceding two relations imply immediately that the rth term in the sum (12.12) is zero since all terms in the sum (12.13) are zero.

4. Now we $r = j$. In this case, again using (12.11),

$$-m_r - m_j + n < -2m_j < -d_j$$

and so the first sum in (12.13) is zero. The second sum is obviously $e^{2\pi i m_j t} D^{\mathrm{per}}_{1,n} P_r(t)$.

Putting the preceding observations together we have

$$D^{\mathrm{per}}_{1,m_j+n} f(t) - D^{\mathrm{per}}_{1,m_j-n-1} f(t) = e^{2\pi i m_j t} D^{\mathrm{per}}_{1,n} P_j(t)$$

for every $j \in \mathbb{Z}_{>0}$, $n \in \{0, 1, \ldots, d_j\}$, and $t \in \mathbb{R}$. Now we claim that for $t \in A$ the sequence $(D^{\mathrm{per}}_{1,n} f(t))_{n \in \mathbb{Z}_{>0}}$ is not Cauchy, and so does not converge. First note that the condition

$$\sup\{\overline{D}^{\mathrm{per}}_1 P_j(t) \mid j \in \mathbb{Z}_{>0}\} = \infty,$$

along with the fact that P_j has degree j, implies that, for every $j_0 \in \mathbb{Z}_{>0}$, there exists $j \geq j_0$ and $n \in \{0, 1, \ldots, d_j\}$ such that $|D^{\mathrm{per}}_{1,n} P_j(t)| \geq 1$. Now let $N \in \mathbb{Z}_{>0}$ and let $j \in \mathbb{Z}_{>0}$ be such that $m_{j-1} \geq N$ and such that $|D^{\mathrm{per}}_{1,n} P_j(t)| \geq 1$ for some $n \in \{0, 1, \ldots, d_j\}$. Then $m_j + n \geq N$ and

$$m_j - n - 1 \geq m_j - d_j - 1 \geq m_j - d_j - d_{j-1} > m_{j-1} \geq N.$$

Thus $m_j + n, m_j - n - 1 \geq N$ have the property that

$$|D^{\mathrm{per}}_{1,m_j+n} f(t) - D^{\mathrm{per}}_{1,m_j-n-1} f(t)| \geq 1.$$

This prohibits the sequence $(D^{\mathrm{per}}_{1,N} f(t))_{N \in \mathbb{Z}_{>0}}$ from Cauchy and so it diverges.

For the converse, we use the following technical lemma.

1 Lemma *For a homogeneous Banach space* $(\mathsf{V}, \|\cdot\|)$ *of* T*-periodic signals and a set of divergence* A *for* V*, there exists* $\mathrm{f} \in \mathsf{V}$ *and a sequence* $(\beta_{\mathrm{j}})_{\mathrm{j} \in \mathbb{Z}_{>0}}$ *in* $\mathbb{R}_{>0}$ *such that* $\beta_{\mathrm{j}+1} > \beta_{\mathrm{j}}$, $\mathrm{j} \in \mathbb{Z}_{>0}$*, and* $\lim_{\mathrm{j} \to \infty} \beta_{\mathrm{j}} = \infty$ *with the property that*

$$\mathrm{card}(\{\mathrm{n} \in \mathbb{Z}_{>0} \mid |D^{\mathrm{per}}_{\mathrm{T,n}} \mathrm{f}(\mathrm{t})| > \beta_{\mathrm{n}}\}) = \infty$$

for every $\mathrm{t} \in \mathrm{A}$.

Proof Since A is a set of divergence, let $g \in \mathsf{V}$ be such that $(D^{\mathrm{per}}_{T,N} g(t))_{N \in \mathbb{Z}_{>0}}$ diverges for each $t \in A$. As in the proof of Lemma 12.2.15, let $(\alpha_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}_{>0}$ that is strictly monotonically increasing and diverging to ∞ for which

$$D^{\mathrm{per}}_{T,n} g(t) - D^{\mathrm{per}}_{T,m} g(t) = \alpha_n^{-1} D^{\mathrm{per}}_{T,n} f(t) - \alpha_{m+1}^{-1} D^{\mathrm{per}}_{T,m} f(t) + \sum_{j=m+1}^{n-1} (\alpha_j^{-1} - \alpha_{j+1}^{-1}) D^{\mathrm{per}}_{T,j} f(t)$$

for every $n, m \in \mathbb{Z}_{>0}$ with $n > m$ and for every $t \in \mathbb{R}$. Now let $(\beta_j)_{j \in \mathbb{Z}_{>0}}$ be a strictly monotonically increasing sequence ▼

By the preceding lemma, let $f \in \mathsf{V}$ and the sequence $(\beta_j)_{j\in\mathbb{Z}_{>0}}$ be such that $(\beta_j)_{j\in\mathbb{Z}_{>0}}$ is strictly monotonically increasing and diverges to ∞ and satisfies

$$\operatorname{card}(\{n \in \mathbb{Z}_{>0} \mid |D^{\mathrm{per}}_{T,n} f(t)| > \beta_n\}) = \infty \tag{12.14}$$

for every $t \in A$. By Lemma 12.2.13 let $(n_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$ such that

$$\|f - F^{\mathrm{per}}_{1,n_j} f\| < 2^{-j}, \qquad j \in \mathbb{Z}_{>0}. \tag{12.15}$$

There exists a strictly monotonically increasing sequence $(m_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ such that

$$\beta_{m_j} > 2\sup\{\overline{D}^{\mathrm{per}}_1 (F^{\mathrm{per}}_{1,n_j} f)(t) \mid t \in A\} \tag{12.16}$$

because the signal $F^{\mathrm{per}}_{1,n_j} f$ is a trigonometric polynomial. Define

$$P_j = V^{\mathrm{per}}_{1,m_{j+1}} * (f - F^{\mathrm{per}}_{1,n_j} * f), \qquad j \in \mathbb{Z}_{>0},$$

where $V^{\mathrm{per}}_{1,N}$, $N \in \mathbb{Z}_{>0}$, denotes the de la Vallée Poussin kernel. Note that by Lemma 12.2.40 it follows that the convolution of the periodic Fejér kernel with a signal in $\mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ is a trigonometric polynomial. Therefore, since the de la Vallée Poussin kernel is a sum of Fejér kernels, it follows that P_j is a trigonometric polynomial. We have, just as in Sublemma 4 from Lemma 12.2.13,

$$\begin{aligned}\|u * g\| &= \Big\|\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} u\Big(\frac{j}{k}\Big)\tau^*_{j/k} g\Big\| \le \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \Big|u\Big(\frac{j}{k}\Big)\Big|\|\tau_j k^* g\|\\ &= \int_0^1 |u(s)|\|\tau^*_s g\|\,\mathrm{d}s = \|u\|_1\|g\|,\end{aligned}$$

where $u \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ satisfies $\int_0^1 u(s)\,\mathrm{d}s = 1$ and where $g \in \mathsf{V}$. Using the preceding formula, (12.13), and the fact that $\|V^{\mathrm{per}}_{1,N}\|_1 = 1$ for each $N \in \mathbb{Z}_{>0}$, we have

$$\sum_{j=1}^{\infty}\|P_j\| = \sum_{j=1}^{\infty}\|V^{\mathrm{per}}_{1,m_{j+1}} * (f - F^{\mathrm{per}}_{1,n_j} * f)\| \le \sum_{j=1}^{\infty}\|f - F^{\mathrm{per}}_{1,n_j} f\| < \infty.$$

Now let $t \in A$ and $n \in \mathbb{Z}_{>0}$ satisfy $|D^{\mathrm{per}}_{T,n} f(t)| > \beta_n$. Let $j \in \mathbb{Z}_{>0}$ be such that $n \in \{\beta_j + 1, \dots, \beta_{j+1}\}$. Note that by Example 12.2.45–5 and Proposition 12.1.19 we have

$$\mathscr{F}_{\mathrm{CD}}(V^{\mathrm{per}}_{1,m} g)(n) = \mathscr{F}_{\mathrm{CD}}(g)(n)$$

for $m \le n$ and for $g \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$. Thus

$$D^{\mathrm{per}}_{1,n} P_j(t) = D^{\mathrm{per}}_{1,n}(f - F^{\mathrm{per}}_{1,n_j} f)(t) = D^{\mathrm{per}}_{1,n} f(t) - D^{\mathrm{per}}_{1,n}(F^{\mathrm{per}}_{1,n_j} f)(t).$$

By (12.16) we have $|D^{\mathrm{per}}_{1,n} P_j(t)| \ge \frac{1}{2}\beta_n$. Since this holds for any $n \in \mathbb{Z}_{>0}$ for which $|D^{\mathrm{per}}_{1,n} f(t)| \ge \beta_n$ and since the sequence $(\beta_n)_{n\in\mathbb{Z}_{>0}}$ diverges to ∞, this part of the theorem follows from (12.14). ■

The following property of sets of divergence is also useful.

12.2.17 Lemma (Countable unions of sets of divergence are sets of divergence) *If* $(\mathsf{V}, \|\cdot\|)$*is a homogeneous Banach space of* T*-periodic signals and if* $(\mathrm{A_j})_{\mathrm{j}\in\mathbb{Z}_{>0}}$ *is a family of sets of divergence for* V*, then* $\cup_{\mathrm{j}\in\mathbb{Z}_{>0}}\mathrm{A_j}$ *is a set of divergence for* V*.*

Proof ***missing stuff*** ■

With the preceding general development, we can state the following theorem which significantly strengthens the conclusions of Example 12.2.10.

12.2.18 Theorem (Continuous signals can have Fourier series diverging on a set of zero measure) *If* $\mathrm{Z} \subseteq [0, \mathrm{T})$ *has Lebesgue measure zero, then* Z *is a set of divergence for* $(\mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C}), \mathrm{T}\|\cdot\|_\infty)$*. Thus there exists* $\mathrm{f} \in \mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *such that* $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f}(\mathrm{t}))_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *diverges for every* $\mathrm{t} \in \mathrm{Z}$*.*

Proof ■

The following general result helps to clarify the nature of sets of divergence.

12.2.19 Theorem (Sets of divergence for classes of signals containing the continuous signals) *If* $(\mathsf{V}, \|\cdot\|)$ *is a homogeneous Banach algebra of* T*-periodic signals containing* $\mathsf{C}^0_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *and if* $\mathrm{A} \subseteq [0, \mathrm{T})$ *is a set of divergence for* V*, then either* A *has zero Lebesgue measure or* $\mathrm{A} = [0, \mathrm{T})$*.*

Proof ■

Using the preceding result we can characterise the set of divergence for the Banach space of integrable signals.

12.2.20 Theorem (Integrable signals can have Fourier series diverging everywhere) *The set* $[0, \mathrm{T})$ *is a set of divergence for* $\mathsf{L}^1_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$*. In particular, there exists* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *such that the sequence* $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f}(\mathrm{t}))_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *does not converge for every* $\mathrm{t} \in \mathbb{R}$*.*

Proof In the proof we freely make use of facts about the CDFT that we have not covered yet. So some forward referencing will be necessary.

Throughout the proof we take $T = 1$ without loss of generality.

We begin with a couple of technical lemmata. The first relies on the CDFT for measures which we discuss in Section **??**.

1 Lemma *There exists* $\mathrm{N} \in \mathbb{Z}_{>0}$*,* $\mathrm{C} \in \mathbb{R}_{>0}$*, and, for each* $\mathrm{n} \in \mathbb{Z}$ *with* $\mathrm{n} \geq \mathrm{N}$*, a periodic measure* μ_n *on* $\mathscr{B}(\mathbb{R})$ *such that*

(i) $\mu_\mathrm{n}([0, 1)) = 1$ *and*

(ii) $\sup\{|\mathrm{D}^{\mathrm{per}}_{1,\mathrm{m}} * \mu_\mathrm{n}(\mathrm{t})| \mid \mathrm{m} \in \mathbb{Z}_{>0}\} \geq \mathrm{C}\log\mathrm{n}$ *for almost every* $\mathrm{t} \in \mathbb{R}$*.*

Proof Let $n \in \mathbb{Z}_{>0}$. Fix $q_1, \dots, q_n \in \mathbb{Q}$ and note that the set

$$\{(t_1, \dots, t_n) \in \mathbb{R}^n \mid q_1 t_1 + \dots + q_n t_n + 1 = 0\}$$

is a hyperplane with normal vector $(q_1, \dots, q_n)$. Such a hyperplane has measure zero. Note that the set

$$C_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid (t_1, \dots, t_n, 1) \text{ is linearly independent over } \mathbb{Q}\}$$

is the same as the set

$$\bigcap_{(q_1,\dots,q_n)\in\mathbb{Q}^n} \{(t_1,\dots,t_n)\in\mathbb{R}^n \mid q_1t_1+\dots+q_nt_n+1=0\}.$$

Thus C_n is the intersection of a countable number of hyperplanes. As each of these hyperplanes has measure zero, so does their intersection.

From the preceding paragraph, choose $t_1,\dots,t_n\in[0,1)$ such that $t_j<t_{j+1}$, $j\in\{1,\dots,n-1\}$, and such that

$$\frac{1}{2n}\le|t_{j+1}-t_j|\le\frac{2}{n},\qquad j\in\{1,\dots,n\},\tag{12.17}$$

where we take $t_{n+1}=t_1+1$. Let

$$A_n=\{t\in[0,1] \mid \{t-t_1,\dots,t-t_n,1\}\text{ is linearly independent over }\mathbb{Q}\}.$$

We claim that $\lambda(A_n)=1$. To prove this, we first claim that

$$[0,1]\setminus\mathbb{Q}[t_1,\dots,t_n,1]\subseteq A_n,$$

where $\mathbb{Q}[t_1,\dots,t_n,1]$ is the field extension of $\mathbb{Q}$ by the linearly independent set $\{t_1,\dots,t_n,1\}$; see Definition **??**. Indeed, if $t\in[0,1]\setminus A_n$ then

$$q_1(t-t_1)+\dots+q_n(t-t_n)+1=0$$

for some $q_1,\dots,q_n\in\mathbb{Q}$. Thus

$$t=q_1t_1+\dots+q_nt_n+\frac{1}{q_1+\dots+q_n},$$

and so $t\in\mathbb{Q}[t_1,\dots,t_n,1]$ as claimed. From this we conclude that $[0,1]\setminus A_n$ is countable and so has measure zero.

Next, with $t_1,\dots,t_n$ as above, define the measure μ_n by

$$\mu_n=\sum_{k\in\mathbb{Z}}\frac{1}{n}\sum_{j=1}^{n}\delta_{t_n+k},$$

noting that μ_n is 1-periodic and that $\mu_n([0,1)=1$. Let $t\in\mathbb{R}$ and $j\in\mathbb{Z}_{>0}$ and compute, using the characterisation Lemma **??** from Example **??** of the Dirichlet kernel,

$$\begin{aligned}|D^{\mathrm{per}}_{1,m}*\mu_n(t)|&=\Big|\sum_{k=-m}^{m}\mathrm{e}^{2\pi\mathrm{i}kt}\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{-2\pi\mathrm{i}kt_j}\Big|=\Big|\frac{1}{n}\sum_{j=1}^{n}D^{\mathrm{per}}_{1,m}(t-t_j)\Big|\\&=\Big|\frac{1}{n}\sum_{j=1}^{n}\frac{\sin(2\pi(m+\frac{1}{2})(t-t_j))}{\sin(\pi(t-t_j))}\Big|\\&=\Big|\frac{1}{n}\sum_{j=1}^{n}\frac{\mathrm{Im}(\mathrm{e}^{2\pi\mathrm{i}(m+\frac{1}{2})(t-t_j)})\,\mathrm{sign}(\sin(\pi(t-t_j)))}{\sin(\pi(t-t_j))}\Big|.\end{aligned}$$

By Theorem **??**, for each $t \in A_n$ there exists $m \in \mathbb{Z}_{>0}$ sufficiently large that

$$|\mathrm{e}^{2\pi \mathrm{i} m(t-t_j)} - \mathrm{i}\mathrm{e}^{2\pi \mathrm{i}\frac{1}{2}(t-t_j)}\,\mathrm{sign}(\sin(\pi(t-t_j)))| < \frac{1}{2}, \qquad j \in \{1,\dots,n\}.$$

This implies that

$$|\mathrm{e}^{2\pi \mathrm{i}(m+\frac{1}{2})(t-t_j)}\,\mathrm{sign}(\sin(\pi(t-t_j))) - \mathrm{i}| < \frac{1}{2}, \qquad j \in \{1,\dots,n\},$$

which in turn implies that

$$\mathrm{Im}(\mathrm{e}^{2\pi \mathrm{i}(m+\frac{1}{2})(t-t_j)}\,\mathrm{sign}(\sin(\pi(t-t_j)))) > \frac{1}{2}, \qquad j \in \{1,\dots,n\}.$$

Since $\sin(\pi x) \leq \pi x$ for $x \in [0,1]$ we then deduce that

$$|D^{\mathrm{per}}_{1,m} * \mu_n(t)| > \frac{1}{2n}\sum_{j=1}^{n} \frac{1}{|\sin(\pi(t-t_j))|} \geq \frac{1}{2\pi n}\sum_{j=1}^{n}\frac{1}{|t-t_j|}.$$

If $t \in [0,1]$ then $t \in [t_{j_0}, t_{j_0+1})$ for some $j_0 \in \{1,\dots,n\}$. Thus, by (12.17),

$$|t-t_j| \leq |t-t_{j_0}| + |t_{j_0}-t_j| \leq (|j-j_0|+1)\frac{2}{n}.$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{n}\frac{1}{|t-t_j|} &\geq \sum_{j=1}^{n}\frac{n}{2}\frac{1}{|j-j_0|+1} \\
&\geq \frac{n}{2}\Big(1 + \sum_{j=1}^{j_0-1}\int_{j-1}^{j}\frac{1}{|x-j_0|+1}\,\mathrm{d}x + \sum_{j=j_0+1}^{n}\int_{j}^{j+1}\frac{1}{|x-j_0|+1}\,\mathrm{d}x\Big) \\
&= \frac{n}{2}(1 + 2\log(2) + \log(n-j_0+2) + \log(j_0+1)).
\end{aligned}$$

Note that

$$\lim_{n\to\infty}\frac{1}{n\log(n)}\frac{n}{2}(1 + 2\log(2) + \log(n-j_0+2) + \log(j_0+1)) = \frac{1}{2}.$$

Thus there exists $N \in \mathbb{Z}_{>0}$ such that

$$\frac{n}{2}(1 + 2\log(2) + \log(n-j_0+2) + \log(j_0+1)) \geq \frac{1}{4}n\log(n)$$

for each $n \geq N$. Thus we have

$$|D^{\mathrm{per}}_{1,m} * \mu_n(t)| \geq C\log(n)$$

upon taking $C = \frac{1}{8\pi}$. ▼

2 Lemma *If* $\mathrm{M} \in \mathbb{R}_{>0}$ *then there exists a* 1*-periodic signal*

$$\mathrm{f_M(t)} = \sum_{\mathrm{k}\in\mathbb{Z}} \mathrm{c_{k,M}} \mathrm{e}^{2\pi \mathrm{ikt}} \tag{12.18}$$

and $\mathrm{A_M} \in \mathscr{L}([0,1))$ *such that*

(i) $\lambda(\mathrm{A_M}) > 1 - 2^{-\mathrm{M}}$,

(ii) the set $\{\mathrm{k} \in \mathbb{Z} \mid \mathrm{c_{k,M}} \neq 0\}$ *is finite,*

(iii) $\|\mathrm{f_M}\|_1 = 1$, *and*

(iv) $\inf\{\sup\{|\mathrm{D}^{\mathrm{per}}_{1,\mathrm{m}} * \mathrm{f_M(t)}| \mid \mathrm{m} \in \mathbb{Z}_{>0}\} \mid \mathrm{t} \in \mathrm{A_M}\} > 2^{\mathrm{M}}$.

Proof Let $M \in \mathbb{R}_{>0}$ and let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $C\log(N) > 2^{M+2}$, where $C \in \mathbb{R}_{>0}$ is as prescribed by Lemma 1. Let μ_N be as prescribed by Lemma 1. By Lemma 1 and Fatou's Lemma we have

$$\begin{aligned} 1 &= \lambda\Big(\Big\{t \in [0,1) \;\Big|\; \lim_{m\to\infty} \sup\{|D^{\mathrm{per}}_{1,j} * \mu_N(t) \geq 2^{M+1}| \mid j \in \{1,\dots,m\}\}\Big\}\Big) \\ &\leq \liminf_{m\to\infty} \lambda(\{t \in [0,1) \mid \sup\{|D^{\mathrm{per}}_{1,j} * \mu_N(t)| \mid j \in \{1,\dots,m\}\}\}). \end{aligned}$$

Therefore, let $m_0 \in \mathbb{Z}_{>0}$ be sufficiently large that, if we take

$$A_M = \{t \in [0,1) \mid \sup\{|D^{\mathrm{per}}_{1,j} * \mu_N(t)| \mid j \in \{1,\dots,m\}\}\},$$

then $\lambda(A_M) \geq 1 - 2^M$. By Theorem 12.2.42(ii), let $k_0 \in \mathbb{Z}_{>0}$ be sufficiently large that

$$\|F^{\mathrm{per}}_{1,k_0} * D^{\mathrm{per}}_{1,j} - D^{\mathrm{per}}_{1,j}\|_\infty \leq 1 \tag{12.19}$$

for $j \in \{1,\dots,m_0\}$. Define $f_M = \mu_N * F^{\mathrm{per}}_{1,k_0}$. By ***missing stuff*** we have that f_M is a finite sum of complex exponential signals. By (12.19) and ***missing stuff*** we have that

$$|D^{\mathrm{per}}_{1,j} * f_M(t) - D^{\mathrm{per}}_{1,j} * \mu_N(t)| \leq \|F^{\mathrm{per}}_{1,k_0} * D^{\mathrm{per}}_{1,j} - D^{\mathrm{per}}_{1,j}\|_\infty \leq 1$$

for each $t \in [0,1)$ and $j \in \{1,\dots,m_0\}$. Therefore, if $t \in A_M$ and if $j \in \{1,\dots,m_0\}$, we have

$$|D^{\mathrm{per}}_{1,j} * f_M(t)| \geq |D^{\mathrm{per}}_{1,j} * \mu_N(t)| - 1 \geq 2^{M+1} - 1 \geq 2^{M+1} > 2^M,$$

using our assumptions on N. Finally, using ***missing stuff***,

$$\|f_M\|_1 = \|\mu_N * F^{\mathrm{per}}_{1,k_0}\|_1 = \|\mu_N\|\|F^{\mathrm{per}}_{1,k_0}\|_1 = 1,$$

since μ_N is a positive measure and since F^{per}_{1,k_0} is positive with unit L^1-norm by our computations of Example 11.3.7–3.

We have verified the four conditions of the lemma. ▼

With these technical lemmata, we are now able to complete the proof.

We inductively construct sequences $(\epsilon_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$, $(M_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$, and $(\delta_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{Z}_{>0}$ as follows. Take $\epsilon_0 = M_0 = \delta_0 = 1$. Suppose that we have defined ϵ_j, M_j, and δ_j for $j \in \{0,1,\dots,n-1\}$ and then take

$$\epsilon_n = 2^{-n}(2\delta_{n-1})^{-1},$$

choose M_n such that

$$\epsilon_n 2^{M_n} \geq 2^n + \delta_{n-1} + 1,$$

and take δ_n to be such that the coefficients c_{k,M_j} in (12.18) are zero for $|k| \geq n$ and $j \in \{1, \dots, n\}$. Note that $\epsilon_j \leq 2^{-j}$ and $\delta_j \leq \delta_{j+1}$ for each $j \in \mathbb{Z}_{\geq 0}$.

Now we define

$$f = \sum_{j=1}^{\infty} \epsilon_j f_{M_j}.$$

Since $\epsilon_j \leq 2^{-j}$ and since $\|f_{M_j}\|_1 = 1$, the sequence of partial sums for this series is a Cauchy sequence in $\mathsf{L}^1_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ and so converges to give $f \in \mathsf{L}^1_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$.

Let $j \in \mathbb{Z}_{>0}$ and let $t \in A_{M_j}$ where the set A_{M_j} is as in the proof of Lemma 2. By Lemma 2 let $m_0 \in \mathbb{Z}_{>0}$ be such that $|D^{\mathrm{per}}_{1,m_0} * f_{M_j}(t)| > 2^{M_j}$ and take $k_0 = \min\{m_0, \delta_j\}$. With these definitions we have

$$\epsilon_j D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t) = D^{\mathrm{per}}_{1,k_0} * f(t) - \sum_{l=1}^{j-1} \epsilon_l D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t) - \sum_{l=j+1}^{\infty} \epsilon_l D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t)$$

and an application of the triangle inequality gives

$$|D^{\mathrm{per}}_{1,k_0} * f(t)| \geq \epsilon_j |D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t)| - \sum_{l=1}^{j-1} \epsilon_l |D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t)| - \sum_{l=j+1}^{\infty} \epsilon_l |D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t)|. \tag{12.20}$$

Note from Lemma **??** from Example **??** that $D^{\mathrm{per}}_{1,l}$ is a finite linear combination of complex exponential signals. Moreover, by Lemma 2, f_{M_j} is also a finite sum of complex exponential signals. Using the orthogonality of the complex exponential signals, cf. Lemma 12.3.2, the definition of k_0, the fact that $\delta_j \leq \delta_l$ for $j < l$, and Proposition 12.1.19, we obtain

$$\begin{aligned} D^{\mathrm{per}}_{1,k_0} * f_{M_j} &= D^{\mathrm{per}}_{1,m_0} * f_{M_j}, \\ D^{\mathrm{per}}_{1,k_0} * f_{M_l} &= D^{\mathrm{per}}_{1,\min(\delta_l,k_0)} * f_{M_l}, \qquad l < j, \\ D^{\mathrm{per}}_{1,k_0} * f_{M_l} &= D^{\mathrm{per}}_{1,\min(\delta_j,k_0)} * f_{M_l}, \qquad l > j. \end{aligned}$$

Now, using the definition of m_0 and the fact that $t \in A_{M_j}$, we have

$$|D^{\mathrm{per}}_{1,k_0} * f_{M_j}(t)| = |D^{\mathrm{per}}_{1,m_0} * f_{M_j}(t)| > 2^{M_j}.$$

Using the fact $\|f_{M_l}\|_1 = 1$, the fact that $\|D^{\mathrm{per}}_{1,k}\|_\infty \leq 2k+1 \leq 3k$, and Theorem 11.2.27, we have

$$|D^{\mathrm{per}}_{1,k_0} * f_{M_l}| \leq 3\delta_l, \qquad l < j,$$

and

$$|D^{\mathrm{per}}_{1,k_0} * f_{M_l}| \leq 3\delta_j, \qquad l > j.$$

Combining the preceding three estimates with (12.20) yields

$$|D^{\mathrm{per}}_{1,k_0} * f(t)| \geq \epsilon_j 2^{M_j} - 3\sum_{l=1}^{j-1} \epsilon_l \delta_l - 3\delta_j \sum_{l=j+1}^{\infty} \epsilon_l.$$

Next we have

$$3\sum_{l=1}^{j-1}\epsilon_l\delta_l = 3\delta_{j-1}\sum_{l=1}^{j-1}\epsilon_l \le \delta_{j-1}\sum_{l=1}^{j-1}\frac{1}{\delta_{l-1}2^l} < \delta_{j-1}$$

and

$$3\delta_j\sum_{l=j+1}^{\infty}\epsilon_l = \sum_{l=j+1}^{\infty}\frac{\delta_j}{\delta_{l-1}2^l} \le \sum_{l=j+1}^{\infty}\frac{1}{2^l} < 1,$$

using the definition of ϵ_l, $l \in \mathbb{Z}_{>0}$, and Example 2.4.2–**??**. Combining the preceding three estimates yields

$$|D^{\mathrm{per}}_{1,k_0} * f(t)| \ge \epsilon_j 2^{M_j} - \delta_{j-1} - 1 \ge 2^j.$$

Thus, for every $j \in \mathbb{Z}_{\ge 0}$ and every $t \in A_{M_j}$ we have

$$\sup\{|D^{\mathrm{per}}_{1,k} * f(t)| \mid k \in \mathbb{Z}_{>0}\} \ge 2^j.$$

It follows that for every $j \in \mathbb{Z}_{>0}$ and $t \in \cup_{l=j}^{\infty} A_{M_j}$ we have

$$\sup\{|D^{\mathrm{per}}_{1,k} * f(t)| \mid k \in \mathbb{Z}_{>0}\} \ge 2^j.$$

Therefore, if we take $A = \cap_{j\in\mathbb{Z}_{>0}} \cup_{l=j}^{\infty} A_{M_l}$ then we have

$$\sup\{|D^{\mathrm{per}}_{1,k} * f(t)| \mid k \in \mathbb{Z}_{>0}\} = \infty$$

whenever $t \in A$. Each of the sets $B_j \triangleq \cup_{l=j}^{\infty} A_{M_l}$, $j \in \mathbb{Z}_{>0}$, has measure 1, and so their complement has measure zero. Therefore,

$$\lambda([0,1) \setminus A) = \lambda\Big([0,1] \setminus \Big(\cap_{j\in\mathbb{Z}_{>0}} B_j\Big)\Big) = \lambda\Big(\cup_{j\in\mathbb{Z}_{>0}}[0,1) \setminus B_j\Big),$$

and so A has full measure. The theorem now follows from Theorem 12.2.19. ■

The signal from the preceding theorem and corollary is admittedly pathological, and one could easily object to it as a counterexample, reasoning that for any "decent" class of signals, maybe the Fourier series does converge pointwise. This is not even the case, as the following example shows.

Let us now consider convergence of Fourier series in the L^1-norm.

12.2.21 Theorem (A signal whose Fourier series diverges in the L^1-norm) *There exists* $\mathsf{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *such that the sequence* $(\mathsf{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathsf{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *does not converge in* $\mathsf{L}^1_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$.

Proof In the proof we freely use some facts we have not yet proved.

For $N \in \mathbb{Z}_{>0}$ let $\mathsf{L}_N \colon \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ be the "Nth partial sum operator," i.e.,

$$\mathsf{L}_N(f)(t) = \frac{1}{T}\sum_{n=-N}^{N}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

By Lemma 12.2.7 and Theorem 11.1.20(i) we thus have

$$\|\mathsf{L}_N(f)\|_1 = \tfrac{1}{T}\|D^{\mathrm{per}}_{T,N} * f\|_1 \le \tfrac{1}{T}\|D^{\mathrm{per}}_{T,N}\|_1\|f\|_1.$$

Therefore, if $\|\cdot\|_{1,1}$ denotes the induced norm for continuous linear maps from $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ to $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ (using the 1-norm for both the domain and codomain), we have $\|\mathsf{L}_N\|_{1,1} \leq \frac{1}{T}\|D^{\mathrm{per}}_{T,N}\|_1$.

Now let $n \geq N$ and let $F^{\mathrm{per}}_{n,T}$ be the periodic Fejér kernel. We now have

$$\|\mathsf{L}_N(F^{\mathrm{per}}_{n,T})\|_1 = \tfrac{1}{T}\|D^{\mathrm{per}}_{N,T} * F^{\mathrm{per}}_{n,T}\|_1 = \tfrac{1}{T}\|F^{\mathrm{per}}_{n,T} * D^{\mathrm{per}}_{N,T}\|_1.$$

By Theorem 12.2.42(ii), $(\frac{1}{T}F^{\mathrm{per}}_{n,T} * D^{\mathrm{per}}_{N,T})_{n\in\mathbb{Z}_{>0}}$ converges uniformly to $D^{\mathrm{per}}_{T,N}$. Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\|\mathsf{L}_N(F^{\mathrm{per}}_{n,T})\|_1 = \tfrac{1}{T}\|D^{\mathrm{per}}_{T,N}\|_1.$$

From this we conclude that $\|\mathsf{L}_N\|_{1,1} = \frac{1}{T}\|D^{\mathrm{per}}_{T,N}\|_1$. Note that $\lim_{N\to\infty}\|\mathsf{L}_N\|_1 = \infty$ by Lemma 1 from Example 11.3.19–5.

Now suppose that $(\mathsf{L}_N(f))_{N\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ for every $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Then, by the Principle of Uniform Boundedness, ***missing stuff***, it follows that there exists $M \in \mathbb{R}_{>0}$ such that $\|\mathsf{L}_N\|_1 \leq M$ for every $N \in \mathbb{Z}_{>0}$. This contradiction of our conclusion from the first part of the proof gives the theorem. ■

12.2.4 Pointwise convergence of Fourier series

Now we consider various forms of convergence of the Fourier series. Given the examples and results from the preceding section, we know that we have to place some sort of stringent conditions on a signal to ensure that its Fourier series has desirable convergence properties.

The most basic form of convergence is pointwise convergence, and so we begin with this. The basic theorem from which all other pointwise convergence theorems are derived is the following.

12.2.22 Theorem (Pointwise convergence of Fourier series) *Let* $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, *let* $t_0 \in \mathbb{R}$, *and let* $s \in \mathbb{C}$. *The following statements are equivalent:*

(i) $\lim_{N\to\infty} f_N(t_0) = s$;

(ii) $\displaystyle\lim_{N\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(f(t_0-t)-s)D^{\mathrm{per}}_{T,N}(t)\,dt = 0$;

(iii) *for each* $\epsilon \in (0, \frac{T}{2}]$ *we have*

$$\lim_{N\to\infty}\frac{1}{T}\int_{-\epsilon}^{\epsilon}(f(t_0-t)-s)D^{\mathrm{per}}_{T,N}(t)\,dt = 0;$$

(iv) *for each* $\epsilon \in (0, \frac{T}{2}]$ *we have*

$$\lim_{N\to\infty}\frac{1}{\pi}\int_{-\epsilon}^{\epsilon}(f(t_0-t)-s)\frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,dt = 0.$$

Proof Applying Lemma 12.2.7 in the case when $f(t) = 1$ gives the formula

$$1 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t. \tag{12.21}$$

Subtracting s from each side of the expression from Lemma 12.2.7 and using (12.21) then gives

$$f_N(t_0) - s = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t.$$

This shows the equivalence of parts (i) and (ii).

Clearly part (iii) is implied by part (ii). To show the converse we proceed as follows. We write

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}&(f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t\\ &= \frac{1}{T}\int_{-\epsilon}^{\epsilon}(f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t + \frac{1}{T}\int_{|t|>\epsilon}(f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t\end{aligned}$$

Define $A_\epsilon = \{t \in [-\frac{T}{2}, \frac{T}{2}] \mid |t| \geq \epsilon\}$. The second integral may be rewritten as

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \chi_{A_\epsilon}(t)\frac{(f(t_0 - t) - s)}{\sin(\pi\frac{t}{T})}\sin((2N+1)\pi\tfrac{t}{T})\,\mathrm{d}t.$$

Note that the function

$$\chi_{A_\epsilon}(t)\frac{(f(t_0 - t) - s)}{\sin(\pi\frac{t}{T})}$$

is integrable on $[-\frac{T}{2}, \frac{T}{2}]$ since the function $t \mapsto \sin(\pi\frac{t}{T})$ is bounded on A_ϵ. From the Riemann–Lebesgue Lemma we then have

$$\lim_{N\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \chi_{A_\epsilon}(t)\frac{(f(t_0 - t) - s)}{\sin(\pi\frac{t}{T})}\sin((2N+1)\pi\tfrac{t}{T})\,\mathrm{d}t = 0,$$

and from this the equivalence of (ii) and (iii) follows.

To show the equivalence of (ii) and (iv) we write

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t &= \frac{1}{\pi}\int_{-\epsilon}^{\epsilon}(f(t_0 - t) - s)\frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t\\ &+ \frac{1}{T}\int_{-\epsilon}^{\epsilon}(f(t_0 - t) - s)\Big(\frac{1}{\sin(\pi\frac{t}{T})} - \frac{T}{\pi t}\Big)\sin((2N+1)\pi\tfrac{t}{T})\,\mathrm{d}t\\ &+ \frac{1}{T}\int_{|t|>\epsilon}(f(t_0 - t) - s)D_{T,N}^{\mathrm{per}}(t)\,\mathrm{d}t.\end{aligned}$$

The last term on the right goes to zero as $N \to \infty$, just as in the preceding part of the proof. The function

$$\chi_{[-\epsilon,\epsilon]}(t)(f(t_0 - t) - s)\Big(\frac{1}{\sin(\pi\frac{t}{T})} - \frac{T}{\pi t}\Big)$$

in integrable and so by the Riemann–Lebesgue Lemma we have

$$\lim_{N\to\infty}\frac{1}{T}\int_{-\epsilon}^{\epsilon}\chi_{[-\epsilon,\epsilon]}(t)(f(t_0-t)-s)\Big(\frac{1}{\sin(\pi\frac{t}{T})}-\frac{T}{\pi t}\Big)\sin((2N+1)\pi\tfrac{t}{T})\,\mathrm{d}t=0,$$

giving the equivalence of (ii) and (iv), as desired. ■

12.2.23 Remark (Localisation) One important upshot of the preceding theorem is that the convergence of $(D^{\mathrm{per}}_{T,N}f(t_0))_{N\in\mathbb{Z}_{>0}}$ only involves the behaviour of f in an arbitrarily small neighbourhood of t_0. This is often referred to as the ***localisation principle***. •

Armed with these computations we proceed to prove a couple of useful conditions for pointwise convergence of Fourier series. The first condition we give is perhaps the easiest and is due to Dini.[1]

12.2.24 Theorem (Dini's test) *Let* $f\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0\in\mathbb{R}$. *If there exists* $\epsilon\in(0,\frac{T}{2}]$ *so that*

$$\int_{-\epsilon}^{\epsilon}\Big|\frac{f(t_0-t)-s}{t}\Big|\,\mathrm{d}t<\infty,$$

then $\lim_{N\to\infty}D^{\mathrm{per}}_{T,N}f(t_0)=s$.

Proof If $t\mapsto\frac{f(t_0-t)-s}{t}$ is in $\mathsf{L}^{(1)}([-\epsilon,\epsilon];\mathbb{C})$ then, by the Riemann–Lebesgue Lemma,

$$\lim_{N\to\infty}\int_{-\epsilon}^{\epsilon}\frac{f(t_0-t)-s}{t}\sin((2N+1)\pi\tfrac{t}{T})\,\mathrm{d}t=0.$$

The result now follows immediately from part (iv) of Theorem 12.2.22. ■

Dini's test has the following useful corollary.

12.2.25 Corollary (Fourier series converge at points of differentiability) *Let* $f\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0\in\mathbb{R}$. *If* f *is differentiable at* t_0 *then* $\lim_{N\to\infty}D^{\mathrm{per}}_{T,N}f(t_0)=f(t_0)$.

Proof If f is differentiable at t_0 then $\lim_{t\to tt_0}\frac{f(t_0-t)-f(t_0)}{t}$ exists and so the function $t\mapsto\frac{f(t_0-t)-f(t_0)}{t}$ is bounded in a neighbourhood of t_0. From this is follows that $\frac{f(t_0-t)-f(t_0)}{t}$ is integrable in a neighbourhood of t_0, and so the corollary follows from Dini's test. ■

There is another version of Dini's test that can sometimes be applied when the theorem above cannot be.

12.2.26 Corollary (An alternative version of Dini's test) *Let* $f\in\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0\in\mathbb{R}$. *If there exists* $\epsilon\in(0,\frac{T}{2}]$ *such that*

$$\int_{0}^{\epsilon}\Big|\frac{\frac{1}{2}(f(t_0+t)+f(t_0-t))-s}{t}\Big|\,\mathrm{d}t<\infty,$$

then $\lim_{N\to\infty}D^{\mathrm{per}}_{T,N}f(t_0)=s$.

Proof This follows from Exercise 12.2.3. ■

Let us look at a couple of examples that illustrate the value and the limitations of Theorem 12.2.24.

[1] Ulisse Dini (1845–1918) was an Italian mathematician who made his main mathematical contributions in the area of real analysis.

12.2.27 Examples (Dini's test)

1. Let us first consider the signal $f(t) = \square_{2,1,0}(t)-1$ introduced in Example 12.1.3–2. At points t_0 that are not integer multiples of $\frac{1}{2}$ we note that f is differentiable, so that Corollary 12.2.25 implies that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges to f at such points. Now let us consider a typical point of discontinuity, say the one at $t = 0$. For $\epsilon \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}$ we compute

$$\int_{-\epsilon}^{\epsilon} \frac{f(-t)-s}{t}\,\mathrm{d}t = \int_{-\epsilon}^{0} (1-s)t^{-1}\,\mathrm{d}t + \int_{0}^{\epsilon} (-1-s)t^{-1}\,\mathrm{d}t = -2\int_{0}^{\epsilon} t^{-1}\,\mathrm{d}t.$$

The last integral diverges, and so we cannot conclude the convergence of $(D^{\mathrm{per}}_{T,N}f(t))_{N\in\mathbb{Z}_{>0}}$ at $t = 0$. The same argument holds at all points of discontinuity of f.

However, the alternative statement of Corollary 12.2.26 works. Indeed, for $t_0 = 0$ we have

$$f(t_0+t) + f(t_0-t) = 0,$$

and so the hypotheses of Corollary 12.2.26 trivially apply, cf. part (c) of Exercise 12.2.3.

2. Next let us consider the signal $g(t) = \triangle_{\frac{1}{2},1,0}(t)$ introduced in Example 12.1.3–3. Note that at points where g is differentiable, that is to say points t_0 that are not integer multiples of $\frac{1}{2}$, $(D^{\mathrm{per}}_{T,N}g(t_0))_{N\in\mathbb{Z}_{>0}}$ converges to $g(t_0)$. Now let us consider a typical point where g is not differentiable, say $t = 0$. In this case we compute

$$\int_{-\epsilon}^{\epsilon} \frac{g(-t)}{t}\,\mathrm{d}t = \int_{-\epsilon}^{0} \mathrm{d}t - \int_{0}^{\epsilon} \mathrm{d}t = -2\epsilon < \infty.$$

We thus conclude that at $t = 0$ the Fourier series for g converges to 0, which is also the value of g at $t = 0$. Therefore, $(D^{\mathrm{per}}_{T,N}g)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to g.

3. Our next signal is a new one, given by the 2π-periodic extension, denoted h, of the signal $t \mapsto (\sin\frac{t}{2})^{1/2}$. In Figure 12.9 we show the graph of h. Note that near $t = 0$ the signal h behaves roughly like $\frac{1}{\sqrt{2}}\sqrt{t}$, and therefore the integral of the function $\frac{h(t)}{t}$ converges near $t = 0$. Thus

$$\int_{-\epsilon}^{\epsilon} \frac{h(0-t)-0}{t}\,\mathrm{d}t < \infty,$$

showing that $(D^{\mathrm{per}}_{T,N}h(t))_{N\in\mathbb{Z}_{>0}}$ converges to 0 at $t = 0$. In a similar way, $(D^{\mathrm{per}}_{T,N}h)_{N\in\mathbb{Z}_{>0}}$ converges to 0 at integer multiples of 2π. At all other points h is differentiable, and so convergence to h at these points is immediate. •

Note that to use Theorem 12.2.24 to conclude things about the convergence of a Fourier series one does not need to compute the CDFT. We shall see this theme illustrated with many of the convergence results we state. Indeed, this is one thing that makes them so useful. This is not to say, however, that there is a disconnect between the CDFT of a signal and the convergence properties of its Fourier series.

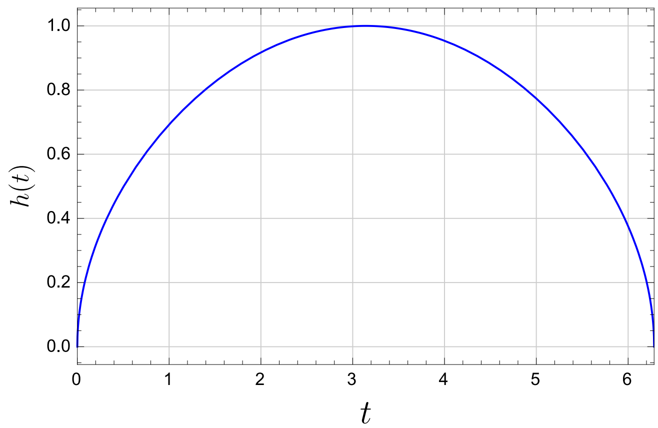


Figure 12.9 The signal h

For example, in Corollary 12.2.35 we shall see directly an instance where a certain property of the CDFT leads to uniform convergence.

The next result we state is historically the first result on convergence of Fourier series. The reader may wish to recall the definition of the left and right limits for a function and its derivative as discussed in Section 3.2.

12.2.28 Theorem (Dirichlet's test) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{C})$ *and suppose that the limits* $\mathsf{f}(\mathsf{t}_0-)$*,* $\mathsf{f}(\mathsf{t}_0+)$*,* $\mathsf{f}'(\mathsf{t}_0-)$*, and* $\mathsf{f}'(\mathsf{t}_0+)$ *exist for* $\mathsf{t}_0 \in \mathbb{R}$*. Then*

$$\lim_{\mathrm{N}\to\infty} \mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f}(\mathrm{t}_0) = \frac{1}{2}\Big(\mathrm{f}(\mathrm{t}_0+) + \mathrm{f}(\mathrm{t}_0-)\Big).$$

Proof Let us make some simplifying assumptions about f that will also be useful in the proof of Theorem 12.2.31. First we define

$$g_f(t) = \tfrac{1}{2}(f(t_0+t) - f(t_0+) + f(t_0-t) - f(t_0-)).$$

Note that g_f is even and that $g_f(0+) = g_f(0-) = 0$. Moreover, by Exercise 12.2.3 $(D^{\mathrm{per}}_{T,N} g_f(0))_{N\in\mathbb{Z}_{>0}}$ converges to zero if and only if $(D^{\mathrm{per}}_{T,N} f(t_0))_{N\in\mathbb{Z}_{>0}}$ converges to $\frac{1}{2}(f(t_0+)+f(t_0-))$. Therefore, without loss of generality we may suppose that f is even, that $f(0+) = f(0-) = 0$, and that we consider convergence of $(D^{\mathrm{per}}_{T,N} f)_{N\in\mathbb{Z}_{>0}}$ to f at $t = 0$.

With these assumptions, we note that the four hypotheses of the theorem imply that f is differentiable from the left and right at $t = 0$. Therefore the limits

$$\lim_{t\uparrow 0} \frac{f(t)}{t}, \quad \lim_{t\downarrow 0} \frac{f(t)}{t}$$

exist. Therefore, there exists $M, \epsilon \in \mathbb{R}_{>0}$ so that $\left|\frac{f(t)}{t}\right| < M$ for $|t| < \epsilon$. The theorem now follows from Theorem 12.2.24. ■

12.2.29 Remark (Dirichlet's test is a special case of Dini's test) In the proof of the preceding theorem we used Dini's test. Indeed, we essentially showed that any signal satisfying the hypotheses of Dirichlet's test also satisfies the hypotheses of Dini's test. Thus Dini's test is more general that Dirichlet's test (we shall see examples below where Dini's test applies but Dirichlet's test does not). Nonetheless, Dirichlet's test is sometimes easier to apply. •

Let us provide examples of signals that satisfy the hypotheses of the theorem and see what their Fourier series look like. These are the same signals considered in Example 12.2.27.

12.2.30 Examples (Dirichlet's test)

1. Next consider the signal $f(t) = \square_{2,1,0}(t) - 1$ introduced in Example 12.1.3–2. We note that f satisfies the hypotheses, and therefore the conclusions, of Theorem 12.2.28 at every point in $[0,1]$, and so the Fourier series converges pointwise. Note that the limit signal is
$$\mathrm{FS}[f](t) = \begin{cases} 0, & t \in \{0, \frac{1}{2}, 1\}, \\ 1, & t \in (0, \frac{1}{2}), \\ -1, & t \in (\frac{1}{2}, 1). \end{cases}$$
Thus we see that in this case, the Fourier series does not converge to the signal whose Fourier series we are computing.
2. We consider the signal $g(t) = \triangle_{\frac{1}{2},1,0}$ introduced in Example 12.1.3–2. This signal is obviously piecewise differentiable. Note that the hypotheses of Theorem 12.2.28 are satisfied by g at each point in $[0,1]$. Thus the Fourier series converges pointwise. Moreover, the limit signal is exactly g in this case.
3. Next we consider the signal h of Example 12.2.27. At points of differentiability, i.e., points t that are not integer multiples of 2π, we may apply Theorem 12.2.28 to conclude the pointwise convergence of $(D^{\mathrm{per}}_{T,N}h(t))_{N\in\mathbb{Z}_{>0}}$ to $h(t)$. However, at integer multiples of 2π the limits $f'(t_0+)$ and $f'(t_0-)$ do not exist and so Theorem 12.2.28 cannot be applied.
Note that Theorem 12.2.24 gives convergence of $(D^{\mathrm{per}}_{T,N}h)_{N\in\mathbb{Z}_{>0}}$ at those points where Theorem 12.2.28 cannot be applied. •

Our last result on the pointwise convergence of Fourier series at a prescribed point is due to Jordan. It is the most general of our results, and it comes the closest of all the results we state here to an assertion of the form, "The Fourier series for any reasonable signal will converge to that signal." Recall from Theorem 3.3.3(**??**) that if f is a signal of bounded variation on a continuous time-domain $\mathbb{T}$ then for each $t_0 \in \mathrm{int}(\mathbb{T})$ the left and right limits for f, denoted $f(t_0-)$ and $f(t_0+)$, exist.

12.2.31 Theorem (Jordan's test) *Let* $f \in L^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, *let* $t_0 \in \mathbb{R}$, *and suppose that there exists a neighbourhood* J *of* t_0 *so that* $f|J$ *has bounded variation. Then*
$$\lim_{N\to\infty} D^{\mathrm{per}}_{T,N}f(t_0) = \frac{1}{2}(f(t_0+) + f(t_0-)).$$

Proof Our proof relies on the Second Mean Value Theorem for integrals, stated as Proposition 3.4.33. According to Exercise 12.2.3, let us first make the assumption that f satisfies the hypotheses of the theorem and that, as well, $t_0 = 0$, f is even, and $f(0) = 0$. Since f has bounded variation in a neighbourhood of t_0, we may write $f = f_+ - f_-$ where f_+ and f_- are monotonically increasing. By applying the argument we give below to each component in the sum, we may without loss of generality also assume that f is monotonically increasing in a neighbourhood of 0 in $[0, \infty)$.

We first claim that there exists $M \in \mathbb{R}_{>0}$ so that

$$\Big|\int_0^t \frac{\sin((2N+1)\pi\frac{\tau}{T})}{\tau}\,\mathrm{d}\tau\Big| \leq M \tag{12.22}$$

for all $t \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$. By a change of variable we have

$$\int_0^t \frac{\sin((2N+1)\pi\frac{\tau}{T})}{\tau}\,\mathrm{d}\tau = \int_0^{(2N+1)\pi t} \frac{\sin u}{u}\,\mathrm{d}u.$$

By Lemma 1 from Example 11.3.7–3 it follows that

$$\lim_{a\to\infty} \int_0^a \frac{\sin u}{u}\,\mathrm{d}u$$

is finite. Thus the function

$$a \mapsto \int_0^a \frac{\sin u}{u}\,\mathrm{d}u$$

is continuous on $[0, \infty)$ and has a limit as $a \to \infty$. Therefore, it is bounded. From this we have boundedness of

$$t \mapsto \int_0^t \frac{\sin((2N+1)\pi\frac{\tau}{T})}{\tau}\,\mathrm{d}\tau,$$

and (12.22) holds, as desired.

Now let M be chosen such that (12.22) holds and let $\epsilon \in \mathbb{R}_{>0}$. Choose $\delta \in \mathbb{R}_{>0}$ so that $f(\delta-) < \frac{\epsilon\pi}{2M}$ and compute, for some $\delta' \in (0, \delta)$ guaranteed by Proposition 3.4.33,

$$\begin{aligned}\Big|\frac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t\Big| &= \Big|\frac{2}{\pi}\int_0^{\delta} f(t)\frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t\Big| \\ &= \Big|\frac{2f(\delta-)}{\pi}\int_{\delta'}^{\delta} \frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t\Big| \\ &\leq \frac{\epsilon\pi}{2M}\frac{2M}{\pi} = \epsilon.\end{aligned}$$

Note that we may apply Proposition 3.4.33 by virtue of Theorem 3.3.3(**??**). The theorem now follows from part (iv) of Theorem 12.2.22. ■

It is not very often easy to verify directly that a signal has bounded variation in a neighbourhood of a point. The way that this is most frequently done is indirectly through some simple characterisation of signals of bounded variation. For this reason, we shall not directly apply this theorem in any of our examples below.

Theorems 12.2.24, 12.2.28, and 12.2.31 give convergence of the Fourier series at a point. They can be make "global" by requiring that their hypotheses hold uniformly at every point in $[0, T]$. To do this for Theorem 12.2.24 it is convenient to introduce a new property for signals. A signal $f\colon \mathbb{T} \to \mathbb{C}$ is ***Lipschitz of order*** $\boldsymbol{\alpha \in \mathbb{R}_{>0}}$***missing stuff*** at $t_0 \in \mathbb{T}$ if there exists $L, \delta \in \mathbb{R}_{>0}$ such that

$$|t - t_0| < \delta \quad \implies \quad |f(t) - f(t_0)| \le L|t - t_0|^{\alpha}.$$

If f is Lipschitz of order α at t_0 it is also continuous at t_0, but the converse is not necessarily true. For example, on $\mathbb{R}$ the signal $f(t) = -\frac{1}{\log|x|}$ is continuous but is not Lipschitz of any order at $t = 0$. A continuous-time signal $f\colon \mathbb{T} \to \mathbb{C}$ is ***uniformly Lipschitz of order*** $\boldsymbol{\alpha \in \mathbb{R}_{>0}}$ if there exists $L \in \mathbb{R}_{>0}$ so that $|f(t) - f(t_0)| \le L|t - t_0|^{\alpha}$ for all $t, t_0 \in \mathbb{T}$. With this notion at hand, we state the following result, whose proof amounts to showing that the hypotheses of Theorems 12.2.24, 12.2.28, and 12.2.31, respectively, hold at each point in $[0, T]$.

12.2.32 Corollary (Conditions for global pointwise convergence) *Let* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R}; \mathbb{C})$ *and suppose that for each* $\mathrm{t} \in \mathbb{R}$ *we have* $\mathrm{f(t)} = \frac{1}{2}(\mathrm{f(t+)} + \mathrm{f(t-)})$. *Then any of the following statements implies that* $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges pointwise to* f*:*

(i) f *is uniformly Lipschitz;*

(ii) $\mathrm{f}|[0, \mathrm{T}] \in \mathsf{C}^1_{\mathrm{pw}}([0, \mathrm{T}]; \mathbb{C})$*;*

(iii) $\mathrm{f}|[0, \mathrm{T}] \in \mathsf{BV}([0, \mathrm{T}]; \mathbb{C})$.

One can look back at Examples 12.2.27 and 12.2.30 to see how Corollary 12.2.32 can be used for the signals f, g, and h to deduce the global convergence properties of their Fourier series.

12.2.5 Uniform convergence of Fourier series

Now, having addressed pointwise convergence with some degree of thoroughness, let us turn to uniform convergence of Fourier series. The first characterisation we provide for uniform convergence provides a useful criterion, and as well gives a relationship between the CDFT of a signal the convergence of its Fourier series.

12.2.33 Theorem (Uniform convergence of Fourier series) *Let* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R}; \mathbb{C})$. *If* $\mathscr{F}_{\mathrm{CD}}(\mathrm{f}) \in \ell^1(\mathbb{Z}(\mathrm{T}^{-1}); \mathbb{C})$ *then the following statements hold:*

(i) $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to a (necessarily continuous)* T*-periodic signal* g*;*

(ii) $\mathrm{f(t)} = \mathrm{g(t)}$ *for almost every* $\mathrm{t} \in \mathbb{R}$.

Proof We use the Weierstrass M-test, Theorem 3.5.15. The nth term in the CDFT for f satisfies

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{-2\pi i n \frac{t}{T}}| = |\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = M_n,$$

and furthermore the series $\sum_{n\in\mathbb{Z}} M_n$ converges. This shows that $(D^{\mathrm{per}}_{T,N} f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly by the Weierstrass M-test, and we denote the limit signal by g. This signal is continuous by Theorem 3.5.8.

To see that f and g are equal almost everywhere we first note that, swapping the sum and the integral using Theorem 3.5.23,

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1}) &= \int_0^T g(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\\ &= \int_0^T \sum_{m\in\mathbb{Z}} \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(f)(mT^{-1})\mathrm{e}^{2\pi \mathrm{i} m\frac{t}{T}}\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t\\ &= \frac{1}{T}\sum_{m\in\mathbb{Z}}\mathscr{F}_{\mathrm{CD}}(f)(mT^{-1})\int_0^T \mathrm{e}^{2\pi \mathrm{i} m\frac{t}{T}}\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}).\end{aligned}$$

The theorem now follows directly from Theorem 12.2.1. ■

It is possible to phrase this result in such a way that it clarifies its relationship to our discussion of the Fourier series as providing a possible inverse for the CDFT. To do so, we will introduce the notation (explained in Section 14.1 below)

$$\overline{\mathscr{F}}_{\mathrm{DC}}(F)(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} F(nT^{-1})\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}},$$

for $F \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$.

12.2.34 Corollary (A case when the Fourier integral in the inverse of the CDFT) *If* $\mathsf{f} \in \mathsf{C}^0_{\mathrm{per},\mathsf{T}}(\mathbb{R};\mathbb{C})$ *has the property that* $\mathscr{F}_{\mathrm{CD}}(\mathsf{f}) \in \ell^1(\mathbb{Z}(\mathsf{T}^{-1});\mathbb{C})$*, then*

$$\overline{\mathscr{F}}_{\mathrm{DC}} \circ \mathscr{F}_{\mathrm{CD}}(\mathsf{f})(\mathsf{t}) = \mathsf{f}(\mathsf{t}), \qquad \mathsf{t} \in \mathbb{R}.$$

The following result is a standard one for uniform convergence of Fourier series, and follows from our more general theorem.

12.2.35 Corollary (A test for uniform convergence) *Let* $\mathsf{f} \in \mathsf{C}^0_{\mathrm{per},\mathsf{T}}(\mathbb{R};\mathbb{C})$ *and suppose that there exists a piecewise continuous signal* $\mathsf{f}' \colon [0,\mathsf{T}] \to \mathbb{C}$ *with the property that*

$$\mathsf{f}(\mathsf{t}) = \mathsf{f}(0) + \int_0^{\mathsf{t}} \mathsf{f}'(\tau)\,\mathrm{d}\tau.$$

Then $(\mathsf{D}^{\mathrm{per}}_{\mathsf{T},\mathsf{N}}\mathsf{f})_{\mathsf{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to* f*. In particular, if* $\mathsf{f} \in \mathsf{C}^1_{\mathrm{per},\mathsf{T}}(\mathbb{R};\mathbb{C})$ *then* $(\mathsf{D}^{\mathrm{per}}_{\mathsf{T},\mathsf{N}}\mathsf{f})_{\mathsf{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to* f*.*

Proof We shall show that the hypotheses of Theorem 12.2.33 hold. Since $(D^{\mathrm{per}}_{N,T}f)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to f by Theorem 12.2.28 and since f is continuous, we may write

$$f(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}}.$$

By Proposition 12.1.12 the CDFT of f' is given by

$$\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1}) = \frac{2\pi \mathrm{i} n}{T}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}).$$

By Bessel's inequality (here we use some ideas from Section 12.3) we then have

$$\frac{1}{T}\sum_{n\in\mathbb{Z}}|\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|^2 \le \|f'\|_2^2 < \infty,$$

so that the sum

$$\sum_{n\in\mathbb{Z}}|\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|^2$$

converges. Now let

$$s_N = \sum_{|n|\le N}|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})|,$$

and note that

$$\begin{aligned} s_N &= |\mathscr{F}_{\mathrm{CD}}(f)(0)| + \sum_{\substack{|n|\le N\\ n\ne 0}}|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = |\mathscr{F}_{\mathrm{CD}}(f)(0)| + \sum_{\substack{|n|\le N\\ n\ne 0}}\frac{|T\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|}{|2\pi i n|} \\ &= |\mathscr{F}_{\mathrm{CD}}(f)(0)| + \frac{T}{2\pi}\sum_{\substack{|n|\le N\\ n\ne 0}}|\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|\Big|\frac{1}{n}\Big| \\ &\le |\mathscr{F}_{\mathrm{CD}}(f)(0)| + \frac{T}{2\pi}\Big(\sum_{\substack{|n|\le N\\ n\ne 0}}|\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|^2\Big)^{1/2}\Big(\sum_{\substack{|n|\le N\\ n\ne 0}}\frac{1}{n^2}\Big)^{1/2}, \end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Now note that both sums

$$\sum_{\substack{n\in\mathbb{Z}\\ n\ne 0}}|\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})|^2, \quad \sum_{\substack{n\in\mathbb{Z}\\ n\ne 0}}\frac{1}{n^2}$$

converge. This shows that

$$\lim_{N\to\infty} s_N < \infty. \tag{12.23}$$

Moreover, since the sequence $(s_N)_{N\in\mathbb{Z}_{>0}}$ is increasing, (12.23) implies that the sequence converges, and this is what we set out to prove. ■

Let us see how these conditions apply to the signals examined in Examples 12.2.27 and 12.2.30.

12.2.36 Examples (Uniform convergence of Fourier series)

1. For the signal $f(t) = \square_{2,1,0}(t) - 1$, we computed its CDFT in Example 12.1.3–2. The computations done in that example give the Fourier series for f as

$$\mathrm{FS}[f](t) = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin(2n\pi t).$$

We may simplify this by only writing the nonzero terms in the series:

$$\mathrm{FS}[f](t) = \sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin(2(2k-1)\pi t).$$

This Fourier series, while converging pointwise, fails to satisfy the hypotheses of Corollary 12.2.35. This does not necessarily imply that the signal fails to satisfy the conclusions of Corollary 12.2.35, however. The fact of the matter is that the Fourier series does not converge uniformly, although this does not quite follow directly from Corollary 12.2.35. To see that the convergence is not uniform, we argue as follows. Recall that if a sequence of continuous signals converges uniformly, the limit signal must be continuous. Therefore, if a sequence of continuous signals converges pointwise to a discontinuous signal, convergence cannot be uniform. The partial sums of a Fourier series do define a sequence of continuous signals. Therefore, if a Fourier series converges pointwise to a discontinuous signal, the convergence cannot be uniform. In Example 12.2.30 we saw that the Fourier series for f converges to a discontinuous signal, so this prohibits uniform convergence of this series. We shall explore this in more detail in Section 12.2.6.

In any event, the 10th partial sum is shown in Figure 12.10, and one can see

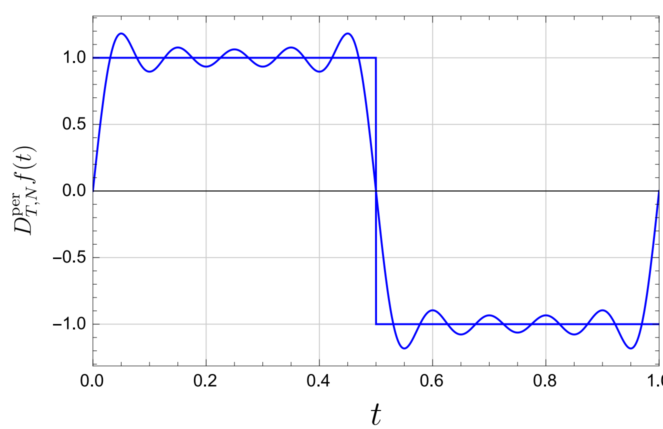


Figure 12.10 The 10th partial sum for f

that the convergence is not all that nice at the points of discontinuity.

2. For the signal $g(t) = \triangle_{\frac{1}{2},1,0}(t)$, we have computed its CDFT in Example 12.1.3–3. Using the computations of Example 12.1.3 we may determine that

$$\mathrm{FS}[g](t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos(2n\pi t).$$

As with f, we can simplify this by only writing the nonzero terms, which gives the series

$$\mathrm{FS}[g](t) = \frac{1}{4} - \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2\pi^2} \cos(2(2k-1)\pi t).$$

Note that g does satisfy the hypotheses for Corollary 12.2.35, so the convergence of this Fourier series is uniform. Furthermore, we see that the CDFT satisfies

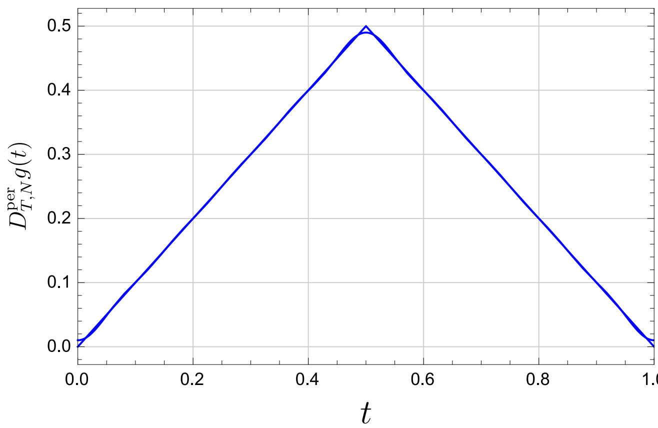


Figure 12.11 The 10th partial sum for g

the hypotheses of Theorem 12.2.33. In Figure 12.11 we show the 10th partial sum for g. It looks pretty nice.

3. Let us consider the signal h introduced in Example 12.2.27. We saw that the sequence of approximations $(D^{\mathrm{per}}_{T,N}h)_{N\in\mathbb{Z}_{>0}}$ converges everywhere to h, and we now wish to see whether this convergence is uniform. Note that h is not in fact continuous and piecewise differentiable, since it fails to possess left and right limits for the derivative at integer multiples of 2π. Therefore, the hypotheses of Corollary 12.2.35 do not hold, and so we cannot deduce uniform convergence using this result. It is possible that we might be able to directly use Theorem 12.2.33. However, this would entail computing the Fourier series for the signal, and this is not so easily done. Also note that since the signal is continuous we cannot immediately exclude uniform convergence. Thus we are left up in the air at this point as concerns the uniform convergence of the Fourier series of h. However, we shall immediately address this in Theorem 12.2.37. •

Now let us provide a rather general condition for uniform convergence, one that builds on our most general pointwise convergence result, Theorem 12.2.31. As with that theorem, the hypotheses of the following theorem may not be so easy to directly validate in practice.

12.2.37 Theorem (Continuous signals of bounded variation have uniformly convergent Fourier series) *If* $\mathsf{f} \in \mathsf{L}^{(1)}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{C})$ *is continuous and if* $\mathsf{f}|[0,\mathrm{T}]$ *has bounded variation then* $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T},\mathrm{N}}\mathsf{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to* f.

Proof By Proposition 3.3.11, $\mathrm{V}(f)$ is continuous. We claim that $\mathrm{V}(f)|[0,T]$ is uniformly continuous. That is to say, we claim that for each $\epsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ so that if $|t_1 - t_2| < \delta$ then $|f(t_1) - f(t_2)| < \epsilon$. To see that this is so, choose $\epsilon \in \mathbb{R}_{>0}$ and for $t_0 \in [0,T]$ define $\delta(t_0) \in \mathbb{R}_{>0}$ so that if $|t - t_0| < \frac{1}{2}\delta(t_0)$, $|f(t) - f(t_0)| < \epsilon$. If we define $I(t_0) = [t_0 - \delta(t_0), t_0 + \delta(t_0)]$ then $[0,T] \subseteq \cup_{t\in[0,T]}U(t)$. Now there exists a finite subset $\{t_1,\dots,t_k\} \subseteq [0,T]$ so that $[0,T] \subseteq \cup_{j=1}^{k} U(t_j)$. Now let $\delta = \frac{1}{2}\min\{\delta(t_1),\dots,\delta(t_k)\}$ and let

$t, t_0 \in [0, T]$ satisfy $|t - t_0| < \delta$, Suppose that $t_0 \in U(t_j)$ and note that

$$|t - t_j| = |t - t_0| + |t_0 - t_j| < \delta + \tfrac{1}{2}\delta(t_j) < \delta(t_2).$$

Therefore we have

$$|f(t) - f(t_0)| = |f(t) - f(t_j) + f(t_j) - f(t_0)| \leq |f(t) - f(t_j)| + |f(t_j) - f(t_0)| < \epsilon.$$

Thus $\mathrm{V}(f)|[0, T]$ is indeed uniformly continuous. As in the proof of Theorem 12.2.31 we let $M \in \mathbb{R}_{>0}$ satisfy

$$\Big|\int_0^t \frac{\sin((2N+1)\pi\frac{\tau}{T})}{\tau}\,\mathrm{d}\tau\Big| \leq M$$

for all $t \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$. Therefore, for $\epsilon \in \mathbb{R}_{>0}$ we may choose $\delta \in \mathbb{R}_{>0}$ so that $|\mathrm{V}(f)(t_1) - \mathrm{V}(f)(t_2)| < \frac{\epsilon\pi}{2M}$ for all $t_1, t_2 \in \mathbb{R}$ satisfying $|t_1 - t_2| < \delta$, this being possible since $\mathrm{V}(f)$ is uniformly continuous. Now the argument of Theorem 12.2.31 can be applied to show that

$$\Big|\int_{-\delta}^{\delta} (f(t_0 - t) - f(t_0))\frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t\Big| < \epsilon,$$

with this holding for each $t_0 \in \mathbb{R}$, giving uniform convergence as desired. ■

With this theorem we can resolve the uniform convergence of the signal h in Example 12.2.36.

12.2.38 Example (An application of the bounded variation test for uniform convergence) We consider the signal h that is the 2π-periodic extension of the function defined by $t \mapsto (\sin\frac{t}{2})^{1/2}$. We claim that this function has bounded variation on $[0, 2\pi]$. To see this we define

$$h_+ = \begin{cases} (\sin\frac{t}{2})^{1/2}, & t \in [0, \pi], \\ 1, & t \in (\pi, 2\pi], \end{cases}$$
$$h_- = \begin{cases} 0, & t \in [0, \pi], \\ 1 - (\sin\frac{t}{2})^{1/2}, & t \in (\pi, 2\pi]. \end{cases}$$

Note that h_+ and h_- are monotonically increasing and that $h = h_+ - h_-$. Therefore, by part (ii) of Theorem 3.3.3 we conclude that h has bounded variation. Therefore Theorem 12.2.37 implies that $(D^{\mathrm{per}}_{T,N} h)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to h. •

12.2.6 Gibbs' phenomenon

In our discussion of convergence of the signal f in the preceding example, we argued that if a Fourier series converges pointwise to a discontinuous signal, then the convergence of this Fourier series cannot be uniform. This does not provide much information about the *way* in which the series is not uniformly convergent. It turns out that for a large class of discontinuities such as often arise in practice, one can give a fairly explicit characterisation of what the partial sums look like. In Exercise 12.2.10 we lead the reader through an investigation of the so-called

Gibbs' phenomenon for the square wave. Here we consider a generalisation of this particular example.

To present the result, we need some notation. Let $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{R})$ and let t_0 be a point of discontinuity of f, and we assume that f possesses left and right limits at t_0. A ***Gibbs sequence*** at t_0 consists of a sequence $(t_j)_{j\in\mathbb{Z}_{>0}}$ with the properties

1. $\lim_{j\to\infty} t_j = t_0$ and
2. $\lim_{j\to\infty} j(t_j - t_0)$ exists.

Thus a Gibbs sequence at t_0 converges to t_0 from one side and not too slowly (this is the upshot of the second of the above conditions). The ***Gibbs set*** for f at t_0 is then

$$\mathrm{G}(f,t_0) = \left\{\lim_{j\to\infty} D^{\mathrm{per}}_{T,j} f(t_j) \;\middle|\; (t_j)_{j\in\mathbb{Z}_{>0}} \text{ is a Gibbs sequence at } t_0\right\}.$$

One way to think of the Gibbs set is as follows. Let $(t_j)_{j\in\mathbb{Z}_{>0}}$ be a Gibbs sequence at t_0. The points in the sequence of $(D^{\mathrm{per}}_{T,j} f(t_j))_{j\in\mathbb{Z}_{>0}}$ correspond to points on the graphs of the finite sums $D^{\mathrm{per}}_{T,j} f$, with the points getting closer to the vertical line $t = t_0$. Thus, one way to understand the Gibbs set is as the collection of points in the graphs of the finite sums that lie close to the vertical line $t = t_0$ in the limit. We shall explore this point of view further later on. For now, let us state the theorem describing the Gibbs set.

12.2.39 Theorem (General Gibbs' phenomenon for Fourier series) *Let* $\mathrm{f} \in \mathsf{L}^{(1)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{R})$*, let* $\mathrm{t}_0 \in \mathbb{R}$*, and suppose that the limits* $\mathrm{f}(\mathrm{t}_0-)$, $\mathrm{f}(\mathrm{t}_0+)$, $\mathrm{f}'(\mathrm{t}_0-)$*, and* $\mathrm{f}'(\mathrm{t}_0+)$ *exist. Denote* $\mathrm{j}(\mathrm{t}_0) = \mathrm{f}(\mathrm{t}_0+) - \mathrm{f}(\mathrm{t}_0-)$ *and denote*

$$\mathrm{I} = \int_0^{\pi} \frac{\sin \mathrm{t}}{\mathrm{t}}\,\mathrm{dt} \approx 1.85194.$$

The Gibbs set for f *at* t_0 *then satisfies*

$$\mathrm{G}(\mathrm{f},\mathrm{t}_0) = [\tfrac{1}{2}(\mathrm{f}(\mathrm{t}_0+) + \mathrm{f}(\mathrm{t}_0-)) - \tfrac{\Delta}{2}, \tfrac{1}{2}(\mathrm{f}(\mathrm{t}_0+) + \mathrm{f}(\mathrm{t}_0-)) + \tfrac{\Delta}{2}],$$

where

$$\Delta = \left|\frac{2\mathrm{I}\mathrm{j}(\mathrm{t}_0)}{\pi}\right| \approx 1.17898|\mathrm{j}(\mathrm{t}_0)|.$$

Proof Let us first prove the theorem in a special case, that for $f = g$ where g is the 2π-periodic extension of the signal $t \mapsto \frac{1}{2}(\pi - t)$ on $[0, 2\pi]$. The signal g has a jump of π at $t = 0$. One verifies by direct computation that

$$\mathrm{FS}[g](t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}.$$

From Lemma **??** in Example **??** and using Euler's formula $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, we have the identity

$$\sum_{j=0}^{n} \cos(nt) = \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}$$

Using this identity we have, for a positive Gibbs sequence $(t_n)_{n\in\mathbb{Z}_{>0}}$,

$$\begin{aligned}D^{\text{per}}_{2\pi,n}g(t_n) &= \sum_{j=1}^{n}\frac{\sin(jt_n)}{j} = \int_0^{t_n}\sum_{j=1}^{n}\cos(jt)\,\mathrm{d}t = \int_0^{t_n}\frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}\,\mathrm{d}t - \frac{t_n}{2}\\ &= \int_0^{t_n}\frac{\sin((n+\frac{1}{2})t)}{t}\,\mathrm{d}t + \int_0^{t_n}\Big(\frac{1}{\sin\frac{t}{2}} - \frac{1}{t}\Big)\sin((n+\tfrac{1}{2})t)\,\mathrm{d}t - \frac{t_n}{2}.\end{aligned}$$

As $n\to\infty$ the last two terms go to zero, the first of these by virtue of the Riemann–Lebesgue Lemma. Thus

$$\lim_{n\to\infty}D^{\text{per}}_{2\pi,n}g(t_n) = \lim_{n\to\infty}\int_0^{t_n}\frac{\sin((n+\frac{1}{2})t)}{t}\,\mathrm{d}t = \lim_{n\to\infty}\int_0^{(n+\frac{1}{2})t_n}\frac{\sin t}{t}\,\mathrm{d}t.$$

By the second property of a Gibbs sequence, the upper limit on the integral converges. Furthermore, by choosing this sequence appropriately, this upper limit can be arbitrarily specified. Thus, if we define

$$\phi(t) = \int_0^t\frac{\sin\tau}{\tau}\,\mathrm{d}\tau,$$

then

$$\mathrm{G}(g,0) = \bigcup_{t\in\mathbb{R}_{>0}}[-\phi(t),\phi(t)].$$

Using Theorem 3.2.16 one may check that the function ϕ has maxima at $t=(2k+1)\pi$ and minima at $2k\pi$, $k\in\mathbb{Z}_{>0}$. One may moreover check that the global maximum occurs at $t=\pi$ and one computes

$$\phi(\pi) = \int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t \approx 1.85194 > \frac{\pi}{2}.$$

This shows that we have $\mathrm{G}(g,0)=[-\phi(\pi),\phi(\pi)]$.

Now we consider a general signal f satisfying the hypotheses of the theorem with a jump discontinuity at t_0. We then define

$$h(t) = f(t) - \frac{j(t_0)}{\pi}g\big(\tfrac{2\pi}{T}(t-t_0)\big).$$

One has

$$\begin{aligned}h(t_0+) &= f(t_0+) - \frac{j(t_0)}{\pi}g(0+) = \tfrac{1}{2}(f(t_0+)+f(t_0-))\\ h(t_0-) &= f(t_0-) - \frac{j(t_0)}{\pi}g(0-) = \tfrac{1}{2}(f(t_0+)+f(t_0-)).\end{aligned}$$

Thus h is continuous at t_0 and the limits $h'(t_0+)$ and $h'(t_0-)$ exist. Therefore the Fourier series for h converges to h at t_0. Using this fact, if $(t_n)_{n\in\mathbb{Z}_{>0}}$ is a Gibbs sequence at t_0 for which $\lim_{n\to\infty}n(t_n-t_0)=\alpha$ then

$$\lim_{n\to\infty}D^{\text{per}}_{T,n}f(t_n) = \tfrac{1}{2}(f(t_0)+f(t_0-)) + \frac{j(t_0)}{\pi}\int_0^{\alpha}\frac{\sin t}{t}\,\mathrm{d}t.$$

Thus the Gibbs set is an interval of length $\left|\frac{2j(t_0)\phi(\pi)}{\pi}\right|$ with centre $\frac{1}{2}(f(t_0+)+f(t_0-))$. ■

The theorem describes the nature of the partial sums $D_{T,N}^{\mathrm{per}} f$ near a discontinuity of f as N becomes large. If one graphs these partial sums, their graph will exhibit some overshoot or undershoot of value of the signal. The graphs as N becomes large tend to approach shapes as exhibited in Figure 12.12. The figure depicts

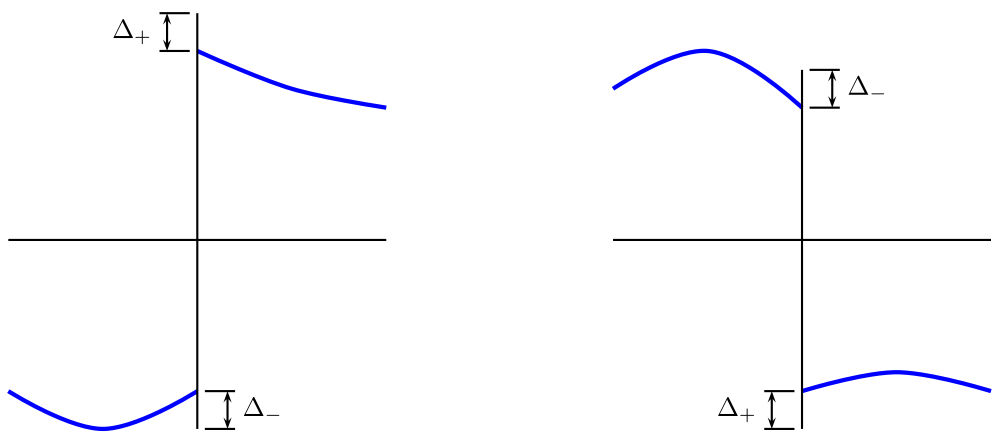


Figure 12.12 Limit of graphs for partial sums near points of discontinuity

qualitatively the overshoot and undershoot exhibited by the partial sums. The exact amount of these is what is given by Theorem 12.2.39. Referring to Figure 12.12, we have

$$\Delta_+ = \Delta_- = |j(t_0)|\Big(\frac{I}{\pi} - \frac{1}{2}\Big) \approx 0.0895|j(t_0)|.$$

Thus the error due to any finite approximation is about 9% of the jump as the approximation gets “better.” Note that what is depicted in Figure 12.12 is *not* the graph of the limit signal! In fact, it is not the graph of *any* single-valued signal. Indeed, it is interesting to read Gibbs’ short 1899 paper in *Nature* as regards this point. As the following excerpt suggests, the matter of uniform convergence caused as much confusion amongst practitioners in that day as it does with students today.

> I think this distinction important; for (with the exception of what relates to my unfortunate blunder described above), whatever differences of opinion have been expressed on this subject seem due, for the most part, to the fact that some writers have had in mind the *limit of the graphs*, and others the *graph of the limit* of the sum. A misunderstanding on this point is a natural consequence of the usage which allows us to omit the word *limit* in certain connections, as when we speak of the sum of an infinite series. [Emphases are Gibbs’.]

The distinction between the “limit of the graphs,” and the “graph of the limit” may be what is confusing you, if you are indeed confused by the notion of uniform convergence. The notion of the Gibbs set is designed to make precise the notion of the “limit of the graph.”

12.2.7 Cesàro summability

As we indicate in Section 12.2.3, the matter of convergence for Fourier series is a rather touchy matter for merely integrable signals, i.e., for signals in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Furthermore, in Example 12.2.10 we saw an example of a continuous signal whose Fourier series diverged. As we have stated all along, this makes the idea of the inversion of the CDFT via Fourier series a little tricky. In this section we investigate an alternate notion of inversion of the CDFT which uses an averaged version of the Fourier series. This notion comes with both advantages and disadvantages. The idea comes to us from the general notion of Cesàro convergence described in Section 6.4.5. The idea is that, given a sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ in a normed vector space $(\mathsf{V},\|\cdot\|)$, we consider, not convergence of the series $\sum_{j\in\mathbb{Z}_{>0}} v_j$, but of the sequence of averaged partial sums $(\bar{S}^1_n)_{n\in\mathbb{Z}_{>0}}$, where

$$\bar{S}^1_n = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{j} v_l.$$

In Theorem 6.4.15 we showed that if a series converges, then its sequence of averaged partial sums converges to the same limit. However, the converse is not necessarily true; see Example 6.4.14–**??**.

Let us apply the above general notion of Cesàro convergence of a series to the Fourier series. The following lemma gives the form for the partial Cesàro sums, just as Lemma 12.2.7 gives the partial sums for the Fourier series. Here we see the periodic Fejér kernel of Example 11.3.19–3 pop up again.

12.2.40 Lemma (Cesàro sums and the periodic Fejér kernel) *For* $\mathrm{f}\in\mathsf{L}^{(1)}_{\mathrm{per},\mathrm{T}}(\mathbb{R};\mathbb{C})$ *we have*

$$\frac{1}{\mathrm{N}}\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\frac{1}{\mathrm{T}}\sum_{\mathrm{j}=-\mathrm{n}}^{\mathrm{n}}\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{j}\mathrm{T}^{-1})\mathrm{e}^{2\pi\mathrm{i}\mathrm{j}\frac{\mathrm{t}}{\mathrm{T}}} = \frac{1}{\mathrm{T}}\int_{-\frac{\mathrm{T}}{2}}^{\frac{\mathrm{T}}{2}}\mathrm{f}(\mathrm{t}-\tau)\mathrm{F}^{\mathrm{per}}_{\mathrm{T,N}}(\tau)\,\mathrm{d}\tau.$$

Proof By Lemma 12.2.7, for $t\in[0,T]$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} D^{\mathrm{per}}_{T,n} f(t) &= \frac{1}{NT}\sum_{n=0}^{N-1}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)D^{\mathrm{per}}_{T,n}(\tau)\,\mathrm{d}\tau\\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)F^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau\\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\big(f(t-\tau)\big)F^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where we have used Lemma **??** from Example **??**–**??**. ■

Note that the lemma gives the Cesàro sums as the T-periodic convolution of f with $F^{\mathrm{per}}_{T,N}$ (see ***missing stuff***).

12.2.41 Notation ($\mathsf{F}^{\mathrm{per}}_{\mathsf{T,N}}\mathsf{f}$) Motivated by the above, for $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and for $N \in \mathbb{Z}_{>0}$ we shall from now on denote the Nth Cesàro sum by

$$F^{\mathrm{per}}_{T,N}f(t) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{T}\sum_{|j|\le n}\mathscr{F}_{\mathrm{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi \mathrm{i} j\frac{t}{T}}.$$

The notation is intended to be suggestive of convolution, just as we did with the periodic Dirichlet kernel in Notation 12.2.8. •

We may now state the main result in this section.

12.2.42 Theorem (Convergence of Cesàro sums) *For* $\mathsf{f} \in \mathsf{L}^{(0)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) if $\mathsf{f} \in \mathsf{L}^{(\mathrm{p})}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *then* $(\mathsf{F}^{\mathrm{per}}_{\mathrm{T,N}}\mathsf{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges to* f *in* $\mathsf{L}^{\mathrm{p}}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$*;*

(ii) if $\mathsf{f} \in \mathsf{C}^{0}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *then* $(\mathsf{F}^{\mathrm{per}}_{\mathrm{T,N}}\mathsf{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to* f*;*

(iii) if $\mathsf{f} \in \mathsf{L}^{(\infty)}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *and if, for* $\mathrm{t}_0 \in \mathbb{R}$*, the limits* $\mathrm{f}(\mathrm{t}_0-)$ *and* $\mathrm{f}(\mathrm{t}_0+)$ *exist then* $(\mathsf{F}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f}(\mathrm{t}_0))_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges to* $\frac{1}{2}(\mathrm{f}(\mathrm{t}_0-)+\mathrm{f}(\mathrm{t}_0+))$.

Proof This follows from Theorems 11.3.14, 11.3.15, and 11.3.16. ■

Note that the theorem tells us that the Cesàro partial sums for the Fourier series of Example 12.2.10 now converge uniformly, although the Fourier series diverges at $t = 0$. Let us further illustrate the advantages of using the Cesàro sums to reconstruct a signal from its CDFT through a simple example.

12.2.43 Example (Cesàro sums) We consider the signal $f(t) = \square_{2,1,0}(t) - 1$ that we have previously dealt with as concerns its convergence. In Figure 12.13 we plot the

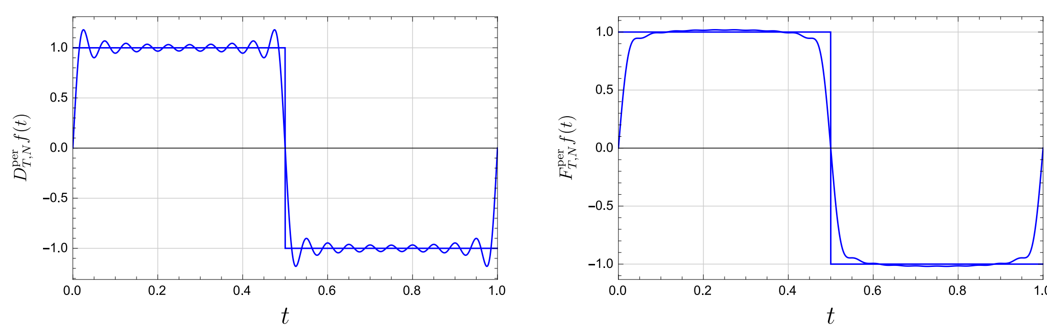


Figure 12.13 $D^{\mathrm{per}}_{T,N}f$ (left) and $F^{\mathrm{per}}_{T,N}f$ (right) for $N = 50$

partial Fourier series sums and the partial Cesàro sums. We note that the Cesàro sums do not exhibit the Gibbs effect at the discontinuity. •

12.2.44 Remarks (Pros and cons of using Cesàro sums)

1. Note that the Cesàro sums provide a genuine left-inverse for the CDFT on $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. That is to say, if we define $\mathscr{I}_{\mathrm{CD}}\colon \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ by
$$\mathscr{I}_{\mathrm{CD}}((F(nT^{-1}))_{n\in\mathbb{Z}}) = \lim_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{T}\sum_{|j|\le n} F(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} \tag{12.24}$$
then this map has the property that $\mathscr{I}_{\mathrm{CD}}\circ\mathscr{F}_{\mathrm{CD}}(f) = f$ for all $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Note that $\mathscr{I}_{\mathrm{CD}}$ as defined is not quite a left-inverse in the usual sense of the word. In particular, $\mathscr{I}_{\mathrm{CD}}((c_n)_{n\in\mathbb{Z}_{>0}})$ is not generally an element of $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ for $(c_n)_{n\in\mathbb{Z}} \in \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$. However, this is a minor problem since we can merely define $\mathscr{I}_{\mathrm{CD}}((c_n)_{n\in\mathbb{Z}}) = 0$ if $(c_n)_{n\in\mathbb{Z}} \notin \mathrm{image}(\mathscr{F}_{\mathrm{CD}})$. This would then give us a *bona fide* left-inverse for $\mathscr{F}_{\mathrm{CD}}$ in the set theoretic sense. The resulting map would not, however, be a linear map from $\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$ to $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. However, it is possible to massage $\mathscr{I}_{\mathrm{CD}}$ as defined by (12.24) in such a way that it is a *linear* left inverse. Note that does not mean that $\mathscr{I}_{\mathrm{CD}}\circ\mathscr{F}_{\mathrm{CD}}(f)(t) = f(t)$ for all $t \in \mathbb{R}$. However, if f is additionally continuous, then the pointwise equality of $\mathscr{I}_{\mathrm{CD}}\circ\mathscr{F}_{\mathrm{CD}}(f)$ with f does hold, and furthermore, the convergence is uniform.
2. It would seem, then, that the Cesàro sums would always be the right thing to use to reconstruct a signal from its CDFT. However, this is not unequivocally so, the reason being that it is frequently the case that convergence of the Cesàro sums is slower than that of the regular Fourier partial sums. This is something we will not get deeply into here, although it is of some importance in some areas of signal processing. •

12.2.8 The CDFT and approximate identities

The reader will have noticed the rôle played by convolution in our inversion of the CDFT, cf. Lemmata 12.2.7 and 12.2.40. A little more precisely, two of the ways in which we have attempted to invert the CDFT—by using Fourier series and by using the Cesàro sums of Fourier series—have turned out to involve approximations by convolution with an appropriate kernel. The kernels we used, the periodic Dirichlet and Fejér kernels, had the special feature of being *finite* linear combinations of harmonic functions. One can ask, however, whether the rôle played by these kernels can be played as well by other functions. Since we saw that the periodic Dirichlet and Fejér kernels arose in our study of periodic approximate identities (the latter kernel did define an approximate identity while the former did not), one may wonder what rôle might be played for the CDFT by general approximate identities. In this section we flesh this out.

12.2.45 Examples (The CDFT of approximate identities)

1. Recall from Example 11.3.19–3 the definition of the periodic Dirichlet kernel:
$$D^{\mathrm{per}}_{T,N}(t) = \begin{cases} \frac{\sin((2N+1)\pi\frac{t}{T})}{\sin(\pi\frac{t}{T})}, & t \notin \mathbb{Z}(T),\\ 2N+1, & t \in \mathbb{Z}(T).\end{cases}$$

Even though we showed in Example 11.3.19–5 that $(D_{T,N})_{N\in\mathbb{Z}_{>0}}$ is not an approximate identity, we shall consider its CDFT here, since it exhibits some of the properties of an approximate identity. By Lemma ?? from Example ?? below we have

$$D^{\mathrm{per}}_{T,N}(t) = \sum_{m=-N}^{N} \mathrm{e}^{2\pi\mathrm{i}m\frac{t}{T}}.$$

Therefore,

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{T,N})(nT^{-1}) &= \int_0^T D^{\mathrm{per}}_{T,N}(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\\ &= \sum_{m=-N}^{N}\int_0^T \mathrm{e}^{2\pi\mathrm{i}m\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \begin{cases} T, & |n|\le N,\\ 0, & \text{otherwise},\end{cases}\end{aligned}$$

the last integral being one that is easily explicit computed, cf. Lemma 12.3.2 below. Thus we have

$$\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{T,N})(nT^{-1}) = \begin{cases} T, & n\in\{-N,\dots,-1,0,1,\dots,N\},\\ 0, & \text{otherwise}.\end{cases}$$

We depict this in Figure 12.14 we show this signal and its CDFT when $T = 1$

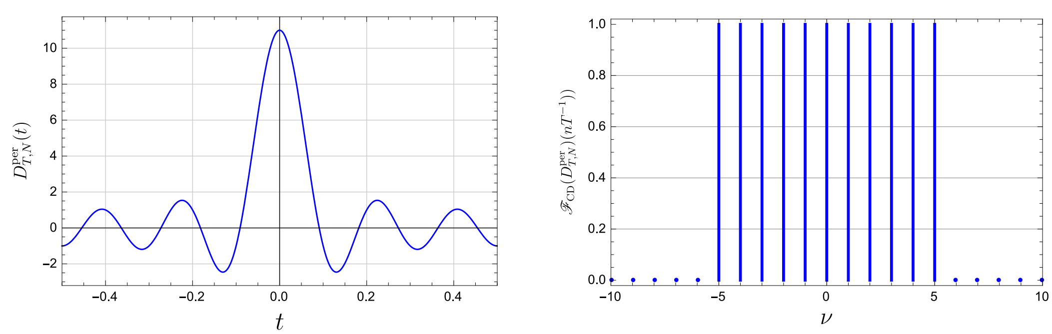


Figure 12.14 The signal $D^{\mathrm{per}}_{T,N}$ (left) and its CDFT (right) for $N = 5$ and $T = 1$

and $N = 5$.

2. We next consider the periodic Poisson kernel

$$P^{\mathrm{per}}_{T,\Omega}(t) = \frac{1-(\mathrm{e}^{-\frac{2\pi}{\Omega T}})^2}{1-2\mathrm{e}^{-\frac{2\pi}{\Omega T}}\cos(2\pi\frac{t}{T}) + (\mathrm{e}^{-\frac{2\pi}{\Omega T}})^2}.$$

from Example 11.3.19–??. As we saw in Example ??–1, we have

$$\mathscr{F}_{\mathrm{CD}}(P^{\mathrm{per}}_{T,\Omega})(nT^{-1}) = T\mathrm{e}^{-\frac{2\pi|n|}{\Omega T}}.$$

We depict the periodic Poisson kernel and its CDFT in Figure 12.15 when $T = 1$ and $\Omega = 5$.

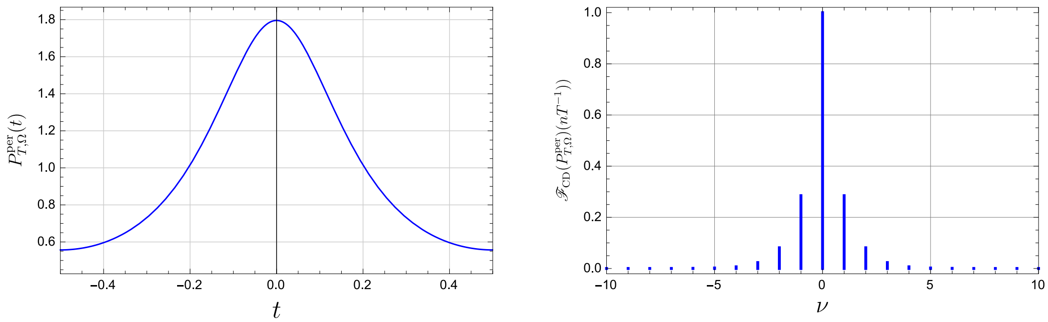


Figure 12.15 The signal $P^{\mathrm{per}}_{T,\Omega}$ (left) and its CDFT (right) for $\Omega = 5$ and $T = 1$

3. The periodic Gauss–Weierstrass kernel was given in Example **??–??** as being determined by the infinite series

$$\mathrm{per}_T(G_\Omega)(t) = \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{4\pi^2\Omega n^2}{T^2}\Big)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

As this series converges uniformly by the Weierstrass M-test, we may swap summation and integration using Theorem 3.5.23 to obtain

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(\mathrm{per}_T(G_\Omega))(nT^{-1}) &= \int_0^T \Big(\sum_{m\in\mathbb{Z}} \exp\Big(-\frac{4\pi^2\Omega m^2}{T^2}\Big)\mathrm{e}^{2\pi\mathrm{i}m\frac{t}{T}}\Big)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}, \mathrm{d}t\\ &= T\exp\Big(-\frac{4\pi^2\Omega n^2}{T^2}\Big),\end{aligned}$$

using the computation of the integral from our determination of $\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{T,N})$. In Figure 12.16 we plot the periodic Gauss–Weierstrass kernel and its CDFT for

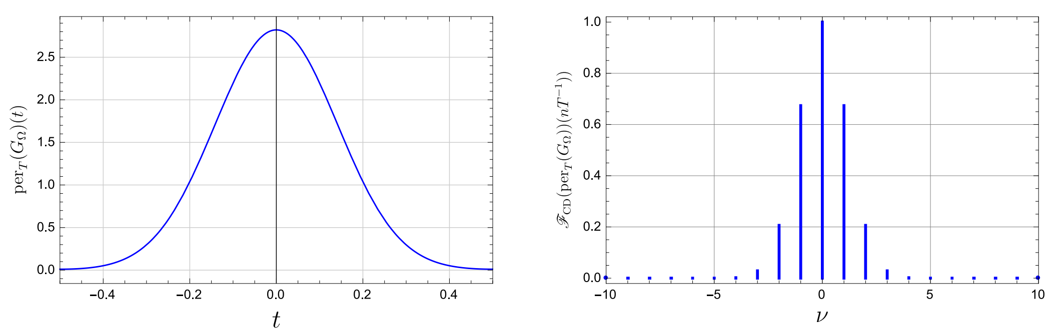


Figure 12.16 The signal $\mathrm{per}_T(G_\Omega)$ (left) and its CDFT (right) for $\Omega = \frac{1}{100}$ and $T = 1$

$T = 1$ and $\Omega = \frac{1}{100}$.

4. Next we consider the CDFT for the periodic Fejér kernel introduced in Example 11.3.19–3:

$$F^{\mathrm{per}}_{T,N}(t) = \begin{cases} \frac{1}{N}\frac{\sin^2(N\pi\frac{t}{T})}{\sin^2(\pi\frac{t}{T})}, & t \notin \mathbb{Z}(T), \\ N, & t \in \mathbb{Z}(T). \end{cases}$$

By Lemmata **??** and **??** from Example **??**–**??** we have

$$F^{\mathrm{per}}_{T,N}(t) = 1 + \sum_{n=-N+1}^{-1}\Big(1 + \frac{n}{N}\Big)e^{2\pi i n\frac{t}{T}} + \sum_{n=1}^{N-1}\Big(1 - \frac{n}{N}\Big)e^{2\pi i n\frac{t}{T}}.$$

As in our computation of $\mathscr{F}_{\mathrm{CD}}(D^{\mathrm{per}}_{N,T})$, we can integrate this expression term-by-term when computing the CDFT to obtain

$$\mathscr{F}_{\mathrm{CD}}(F^{\mathrm{per}}_{T,N})(nT^{-1}) = T\begin{cases} 1 - \frac{|n|}{N}, & |n| \in \{0,1,\dots,(N-1)\}, \\ 0, & \text{otherwise}. \end{cases}$$

We depict $F^{\mathrm{per}}_{T,N}$ and its CDFT in Figure 12.17 when $T = 1$ and $N = 5$.

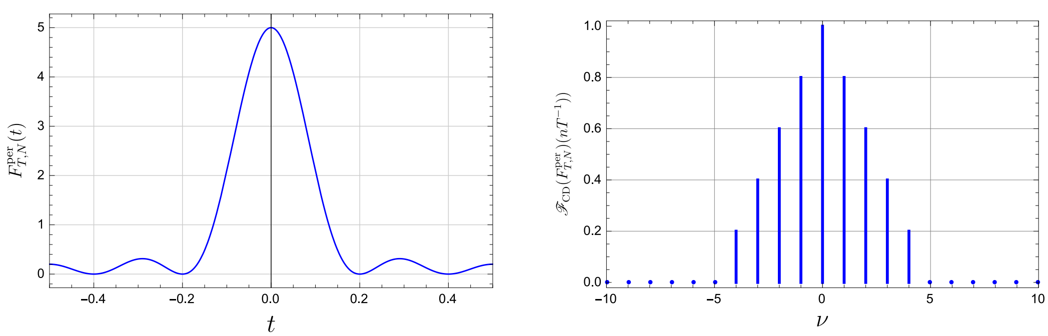


Figure 12.17 The signal $F^{\mathrm{per}}_{T,N}$ (left) and its CDFT (right) for $N = 5$ and $T = 1$

5. Next we determine the CDFT of the periodic de la Vallée Poussin kernel

$$V^{\mathrm{per}}_{T,N}(t) = 2F^{\mathrm{per}}_{T,N}(t) - F^{\mathrm{per}}_{T,N}(t).$$

As we have just computed the CDFT of the Fejér kernel, we can use linearity of the CDFT to compute

$$\mathscr{F}_{\mathrm{CD}}(V^{\mathrm{per}}_{T,N}(nT^{-1}) = T\begin{cases} 1, & |n| \in \{0,1,\dots,N-1\}, \\ 2 - \frac{|n|}{N}, & |n| \in \{N, N+1,\dots,2N-1\}, \\ 0, & |n| \geq 2N. \end{cases}$$

We depict $V^{\mathrm{per}}_{T,N}$ and its CDFT in 12.18 when $T = 1$ and $N = 5$. •

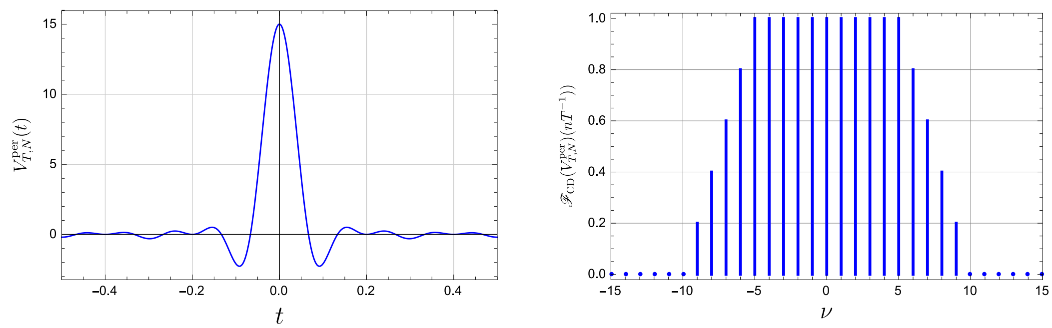


Figure 12.18 The signal $V_{T,N}^{\mathrm{per}}$ (left) and its CDFT (right) for $N = 5$ and $T = 1$

12.2.9 Notes

Theorem 12.2.28 was published by **JPGLD:29** and was the first convergence theorem for Fourier series. Thus Dirichlet's test was the first vindication of Fourier's idea of writing a periodic function as an infinite sum of harmonic functions.

In Example 12.2.10 we give an example of a continuous signal with a Fourier series that diverges at a point. The first such example was given by **PdBR:76**,[2] after which time many such example were produced. The example we give is from [**GHH/WWR:44**]. Our treatment of sets of divergence follows the excellent presentation of **YK:04**.

Theorem 12.2.20 is due to **ANK:23**. This result was improved by **ANK:26** to show that there exists $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ whose Fourier series diverges everywhere, not just almost everywhere.

Gibbs' phenomenon was reported by **JWG:99**[3] in response to experimental phenomenon reported by the physicist Albert Abraham Michelson (1852–1931). Although Gibbs gets the credit here, the essential ideas were noticed as early as 1848 by a Cambridge mathematician Henry Wilbraham. A discussion of such matters may be found in [**HSC:30**]. The general case presented in Theorem 12.2.39 was worked out by **MB:06**.[4]

The use of the Fejér kernel in the study of Fourier series was introduced by **LF:04** where uniform convergence of the Cesàro means were shown for continuous signals. One can define alternate forms of Cesàro summability. An extensive discussion of these issues for Fourier series may be found in the treatise of **AZ:59**.

A rigorous discussion of the tradeoffs encountered in using Cesàro sums may

[2]Paul David Gustav du Bois-Reymond (1831–1889) was a German mathematician who made to mathematical analysis.

[3]Josiah Willard Gibbs (1839–1903) was an American mathematical physicist. In the realm of mathematics, his most significant contribution is the Fourier series phenomenon bearing his name. In physics, he is also known for his work in the area of thermodynamics where his name appears by virtue of Gibbs' free energy.

[4]Maxime Bôcher (1867–1918) was an American mathematician who made mathematical contributions to linear algebra, differential equations, and analysis.

be found in [**MAP:09**].

Exercises

12.2.1 Let $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Show that for every $N \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned}&\frac{1}{T}\sum_{n=-N}^{N}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\\&=\frac{1}{2T}\mathscr{C}_{\mathrm{CD}}(f)(0)+\frac{1}{T}\sum_{n=1}^{N}\Big(\mathscr{C}_{\mathrm{CD}}(f)(nT^{-1})\cos(2\pi n\tfrac{t}{T})+\mathscr{S}_{\mathrm{CD}}(f)(nT^{-1})\sin(2\pi n\tfrac{t}{T})\Big).\end{aligned}$$

12.2.2 Let $f\colon [0,1] \to \mathbb{R}$ given by $f(t) = t$.

(a) Using Exercise 12.1.4, compute the CDCT and the CDST f_{even} and f_{odd}.

(b) Plot the first few partial sums for both f_{even} and f_{odd} and comment on their relative merits.

Our definition of partial sums for Fourier series is symmetric about zero, i.e., we take the Nth partial sum to be the terms $-N,-N+1,\dots,-1,0,1,\dots,N-1,N$ in the series. In the following exercise you will explore one of the consequences of this somewhat arbitrary choice of definition for the partial sums.

12.2.3 Let $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and, for $t_0 \in \mathbb{R}$ and $s \in \mathbb{C}$, define $e_{f,t_0,s}\colon \mathbb{R} \to \mathbb{C}$ by

$$e_{f,t_0,s}(t) = \tfrac{1}{2}(f(t_0+t)+f(t_0-t)) - s.$$

(a) Show that $e_{f,t_0,s} \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

(b) Show that the Fourier series for f converges to $s \in \mathbb{C}$ at t_0 if and only if the Fourier series for $e_{f,t_0,s}$ converges to 0 at 0.

(c) Show that, if there exists a neighbourhood U of 0 for which $f(t_0+t) = -f(t_0-t)$ for every $t \in U$, then it holds that the Fourier series for f converges to zero at t_0.

(d) Sketch the graph of a typical function from part (c).

12.2.4 Answer the following questions.

(a) Is the function

$$n \mapsto \frac{(-1)^{n+1}}{2n-1}$$

in $\ell^1(\mathbb{Z}_{>0};\mathbb{R})$?

(b) Show that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$

Hint: *Use Example 12.1.3–2 and Theorem 12.2.28.*

12.2.5 Answer the following questions.

(a) Is the function

$$n \mapsto \frac{1}{(2n-1)^2}$$

in $\ell^1(\mathbb{Z}_{>0};\mathbb{R})$?

(b) Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Hint: *Use Example 12.1.3–3 and Theorem 12.2.28.*

12.2.6 Give a signal $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ such that $\mathscr{F}_{\mathrm{CD}}(f) \notin \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$. Explain why your example works without doing any computations.

12.2.7 For the 1-periodic extensions f_{per} of the following signals $f \in \mathsf{C}^0([0,1];\mathbb{R})$, do the following:

1. sketch one period of the graph of f_{per};
2. determine, without computation, which (if any) of the terms $\mathscr{C}_{\mathrm{CD}}(f)(n)$, $n \in \mathbb{Z}_{\geq 0}$, and $\mathscr{S}_{\mathrm{CD}}(f)(n)$, $n \in \mathbb{Z}_{>0}$, are zero;
3. indicate whether the sequence $\{D^{\mathrm{per}}_{T,N} f\}_{N\in\mathbb{Z}_{>0}}$ converges pointwise, and if it does, to which signal does it pointwise converge;
4. indicate whether the sequence $\{D^{\mathrm{per}}_{T,N} f\}_{N\in\mathbb{Z}_{>0}}$ converges uniformly, and if it does, to which signal does it uniformly converge.

The signals are:

(a) $f(t) = t(1-t)$;

(b) $f(t) = \begin{cases} 1, & t \in [0,\frac{1}{2}], \\ 0, & t \in (\frac{1}{2},1]; \end{cases}$

(c) $f(t) = \mathrm{e}^t$.

12.2.8 Define $f, g\colon [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \sqrt{\frac{1}{2}-t}, & t \in [0,\frac{1}{2}], \\ \sqrt{t-\frac{1}{2}}, & t \in (\frac{1}{2},1], \end{cases}$$

$$g(t) = \begin{cases} \sqrt{\frac{1}{2}-t}, & t \in [0,\frac{1}{2}], \\ -\sqrt{t-\frac{1}{2}}, & t \in (\frac{1}{2},1], \end{cases}$$

and let $f_{\mathrm{per}}, g_{\mathrm{per}}\colon \mathbb{R} \to \mathbb{R}$ be the signals of period 1 obtained by periodically extending f and g, respectively. For each of the signals f and g, answer the following questions.

(a) Sketch the graph of the signal on the interval $[0,1]$.

(b) If possible determine the points at which the Fourier series converges, and indicate to what value it converges.

(c) If possible using results from the course, determine whether the Fourier series converges uniformly, and if so to what signal.

12.2.9 For the given signals with their CDFT's, do the following:

1. plot one fundamental period of the signal;
2. plot the 10th partial sum for the Fourier series and comment on the quality of the approximation;
3. indicate whether the Fourier series converges pointwise, and if so indicate to what signal it converges;
4. indicate whether the Fourier series converges uniformly, and if so indicate to what signal it converges.

The signals, defined for a single fundamental period, and their CDFT's are:

(a) $f(t) = \begin{cases} 0, & t \in [0,1], \\ 1, & t \in (1,2], \\ 0, & t \in (2,3], \end{cases}$
$\mathscr{F}_{\mathrm{CD}}(f)(0) = 1,$
$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \frac{3\mathrm{i}}{2n\pi}\left(\mathrm{e}^{-\mathrm{i}n\pi\frac{4}{3}} - \mathrm{e}^{-\mathrm{i}n\pi\frac{2}{3}}\right), n \in \mathbb{Z} \setminus \{0\};$

(b) $f(t) = \begin{cases} \mathrm{e}^t - 1, & t \in [0,1], \\ (1-e)t + 2(e-1), & t \in (1,2], \end{cases}$
$a_0(f) = 3\mathrm{e} - 5,$
$\mathscr{C}_{\mathrm{CD}}(f)(nT^{-1}) = 2\frac{1-(-1)^n-(-1)^n n^2\pi^2+\mathrm{e}\left(-1+(-1)^n+\left(-1+2(-1)^n\right)n^2\pi^2\right)}{n^2\pi^2+n^4\pi^4}, n \in \mathbb{Z}_{>0},$
$\mathscr{S}_{\mathrm{CD}}(f)(nT^{-1}) = 2\frac{-1+(-1)^n\mathrm{e}}{n\pi+n^3\pi^3}, n \in \mathbb{Z}_{>0};$

(c) $f(t) = \begin{cases} t, & t \in [0,\pi], \\ 0, & t \in (\pi,2\pi], \end{cases}$
$\mathscr{F}_{\mathrm{CD}}(f)(0) = \frac{\pi^2}{2},$
$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \frac{(-1)^n-1}{n^2} + \pi\mathrm{i}\frac{(-1)^n}{n}, n \in \mathbb{Z} \setminus \{0\};$

(d) $f(t) = \begin{cases} -t^2, & t \in [0,\pi], \\ \pi t - 2\pi^2, & t \in (\pi,2\pi], \end{cases}$
$\mathscr{F}_{\mathrm{CD}}(f)(0) = -\frac{10\pi^3}{12},$
$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1} = \pi\frac{(1-3(-1)^n)}{n^2} + 2\mathrm{i}\frac{(-1)^n-1}{n^3}, n \in \mathbb{Z} \setminus \{0\}.$

In the following exercise you will verify the famous Gibbs phenomenon for the square wave.

12.2.10 Let $f\colon [0,2\pi] \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 1, & t \in [0,\pi], \\ -1, & t \in (\pi,2\pi]. \end{cases}$$

(a) Show that

$$\mathrm{FS}[f](t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)t)}{2n-1}.$$

(b) Does the Fourier series converge pointwise? Uniformly?

Let

$$f_N(t) = D^{\mathrm{per}}_{2\pi,N} f(t) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin((2n-1)t)}{2n-1}$$

be the Nth partial sum.

(c) Show that $\pi \sin t f_N'(t) = 2\sin(2Nt)$.
Hint: *Directly differentiate* $\mathrm{D}^{\mathrm{per}}_{2\pi,\mathrm{N}}\mathrm{f}$, *and use mathematical induction with some trig identities.*

(d) Show that the maximum value of $D^{\mathrm{per}}_{2\pi,N} f$ on the interval $[0,\pi]$ is

$$\frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{2n-1}\sin\Big(\frac{(2n-1)\pi}{2N}\Big). \tag{12.25}$$

(e) Show that the sum (12.25) is the approximation of the integral

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t$$

by N rectangles of equal width.

(f) Take the limit as $N \to \infty$ to show that the maximum value of $D^{\mathrm{per}}_{2\pi,N} f$ on the interval $[0,\pi]$ approaches

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t$$

as $N \to \infty$.

(g) Evaluate or look up the value of the integral to obtain the maximum value of $D^{\mathrm{per}}_{2\pi,N} f$ on the interval $[0,\pi]$ as $N \to \infty$.

(h) How does this reflect on uniform convergence of the Fourier series.

(i) Plot a few partial sums to check your analysis.

12.2.11 Use Bôcher's theorem, Theorem 12.2.39, to draw the limit of the graph of the Fourier series for the following signals. Make sure that you assign numbers to appropriate bits of the graph.

(a) $f(t) = \begin{cases} t, & t \in [0,\frac{1}{2}], \\ 1-t, & t \in (\frac{1}{2},1]. \end{cases}$

(b) $f(t) = \begin{cases} 1, & t \in [0,\frac{1}{2}], \\ -1, & t \in (\frac{1}{2},1]. \end{cases}$

(c) $f(t) = \begin{cases} t, & t \in [0,\pi], \\ t-\pi, & t \in (\pi,2\pi]. \end{cases}$

(d) $f(t) = \mathrm{e}^t$, $t \in [0,1]$.

Section 12.3

The L^2-CDFT

Since $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \subseteq \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ by part (iv) of Theorem 8.3.11, we may define the CDFT on $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ simply by restriction; we call this the **L^2-*CDFT***. Thus, everything we have said thus far can be applied in particular to $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. However, the L^2-CDFT has many interesting properties not possessed by the more general L^1-CDFT. Many of these properties are a consequence of the Hilbert space structure of $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, whereas $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is a more general Banach space. One of the useful features present in Hilbert spaces that is not present in a general Banach space is the existence of Hilbert bases which give series representations of all elements in the Hilbert space. If the world is a sane place then, given our results in Section 12.2, it ought to hold that the harmonic signals are a Hilbert basis (after normalisation). We show in Section 12.3.1 that this is indeed the case. For the remainder of the section we explore some of the additional structure present for the CDFT that arises from the additional structure of $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

While the additional structure added to the CDFT by assuming signals to be in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ may seem to be a little esoteric, in fact, the L^2-CDFT is often what one normally studies. Indeed, it is very often the case that the fact that the natural domain of definition for the CDFT is $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is ignored, and it is assumed that the only signals of interest are in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Thus the theory of Fourier series is often presented as an L^2-theory, which seems a little unnatural if one thinks about it for a moment. However, what *is* true is that for the purposes of applications, there are precious few interesting signals in $\mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \setminus \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, and so the simplification to $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is justified on these grounds.

Do I need to read this section? The L^2-theory of the CDFT is central, so if you are reading this chapter, then read this section. •

12.3.1 The Hilbert basis of harmonic signals

We know from Theorem 6.7.56 that $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is a $\mathbb{C}$-Hilbert space, and from Proposition 6.7.58 that it is separable. From Theorem 7.3.21 we can then assert the existence of a countable Hilbert basis for $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. In this section we show that we already have at hand a Hilbert basis: the collection of harmonic signals.

For the purposes of this section it is convenient to define, for $a \in \mathbb{C}$, the signal $\mathsf{E}_a\colon \mathbb{R} \to \mathbb{C}$ by $\mathsf{E}_a(t) = \mathrm{e}^{at}$. For $a \in \mathbb{R}$ let us also define $\mathsf{C}_a, \mathsf{S}_a\colon \mathbb{R} \to \mathbb{R}$ by $\mathsf{C}_a(t) = \cos(at)$ and $\mathsf{S}_a(t) = \sin(at)$. Note that C_0 is the constant function $t \mapsto 1$. For signals in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the CDFT is expressible in terms of the L^2-inner product

$$\langle f, g\rangle_2 = \int_0^T f(t)\bar{g}(t)\,\mathrm{d}t.$$

We have the following obvious result.

12.3.1 Lemma (The CDFT using inner products) *For* $f \in \mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *we have*

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) &= \langle f, \mathsf{E}_{2\pi i n T^{-1}}\rangle_2, && n \in \mathbb{Z},\\ \mathscr{C}_{\mathrm{CD}}(f)(nT^{-1}) &= \langle f, \mathsf{C}_{2\pi n T^{-1}}\rangle_2, && n \in \mathbb{Z}_{\geq 0},\\ \mathscr{S}_{\mathrm{CD}}(f)(nT^{-1}) &= \langle f, \mathsf{S}_{2\pi n T^{-1}}\rangle_2, && n \in \mathbb{Z}_{>0}.\end{aligned}$$

While the previous result is obvious, note that it does not make sense for $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. However, its neat connection to Hilbert space geometry explains why many authors are lured into making this their definition of the CDFT (or CDCT and CDST), thus immediately excluding any discussion of the L^1-CDFT.

Let us record the following facts about the sets of harmonic signals.

12.3.2 Lemma (Orthogonality of harmonics) *The sets*

$$\{\mathsf{E}_{2\pi i n T^{-1}}\}_{n\in\mathbb{Z}}, \qquad \{\mathsf{C}_{2\pi n T^{-1}}\}_{n\in\mathbb{Z}_{\geq 0}} \cup \{\mathsf{S}_{2\pi n T^{-1}}\}_{n\in\mathbb{Z}_{>0}}$$

are orthogonal in $(\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R},\mathbb{C}), \|\cdot\|_2)$. *Moreover, the sets*

$$\left\{\tfrac{1}{\sqrt{T}}\mathsf{E}_{2\pi i n T^{-1}}\right\}_{n\in\mathbb{Z}}, \qquad \left\{\tfrac{1}{\sqrt{T}}\right\} \cup \left\{\sqrt{\tfrac{2}{T}}\mathsf{C}_{2\pi n T^{-1}}\right\}_{n\in\mathbb{Z}_{>0}} \cup \left\{\sqrt{\tfrac{2}{T}}\mathsf{S}_{2\pi n T^{-1}}\right\}_{n\in\mathbb{Z}_{>0}}$$

are orthonormal.

Proof This follows from the following computations:

$$\begin{aligned}\int_0^T e^{2\pi i m \frac{t}{T}} e^{-2\pi i n \frac{t}{T}}\,dt &= \begin{cases} T, & m = n,\\ 0, & m \neq n,\end{cases}\\ \int_0^T \cos(2\pi m \tfrac{t}{T})\cos(2\pi n \tfrac{t}{T})\,dt &= \begin{cases} \frac{T}{2}, & m = n,\\ 0, & m \neq n,\end{cases}\\ \int_0^T \sin(2\pi m \tfrac{t}{T})\sin(2\pi n \tfrac{t}{T})\,dt &= \begin{cases} \frac{T}{2}, & m = n,\\ 0, & m \neq n,\end{cases}\\ \int_0^T \cos(2\pi m \tfrac{t}{T})\sin(2\pi n \tfrac{t}{T})\,dt &= 0,\end{aligned}$$

for $m, n \in \mathbb{Z}_{\geq 0}$. ■

From the preceding two lemmata we have that the Fourier series for $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ can be written as

$$\mathrm{FS}[f] = \sum_{n\in\mathbb{Z}_{>0}} \left\langle f, \tfrac{1}{\sqrt{T}}\mathsf{E}_{2\pi i n T^{-1}}\right\rangle_2 \tfrac{1}{\sqrt{T}}\mathsf{E}_{2\pi i n T^{-1}}$$

or

$$\mathrm{FS}[f] = \left\langle f, \tfrac{1}{\sqrt{T}}\mathsf{C}_0\right\rangle_2 \tfrac{1}{\sqrt{T}}\mathsf{C}_0 + \sum_{n=1}^{\infty}\left\langle f, \tfrac{2}{\sqrt{T}}\mathsf{C}_{2\pi nT^{-1}}\right\rangle_2 \tfrac{2}{\sqrt{T}}\mathsf{C}_{2\pi nT^{-1}} + \sum_{n=1}^{\infty}\left\langle f, \tfrac{2}{\sqrt{T}}\mathsf{S}_{2\pi nT^{-1}}\right\rangle_2 \tfrac{2}{\sqrt{T}}\mathsf{S}_{2\pi nT^{-1}}.$$

We shall stick primarily to the first of these for our general discussion since it is simpler to manage notationally. The main point is that the Fourier series looks like one is writing f as an orthonormal expansion using the orthonormal set $\{\frac{1}{\sqrt{T}}\mathsf{E}_{2\pi i nT^{-1}}\}_{n\in\mathbb{Z}}$. The interesting question is then, "Does this series converge in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$?" As we have seen in Section 7.3, this question tantamount to asking whether $\{\frac{1}{\sqrt{T}}\mathsf{E}_{2\pi i nT^{-1}}\}_{n\in\mathbb{Z}}$ is a Hilbert basis.

That this is in fact the case is the next result.

12.3.3 Theorem (Harmonic signals form a Hilbert basis for $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$) *In the Hilbert space $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the set of signals $\{\frac{1}{\sqrt{T}}\mathsf{E}_{2\pi i nT^{-1}}\}_{n\in\mathbb{Z}}$ is a Hilbert basis.*

Proof We let $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and prove the theorem by showing that there exists a sequence $(p_j)_{j\in\mathbb{Z}}$ in $\mathrm{span}_{\mathbb{C}}(\mathsf{E}_{2\pi i nT^{-1}})$ which converges to f in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$:

$$\lim_{j\to\infty}\|f - p_j\|_2 = 0. \tag{12.26}$$

Let us perform some simple preliminary computations. By Bessel's inequality, Theorem 7.3.24, we have

$$\sum_{n\in\mathbb{Z}}\left|\left\langle f, \tfrac{1}{\sqrt{T}}\mathsf{E}_{2\pi i n}\right\rangle\right|_2^2 = \frac{1}{T}\sum_{n\in\mathbb{Z}}\left|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\right|^2 \le \|f\|_2^2. \tag{12.27}$$

Thus, f being in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, the series in the middle converges. Now, as we saw in the proof of Theorem 7.3.24 (among other places), the signals f_N and $f - f_N$ are orthogonal. The Pythagorean identity (Exercise 7.1.12) then gives

$$\sum_{|n|\le N}|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})|^2 + \int_0^T |f(t) - f_N(t)|^2\,\mathrm{d}t = \int_0^T |f(t)|^2\,\mathrm{d}t. \tag{12.28}$$

Now suppose for the moment that f is continuous and define $\phi\colon \mathbb{R}\to\mathbb{R}$ by

$$\phi(t) = \int_0^T f(t+\tau)\bar{f}(\tau)\,\mathrm{d}\tau.$$

Since f has period T, so too does ϕ. What's more, ϕ is locally absolutely continuous by Theorem 5.9.31, and so continuous by Proposition 5.9.24. Let us compute the Fourier coefficients of ϕ:

$$\begin{aligned}
\mathscr{F}_{\mathrm{CD}}(\phi)(nT^{-1}) &= \int_0^T \phi(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\
&= \int_0^T \Big(\int_0^T f(t+\tau)\bar{f}(\tau)\,\mathrm{d}\tau\Big)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t \\
&= \int_0^T \bar{f}(\tau)\Big(\int_0^T f(t+\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\Big)\,\mathrm{d}\tau \\
&= \int_0^T \bar{f}(\tau)\Big(\int_\tau^{\tau+T} f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\Big)\mathrm{e}^{2\mathrm{i}n\pi\frac{\tau}{T}}\,\mathrm{d}\tau \\
&= \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\int_0^T \bar{f}(\tau)\mathrm{e}^{2\mathrm{i}n\pi\frac{\tau}{T}}\,\mathrm{d}\tau \\
&= \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\overline{\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t} \\
&= |\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})|^2,
\end{aligned}$$

where we have used Fubini's Theorem. By (12.27) this implies that the Fourier coefficients of ϕ must have the property that

$$\sum_{n\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}}(\phi)(nT^{-1})| < \infty,$$

i.e., $\mathscr{F}_{\mathrm{CD}}(\phi) \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$. By Theorem 12.2.33 it follows that $\mathrm{FS}[\phi] = \psi$ for some continuous function ψ. Moreover, also by Theorem 12.2.33, $\phi(t) = \psi(t)$ for almost every $t \in \mathbb{R}$. By Exercise 5.9.8 we can then conclude that $\psi = \phi$. This shows that for $t \in \mathbb{R}$ we have

$$\phi(t) = \int_0^T f(t+\tau)\bar{f}(\tau)\,\mathrm{d}\tau = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(\phi)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

In the case when $t = 0$ this reads

$$\int_0^T |f(\tau)|^2\,\mathrm{d}\tau = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(\phi)(nT^{-1}) = \sum_{n\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})|^2.$$

Applying this equality to (12.28) gives

$$\lim_{N\to\infty} \int |f(t) - f_N(t)|^2\,\mathrm{d}t = 0,$$

giving (12.26) in the case when f is continuous (take the sequence $(p_j)_{j\in\mathbb{Z}_{>0}}$ to be the sequence of partial sums for $\mathrm{FS}[f]$).

If $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is not necessarily continuous, then by part (ii) of Theorem 8.3.11, for any $\epsilon \in \mathbb{R}_{>0}$ there exists a continuous signal $g_\epsilon \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ with the property that $\|f - g_\epsilon\|_2 < \frac{\epsilon}{2}$. Let $g_{\epsilon,j}$ be the jth partial sum for the Fourier series of g_ϵ. For any $\epsilon \in \mathbb{R}_{>0}$ there exists $N_\epsilon \in \mathbb{Z}_{>0}$ so that $\|g_\epsilon - g_{\epsilon,j}\|_2 < \frac{\epsilon}{2}$ provided that $j \geq N_\epsilon$. Thus, by the triangle inequality

$$\begin{aligned}
\|f - g_{j,\epsilon}\|_2 &= \|(f - g_\epsilon) + (g_\epsilon - g_{\epsilon,j})\|_2 \\
&\leq \|f - g_\epsilon\|_2 + \|g_\epsilon - g_{\epsilon,j}\|_2 = \epsilon,
\end{aligned}$$

for any $j \geq N_\epsilon$. Taking $p_j = g_{j^{-1},N_j}$ gives

$$\lim_{j\to\infty} \|f - p_j\|_2 = 0.$$

Thus this shows that the collection of finite sums of $\{\mathsf{E}_{2\pi i n T^{-1}}\}_{n\in\mathbb{Z}}$ are dense in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, implying that the signals $\{\frac{1}{\sqrt{T}}\mathsf{E}_{2\pi i n T^{-1}}\}_{n\in\mathbb{Z}}$ are a Hilbert basis by virtue of Theorem 7.3.25. ■

Since the theorem holds, so too do all the equivalent conditions of Theorem 7.3.25. Let us record these here for convenience.

12.3.4 Corollary (Further properties of the harmonic basis) *The following equivalent statements hold:*

(i) $\mathrm{cl}(\mathrm{span}_{\mathbb{C}}(\{\mathsf{E}_{2\pi\mathrm{inT}^{-1}}\}_{\mathrm{n}\in\mathbb{Z}})) = \mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$;

(ii) for all $\mathrm{f} \in \mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *we have*

$$\frac{1}{\mathrm{T}}\sum_{\mathrm{n}\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})|^2 = \int_0^{\mathrm{T}} |\mathrm{f}(\mathrm{t})|^2\,\mathrm{dt}$$

*(**Parseval's equality**);*

(iii) for all $\mathrm{f},\mathrm{g} \in \mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *we have*

$$\frac{1}{\mathrm{T}}\sum_{\mathrm{n}\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})\overline{\mathscr{F}_{\mathrm{CD}}(\mathrm{g})(\mathrm{nT}^{-1})} = \int_0^{\mathrm{T}} \mathrm{f}(\mathrm{t})\bar{\mathrm{g}}(\mathrm{t})\,\mathrm{dt};$$

(iv) $\{\mathsf{E}_{2\pi\mathrm{inT}^{-1}}\}^{\perp}_{\mathrm{n}\in\mathbb{Z}} = \{0\}$;

(v) if $\mathscr{B}$ *is any orthonormal set in* $\mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *containing* $\left\{\frac{1}{\sqrt{\mathrm{T}}}\mathsf{E}_{2\pi\mathrm{inT}^{-1}}\right\}_{\mathrm{n}\in\mathbb{Z}}$*, then* $\mathscr{B} = \left\{\frac{1}{\sqrt{\mathrm{T}}}\mathsf{E}_{2\pi\mathrm{inT}^{-1}}\right\}_{\mathrm{n}\in\mathbb{Z}}$.

From Theorem 7.1.26, along with Theorem 12.3.3, now follows this result.

12.3.5 Corollary (Partial Fourier sums minimise distance) *If* $\mathrm{f} \in \mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *then the unique point* $\mathrm{f}_{\mathrm{N}} \in \mathrm{span}_{\mathbb{C}}(\mathsf{E}_{2\pi\mathrm{inT}^{-1}})_{|\mathrm{n}|\leq\mathrm{N}}$ *for which the distance* $\|\mathrm{f}-\mathrm{f}_{\mathrm{N}}\|_2$ *is minimised is the partial sum*

$$\mathrm{f}_{\mathrm{N}}(\mathrm{t}) = \frac{1}{\mathrm{T}}\sum_{|\mathrm{n}|\leq\mathrm{N}} \mathscr{F}_{\mathrm{CD}}(\mathrm{f})(\mathrm{nT}^{-1})\mathrm{e}^{2\pi\mathrm{in}\frac{\mathrm{t}}{\mathrm{T}}}.$$

Furthermore, $\lim_{\mathrm{N}\to\infty}\|\mathrm{f}-\mathrm{f}_{\mathrm{N}}\|_2 = 0$.

This result gives geometric meaning to our choice of Fourier coefficients. No matter what the signal is, as long as it is in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, the partial sum f_N is the closest point in $\mathrm{span}_{\mathbb{C}}(\mathsf{E}_{2\pi i n T^{-1}})_{|n|\leq N}$ to f. This is, you must agree, a nifty geometric interpretation.

Another neat consequence of Theorem 12.3.3 is the following property of the CDFT.

12.3.6 Corollary (The L^2-CDFT as a continuous map) *If* $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *then* $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$. *Moreover,* $\mathscr{F}_{\mathrm{CD}}|\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R}\mathbb{C})\colon \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$ *is continuous.*

Proof That $\mathscr{F}_{\mathrm{CD}}(\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})) \subseteq \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$ follows immediately from Parseval's equality. Moreover, this equality also gives

$$\|\mathscr{F}_{\mathrm{CD}}(f)\|_2 = \|f\|_2,$$

recalling that

$$\|F\|_2^2 = \frac{1}{T}\sum_{n\in\mathbb{Z}} |F(n)|_2^2$$

is the norm on $\ell^2(\mathbb{Z}(T^{-1}),\mathbb{C})$. This gives continuity of the CDFT by virtue of Theorem 6.5.8. ■

This raises the question about whether the same sort of thing can be said for the restriction of the CDFT to the other L^p-spaces. It turns out that this is not the case, and the only instance for which $\mathscr{F}_{\mathrm{CD}}(\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{C})) \subseteq \ell^p(\mathbb{Z}(T^{-1});\mathbb{C})$ is $p = 2$. This gives further hints of the "magic" that happens in this case. The following example illustrates the failure of the general proposition for $p = 1$.

12.3.7 Example (The CDFT only preserves p = 2) We let f be the 2π-periodic extension of the signal on $[0, 2\pi]$ defined by $t \mapsto (\pi - t)$. Clearly $f \in \mathsf{L}^{(2)}_{\mathrm{per},2\pi}(\mathbb{R};\mathbb{C})$, and so $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. We also compute

$$\mathscr{F}_{\mathrm{CD}}(f)(n(2\pi)^{-1}) = \begin{cases} 0, & n = 0,\\ -\frac{\mathrm{i}}{n}, & n \neq 0.\end{cases}$$

Thus we see that $\mathscr{F}_{\mathrm{CD}}(f) \notin \ell^1(\mathbb{Z}((2\pi)^{-1});\mathbb{C})$ although $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^2(\mathbb{Z}((2\pi)^{-1});\mathbb{C})$. •

12.3.2 The inverse L^2-CDFT

Now we study invertibility of the L^2-CDFT. As we saw in Theorem 12.2.20, invertibility of the L^1-CDFT by using Fourier series is hopeless. Moreover, we have seen in Example 12.2.10 that invertibility in the sense of pointwise convergence of Fourier series is not achievable, even for continuous signals. Since continuous signals are in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, this means that for signals in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ we cannot rule out their Fourier series diverging pointwise.

Nonetheless, there is a weaker form of convergence using Fourier series that works for $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, and this is what we consider in this section. The key is Corollary 12.3.6 which tells us that the L^2-CDFT is ℓ^2-valued.

12.3.8 Theorem (The L^2-CDFT is an isomorphism) *The map* $\mathscr{F}_{\mathrm{CD}}\colon \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$ *is an isomorphism of vector spaces with inverse*

$$\mathscr{F}_{\mathrm{CD}}^{-1}(F) = \frac{1}{T}\sum_{n\in\mathbb{Z}} F(nT^{-1})\mathsf{E}_{2\pi i n T^{-1}}.$$

Moreover, $\mathscr{F}_{\mathrm{CD}}$ *is a Hilbert space isomorphism from* $(\mathsf{L}^2_{\mathrm{per,T}}(\mathbb{R};\mathbb{C}),\langle\cdot,\cdot\rangle_2)$ *to* $(\ell^2(\mathbb{Z}(\mathrm{T}^{-1});\mathbb{C}),\langle\cdot,\cdot\rangle_2)$.

Proof By a direct computation, using the definition of the norm

$$\|F\|_2^2 = \frac{1}{T}\sum_{n\in\mathbb{Z}}|F(n)|_2^2$$

on $\ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$, we have

$$\|\mathscr{F}_{\mathrm{CD}}(f)\|_2^2 = \frac{1}{T}\sum_{n\in\mathbb{Z}}|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})|^2 = \int_0^T |f(t)|^2\,\mathrm{d}t = \|f\|^2.$$

Thus $\mathscr{F}_{\mathrm{CD}}$ is norm-preserving and so inner product preserving by ***missing stuff***. Moreover, $\mathscr{F}_{\mathrm{CD}}$ maps the Hilbert basis $\{\frac{1}{\sqrt{T}}\mathsf{E}_{2\pi\mathrm{i}nT^{-1}}\}_{n\in\mathbb{Z}}$ for $(\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}),\langle\cdot,\cdot\rangle_{2,T})$ to the Hilbert basis $\{\sqrt{T}\boldsymbol{e}_n\}_{n\in\mathbb{Z}}$ (here $\{\boldsymbol{e}_n\}_{n\in\mathbb{Z}}$ is the standard basis for $\mathbb{C}_0^{\mathbb{Z}}$) for $(\ell^2(\mathbb{Z}(T^{-1});\mathbb{C}),\langle\cdot,\cdot\rangle_2)$. Therefore, it follows from Corollary 7.3.35 that the map

$$F \mapsto \sum_{n\in\mathbb{Z}}\frac{1}{T}F(nT^{-1})\mathsf{E}_{2\pi\mathrm{i}nT^{-1}}$$

is a Hilbert space isomorphism from $(\ell^2(\mathbb{Z}(T^{-1});\mathbb{C}),\langle\cdot,\cdot\rangle_2)$ to $(\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}),\langle\cdot,\cdot\rangle_2)$. The inverse of this isomorphism is then $\mathscr{F}_{\mathrm{CD}}$ by Proposition 7.3.23. ■

The above results show that the Fourier series provides a left-inverse for the CDFT restricted to $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. The inverse is defined only in the L^2-sense, however, not pointwise. For pointwise convergence one still must revert to the results of Sections 12.2.4, 12.2.5, or 12.2.7.

Note that the inverse of the L^2-CDFT is not to a signal, but to an equivalence class of signals in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. One can then ask, “What is the relationship between the inversion of the L^2-CDFT as given by Theorem 12.3.8 and the inversion in terms of pointwise convergence of Fourier series?” This is what we now address. As we saw in Theorem 12.2.20, the prospect of inversion of the L^1-CDFT is hopeless, since there are signals in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ whose Fourier series diverge everywhere. Furthermore, our example in Example 12.2.10 shows that even for continuous T-periodic signals one cannot expect convergence everywhere for Fourier series. Note that this also precludes convergence everywhere for signals in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. Therefore, it is at this point in our presentation an open question as concerns the inversion of the CDFT for signals in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. The question can also be asked in the context of signals in $\mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ for $p\in(1,\infty)$. The following result is famous and difficult, and we devote Section **??** to its proof.

12.3.9 Theorem (Fourier series for signals in L^2 converge almost everywhere) *Let* $\mathrm{p}\in(1,\infty)$. *If* $\mathrm{f}\in\mathsf{L}^{(\mathrm{p})}_{\mathrm{per,T}}(\mathbb{R};\mathbb{C})$ *then the sequence* $(\mathrm{D}^{\mathrm{per}}_{\mathrm{T,N}}\mathrm{f}(\mathrm{t}))_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges for almost every* $\mathrm{t}\in[0,\mathrm{T}]$ *and*

$$\lambda\{\mathrm{t}\in[0,\mathrm{T}] \mid \mathrm{FS}[\mathrm{f}](\mathrm{t})-\mathrm{f}(\mathrm{t})\neq 0\} = 0.$$

Thus $\mathrm{FS}[\mathrm{f}]$ *differs from* f *only on a set of measure zero.*

As we know from Theorem 12.2.18, the conclusions of the theorem cannot be strengthened to assert convergence everywhere, even for continuous signals.

12.3.3 The relationship between various notions of convergence

We have seen thus far three notions of convergence for Fourier series: pointwise convergence as discussed in Section 12.2.4, uniform convergence as discussed in Section 12.2.5, and convergence in L^2 as given by Theorem 12.3.8. The latter sort of convergence is often referred to as ***mean convergence*** since it is convergence in the sense that the integral of the square of the error goes to zero. We will now summarise the relationships between these types of convergence.

1. Suppose that $f \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ and that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to f. Clearly $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to f. It is also easy to show that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges in mean to f, and the reader is invited to verify this in Exercise 12.3.5.
2. Suppose that $f \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ and that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to f. Then it is not necessary that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converge uniformly to f (consider the signal $f(t) = \square_{2,1,0}(t) - 1$ discussed at various points throughout this chapter.) It is also true that pointwise convergence does not imply mean convergence. We demonstrate this with an example. We consider the signal f that is the odd extension of the signal on $\in [0, 2\pi]$ given by

$$(f|[0,2\pi]) = \begin{cases} -\log|\sin\frac{t}{2}|, & t \in (0, 2\pi), \\ 0, & t \in \{0, 2\pi\}. \end{cases}$$

In Figure 12.19 we plot one period of f. Let us make some comments concerning

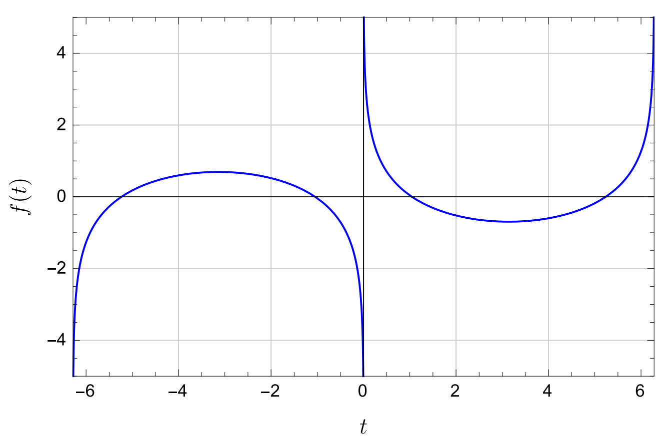


Figure 12.19 A signal whose Fourier series converges pointwise but not in mean

this signal.

(a) The signal f is not bounded. Indeed, to show what we want, it cannot be bounded since bounded signals in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ are also in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, and so then necessarily have mean convergent Fourier series. (The reader may wish to check that they understand this statement.)

(b) We do have $f \in \mathsf{L}^{(1)}_{\mathrm{per},4\pi}(\mathbb{R};\mathbb{C})$. To see that this is so we note that the singularities of f at integer multiples of 2π are logarithmic, and also note that $t \mapsto |\log|t||$ is integrable near $t = 0$. Thus we can compute the CDFT of f.

(c) Since f is odd its Fourier series is

$$\mathrm{FS}[f](t) = \frac{1}{2\pi}\sum_{n=1}^{\infty} \mathscr{S}_{\mathrm{CD}}(f)(\tfrac{n}{4\pi}) \sin \tfrac{nt}{2}$$

where

$$\mathscr{S}_{\mathrm{CD}}(f)(\tfrac{n}{4\pi}) = 2\int_0^{2\pi} f(t) \sin \tfrac{nt}{2}\, \mathrm{d}t, \qquad n \in \mathbb{Z}_{>0}.$$

(d) To see that $(D^{\mathrm{per}}_{4\pi,N} f)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to f we note that at points that are not integer multiples of 2π the series converges pointwise since f is differentiable at these points. At integer multiples of 2π we note that $D^{\mathrm{per}}_{4\pi,N} f(2n\pi) = 0$, $n \in \mathbb{Z}$, so giving convergence at these points as well.

Thus we see that f has a Fourier series converging pointwise, but not in mean, just as desired.

3. Finally, suppose that $f \in \mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, which by the Riesz-Fischer theorem is equivalent to $(D^{\mathrm{per}}_{T,N} f)_{N\in\mathbb{Z}_{>0}}$ converging in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ to f. By Theorem 12.3.9 we know that $(D^{\mathrm{per}}_{T,N} f)_{N\in\mathbb{Z}_{>0}}$ converges almost everywhere to f. Obviously uniform convergence is not guaranteed.

The preceding discussion is summarised in Figure 12.20. Note that uniform

$$\begin{tikzcd}
\text{pointwise convergence} & & \text{uniform convergence} \arrow[ll, Rightarrow] \arrow[dl, Rightarrow] \\
& \text{mean convergence} \arrow[ul, Rightarrow, "\text{except on set of measure zero}"] &
\end{tikzcd}$$

Figure 12.20 Relationship between pointwise, uniform, and mean convergence

convergence is the strongest flavour, and is the one most desired in practise.

12.3.4 Convolution, multiplication, and the L^2-CDFT

missing stuff

12.3.5 The Uncertainty Principle for the CDFT

missing stuff

12.3.6 Notes

The L^2-version of Theorem 12.3.9 is due to **LC:66**[5] and the version for general p is due to **RAH:68**. The result of **LC:66** answered a 1913 conjecture of Luzin,[6] and the passage of the fifty-three years from the formal announcement of the conjecture to its resolution is a reflection of the difficulty of the result. Indeed, the predominant thinking at the time **LC:66** published his result was that it was false. Our Theorem 12.2.18 was proved by **JPK/YK:66**.

Exercises

12.3.1 Answer the following two questions.

(a) Find a countable orthonormal set in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ that is not a Hilbert basis.

(b) For the set you found in part (a), find a signal for which the corresponding "Fourier series" does not converge in mean to the signal.

12.3.2 For each of the following signals, defined on the interval $[0,1]$,

(a) $f(t) = \begin{cases} 1, & t \in [0,\frac{1}{2}], \\ 4(t-\frac{1}{2})^2, & t \in (\frac{1}{2},1]. \end{cases}$

(b) $f(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{\sqrt{t}}, & t \in (0,1]. \end{cases}$

(c) $f(t) = \begin{cases} t^3, & t \in [0,\frac{1}{2}], \\ \frac{1}{2}(t-1)^2, & t \in (\frac{1}{2},1]. \end{cases}$

answer each of the following questions.

1. Sketch the graph of the signal.
2. Does the Fourier series for the signal f_{per} converge pointwise?
3. If the Fourier series converges pointwise, what is the limit signal?
4. Does the Fourier series for the signal f_{per} converge uniformly?
5. Does the Fourier series for the signal f_{per} converge in mean?

[5] Lennart Axel Edvard Carleson (1928–) is a Swedish mathematician who has made importanrt contributions to harmonic analysis. Aside from the theorem concerning pointwise convergence of L^2-Fourier series that we give here, he also proved a theorem known as Carleson's Corona Theorem which has to do with ideals in the set of bounded functions on the closed unit disk in $\mathbb{C}$, analytic in the interior.

[6] Nikolai Nikolaevich Luzin (1883–1950) was born in Russia. His mathematical work was mainly in the areas of set theory and analysis.

6. Which of the following assertions,

$$\textbf{(i)}\ \lim_{n\to\infty} \mathscr{F}_{\mathrm{CD}}(f)(n) = 0 \qquad \textbf{(ii)}\ \sum_{n\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}}(f)(n)|^2 < \infty$$

$$\textbf{(iii)}\ \sum_{n\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}}(f)(n)| < \infty,$$

represents the strongest that can be made for the Fourier coefficients of the signal. (Note that **(iii)** $\implies$ **(ii)** $\implies$ **(i)**.)

Do not compute the Fourier series for any of these signals!

12.3.3 Consider three signals $f_1, f_2, f_3 \colon \mathbb{R} \to \mathbb{R}$ that are periodic with period 1 and which satisfy

$$\mathscr{F}_{\mathrm{CD}}(f_1)(n) = \begin{cases} 0, & n = 0, \\ \frac{1}{|n|}, & n \neq 0, \end{cases}$$

$$\mathscr{F}_{\mathrm{CD}}(f_2)(n) = \begin{cases} 0, & n = 0, \\ \frac{1}{\sqrt{|n|}}, & n \neq 0, \end{cases}$$

$$\mathscr{F}_{\mathrm{CD}}(f_3)(n) = \begin{cases} 0, & n = 0, \\ \frac{1}{n^2}, & n \neq 0. \end{cases}$$

Answer the following questions.

(a) For each of the signals f_1, f_2, and f_3, indicate whether it is continuous.

(b) For each of the signals f_1, f_2, and f_3, indicate whether it is in $\mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{R})$.

(c) For each of the signals f_1, f_2, and f_3, indicate whether it is in $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{R})$.

(d) For each of the signals f_1, f_2, and f_3, indicate whether it has a uniformly convergent Fourier series.

12.3.4 Find a signal $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ such that the sum

$$\sum_{n\in\mathbb{Z}} |\mathscr{F}_{\mathrm{CD}} f(nT^{-1})|^2$$

does not converge.

12.3.5 Show that if the Fourier series for $f \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ converges uniformly to f, then it also converges in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ to f.

Suppose that an electric circuit is provided with a voltage $t \mapsto V(t)$ and a resulting current $t \mapsto I(t)$. The average power supplied to the circuit on the time interval $[t_0, t_1]$ is

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} V(t) I(t) \, \mathrm{d}t.$$

In particular, if V and I are T-periodic, then the average power over one period is

$$\frac{1}{T} \int_0^T V(t) I(t) \, \mathrm{d}t.$$

12.3.6 Show that the average power over one period in a circuit with T-periodic voltage $t \mapsto V(t)$ and T-periodic current $t \mapsto I(t)$ is

$$\frac{1}{T^2} \sum_{n \in \mathbb{Z}_{>0}} \mathscr{F}_{\mathrm{CD}}(V)(nT^{-1}) \mathscr{F}_{\mathrm{CD}}(I)(-nT^{-1}).$$

12.3.7 Answer the following questions.

(a) Is the function

$$n \mapsto \frac{1}{2n-1}$$

in $\ell^2(\mathbb{Z}_{>0}; \mathbb{R})$?

(b) Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Hint: *Use Example 12.1.3–2 and Parseval's equality.*

12.3.8 For each of the following six signals $F \colon \mathbb{Z} \to \mathbb{C}$, *if directly possible using what you have learned in this book*, answer the following questions with concise explanations:

1. is $F \in \mathsf{c}_{\mathrm{fin}}(\mathbb{Z}; \mathbb{C})$?
2. is $F \in \mathsf{c}_0(\mathbb{Z}; \mathbb{C})$?
3. is $F \in \ell^\infty(\mathbb{Z}; \mathbb{C})$?
4. is $F \in \ell^2(\mathbb{Z}; \mathbb{C})$?
5. is $F \in \ell^1(\mathbb{Z}; \mathbb{C})$?
6. is $F = \mathscr{F}_{\mathrm{CD}}(f)$ for $f \in \mathsf{L}^1_{\mathrm{per},1}(\mathbb{R}; \mathbb{C})$?
7. is $F = \mathscr{F}_{\mathrm{CD}}(f)$ for $f \in \mathsf{L}^2_{\mathrm{per},1}(\mathbb{R}; \mathbb{C})$?
8. is $F = \mathscr{F}_{\mathrm{CD}}(f)$ for $f \in \mathsf{C}^0_{\mathrm{per},1}(\mathbb{R}; \mathbb{C})$?

Here are the signals:

(a) $F(n) = \begin{cases} 1, & |n| \le 10^{100}, \\ 0, & \text{otherwise}; \end{cases}$

(b) $F(n) = \begin{cases} |n|, & n \neq 0, \\ 0, & \text{otherwise}; \end{cases}$

(c) $F(n) = \begin{cases} |n|^{-1}, & n \neq 0, \\ 0, & \text{otherwise}; \end{cases}$

(d) $F(n) = \begin{cases} |n|^{-1/2}, & n \neq 0, \\ 0, & \text{otherwise}; \end{cases}$

(e) $F(n) = \begin{cases} |n|^{-2}, & n \neq 0, \\ 0, & \text{otherwise}; \end{cases}$

(f) $F(n) = \mathrm{e}^{-n^2}$.

12.3.9 Answer the following questions

(a) Is the function

$$n \mapsto \frac{1}{(2n-1)^2}$$

in $\ell^2(\mathbb{Z}_{>0};\mathbb{R})$?

(b) Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

Hint: *Use Example 12.1.3–3 and Parseval's equality.*

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Chapter 13

The continuous-continuous Fourier transform

The preceding chapter deals with frequency-domain representations of periodic signals. Now we consider frequency-domain representations of signals that are not periodic. While frequency-domain representations of periodic signals seem somehow natural, this is less so for aperiodic signals. In Section 9.6.2 we attempt to motivate this idea by adapting, in an utterly non-rigorous way, the idea of periodic frequency-domain representations in the limit as the period gets large. However, at some point it seems as if the CCFT that we consider in this section is something that one must just get used to. (Alternatively, one might try to understand the idea of a Fourier transform by making the generalisation to locally compact groups. This is an idea we will not explore in these volumes.)

Some comfort should be afforded by the fact that readers having already studied the CDFT in Chapter 12 will see many similarities between the CCFT we discuss in this chapter and the CDFT. Indeed, we try to emphasise this similarity as much as possible. This allows the treatment of one to reinforce that of the other. By way of warning, we mention that one significant area of difference if the manner in which the L^2-theory is developed in each case. These differences are pointed out in the subsequent text.

Finally, we comment that the transform we consider in this chapter is most often known simply as *the* "Fourier transform." Thus our terminology departs from the standard terminology since we do not distinguish this transform as being any more or less important than the other three Fourier transforms we consider.

Do I need to read this chapter? If you are learning Fourier transform theory, then you must read this chapter. •

Contents

Section 13.1

The L^1-CCFT

In this section we present the CCFT in its most natural setting, at least from the mathematical point of view. For applications, other forms of the CCFT are actually the more useful. In particular, the L^2-CCFT of Section 13.3 is what is most often of use in signals and systems theory. Indeed, very often one sees *only* the L^2-CCFT presented. However, as we shall see, this is actually incoherent. The very definition of the L^2-CCFT rests in a crucial way on the *a priori* understanding of the L^1-CCFT.

Do I need to read this section? If you are reading this chapter then you are reading this section. •

13.1.1 Definitions and computations

Let us give the basic definition.

13.1.1 Definition (CCFT) The ***continuous-continuous Fourier transform*** or ***CCFT*** assigns to $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{F}_{\mathrm{CC}}(f)\colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t. \qquad \bullet$$

13.1.2 Remarks (Comments on the definition of the CCFT)

1. Note that the expression for $\mathscr{F}_{\mathrm{CC}}(f)$ makes sense if and only if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, so the CCFT is most naturally defined on such signals.
2. Note that if $f_1, f_2 \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ have the property that $f_1(t) = f_2(t)$ for almost every $t \in \mathbb{R}$, then we have $\mathscr{F}_{\mathrm{CC}}(f_1) = \mathscr{F}_{\mathrm{CC}}(f_2)$ by Proposition 5.7.11. Therefore, $\mathscr{F}_{\mathrm{CC}}$ is well-defined as a map from equivalence classes in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$. Frequently we shall be interested in this equivalence class version of the CCFT, and we shall explicitly indicate that we are working with $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ rather than $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ in such cases. However, we shall adhere to our convention of denoting equivalence classes of signals in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ by f rather than with some more cumbersome notation.
3. We comment that there are many slightly different versions of the CCFT, mostly having to do with the replacing of $2\pi\nu$ with other expressions. Some people prefer one over the other with ferocious devotion. For example, it is common to use ω rather than $2\pi\nu$. This corresponds to using angular frequency rather than frequency. In Section 13.1.6 we explore these alternative formulae in a general way. It is important to note that in terms of the mathematics, these formulae have the same properties, although the details of the computations will vary in each case.

4. As we did in our development of the CDFT, we shall consider $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ as a subspace of $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, so that our development is conveniently made assuming all signals to be complex. •

Let us look at some examples.

13.1.3 Examples (Computing the CCFT)

1. For $a \in \mathbb{C}$ with $\operatorname{Re}(a) \in \mathbb{R}_{>0}$, note that $f(t) = 1(t)\mathrm{e}^{-at}$ is a signal in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. We then compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CC}}(f)(\nu) &= \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(a+2\pi\mathrm{i}\nu)t}\,\mathrm{d}t \\ &= -\frac{\mathrm{e}^{-(a+2\pi\mathrm{i}\nu)t}}{a+2\pi\mathrm{i}\nu}\Big|_0^\infty = \frac{1}{a+2\pi\mathrm{i}\nu}.\end{aligned}$$

In Figure 13.1 we show the signal and its CCFT.

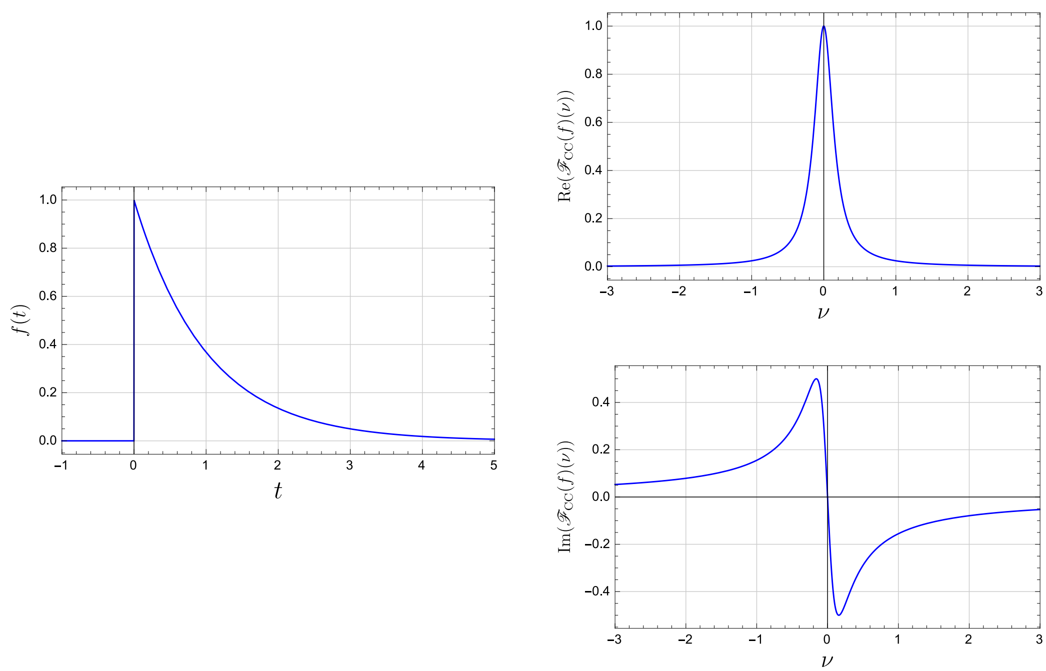


Figure 13.1 The signal $t \mapsto 1(1)\mathrm{e}^{-at}$ with $a = 1$ (left) and its CCFT (right)

2. If we take $f(t) = 1(t)\mathrm{e}^{-at} + 1(-t)\mathrm{e}^{bt}$ for $a, b \in \mathbb{C}$ with $\operatorname{Re}(a), \operatorname{Re}(b) \in \mathbb{R}_{>0}$, then we ascertain that $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. It is then easy to compute

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \frac{1}{a+2\pi\mathrm{i}\nu} + \frac{1}{b-2\pi\mathrm{i}\nu}.$$

In Figure 13.2 we show the signal and its CCFT.

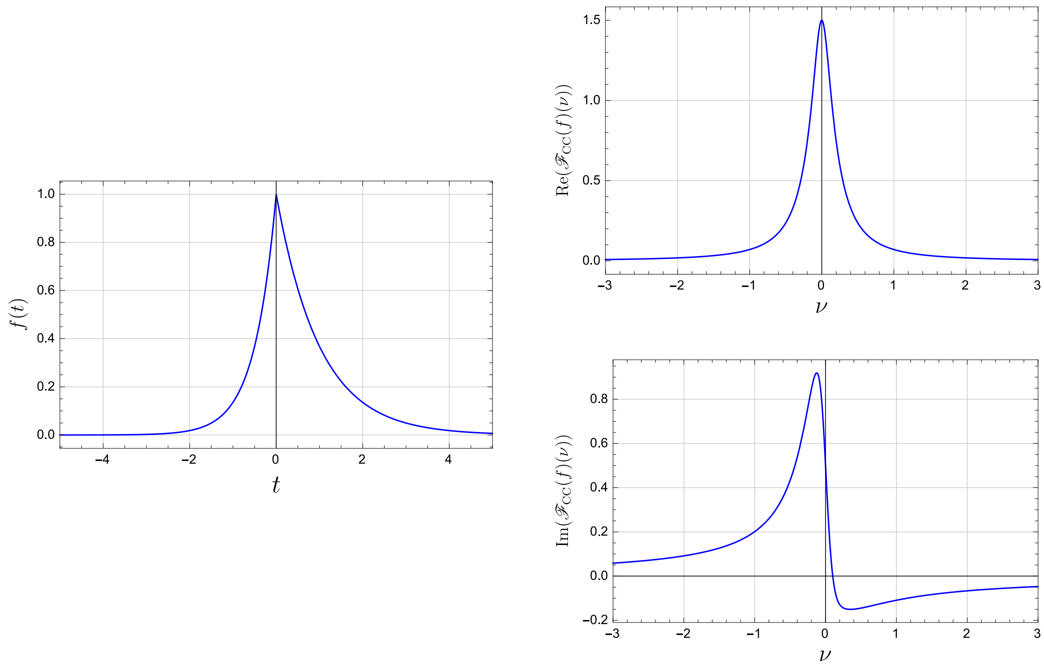


Figure 13.2 The signal $1 \mapsto 1(1)\mathrm{e}^{-at} + 1(-1)\mathrm{e}^{bt}$ with $a = 1, b = 2$ (left) and its CCFT (right)

3. Let $a \in \mathbb{R}_{>0}$ and consider the signal $f = \chi_{[-a,a]}$ given by the characteristic function of $[-a, a]$. We then compute

$$\mathscr{F}_{\mathrm{CC}}(\sigma)(\nu) = \int_{-a}^{a} \mathrm{e}^{-2\pi \mathrm{i} \nu t}\, \mathrm{d}t = -\frac{\mathrm{e}^{-2\pi \mathrm{i}\nu t}}{2\pi \mathrm{i}\nu}\bigg|_{-a}^{a} = \frac{\sin(2\pi a \nu)}{\pi\nu}.$$

In Figure 13.3 we plot the signal along with its CCFT, which happens to be real

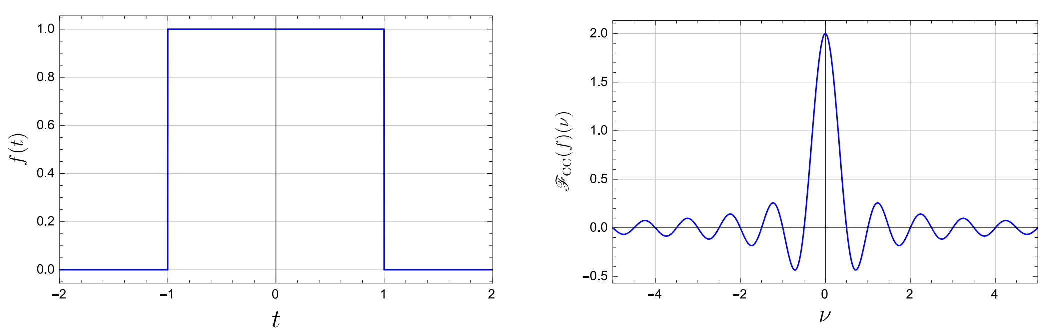


Figure 13.3 The characteristic function of $[-1, 1]$ (left) and its CCFT (right)

in this case.

4. Here we consider $f\colon \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) = \begin{cases} 1 + \frac{t}{a}, & t \in [-a, 0], \\ 1 - \frac{t}{a}, & t \in (0, a], \\ 0, & \text{otherwise}, \end{cases}$$

for $a \in \mathbb{R}_{>0}$. We then compute

$$\begin{aligned} \mathscr{F}_{\mathrm{CC}}(f)(\nu) &= \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t \\ &= \int_{-a}^{0} (1 + \tfrac{t}{a})\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t + \int_{0}^{a} (1 - \tfrac{t}{a})\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t \\ &= \int_{-a}^{a} \mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t - \frac{2}{a}\int_{0}^{a} t\cos(2\pi\nu t)\,\mathrm{d}t \\ &= \frac{\sin(2\pi\nu a)}{\pi\nu} - \frac{2}{a}\Big(\frac{t\sin(2\pi\nu t)}{2\pi\nu}\Big|_{t=0}^{t=a} - \frac{1}{2\pi\nu}\int_{0}^{a}\sin(2\pi\nu t)\,\mathrm{d}t\Big) \\ &= -\frac{2}{a}\frac{\cos(2\pi\nu t)}{(2\pi\nu)^2}\Big|_{t=0}^{t=a} = \frac{1-\cos(2\pi a\nu)}{2\pi a\nu^2}\Big|_{t=0}^{t=a} = \frac{\sin^2(\pi a\nu)}{\pi^2 a\nu^2}. \end{aligned}$$

In Figure 13.4 we show f and its CCFT.

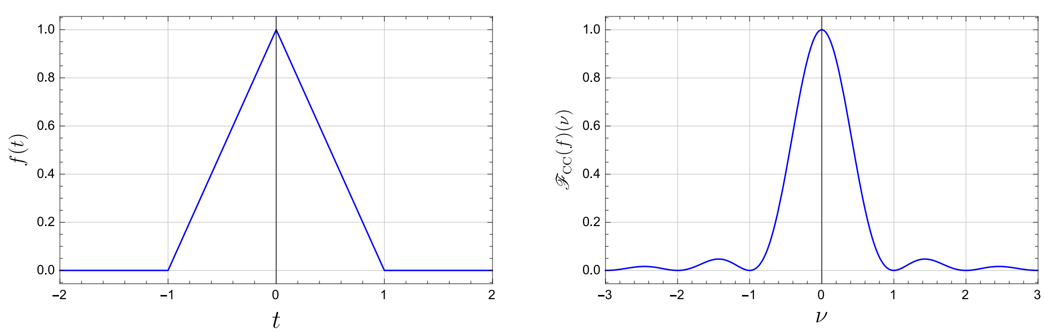


Figure 13.4 The signal from part 4 with $a = 1$ (left) and its CCFT (right)

5. Let $f(t) = \mathrm{e}^{-a|t|}$ for $a \in \mathbb{R}_{>0}$. We then compute

$$\mathscr{F}_{\mathrm{CC}}(f) = \frac{2a}{a^2 + 4a^2\nu^2}.$$

In Figure 13.5 we plot the signal along with its CCFT, which again is real in this case. •

As with the CDFT, there are cosine and sine transforms associated with the CCFT.

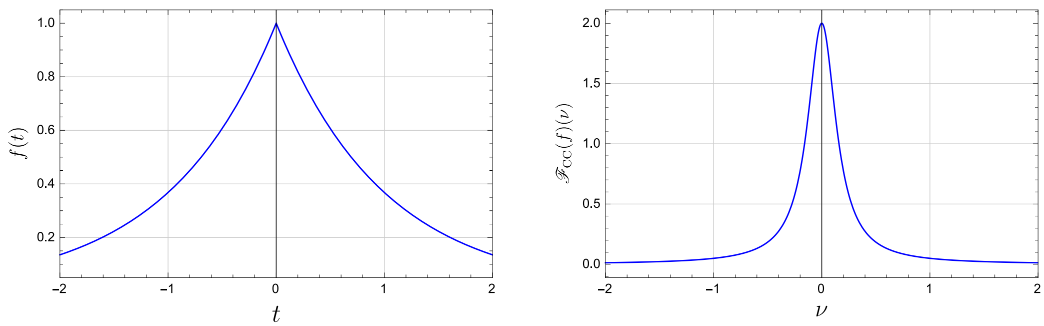


Figure 13.5 The signal of $e^{-a|t|}$ with $a = 1$ (left) and its CCFT (right)

13.1.4 Definition (CCCT and CCST)

(i) The ***continuous-continuous cosine transform*** or ***CCCT*** assigns to $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{C}_{\mathrm{CC}}(f)\colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ by

$$\mathscr{C}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t)\cos(2\pi\nu t)\,\mathrm{d}t, \qquad \nu \in \mathbb{R}_{\geq 0}.$$

(ii) The ***continuous-continuous sine transform*** or ***CCST*** assigns to $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{S}_{\mathrm{CC}}(f)\colon \mathbb{R}_{>0} \to \mathbb{C}$ by

$$\mathscr{S}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t)\sin(2\pi\nu t)\,\mathrm{d}t, \qquad \nu \in \mathbb{R}_{>0}. \qquad \bullet$$

The same sorts of relationships hold between the CCFT, and the CCCT and CCST as hold in the periodic case.

13.1.5 Proposition (The CCFT, and the CCCT and the CCST) *For* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) $\mathscr{F}_{\mathrm{CC}}(0) = \mathscr{C}_{\mathrm{CC}}(\mathrm{f})(0)$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\nu) = \mathscr{C}_{\mathrm{CC}}(\mathrm{f})(\nu) - \mathrm{i}\mathscr{S}_{\mathrm{CC}}(\mathrm{f})(\nu)$ *and*
$\mathscr{F}_{\mathrm{CC}}(\mathrm{f})(-\nu) = \mathscr{C}_{\mathrm{CC}}(\mathrm{f})(\nu) + \mathrm{i}\mathscr{S}_{\mathrm{CC}}(\mathrm{f})(\nu)$ *for every* $\nu \in \mathbb{R}_{>0}$;

(iii) $\mathscr{C}_{\mathrm{CC}}(\mathrm{f})(\nu) = \frac{1}{2}(\mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\nu) + \mathscr{F}_{\mathrm{CC}}(\mathrm{f})(-\nu))$ *for every* $\nu \in \mathbb{R}_{\geq 0}$;

(iv) $\mathscr{S}_{\mathrm{CC}}(\mathrm{f})(\nu) = \frac{\mathrm{i}}{2}(\mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\nu) - \mathscr{F}_{\mathrm{CC}}(\mathrm{f})(-\nu))$ *for every* $\nu \in \mathbb{R}_{>0}$.

As with the CDFT, it might sometimes be easier to compute the CCFT through the CCCT and/or the CCST. Since cosine is even and sine is odd we can write

$$\mathscr{C}_{\mathrm{CC}}(f)(\nu) = 2\int_0^{\infty} f_{\mathrm{even}}(t)\cos(2\pi\nu t)\,\mathrm{d}t,$$
$$\mathscr{S}_{\mathrm{CC}}(f)(\nu) = 2\int_0^{\infty} f_{\mathrm{odd}}(t)\sin(2\pi\nu t)\,\mathrm{d}t,$$

where

$$f_{\mathrm{even}}(t) = \tfrac{1}{2}(f(t) + f(-t)), \quad f_{\mathrm{odd}}(t) = \tfrac{1}{2}(f(t) - f(-t)).$$

For this reason, the cosine and sine transforms are often defined only for signals that are zero on $\mathbb{R}_{<0}$.

13.1.2 Properties of the CCFT

First let us give some of the more elementary properties of the CCFT. Recall from Example 8.1.6 that $\sigma(t) = -t$, so that $\sigma^* f(t) = f(-t)$. Clearly, if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then $\sigma^* f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Also, if $a \in \mathbb{R}$ then $\tau_a^* f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ denotes the signal defined by $\tau_a^* f(t) = f(t-a)$. In like manner, if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then $\bar{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ denotes the signal defined by $\bar{f}(t) = \overline{f(t)}$. For $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ let us also define the signal $\overline{\mathscr{F}}_{\mathrm{CC}}(f)\colon \mathbb{R} \to \mathbb{C}$ by

$$\overline{\mathscr{F}}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t) \mathrm{e}^{2\pi \mathrm{i}\nu t} \, \mathrm{d}t.$$

The proof of the following mirrors that of Proposition 12.1.6 for the CDFT.

13.1.6 Proposition (Elementary properties of the CCFT) *For* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{CC}}(f)} = \overline{\mathscr{F}}_{\mathrm{CC}}(\bar{f})$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\sigma^* f) = \sigma^*(\mathscr{F}_{\mathrm{CC}}(f)) = \overline{\mathscr{F}}_{\mathrm{CC}}(f)$;

(iii) *if* f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{CC}}(f)$ *is even (resp. odd);*

(iv) *if* f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{CC}}(f)$ *is real and even (resp. imaginary and odd);*

(v) $\mathscr{F}_{\mathrm{CC}}(\tau_a^* f)(\nu) = \mathrm{e}^{-2\pi \mathrm{i} a \nu} \mathscr{F}_{\mathrm{CC}}(f)(\nu)$.

The next result is the most general result concerning the basic behaviour of the CCFT, and gives the analogue of the Riemann–Lebesgue Lemma (Theorem 12.1.8) for the CCFT.

13.1.7 Theorem (The Riemann–Lebesgue Lemma for the CCFT) *For* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$

(i) $\mathscr{F}_{\mathrm{CC}}(f)$ *is a bounded, uniformly continuous function and*

(ii) $\lim_{|\nu|\to\infty} |\mathscr{F}_{\mathrm{CC}}(f)(\nu)| = 0$.

Proof (i) Let $t \in \mathbb{R}$ and compute

$$\begin{aligned}
|\mathscr{F}_{\mathrm{CC}}(f)(\nu + h) - \mathscr{F}_{\mathrm{CC}}(f)(\nu)| &= \Big(\int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi \mathrm{i}\nu t} (\mathrm{e}^{-2\pi \mathrm{i} h t} - 1) \, \mathrm{d}t \Big) \\
&\le \int_{\mathbb{R}} |f(t)| |\mathrm{e}^{-2\pi \mathrm{i} h t} - 1| \, \mathrm{d}t \\
&= \int_{|t| \le T} |f(t)| |\mathrm{e}^{-2\pi \mathrm{i}\nu h} - 1| \, \mathrm{d}t + \int_{|t| > T} |f(t)| |\mathrm{e}^{-2\pi \mathrm{i} h t} - 1| \, \mathrm{d}t
\end{aligned}$$

for $T \in \mathbb{R}_{>0}$. The signal $t \mapsto \mathrm{e}^{-2\pi \mathrm{i} h t} - 1$ is uniformly bounded in t, and since $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ we may choose $T \in \mathbb{R}_{>0}$ sufficiently large that

$$\int_{|t| > T} |f(t)| |\mathrm{e}^{-2\pi \mathrm{i} h t} - 1| \, \mathrm{d}t < \epsilon$$

for any given $\epsilon \in \mathbb{R}_{>0}$. Using the Taylor series expansion for $e^{-2\pi i h t}$ we have

$$\int_{|t|\leq T} |f(t)||e^{-2\pi i \nu h} - 1| \, dt \leq 2\pi |h| \int_{|t|\leq T} |t f(t)| \, dt = Ch,$$

so defining $C \in \mathbb{R}_{>0}$. Therefore

$$\lim_{h\to 0} \sup\{|\mathscr{F}_{CC}(f)(\nu + h)| t \in \mathbb{R} - \mathscr{F}_{CC}(f)(\nu)\,| \;\; \leq\} \lim_{h\to 0} \sup(Ch + \epsilon) = \epsilon,$$

giving uniform continuity, as stated.

(ii) We shall prove this part of the result first for step functions, then for continuous signals with compact support, then for arbitrary integrable signals. Let $I \subseteq \mathbb{R}$ be a compact interval and let $[-a, a]$ be that interval symmetric about 0 for which $\lambda(I) = 2a$. Then we have, for some $\alpha \in \mathbb{R}$,

$$|\mathscr{F}_{CC}(\chi_I)(\nu)| = \left|\int_I e^{-2\pi i \nu t} \, dt\right| = \left|e^{i\alpha} \int_{-a}^{a} e^{-2\pi i \nu t} \, dt\right| \leq |2\nu^{-1}|$$

by Example 13.1.3–3. This shows that this part of the theorem is true for characteristic functions of compact intervals, and is therefore true for any step function with compact support. Now let f be a continuous signal with compact support. There then exists a sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ of step functions for which $\lim_{j\to\infty}\|f - g_j\|_1 = 0$. We have $\lim_{|\nu|\to\infty}|\mathscr{F}_{CC}(g_j)(\nu)| = 0$ for each $j \in \mathbb{Z}_{>0}$, and by part (i) we have $|f(\nu) - g_j(\nu)| \leq \|f - g\|_1$. Taking the limit as $j \to \infty$ in this last expression gives the result for continuous signals with compact support. By Theorem 8.3.11(ii) $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{C})$ is dense in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$. Therefore, if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ there exists a sequence $(g_j)_{j\in\mathbb{Z}_{>0}} \subseteq \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{C})$ for which $\lim_{j\to\infty}\|f - g_j\|_1 = 0$. The argument above may now be repeated to give this part of the theorem. ■

Recall from Section 8.3.2 that $\mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ denotes the set of continuous signals on $\mathbb{R}$ that decay to zero at infinity. The following result provides an important interpretation of Theorem 13.1.7 in terms of the ideas introduced in Chapter **??**.

13.1.8 Corollary (The CCFT is continuous) *$\mathscr{F}_{CC}$ is a continuous linear mapping from $(\mathsf{L}^1(\mathbb{R};\mathbb{C}), \|\cdot\|_1)$ to $(\mathsf{C}^0_0(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$.*

Proof Linearity of $\mathscr{F}_{CC}$ follows from linearity of the integral. By Theorem 13.1.7 we know that $\mathscr{F}_{CC}(f) \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ for $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. Continuity of $\mathscr{F}_{CC}$ follows from Theorem 6.5.8 along with the estimate

$$|\mathscr{F}_{CC}(f)(\nu)| = \left|\int_{\mathbb{R}} f(t) e^{-2\pi i \nu t} \, dt\right| \leq \int_{\mathbb{R}} |f(t)| \, dt = \|f\|_1. \qquad ■$$

The final result in this section often comes in handy when dealing with the CCFT.

13.1.9 Proposition (Fourier Reciprocity Relation for the CCFT) *If $f, g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then $f\mathscr{F}_{CC}(g), \mathscr{F}_{CC}(f)g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and*

$$\int_{\mathbb{R}} f(\xi)\mathscr{F}_{CC}(g)(\xi) \, d\xi = \int_{\mathbb{R}} \mathscr{F}_{CC}(f)(\xi) g(\xi) \, d\xi.$$

Proof First note that since $\mathscr{F}_{CC}(g)$ is continuous and decays to zero at infinity by Theorem 13.1.7, it follows that $\mathscr{F}_{CC}(g)$ is bounded. Therefore we have

$$\int_{\mathbb{R}} f(\xi)\mathscr{F}_{CC}(g)(\xi)\,d\xi \leq \|\mathscr{F}_{CC}(g)\|_\infty \int_{\mathbb{R}} f(\xi)\,d\xi < \infty,$$

showing that $f\mathscr{F}_{CC}(g) \in L^{(1)}(\mathbb{R};\mathbb{C})$ (and, of course, that $\mathscr{F}_{CC}(f)g \in L^{(1)}(\mathbb{R};\mathbb{C})$). We also have

$$\begin{aligned}\int_{\mathbb{R}} f(\xi)\mathscr{F}_{CC}(g)(\xi)\,d\xi &= \int_{\mathbb{R}} f(\xi)\Big(\int_{\mathbb{R}} g(\eta)e^{-2\pi i\xi\eta}\,d\eta\Big)\,d\xi\\ &= \int_{\mathbb{R}} g(\eta)\Big(\int_{\mathbb{R}} f(\xi)e^{-2\pi i\eta\xi}\,d\xi\Big)\,d\eta\\ &= \int_{\mathbb{R}} \mathscr{F}_{CC}(f)(\eta)g(\eta)\,d\eta,\end{aligned}$$

where we have used Fubini's Theorem, whose hypotheses are satisfied by virtue of Corollary 5.8.8. ■

13.1.3 Differentiation, integration, and the CCFT

Next let us turn to signals with more structure and see what we can say about their character relative to the CCFT. The ideas here, as with their analogues for the CDFT, are important in that they show that information about a signal can be obtained from its transform.

13.1.10 Proposition (The CCFT and differentiation) *Suppose that* $f \in C^0(\mathbb{R};\mathbb{C}) \cap L^{(1)}(\mathbb{R};\mathbb{C})$ *and that there exists a signal* $f' \colon \mathbb{R} \to \mathbb{C}$ *with the following properties:*

(i) for every $T \in \mathbb{R}_{>0}$*,* f' *is piecewise continuous on* $[-T, T]$*;*

(ii) f' *is discontinuous at a finite number of points;*

(iii) $f' \in L^{(1)}(\mathbb{R};\mathbb{C})$*;*

(iv) $f(t) = f(0) + \int_0^t f'(\tau)\,d\tau.$

Then

$$\mathscr{F}_{CC}(f')(\nu) = (2\pi i\nu)\mathscr{F}_{CC}(f)(\nu).$$

Proof By Exercise 8.3.18 we have $\lim_{|t|\to\infty} f(t) = 0$. Now we let T be sufficiently large that all discontinuities of f' are contained in $(-T, T)$. Let us denote the points of discontinuity of f' by $\{t_1, \dots, t_k\}$, denote $t_0 = -T$ and $t_{k+1} = T$, and compute

$$\begin{aligned}\int_{-T}^{T} f'(t)e^{-2\pi i\nu t}\,dt &= \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} f'(t)\,dt\\ &= \sum_{j=0}^{k} f(t)e^{-2\pi i\nu t}\Big|_{t_j}^{t_{j+1}} + \sum_{j=0}^{k}(2\pi i\nu)\int_{t_j}^{t_{j+1}} f(t)e^{-2\pi i\nu t}\,dt\\ &= f(T)e^{-2\pi i\nu T} - f(-T)e^{2\pi i\nu t} + (2\pi i\nu)\int_{-T}^{T} f(t)e^{-2\pi i\nu t}\,dt,\end{aligned}$$

using the fact that f is continuous. The result now follows by letting $T \to \infty$. ■

As with the corresponding results for the CDFT, we may extend the result for signals that have more differentiability.

13.1.11 Corollary (The CCFT and higher-order derivatives) *If* $f \in C^{r-1}(\mathbb{R};\mathbb{C}) \cap L^{(1)}(\mathbb{R};\mathbb{C})$ *for* $r \in \mathbb{Z}_{>0}$ *and suppose that there exists a signal* $f^{(r)}: \mathbb{R} \to \mathbb{C}$ *with the following properties:*

(i) *for every* $T \in \mathbb{R}_{>0}$, $f^{(r)}$ *is piecewise continuous on* $[-T, T]$*;*

(ii) $f^{(r)}$ *is discontinuous at a finite number of points;*

(iii) $f^{(j)} \in L^{(1)}(\mathbb{R};\mathbb{C})$ *for* $j \in \{1, \dots, r\}$*;*

(iv) $f^{(r-1)}(t) = f^{(r-1)}(0) + \int_0^t f^{(r)}(\tau)\,d\tau.$

Then

$$\mathscr{F}_{CC}(f^{(r)})(\nu) = (2\pi i\nu)^r \mathscr{F}_{CC}(f)(\nu).$$

For the CCFT we can also talk about the differentiability of the transform. Note that this is something that is not possible for the CDFT.

13.1.12 Proposition (Differentiability of transformed signals) *For* $f \in L^{(1)}(\mathbb{R};\mathbb{C})$*, if* $t \mapsto t^k f(t) \in L^{(1)}(\mathbb{R};\mathbb{C})$*, then* $\mathscr{F}_{CC}(f)$ *is* k*-times continuously differentiable and*

$$\mathscr{F}_{CC}(f)^{(k)}(\nu) = \int_{\mathbb{R}} (-2\pi i t)^k f(t) e^{-2\pi i\nu t}\,dt.$$

Proof For fixed t the signal $f(t)e^{-2\pi i\nu t}$ is infinitely differentiable with respect to ν. Furthermore, the kth derivative is bounded in magnitude by $2\pi|t^k f(t)|$. As this signal is assumed to be in $L^{(1)}(\mathbb{R};\mathbb{C})$ we may apply Theorem 5.9.16(**??**) to conclude that $\mathscr{F}_{CC}(f)$ is k-times continuously differentiable and that the derivative and the integral may be swapped, giving the stated formula. ■

The theorem has the following immediate corollary.

13.1.13 Corollary (Signals with compact support have infinitely differentiable transforms) *If* $f \in L^{(1)}(\mathbb{R};\mathbb{C})$ *has compact support then* $\mathscr{F}_{CC}(f)$ *is infinitely differentiable.*

Proof We leave it to the reader as Exercise 13.1.4 to show how this follows from Proposition 13.1.12. ■

Let us give some examples that illustrate Theorem 13.1.7 and Propositions 13.1.10 and 13.1.12, and which also illustrate what can happen when the hypotheses do not hold.

13.1.14 Examples (The CCFT and differentiation)

1. Let us first take the signal $f(t) = 1(t)e^{-at}$ with $\operatorname{Re}(a) \in \mathbb{R}_{>0}$, and for which we computed in Example 13.1.3–1

$$\mathscr{F}_{CC}(f)(\nu) = \frac{1}{a + 2\pi i\nu}.$$

We see that $\mathscr{F}_{CC}(f)$ does indeed satisfy the conclusions of Theorem 13.1.7. Note that this signal satisfies the hypotheses of Proposition 13.1.12 for any $k \in \mathbb{Z}_{>0}$.

Therefore we expect that $\mathscr{F}_{CC}(\sigma)$ will be infinitely differentiable, which it indeed is. Note that this shows that it is not necessary that f have compact support in order that $\mathscr{F}_{CC}(f)$ be infinitely differentiable.

2. For the signal $\sigma(t) = 1(t)e^{-at} + 1(-t)e^{bt}$, $\mathrm{Re}(a), \mathrm{Re}(b) \in \mathbb{R}_{>0}$, we have
$$\mathscr{F}_{CC}(f)(\nu) = \frac{1}{a + 2\pi i\nu} + \frac{1}{b - 2\pi i\nu}.$$
Note again that the CCFT is infinitely differentiable because the signal decays exponentially at infinity.

3. For the Gaussian $\gamma_a(t) = e^{-at^2}$ note that the CCFT $\mathscr{F}_{CC}(\gamma_a)$ has exactly the behaviour of γ_a. That is to say, it is infinitely differentiable and decays faster than any polynomial at infinity. This will make sense in the context of Section 13.4.2.

4. Let us consider the signal $\sigma(t) = \frac{1}{t^2+1}$. For this signal the signal $t^k f(t)$ is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ if and only if $k = 0$ (cf. Exercise 8.3.9). Thus all we can deduce from Proposition 13.1.12 is that $\mathscr{F}_{CC}(f)$ is continuous. Indeed, we compute
$$\mathscr{F}_{CC}(f) = \frac{\pi}{e^{2\pi|\nu|}},$$
which is continuous but not differentiable. In Figure 13.6 we show the CCFT

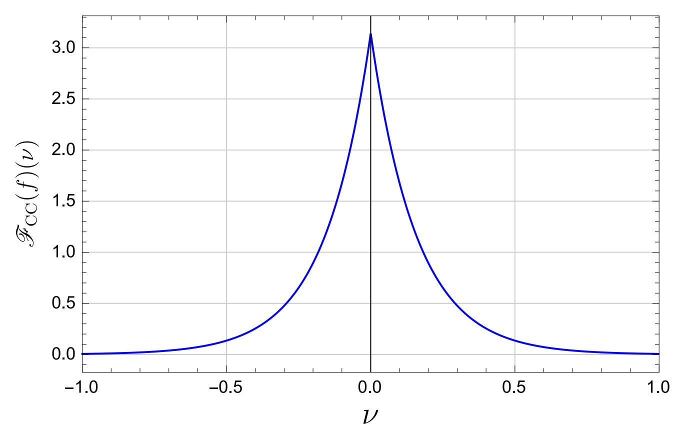


Figure 13.6 The CCFT for a slowly decaying signal

of f. •

Next let us consider the behaviour of the CCFT relative to integration.

13.1.15 Proposition (The CCFT and integration) *Let* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and define*
$$g(t) = \int_0^t f(\tau)\,d\tau.$$
If $g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *then*
$$\mathscr{F}_{CC}(g)(\nu) = \begin{cases} \frac{1}{2\pi i\nu}\mathscr{F}_{CC}(f)(\nu), & \nu \neq 0, \\ -\int_{\mathbb{R}} tf(t)\,dt, & \nu = 0. \end{cases}$$

Proof Fix $\nu \neq 0$ and compute, using integration by parts,

$$\begin{aligned}\mathscr{F}_{\mathrm{CC}}(g)(\nu) &= \lim_{T\to\infty}\int_{-T}^{T} g(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t \\ &= \lim_{T\to\infty} -g(t)\frac{\mathrm{e}^{-2\pi\mathrm{i}\nu t}}{2\pi\mathrm{i}\nu}\Big|_{-T}^{T} + \frac{1}{2\pi\mathrm{i}\nu}\lim_{T\to\infty}\int_{-T}^{T} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t.\end{aligned}$$

By Exercise 8.3.18, $\lim_{T\to\pm\infty} g(T) = 0$, and so the result follows for $\nu \neq 0$. For $\nu = 0$ the result follows directly from an integration by parts. ■

13.1.4 Decay of the CCFT

As discussed for the CDFT in Section 12.1.4, there are relationships between signals and the rate of decay of their CCFT. Indeed, we have already encountered the following facts.

1. If $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then the CCFT satisfies
$$\lim_{|\nu|\to\infty} |\mathscr{F}_{\mathrm{CC}}(f)(\nu)| = 0.$$
This is the Riemann–Lebesgue Lemma, Theorem 13.1.7.
2. If $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. This is nontrivial and will be discussed in Section 13.3, also cf. Proposition 12.1.11.
3. If f satisfies the conditions of Proposition 13.1.10 then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, as we shall show in Corollary 13.2.28. In particular, if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ is differentiable and if $f' \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.
4. If $f \in \mathsf{C}^r(\mathbb{R};\mathbb{C})$ and if $f^{(k)} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for $k \in \{0,1,\dots,r\}$ then the CCFT of f has the property that
$$\lim_{|\nu|\to\infty} \nu^j \mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$$
for $j \in \{0,1,\dots,r\}$. This is a consequence of Corollary 13.1.11.
5. A sort of converse of the preceding statement is provided by Proposition 13.1.12, along with the inversion theorem Theorem 13.2.26 below. Precisely, $\nu \mapsto \nu^{r+1+\epsilon}\mathscr{F}_{\mathrm{CC}}(f)(\nu) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for some $\epsilon \in \mathbb{R}_{>0}$, then $f(t) = g(t)$ for almost every $t \in \mathbb{R}$, where $g \in \mathsf{C}^r(\mathbb{R};\mathbb{C})$ and $g^{(k)} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for $k \in \{0,1,\dots,r\}$.
6. If $f \in \mathsf{C}^\infty(\mathbb{R};\mathbb{C})$ and if $f^{(k)} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for $k \in \mathbb{Z}_{\geq 0}$ then the CCFT of f has the property that
$$\lim_{|\nu|\to\infty} \nu^k \mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$$
for any $k \in \mathbb{Z}_{\geq 0}$. This follows from a repeated application of Corollary 13.1.11.
7. If $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and if the signal $t \mapsto t^r f(t) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{C}^k(\mathbb{R};\mathbb{C})$. This is Proposition 13.1.12.
8. A converse to the preceding situation is furnished by
Precisely, if $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{C}^r$
9. If $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and if $t \mapsto t^r f(t) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, $r \in \mathbb{Z}_{\geq 0}$, then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{C}^\infty(\mathbb{R};\mathbb{C})$. This follows from a repeated application of Proposition 13.1.12.

Using the CDFT, in Theorem 12.1.17 we precisely characterised periodic real analytic signals. For integrable real analytic signals, one imagines that it is possible to use the CCFT. However, for the CCFT the situation is more complicated because of the noncompactness of the domain of the signals being transformed. Indeed, it turns out that the CCFT is not the proper tool for distinguishing real analytic signals from general signals, at least in terms of their decay rate. What one *can* prove is the following.

13.1.16 Proposition (A sufficient condition on the CCFT for a signal to be real analytic) *If* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *satisfies the condition that*

$$|\mathscr{F}_{\mathrm{CC}}(f)(\nu)| \le C e^{-\alpha t}$$

for some $C, \alpha \in \mathbb{R}_{>0}$, *then* f *is almost everywhere equal to a real analytic signal.*

Proof By Theorem 13.2.26 below we have $f = \overline{\mathscr{F}}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(f)$. Note that by applying Proposition 13.1.12 to the transform $\overline{\mathscr{F}}_{\mathrm{CC}}$ instead of to $\mathscr{F}_{\mathrm{CC}}$ we have that f is almost everywhere equal to an infinitely differentiable signal. Without loss of generality we assume that f is itself infinitely differentiable. We then have, again by Proposition 13.1.12,

$$f^{(k)}(t) = (2\pi \mathrm{i})^k \int_{\mathbb{R}} \nu^k \mathscr{F}_{\mathrm{CC}}(f)(\nu) e^{2\pi \mathrm{i}\nu t} \mathrm{d}\nu, \qquad t \in \mathbb{R},\ k \in \mathbb{Z}_{\ge 0}.$$

Thus we have

$$|f^{(k)}(t)| \le C(2\pi)^k \int_{\mathbb{R}} |\nu|^k e^{-\alpha\nu}\, \mathrm{d}\nu = 2C(2\pi)^k \int_0^\infty \nu^k e^{-\alpha\nu}\, \mathrm{d}\nu = 2C(2\pi)^k \frac{1}{\alpha}\frac{k!}{\alpha^k},$$

the last step following by a repeated integration by parts. Taking $M = \frac{2C}{\alpha}$ and $r = \frac{2\pi}{\alpha}$ we have

$$|f^{(k)}(t)| \le M k! r^{-k}, \qquad t \in \mathbb{R},$$

giving real analyticity of f by Theorem **??**. ■

The sufficient condition of the preceding result on the CCFT of a signal for it to be real analytic is not necessary. We refer to Section 13.1.7 for further discussion.

As with the CDFT (see Theorem 12.1.18), it is not possible to prescribe a rate of decay for the CCFT of a general signal in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. The following result makes this precise.

13.1.17 Theorem (The CDFT decays arbitrarily slowly generally) *If* $G \in \mathsf{C}^0_0(\mathbb{R};\mathbb{R}_{\ge 0})$ *then there exists* $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *such that, for any* $\Omega \in \mathbb{R}_{>0}$, *there exists* $\nu_+, \nu_- \ge \Omega$ *for which*

$$|\mathscr{F}_{\mathrm{CC}}(f)(\nu_+)| \ge G(\nu_+), \quad |\mathscr{F}_{\mathrm{CC}}(f)(-\nu_-)| \ge G(-\nu_-).$$

Proof Define $F \in \mathsf{c}_0(\mathbb{Z};\mathbb{R}_{\ge 0})$ by $F(n) = 2G(n)$ for each $n \in \mathbb{Z}$. By Theorem 12.1.18 let $g \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ be such that

$$|\mathscr{F}_{\mathrm{CD}}(g)(n)| \ge F(n), \qquad n \in \mathbb{Z}.$$

Let $f = g\chi_{[0,T]}$. Then, for $\Omega \in \mathbb{R}_{>0}$, let $\nu_+ = \nu_-$ be the smallest integer greater than Ω. By Exercise **??** we have

$$|\mathscr{F}_{\mathrm{CC}}(f)(\nu_+)| = |\mathscr{F}_{\mathrm{CD}}(g)(\nu_+)| \ge F(\nu_+) > G(\nu_+),$$

and similarly $|\mathscr{F}_{\mathrm{CC}}(f)(-\nu_-)| > G(-\nu_-)$, giving the theorem. ■

13.1.5 Convolution, multiplication, and the L^1-CCFT

In this section we consider the interaction of convolution with the L^1-CCFT. Results for the L^2-CCFT are given in Section 13.3.2. As we saw in Proposition 12.1.19 for the CDFT, the result is a simple one: The CCFT of a convolution is the product of the convolutions.

13.1.18 Proposition (The L^1-CCFT of a convolution is the product of the L^1-CCFT's) *If* $f, g \in L^{(1)}(\mathbb{R};\mathbb{C})$ *then*

$$\mathscr{F}_{CC}(f * g)(\nu) = \mathscr{F}_{CC}(f)(\nu)\mathscr{F}_{CC}(g)(\nu)$$

for all $\nu \in \mathbb{R}$.

Proof This is a fairly straightforward application of Fubini's Theorem, the change of variables theorem, and periodicity of f:

$$\begin{aligned}
\mathscr{F}_{CD}(f * g)(\nu) &= \int_{\mathbb{R}} f * g(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s\Big)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} g(s)\Big(\int_{\mathbb{R}} f(t-s)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&= \int_{\mathbb{R}} g(\sigma)\Big(\int_{\mathbb{R}} f(\tau)\mathrm{e}^{-2\pi\mathrm{i}\nu(\sigma+\tau)}\,\mathrm{d}\tau\Big)\,\mathrm{d}\sigma \\
&= \Big(\int_{\mathbb{R}} g(\sigma)\mathrm{e}^{-2\pi\mathrm{i}\nu\sigma}\,\mathrm{d}\sigma\Big)\Big(\int_{\mathbb{R}} f(\tau)\mathrm{e}^{-2\pi\mathrm{i}\nu\tau}\,\mathrm{d}\tau\Big) \\
&= \mathscr{F}_{CD}(f)(\nu)\mathscr{F}_{CD}(g)(\nu).
\end{aligned}$$

(The reader may wish to compare this computation to that performed at some length in the proof of Theorem 11.1.5.) ■

The principle value of the preceding result is theoretical rather than computational. In ***missing stuff*** we shall see that convolution plays a crucial rôle in the theory of linear systems. However, on occasion, the result can be used to compute a CCFT, as the following example shows.

13.1.19 Example (The CCFT of a convolution) Define $f, g \in L^{(1)}(\mathbb{R};\mathbb{C})$ by

$$f(t) = \begin{cases} 1, & t \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{otherwise,} \end{cases} \qquad g(t) = \begin{cases} 1+t, & t \in [-1, 0], \\ 1-t, & t \in (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

One computes directly that $g = f * f$. Moreover, in Examples 13.1.3–3 and 13.1.3–4 we computed

$$\mathscr{F}_{CC}(f)(\nu) = \begin{cases} \frac{\sin(\pi\nu)}{\pi\nu}, & \nu \neq 0, \\ 1, & \nu = 0, \end{cases} \qquad \mathscr{F}_{CC}(g)(\nu) = \begin{cases} \frac{\sin^2(\pi\nu)}{\pi^2\nu^2}, & \nu \neq 0, \\ 1, & \nu = 0. \end{cases}$$

As predicted by the previous result, the CCFT of the convolution is the convolution of the CCFT's. •

The previous result can also be turned around.

13.1.20 Proposition (The L^1-CCFT of a product is the convolution of the L^1-CCFT's) *If* $f, g \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and if* $\mathscr{F}_{CC}(f), \mathscr{F}_{CC}(g) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$*, then*

$$\mathscr{F}_{CC}(fg)(\nu) = \mathscr{F}_{CC}(f) * \mathscr{F}_{CC}(g)(\nu), \qquad \nu \in \mathbb{R}.$$

Proof Our proof relies on some facts about the inverse of the CCFT presented in Section 13.2.

By Theorem 13.2.26 it follows that f and g are almost everywhere equal to continuous signals. Let us without loss of generality assume that f and g are continuous.

We use the fact, resulting from Theorem 13.2.26 below, that if $F \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{C}^0(\mathbb{R};\mathbb{C})$ has the property that $\mathscr{F}_{CC}(F) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, then

$$\overline{\mathscr{F}}_{CC} \circ \mathscr{F}_{CC}(F) = \mathscr{F}_{CC} \circ \overline{\mathscr{F}}_{CC}(F) = F.$$

That is, $\mathscr{F}_{CC}^{-1} = \overline{\mathscr{F}}_{CC}$ when restricted to signals having these properties.

Note that

$$\mathscr{F}_{CC}(f) * \mathscr{F}_{CC}(g) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$$

by Theorem 11.2.1. By Proposition 13.1.6(ii), Proposition 13.1.18, and the fact that $\overline{\mathscr{F}}_{CC} = \mathscr{F}_{CC}^{-1}$ for the signals with which we are dealing, we have

$$\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f) * \mathscr{F}_{CC}(g)) = fg.$$

Since $f = \overline{\mathscr{F}}_{CC} \circ \mathscr{F}_{CC}(f)$ and since $\mathscr{F}_{CC}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, it follows by Theorem 13.1.7 (essentially) that $f \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$. Thus $f \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{C})$ and so $fg \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ (why?). Thus we can take the CCFT of both sides of the preceding equation to get the result. ■

13.1.6 Alternative formulae for the CCFT

In this section we briefly and concisely provide alternative formulae for the CCFT and its inverse. The notion of the inverse of the CCFT is discussed in the next section, and it will be seen there that what we denoted prior to the statement of Proposition 13.1.6 as $\overline{\mathscr{F}}_{CC}$ serves, in some sense, as an inverse. Of course, the alternative definitions must be made so that all the relevant theorems hold. That is to say, the "inverse" must be the inverse, when it is actually defined (e.g., in the L^2-theory of Section 13.3). In this section we briefly characterise these possible alternate definitions so that they are available for easy reference when looking at other work.

13.1.21 Definition (Alternative formulae for the CCFT) Let $a, b \in \mathbb{R}$ with $b \neq 0$. For $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ define

$$\mathscr{F}_{CC}^{(a,b)}(f)(\nu) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{\mathbb{R}} f(t) e^{ib\nu t} \, dt$$
$$\overline{\mathscr{F}}_{CC}^{(a,b)}(f)(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{\mathbb{R}} f(\nu) e^{-ib\nu t} \, d\nu. \qquad \bullet$$

Note that we have $\mathscr{F}_{CC} = \mathscr{F}_{CC}^{(0,-2\pi)}$ and $\overline{\mathscr{F}}_{CC} = \overline{\mathscr{F}}_{CC}^{(0,-2\pi)}$. Other popular choices are $(a, b) = (0, 1)$ (this is used in Mathematica®), $(a, b) = (0, -1)$ (a common choice of physicists), $(a, b) = (1, -1)$ (a common choice of mathematicians and systems engineers), $(a, b) = (-1, 1)$ (used by ancient physicists), and $(a, b) = (1, 1)$ (used in some areas of probability theory). The choice $(a, b) = (0, -2\pi)$ used in this text is sometimes used in signal processing. There is no real difference between these choices in the sense that there are no important theorems that hold with one choice of (a, b) but not with another. However, it is true that in certain disciplines there are often reasons of convenience for using a particular (a, b). One place where one should use care is with Parseval's equality (see Theorem 13.3.3(ii)). For the $\mathscr{F}_{\mathrm{CC}}^{(a,b)}$ transform this reads

$$\|\mathscr{F}_{\mathrm{CC}}^{(a,b)}(f)\|_2^2 = (2\pi)^a \|f\|_2^2.$$

13.1.7 Notes

In Proposition 13.1.16 we gave a sufficient condition on the CCFT of a signal for it to be real analytic. This condition is not necessary. However, there is a sharp condition of a transform of a signal in $\mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ for it to be real analytic, but the transform is not the CCFT, but the so-called ***FBI transform***.[1] This is actually a family of transforms $\mathscr{F}_a$, $a \in \mathbb{R}_{>0}$, assigning to $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ a map $\mathscr{F}_a(f) \colon \mathbb{R}^2 \to \mathbb{C}$ by the formula

$$\mathscr{F}_a(f)(s, \nu) = \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi \mathrm{i} \nu t} \mathrm{e}^{-\pi a (t-s)^2} \, \mathrm{d}t.$$

One then has the following theorem.

Theorem [DI:75] *For* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ *and* $\mathrm{t}_0 \in \mathbb{R}$, *the following statements are equivalent:*

(i) f *is real analytic at* t_0;

(ii) *there exists* $\mathrm{C}, \mathrm{M}, \alpha \in \mathbb{R}_{>0}$ *and a neighbourhood* U *of* t_0 *such that*

$$|\mathscr{F}_{\mathrm{a}}(\mathrm{f})(\mathrm{s}, \mathrm{a}\nu)| \leq \mathrm{C}\mathrm{e}^{-\alpha \mathrm{a}}$$

for all $\mathrm{a} \in \mathbb{R}_{>0}$, $\mathrm{s} \in \mathrm{U}$, *and* $\nu \in \mathbb{R}$ *such that* $|\nu| \geq \mathrm{M}$.

Exercises

13.1.1 Let $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$.

(a) For $a \in \mathbb{R}$, show that $|\mathscr{F}_{\mathrm{CC}}(\tau_a^* f)(\nu)| = |\mathscr{F}_{\mathrm{CC}}(f)(\nu)|$ for each $\nu \in \mathbb{R}$.

(b) For which values of $a \in \mathbb{R}$ is it true that $\arg(\mathscr{F}_{\mathrm{CC}}(\tau_a^* f)(\nu)) = \arg(\mathscr{F}_{\mathrm{CC}}(f)(\nu))$ for every $\nu \in \mathbb{R}$? Does your conclusion depend on f?

(c) Find a codomain transformation $\phi \colon \mathbb{C} \to \mathbb{C}$ such that $\arg(\mathscr{F}_{\mathrm{CC}}(\phi \circ f)(\nu)) = \arg(\mathscr{F}_{\mathrm{CD}}(f)(\nu))$ for every $\nu \in \mathbb{R}$?

13.1.2 Let $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ and let $a \in \mathbb{R}$. Define $f_a(t) = \mathrm{e}^{2\pi \mathrm{i} a t} f(t)$.

(a) Show that $f_a \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$.

(b) Show that $\mathscr{F}_{\mathrm{CC}}(f_a)(\nu) = \mathscr{F}_{\mathrm{CC}}(f)(\nu - a)$.

[1] Named after Fourier, as well as Jacques Bros and Daniel Iagolnitzer.

13.1.3 Let $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and let $\lambda \in \mathbb{R}_{>0}$. Define $f_\lambda(t) = f(\lambda t)$.

(a) Show that $f_\lambda \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.

(b) Show that $\mathscr{F}_{\mathrm{CC}}(f_\lambda)(\nu) = \lambda^{-1}\mathscr{F}_{\mathrm{CC}}(f)(\lambda^{-1}\nu)$.

13.1.4 Prove Corollary 13.1.13. That is, show that if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ has compact support then $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable.

13.1.5 In Table 13.1 you are given the expressions for four signals, all defined on $\mathbb{R}$, along with the graphs of four CCFT's. Match the signal with the appropriate CCFT.

13.1.6 Answer the following questions. Note that "nonzero signal" means "signal that is nonzero on a set of positive measure." The last two parts of the question can only be answered after the material from Section 13.3 has been understood.

(a) Find a nonzero signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ with the property that $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ but that the signal $\nu \mapsto \nu\mathscr{F}_{\mathrm{CC}}(f)(\nu)$ is not in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.

(b) Find a nonzero signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ with the property that $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{C}^0(\mathbb{R};\mathbb{C})$ but that $\mathscr{F}_{\mathrm{CC}}(f) \notin \mathsf{C}^1(\mathbb{R};\mathbb{C})$.

(c) Find a nonzero signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ with the property that the signal $\nu \mapsto \nu^k\mathscr{F}_{\mathrm{CC}}(f)(\nu)$ is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for each $k \in \mathbb{Z}_{>0}$.

(d) Find a nonzero signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ with the property that $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable.

(e) Find a nonzero signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ with the property that $\mathscr{F}_{\mathrm{CC}}(f) \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and that $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$.

(f) For the signal f from part (e), explain the meaning of the expression

$$f(t) \text{ “=” } \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu.$$

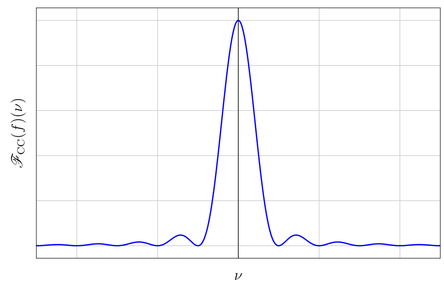
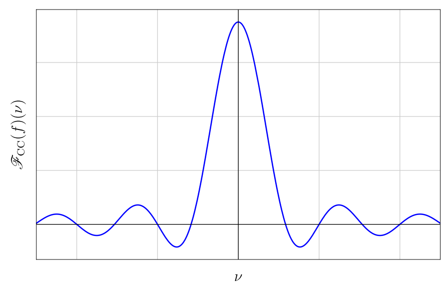
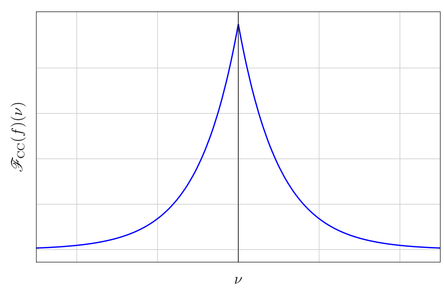
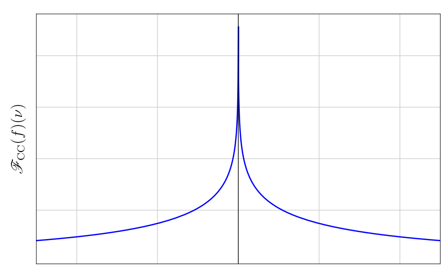
13.1.7 For the following five signals, list them in order of smoothness of their CCFT's, the least smooth being first and the most smooth being fifth:

1. $f_1(t) = \mathrm{e}^{-|t|}$;
2. $f_2(t) = \frac{1}{1+|t|}$;
3. $f_3(t) = \mathrm{e}^{-t^2}$;
4. $f_4(t) = \frac{1}{1+t^2}$;
5. $f_5(t) = \frac{t}{1+|t|^3}$.

13.1.8 For the following five CCFT's, list them in order of the rate of decay at infinity of the corresponding signals, the slowest decaying being first and the most rapidly decaying being fifth:

1. $\mathscr{F}_{\mathrm{CC}}(f_1)(\nu) = 1(\nu)\mathrm{e}^{-\nu}$, where $\nu \mapsto 1(\nu)$ is the step signal which is 1 for $\nu \geq 0$ and 0 for $\nu < 0$;
2. $\mathscr{F}_{\mathrm{CC}}(f_2)(\nu) = \mathrm{e}^{-|\nu|}$;
3. $\mathscr{F}_{\mathrm{CC}}(f_3)(\nu) = \mathrm{e}^{-\nu^2}$;

Table 13.1 Table of signals and graphs of CCFT's

Signals	Graphs of CCFT's
1. $t \mapsto \begin{cases} 1+t, & t \in [-\frac{1}{2}, 0), \\ 1-t, & t \in [0, \frac{1}{2}], \\ 0, & \text{otherwise} \end{cases}$	**1.** 
2. $t \mapsto \begin{cases} 1+t, & t \in [-1, 0), \\ 1-t, & t \in [0, 1], \\ 0, & \text{otherwise} \end{cases}$	**2.** 
3. $t \mapsto \dfrac{1}{1+80\lvert t\rvert}$	**3.** 
4. $t \mapsto \dfrac{1}{1+40t^2}$	**4.** 

4. $\mathscr{F}_{\mathrm{CC}}(f_4)(\nu) = \begin{cases} 0, & \lvert\nu\rvert > 1, \\ 1+\nu, & \nu \in [-1, 0], \\ 1-\nu, & \nu \in (0, 1]; \end{cases}$

5. $\mathscr{F}_{\mathrm{CC}}(f_5)(\nu) = \begin{cases} 0, & \lvert\nu\rvert > 1, \\ \nu, & \nu \in [-1, 0], \\ -\nu, & \nu \in (0, 1]. \end{cases}$

Section 13.2

Inversion of the CCFT

As with the CDFT, the matter of the invertibility of the CCFT (in an appropriate sense) is of vital importance if the transform is to have meaning. In this section we turn to precisely this matter. The discussion bears much similarity to that for the CDFT, although there are some technical distinctions that crop up to make life more difficult.

Do I need to read this section? If you are reading this chapter, then this section is a very important part of it. •

13.2.1 Preparatory work

In order to begin to think of an inverse for the CCFT, we need to be sure an inverse exists in an appropriate sense. The following result then serves to make sense of what we are doing in this section.

13.2.1 Theorem (The CCFT is injective) *The map* $\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ *is injective.*

Proof Recall from Example 11.3.7–3 the definition of the Fejér kernel:

$$F_\Omega(t) = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \neq 0, \\ \Omega, & t = 0. \end{cases}$$

Suppose that $\mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$ for every $\nu \in \mathbb{R}$. By Theorem 5.9.35 the theorem will follow if we can show that $f(t) = 0$ for every Lebesgue point t for f. So suppose that $t_0 \in \mathbb{R}$ is a Lebesgue point for f. By Proposition 13.1.6(v) we may without loss of generality assume that $t_0 = 0$. Since $\mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$ for every $\nu \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = 0$$

for every $\nu \in \mathbb{R}$. Then, using Fubini's Theorem,

$$\int_{-a}^{a}\Big(\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t\Big)\,\mathrm{d}\nu = \int_{\mathbb{R}} f(t)\Big(\int_{-a}^{a}\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\,\mathrm{d}t = \int_{\mathbb{R}} f(t)\frac{\sin(2\pi a t)}{\pi t}\,\mathrm{d}t = 0$$

for every $a \in \mathbb{R}_{>0}$ (see Example 13.1.3–3 for the easy integral computed in the above computation). Therefore, arguing in the same way,

$$\begin{aligned}\frac{1}{\Omega}\int_0^{\Omega}\Big(\int_{\mathbb{R}} f(t)\frac{\sin(2\pi a t)}{\pi t}\,\mathrm{d}t\Big)\,\mathrm{d}a &= \frac{1}{\Omega}\int_{\mathbb{R}} f(t)\Big(\int_0^{\Omega}\frac{\sin(2\pi a t)}{\pi t}\,\mathrm{d}a\Big)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f(t)\frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2} = \int_{\mathbb{R}} f(t)F_\Omega(t)\,\mathrm{d}t = 0\end{aligned}$$

for every $\Omega \in \mathbb{R}_{>0}$, using Lemma 4 from Example 11.3.7–3. Since F_Ω is even we have

$$\int_{\mathbb{R}} f(t)F_\Omega(t)\,\mathrm{d}t = 0 \quad\iff\quad \int_{\mathbb{R}}(f(t) + f(-t))F_\Omega\,\mathrm{d}t = 0.$$

We may thus suppose without loss of generality that f is even and so that

$$\int_0^\infty f(t)F_\Omega(t)\,\mathrm{d}t = 0$$

for all $\Omega \in \mathbb{R}_{>0}$. Since

$$\int_{\mathbb{R}} F_\Omega(t)\,\mathrm{d}t = 1$$

(as shown in Example 11.3.7–3), and since F_Ω is even, we have

$$\tfrac{1}{2}f(0) = \int_0^\infty f(0)F_\Omega(t)\,\mathrm{d}t \quad \implies \quad \tfrac{1}{2}f(0) = \int_0^\infty (f(t)-f(0))F_\Omega(t)\,\mathrm{d}t.$$

Thus, to show that $f(0) = 0$ it suffices to show that

$$\int_0^\infty (f(t)-f(0))F_\Omega(t)\,\mathrm{d}t,$$

and this is what we shall do.

Let $\epsilon \in \mathbb{R}_{>0}$ and for brevity in what follows let us denote $\epsilon_1 = \frac{\pi^2}{2(4+\pi^2)}\epsilon$. Since 0 is a Lebesgue point for f we have

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(t)-f(0)|\,\mathrm{d}t = 0.$$

Thus choose $h_0 \in \mathbb{R}_{>0}$ such that

$$\frac{1}{h}\int_0^h |f(t)-f(0)|\,\mathrm{d}t < \epsilon_1$$

for all $h \in (0, h_0]$.

Let $\Omega_1 \in \mathbb{R}_{>0}$ be such that $\frac{1}{\Omega_1} < h_0$. For $x \in \mathbb{R}_{>0}$ we have

$$\sin(x) \le x \quad \implies \quad \sin^2(\pi\Omega t) \le \pi^2\Omega^2t^2$$

for all $t, \Omega \in \mathbb{R}_{>0}$. Thus we have

$$\int_0^{1/\Omega_1} |f(t)-f(0)|F_{\Omega_1}(t)\,\mathrm{d}t = \Omega_1\int_0^{1/\Omega_1} |f(t)-f(0)|\frac{\sin^2(\pi\Omega_1 t)}{\pi^2\Omega_1^2t^2}\,\mathrm{d}t \le \epsilon_1. \tag{13.1}$$

We also have

$$\begin{aligned}\int_{1/\Omega_1}^{h_0} |f(t)-f(0)|F_{\Omega_1}(t)\,\mathrm{d}t &\le \int_{1/\Omega_1}^{h_0} \frac{|f(t)-f(0)|}{\pi^2\Omega_1 t^2}\,\mathrm{d}t \\ &= \frac{1}{\pi^2\Omega_1}\frac{1}{t^2}\int_0^t |f(\tau)-f(0)|\,\mathrm{d}\tau\Big|_{t=1/\Omega_1}^{t=h_0} + \frac{2}{\pi^2\Omega_1}\int_{1/\Omega_1}^{h_0}\frac{\int_0^t|f(\tau)-f(0)|\,\mathrm{d}\tau}{t^3}\,\mathrm{d}t\end{aligned}$$

using integration by parts. Now we have

$$\frac{1}{\pi^2\Omega_1}\frac{1}{h_0^2}\int_0^{h_0} |f(\tau)-f(0)|\,\mathrm{d}\tau \le \frac{1}{\pi^2h_0}\int_0^{h_0} |f(\tau)-f(0)|\,\mathrm{d}\tau \le \frac{\epsilon_1}{\pi^2} \tag{13.2}$$

and

$$\frac{\Omega_1}{\pi^2}\int_0^{1/\Omega_1} |f(\tau) - f(0)|\,\mathrm{d}\tau \le \frac{\epsilon_1}{\pi^2} \tag{13.3}$$

by definition of h_0 and Ω_1. We also have

$$\frac{2}{\pi^2\Omega_1}\int_{1/\Omega_1}^{h_0} \frac{\int_0^t |f(\tau) - f(0)|\,\mathrm{d}\tau}{t^3}\,\mathrm{d}t \le \frac{2}{\pi^2\Omega_1}\int_{1/\Omega_1}^{h_0} \frac{\epsilon_1}{t^2}\,\mathrm{d}t \le \frac{2\epsilon_1}{\pi^2}, \tag{13.4}$$

again using the properties of h_0 and Ω_1. Combining (13.2), (13.3), and (13.4) we have

$$\int_{1/\Omega_1}^{h_0} |f(t) - f(0)| F_{\Omega_1}(t)\,\mathrm{d}t \le \frac{4\epsilon_1}{\pi^2}. \tag{13.5}$$

Since $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ let $\Omega_2 \in \mathbb{R}_{>0}$ be such that $\frac{1}{\pi^2\Omega_2\alpha^2} < \frac{\epsilon}{2\|f\|_1}$. Then $|F_\Omega(t)| < \frac{\epsilon}{2\|f\|_1}$ for all $t \in (-\infty, -\Omega_2] \cup [\Omega_2, \infty)$. Then

$$\left|\int_{|t|\ge h_0} f(t)F_\Omega(t)\,\mathrm{d}t\right| \le \int_{|t|\ge h_0} |f(t)|F_\Omega(t) \le \|f\|_1 \sup\{F_\Omega(t) \mid |t| \ge h_0\} < \frac{\epsilon}{2} \tag{13.6}$$

for $\Omega \ge \Omega_2$.

Now, for $\Omega \ge \max\{\Omega_1, \Omega_2\}$, combining (13.1), (13.5), and (13.6) gives

$$\left|\int_0^\infty (f(t) - f(0))F_\Omega(t)\,\mathrm{d}t\right| \le \int_0^\infty |f(t) - f(0)|F_\Omega(t)\,\mathrm{d}t < \epsilon,$$

so proving the theorem. ■

As with the corresponding Theorem 12.2.1 for the CDFT, the preceding theorem is a little technical, and it deserves to be since it is telling us that the CCFT faithfully preserves signals in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$. During the course of the proof we used the continuous Fejér kernel first described in Example 11.3.7–3. This kernel is plotted in Figure 11.19 for a few values of Ω. As with the discrete Fejér kernel, the essential feature is that as Ω becomes large, the signal becomes “concentrated” around the origin. Indeed, as we showed in Example 11.3.7–3, the family $(F_\Omega)_{\Omega\in\mathbb{R}_{>0}}$ is an approximate identity.

13.2.2 Example (The CCFT is not onto $\mathsf{C}^0_0(\mathbb{R};\mathbb{C})$) The holding of Theorem 13.2.1 makes one wonder if $\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ is surjective. It is not. We shall demonstrate this with a counterexample, although we will refer ahead to other results proved in this section to verify what we assert. Consider the signal

$$F\colon \nu \mapsto \frac{\nu}{(1+|\nu|)\log(2+|\nu|)}$$

which is easily verified to $\mathsf{C}^0_0(\mathbb{R};\mathbb{C})$. Suppose that $F = \mathscr{F}_{\mathrm{CC}}(f)$ for $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. Let $g = \chi_{[0,1]}$ so that

$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \frac{1 - \mathrm{e}^{-2\pi\mathrm{i}\nu}}{2\pi\mathrm{i}\nu},$$

as may be directly computed. By Theorem 13.2.24 below it follows that, with the exception of the points $t = 0$ and $t = 1$,

$$g(t) = \lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} g(t-\tau) D_\Omega(\tau)\,\mathrm{d}\tau = \lim_{\Omega\to\infty} D_\Omega g(t),$$

where D_Ω is the continuous Dirichlet kernel defined in Example 11.3.7–5. Moreover, by Theorem 13.2.33 the convergence is bounded. Thus, by the Dominated Convergence Theorem, Fubini's Theorem, and the change of variable formula

$$\begin{aligned}
\int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t &= \lim_{\Omega\to\infty} \int_{\mathbb{R}} f(t) D_\Omega g(t)\,\mathrm{d}t = \lim_{\Omega\to\infty} \int_{\mathbb{R}} f(t)\Big(\int_{-\Omega}^{\Omega} g(t-\tau) D_\Omega(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}t \\
&= \lim_{\Omega\to\infty} \int_{\mathbb{R}} f(t)\Big(\int_{t-\Omega}^{t+\Omega} g(s) D_\Omega(t-s)\,\mathrm{d}s\Big)\,\mathrm{d}t \\
&= \lim_{\Omega\to\infty} \int_{\mathbb{R}} g(s)\Big(\int_{s-\Omega}^{s+\Omega} f(t) D_\Omega(t-s)\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&= \lim_{\Omega\to\infty} \int_{\mathbb{R}} g(s)\Big(\int_{-\Omega}^{\Omega} f(s-\tau) D_\Omega(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = \lim_{\Omega\to\infty} \int_{\mathbb{R}} g(t) D_\Omega f(t)\,\mathrm{d}t.
\end{aligned}$$

By Lemma 13.2.7 we have

$$D_\Omega f(t) = \int_{-\Omega}^{\Omega} F(\nu) e^{2\pi \mathrm{i}\nu t}\,\mathrm{d}\nu.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t &= \lim_{\Omega\to\infty} \int_{\mathbb{R}} g(t)\Big(\int_{-\Omega}^{\Omega} F(\nu) e^{2\pi \mathrm{i}\nu t_0}\,\mathrm{d}\nu\Big)\,\mathrm{d}t \\
&= \lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} F(\nu)\Big(\int_{\mathbb{R}} g(t) \mathrm{e}^{2\pi \mathrm{i}\nu t}\,\mathrm{d}t\Big)\,\mathrm{d}\nu \\
&= \int_{-\Omega}^{\Omega} F(\nu) \mathscr{F}_{\mathrm{CC}}(g)(-\nu)\,\mathrm{d}\nu = \int_{-\Omega}^{\Omega} F(-\nu) \mathscr{F}_{\mathrm{CC}}(g)(\nu)\,\mathrm{d}\nu.
\end{aligned}$$

The real part of this last integral is precisely

$$\int_{-\Omega}^{\Omega} \frac{1-\cos(2\pi\nu)}{(1+|\nu|)\log(2+|\nu|)}\,\mathrm{d}\nu$$

which diverges as $\Omega \to \infty$. However, clearly $fg \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ and the resulting contradiction implies that F cannot be the CCFT of any signal in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$. •

Let us close this section by making some remarks mirroring those we made for the CDFT.

13.2.3 Remarks (On inversion of the CCFT)

1. Theorem 13.2.1 ensures that there exists a map $\mathscr{I}_{\mathrm{CC}}\colon \mathsf{C}^0_0(\mathbb{R};\mathbb{C}) \to \mathsf{L}^1(\mathbb{R};\mathbb{C})$ such that $\mathscr{I}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(f) = f$ for $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$, i.e., a left-inverse for $\mathscr{F}_{\mathrm{CC}}$. Indeed, there will be many such inverses, even linear ones. However, the existence of a left-inverse is not of much use. One would instead like to have a left-inverse with some properties one enjoys.
2. Another approach to inversion of the CCFT is to propose an inverse and see which signals can be recovered from their CCFT by the proposed inverse. We shall spend a good deal of effort in this section doing precisely this for various left-inverses. •

13.2.2 The Fourier integral

With the CDFT one can ask whether a signal can be recovered by its Fourier series, and this focused our discussion of the inverse of the CDFT in Section 12.2. For the CCFT, the natural inverse is produced by another integral transformation, in this case it turns out to be the transformation $\overline{\mathscr{F}}_{\mathrm{CC}}$ defined in the discussion preceding Proposition 13.1.6. That is, we propose that the map $\mathscr{F}^{-1}_{\mathrm{CC}}\colon \mathsf{C}^0_0(\mathbb{R};\mathbb{C}) \to \mathsf{L}^1(\mathbb{R};\mathbb{C})$ defined by

$$\mathscr{F}^{-1}_{\mathrm{CC}}(F)(t) = \int_{\mathbb{R}} F(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

is an inverse for $\mathscr{F}_{\mathrm{CC}}$. Of course, this is absurd since the integral will not be defined for general frequency-domain signals $F \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$. But possibly it holds that for every $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ we have

$$f(t) = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu.$$

However, just as with the CDFT, this idea fails spectacularly, and we discuss some of the fascinating reasons for this in Section 13.2.9. Nonetheless, we will attempt to describe signals for which the preceding formula holds.

With the above as motivation, we introduce the following terminology.

13.2.4 Definition (Fourier integral) For $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ the ***Fourier integral*** for f is

$$\mathrm{FI}[f](t) = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu,$$

disregarding whether the integral converges. The ***real Fourier integral*** for f is

$$\mathrm{FI}[f](t) = \int_0^\infty \mathscr{C}_{\mathrm{CC}}(f)(\nu)\cos(2\pi\nu t)\,\mathrm{d}\nu + \int_0^\infty \mathscr{S}_{\mathrm{CC}}(f)(\nu)\sin(2\pi\nu t)\,\mathrm{d}\nu,$$

again disregarding convergence of the integral. •

The use of the same symbol $\mathrm{FI}[f]$ for two possibly different things is justified since they are actually the same; see Exercise 13.2.3.

13.2.5 Remark (The meaning of "disregarding whether the integral converges") The expression "disregarding whether the integral converges" in the preceding definition is admittedly vague. What we mean is that we will consider conditions under which the integral makes sense, and the manner in which it makes sense. If one want to attach precise meaning to the Fourier integral at this point, one could say that $\mathrm{FI}[f]$ is the element of $\mathscr{S}'(\mathbb{R};\mathbb{C})$ defined by

$$\mathrm{FI}[f](\phi) = \theta_{\mathscr{F}_{\mathrm{CC}}(f)}(\overline{\mathscr{F}}_{\mathrm{CC}}(\phi)) = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\Big(\int_{\mathbb{R}} \phi(t)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t\Big)\,\mathrm{d}\nu \tag{13.7}$$

for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. Note that $\overline{\mathscr{F}}_{\mathrm{CC}}(f) \in \mathscr{S}(\mathbb{R};\mathbb{C})$ (as we shall prove in Theorem 13.4.1 below), and so the above integral makes sense as written. However, it is *not* generally equal to

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\phi(t)\,\mathrm{d}t$$

since the inner integral does not generally exist. The formula for $\mathrm{FI}[f]$ as a tempered distribution has its justification in the Fourier Reciprocity Relation, Proposition 13.1.9, applied to $\overline{\mathscr{F}}_{\mathrm{CC}}$. Precisely, suppose that $\mathrm{FI}[f](t)$ is well-defined as an integral for each $t \in \mathbb{R}$ (as in the definition) and that $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. Then, according to Proposition 13.1.9,

$$\int_{\mathbb{R}} \mathrm{FI}[f](t)\phi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \overline{\mathscr{F}}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(f))(t)\phi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\overline{\mathscr{F}}_{\mathrm{CC}}(\phi)(\nu)\,\mathrm{d}\nu,$$

which is our defining formula (13.7) in this case. Note, however, that (13.7) makes sense for all $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. This all being said, we shall not make use of this precise characterisation of $\mathrm{FI}[f]$. •

We advise the reader to compare the definition of the Fourier integral with the definition of Fourier series and convince themselves that these are really two manifestations of the same thing. The reader should not sleep until they are reconciled to this idea.

13.2.6 Remark (The usual rôle of the Fourier integral) While our presentation exploits the similarities between the CDFT and the CCFT as much as possible, this is not the usual way of doing things. The usual presentation of the material goes under the names of "Fourier series" and "Fourier transform." Note that these are actually the opposite concepts! Fourier series has to do with inversion of the CDFT and the usual Fourier transform is exactly the CCFT. In the typical presentation, the Fourier integral is presented as we have presented it: as having to do with inverting the CCFT. Careless treatments will simply say that the Fourier integral *is* the inverse of the Fourier transform, and leave it at that, seemingly not caring that the formula makes no sense. •

We will study the convergence of the Fourier integral. In order to do this we define the partial sums (we hope the reader will forgive us calling these partial sums, even though they are integrals)

$$f_\Omega(t) = \int_{-\Omega}^{\Omega} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\, d\nu$$

and

$$f_\Omega(t) = \int_0^{\Omega} \mathscr{C}_{CC}(f)(\nu)\cos(2\pi\nu t)\, d\nu + \int_0^{\Omega} \mathscr{S}_{CC}(f)(\nu)\sin(2\pi\nu t)\, d\nu,$$

for $\Omega \in \mathbb{R}_{>0}$. The reader can show in Exercise 13.2.3 that the use of f_Ω in both equations is justified.

To give a useful formula for these partial sums we recall the continuous Dirichlet kernel first introduced in Example 11.3.7–5:

$$D_\Omega(t) = \begin{cases} \frac{\sin(2\pi\Omega t)}{\pi t}, & t \neq 0, \\ 2\Omega, & t = 0. \end{cases}$$

In Figure 11.22 we plot D_Ω for a few values of Ω. We note the similarity with the behaviour of D_Ω with that of $D^{\mathrm{per}}_{T,N}$. One of the fundamental differences is that the Dirichlet kernel is periodic, befitting its use for the CDFT, but the continuous Dirichlet kernel is not. We shall investigate this further in Section **??**. As with the discrete Dirichlet kernel, the continuous Dirichlet kernel does not define an approximate identity.

The following lemma gives a useful formula for these partial sums.

13.2.7 Lemma (Partial sums and the Dirichlet kernel) *For* $f \in L^{(1)}(\mathbb{R}; \mathbb{C})$ *we have*

$$f_\Omega(t) = \int_{\mathbb{R}} f(t-\tau) D_\Omega(\tau)\, d\tau$$

for every $\Omega \in \mathbb{R}_{>0}$.

Proof We compute, using Fubini's Theorem and the computations of Example 13.1.3–3,

$$\begin{aligned}
\int_{-\Omega}^{\Omega} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\, d\nu &= \int_{-\Omega}^{\Omega}\Big(\int_{\mathbb{R}} f(\tau)e^{-2\pi i\nu\tau}\, d\tau\Big)e^{2\pi i\nu t}\, d\nu \\
&= \int_{\mathbb{R}} f(\tau)\Big(\int_{-\Omega}^{\Omega} e^{2\pi i\nu(t-\tau)}\, d\nu\Big)\, d\tau \\
&= \int_{\mathbb{R}} f(t-\tau)\Big(\int_{-\Omega}^{\Omega} e^{2\pi i\nu\tau}\, d\nu\Big)\, d\tau \\
&= \int_{\mathbb{R}} f(t-\tau)\frac{\sin(2\pi\Omega\tau)}{\pi\tau}\, d\tau,
\end{aligned}$$

as desired. ■

We comment that (f, D_Ω) is convolvable by Corollary 11.2.10, and that $f * D_\Omega$ is continuous and bounded. Based on this we use the following notation.

13.2.8 Notation ($D_\Omega f$) For $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and for $\Omega \in \mathbb{R}_{>0}$ we shall denote

$$D_\Omega f(t) = \int_{\mathbb{R}} f(t-\tau) D_\Omega(\tau)\,\mathrm{d}\tau.$$

The notation is intended to suggest the rôle of convolution in the partial sums. •

Based on the preceding lemma and notation, when we talk about convergence of the Fourier integral we will speak of convergence of the net $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$. As with sequences of signals, there are various flavours of convergence and we refer to Section 3.5 for these. It is also important to point out that if the limit

$$\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

exists, this does *not* mean that the integral

$$\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

exists in the usual sense. It is important to keep this in mind.

Let us give a quick example of a Fourier integral and examine its partial sums.

13.2.9 Example (A sample Fourier integral) We consider the signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ defined by

$$f(t) = \begin{cases} t, & t \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

We may compute

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \mathrm{i}\frac{2\pi\nu\cos(2\pi\nu) - \sin(2\pi\nu)}{2\pi^2\nu^2}.$$

Thus

$$FI[f](t) = \frac{\mathrm{i}}{2\pi^2}\int_{\mathbb{R}} \frac{2\pi\nu\cos(2\pi\nu) - \sin(2\pi\nu)}{\nu^2}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu.$$

In Figure 13.7 we show the signal and a few partial sums. Let us comment on the behaviour of the partial sums as a preview for the kinds of things that will be of interest for us.

1. At points of continuity of f, the partial sums appear to be converging nicely. As with Fourier series, it turns out that it is differentiability, not continuity, that is tied to pointwise convergence of the Fourier integral. This will be proved in Corollary 13.2.18.
2. At points of discontinuity of f the region of approximation for the partial sums get larger, but the approximation does not seem to get better. There is a theorem that describes this, and we consider this in Section 13.2.6. •

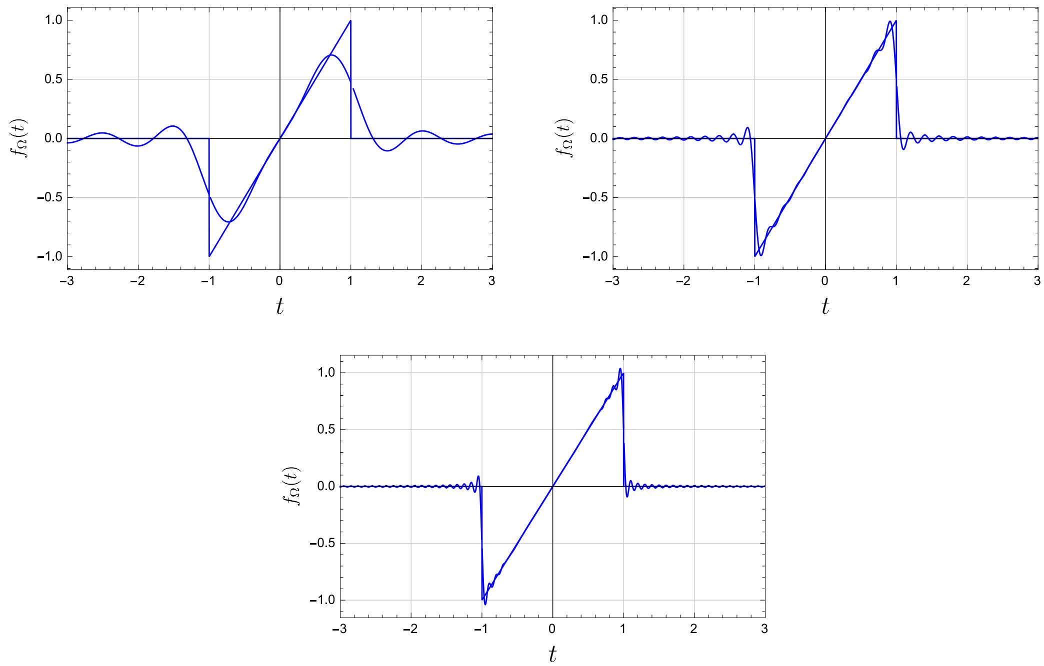


Figure 13.7 The partial sums for the Fourier integral for $\Omega = 1$ (top left), $\Omega = 5$ (top right), and $\Omega = 10$ (bottom)

13.2.3 Divergence of Fourier integrals

Before we consider specific conditions under which Fourier integral converge, let us first show that generally they will not converge. Fortunately, a lot of the work has been done for us when we considered such results for the CDFT in Section 12.2.3. Here we are able to adapt many of the constructions from that section.

We begin with a continuous signal whose Fourier integral vanishes at a point.

13.2.10 Example (A continuous signal whose Fourier integral diverges at a point) In Example 12.2.10 we considered a continuous signal g (denoted by f in Example 12.2.10) of period 2π whose Fourier series diverged at $t = 0$. Define $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ by

$$f(t) = \begin{cases} f(t), & t \in [-\pi, \pi] \\ 0, & \text{otherwise.} \end{cases}$$

Since $g(-\pi) = g(\pi) = 0$ it follows that f is continuous. By Theorem 12.2.22 this means that there exists $\epsilon \in (0, \pi)$ such that the limit

$$\lim_{N\to\infty} \int_{-\epsilon}^{\epsilon} f(-t) \frac{\sin\left((2N+1)\pi \frac{t}{T}\right)}{t} \, dt$$

does not exist. By Theorem 13.2.14 and Proposition 2.3.29 we have that $(D_\Omega f(t_0))_{\Omega \in \mathbb{R}_{>0}}$ does not converge. •

Next we show that there can be continuous signals whose Fourier integrals vanish on a given set of measure zero.

13.2.11 Theorem (Continuous signals can have Fourier series diverging on a set of measure zero) *If* $\mathsf{Z} \subseteq \mathbb{R}$ *has Lebesgue measure zero, then there exists* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{C}^0(\mathbb{R};\mathbb{C})$ *such that* $(\mathrm{D}_\Omega \mathrm{f}(\mathrm{t}))_{\Omega\in\mathbb{R}_{>0}}$ *diverges for every* $\mathrm{t} \in \mathbb{R}$.

Proof Let $Z_n = Z \cap [n, n+1)$ and let

$$A_n = \{n + t \mid t \in Z_n\}, \quad A = \cup_{n\in\mathbb{Z}_{>0}} A_n.$$

Note that $A \subseteq [0, 1)$ has measure zero, being a countable union of sets of measure zero. By Theorem 12.2.18 there exists $g \in \mathsf{C}^0_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ such that $(D^{\mathrm{per}}_{1,N} g(t))_{N\in\mathbb{Z}_{>0}}$ diverges for every $t \in [0, 1)$. Define $f : \mathbb{R} \to \mathbb{C}$ by asking that $f(t) = \frac{1}{2^{|n|}} g(t)$ if $t \in [n, n + 1)$. Note that

$$\int_{\mathbb{R}} |f(t)|\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} \frac{1}{2^{|n|}} \int_0^1 |g(t)|\,\mathrm{d}t < \infty$$

missing stuff ■

Our first result deals with pointwise divergence of Fourier integrals.

13.2.12 Theorem (Integrable signals can have Fourier series diverging everywhere) *There exists* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *such that* $(\mathrm{D}_\Omega \mathrm{f}(\mathrm{t}))_{\Omega\in\mathbb{R}_{>0}}$ *diverges for almost every* $\mathrm{t} \in \mathbb{R}$.

Proof By Theorem 12.2.20 let $g \in \mathsf{L}^{(1)}_{\mathrm{per},1}(\mathbb{R};\mathbb{C})$ be such that $(D^{\mathrm{per}}_{1,N} g(t))_{N\in\mathbb{Z}_{>0}}$ diverges for almost every $t \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{C}$ by asking that $f(t) = \frac{1}{2^{|n|}} g(t-n)$ if $t \in [n, n+1)$. Note that

$$\int_{\mathbb{R}} |f(t)|\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} \frac{1}{2^{|n|}} \int_0^1 |g(t)|\,\mathrm{d}t < \infty$$

by Example 2.4.2–**??**. Thus $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Let $t_0 \in \mathbb{R}$ be a time for which $t \in (n, n + 1)$ for some $n \in \mathbb{Z}$ and for which $(D^{\mathrm{per}}_{1,N} g(t_0))_{N\in\mathbb{Z}_{>0}}$ diverges. Note that the set of all such times has a complement whose measure is zero. By Theorem 12.2.22 it follows that there exists $\epsilon \in \mathbb{R}_{>0}$ such that the limit

$$\lim_{N\to\infty} \int_{-\epsilon}^{\epsilon} g(t_0 - t) \frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t$$

does not exist. Let us assume that ϵ is sufficiently small that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq (n, n + 1)$. By definition of f it then follows that the limit

$$\lim_{N\to\infty} \int_{-\epsilon}^{\epsilon} f(t_0 - t) \frac{\sin((2N+1)\pi\frac{t}{T})}{t}\,\mathrm{d}t$$

does not exist. Then it follows from Theorem 13.2.14 and Proposition 2.3.29 that $(D_\Omega f(t_0))_{\Omega\in\mathbb{R}_{>0}}$ does not converge, as desired. ■

Next we have an example that illustrates that "nice" signals can give rise to divergent Fourier integrals.

Finally we consider the lack of norm convergence of Fourier integrals.

13.2.13 Example (A signal whose Fourier integral diverges in the L^1-norm) Let us take $f = \chi_{[-1,1]}$. By Example 13.1.3–3 we have $\mathscr{F}_{\mathrm{CC}}(f) = D_1$. By Theorem 13.2.21 we have that $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ converges pointwise to the signal

$$g(t) = \begin{cases} 1, & t \in (-1,1), \\ \frac{1}{2}, & t \in \{-1,1\}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $g(t) = f(t)$ for almost every $t \in \mathbb{R}$, it follows that if $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ converges in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$, it must necessarily converge to f. Thus, supposing that this convergence to f holds,

$$\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} D_\Omega$$

missing stuff

13.2.4 Pointwise convergence of Fourier integrals

As with our treatment of the CDFT, we begin with a discussion of the general conditions that ensure convergence of the inverse CCFT at a point $t \in \mathbb{R}$. The basic result is the following, recalling from Definition 5.9.9 the conditional Lebesgue integral.

13.2.14 Theorem (Pointwise convergence of Fourier integrals) *Let* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$*, let* $\mathrm{t}_0 \in \mathbb{R}$*, and let* $\mathrm{s} \in \mathbb{C}$*. The following statements are equivalent:*

(i) $\displaystyle\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu\mathrm{t}_0}\,\mathrm{d}\nu = \mathrm{s}$*;*

(ii) $\displaystyle\lim_{\Omega\to\infty} \int_{\mathbb{R}} \mathrm{f}(\mathrm{t}_0 - \mathrm{t})\mathrm{D}_\Omega(\mathrm{t})\,\mathrm{d}\mathrm{t} = \mathrm{s}$*;*

(iii) $\displaystyle\lim_{\Omega\to\infty} \mathrm{C}\int_{\mathbb{R}} (\mathrm{f}(\mathrm{t}_0 - \mathrm{t}) - \mathrm{s})\mathrm{D}_\Omega(\mathrm{t})\,\mathrm{d}\mathrm{t} = 0$*;*

(iv) for each $\epsilon \in \mathbb{R}_{>0}$ *we have* $\displaystyle\lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} (\mathrm{f}(\mathrm{t}_0 - \mathrm{t}))\mathrm{D}_\Omega(\mathrm{t})\,\mathrm{d}\mathrm{t} = \mathrm{s}$*;*

(v) for each $\epsilon \in \mathbb{R}_{>0}$ *we have* $\displaystyle\lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} (\mathrm{f}(\mathrm{t}_0 - \mathrm{t}) - \mathrm{s})\mathrm{D}_\Omega(\mathrm{t})\,\mathrm{d}\mathrm{t} = \mathrm{s}$*.*

Proof The equivalence of parts (i) and (ii) is obvious, given Lemma 13.2.7.

The equivalence of parts (i) and (iii) follows after we use from Lemma 1 from Example 11.3.7–3, along with a change of variable, to see that

$$C\int_{\mathbb{R}} D_\Omega(t)\,\mathrm{d}t = 1.$$

Let us next prove that

$$\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = \lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} f(t_0 - t)D_\Omega(t)\,\mathrm{d}t \tag{13.8}$$

for every $\epsilon \in \mathbb{R}_{>0}$. To see this we note from Lemma 13.2.7 that

$$\lim_{\Omega \to \infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu) e^{2\pi i \nu t} \, d\nu = \lim_{\Omega \to \infty} \int_{\mathbb{R}} f(t_0 - t) D_\Omega(t) \, dt.$$

Now we write

$$\int_{\mathbb{R}} f(t_0 - t) D_\Omega(t) \, dt = \int_{-\infty}^{-\epsilon} f(t_0 - t) D_\Omega(t) \, dt + \int_{-\epsilon}^{\epsilon} f(t_0 - t) D_\Omega(t) \, dt + \int_{\epsilon}^{\infty} f(t_0 - t) D_\Omega(t) \, dt.$$

Since $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ and since $t \mapsto \frac{1}{t}$ is bounded on $(-\infty, -\epsilon]$ it follows that $t \mapsto \frac{f(t)}{\pi t}$ is in $\mathsf{L}^{(1)}((-\infty, -\epsilon]; \mathbb{C})$. Thus, by the Riemann–Lebesgue Lemma,

$$\lim_{\Omega \to \infty} \int_{-\infty}^{-\epsilon} f(t_0 - t) D_\Omega(t) \, dt = \lim_{\Omega \to \infty} \int_{-\infty}^{-\epsilon} \frac{f(t_0 - t)}{\pi t} \sin(2\pi \Omega t) \, dt = 0.$$

Similarly,

$$\lim_{\Omega \to \infty} \int_{\epsilon}^{\infty} f(t_0 - t) D_\Omega(t) \, dt = 0.$$

This proves (13.8).

From this immediately follows the equivalence of parts (i) and (iv).

The equivalence of parts (i) and (v) follows from the formula

$$\lim_{\Omega \to \infty} \int_{-\epsilon}^{\epsilon} D_\Omega(t) \, dt = 1,$$

which the reader can verify as Exercise 13.2.4. ■

13.2.15 Remark (The "conditional" in the preceding theorem) The appearance of the conditional Lebesgue integral in the previous theorem—as compared to its absence from the corresponding Theorem 12.2.22 for the CDFT—arises because $D_\Omega \notin \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ (see Lemma 3 from Example 11.3.7–3), whereas $D_{T,N}^{\mathrm{per}} \in \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R}; \mathbb{F})$. Nonetheless, the Dirichlet kernel D_Ω *is* conditionally Lebesgue integrable (see Lemma 1 from Example 11.3.7–3). •

13.2.16 Remark (Localisation) As with the recovery of a signal from its CDFT, we see from part (v) that the recovery of a signal from its CCFT is a local matter. That is to say, to ascertain the behaviour of inverse Fourier transform at t_0 one only cares about the behaviour of f in an arbitrarily small neighbourhood of t_0. This is another instance of the ***localisation principle***. •

We may now state results for the inversion of the CCFT that are analogous to those stated for the CDFT in Section 12.2.4. The first result is a Dini-type test.

13.2.17 Theorem (Dini's test) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and let* $\mathsf{t}_0 \in \mathbb{R}$. *If there exists* $\epsilon \in \mathbb{R}_{>0}$ *so that*

$$\int_{-\epsilon}^{\epsilon} \left|\frac{\mathsf{f}(\mathsf{t}_0 - \mathsf{t}) - \mathsf{s}}{\mathsf{t}}\right| \mathrm{d}\mathsf{t} < \infty,$$

then $(D_\Omega \mathsf{f}(\mathsf{t}_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* s.

Proof This follows immediately from part (v) of Theorem 13.2.14 since we have

$$\lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} \frac{f(t_0 - t) - s}{\pi t} \sin(2\pi\Omega t)\,\mathrm{d}t = 0$$

by the Riemann–Lebesgue Lemma. ■

The following consequence of Dini's test is very often useful.

13.2.18 Corollary (Fourier integrals converge at points of differentiability) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and let* $\mathsf{t}_0 \in \mathbb{R}$. *If* f *is differentiable at* t_0 *then* $(D_\Omega \mathsf{f}(\mathsf{t}_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* $\mathsf{f}(\mathsf{t}_0)$.

Proof This follows from Theorem 13.2.17 exactly as Corollary 12.2.25 follows from Theorem 12.2.24. ■

The following alternative version of Dini's test is sometimes useful.

13.2.19 Corollary (An alternative version of Dini's test) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and let* $\mathsf{t}_0 \in \mathbb{R}$. *If there exists* $\epsilon \in \mathbb{R}_{>0}$ *so that*

$$\int_{0}^{\epsilon} \left|\frac{\frac{1}{2}(\mathsf{f}(\mathsf{t}_0 + \mathsf{t}) + \mathsf{f}(\mathsf{t}_0 - \mathsf{t})) - \mathsf{s}}{\mathsf{t}}\right| \mathrm{d}\mathsf{t} < \infty,$$

then $(D_\Omega \mathsf{f}(\mathsf{t}_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* s.

Proof This follows from Exercise 13.2.5. ■

The following examples illustrate the utility of the Dini test.

13.2.20 Examples (Dini's test) We introduce a collection of signals on $\mathbb{R}$ that will allow us to understand the workings of the CCFT as it relates to the workings of the CDFT.

1. Our first signal is given by $f(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\square_{2,1,1}(t) - 1)$, which we plot in Figure 13.8. We readily compute the CCFT of f to be

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \mathrm{i}\frac{1 - \cos(\pi\nu)}{\pi\nu},$$

and in Figure 13.8 the CCFT is also shown. As concerns Theorem 13.2.17 we note that the analysis goes just as it did for Example 12.2.27–1. The conclusion is that Theorem 13.2.17 predicts the convergence of $(D_\Omega f(t))_{\Omega\in\mathbb{R}_{>0}}$ to $f(t)$ at points where f is continuous. At points of discontinuity, in this case the points $t \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$, Theorem 13.2.17 does not directly predict convergence of the inverse CCFT. However, the alternative version of Corollary 13.2.19 does indeed work since, for $t_0 = 0$,

$$f(t_0 + t) + f(t_0 - t) = 0.$$

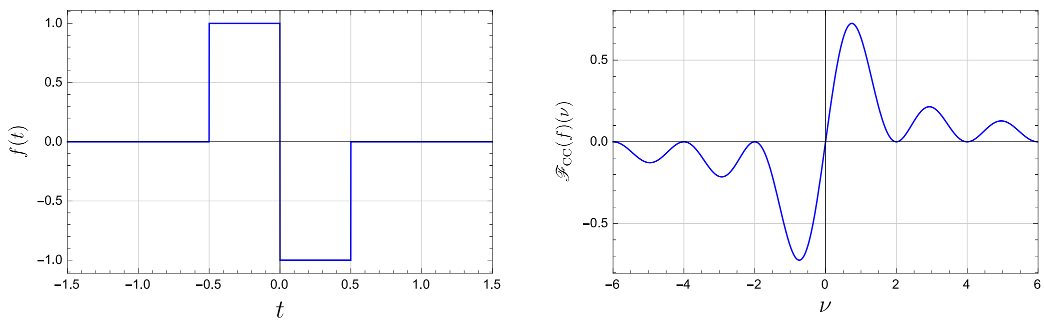


Figure 13.8 The signal $\chi_{[-\frac{1}{2},\frac{1}{2}]}(\square_{2,1,1} - 1)$ (top) and the imaginary part of its CCFT (bottom)

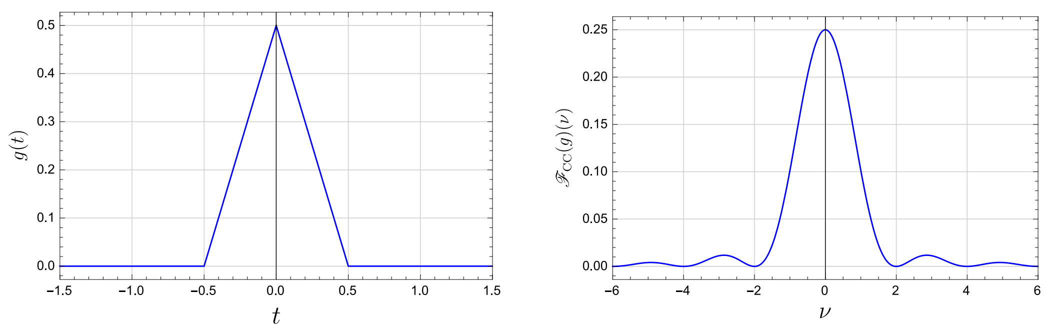


Figure 13.9 The signal $\chi_{[-\frac{1}{2},\frac{1}{2}]}\triangle_{\frac{1}{2},1,1}$ (top) and its CCFT (bottom)

2. Next we look at the signal $g(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\triangle_{\frac{1}{2},1,1}(t)$, which we plot in Figure 13.9. The CCFT of g is computed to be

$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \frac{1 - \cos(\pi\nu)}{2\pi^2\nu^2},$$

and this is plotted in Figure 13.9. As we saw with Example 12.2.27–2, Theorem 13.2.17 allows us to assert that $(D_\Omega g(t))_{\Omega\in\mathbb{R}_{>0}}$ converges to $f(t)$ for all $t \in \mathbb{R}$.

3. The last signal we consider in this example is given by

$$h(t) = \begin{cases} \sqrt{\sin\frac{t+\pi}{2}}, & |t| \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 13.10 we plot this signal. The analysis for the applicability of Theorem 13.2.17 to this signal proceeds as we saw in Example 12.2.27–3. Thus we conclude that $(D_\Omega h(t))_{\Omega\in\mathbb{R}_{>0}}$ converges to $f(t)$ for all $t \in \mathbb{R}$. •

The value of tests like Dini's test is that one does not actually need to compute the CCFT to determine whether the Fourier integral converges.

Next we state the analogue of the Dirichlet test for Fourier series, stated as Theorem 12.2.28.

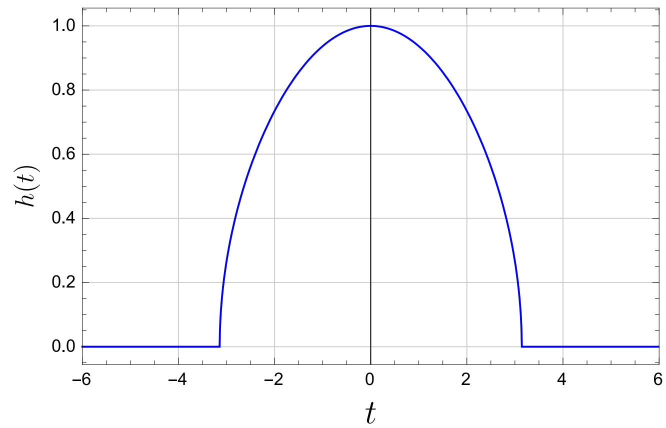


Figure 13.10 The signal h

13.2.21 Theorem (Dirichlet's test) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and suppose that the limits* $\mathsf{f}(\mathsf{t}_0-)$, $\mathsf{f}(\mathsf{t}_0+)$, $\mathsf{f}'(\mathsf{t}_0-)$, *and* $\mathsf{f}'(\mathsf{t}_0+)$ *exist for* $\mathsf{t}_0 \in \mathbb{R}$. *Then* $(\mathsf{D}_\Omega\mathsf{f}(\mathsf{t}_0))_{\Omega\in\mathbb{R}_{>0}}$ *converges to* $\frac{1}{2}(\mathsf{f}(\mathsf{t}_0+) + \mathsf{f}(\mathsf{t}_0-))$.

Proof We first define

$$g_f(t) = \tfrac{1}{2}(f(t_0 + t) - f(t_0+) + f(t_0 - t) - f(t_0-))$$

and note that g_f is even and $g_f(0+)g_f(0-) = 0$. Moreover, by Exercise 13.2.5 $(D_\Omega g_f(0))_{\Omega\in\mathbb{R}_{>0}}$ converges to zero if and only if $(D_\Omega f(t_0))_{\Omega\in\mathbb{R}_{>0}}$ converges to $\frac{1}{2}(f(t_0+) + f(t_0-))$. Therefore, without loss of generality we may suppose that f is even, that $f(0+) = f(0-) = 0$, and that we consider convergence of $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ to f at $t = 0$.

With these assumptions, we note that the four hypotheses of the theorem imply that f is differentiable from the left and right at $t = 0$. Therefore the limits

$$\lim_{t\uparrow 0}\frac{f(t)}{t}, \quad \lim_{t\downarrow 0}\frac{f(t)}{t}$$

exist. Therefore, there exists $M, \epsilon \in \mathbb{R}_{>0}$ so that $\left|\frac{f(t)}{t}\right| < M$ for $|t| < \epsilon$. The theorem now follows from Theorem 13.2.17. ■

13.2.22 Remark (Dirichlet's test is a special case of Dini's test) From the proof of Theorem 13.2.21 we see that Dirichlet's test follows from Dini's test. However, it is often easier in practice to directly verify the hypotheses of Dirichlet's test. •

A couple of examples are useful in illustrating the relationship between Theorem 13.2.17 and Theorem 13.2.21.

13.2.23 Examples (Dirichlet's test)

1. Consider the signal $f(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\square_{2,1,1}(t) - 1)$ considered in Example 13.2.20–1. Mimicking the analysis of Example 12.2.30–1, we conclude from Theorem 13.2.21 that at all points except $t \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ we have

$\lim_{\Omega\to\infty} D_\Omega f(t) = f(t)$. At the exceptional points we have

$$\lim_{\Omega\to\infty} D_\Omega f(-\tfrac{1}{2}) = \frac{1}{2},$$
$$\lim_{\Omega\to\infty} D_\Omega f(0) = 0,$$
$$\lim_{\Omega\to\infty} D_\Omega f(\tfrac{1}{2}) = -\frac{1}{2},$$

these values simply being the average of the left and right limits of the signal at these points, as predicted by Theorem 13.2.21.

2. Next we consider the signal $g(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\triangle_{\frac{1}{2},1,1}(t)$ first discussed in Example 13.2.20–2. From Theorem 13.2.21 we conclude that for each $t \in \mathbb{R}$ we have $\lim_{\Omega\to\infty} D_\Omega g(t) = g(t)$.
3. Finally, we look again at the signal h discussed in Example 13.2.20–3. Since this signal is differentiable for $t \notin \{-\frac{1}{2}, \frac{1}{2}\}$, and so at such values of t we have $\lim_{\Omega\to\infty} D_\Omega g(t) = g(t)$, from Theorem 13.2.21. This result cannot be used to predict the value of $\lim_{\Omega\to\infty} D_\Omega g(t)$ when $t \in \{-\frac{1}{2}, \frac{1}{2}\}$. •

Next we state our most powerful test concerning the inversion of the Fourier transform.

13.2.24 Theorem (Jordan's test) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, *let* $\mathsf{t}_0 \in \mathbb{R}$, *and suppose that there exists a neighbourhood* J *of* t_0 *so that* $\mathsf{f}|\mathsf{J}$ *has bounded variation. Then* $(\mathsf{D}_\Omega \mathsf{f}(\mathsf{t}_0))_{\Omega\in\mathbb{R}_{>0}}$ *converges to* $\frac{1}{2}(\mathsf{f}(\mathsf{t}_0+) + \mathsf{f}(\mathsf{t}_0-))$.

Proof By Exercise 13.2.5 let us first make the assumption that f satisfies the hypotheses of the theorem and that, as well, $t_0 = 0$, f is even, and $f(0) = 0$. Since f has bounded variation in a neighbourhood of t_0, we may write $f = f_+ - f_-$ where f_+ and f_- are monotonically increasing (part (i) of Theorem 3.3.3). By applying the argument we give below to each component in the sum, we may without loss of generality also assume that f is monotonically increasing in a neighbourhood of 0 in $[0,\infty)$.

We first note that as in the proof of Theorem 12.2.31 there exists $M \in \mathbb{R}_{>0}$ so that

$$\Big|\int_0^t \frac{\sin(2\pi\Omega\tau)}{\tau}\,d\tau\Big| \le M$$

for all $t, \Omega \in \mathbb{R}_{>0}$. Now let M be so chosen and let $\epsilon \in \mathbb{R}_{>0}$. Choose $\delta \in \mathbb{R}_{>0}$ so that $f(\delta-) < \frac{\epsilon}{2M}$ and compute, for some $\delta' \in (0,\delta)$ guaranteed by Proposition 3.4.33,***missing stuff***

$$\begin{aligned}\Big|\int_{-\delta}^{\delta} f(t)\frac{\sin(2\pi\Omega t)}{t}\,dt\Big| &= \Big|2\int_0^{\delta} f(t)\frac{\sin(2\pi\Omega t)}{t}\,dt\Big| \\ &= \Big|2f(\delta-)\int_{\delta'}^{\delta} \frac{\sin(2\pi\Omega t)}{t}\,dt\Big| \\ &\le \frac{\epsilon}{2M}2M = \epsilon.\end{aligned}$$

The theorem now follows from part (v) of Theorem 13.2.14. ■

The pointwise convergence tests of Theorems 13.2.17, 13.2.21, and 13.2.24 can be made global by asking that their hypotheses hold in a global sense. Doing this gives the following analog of Corollary 12.2.32.

13.2.25 Corollary (Conditions for global pointwise convergence) *Let* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and suppose that for each* $\mathsf{t} \in \mathbb{R}$ *we have* $\mathsf{f}(\mathsf{t}) = \frac{1}{2}(\mathsf{f}(\mathsf{t}+) + \mathsf{f}(\mathsf{t}-))$*. Then either of the following statements implies that* $(\mathrm{D}_\Omega \mathsf{f})_{\Omega\in\mathbb{R}_{>0}}$ *converges pointwise to* f *as* $\Omega \to \infty$*:*

(i) f *is uniformly Lipschitz;*

(ii) $\mathsf{f} \in \mathsf{C}^1_{\mathrm{pw}}(\mathbb{R};\mathbb{C})$*;*

(iii) $\mathsf{f} \in \mathsf{BV}(\mathbb{R};\mathbb{C})$*.*

As with Fourier series, the above results for Fourier integrals suggest that it may be that Fourier integrals will diverge for commonplace signals. Indeed, just as in Example 12.2.10, there are continuous signals in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ for which the Fourier integral for the signal diverges at a point. Indeed, if one simply extends the example of Example 12.2.10 to $\mathbb{R}$ by asking that it be zero outside the interval $[-\pi,\pi]$, then this signal's Fourier integral will diverge at $t = 0$, essentially by virtue of Remark 13.2.16.

13.2.5 Uniform convergence of Fourier integrals

In this section we establish results analogous to those for uniform convergence for Fourier series. The central result we give is the following.

13.2.26 Theorem (Uniform convergence of Fourier integrals) *If* $\mathsf{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *and if* $\mathscr{F}_{\mathrm{CC}}(\mathsf{f}) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$*, then the following statements hold:*

(i) $(\mathrm{D}_\Omega \mathsf{f})_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly to a (necessarily continuous) signal* g *as* $\Omega \to \infty$*;*

(ii) $\mathsf{f}(\mathsf{t}) = \mathsf{g}(\mathsf{t})$ *for almost every* $\mathsf{t} \in \mathbb{R}$*.*

Proof Just as $\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ is continuous when using the norm $\|\cdot\|_1$ on $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ and the norm $\|\cdot\|_\infty$ on $\mathsf{C}^0_0(\mathbb{R};\mathbb{C})$, the map $\overline{\mathscr{F}}_{\mathrm{CC}}$ will have these same properties. Indeed, only trivial modifications need be made to the proof of Theorem 13.1.7 to show this. Now note that the net $(\chi_{[-\Omega,\Omega]}\mathscr{F}_{\mathrm{CC}}(f))_{\Omega\in\mathbb{R}_{>0}}$ converges to $\mathscr{F}_{\mathrm{CC}}(f)$ in the L^1-norm if $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. If we let

$$g(t) = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu$$

then it follows that $g \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ and that if

$$g_\Omega(t) = \int_{\mathbb{R}} \chi_{[-\Omega,\Omega]}(\nu)\mathscr{F}_{\mathrm{CC}}(f)(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu$$

then the net $(g_\Omega)_{\Omega\in\mathbb{R}_{>0}}$ converges to g in the L_∞-norm. That is to say, it converges uniformly to g since both g and the approximations g_Ω are continuous. The first part of the result follows since $g_\Omega = D_\Omega f$.

The second part of the theorem follows from Theorem 13.2.1 along with Exercise 5.9.8. ■

This result has the following immediate corollary, recalling the notation

$$\overline{\mathscr{F}}_{\mathrm{CC}}(F)(t) = \int_{\mathbb{R}} F(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

for $F \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.

13.2.27 Corollary (A case when the Fourier integral in the inverse of the CCFT) *If* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{C}^0(\mathbb{R};\mathbb{C})$ *has the property that* $\mathscr{F}_{\mathrm{CC}}(\mathrm{f}) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$*, then*

$$\overline{\mathscr{F}}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\mathrm{t}) = \mathrm{f}(\mathrm{t}), \qquad \mathrm{t} \in \mathbb{R}.$$

Let us now state a result that is analogous to Corollary 12.2.35 for the CDFT. The proof here requires a little work to ensure that it follows from Theorem 13.2.26.

13.2.28 Corollary (A test for uniform convergence) *Let* $\mathrm{f} \in \mathsf{C}^0(\mathbb{R};\mathbb{C})$ *and suppose that there exists a signal* $\mathrm{f}' \colon \mathbb{R} \to \mathbb{C}$ *such that*

(i) *for every* $\mathrm{T} \in \mathbb{R}_{>0}$*,* f' *is piecewise continuous on* $[-\mathrm{T}, \mathrm{T}]$*,*

(ii) f' *is discontinuous at a finite number of points,*

(iii) $\mathrm{f}' \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$*, and*

(iv) $\mathrm{f}(\mathrm{t}) = \displaystyle\int_{-\infty}^{\mathrm{t}} \mathrm{f}'(\tau)\,\mathrm{d}\tau.$

Then $(\mathrm{D}_\Omega \mathrm{f})_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly to* f*. In particular, if* $\mathrm{f}, \mathrm{f}^{(1)}, \mathrm{f}^{(2)} \in \mathsf{C}^0(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ *then* $(\mathrm{D}_\Omega \mathrm{f})_{\mathrm{N}\in\mathbb{Z}_{>0}}$ *converges uniformly to* f*.*

Proof The hypotheses of the corollary ensure that the limits $f(t+)$, $f(t-)$, $f'(t+)$ and $f'(t-)$ exist for each $t \in \mathbb{R}$ so that Theorem 13.2.21 implies

$$f(t) = \lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(t) e^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

for each $t \in \mathbb{R}$.

By Proposition 13.1.10 we have

$$\mathscr{F}_{\mathrm{CC}}(f)'(\nu) = 2\pi\mathrm{i}\nu\mathscr{F}_{\mathrm{CC}}(f)(\nu).$$

We then have

$$\int_{-\Omega}^{\Omega} |\mathscr{F}_{\mathrm{CC}}(f)(\nu)|\,\mathrm{d}\nu = \int_{-\Omega}^{\Omega} \left|\frac{\mathscr{F}_{\mathrm{CC}}(f)'(\nu)}{2\pi\mathrm{i}\nu}\right|\mathrm{d}\nu.$$

Since $\chi_{[-\Omega,\Omega]}\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ the integral on the right exists for any finite Ω. Since we are interested in the behaviour as $\Omega \to \infty$ let us write

$$\int_{-\Omega}^{\Omega} \left|\frac{\mathscr{F}_{\mathrm{CC}}(f)'(\nu)}{2\pi\mathrm{i}\nu}\right|\mathrm{d}\nu = \int_{|\nu|\le 1} \left|\frac{\mathscr{F}_{\mathrm{CC}}(f)'(\nu)}{2\pi\mathrm{i}\nu}\right|\mathrm{d}\nu + \int_{1\le|\nu|\le\Omega} \left|\frac{\mathscr{F}_{\mathrm{CC}}(f)'(\nu)}{2\pi\mathrm{i}\nu}\right|\mathrm{d}\nu,$$

and note that the limit as $\Omega \to \infty$ exists on the left if and only if it exists for the second integral on the right. For this integral we use the Cauchy–Bunyakovsky–Schwarz inequality to deduce

$$\int_{1\le|\nu|\le\Omega} \left|\frac{\mathscr{F}_{\mathrm{CC}}(f)'(\nu)}{2\pi\mathrm{i}\nu}\right|\mathrm{d}\nu \le \Big(\int_{1\le|\nu|\le\Omega} |\mathscr{F}_{\mathrm{CC}}(f)'(\nu)|^2\,\mathrm{d}\nu\Big)^{1/2}\Big(\int_{1\le|\nu|\le\Omega} \left|\frac{1}{2\pi\mathrm{i}\nu}\right|^2\mathrm{d}\nu\Big)^{1/2}.$$

By Lemma 13.3.1 below the first integral on the right converges as $\Omega \to \infty$, and a direct computation shows that the second integral on the right also converges as $\Omega \to \infty$. This shows that $\mathscr{F}_{CC}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, and the main statement in the result now follows from Theorem 13.2.26.

For the last statement of the corollary note that by Proposition 13.1.10 it follows that if $f, f^{(1)}, f^{(2)} \in \mathsf{C}^0(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{CC}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, and so Theorem 13.2.26 again applies. ■

13.2.29 Remark (The L²-condition in Corollary 12.2.35) Note that the assumption that $f' \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ allows us to use Lemma 13.3.1 below. This lemma plays the rôle that is played by Bessel's inequality in the proof of Corollary 12.2.35. This is yet another instance of the similarity of the development of the CDFT and the CCFT, provided one makes suitable modifications. •

Let us consider some of our examples in light of this result on uniform convergence.

13.2.30 Examples (Uniform convergence of Fourier integrals)

1. We first consider the signal $f(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\square_{2,1,1}(t) - 1)$ considered previously as concerns its pointwise convergence. Since this signal is not continuous (more precisely, it is not equal almost everywhere to a continuous signal), Theorem 13.2.26 implies that $\mathscr{F}_{CC}(f) \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. One of the consequences of this is that $f \neq \mathrm{FI}[f]$ since the latter integral does not exist. What *does* exist is the improper integral
$$\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{CC}(f)(\nu) e^{2\pi i \nu t}\, d\nu.$$
Furthermore, we have seen that this integral converges to the signal
$$t \mapsto \begin{cases} f(t), & t \notin \{-1, 0, 1\}, \\ \frac{1}{2}, & t = -1, \\ 0, & t = 0 \\ -\frac{1}{2}, & t = 1. \end{cases}$$
In Figure 13.11 we show an approximation of f by one of the partial sums $D_\Omega f$. Note that it illustrates the same sort of oscillatory behaviour near the discontinuities as we have seen with Fourier series.
2. Next we consider $g(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\triangle_{\frac{1}{2},1,1}(t)$, another of the signals considered above for pointwise convergence. We saw that the net $(D_\Omega g)_{\Omega\in\mathbb{R}_{>0}}$ converged pointwise to f. In Example 13.2.20–2 we produced the CCFT for g, and one can see that $\mathscr{F}_{CC}(g) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Therefore, Theorem 13.2.26 indicates that $(D_\Omega g)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to g, since g is continuous. In Figure 13.12 we show an approximation of g by $D_\Omega g$ for $\Omega = 20$. Note that the behaviour of the approximation is quite good. Also notice that in this case we may directly use the definition of the Fourier integral and write $g = \mathrm{FI}[g]$. Note that it is not always the case that one can do this!

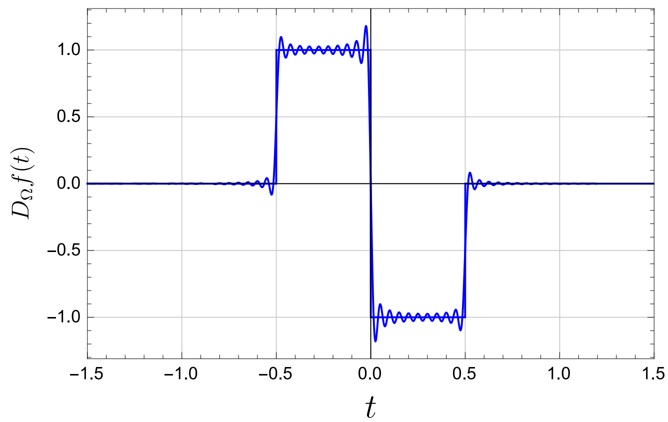


Figure 13.11 The approximation $D_\Omega f$ to f when $\Omega = 20$

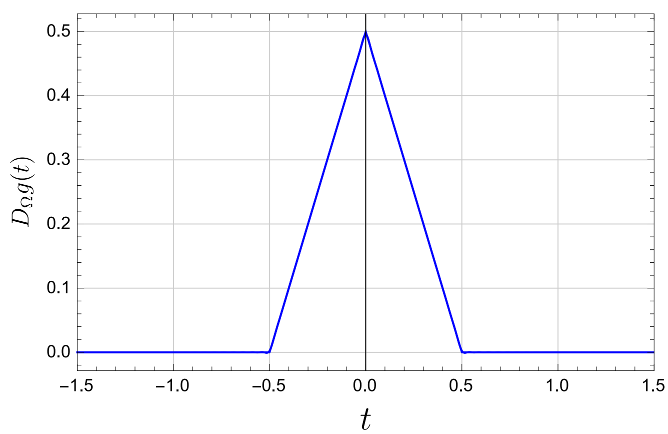


Figure 13.12 The approximation $D_\Omega g$ to g when $\Omega = 20$

3. Finally we consider the signal h considered first in Example 13.2.20–3. This signal does not satisfy the hypotheses of Corollary 13.2.28. And, without actually computing the CCFT, we are not in a position to apply Theorem 13.2.26. Thus we are a bit up in the air at this point as concerns the uniform convergence of $(D_\Omega h)_{\Omega\in\mathbb{R}_{>0}}$ to h. However, this will be taken care of by Theorem 13.2.31 below. •

We now give the analogue of Theorem 12.2.37 for the CCFT. The result we state here has two parts, coinciding with the fact that a signal on $\mathbb{R}$ may have its variation constrained in at least two different ways.

13.2.31 Theorem (Continuous signals of finite variation have uniformly convergent Fourier integrals) *If* $f \in L^{(1)}(\mathbb{R};\mathbb{C})$ *is continuous then the following statements hold:*

(i) if f *has finite variation then* $(D_\Omega f|K)_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly for any compact subset* $K \subseteq \mathbb{R}$*;*

(ii) if f *has bounded variation then* $(\mathrm{D}_\Omega \mathrm{f})_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly on* $\mathbb{R}$.

Proof As in the proof of Theorem 13.2.24 (without the modifications necessary to make the assumptions that $t_0 = 0$, $f(t_0) = 0$, and evenness of f), uniform convergence will follow if we can show that for $\epsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ so that

$$\Big|\int_{-\delta}^{\delta}(f(t_0 - t) - f(t_0))\frac{\sin(2\pi\Omega t)}{t}\,\mathrm{d}t\Big| < \epsilon$$

for all $\Omega \in \mathbb{R}_{>0}$. As in the proof of Theorem 12.2.37 this will involve showing uniform continuity of the variation V.

For part (i) let $T \in \mathbb{R}_{>0}$ be chosen so that $K \subseteq [-T, T]$. We prove uniform continuity of V on $[-T, T]$ for any $T \in \mathbb{R}_{>0}$ just as was done in Theorem 12.2.37, this being valid by compactness of $[-T, T]$. We let M satisfy

$$\Big|\int_0^t \frac{\sin(2\pi\Omega\tau)}{\tau}\,\mathrm{d}\tau\Big| \le M \tag{13.9}$$

for all $t, \Omega \in \mathbb{R}_{>0}$. Now choose δ so that $|\mathrm{V}(f)(t_1) - \mathrm{V}(f)(t_2)| < \frac{\epsilon\pi}{2M}$ for all $t_1, t_2 \in [-T, T]$ satisfying $|t_1 - t_2| < \delta$. The estimate (13.9) then holds for all $t_0 \in K$.

For part (ii) we note that bounded variation of f implies that the limits $\lim_{t\to-\infty} \mathrm{V}(t)$ and $\lim_{t\to\infty} \mathrm{V}(t)$ exist. This follows immediately from monotonicity of V and from the fact that the closure of $\mathrm{V}(\mathbb{R})$ is a compact interval. Now, for $\epsilon \in \mathbb{R}_{>0}$ choose $T \in \mathbb{R}_{>0}$ so that $|\mathrm{V}(t) - \mathrm{V}(-\infty)| < \epsilon$ for $t < -T$ and $|\mathrm{V}(t) - \mathrm{V}(\infty)| < \epsilon$ for $t > T$, and choose δ so that $|\mathrm{V}(f)(t_1) - \mathrm{V}(f)(t_2)| < \frac{\epsilon\pi}{2M}$ for all $t_1, t_2 \in [-T, T]$ satisfying $|t_1 - t_2| < \delta$. The estimate (13.9) again holds, but now for all $t_0 \in \mathbb{R}$. ■

We can apply the theorem to an example to see how it can be used to verify uniform convergence.

13.2.32 Example (An application of the bounded variation test for uniform convergence) We consider the signal h from Example 13.2.20–3. If we define

$$h_+ = \begin{cases} 0, & t \in (-\infty, -\pi),\\ (\sin(\frac{t+\pi}{2})^{1/2}, & t \in [-\pi, 0],\\ 1, & t \in (0, \infty] \end{cases}$$

$$h_- = \begin{cases} 0, & t \in (-\infty, 0),\\ 1 - (\sin\frac{t+\pi}{2})^{1/2}, & t \in [0, \pi],\\ 1, & t \in [\pi, \infty). \end{cases}$$

We see that both h_+ and h_- are monotonically increasing and that $h = h_+ - h_-$. Therefore, by part (ii) of Theorem 3.3.3 we conclude that h has finite variation. Moreover, since h is constant outside the compact interval $[-\pi, \pi]$, h actually has bounded variation. Therefore Theorem 12.2.37 implies that $(D_\Omega h)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to h. •

The reader is encouraged to consider the examples of Section 13.1.1 to see which are eligible to make valid the formula $f = \mathrm{FI}[f]$. You should find that Examples 13.1.3–1 and 3 are ineligible, and that Examples 13.1.3–2, 4, and 5 are eligible.

13.2.6 Gibbs' phenomenon

Just as for Fourier series, when determining a discontinuous signal from its CCFT, there can arise difficulties with convergence at points of discontinuity, even though the Fourier integral converges pointwise. In this section we quantify this.

The first thing to do is introduce notation that mirrors that used for the Gibbs' phenomenon for the inverse CDFT. Let $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$ and let t_0 be a point of discontinuity of f, and we assume that f possesses left and right limits at t_0. We refer the reader to Section 12.2.6 for the notion of a Gibbs sequence. The ***Gibbs set*** for f at t_0 is

$$\mathrm{G}(f,t_0) = \left\{ \lim_{j\to\infty} D_j f(t_j) \;\middle|\; (t_j)_{j\in\mathbb{Z}_{>0}} \text{ is a Gibbs sequence at } t_0 \right\}.$$

The interpretation of the Gibbs set is essentially the same as it was for the CDFT. That is to say, the Gibbs set can be interpreted as the collection of points in the graphs of the approximations to f that lie close to the vertical line $t = t_0$ in the limit. The following result characterises the Gibbs set.

13.2.33 Theorem (General Gibbs' phenomenon for Fourier integrals) *Let* $\mathrm{f}\colon [0,\mathrm{T}] \to \mathbb{R}$ *satisfy the conditions of Corollary 13.2.25(ii), and for* $\mathrm{t}_0 \in \mathbb{R}$ *let* $\mathrm{j}(\mathrm{t}_0) = \mathrm{f}(\mathrm{t}_0+) - \mathrm{f}(\mathrm{t}_0-)$*. Also denote*

$$\mathrm{I} = \int_0^{\pi} \frac{\sin \mathrm{t}}{\mathrm{t}}\,\mathrm{dt} \approx 1.85194.$$

The Gibbs set then satisfies

$$\mathrm{G}(\mathrm{f},\mathrm{t}_0) = [\tfrac{1}{2}(\mathrm{f}(\mathrm{t}_0+) + \mathrm{f}(\mathrm{t}_0-)) - \tfrac{\Delta}{2}, \tfrac{1}{2}(\mathrm{f}(\mathrm{t}_0+) + \mathrm{f}(\mathrm{t}_0-)) + \tfrac{\Delta}{2}],$$

where

$$\Delta = \left|\frac{2\mathrm{I}\mathrm{j}(\mathrm{t}_0)}{\pi}\right| \approx 1.17898|\mathrm{j}(\mathrm{t}_0)|.$$

Proof The strategy for the proof follows that of Theorem 12.2.39. Thus we establish the theorem for a special signal for which the computations can be performed directly, then we reduce the general case to this special one. The special signal we choose is $g = \chi_{[0,1]} - \chi_{[-1,0]}$. This signal has a jump discontinuity at $t_0 = 1$ of magnitude 2. We compute

$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \frac{2 - e^{-2\pi\mathrm{i}\nu} - e^{2\pi\mathrm{i}\nu}}{2\pi\mathrm{i}\nu}.$$

We then have

$$\begin{aligned} D_\Omega g(t) &= \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(g)(\nu) e^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu \\ &= \int_{-\Omega}^{\Omega} \frac{2e^{2\pi\mathrm{i}\nu t} - e^{2\pi\mathrm{i}\nu(t-1)} - e^{2\pi\mathrm{i}\nu(t+1)}}{2\pi\mathrm{i}\nu}\,\mathrm{d}\nu \\ &= \int_0^{\Omega} \frac{2\sin(2\pi\nu t) - \sin(2\pi\nu(t-1)) - \sin(2\pi\nu(t+1))}{\pi\nu}\,\mathrm{d}\nu, \end{aligned}$$

using the fact that the integral of an odd function over the domain $[-\Omega, \Omega]$ will vanish. Now let $(t_j)_{j\in\mathbb{Z}_{>0}}$ be a Gibbs sequence at $t_0 = 0$. We then have

$$\begin{aligned}
\lim_{n\to\infty} D_n g(t_n) &= \lim_{n\to\infty} \int_0^n \frac{\sin(2\pi\nu(t_n-1)) + \sin(2\pi\nu(t_n+1)) - 2\sin(2\pi\nu t_n)}{2\pi\nu}\,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \frac{1}{\pi}\int_0^{2\pi n t_n} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi \\
&\quad - \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi n(t_n-1)} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi - \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi n(t_n+1)} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi
\end{aligned} \tag{13.10}$$

The first integral in (13.10) can have the form

$$\frac{1}{\pi}\int_0^{\alpha} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi$$

for any $\alpha \in \mathbb{R}_{>0}$ by an appropriate choice of Gibbs sequence. As we showed in the proof of Theorem 12.2.39, the maximum value occurs when $\alpha = \pm\pi$. The second integral in (13.10) has the value $-\frac{1}{2}$ in the limit, and the third has the value $\frac{1}{2}$ in the limit since

$$C\int_0^{\infty} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}$$

by Lemma 1 from Example 11.3.7–3. Thus we have the result in the case when $f = g$.

To complete the proof we define

$$h(t) = f(t) + j(t_0)g(t_0),$$

noting that h is continuous at t_0, and that the limits $h'(t_0+)$ and $h'(t_0-)$ exist. The remainder of the proof now goes like the final steps in Theorem 12.2.39. ■

The reader may refer to the discussion following Theorem 12.2.39 for a discussion of how one may interpret the Gibbs set, and for a discussion of the distinction of the difference between the limit of a graph and the graph of a limit.

13.2.7 Cesàro means

For the CDFT, we saw that the Cesàro sums gave us a means of explicitly inverting the CDFT for general classes of signals, where as the use of the Fourier partial sums did not allow this. The same phenomenon happens for the CCFT, where we can define an analogue of the Cesàro sums, and show that these will always recover the signal, under extremely weak hypotheses. In the course of the development we shall come across the continuous Fejér kernel F_Ω which came up during the course of the proof of Theorem 13.2.1. We recall that

$$F_\Omega(t) = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \neq 0, \\ \Omega, & t = 0. \end{cases}$$

To see how this kernel might be related to Cesàro means as they arise in Fourier integrals we give the following lemma.

13.2.34 Lemma (Cesàro means and the Fejér kernel) *For* $f \in L^{(1)}(\mathbb{R};\mathbb{C})$ *we have*

$$\frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\omega}^{\omega}\mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu\Big)\,d\omega = \int_{\mathbb{R}} f(t-\tau)F_\Omega(\tau)\,d\tau.$$

Proof We compute, using Lemma 13.2.7,

$$\begin{aligned}\frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\omega}^{\omega}\mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu\Big)\,d\omega &= \frac{1}{\Omega}\int_0^\Omega D_\omega f(t)\,d\omega\\ &= \frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\omega}^{\omega}\mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu\Big)\,d\omega\\ &= \frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\omega}^{\omega}\Big(\int_{\mathbb{R}} f(\tau)e^{-2\pi i\nu\tau}\,d\tau\Big)e^{2\pi i\nu t}\,d\nu\Big)\,d\omega\\ &= \frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\mathbb{R}}\Big(\int_{-\omega}^{\omega} f(\tau)e^{2\pi i\nu(t-\tau)}\,d\nu\Big)\,d\tau\Big)\,d\omega\\ &= \frac{1}{\Omega}\int_0^\Omega\Big(\int_{\mathbb{R}} f(\tau)\frac{\sin(2\pi\omega(t-\tau))}{\pi(t-\tau)}\,d\tau\Big)\,d\omega\\ &= \frac{1}{\Omega}\int_{\mathbb{R}} f(\tau)\Big(\int_0^\Omega\frac{\sin(2\pi\omega(t-\tau))}{\pi(t-\tau)}\,d\omega\Big)\,d\tau\\ &= \int_{\mathbb{R}} f(t-\tau)F_\Omega(\tau)\,d\tau,\end{aligned}$$

twice using Fubini's Theorem. ■

Thus the Cesàro means appear as the convolution with the Fejér kernel.

13.2.35 Notation ($F_\Omega f$) For $f \in L^{(1)}(\mathbb{R};\mathbb{C})$ and for $\Omega \in \mathbb{R}_{>0}$ we denote

$$F_\Omega f(t) = \frac{1}{\Omega}\int_0^\Omega\Big(\int_{-\omega}^{\omega}\mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu\Big)\,d\omega.$$

This suggests the rôle of convolution in these Cesàro means. •

The following result is what now we are after.

13.2.36 Theorem (Convergence of Cesàro means) *For* $f \in L^{(0)}(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) *if* $f \in L^{(p)}(\mathbb{R};\mathbb{C})$ *then* $(F_\Omega f)_{\Omega\in\mathbb{R}}$ *converges to* f *in* $L^p(\mathbb{R};\mathbb{C})$*;*

(ii) *if* $f \in C^0_{bdd}(\mathbb{R};\mathbb{C})$ *then* $(F_\Omega f|K)_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly to* $f|K$ *for every compact subset* $K \subseteq \mathbb{R}$*;*

(iii) *if* $f \in C^0_{unif,bdd}(\mathbb{R};\mathbb{C})$ *is then* $(F_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ *converges uniformly to* f*;*

(iv) *if* $f \in L^{(\infty)}(\mathbb{R};\mathbb{C})$ *and if, for* $t_0 \in \mathbb{R}$*, the limits* $f(t_0-)$ *and* $f(t_0+)$ *exist then* $(F_\Omega f(t_0))_{\Omega\in\mathbb{R}_{>0}}$ *converges to* $\frac{1}{2}(f(t_0-)+f(t_0+))$.

Proof This follows from Theorems 11.3.2, 11.3.3, 11.3.4, and 11.3.5. ■

An example is helpful in illustrating the smoothing effect of the Cesaro means in recovering a signal from its CCFT.

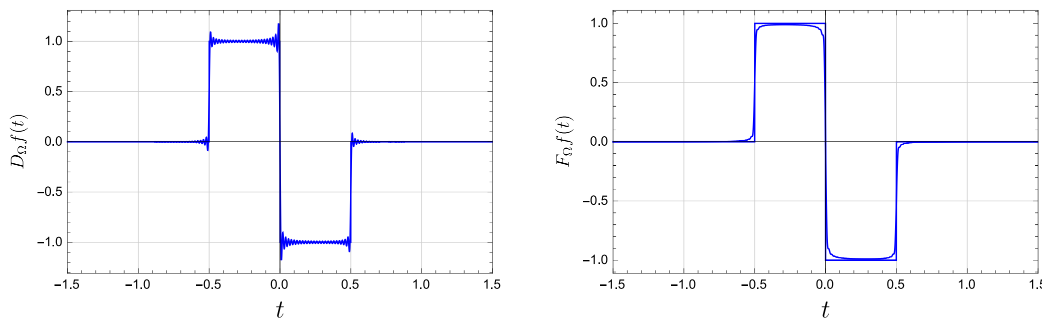


Figure 13.13 $D_\Omega f$ (left) and $F_\Omega f$ (right) for $\Omega = 50$

13.2.37 Example (Cesàro means) We consider the signal $f(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\square_{2,1,1}(t) - 1)$ first discussed in Example 13.2.20. In Figure 13.13 we show the approximations to f using the Dirichlet and Fejér kernels. As expected, the Cesàro means do not exhibit the Gibbs effect. •

13.2.38 Remarks (Pros and cons of using Cesàro means)

1. Note that the Cesàro means provide us with a left-inverse of $\mathscr{F}_{\mathrm{CC}}$. Indeed, if we define $\mathscr{I}_{\mathrm{CC}}\colon \mathsf{C}^0_0(\mathbb{R};\mathbb{C}) \to \mathsf{L}^1(\mathbb{R};\mathbb{C})$ by

$$\mathscr{I}_{\mathrm{CC}}(F) = \lim_{\Omega\to\infty} \frac{1}{\Omega}\int_0^\Omega \Big(\int_{-\omega}^{\omega} F(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\,\mathrm{d}\omega,$$

then we have $\mathscr{I}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(f) = f$ for all $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. Of course, $\mathscr{I}_{\mathrm{CC}}$ is not actually defined for all $F \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$, but this can be kludged for the purposes of the left-inverse conversation by setting $\mathscr{I}_{\mathrm{CC}}(F) = 0$ for all $F \notin \mathrm{image}(\mathscr{F}_{\mathrm{CC}})$.

2. As with the use of Cesàro sums with the CDFT, the use of Cesàro means for the CCFT has some drawbacks. We refer to Remark 12.2.44 for an account of some of this. •

13.2.8 The CCFT and approximate identities

In this section we investigate the CCFT as it applies to approximate identities on $\mathbb{R}$. We have already seen in this chapter the important rôle of approximate identities. The Dirichlet kernel (okay, it is not an approximate identity) is linked to the Fourier integral in Lemma 13.2.7, and the Fejér kernel is linked to the Cesàro means for Fourier integrals in Lemma 13.2.34. In this section we see the rôle played by general approximate identities with respect to the CCFT.

Lemma 2.2.2 in Pinsky

Next we compute the CCFT for a few important approximate identities. In all of the examples we consider, the approximate identity and its CCFT are continuous and integrable and so, by Theorem 13.2.26, it follows that the corresponding Fourier integral converge uniformly. Therefore, in these cases, it is possible to use the similarity of the CCFT with the Fourier integral to compute the CCFT of the approximate identity by finding a signal whose CCFT is equal to the approximate identity. Moreover, in all cases we have actually found such signals in Example 13.1.3. However, here we shall directly compute the CCFT's using complex analysis in order to illustrate the relationships between the CCFT and contour integration. In Exercise 13.2.7 the reader can explore the easier way of computing these CCFT's.

13.2.39 Examples (The CCFT of approximate identities)

1. Recall from Example 11.3.7–1 the definition of the Poisson kernel:

$$P_\Omega(t) = \frac{1}{\pi}\frac{\Omega}{1+\Omega^2 t^2}.$$

We claim that

$$\mathscr{F}_{\mathrm{CC}}(P_\Omega)(\nu) = \mathrm{e}^{-\frac{2\pi|\nu|}{\Omega}}.$$

To verify this formula, let us recall the contours γ_T, $C_{-,T}$, and $C_{+,T}$ from the proof of Lemma 1 in Example 11.3.7–3. For $\nu \in \mathbb{R}$ let us define $F_\nu \colon \mathbb{C} \to \mathbb{C}$ by

$$F_\nu(z) = -\mathrm{i}\frac{\Omega}{\pi}\frac{\mathrm{e}^{-2\pi\nu z}}{1-\Omega^2 z^2},$$

noting that F_ν has simple poles at $\pm\Omega^{-1}$. Note that

$$\int_{-T}^{T} P_\Omega(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \int_{\gamma_T} F_\nu(z)\,\mathrm{d}z.$$

First let $\nu \in \mathbb{R}_{>0}$. By the Residue Theorem,***missing stuff***

$$\begin{aligned}-\int_{\gamma_T} F_\nu(z)\,\mathrm{d}z + \int_{C_{+,T}} F_\nu(z)\,\mathrm{d}z &= 2\pi\mathrm{i}\,\mathrm{Res}(F_\nu,\Omega^{-1})\\ &= \lim_{z\to\Omega^{-1}} 2\pi\mathrm{i}(z-\Omega^{-1})F_\nu(z) = -\mathrm{e}^{-2\pi\nu/\Omega}.\end{aligned}$$

Similarly, if $\nu \in \mathbb{R}_{<0}$, we have

$$\begin{aligned}\int_{\gamma_T} F_\nu(z)\,\mathrm{d}z + \int_{C_{-,T}} F_\nu(z)\,\mathrm{d}z &= 2\pi\mathrm{i}\,\mathrm{Res}(F_\nu,-\Omega^{-1})\\ &= \lim_{z\to-\Omega^{-1}} 2\pi\mathrm{i}(z+\Omega^{-1})F_\nu(z) = \mathrm{e}^{-2\pi|\nu|/\Omega}.\end{aligned}$$

By Jordan's Lemma***missing stuff*** we have

$$\lim_{T\to\infty}\int_{C_{+,T}} F_\nu(z)\,\mathrm{d}z = \lim_{T\to\infty}\int_{C_{-,T}} F_\nu(z)\,\mathrm{d}z = 0,$$

noting that we take $\nu \in \mathbb{R}_{>0}$ in the first case and $\nu \in \mathbb{R}_{<0}$ in the second. Thus we have

$$\mathscr{F}_{\mathrm{CC}}(P_\Omega)(\nu) = \lim_{T\to\infty} P_\Omega\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \lim_{T\to\infty}\int_{\gamma_T} F_\nu(z)\,\mathrm{d}z = \mathrm{e}^{-2\pi|\nu|/\Omega},$$

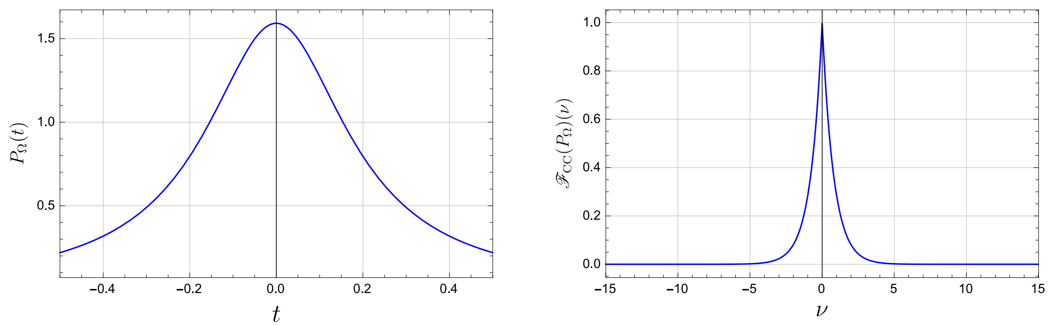


Figure 13.14 The signal P_Ω (left) and its CCFT (right) for $\Omega = 5$

for $\nu \in \mathbb{R} \setminus \{0\}$. We also have $\mathscr{F}_{\mathrm{CC}}(P_\Omega)(0) = 1$ since the CCFT of an integrable signal is continuous (Theorem 13.1.7). This gives the desired formula. In Figure 13.14 we show the Poisson kernel and its CCFT for $\Omega = 5$.

2. Recall from Example 11.3.7–2 the Gauss–Weierstrass kernel

$$G_\Omega(t) = \frac{\exp(-\frac{t^2}{4\Omega})}{\sqrt{4\pi\Omega}}.$$

We claim that

$$\mathscr{F}_{\mathrm{CC}}(G_\Omega)(\nu) = \exp(-4\pi^2\Omega\nu^2).$$

To verify this formula, we shall compute the CCFT of the general Gaussian $\gamma_a(t) = \mathrm{e}^{-at^2}$ where $a \in \mathbb{R}_{>0}$. We claim that

$$\mathscr{F}_{\mathrm{CC}}(\gamma_a)(\nu) = \sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{\pi^2\nu^2}{a}}.$$

To verify this, we compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CC}}(\gamma_a)(\nu) &= \int_{\mathbb{R}} \mathrm{e}^{-at^2-2\pi\mathrm{i}\nu t}\,\mathrm{d}t\\ &= \int_{\mathbb{R}} \mathrm{e}^{-at^2-2\pi\mathrm{i}\nu t+\frac{\pi^2\nu^2}{a}}\mathrm{e}^{-\frac{\pi^2\nu^2}{a}}\,\mathrm{d}t\\ &= \mathrm{e}^{-\frac{\pi^2\nu^2}{a}}\int_{\mathbb{R}} \mathrm{e}^{-a(t+\mathrm{i}\frac{\pi\nu}{a})^2}\,\mathrm{d}t.\end{aligned}$$

This last integral is an integral along the line through $\mathrm{i}\frac{\pi\nu}{a} \in \mathbb{C}$ and parallel to the real axis. To perform this integral we use contour integration in $\mathbb{C}$. Let us take the case of $\nu \in \mathbb{R}_{>0}$ first. We define a contour Γ_R given by

$$\begin{aligned}\Gamma_R = \{(x,0) \mid x \in [-R,R]\} &\cup \left\{(R,y) \;\middle|\; y \in [0,\tfrac{\pi\nu}{a}]\right\}\\ &\cup \left\{(x,\tfrac{\pi\nu}{a}) \;\middle|\; x \in [-R,R]\right\} \cup \left\{(-R,y) \;\middle|\; y \in [0,\tfrac{\pi\nu}{a}]\right\},\end{aligned}$$

and we take the counterclockwise sense for performing the integration. Since the function $z \mapsto e^{-az^2}$ is analytic in $\mathbb{C}$ we have

$$\begin{aligned}0 &= \int_{\Gamma_R} e^{-az^2}\,dz \\ &= \int_{-R}^{R} e^{-ax^2}\,dx + \int_0^{\frac{\pi\nu}{a}} e^{-a(R+iy)^2}\,dy + \int_R^{-R} e^{-a(x+i\frac{\pi\nu}{a})^2}\,dx + \int_{\frac{\pi\nu}{a}}^{0} e^{-a(-R+iy)^2}\,dy.\end{aligned}$$

We claim that the second and fourth integrals are zero in the limit as $R \to \infty$. To see this for the second integral, note that

$$|e^{-a(R+iy)^2}| = |e^{-aR^2}e^{-2aiRy}e^{-ay^2}| \le e^{-aR^2}e^{\frac{\pi^2\nu^2}{a}},$$

Thus

$$\left|\int_0^{\frac{\pi\nu}{a}} e^{-a(R+iy)^2}\,dy\right| \le \int_0^{\frac{\pi\nu}{a}} |e^{-a(R+iy)^2}|\,dy \le e^{\frac{\pi^2\nu^2}{a}} \int_0^{\frac{\pi\nu}{a}} e^{-aR^2}\,dy.$$

We then compute

$$\lim_{R\to\infty} \int_0^{\frac{\pi\nu}{a}} e^{-aR^2}\,dy = \lim_{R\to\infty} \frac{\pi\nu}{a} e^{-aR^2} = 0.$$

This gives the vanishing of the second integral as $R \to \infty$. The same sort of argument gives the same conclusion as regards the fourth integral. Therefore, we get

$$\int_{\mathbb{R}} e^{-a(t+i\frac{\pi\nu}{a})^2}\,dt = \lim_{R\to\infty} \int_{-R}^{R} e^{-a(t+i\frac{\pi\nu}{a})^2}\,dt = \lim_{R\to\infty} \int_{-R}^{R} e^{-at^2}\,dt.$$

Thus we have

$$\mathscr{F}_{CC}(\gamma_a)(\nu) = e^{-\frac{\pi^2\nu^2}{a}} \int_{\mathbb{R}} e^{-at^2}\,dt,$$

this being valid for $\nu \in \mathbb{R}_{>0}$. A similar analysis to the above gives the same formula for $\nu \in \mathbb{R}_{<0}$. To evaluate the integral on the right we use Lemma 1 from Example 5.3.32–**??** to give its value as $\sqrt{\frac{\pi}{a}}$. This gives $\mathscr{F}_{CC}(\gamma_a)$ as stated. Thus we see that the Gaussian has the feature that its CCFT is almost equal to itself. Indeed, if $a = \pi$ then the CCFT is *exactly* equal to the original signal. Moreover, we also have $\mathscr{F}_{CC}(G_\Omega)$ as stated. In Figure 13.15 we show the Gaussian kernel and its CCFT for $\Omega = 5$.

3. Next we consider the Fejér kernel from Example 11.3.7–3.

$$F_\Omega = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \ne 0, \\ \Omega, & t = 0. \end{cases}$$

We claim that

$$\mathscr{F}_{CC}(F_\Omega)(\nu) = \begin{cases} 1 - \frac{|\nu|}{\Omega}, & |\nu| \in [0, \Omega], \\ 0, & \text{otherwise.} \end{cases}$$

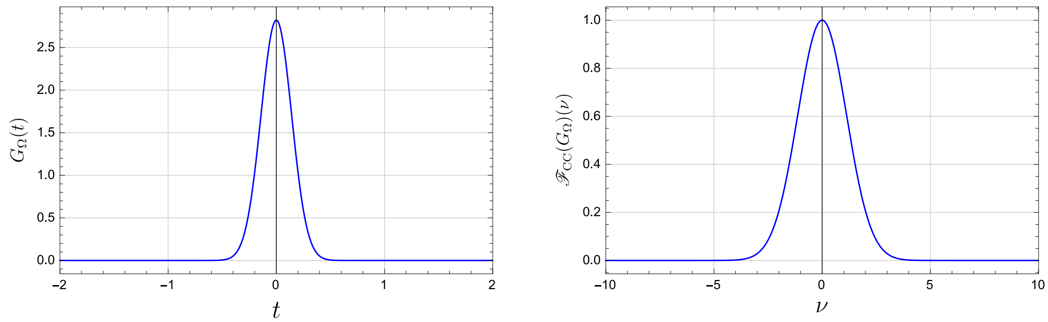


Figure 13.15 The signal G_Ω (left) and its CCFT (right) for $\Omega = 5$

Let us verify this expression. Fix $\Omega \in \mathbb{R}_{>0}$. Let $R \in \mathbb{R}_{>0}$. We recall from the proof of Lemma 1 from Example 11.3.7–3 the definitions of the contours γ_R, γ'_R, $C_{+,R}$, $C_{-,R}$, $\Gamma_{+,R}$, and $\Gamma_{-,R}$. For $\nu \in \mathbb{R}$ define $F_\nu \colon \mathbb{C} \to \mathbb{C}$ by

$$F_\nu(z) = \mathrm{i}\frac{1 - \frac{1}{2}(\mathrm{e}^{2\pi\Omega z} + \mathrm{e}^{-2\pi\Omega z})}{2\pi^2\Omega z^2}\mathrm{e}^{-2\pi\nu z},$$

noting that F_ν is obviously analytic on $\mathbb{C} \setminus \{0\}$. Moreover, one checks that $\lim_{z\to 0} F_\nu(z) = \Omega$, and so F_ν is entire by ***missing stuff***. Note that a direct computation using the identity $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)$ verifies that

$$\int_{-R}^{R} F_\Omega(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \int_{\gamma_R} F_\nu(z)\,\mathrm{d}z.$$

Since F_ν is entire, we also have

$$\int_{\gamma_R} F_\nu(z)\,\mathrm{d}z = \int_{\gamma'_R} F_\nu(z)\,\mathrm{d}z$$

by ***missing stuff***.

Let us define $f_{\nu,1}, f_{\nu,2}, f_{\nu,3} \colon \mathbb{C} \to \mathbb{C}$ by

$$f_{\nu,1}(z) = \frac{\mathrm{e}^{-2\pi\nu z}}{z^2}, \quad f_{\nu,2}(z) = \frac{\mathrm{e}^{2\pi(\Omega-\nu)z}}{z^2}, \quad f_{\nu,3}(z) = \frac{\mathrm{e}^{2\pi(-\Omega-\nu)z}}{z^2},$$

noting that these functions all have a pole of degree 2 at the origin.

We next compute a few contour integrals using Cauchy's Theorem and the Residue Theorem.

(a) Suppose that $\nu \in \mathbb{R}_{>0}$. Then

$$-\int_{\gamma'_R} f_{\nu,1}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{\nu,1}(z)\,\mathrm{d}z = \int_{\Gamma_{+,R}} f_{\nu,1}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. By Jordan's Lemma we then have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,1}(z)\,\mathrm{d}z = 0.$$

(b) Suppose that $\nu \in \mathbb{R}_{<0}$. Then

$$\int_{\gamma'_R} f_{\nu,1}(z)\,\mathrm{d}z + \int_{C_{-,R}} f_{\nu,1}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{\nu,1}(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f_{\nu,1}, 0) = -4\mathrm{i}\pi^2\nu$$

by the Residue Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,1}(z)\,\mathrm{d}z = -4\mathrm{i}\nu\pi^2.$$

(c) Suppose that $\Omega - \nu \in \mathbb{R}_{>0}$. Then

$$\int_{\gamma'_R} f_{\nu,2}(z)\,\mathrm{d}z + \int_{C_{-,R}} f_{\nu,2}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{\nu,2}(z)\,\mathrm{d}z = -4\mathrm{i}\pi^2(\nu - \Omega).$$

by the Residue Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,2}(z)\,\mathrm{d}z = -4\mathrm{i}\pi^2(\nu - \Omega).$$

(d) Suppose that $\Omega - \nu \in \mathbb{R}_{<0}$. Then

$$-\int_{\gamma'_R} f_{\nu,2}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{\nu,2}(z)\,\mathrm{d}z = \int_{\Gamma_{+,R}} f_{\nu,2}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,2}(z)\,\mathrm{d}z = 0.$$

(e) Suppose that $-\Omega - \nu \in \mathbb{R}_{>0}$. Then

$$\int_{\gamma'_R} f_{\nu,3}(z)\,\mathrm{d}z + \int_{C_{-,R}} f_{\nu,3}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{\nu,3}(z)\,\mathrm{d}z = -4\mathrm{i}\pi^2(\nu + \Omega)$$

by the Residue Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,3}(z)\,\mathrm{d}z = -4\mathrm{i}\pi^2(\nu + \Omega).$$

(f) Suppose that $-\Omega - \nu \in \mathbb{R}_{<0}$. Then

$$-\int_{\gamma'_R} f_{\nu,3}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{\nu,3}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{\nu,3}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{\nu,3}(z)\,\mathrm{d}z = 0.$$

Now, using these calculations and noting that

$$F_\nu(z) = \frac{\mathrm{i}}{2\pi^2\Omega}(f_{\nu,1}(z) - \tfrac{1}{2}f_{\nu,2}(z) - \tfrac{1}{2}f_{\nu,3}(z)),$$

we have the following cases.

(a) $\nu < -\Omega$: In this case we have $\nu < 0$, $\Omega - \nu > 0$, and $-\Omega - \nu > 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi^2\Omega}(-4\mathrm{i}\pi^2\nu + 2\mathrm{i}\pi^2(\nu-\Omega) + 2\mathrm{i}\pi^2(\nu+\Omega)) = 0.$$

(b) $-\Omega < \nu < 0$: In this case we have $\nu < 0$, $\Omega - \nu > 0$, and $-\Omega - \nu < 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi^2\Omega}(-4\mathrm{i}\pi^2\nu + 2\mathrm{i}\pi^2(\nu-\Omega) + 0) = 1 + \frac{\nu}{\Omega}.$$

(c) $0 < \nu < \Omega$: In this case we have $\nu > 0$, $\Omega - \nu > 0$, and $-\Omega - \nu < 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi^2\Omega}(0 + 2\mathrm{i}\pi^2(\nu-\Omega) + 0) = \frac{1}{-}\frac{\nu}{\Omega}.$$

(d) $\nu > \Omega$: In this case we have $\nu > 0$, $\Omega - \nu < 0$, and $-\Omega - \nu < 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi^2\Omega}(0 + 0 + 0) = 0.$$

Now, since $\mathscr{F}_{\mathrm{CC}}(F_\Omega)$ is continuous by Theorem 13.1.7, our formula for $\mathscr{F}_{\mathrm{CC}}(F_\Omega)$ is as stated.

In Figure 13.16 we show the Fejér kernel and its CCFT for $\Omega = 5$.

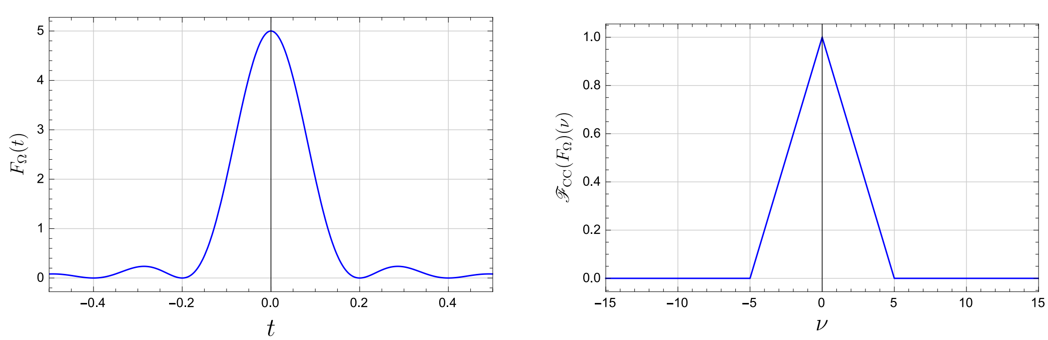


Figure 13.16 The signal V_Ω (left) and its CCFT (right) for $\Omega = 5$

4. Let us next determine the CCFT of the de la Vallée Poussin kernel defined by

$$V_\Omega(t) = 2F_{2\Omega}(t) - F_\Omega(t).$$

As we have computed $\mathscr{F}_{\mathrm{CC}}(F_\Omega)$ above, we easily use linearity of the CCFT to determine

$$\mathscr{F}_{\mathrm{CC}}(V_\Omega)(t) = \begin{cases} 1, & |\nu| \in [0, \Omega], \\ 2 - \frac{|\nu|}{\Omega}, & |\nu| \in (\Omega, 2\Omega], \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 13.17 we show the de la Vallée Poussin kernel and its CCFT for $\Omega = 5$.

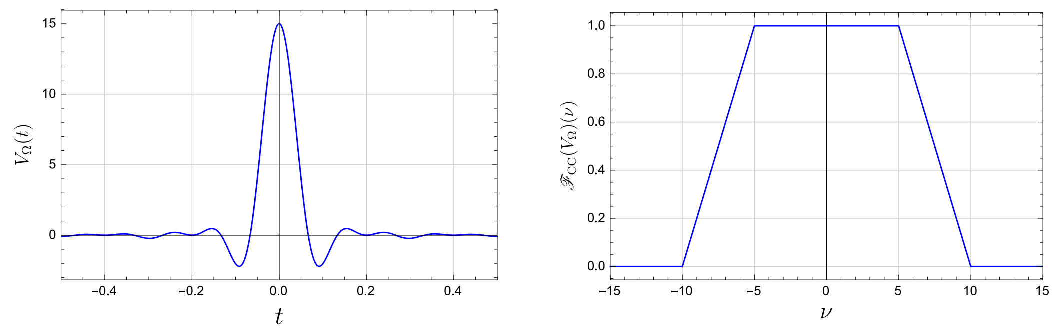


Figure 13.17 The signal V_Ω (left) and its CCFT (right) for $\Omega = 5$

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13.2.9 Notes

Our Example 13.2.2 appears in [**MAP:09**].

Exercises

13.2.1 Let $f \in \mathsf{L}^{(1)}(\mathbb{R}; \mathbb{C})$ and denote its CCFT by $\mathscr{F}_{\mathrm{CC}}(f)$.

(a) Write the integral version of the expression $\mathrm{FI}[f]$.

(b) Naïvely, what would have to be true for it to hold that

$$\mathrm{FI}[f](t) = f(t), \qquad t \in \mathbb{R}?$$

13.2.2 Give a proof along the lines of the proof of Proposition 12.2.2 that $\mathscr{F}_{\mathrm{CC}}$ is not onto $\mathsf{C}^0_0(\mathbb{R}; \mathbb{C})$.

13.2.3 Show that for each $\Omega \in \mathbb{R}_{>0}$ we have

$$\int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu) \mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = \int_0^{\Omega} \mathscr{C}_{\mathrm{CC}}(f)(\nu)\cos(2\pi\nu t)\,\mathrm{d}\nu + \int_0^{\Omega} \mathscr{S}_{\mathrm{CC}}(f)(\nu)\sin(2\pi\nu t)\,\mathrm{d}\nu.$$

13.2.4 Show that for every $\epsilon \in \mathbb{R}_{>0}$ we have

$$\lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} D_\Omega(t)\,\mathrm{d}t = 1.$$

Hint: *Use Lemma 1 from Example 11.3.7–3.*

13.2.5 Let $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and, for $t_0 \in \mathbb{R}$ and $s \in \mathbb{C}$, define $e_{f,t_0,s} \colon \mathbb{R} \to \mathbb{C}$ by

$$e_{f,t_0,s}(t) = \tfrac{1}{2}(f(t_0+t) + f(t_0-t)) - s.$$

(a) Show that $e_{f,t_0,s} \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$.

(b) Show that the Fourier integral for f converges to $s \in \mathbb{C}$ at t_0 if and only if the Fourier integral for $e_{f,t_0,s}$ converges to 0 at 0.

(c) Show that, if there exists a neighbourhood U of t_0 for which $f(t_0+t) = -f(t_0-t)$ for every $t \in U$, then it holds that the Fourier integral for f converges to zero at t_0.

(d) Sketch the graph of a typical function from part (c).

13.2.6 Give a signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ such that $\mathscr{F}_{\mathrm{CC}}(f) \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Explain why your example works without doing any computations.

13.2.7 Using Theorem 13.2.26 and examples given in the text, compute the CCFT's of P_Ω and F_Ω.

13.2.8 Answer the following questions.

(a) Is the function

$$x \mapsto \frac{(1-\cos x)\sin x}{x}$$

in $\mathsf{L}^{(1)}(\mathbb{R}_{>0};\mathbb{R})$?

(b) Show that

$$\lim_{R\to\infty} \int_0^R \frac{(1-\cos x)\sin x}{x}\,\mathrm{d}x = \frac{\pi}{4}.$$

Hint: *Use Example 13.2.20–1.*

13.2.9 Answer the following questions.

(a) Is the function

$$x \mapsto \frac{1-\cos x}{x^2}$$

in $\mathsf{L}^{(1)}(\mathbb{R}_{>0};\mathbb{R})$?

(b) Show that

$$\lim_{R\to\infty} \int_0^R \frac{1-\cos x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

Hint: *Use Example 13.2.20–2.*

Section 13.3

The L^2-CCFT

As with the CDFT, the L^2-theory of the CCFT is extremely important. The reason for this is that, as we shall see in this section, the L^2-CCFT has nice inversion properties as a consequence of the more friendly geometry of the Hilbert space $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ versus the Banach space geometry of $\mathsf{L}^1(\mathbb{R};\mathbb{C})$.

One of the places where the CCFT diverges in its development from the CDFT is with the L^2-theory. For the CDFT the L^2-transform is obtained by restriction since $\mathsf{L}^{(2)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \subseteq \mathsf{L}^{(1)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. However, as can be recalled from Figure 8.21, $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C}) \not\subset \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Thus a direct definition of the L^2-CCFT is not possible. Indeed, it is not immediately clear that *any* definition is possible. That this *is* possible is a consequence of some nice interrelations between $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$, and the CCFT. These interrelationships are what this section is concerned with.

Do I need to read this section? If you are learning about the Fourier transform, then this section is required reading. •

13.3.1 Definition of the L^2-CCFT

Our construction of the L^2-CCFT is a little indirect. However, at the end of the day we do end up with a computable theory; see Section 13.3.4.

If one looks back at the examples we used in Sections 13.1 and 13.2 it can be seen that all of the signals used had the property that they were not only in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, but were also in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Indeed, signals in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) - \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ tend to be a little unfriendly (try doing Exercise 8.3.15). The following result records what happens with the CCFT in the intersection of $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$.

13.3.1 Lemma ($\mathscr{F}_{\mathrm{CC}}(\mathsf{L}^1 \cap \mathsf{L}^2) \subseteq \mathsf{L}^2$) *If* $\mathrm{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ *then* $\|\mathscr{F}_{\mathrm{CC}}(\mathrm{f})\|_2 = \|\mathrm{f}\|_2$. *In particular,* $\mathscr{F}_{\mathrm{CC}}(\mathrm{f}) \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$.

Proof Let $t \in \mathbb{R}$. Since $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$, $\tau^*_{-t}f, \bar{f} \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Thus $\tau^*_{-t}f\bar{f} \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ by Hölder's inequality. Define

$$\phi(t) = \int_{\mathbb{R}} f(t+\tau)\bar{f}(\tau)\,\mathrm{d}\tau.$$

We first claim that $\phi \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$, and that ϕ is in fact uniformly continuous. For $a, t \in \mathbb{R}$ we compute

$$\begin{aligned}|\phi(t+a) - \phi(t)| &\le \int_{\mathbb{R}} |f(t+a+\tau) - f(t+\tau)||f(\tau)|\,\mathrm{d}\tau \\ &\le \Big(\int_{\mathbb{R}} |f(a+\tau) - f(\tau)|^2\,\mathrm{d}\tau\Big)^{1/2}\|f\|_2 \\ &= \|\tau^*_{-a}f - f\|_2\|f\|_2\end{aligned}$$

by the Cauchy–Bunyakovsky–Schwarz inequality. By Lemma 1 from Corollary 11.2.10, for $\epsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ so that if $|a| < \delta$ then $\|\tau_{-a}^* f - f\|_2 < \frac{\epsilon}{\|f\|_2}$. Uniform continuity of ϕ now follows. We now show that $\lim_{|t|\to\infty} \phi(t) = 0$. Let $\epsilon \in \mathbb{R}_{>0}$ and let $T \in \mathbb{R}_{>0}$ be sufficiently large that

$$\Big(\int_{\mathbb{R}\setminus[-T,T]} |f(t)|^2 \,\mathrm{d}t\Big)^{1/2} < \frac{\epsilon}{2\|f\|_2},$$

this being possible since $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Now let $t \in \mathbb{R}$ satisfy $|t| > 2T$. Then

$$\begin{aligned}
|\phi(t)| &\le \int_{-T}^{T} |f(t+\tau)||f(\tau)| \,\mathrm{d}\tau + \int_{\mathbb{R}\setminus[-T,T]} |f(t+\tau)||f(\tau)| \,\mathrm{d}\tau \\
&\le \Big(\int_{-T}^{T} |f(t+\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2}\Big(\int_{-T}^{T} |f(\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2} \\
&\quad + \Big(\int_{\mathbb{R}\setminus[-T,T]} |f(t+\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2}\Big(\int_{\mathbb{R}\setminus[-T,T]} |f(\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2} \\
&\le \|f\|_2\Big(\int_{t-T}^{t+T} |f(\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2} + \|f\|_2\Big(\int_{\mathbb{R}\setminus[-T,T]} |f(\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2} \\
&\le 2\|f\|_2\Big(\int_{\mathbb{R}\setminus[-T,T]} |f(\tau)|^2 \,\mathrm{d}\tau\Big)^{1/2} < \epsilon,
\end{aligned}$$

where we have used the Cauchy–Bunyakovsky–Schwarz inequality, the change of variable formula, and the fact that $[t-T, t+T] \subseteq \mathbb{R} \setminus [-T, T]$. Thus $\phi(t)$ goes to zero as $|t| \to \infty$ as claimed.

We claim that $\phi \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}} |\phi(t)| \,\mathrm{d}t &\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |f(t+\tau)\bar{f}(\tau)| \,\mathrm{d}\tau\Big) \mathrm{d}t \\
&= \int_{\mathbb{R}} |f(\tau)|\Big(\int_{\mathbb{R}} |f(t+\tau)| \,\mathrm{d}t\Big) \mathrm{d}\tau = \|f\|_1^2,
\end{aligned}$$

using Fubini's Theorem. We then compute

$$\begin{aligned}
\mathscr{F}_{\mathrm{CC}}(\phi)(\nu) &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t+\tau)\bar{f}(\tau) \,\mathrm{d}\tau\Big) \mathrm{e}^{-2\pi\mathrm{i}\nu t} \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \bar{f}(\tau)\Big(\int_{\mathbb{R}} f(t+\tau)\mathrm{e}^{-2\pi\mathrm{i}\nu t} \,\mathrm{d}t\Big) \mathrm{d}\tau \\
&= \int_{\mathbb{R}} \bar{f}(\tau)\Big(\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t} \,\mathrm{d}t\Big) \mathrm{e}^{2\pi\mathrm{i}\nu\tau} \,\mathrm{d}\tau \\
&= \mathscr{F}_{\mathrm{CC}}(f)(\nu) \int_{\mathbb{R}} \bar{f}(\tau)\mathrm{e}^{2\mathrm{i}n\pi\frac{\tau}{T}} \,\mathrm{d}\tau \\
&= \mathscr{F}_{\mathrm{CC}}(f)(\nu)\overline{\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t} \,\mathrm{d}t} = |\mathscr{F}_{\mathrm{CC}}(f)(\nu)|^2.
\end{aligned}$$

From Example 13.2.39–3, and using Proposition 13.1.6(ii) along with the fact that $\sigma^* F_\Omega = F_\Omega$, we have

$$\overline{\mathscr{F}}_{\mathrm{CC}}(F_\Omega)(\nu) = \mathscr{F}_{\mathrm{CC}}(\sigma^* F_\Omega)(\nu) = \mathscr{F}_{\mathrm{CC}}(F_\Omega)(\nu) = \begin{cases} 1 + \frac{t}{\Omega}, & t \in [-\Omega, 0], \\ 1 - \frac{t}{\Omega}, & t \in (0, \Omega], \\ 0, & \text{otherwise.} \end{cases}$$

We also note that $\lim_{\Omega\to\infty} \mathscr{F}_{CC}(F_\Omega)(\nu) = 1$ for every $\nu \in \mathbb{R}$.

We saw in Theorem 13.2.36 that $(F_\Omega \sigma^*\phi)_{\Omega\in\mathbb{R}_{>0}}$ converges to $\sigma^*\phi$ in $(\mathsf{C}^0_{\mathrm{unif,bdd}}(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$. Thus

$$\begin{aligned}\sigma^*\phi(t) &= \lim_{\Omega\to\infty} \int_{\mathbb{R}} \sigma^*\phi(t-\tau)F_\Omega(\tau)\,\mathrm{d}\tau\\ &= \lim_{\Omega\to\infty} \int_{\mathbb{R}} \sigma^*\phi(t-\tau)\mathscr{F}_{CC}\circ\overline{\mathscr{F}}_{CC}(F_\Omega(\tau))\,\mathrm{d}\tau\\ &= \lim_{\Omega\to\infty} \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\nu t}\mathscr{F}_{CC}(\phi)(\nu)\overline{\mathscr{F}}_{CC}(F_\Omega)(\nu)\,\mathrm{d}\nu\\ &= \int_{\mathbb{R}} |\mathscr{F}_{CC}(f)(\nu)|^2 \mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu,\end{aligned}$$

using Proposition 13.1.6(ii) and (v), Fourier Reciprocity, and the Dominated Convergence Theorem. Setting $t = 0$ gives $\|f\|_2 = \|\mathscr{F}_{CC}(f)\|_2$, as desired. ■

The reader may want to check that the lemma is satisfied for all of the signals we encountered in Sections 13.1 and 13.2.

The next result says that, by knowing $\mathscr{F}_{CC}|\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$, we know the CCFT on a "large" subspace of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. This will allow us to extend the CCFT to all of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

13.3.2 Lemma ($\mathsf{L}^1 \cap \mathsf{L}^2$ is dense in L^2) *$\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ is dense in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.*

Proof Let $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$, let $\epsilon \in \mathbb{R}_{>0}$, and choose $T \in \mathbb{R}_{>0}$ sufficiently large that

$$\int_{\mathbb{R}\setminus[-T,T]} |f(t)|^2\,\mathrm{d}t < \epsilon^2,$$

this being possible since $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Then it is clear that, if $g = \chi_{[-T,T]}f$, we have $g \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and $\|f-g\|_2 < \epsilon$. Since $f|[-T,T] \in \mathsf{L}^2([-T,T];\mathbb{C})$ and since $\mathsf{L}^2([-T,T];\mathbb{C}) \subseteq \mathsf{L}^1([-T,T];\mathbb{C})$ (by Theorem 8.3.11(iv)), it follows that $g \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. ■

From this we have the following result.

13.3.3 Theorem (Plancherel's[2] Theorem) *There exists a unique continuous linear map $\tilde{\mathscr{F}}_{CC}\colon \mathsf{L}^2(\mathbb{R};\mathbb{C}) \to \mathsf{L}^2(\mathbb{R};\mathbb{C})$ with the properties*

(i) $\tilde{\mathscr{F}}_{CC}(f) = \mathscr{F}_{CC}(f)$ for $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and

*(ii) $\|\tilde{\mathscr{F}}_{CC}(f)\|_2 = \|f\|_2$ (**Parseval's equality** or **Plancherel's equality**).*

Furthermore, if $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and if $(f_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ for which $\lim_{j\to\infty}\|f - f_j\|_2 = 0$, then $\lim_{j\to\infty}\|\tilde{\mathscr{F}}_{CC}(f) - \mathscr{F}_{CC}(f_j)\|_2 = 0$.

Proof The existence and uniqueness of $\tilde{\mathscr{F}}_{CC}$ follows from Proposition 6.5.11. The final assertion in the theorem follows from continuity of $\tilde{\mathscr{F}}_{CC}$. Part (ii) follows from the last assertion and, by Lemma 13.3.1, the fact that $\|\tilde{\mathscr{F}}_{CC}(f)\|_2 = \|f\|_2$ for $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. ■

The theorem makes sense of the following definition.

[2] Michel Plancherel (1885–1967) was a Swiss mathematician who made contributions to the fields of analysis, algebra, and mathematical physics.

13.3.4 Definition (L^2-CCFT) The map $\tilde{\mathscr{F}}_{\mathrm{CC}}\colon \mathsf{L}^2(\mathbb{R};\mathbb{C}) \to \mathsf{L}^2(\mathbb{R},\mathbb{C})$ of Theorem 13.3.3 is the **L^2-*CCFT***. We shall write the L^2-CCFT simply as $\mathscr{F}_{\mathrm{CC}}$. •

13.3.5 Remarks (Attributes of the L^2-CCFT)

1. Note that the L^2-CCFT differs in spirit from the CCFT for L^1 signals. Indeed, the definition of the L^1-CCFT explicitly defines a function of frequency ν. However, the L^2-CCFT defines only an equivalence class of functions of frequency.
2. There are many ways to define a sequence in $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ converging to $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. Two common ones are as follows:
   (a) $f_j(t) = \chi_{[-j,j]}(t)f(t)$ (cf. the proof of Lemma 13.3.2);
   (b) $f_j(t) = \mathrm{e}^{-t^2/j}f(t)$ (use the Cauchy–Bunyakovsky–Schwarz inequality to show that this signal is in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$).
3. The above arguments can be carried out *mutatis mutandis* for the transform $\overline{\mathscr{F}}_{\mathrm{CC}}$ to extend it from $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ to $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. The properties of this transformation will be explored in Section 13.3.3. However, in the next section we shall tacitly suppose that the map $\overline{\mathscr{F}}_{\mathrm{CC}}\colon \mathsf{L}^2(\mathbb{R};\mathbb{C}) \to \mathsf{L}^2(\mathbb{R};\mathbb{C})$ is defined. •

13.3.2 Properties of the L^2-CCFT

In this section we shall develop a few consequences of the definition of the L^2-CCFT. First we have the following basic properties, just as we do for the L^1-CCFT.

13.3.6 Proposition (Elementary properties of the L^2-CCFT) *For* $\mathrm{f} \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ *the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{CC}}(\mathrm{f})} = \overline{\mathscr{F}}_{\mathrm{CC}}(\bar{\mathrm{f}})$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\sigma^*\mathrm{f}) = \sigma^*(\mathscr{F}_{\mathrm{CC}}(\mathrm{f})) = \overline{\mathscr{F}}_{\mathrm{CC}}(\mathrm{f})$;

(iii) if f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{CC}}(\mathrm{f})$ *is even (resp. odd);*

(iv) if f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{CC}}(\mathrm{f})$ *is real and even (resp. imaginary and odd);*

(v) $\mathscr{F}_{\mathrm{CC}}(\tau_{\mathrm{a}}^*\mathrm{f})(\nu) = \mathrm{e}^{-2\pi\mathrm{i}\mathrm{a}\nu}\mathscr{F}_{\mathrm{CC}}(\mathrm{f})(\nu)$.

Proof If $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ then the assertions of the result hold by virtue of Proposition 13.1.6. For $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$, by Lemma 13.3.2 we have a sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ converging to f in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. By continuity of the L^2-CCFT and by continuity of the operations involved in the statement of the proposition (cf. Proposition 6.5.4), since the assertions hold for each of the signals f_j, it also holds for f by Theorem 6.5.2. ■

The relationship between the CCFT and differentiation also carries over to the L^2-CCFT. The following result is a special case of Proposition 13.4.5. While it is possible to prove this result without the aid of distributions, we will not do so.

13.3.7 Proposition (The L^2-CCFT and differentiation) *Suppose that* $f \in C^0(\mathbb{R};\mathbb{C}) \cap L^{(2)}(\mathbb{R};\mathbb{C})$ *and that there exists a signal* $f' \colon \mathbb{R} \to \mathbb{C}$ *with the following properties:*

(i) for every $T \in \mathbb{R}_{>0}$, f' *is piecewise continuous on* $[-T, T]$*;*

(ii) f' *is discontinuous at a finite number of points;*

(iii) $f' \in L^{(1)}(\mathbb{R};\mathbb{C})$*;*

(iv) $f(t) = f(0) + \int_0^t f'(\tau)\,d\tau$.

Then

$$\mathscr{F}_{CC}(f')(\nu) = (2\pi i\nu)\mathscr{F}_{CC}(f)(\nu).$$

Proof By Exercise 8.3.18 we have $\lim_{|t|\to\infty} f(t) = 0$. The remainder of the proof now follows Proposition 13.1.10. ■

The result concerning the differentiability of the CCFT also holds. Again, this result will be proved in the setting of tempered distributions, so we will not prove it here in this less general setting.

13.3.8 Proposition (Differentiability of transformed signals) *For* $f \in L^{(2)}(\mathbb{R};\mathbb{C})$*, if the signals* $t \mapsto t^j f(t)$, $j \in \{0, 1, \dots, k\}$*, are in* $L^{(2)}(\mathbb{R};\mathbb{C})$ *then* $\mathscr{F}_{CC}(f)$ *is* k*-times continuously differentiable and* $\mathscr{F}_{CC}(f)^{(k)}(\nu) = \mathscr{F}_{CC}(f_k)(\nu)$*, where* $f_k(t) = (-2\pi i t)^k f(t)$.

Proof For fixed t the signal $f(t)e^{-2\pi i\nu t}$ is infinitely differentiable with respect to ν. Furthermore, the kth derivative is bounded in magnitude by $2\pi|t^k f(t)|$. As this signal is assumed to be in $L^{(2)}(\mathbb{R};\mathbb{C})$, it is locally integrable, cf. Exercise 8.3.8. We may apply Theorem 5.9.17 to conclude that, for every $j \in \mathbb{Z}_{>0}$, the signal

$$\nu \mapsto \int_{\mathbb{R}} f(t)e^{-2\pi i\nu t}\,dt$$

is k-times continuously differentiable and that its kth derivative is

$$\nu \mapsto \int_{\mathbb{R}} (-2\pi i t)^k f(t)e^{-2\pi i\nu t}\,dt,$$

which is the result. ■

Next we prove that the Fourier Reciprocity Relation holds for signals in $L^2(\mathbb{R};\mathbb{C})$.

13.3.9 Proposition (Fourier Reciprocity Relation for the L^2-CCFT) *If* $f, g \in L^2(\mathbb{R};\mathbb{C})$ *then* $\mathscr{F}_{CC}(f)g, f\mathscr{F}_{CC}(g) \in L^1(\mathbb{R};\mathbb{C})$ *and*

$$\int_{\mathbb{R}} f(\xi)\mathscr{F}_{CC}(g)(\xi)\,d\xi = \int_{\mathbb{R}} \mathscr{F}_{CC}(f)(\xi)g(\xi)\,d\xi.$$

Proof By the Cauchy–Bunyakovsky–Schwarz inequality, the product of two signals in $L^2(\mathbb{R};\mathbb{C})$ gives a signal in $L^1(\mathbb{R};\mathbb{C})$. For $f, g \in L^2(\mathbb{R};\mathbb{C})$ let $(f_j)_{j\in\mathbb{Z}_{>0}}$ and $(g_j)_{j\in\mathbb{Z}_{>0}}$ be sequences in $L^1(\mathbb{R};\mathbb{C}) \cap L^2(\mathbb{R};\mathbb{C})$ converging, respectively, to f and g. By Proposition 13.1.9 it follows that

$$\int_{\mathbb{R}} f_j(\xi)\mathscr{F}_{CC}(g_j)(\xi)\,d\xi = \int_{\mathbb{R}} \mathscr{F}_{CC}(f_j)(\xi)g_j(\xi)\,d\xi$$

for all $j \in \mathbb{Z}_{>0}$. For the expression on the left we have

$$\begin{aligned}\Big|\int_{\mathbb{R}} f(\xi)\mathscr{F}_{\mathrm{CC}}(g)(\xi)\,\mathrm{d}\xi - \int_{\mathbb{R}} f_j(\xi)\mathscr{F}_{\mathrm{CC}}(g_j)(\xi)\,\mathrm{d}\xi\Big| \\ &\leq \Big|\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f-f_j)(\xi)g(\xi)\,\mathrm{d}\xi\Big| + \Big|\int_{\mathbb{R}} f(\xi)\mathscr{F}_{\mathrm{CC}}(g-g_j)(\xi)\,\mathrm{d}\xi\Big| \\ &\leq \|f-f_j\|_2\|g\|_2 + \|f\|_2\|g-g_j\|_2,\end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Taking the limit as $j \to \infty$ gives

$$\int_{\mathbb{R}} f(\xi)\mathscr{F}_{\mathrm{CC}}(g)(\xi)\,\mathrm{d}\xi = \lim_{j\to\infty} \int_{\mathbb{R}} f_j(\xi)\mathscr{F}_{\mathrm{CC}}(g_j)(\xi)\,\mathrm{d}\xi.$$

Similarly we have

$$\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\xi)g(\xi)\,\mathrm{d}\xi = \lim_{j\to\infty} \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f_j)(\xi)g_j(\xi)\,\mathrm{d}\xi.$$

From this the result follows. ■

13.3.3 The inverse L^2-CCFT

In this section we establish the important fact that $\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^2(\mathbb{R};\mathbb{C}) \to \mathsf{L}^2(\mathbb{R};\mathbb{C})$ is an isomorphism. This mirrors the corresponding fact for $\mathscr{F}_{\mathrm{CD}}\colon \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$.

13.3.10 Theorem (The L^2-CCFT is an isomorphism) *The L^2-CCFT is a Hilbert space isomorphism. That is to say,*

(i) it is a linear bijection and

(ii) it preserves the L^2-inner product, i.e., $\langle \mathscr{F}_{\mathrm{CC}}(\mathrm{f}), \mathscr{F}_{\mathrm{CC}}(\mathrm{g})\rangle_2 = \langle \mathrm{f}, \mathrm{g}\rangle_2$ for each $\mathrm{f}, \mathrm{g} \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$.

Proof To show that the L^2-CCFT is injective, suppose that $\mathscr{F}_{\mathrm{CC}}(f) = 0$ for $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. Then by part (ii) of Theorem 13.3.3 it follows that $\|f\|_2 = 0$, or that $f = 0$. Thus the L^2-CCFT is injective. We next claim that the image of the L^2-CCFT is a closed subspace of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. Indeed, if $(\mathscr{F}_{\mathrm{CC}}(f_j))_{j\in\mathbb{Z}_{>0}}$ is a sequence in the image of the L^2-CCFT converging to $g \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ then, since

$$\|f_j - f_k\|_2 = \|\mathscr{F}_{\mathrm{CC}}(f_j) - \mathscr{F}_{\mathrm{CC}}(f_k)\|_2,$$

it follows that $(f_j)_{j\in\mathbb{Z}_{>0}}$ is itself a Cauchy sequence, and so converges to $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. It remains to show that $g = \mathscr{F}_{\mathrm{CC}}(f)$. For this we have

$$\|\mathscr{F}_{\mathrm{CC}}(f) - g\|_2 \leq \|\mathscr{F}_{\mathrm{CC}}(f) - \mathscr{F}_{\mathrm{CC}}(f_j)\|_2 + \|g - \mathscr{F}_{\mathrm{CC}}(f_j)\|_2.$$

In the limit as $j \to \infty$ the first term on the right vanishes by continuity of the L^2-CCFT, and the second term on the right vanishes by definition of g. This shows that the image of the L^2-CCFT is a closed subspace. Therefore, to show that this closed subspace is all of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ it suffices by Theorem 7.1.19 to show that, if $\langle \mathscr{F}_{\mathrm{CC}}(f), g\rangle_2 = 0$ for every $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$, then $g = 0$. By Proposition 13.3.9 we have

$$\langle \mathscr{F}_{\mathrm{CC}}(f), g\rangle = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\xi)\bar{g}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} f(\xi)\mathscr{F}_{\mathrm{CC}}(\bar{g})(\xi)\,\mathrm{d}\xi.$$

If this is to vanish for each $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ it must hold that $\mathscr{F}_{\mathrm{CC}}(\bar{g}) = 0$. By part (ii) of Theorem 13.3.3 this implies that $\bar{g} = 0$, so giving surjectivity of the L^2-CCFT.

Next we verify that the L^2-CCFT preserves the L^2-inner product. For $f, g \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ we have

$$\langle \mathscr{F}_{\mathrm{CC}}(f+g), \mathscr{F}_{\mathrm{CC}}(f+g)\rangle = \langle f, f\rangle + \langle g, g\rangle + \langle \mathscr{F}_{\mathrm{CC}}(f), \mathscr{F}_{\mathrm{CC}}(g)\rangle + \langle \mathscr{F}_{\mathrm{CC}}(g), \mathscr{F}_{\mathrm{CC}}(f)\rangle,$$

using part (ii) of Theorem 13.3.3. Again by part (ii) of Theorem 13.3.3 we have

$$\langle \mathscr{F}_{\mathrm{CC}}(f+g), \mathscr{F}_{\mathrm{CC}}(f+g)\rangle = \langle f+g, f+g\rangle = \langle f, f\rangle + \langle g, g\rangle + \langle f, g\rangle + \langle g, f\rangle.$$

Combining these two expressions we get

$$\langle \mathscr{F}_{\mathrm{CC}}(f), \mathscr{F}_{\mathrm{CC}}(g)\rangle + \langle \mathscr{F}_{\mathrm{CC}}(g), \mathscr{F}_{\mathrm{CC}}(f)\rangle = \langle f, g\rangle + \langle g, f\rangle, \qquad fg \in \mathsf{L}^2(\mathbb{R};\mathbb{C}). \tag{13.11}$$

Using the symmetry property of the inner product, (13.11) implies that $\mathrm{Re}(\langle \mathscr{F}_{\mathrm{CC}}(f), \mathscr{F}_{\mathrm{CC}}(g)\rangle) = \mathrm{Re}(\langle f, g\rangle)$. Applying (13.11) to $-\mathrm{i}f$ and g similarly gives $\mathrm{Im}(\langle \mathscr{F}_{\mathrm{CC}}(f), \mathscr{F}_{\mathrm{CC}}(g)\rangle) = \mathrm{Im}(\langle f, g\rangle)$. This gives the result. ■

Next let us establish explicitly the inverse of the L^2-CCFT. Note that for the map $\overline{\mathscr{F}}_{\mathrm{CC}} \colon \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ defined by

$$\overline{\mathscr{F}}_{\mathrm{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t$$

we may apply the same sort of arguments as were applied to $\mathscr{F}_{\mathrm{CC}}$ in developing the L^2-CCFT. In doing so, one arrives at the following conclusions:

1. if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ then $\overline{\mathscr{F}}_{\mathrm{CC}}(f) \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$;
2. the map $\overline{\mathscr{F}}_{\mathrm{CC}}$ extends uniquely from $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ to a linear map $\overline{\mathscr{F}}_{\mathrm{CC}} \colon \mathsf{L}^2(\mathbb{R};\mathbb{C}) \to \mathsf{L}^2(\mathbb{R};\mathbb{C})$ having the property that $\|\overline{\mathscr{F}}_{\mathrm{CC}}(f)\|_2 = \|f\|_2$;
3. $\overline{\mathscr{F}}_{\mathrm{CC}}$ is a Hilbert space isomorphism.

The proof of these facts requires only sign modifications in the arguments used for $\mathscr{F}_{\mathrm{CC}}$, and we leave this to the reader to fill in. Next we establish that for the L^2-CCFT, its inverse is exactly $\overline{\mathscr{F}}_{\mathrm{CC}}$, or more precisely the extension of $\overline{\mathscr{F}}_{\mathrm{CC}}$ from $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ to $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

13.3.11 Theorem ($\overline{\mathscr{F}}_{\mathrm{CC}}$ is the inverse of $\mathscr{F}_{\mathrm{CC}}$ for the L^2-CCFT) *$\mathscr{F}_{\mathrm{CC}} \circ \overline{\mathscr{F}}_{\mathrm{CC}}(\mathrm{f}) = \mathrm{f}$ and $\overline{\mathscr{F}}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(\mathrm{f}) = \mathrm{f}$ for all $\mathrm{f} \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$.*

Proof Recall that $\mathscr{S}(\mathbb{R};\mathbb{C})$ denotes the set of Schwartz signals, i.e., those signal which, along with all of their derivatives, decay rapidly. In Theorem 13.4.1 below we shall show that

$$\mathscr{F}_{\mathrm{CC}} \circ \overline{\mathscr{F}}_{\mathrm{CC}}(\phi)(t) = \overline{\mathscr{F}}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(\phi)(t) = \phi(t)$$

for every $t \in \mathbb{R}$ and $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. In ***missing stuff*** we showed that $\mathscr{S}(\mathbb{R};\mathbb{C})$ is a dense subspace of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. It, therefore, follows from Proposition 6.5.12 that

$$\mathscr{F}_{\mathrm{CC}} \circ \overline{\mathscr{F}}_{\mathrm{CC}}(f) = \overline{\mathscr{F}}_{\mathrm{CC}} \circ \mathscr{F}_{\mathrm{CC}}(f) = f$$

(equality being of equivalence classes!) for every $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$, as desired. ■

13.3.12 Remarks (The character of the L^2-CCFT) It might seem as if the CCFT in the L^2-setting achieves Fourier Nirvana. However, one must be a little careful since there is nothing in the theory about the pointwise properties of the transform or its inverse. The reader should refer to the discussion concerning pointwise convergence in Section 13.2.9. That *caveat* being stated, for practical purposes the L^2-CCFT is often of great utility. •

We close this section with an example that illustrates that the use of the Lebesgue integral in the above definition of the L^2-CCFT is essential.

13.3.13 Example (The Riemann integral cannot be used for the L^2-CCFT) Since we wish to distinguish between the Riemann and Lebesgue integrals on $\mathbb{R}$, we shall denote these integrals by

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t, \qquad \int_{\mathbb{R}} f\,\mathrm{d}\lambda,$$

respectively. We denote by $\mathsf{R}^{(p)}(\mathbb{R};\mathbb{C})$ the collection of functions $f\colon\mathbb{R}\to\mathbb{C}$ which satisfy

$$\int_{-\infty}^{\infty} |f(t)|^p\,\mathrm{d}t < \infty,$$

where we use the Riemann integral for possibly unbounded functions defined on unbounded domains; see Section **??**. We also define

$$\mathsf{R}_0(\mathbb{R};\mathbb{C}) = \Big\{ f \in \mathsf{R}^p(\mathbb{R};\mathbb{C}) \;\Big|\; \int_{-\infty}^{\infty} |f(t)|^p\,\mathrm{d}t = 0 \Big\}$$

and denote

$$\mathsf{R}^p(\mathbb{R};\mathbb{C}) = \mathsf{R}^{(p)}(\mathbb{R};\mathbb{C})/\mathsf{R}_0(\mathbb{R};\mathbb{C}).$$

As we have done when defining the Lebesgue integral, we denote $[f] = f + \mathsf{R}_0(\mathbb{R};\mathbb{C})$. If we define

$$\|[f]\|_p = \Big(\int_{-\infty}^{\infty} |f(t)|^p\,\mathrm{d}t \Big)^{1/p},$$

then $(\mathsf{R}^p(\mathbb{R};\mathbb{C}), \|\cdot\|_p)$ is a normed vector space. It is not a Banach space since the example of Proposition 5.1.12 can be extended to $\mathsf{R}^p(\mathbb{R};\mathbb{C})$ by taking all functions to be zero outside the interval $[0,1]$.

Let us show that $\mathscr{F}_{\mathrm{CC}}|\mathsf{R}^2(\mathbb{R};\mathbb{C})$ does not take values in $\mathsf{R}^2(\mathbb{R};\mathbb{C})$, and thus show that the "R^2-Fourier transform" is not well-defined. We denote by F the function defined (and denoted by f) in Proposition 5.1.12, but now extended to be defined on $\mathbb{R}$ by taking it to be zero off $[0,1]$. We have $F \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ since F is bounded and measurable with compact support. Now define $f\colon\mathbb{R}\to\mathbb{C}$ by

$$f(t) = \int_{\mathbb{R}} F\mathsf{E}_{2\pi i t}\,\mathrm{d}\lambda;$$

thus f is the inverse Fourier transform of F. Since $F \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ it follows that $f \in \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$. Therefore, $f|[-R,R]$ is continuous and bounded, and hence Riemann integrable, for every $R \in \mathbb{R}_{>0}$. Since $F \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ we have $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ which implies that

$$\int_{-R}^{R} |f(t)|^2\,\mathrm{d}t = \int_{[-R,R]} |f|^2\,\mathrm{d}\lambda \le \int_{\mathbb{R}} |f|^2\,\mathrm{d}\lambda, \qquad R \in \mathbb{R}_{>0}.$$

Thus the limit

$$\lim_{R\to\infty} \int_{-R}^{R} |f(t)|^2\,\mathrm{d}t$$

exists. This is exactly the condition for Riemann integrability of f as a function on an unbounded domain as in Section **??**. Now, since $[f] = \mathscr{F}_{\mathrm{CC}}^{-1}([F])$ by definition, we have $\mathscr{F}_{\mathrm{CC}}([f]) = [F]$, where here $\mathscr{F}_{\mathrm{CC}}$ denotes the L^2-CCFT. In Proposition 5.1.12 we showed that $[F]|[0,1] \notin \mathsf{R}^1([0,1];\mathbb{C})$. From this we conclude that $[F] \notin \mathsf{R}^1(\mathbb{R};\mathbb{C})$ and, since $|F|^2 = F$, $[F] \notin \mathsf{R}^2(\mathbb{R};\mathbb{C})$. Thus $\mathscr{F}_{\mathrm{CC}}(\mathsf{R}^2(\mathbb{R};\mathbb{C})) \not\subset \mathsf{R}^2(\mathbb{R};\mathbb{C})$, as it was desired to show. •

13.3.4 Computation of the L^2-CCFT

The preceding discussion of the L^2-CCFT is somewhat abstract, and hides somewhat its value in practice. In this section we therefore look at some simple and illustrative examples that show how the L^2-CCFT can be used to give a coherent discussion of the CCFT for a large variety of signals, some of which are a little problematic in the L^1 theory.

13.3.14 Examples (L^2-CCFT)

1. Let us consider the Dirichlet kernel:

$$D_\Omega(t) = \begin{cases} \frac{\sin(2\pi\Omega t)}{\pi t}, & t \neq 0, \\ 2\Omega, & t = 0. \end{cases}$$

By Lemma 3 from Example 11.3.7–3 it follows that $D_\Omega \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, but that $D_\Omega \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Thus we can use the L^2-CCFT to compute $\mathscr{F}_{\mathrm{CC}}(D_\Omega)$, but the L^1-CCFT does not apply.

Let us do the computations, using contour integration. For $\nu \in \mathbb{R}$ define

$$F_\nu(z) = \mathrm{i}\frac{\mathrm{e}^{-2\pi\Omega z} - \mathrm{e}^{2\pi\Omega z}}{2\pi z}\mathrm{e}^{-2\pi\nu z}.$$

It is clear that F_ν is analytic on $\mathbb{C} \setminus \{0\}$, and by ***missing stuff*** it is entire since $\lim_{z\to 0} F_\nu(z) = 2\Omega$. We recall from the proof of Lemma 1 from Example 11.3.7–3 the definitions of the contours γ_R, γ'_R, $C_{+,R}$, $C_{-,R}$, $\Gamma_{+,R}$, and $\Gamma_{-,R}$. One checks directly that

$$\int_{-R}^{R} D_\Omega(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \int_{\gamma_R} F_\nu(z)\,\mathrm{d}z.$$

Since F_ν is entire, we also have

$$\int_{\gamma_R} F_\nu(z)\,\mathrm{d}z = \int_{\gamma'_R} F_\nu(z)\,\mathrm{d}z$$

by ***missing stuff***.

Let us define $f_{\nu,1}, f_{\nu,2}\colon \mathbb{C} \to \mathbb{C}$ by

$$f_{\nu,1}(z) = \frac{\mathrm{e}^{2\pi(-\Omega-\nu)z}}{z}, \quad f_{\nu,2}(z) = \frac{\mathrm{e}^{2\pi(\Omega-\nu)z}}{z},$$

noting that these functions all have a simple pole at the origin.

We next compute a few contour integrals using Cauchy's Theorem and the Residue Theorem.

(a) Suppose that $-\Omega - \nu \in \mathbb{R}_{>0}$. Then

$$\int_{\gamma'_R} f_{1,\nu}(z)\,\mathrm{d}z + \int_{C_{-,R}} f_{1,\nu}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{1,\nu}(z)\,\mathrm{d}z = 2\pi\mathrm{i}$$

by the Residue Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{1,\nu}(z)\,\mathrm{d}z = 2\pi\mathrm{i}.$$

(b) Suppose that $-\Omega - \nu = 0$. Then note that $f_{1,-\Omega}(z) = \frac{1}{z}$ and so, by direct computation,

$$\int_{C_{+,R}} f_{1,-\Omega}(z)\,\mathrm{d}z = \mathrm{i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta = \pi\mathrm{i}.$$

Also,

$$-\int_{\gamma'_R} f_{1,-\Omega}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{1,-\Omega}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. Thus we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{1,-\Omega}(z)\,\mathrm{d}z = -\pi\mathrm{i}.$$

(c) Suppose that $-\Omega - \nu \in \mathbb{R}_{<0}$. Then

$$-\int_{\gamma'_R} f_{1,\nu}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{1,\nu}(z)\,\mathrm{d}z = \int_{\Gamma_{+,R}} f_{1,\nu}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{1,\nu}(z)\,\mathrm{d}z = 0.$$

(d) Suppose that $\Omega - \nu \in \mathbb{R}_{>0}$. Then

$$\int_{\gamma'_R} f_{2,\nu}(z)\,\mathrm{d}z + \int_{C_{-,R}} f_{2,\nu}(z)\,\mathrm{d}z = \int_{\Gamma_{-,R}} f_{2,\nu}(z)\,\mathrm{d}z = 2\pi\mathrm{i}$$

by the Residue Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{\gamma'_R} f_{2,\nu}(z)\,\mathrm{d}z = 2\pi\mathrm{i}.$$

(e) Suppose that $\Omega - \nu = 0$. Then note that $f_{2,\Omega}(z) = \frac{1}{z}$ and so, by direct computation,

$$\int_{C_{+,R}} f_{2,\Omega}(z)\,\mathrm{d}z = \mathrm{i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta = \pi\mathrm{i}.$$

Also,

$$-\int_{\gamma'_R} f_{2,\Omega}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{2,\Omega}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. Thus we have

$$\lim_{R\to\infty}\int_{\gamma'_R} f_{2,\Omega}(z)\,\mathrm{d}z = -\pi\mathrm{i}.$$

(f) Suppose that $-\Omega - \nu \in \mathbb{R}_{<0}$. Then

$$-\int_{\gamma'_R} f_{2,\nu}(z)\,\mathrm{d}z + \int_{C_{+,R}} f_{2,\nu}(z)\,\mathrm{d}z = \int_{\Gamma_{+,R}} f_{2,\nu}(z)\,\mathrm{d}z = 0$$

by Cauchy's Theorem. By Jordan's Lemma we have

$$\lim_{R\to\infty}\int_{\gamma'_R} f_{2,\nu}(z)\,\mathrm{d}z = 0.$$

Now, using these calculations and noting that

$$F_\nu(z) = \frac{\mathrm{i}}{2\pi}(f_{\nu,1}(z) - f_{\nu,2}(z)),$$

we have the following cases.

(a) $\nu < -\Omega$: In this case we have $-\Omega - \nu > 0$ and $\Omega - \nu > 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi}(2\pi\mathrm{i} - 2\pi\mathrm{i}) = 0.$$

(b) $\nu = -\Omega$: In this case we have $\Omega - \nu > 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi}(\pi\mathrm{i} - 2\pi\mathrm{i}) = \frac{1}{2}.$$

(c) $-\Omega < \nu < \Omega$: In this case we have $-\Omega - \nu < 0$ and $\Omega - \nu > 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi}(0 - 2\pi\mathrm{i}) = 1.$$

(d) $\nu = \Omega$: In this case we have $-\Omega - \nu < 0$. Thus we get

$$\lim_{R\to\infty}\int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi}(0 - \pi\mathrm{i}) = \frac{1}{2}.$$

(e) $\nu > \Omega$: In this case we have $-\Omega - \nu < 0$ and $\Omega - \nu < 0$. Thus we get

$$\lim_{R\to\infty} \int_{-R}^{R} F_\nu(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2\pi}(0-0) = 0.$$

Thus, putting the above computations all together,

$$\lim_{j\to\infty} \int_{-j}^{j} D_\Omega(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t = \begin{cases} 1, & \nu \in (-\Omega, \Omega), \\ \frac{1}{2}, & t \in \{-\Omega, \Omega\}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus $\mathscr{F}_{\mathrm{CC}}(D_\Omega)$ is equal to the equivalence class of the frequency-domain signal on the right in the preceding expression. We have described this equivalence class by taking a particular sequence, namely the sequence $(D_\Omega \chi_{[-j,j]})_{j\in\mathbb{Z}_{>0}}$, in $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ that converges to D_Ω.

2. We take $f = \chi_{[-a,a]}$ whose CCFT we computed in Example 13.1.3–3 to be

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \frac{\sin(2\pi a\nu)}{\pi\nu} = D_a(t).$$

Note that from this it immediately follows that $D_a \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and that Parseval's equality gives

$$\int_{\mathbb{R}} \Big(\frac{\sin(2\pi a\nu)}{\pi\nu}\Big)^2 \mathrm{d}\nu = \|f\|_2^2 = \int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t = 4a^2.$$

One can use this technique to generate all kinds of interesting integrals.

The recovery of f from $\mathscr{F}_{\mathrm{CC}}(f)$ in this case is determined, except on a set of measure zero, by computing the limit

$$\lim_{j\to\infty} \int_{-j}^{j} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu. \tag{13.12}$$

In this case, because $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$, this also follows from the developments of Section 13.2.4 (e.g., Theorem 13.2.21). What the L^2-theory gives us in this case that we did not have before is that the convergence to f is valid in the L^2-sense. Thought of in this way, $\overline{\mathscr{F}}_{\mathrm{CC}}$ *does* recover f directly from its CCFT, albeit only as an equivalence class in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

Let us illustrate another limiting process that recovers f from its CCFT. We consider the limit

$$\lim_{j\to\infty} \int_{\mathbb{R}} \mathrm{e}^{-\nu^2/j} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu \tag{13.13}$$

(cf. Remark 13.3.5–2). In Figure 13.18 we show the approximations using (13.12) (left) and (13.13) (right). This emphasises that there are many ways of converging to an element in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ with a sequence in $\mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$. Note that the L^2-CCFT differs from the L^1-CCFT in that the pointwise behaviour of the two limits shown in Figure 13.18 are irrelevant. What matters is the limit signal as an equivalence class in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

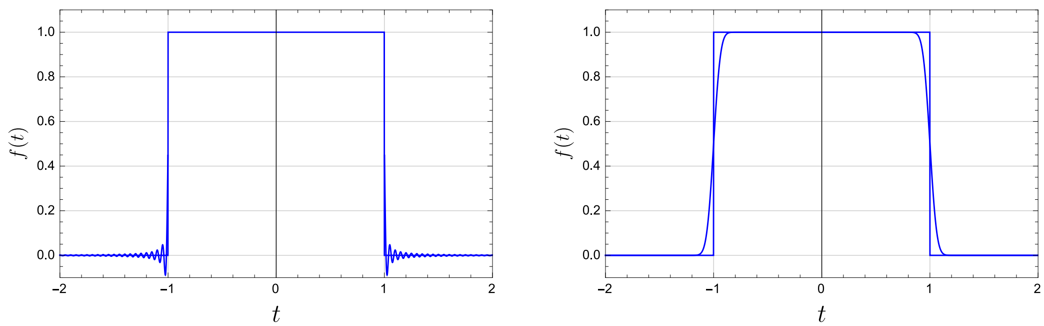


Figure 13.18 Two L^2-approximations of $\chi_{[-1,1]}$ ($j = 20$)

3. The signal $f(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\square_{2,1,1}(t) - 1)$ is one in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Thus its CCFT, which was given in Example 13.2.20–1 as
$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \mathrm{i}\frac{1 - \cos(\pi\nu)}{\pi\nu},$$
is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. However, since f is not almost everywhere equivalent to a continuous signal, it cannot be that $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. Thus we are in a situation entirely like that considered in the preceding example where one can reconstruct f, as an equivalence class in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$, from its CCFT using $\overline{\mathscr{F}}_{\mathrm{CC}}$. This could be done explicitly using contour integration, or since f satisfied the hypotheses of (say) Dirichlet's Test, we can use the inversion results from Section 13.2.4.
4. Next we consider the signal $g(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\triangle_{\frac{1}{2},1,1}(t)$ whose CCFT was given in Example 13.2.20–2 as
$$\mathscr{F}_{\mathrm{CC}}(g)(\nu) = \frac{1 - \cos(\pi\nu)}{2\pi^2\nu^2}.$$
Thus g is a signal in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ whose CCFT is also in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Furthermore, the sequence $(D_\Omega g)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to g (or, more precisely, to a signal almost everywhere equal to g, cf. Theorem 13.2.26). Thus this is an example of a signal for which the L^1-CCFT works quite satisfactorily. Nevertheless, the L^2-CCFT may still be applied, provided one accepts that it deals in equivalence classes of signals, not with the signals themselves.
5. The final signal we consider in our list of examples is the signal
$$h(t) = \begin{cases} \sqrt{\sin\frac{t+\pi}{2}}, & |t| \leq \pi, \\ 0, & \text{otherwise} \end{cases}$$
considered in Example 13.2.20–3. This signal is one in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$, so its CCFT must be in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Nothing that we have presented thus far allows us to conclude that the CCFT of f is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$. However, we did show using Theorem 13.2.31 that $(D_\Omega h)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to h. •

13.3.5 Convolution, multiplication, and the L^2-CCFT

In Section 13.1.5 we considered the relationships between convolution and the CCFT in the L^1-setting. In this section we carry this out in the L^2-setting. Recall from Corollary 11.2.10 that the convolution of signals in $L^{(2)}(\mathbb{R};\mathbb{C})$ is a signal in $C^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{C})$. Generally, the CCFT of signals in $C^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{C})$ is not defined, and indeed there exist signals in $L^2(\mathbb{R};\mathbb{C})$ whose convolution is in the domain of neither the L^1- nor the L^2-CCFT. However, it is still possible to state a result in this case.

13.3.15 Proposition (The convolution of L^2-signals of the inverse CCFT of the product of the L^2-CCFT's) *If* $f, g \in L^{(2)}(\mathbb{R};\mathbb{C})$ *then*

$$f * g(t) = \overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g))(t)$$

for all $t \in \mathbb{R}$.

Proof Define $f_j = \chi_{[-j,j]}f$ and $g_j = \chi_{[-j,j]}g$. As shown in the proof of Lemma 13.3.2, the sequences $(f_j)_{j\in\mathbb{Z}_{>0}}$ and $(g_j)_{j\in\mathbb{Z}_{>0}}$ are in $L^{(1)}(\mathbb{R};\mathbb{C})\cap L^{(2)}(\mathbb{R};\mathbb{C})$ and converge in $L^2(\mathbb{R};\mathbb{C})$ to f and g, respectively. Moreover, since f_j and g_j have compact support, by Proposition 11.1.8 it follows that $f_j * g_j$ has compact support for each $j \in \mathbb{Z}_{>0}$. Since $f_j * g_j$ is in $L^{(1)}(\mathbb{R};\mathbb{C})$ by Theorem 11.2.1, it follows from Theorem 8.3.11(iv) that $f_j * g_j \in L^{(2)}(\mathbb{R};\mathbb{C})$. Then, according to Proposition 13.1.18,

$$\begin{aligned} &\mathscr{F}_{CC}(f_j * g_j) = \mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)\\ \implies\quad & f_j * g_j = \overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)) \qquad j \in \mathbb{Z}_{>0}, \end{aligned}$$

because $f_j * g_j \in L^{(2)}(\mathbb{R};\mathbb{C})$ and since $\overline{\mathscr{F}}_{CC}\circ\mathscr{F}_{CC}$ is the identity on $L^2(\mathbb{R};\mathbb{C})$, as we showed in Theorem 13.3.11.

By ***missing stuff*** the sequence $((f_j, g_j))_{j\in\mathbb{Z}_{>0}}$ converges to (f, g) in the product topology on $L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$. By Corollary 11.2.12 the sequence $(f_j * g_j)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $f * g$.

We claim that $(\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)))_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g))$. Indeed, we have

$$\begin{aligned} \|\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g)-\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)\|_1 &\le \|\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g) - \mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g)\|_2\\ &\quad + \|\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g) - \mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)\|_2\\ &\le \|\mathscr{F}_{CC}(f) - \mathscr{F}_{CC}(f_j)\|_2\|\mathscr{F}_{CC}(g)\|_2 + \|\mathscr{F}_{CC}(f_j)\|\|\mathscr{F}_{CC}(g) - \mathscr{F}_{CC}(g_j)\|_2\\ &= \|f - f_j\|_2\|g\|_2 + \|f_j\|\|g - g_j\|_2 \end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality and Parseval's equality. Thus

$$\lim_{j\to\infty}\|\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g) - \mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)\|_1 = 0.$$

By Corollary 13.1.8 (applied to $\overline{\mathscr{F}}_{CC}$ rather than $\mathscr{F}_{CC}$) it then follows that $(\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)))_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g))$, as desired.

Thus we have

$$\lim_{j\to\infty} f_j g_j = fg, \quad \lim_{j\to\infty}\overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f_j)\mathscr{F}_{CC}(g_j)) = \overline{\mathscr{F}}_{CC}(\mathscr{F}_{CC}(f)\mathscr{F}_{CC}(g)),$$

with both limits being with respect to the ∞-norm. From this the result follows. ■

For the relationship between products and the L^2-CCFT, the result is the same as in the L^1-case, but now we do not need any restrictions on the character of the CCFT's of the signals

13.3.16 Proposition (The L^2-CCFT of a product is the convolution of the L^2-CCFT's) *If* $f, g \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ *then*

$$\mathscr{F}_{\mathrm{CC}}(fg)(\nu) = \mathscr{F}_{\mathrm{CC}}(f) * \mathscr{F}_{\mathrm{CC}}(g)(\nu),$$

for almost every $\nu \in \mathbb{R}$.

Proof As in the proof of Proposition 13.3.15, let $(f_j)_{j\in\mathbb{Z}_{>0}}$ and $(g_j)_{j\in\mathbb{Z}_{>0}}$ be sequences of compactly supported signals in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ converging in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ to f and g, respectively. By Proposition 13.1.20 we have

$$\mathscr{F}_{\mathrm{CC}}(f_j g_j)(\nu) = \mathscr{F}_{\mathrm{CC}}(f_j) * \mathscr{F}_{\mathrm{CC}}(g_j)(\nu)$$

for every $j \in \mathbb{Z}_{>0}$ and for every $\nu \in \mathbb{R}$.

By continuity of the L^2-CCFT it follows that the sequences $(\mathscr{F}_{\mathrm{CC}}(f_j))_{j\in\mathbb{Z}_{>0}}$ and $(\mathscr{F}_{\mathrm{CC}}(g_j))_{j\in\mathbb{Z}_{>0}}$ converge in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ to $\mathscr{F}_{\mathrm{CC}}(f)$ and $\mathscr{F}_{\mathrm{CC}}(g)$, respectively. Thus, by ***missing stuff***, the sequence $((\mathscr{F}_{\mathrm{CC}}(f_j), \mathscr{F}_{\mathrm{CC}}(f_j)))_{j\in\mathbb{Z}_{>0}}$ converges in $\mathsf{L}^2(\mathbb{R};\mathbb{C}) \times \mathsf{L}^2(\mathbb{R};\mathbb{C})$ to $(\mathscr{F}_{\mathrm{CC}}(f), \mathscr{F}_{\mathrm{CC}}(g))$ with the product topology. By Corollary 11.2.12 it follows that the sequence $(\mathscr{F}_{\mathrm{CC}}(f_j) * \mathscr{F}_{\mathrm{CC}}(g_j))_{j\in\mathbb{Z}_{>0}}$ converges to $\mathscr{F}_{\mathrm{CC}}(f) * \mathscr{F}_{\mathrm{CC}}(g)$ uniformly.

We claim that the sequence $(\mathscr{F}_{\mathrm{CC}}(f_j g_j))$ converges uniformly to $\mathscr{F}_{\mathrm{CC}}(fg)$. Indeed, we have

$$\|fg - f_j g_j\|_1 \le \|fg - f_j g\|_2 + \|f_j g - f_j g_j\| \le \|f - f_j\|_2 \|g\|_2 + \|f_j\| \|g - g_j\|_2,$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Thus

$$\lim_{j\to\infty} \|fg - f_j g_j\|_1 = 0.$$

By Corollary 13.1.8 it then follows that $(\mathscr{F}_{\mathrm{CC}}(f_j g_j))_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\mathscr{F}_{\mathrm{CC}}(fg)$, as desired.

Thus

$$\lim_{j\to\infty} \mathscr{F}_{\mathrm{CC}}(f_j g_j) = \mathscr{F}_{\mathrm{CC}}(fg), \quad \lim_{j\to\infty} \mathscr{F}_{\mathrm{CC}}(f_j) * \mathscr{F}_{\mathrm{CC}}(g_j) = \mathscr{F}_{\mathrm{CC}}(f)\mathscr{F}_{\mathrm{CC}}(g),$$

with convergence being uniform in each case. This gives the result. ■

13.3.6 One version of the Paley–Weiner Theorem

In this section we consider the L^2-CCFT restricted to signals with compact support. It turns out that the CCFT of such signals has a very particular structure, and this structure will be of some interest to us. We first recall from Corollary 13.1.13 that if $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ has compact support then $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable. If we further have $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ then there is more we can say about the character of $\mathscr{F}_{\mathrm{CC}}(f)$. To discuss this thoroughly, we introduce some notation that is essential in describing the image of the CCFT.

Recall from ***missing stuff*** that we denote by $\mathsf{H}(\mathbb{C};\mathbb{C})$ the set of entire functions, i.e., the set of holomorphic $\mathbb{C}$-valued functions on $\mathbb{C}$.

13.3.17 Definition (Entire function of exponential type) An entire function $F \in \mathsf{H}(\mathbb{C};\mathbb{C})$ is of ***exponential type*** is there exist $M, \alpha \in \mathbb{R}_{>0}$ such that $|F(z)| \leq M e^{\alpha|z|}$ for $z \in \mathbb{C}$. When this inequality holds for a certain $\alpha \in \mathbb{R}_{>0}$, we say the function is of ***exponential type α***. The set of entire functions of exponential type is denoted by $\mathsf{H}_{\exp}(\mathbb{C};\mathbb{C})$ and the set of entire functions of exponential type α is denoted by $\mathsf{H}_{\exp,\alpha}(\mathbb{C};\mathbb{C})$. •

We begin by characterising the CCFT of signals that are in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ and with compact support.

13.3.18 Theorem (Paley–Wiener Theorem I) *For* $f\colon \mathbb{R} \to \mathbb{C}$ *and for* $T \in \mathbb{R}_{>0}$*, the following two statements are equivalent:*

(i) $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ *and* $\mathrm{supp}(f) \subseteq [-T, T]$*;*

(ii) there exists $F \in \mathsf{H}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$ *such that*

  (a) $\int_{\mathbb{R}} |F(\nu + i0)|^2 \, d\nu < \infty$ *and*

  (b) $\mathscr{F}_{CC}(f)(\nu) = F(\nu + i0)$ *for all* $\nu \in \mathbb{R}$*.*

Moreover, with f *and* F *as above, we have*

$$F(z) = \int_{\mathbb{R}} f(t) e^{2\pi i z t} \, dt$$

for every $z \in \mathbb{C}$*.*

Proof First we assume that $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ and that $\mathrm{supp}(f) \subseteq [-T, T]$. We define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_{-T}^{T} f(t) e^{-2\pi i z t} \, dt.$$

Let us denote $G(t, z) = f(t) e^{-2\pi i z t}$. Since $f \in \mathsf{L}^{(2)}([-T, T];\mathbb{C})$ and since $t \mapsto e^{-2\pi i z t}$ is in $\mathsf{L}^{(2)}([-T, T];\mathbb{C})$ for every $z \in \mathbb{C}$, it follows from the Cauchy–Bunyakovsky–Schwarz inequality that $t \mapsto G(t, z)$ is in $\mathsf{L}^{(1)}([-T, T];\mathbb{C})$ for every $z \in \mathbb{C}$. It is also evident that the function $z \mapsto e^{-2\pi i z t}$ is entire. Let $z_0 \in \mathbb{C}$ and let $U = \mathsf{B}(1, z_0)$. Define

$$\eta = \sup\{|\mathrm{Im}(z)| \mid z \in U\}$$

and note that for $z = x + iy \in U$ we have

$$|G(t, z)| = |f(t)||e^{2\pi y t}| \leq |f(t)||e^{2\pi \eta t}|.$$

Since $t \mapsto |f(t)|$ and $t \mapsto |e^{2\pi\eta t}|$ are both in $\mathsf{L}^{(2)}([-T, T];\mathbb{C})$, it follows from the Cauchy–Bunyakovsky–Schwarz inequality that $t \mapsto |f(t)||e^{2\pi\eta t}|$ is in $\mathsf{L}^{(1)}([-T, T];\mathbb{C})$. The lemma above then immediately shows that $F \in \mathsf{H}(\mathbb{C};\mathbb{C})$.

Note that for $\nu \in \mathbb{R}$ we have

$$F(\nu + i0) = \int_{\mathbb{R}} f(t) e^{-2\pi i \nu t} \, dt = \mathscr{F}_{CC}(f)(\nu).$$

Since $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ it follows that the function $\nu \mapsto F(\nu + i0)$ is also in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. Moreover, for $z = x + iy \in \mathbb{C}$ we have

$$|e^{-2\pi i(x+iy)t}| = |e^{2\pi y t}| \leq |e^{2\pi|y||t|}| \leq |e^{2\pi i|z||t|}|$$

This then gives

$$|F(z)| \leq \int_{-T}^{T} |f(t)\mathrm{e}^{-2\pi \mathrm{i} z t}| \,\mathrm{d}t \leq \int_{-T}^{T} |f(t)||\mathrm{e}^{2\pi \mathrm{i}|z|t}| \,\mathrm{d}t \leq \mathrm{e}^{2\pi|z|T} \int_{-T}^{T} |f(t)| \,\mathrm{d}t,$$

which shows that $F \in \mathsf{H}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$, as desired.

The proof of the converse might be seen as being rather indirect. We let $F \in \mathsf{H}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$ be such that its restriction to the real axis is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ and such that this restriction is the CCFT of f. This immediately implies, by invertibility of the L^2-CCFT, that $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. We must now show that $\mathrm{supp}(f) \subseteq [-T, T]$.

Let $\theta \in \mathbb{R}$ and define

$$\rho_\theta = \{r\mathrm{e}^{-\mathrm{i}\theta} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}\}$$

and

$$P_{\theta,T} = \{z \in \mathbb{C} \mid \operatorname{Re}(z\mathrm{e}^{-\mathrm{i}\theta}) < -T\}.$$

We depict these subsets of the complex plane in Figure 13.19. Now, for $\theta \in \mathbb{R}$ and for

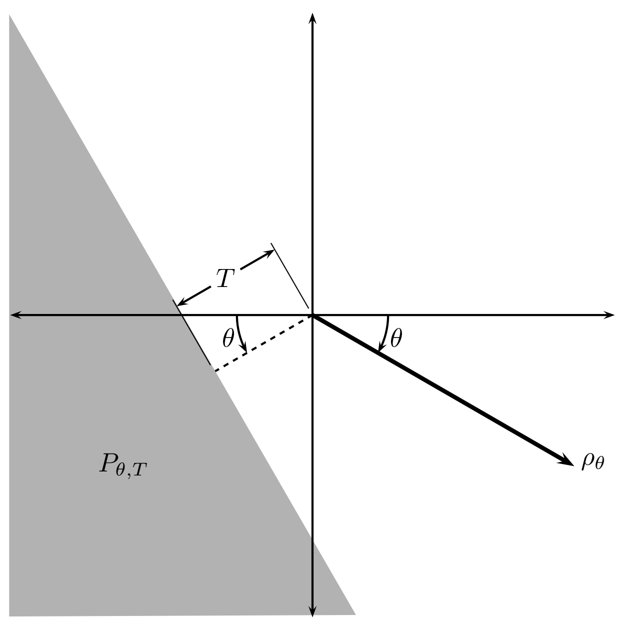


Figure 13.19 Sets used in the proof of the Paley-Wiener Theorem

$w \in \mathbb{C}$ let us denote

$$G_\theta(w) = \int_{\rho_\theta} F(z)\mathrm{e}^{2\pi w z} \,\mathrm{d}z.$$

This function is not well defined for every $w \in \mathbb{C}$, so let us record when it *is* defined, and give some of its properties.

1 Lemma *The functions* G_θ *have the following properties:*

(i) for $\theta \in \mathbb{R}$, G_θ *is well-defined and holomorphic on* $\mathrm{P}_{\theta,\mathrm{T}}$*;*

(ii) G_0 *is well-defined and holomorphic on* $\mathbb{C}_{<0}$ *and* G_π *is well-defined and holomormorphic on* $\mathbb{C}_{>0}$*;*

(iii) *for* $w \in P_{\theta_1,T} \cap P_{\theta_2,T}$*,* $G_{\theta_1}(w) = G_{\theta_2}(w)$*;*

(iv) *for* $\epsilon \in \mathbb{R}_{>0}$*,*

$$\int_{\mathbb{R}} F(\nu + i0)e^{-2\pi\epsilon|\nu|}e^{2\pi i\nu t}\,d\nu = G_0(-\epsilon + it) - G_\pi(\epsilon + it),$$

for $t \neq 0$*.*

Proof (i) Note that

$$G_\theta(w) = e^{-i\theta}\int_0^\infty F(re^{-i\theta})\exp(2\pi wre^{-i\theta})\,dr. \tag{13.14}$$

We have

$$\begin{aligned}|F(re^{-i\theta})\exp(2\pi wre^{-i\theta})| &\le M\exp(2\pi T|re^{-i\theta}|)\exp(\mathrm{Re}(2\pi wre^{-i\theta}))\\ &= M\exp(2\pi r(T + \mathrm{Re}(we^{-i\theta}))).\end{aligned} \tag{13.15}$$

Therefore, for $w \in P_{\theta,T}$ the integral (13.14) defining $G_\theta(w)$ is well-defined since

$$r \mapsto F(re^{-i\theta})\exp(2\pi wre^{-i\theta})$$

is in $\mathsf{L}^{(1)}(\mathbb{R}_{>0}, \mathbb{C})$ when $w \in P_{\theta,T}$. Also, the function

$$w \mapsto F(re^{-i\theta})\exp(2\pi wre^{-i\theta})$$

is in $\mathsf{H}(P_{\theta,T}; \mathbb{C})$ for every $r \in \mathbb{R}_{>0}$ (indeed, it is in $\mathsf{H}(\mathbb{C}; \mathbb{C})$ for every $r \in \mathbb{R}_{>0}$). Moreover, if $w_0 \in P_{\theta,T}$ then let $\epsilon \in \mathbb{R}_{>0}$ be such that $\mathsf{B}(2\epsilon, w_0) \subseteq P_{\theta,T}$. Then denote

$$\alpha = \sup\{T + \mathrm{Re}(we^{-i\theta}) \mid w \in \mathsf{B}(2\epsilon, w_0)\},$$

noting that this quantity is finite since $w \mapsto \mathrm{Re}(we^{-i\theta})$ is a continuous function of w and is negative since $\overline{\mathsf{B}}(\epsilon, w_0) \subseteq P_{\theta,T}$. We then have

$$|F(re^{-i\theta})\exp(2\pi wre^{-i\theta})| \le M\exp(2\pi r\alpha).$$

This shows that the hypotheses of Theorem 5.9.18 above hold, and so G_θ is holomorphic on $P_{\theta,T}$.

(ii) We shall prove the assertion for G_0, the proof for G_π following in a similar vein. We have

$$G_0(w) = \int_0^\infty F(r + i0)e^{2\pi wr}\,dr.$$

By hypothesis, $r \mapsto F(r + i0)$ is in $\mathsf{L}^{(2)}(\mathbb{R}_{>0}; \mathbb{C})$. For $w \in \mathbb{C}_{<0}$ we also have $r \mapsto e^{2\pi wr}$ in $\mathsf{L}^{(2)}(\mathbb{R}_{>0}; \mathbb{C})$. By the Cauchy–Bunyakovsky–Schwarz inequality, $r \mapsto F(r + i0)e^{2\pi wr}$ is in $\mathsf{L}^{(1)}(\mathbb{R}_{>0}; \mathbb{C})$. As above,

$$w \mapsto F(r + i0)e^{2\pi wr}$$

is in $\mathsf{H}(\mathbb{C}; \mathbb{C})$ and so in $\mathsf{H}(\mathbb{C}_{<0}; \mathbb{C})$. If $w_0 \in \mathbb{C}_{<0}$, let $\epsilon \in \mathbb{R}_{>0}$ be such that $\mathsf{B}(2\epsilon, w_0) \subseteq \mathbb{C}_{<0}$. Then let

$$\alpha = \sup\{\mathrm{Re}(w) \mid w \in \mathsf{B}(\epsilon, w_0)\},$$

which then gives

$$|F(r + i0)e^{2\pi wr}| \le F(r + i0)e^{2\pi\alpha r}$$

for $w \in \mathsf{B}(\epsilon, w_0)$. Thus the hypotheses of Theorem 5.9.18 hold and so G_0 is holomorphic in $\mathbb{C}_{<0}$.

(iii) Suppose that $\theta_1 \neq \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$. If $P_{\theta_1,T} \cap P_{\theta_2,T} \neq \emptyset$, then without loss of generality we may assume that $\theta_2 > \theta_1$ and that $\theta_2 - \theta_1 < \pi$. This being the case, let us define

$$\phi = \tfrac{1}{2}(\theta_1 + \theta_2), \qquad \psi = \tfrac{1}{2}(\theta_2 - \theta_1).$$

Consider $w \in \mathbb{C}$ of the form $w = -re^{i\phi}$ for $r \in \mathbb{R}_{>0}$. We then directly compute

$$\operatorname{Re}(we^{-i\theta_1}) = \operatorname{Re}(we^{-i\theta_2}) = -r\cos\psi.$$

Thus the ray

$$\rho_{\theta_1,\theta_2,T} = \{-re^{i\phi} \mid -r\cos\psi < -T\}$$

is contained in $w \in P_{\theta_1,T} \cap P_{\theta_2,T}$.

Now, for $R \in \mathbb{R}_{>0}$, define

$$\begin{aligned}\Gamma_{\theta_1,R} &= \{re^{-i\theta_1} \mid r \in [0,R]\},\\ \Gamma_{\theta_2,R} &= \{re^{-i\theta_2} \mid r \in [0,R]\},\\ \Gamma_{\theta_1,\theta_2,R} &= \{Re^{-i\alpha} \mid \alpha \in [\theta_1,\theta_2]\}.\end{aligned}$$

Note that $\Gamma = \Gamma_{\theta_1,R} \cup \Gamma_{\theta_2,R} \cup \Gamma_{\theta_1,\theta_2,R}$ is a closed contour and so, taking the contour with positive (i.e., counterclockwise) orientation,

$$\int_\Gamma F(z)e^{2\pi wz}\,dz = \int_{\Gamma_{\theta_1,R}} F(z)e^{2\pi wz}\,dz + \int_{\Gamma_{\theta_1,\theta_2,R}} F(z)e^{2\pi wz}\,dz - \int_{\Gamma_{\theta_2,R}} F(z)e^{2\pi wz}\,dz = 0$$

by Cauchy's Theorem. Let us examine the middle integral on the right. We can write it as

$$\int_{\Gamma_{\theta_1,\theta_2,R}} F(z)e^{2\pi wz}\,dz = -iR\int_{\theta_1}^{\theta_2} F(Re^{-i\alpha})e^{2\pi wRe^{-i\alpha}}\,d\alpha.$$

Estimating the integrand gives

$$|F(Re^{-i\alpha})\exp(2\pi wRe^{-i\alpha})| \leq M\exp(2\pi R(T + \operatorname{Re}(we^{-i\alpha}))),$$

just as in (13.15). Taking $w = -re^{i\phi}$ as above we have

$$\operatorname{Re}(we^{-i\alpha}) = -r\cos(\phi - \alpha).$$

As a function of α, the expression on the right is monotonically decreasing on $[\theta_1, \phi]$ and monotonically increasing on $[\phi, \theta_2]$, and so achieves its maximum either when $\alpha = \theta_1$ or when $\alpha = \theta_2$. However, we have

$$\cos(\phi - \theta_1) = \cos(\phi - \theta_2) = \cos\psi.$$

Therefore, $\operatorname{Re}(we^{-i\alpha}) \leq -r\cos\psi$ and so

$$|F(Re^{-i\alpha})\exp(2\pi wRe^{-i\alpha})| \leq M\exp(2\pi R(T - r\cos\psi)).$$

Thus, provided that $w \in \rho_{\theta_1,\theta_2,T}$, we have

$$\lim_{R\to\infty}\int_{\Gamma_{\theta_1,\theta_2,R}} F(z)e^{2\pi wz}\,dz = 0.$$

Therefore, again for $w \in \rho_{\theta_1,\theta_2,T}$, and taking orientations into account,

$$\lim_{R\to\infty} \int_{\Gamma_{\theta_1,R}} F(z)\mathrm{e}^{2\pi wz}\,\mathrm{d}z = \lim_{R\to\infty} \int_{\Gamma_{\theta_2,R}} F(z)\mathrm{e}^{2\pi wz}\,\mathrm{d}z.$$

From the definition of G_θ we have $G_{\theta_1}(w) = G_{\theta_2}(w)$ when $w \in \rho_{\theta_1,\theta_2,T}$. Since $P_{\theta_1,T} \cap P_{\theta_2,T}$ is a connected open set containing the ray $\rho_{\theta_1,\theta_2,T}$, it follows from ***missing stuff*** that G_{θ_1} and G_{θ_2} agree on $P_{\theta_1,T} \cap P_{\theta_2,T}$ as desired.

(iv) This is a direct computation using the formula (13.14) for G_θ for $\theta \in \{0, \pi\}$. ▼

Now we may complete the proof. From part (iv) of Lemma 1 we have

$$\int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = G_0(-\epsilon + \mathrm{i}t) - G_\pi(\epsilon + \mathrm{i}t),$$

for $\epsilon \in \mathbb{R}_{>0}$ and for $|t| > T$. Note that for $\epsilon \in \mathbb{R}_{>0}$ we have $-\epsilon + \mathrm{i}t \in \mathbb{C}_{<0} \cap P_{-\frac{\pi}{2},T}$ when $t > T$ and $-\epsilon + \mathrm{i}t \in \mathbb{C}_{<0} \cap P_{\frac{\pi}{2},T}$ when $t < -T$. The argument used in Lemma 1(iii) shows that G_0 and $G_{-\frac{\pi}{2}}$ agree on $\mathbb{C}_{<0} \cap P_{-\frac{\pi}{2},T}$ since they agree on the ray $\rho_{-\frac{\pi}{2},0,T}$. In a similar manner, G_0 and $G_{\frac{\pi}{2}}$ agree on $\mathbb{C}_{<0} \cap P_{\frac{\pi}{2},T}$ since they agree on the ray $\rho_{0,\frac{\pi}{2},T}$. Corresponding statements hold with G_π in place of G_0 and $\mathbb{C}_{>0}$ in place of $\mathbb{C}_{<0}$. Thus we can write

$$\int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = G_{-\frac{\pi}{2}}(-\epsilon + \mathrm{i}t) - G_{-\frac{\pi}{2}}(\epsilon + \mathrm{i}t)$$

when $t > T$ and

$$\int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = G_{\frac{\pi}{2}}(-\epsilon + \mathrm{i}t) - G_{\frac{\pi}{2}}(\epsilon + \mathrm{i}t)$$

when $t < -T$. In any case, we have

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = 0$$

whenever $|t| > T$.

Now note that as $\epsilon \to 0$ the signal

$$\nu \mapsto F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|} \tag{13.16}$$

converges in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ to the signal

$$\nu \mapsto F(\nu + \mathrm{i}0), \tag{13.17}$$

cf. Remark 13.3.5–1. Since $\mathscr{F}_{\mathrm{CC}}^{-1}$ is a continuous isomorphism of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$, it follows that the inverse CCFT of the signal (13.16) converges in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ to the inverse CCFT of the signal (13.17). Since the inverse CCFT of the signal (13.17) is f by hypothesis, it follows that

$$f(t) = \int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu = \lim_{\epsilon\to 0} \int_{\mathbb{R}} F(\nu + \mathrm{i}0)\mathrm{e}^{-2\pi\epsilon|\nu|}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

for almost every $t \in \mathbb{R}$. In particular, $f(t) = 0$ for almost every t for which $|t| > T$. ■

13.3.19 Remarks (On the Paley–Wiener Theorem)

1. There is another version of the Paley–Wiener Theorem which we state as Theorem **??** when we discuss the Laplace transform.
2. Of course, the statement of the Paley–Wiener Theorem holds if "$\mathscr{F}_{\mathrm{CC}}$" is replaced with "$\mathscr{F}_{\mathrm{CC}}^{-1}$" in the statement. •

Let us give some examples so that we can explicitly see how the Paley–Wiener Theorem works.

13.3.20 Examples (Paley–Wiener Theorem)

1. Let $a \in \mathbb{R}_{>0}$ and consider $f = \chi_{[-a,a]}$. In Example 13.1.3–3 we computed $\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \frac{\sin(2\pi a\nu)}{\pi\nu}$. We note that f is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ and that $\operatorname{supp}(f) \subseteq [-a, a]$. Note that if we define

$$F(z) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi \mathrm{i} zt}\,\mathrm{d}t = \frac{\sin(2\pi az)}{\pi z},$$

then $F(\nu + \mathrm{i}0) = \mathscr{F}_{\mathrm{CC}}(f)(\nu)$. We should verify that $F \in \mathsf{H}_{\exp,2\pi a}(\mathbb{C};\mathbb{C})$. By verifying the Cauchy–Riemann equations, we can check that F is holomorphic. Let

$$C = \sup\{|F(z)| \mid z \in \overline{\mathsf{B}}(1, 0)\},$$

noting that $C < \infty$. If $|z| > 1$ we then have, using Euler's formula,

$$|F(z)| \le \left|\frac{\mathrm{e}^{2\pi a\mathrm{i}z} - \mathrm{e}^{-2\pi a\mathrm{i}z}}{2\pi\mathrm{i}z}\right| \le \frac{1}{2\pi}(\mathrm{e}^{2\pi a|z|} + \mathrm{e}^{-2\pi a|z|}) \le \frac{1}{\pi}\mathrm{e}^{2\pi a|z|}.$$

Thus, if we take $M = \max\{C, \frac{1}{\pi}\}$, $|F(z)| \le M\mathrm{e}^{2\pi a|z|}$. This then verifies the conclusions of the Paley–Wiener Theorem.

2. Here we take f defined by

$$f(t) = \begin{cases} 1 + \frac{t}{a}, & t \in [-a, 0], \\ 1 - \frac{t}{a}, & t \in (0, a], \\ 0, & \text{otherwise}, \end{cases}$$

for $a \in \mathbb{R}_{>0}$. In Example 13.1.3–4 we computed $\mathscr{F}_{\mathrm{CC}}(f) = \frac{\sin^2(\pi a\nu)}{\pi^2 a\nu^2}$. In this case we note that

$$F(z) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi \mathrm{i} zt}\,\mathrm{d}t = \frac{\sin^2(\pi az)}{\pi^2 az^2}$$

has the property that $\mathscr{F}_{\mathrm{CC}}(f)(\nu) = F(\nu + \mathrm{i}0)$. Note that $\operatorname{supp}(f) = [-a, a]$. To verify that $F \in \mathsf{H}_{\exp,2\pi a}(\mathbb{C};\mathbb{C})$ we note that

$$\sin^2(\pi az) = \left(\tfrac{1}{2\mathrm{i}}(\mathrm{e}^{\pi a\mathrm{i}z} - \mathrm{e}^{-\pi a\mathrm{i}z})\right)^2 = -\tfrac{1}{4}(\mathrm{e}^{2\pi a\mathrm{i}z} + \mathrm{e}^{-2\pi a\mathrm{i}z} + 2).$$

Now a computation just like the one in the previous example gives $|F(z)| \le M\mathrm{e}^{2\pi a|z|}$ for an appropriately chosen M. •

13.3.7 Notes

Exercises

13.3.1 Find a signal $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ such that the integral

$$\int_{\mathbb{R}} |\mathscr{F}_{\mathrm{CC}} f(\nu)|^2 \,\mathrm{d}\nu$$

does not exist.

13.3.2 Let $f(t) = \frac{1}{1+t^2}$.

(a) Show that $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.

(b) Compute the CCFT of f directly from the definition.

(c) Perform a complex partial fraction decomposition of f to write it as a sum of two signals, each with a denominator linear in t.

(d) Are the components in the partial sum in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$?

(e) Show that the components in the partial sum are in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$.

(f) Compute the L^2-CCFT of each component, and show that the sum of the resulting CCFT's equals the CCFT of f.

13.3.3 Let $f(t) = \frac{t}{1+|t|}$.

(a) Show that $f \notin \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$.

(b) Show that $f \in \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$.

(c) Compute the L^2-CCFT of f.

13.3.4 Consider four signals $f_1, f_2, f_3, f_4\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f_1(t) = \begin{cases} 1, & t \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$f_2(t) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right), & t \in (-1,1), \\ 0, & \text{otherwise}, \end{cases}$$

$$f_3(t) = \frac{1}{1+\sqrt{|t|}},$$

$$f_4(t) = \frac{1}{1+|t|}.$$

Answer the following questions.

(a) For each of the transforms $\mathscr{F}_{\mathrm{CC}}(f_1)$, $\mathscr{F}_{\mathrm{CC}}(f_2)$, $\mathscr{F}_{\mathrm{CC}}(f_3)$, and $\mathscr{F}_{\mathrm{CC}}(f_4)$, indicate whether it exists in the sense that

$$\mathscr{F}_{\mathrm{CC}}(f_a)(\nu) = \int_{\mathbb{R}} f_a(t)\mathrm{e}^{-2\pi \mathrm{i}\nu t}\,\mathrm{d}t,$$

for $a \in \{1,2,3,4\}$.

(b) For each of the transforms $\mathscr{F}_{\mathrm{CC}}(f_1)$, $\mathscr{F}_{\mathrm{CC}}(f_2)$, $\mathscr{F}_{\mathrm{CC}}(f_3)$, and $\mathscr{F}_{\mathrm{CC}}(f_4)$, indicate whether it is continuous.

(c) For each of the transforms $\mathscr{F}_{CC}(f_1)$, $\mathscr{F}_{CC}(f_2)$, $\mathscr{F}_{CC}(f_3)$, and $\mathscr{F}_{CC}(f_4)$, indicate whether it is differentiable.
(d) For each of the transforms $\mathscr{F}_{CC}(f_1)$, $\mathscr{F}_{CC}(f_2)$, $\mathscr{F}_{CC}(f_3)$, and $\mathscr{F}_{CC}(f_4)$, indicate whether it is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{R})$.
(e) For each of the transforms $\mathscr{F}_{CC}(f_1)$, $\mathscr{F}_{CC}(f_2)$, $\mathscr{F}_{CC}(f_3)$, and $\mathscr{F}_{CC}(f_4)$, indicate whether it is in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{R})$.

13.3.5 Let $f = \chi_{[-1,1]}$.
(a) Compute $\mathscr{F}_{CC}(f)$.

We propose to recover f from its CCFT by computing

$$\int_{-\Omega_1}^{\Omega_2} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu,$$

and letting Ω_1 and Ω_2 tend to infinity separately. The resulting limit will exist if and only if the two limits

$$\lim_{\Omega_1\to\infty}\int_{-\Omega_1}^{0} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu, \quad \lim_{\Omega_2\to\infty}\int_{0}^{\Omega_2} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu, \tag{13.18}$$

exist.
(b) Show that the imaginary parts of the two integrals in (13.18) converge in the limit.
(c) Show that the real parts of the two integrals in (13.18) do not converge in the limit.
(d) What is the point of this exercise?

13.3.6 Answer the following questions.
(a) Is the function

$$x \mapsto \frac{(1-\cos x)}{x}$$

in $\mathsf{L}^{(2)}(\mathbb{R}_{>0};\mathbb{R})$?
(b) Show that

$$\lim_{R\to\infty}\int_0^R \frac{(1-\cos x)^2}{x^2}\,dx = \frac{\pi}{2}.$$

Hint: *Use Example 13.2.20–1 and Parseval's equality.*

13.3.7 Answer the following questions.
(a) Is the function

$$x \mapsto \frac{(1-\cos x)}{x^2}$$

in $\mathsf{L}^{(2)}(\mathbb{R}_{>0};\mathbb{R})$?
(b) Show that

$$\lim_{R\to\infty}\int_0^R \frac{(1-\cos x)^2}{x^4}\,dx = \frac{\pi}{6}.$$

Hint: *Use Example 13.2.20–2 and Parseval's equality.*

In the next exercise you will be led through the proof of a simple version of the so-called Sampling Theorem. This result will be discussed in detail and in more generality in Section **??**.

13.3.8 Consider the following result.

Theorem *If* $f \in C^0(\mathbb{R};\mathbb{C}) \cap L^{(2)}(\mathbb{R};\mathbb{C})$ *is such that* $\mathscr{F}_{CC}(f)$ *is band-limited with* $\operatorname{supp}(\mathscr{F}_{CC}(f)) \subseteq [-\Omega,\Omega]$, *then*

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{2\Omega}\Big)\frac{\sin(\pi(2\Omega t - n))}{\pi(2\Omega t - n)}$$

for all $t \in \mathbb{R}$.

Prove the theorem along the following lines, filling in the gaps and justifying all the steps.

(a) Prove that $\mathscr{F}_{CC}(f) \in L^{(1)}(\mathbb{R};\mathbb{C}) \cap L^{(2)}(\mathbb{R};\mathbb{C})$.

(b) Conclude that if we define

$$g(t) = \int_{-\Omega}^{\Omega} \mathscr{F}_{CC}(f)(\nu)e^{2\pi i\nu t}\,d\nu, \tag{13.19}$$

then $g(t) = f(t)$ for almost every $t \in \mathbb{R}$.

(c) Also conclude that continuity of f allows us to assert that $g(t) = f(t)$ for every $t \in \mathbb{R}$.
Hint: *Use the fact that* $\overline{\mathscr{F}}_{CC}$ *is a continuous map from* $(L^1(\mathbb{R};\mathbb{C}), \|\cdot\|_1)$ *into* $(C^0_0(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$ *by Corollary 13.1.8, and then apply Theorems 6.5.2 and 6.7.39 and Exercise 5.9.8.*

(d) Note that $\mathscr{F}_{CC}(f) \in L^{(2)}([-\Omega,\Omega];\mathbb{C})$. Write the Fourier series for $\mathscr{F}_{CC}(f)|[-\Omega,\Omega]$:

$$\mathrm{FS}[\mathscr{F}_{CC}(f)|[-\Omega,\Omega]](\nu) = \sum_{n\in\mathbb{Z}} c_n e^{-\pi i n\frac{\nu}{\Omega}},$$

where

$$c_n = \frac{1}{2\Omega} f\Big(\frac{n}{2\Omega}\Big).$$

(e) Perform a computation to finish the proof.

Section 13.4

The CCFT for tempered distributions

Now we turn to our first development of the CCFT for distributions. We begin by considering the CCFT for tempered distributions. This has the limitation that it does not include signals that grow faster than polynomials at infinity, e.g., it does not allow signals with ubiquitous exponential growth. However, as we shall see, the CCFT for tempered distributions has a very attractive symmetry that makes these distributions somehow natural to the CCFT.

Do I need to read this section? The material in this section is essential if one is to tie together the four Fourier transforms we consider; see ***missing stuff***. However, it can perhaps be bypassed on a first superficial treatment of Fourier analysis. •

13.4.1 The strategy for defining the CCFT of a distribution

The motivation for our methodology of defining the CCFT for a distribution is the equality $\theta_{\mathscr{F}_{\mathrm{CC}}(f)}(\phi) = \theta_f(\mathscr{F}_{\mathrm{CC}}(\phi))$, which is valid for all $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ and $f \in \mathsf{L}^{(1)}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$ for which $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$:

$$\begin{aligned}\theta_{\mathscr{F}_{\mathrm{CC}}(f)}(\phi) &= \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\phi(\nu)\,\mathrm{d}\nu = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t\Big)\phi(\nu)\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}}\Big(\phi(\nu)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\,\mathrm{d}t = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(\phi)(t)f(t)\,\mathrm{d}t,\end{aligned}$$

using Fubini's Theorem. This suggests that a good approach for defining the CCFT on a general distribution $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{C})$ is to take $\mathscr{F}_{\mathrm{CC}}(\theta)(\phi) = \theta(\mathscr{F}_{\mathrm{CC}}(\phi))$. This approach has one immediate drawback, namely that it is generally not true that $\mathscr{F}_{\mathrm{CC}}(\phi) \in \mathscr{D}(\mathbb{R};\mathbb{C})$ when $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$. One can then take two approaches to resolve the problem. The first is work with a class of test signals that is invariant under the CCFT. The other is to consider a set of distributions defined using test signals that are the CCFT's of signals from $\mathscr{D}(\mathbb{R};\mathbb{C})$. We shall pursue both approaches, the first in this section being the more standard, and the most widely applicable. The second approach we pursue in Section **??**.

13.4.2 The Fourier transform of Schwartz test signals

It turns out that the set $\mathscr{S}(\mathbb{R};\mathbb{C})$ of Schwartz signals gives a space of test signals invariant under $\mathscr{F}_{\mathrm{CC}}$. Indeed, $\mathscr{F}_{\mathrm{CC}}|\mathscr{S}(\mathbb{R};\mathbb{C})$ has many remarkable features that we will exploit. The following result is the basic one.

13.4.1 Theorem (The CCFT is an isomorphism of the Schwartz test signals) *$\mathscr{F}_{\mathrm{CC}}|\mathscr{S}(\mathbb{R};\mathbb{C})$ is a Hilbert space isomorphism from $\mathscr{S}(\mathbb{R};\mathbb{C})$ to itself. That is to say,*

(i) $\mathscr{F}_{\mathrm{CC}}(\mathscr{S}(\mathbb{R};\mathbb{C})) \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$,

(ii) $\mathscr{F}_{CC}|\mathscr{S}(\mathbb{R};\mathbb{C})$ *is a bijection onto* $\mathscr{S}(\mathbb{R};\mathbb{C})$*, and*

(iii) $\|\phi\|_2 = \|\mathscr{F}_{CC}(\phi)\|_2$, $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

Proof Let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. The infinite differentiability of $\mathscr{F}_{CC}(\phi)$ for follows from Proposition 13.1.12 and the fact that ϕ is rapidly decreasing. That $\mathscr{F}_{CC}(\phi)$ is rapidly decreasing follows from Proposition 13.1.10 and the fact that ϕ is infinitely differentiable. To show that $\mathscr{F}_{CC}(\phi)^{(k)}$ is rapidly decreasing note that the signal $\phi_{m,k}\colon t \mapsto ((-2\pi i t)^m \phi(t))^{(k)}$ is in $\mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ since ϕ and all of its derivatives are rapidly decreasing. Now note that by Proposition 13.1.10 we have

$$\Big(\frac{1}{2\pi i}\Big)^k \mathscr{F}_{CC}(\phi_{m,k})(\nu) = \nu^k \mathscr{F}_{CC}(\phi_{m,0})(\nu) = \nu^k \mathscr{F}_{CC}(\phi)^{(m)}(\nu).$$

The leftmost expression tends to zero as $\nu \to \infty$ by the Riemann–Lebesgue Lemma, and this gives the rapid decrease of $\mathscr{F}_{CC}(\phi)^{(m)}$ for any m, since $k \in \mathbb{Z}_{>0}$ is arbitrary. This shows that $\mathscr{F}_{CC}(\mathscr{S}(\mathbb{R};\mathbb{C})) \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$.

Since $\mathscr{S}(\mathbb{R};\mathbb{C}) \subseteq \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$ and since Schwartz signals are continuous, it follows from Theorem 13.2.26 that $\overline{\mathscr{F}}_{CC} \circ \mathscr{F}_{CC}(\phi) = \mathscr{F}_{CC} \circ \overline{\mathscr{F}}_{CC}(\phi) = \phi$ for all $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

The final statement of the theorem follows from part (ii) of Theorem 13.3.3. ■

13.4.3 Definitions and computations

Now we may define the CCFT for tempered distributions using the fact that $\mathscr{S}(\mathbb{R};\mathbb{C})$ is invariant under the CCFT.

13.4.2 Definition (The CCFT for tempered distributions) The ***continuous-continuous Fourier transform*** or ***CCFT*** assigns to $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ the element $\mathscr{F}_{CC}(\theta) \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ defined by $\mathscr{F}_{CC}(\theta)(\phi) = \theta(\mathscr{F}_{CC}(\phi))$, $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. •

Of course, we can similarly define $\overline{\mathscr{F}}_{CC}\colon \mathscr{S}'(\mathbb{R};\mathbb{C}) \to \mathscr{S}'(\mathbb{R};\mathbb{C})$ by $\overline{\mathscr{F}}_{CC}(\theta)(\phi) = \theta(\overline{\mathscr{F}}_{CC}(\phi))$.

Before we embark on a discussion of the various properties of the CCFT for tempered distributions, let us look at some examples.

13.4.3 Examples (The CCFT for tempered distributions)

1. If $f \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{CC}(\theta)_f = \theta_{\mathscr{F}_{CC}(f)}$. First of all, note that $\theta_f \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ by part (ii) of Proposition 10.8.4. Also, since $\mathscr{F}_{CC}(f)$ is bounded it follows from Proposition 10.3.17 that $\theta_{\mathscr{F}_{CC}(f)}$ is indeed a tempered distribution. Now, by Fourier Reciprocity, Proposition 13.1.9, we have

$$\begin{aligned}\theta_{\mathscr{F}_{CC}(f)}(\phi) &= \int_{\mathbb{R}} \mathscr{F}_{CC}(f)(\xi)\phi(\xi)\,d\xi = \int_{\mathbb{R}} f(\xi)\mathscr{F}_{CC}(\phi)(\xi)\,d\xi \\ &= \theta_f(\mathscr{F}_{CC}(\phi)) = \mathscr{F}_{CC}(\theta)_f(\phi)\end{aligned}$$

for any $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. Thus the CCFT for tempered distributions agrees with the L^1-CCFT in the cases where they are both defined.

2. We also claim that CCFT for tempered distributions generalises the L^2-CCFT. That is, for $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ we claim that $\mathscr{F}_{\mathrm{CC}}(\theta)_f = \theta_{\mathscr{F}_{\mathrm{CC}}(f)}$. It follows from part (ii) of Proposition 10.8.4 that $\theta_f, \theta_{\mathscr{F}_{\mathrm{CC}}(f)} \in \mathscr{S}'(\mathbb{R};\mathbb{C})$. The same sequence of computations as above, using Fourier Reciprocity for the L^2-CCFT (Proposition 13.3.9), gives the claim in this case.

The above two examples show that the CCFT for tempered distributions does generalise both the L^1 and L^2-CCFT. Let us now show that there are signals whose CCFT can be computed as tempered distributions, but which do not fit the theory for the CCFT developed during the preceding sections.

3. Let us compute the CCFT of the delta-signal, δ_a, for $a \in \mathbb{R}$. We have

$$\mathscr{F}_{\mathrm{CC}}(\delta)_a(\phi) = \delta_a(\mathscr{F}_{\mathrm{CC}}(\phi)) = \mathscr{F}_{\mathrm{CC}}(\phi)(a) = \int_{\mathbb{R}} \phi(t)\mathrm{e}^{-2\pi\mathrm{i}at}\,\mathrm{d}t.$$

Therefore, $\mathscr{F}_{\mathrm{CC}}(\delta)_a = \theta_{\mathsf{E}_{-2\pi\mathrm{i}a}}$.

4. Consider $f(t) = \mathrm{e}^{2\pi\mathrm{i}at}$. As a signal of slow growth, this signal qualifies as a tempered distribution. We compute its CCFT by

$$\mathscr{F}_{\mathrm{CC}}(\theta)_f(\phi) = \theta_f(\mathscr{F}_{\mathrm{CC}}(\phi)) = \int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}a\xi}\mathscr{F}_{\mathrm{CC}}(\phi)(\xi)\,\mathrm{d}\xi = \overline{\mathscr{F}}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(\phi))(a) = \phi(a),$$

using the fact that $\overline{\mathscr{F}}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(\phi))$ recovers ϕ by Theorem 13.2.26 and since $\mathscr{F}_{\mathrm{CC}}(\phi) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ by part (ii) of Proposition 10.8.4. Thus $\mathscr{F}_{\mathrm{CC}}(f) = \delta_a$. •

13.4.4 Properties of the CCFT for tempered distributions

The CCFT for tempered distributions has the same basic properties of the L^1-CCFT as outlined in Section 13.1.2. We state these here for completeness. For the following result, recall from Exercise 10.2.5 the definition of $\tau^*\theta$ for $\theta \in \mathscr{D}(\mathbb{R};\mathbb{C})$. Also recall from the preliminary remarks of Section 10.7.2 the definition of $\tau_a^*\theta$ for $\theta \in \mathscr{D}(\mathbb{R};\mathbb{C})$. We also use the notation $\bar{\theta} \in \mathscr{D}'(\mathbb{R};\mathbb{C})$ to define the distribution $\bar{\theta}(\phi) = \overline{\theta(\bar{\phi})}$.

13.4.4 Proposition (Elementary properties of the CCFT for tempered distributions) *For $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{CC}}(\theta)} = \overline{\mathscr{F}}_{\mathrm{CC}}(\bar{\theta})$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\sigma^*\theta) = \sigma^*(\mathscr{F}_{\mathrm{CC}}(\theta)) = \overline{\mathscr{F}}_{\mathrm{CC}}(\theta)$;

(iii) if θ is even (resp. odd) then $\mathscr{F}_{\mathrm{CC}}(\theta)$ is even (resp. odd);

(iv) if θ is real and even (resp. real and odd) then $\mathscr{F}_{\mathrm{CC}}(\theta)$ is real and even (resp. imaginary and odd);

(v) $\mathscr{F}_{\mathrm{CC}}(\tau_{\mathrm{a}}^*\theta)(\nu) = \mathsf{E}_{-2\pi\mathrm{ia}}\mathscr{F}_{\mathrm{CC}}(\theta)(\nu)$.

The proof is a matter of working through the definitions, and makes an excellent exercise (see Exercise 13.4.2). The properties of the CCFT for tempered distributions and differentiation also mirror those for the L^1-CCFT.

13.4.5 Proposition (The CCFT for tempered distributions and differentiation) *For $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ we have $\mathscr{F}_{\mathrm{CC}}(\theta^{(\mathrm{k})}) = (2\pi\mathrm{i}\rho)^{\mathrm{k}}\mathscr{F}_{\mathrm{CC}}(\theta)$, where $\rho(\nu) = \nu$.*

Proof For $k = 1$ and for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ we have

$$\mathscr{F}_{\mathrm{CC}}(\theta')(\phi) = \theta'(\mathscr{F}_{\mathrm{CC}}(\phi)) = -\theta(\mathscr{F}_{\mathrm{CC}}(\phi)') = -\theta(\mathscr{F}_{\mathrm{CC}}((-2\pi\mathrm{i}\rho)\phi)) = \mathscr{F}_{\mathrm{CC}}(\theta)(2\pi\mathrm{i}\rho\phi),$$

where we have used Proposition 13.1.12. The result for general k follows by a trivial induction. ■

13.4.6 Proposition (The CCFT for tempered distributions and differentiation of the transform) *For $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ we have $\mathscr{F}_{\mathrm{CC}}(\theta)^{(\mathrm{k})} = \mathscr{F}_{\mathrm{CC}}((-2\pi\mathrm{i}\rho)^{\mathrm{k}}\theta)$, where $\rho(\mathrm{t}) = \mathrm{t}$.*

Proof For $k = 1$ and for $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ we have

$$\mathscr{F}_{\mathrm{CC}}(\theta)'(\phi) = -\mathscr{F}_{\mathrm{CC}}(\theta)(\phi') = -\theta(\mathscr{F}_{\mathrm{CC}}(\phi')) = -\theta(2\pi\mathrm{i}\rho\mathscr{F}_{\mathrm{CC}}(\phi)) = \mathscr{F}_{\mathrm{CC}}((-2\pi\mathrm{i}\rho)\theta)(\phi),$$

where we have used Proposition 13.1.10. The general case follows by an easy induction. ■

Let's use the above computations to derive the CCFT for the step signal. Interestingly, this is not so easily done.

13.4.7 Example (The CCFT of the step signal) The unit step signal is denoted by 1. We wish to compute the CCFT of 1. To do so requires some work. We begin with the easy part. Note that $\theta_1' = \delta_0$. Thus, by Example 13.4.3–3, $\mathscr{F}_{\mathrm{CC}}(\theta_1')$ is the tempered distribution associated with the frequency signal $\nu \mapsto 1$. The us denote this signal by u in our discussion. By Proposition 13.1.10, $2\pi\mathrm{i}\rho\mathscr{F}_{\mathrm{CC}}(\theta_1) = \mathscr{F}_{\mathrm{CC}}(\theta_1')$ where $\rho(\nu) = \nu$. We have to solve this equation for $\mathscr{F}_{\mathrm{CC}}(\theta_1)$, and this is where the hard part comes in. We break this into several steps. Throughout the ensuing discussion, ρ is the signal $\rho(t) = t$.

1. *Characterise elements of $\mathscr{D}(\mathbb{R};\mathbb{C})$ vanishing at $t = 0$:* We claim that $\chi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ satisfies $\chi^{(j)}(0) = 0$, $j \in \{0, 1, \dots, k-1\}$, if and only if there exists $\tilde{\chi} \in \mathscr{D}(\mathbb{R};\mathbb{C})$ such that $\chi(t) = t^k\tilde{\chi}(t)$. Clearly if $\chi(t) = t^k\tilde{\chi}(t)$ then the first $k-1$ derivatives of χ vanish at $t = 0$. Now suppose that the first $k-1$ derivatives so vanish. Note that the function $\tilde{\chi}(t) = t^{-k}\chi(t)$ is infinitely differentiable away from $t = 0$, and has compact support. We may then compute the derivative of $\tilde{\chi}$ away from $t = 0$ using the product rule:

$$\tilde{\chi}^{(m)}(t) = \sum_{j=0}^{m}\binom{m}{j}\chi^{(j)}(t)\frac{\mathrm{d}^{m-j}}{\mathrm{d}t^{m-j}}\frac{1}{t^k}.$$

Since χ is infinitely differentiable, for any $j \in \mathbb{Z}_{>0}$ we may write

$$\chi(t) = \sum_{j=0}^{r}\frac{\chi^{(j)}(0)t^j}{j!} + R_m(t),$$

where $|R_m(t)| \le Kt^{m+1}$ for some $K \in \mathbb{R}_{\ge 0}$. This is Taylor's Theorem with remainder. Taking j sufficiently large we see that $\tilde{\chi}$ will be infinitely differentiable at $t = 0$ provided we take $\tilde{\chi}(0) = \frac{\psi^{(k)}(0)}{k!}$.

2. *Characterise elements of $\mathscr{D}(\mathbb{R};\mathbb{C})$ taking the value 1 at $t = 0$:* Here we claim that if $\psi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ satisfies $\psi(0) = 1$ and $\psi^{(j)}(0) = 0$, $j \in \{1,\dots,k-1\}$, then for any $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ we can write
$$\phi(t) = \psi(t)\sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)t^j}{j!} + \chi(t), \tag{13.20}$$
where $\chi(t) \in \mathscr{D}(\mathbb{R};\mathbb{C})$ satisfies $\chi^{(j)}(0) = 0$, $j \in \{0,1,\dots,k\}$. To see this, first note that (13.20) uniquely determines χ, and that χ is clearly an element of $\mathscr{D}(\mathbb{R};\mathbb{C})$. To show that the first k derivatives of χ vanish, one merely differentiates (13.20), using the properties of ψ.
3. *Solve $\rho^k\theta = 0$ for $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{C})$:* We claim that the solutions to this equation are of the form
$$\theta = \sum_{j=0}^{k-1} c_j\delta_0^{(j)}, \qquad c_0, c_1, \dots, c_{k-1} \in \mathbb{C}. \tag{13.21}$$
If $\chi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ satisfies $\chi^{(j)}(0) = 0$, $j \in \{0,1,\dots,k\}$, we have $\chi(t) = t^k\tilde{\chi}(t)$ for $\tilde{\chi} \in \mathscr{D}(\mathbb{R};\mathbb{C})$, and therefore, $\theta(\chi) = 0$. Now fix such a test signal χ and write $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ as in (13.20). Then
$$\theta(\phi) = \sum_{j=1}^{k-1}\frac{\phi^{(j)}(0)}{j!}\theta(\rho^j\chi).$$
The representation (13.21) follows by taking
$$c_j = \frac{(-1)^j}{j!}\theta(\rho^j\chi).$$
4. *The distribution $\mathrm{pv}(\rho^{-1})$:* Note that the signal $f(t) = t^{-1}$ is not locally integrable, so does not define a distribution in a direct manner. However, its primitive $f^{-1}(t) = \log|t|$ *is* locally integrable. Therefore, the derivative of the primitive defines a distribution, and this distribution we denote by $\mathrm{pv}(\rho^{-1})$. Let us try to better understand this distribution (and in so doing, make sensible the notation we have used to denote it). Let us denote $g(t) = \log|t|$ so that $\mathrm{pv}(\rho^{-1}) = g'$. We then have, for $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$,
$$\begin{aligned}
\mathrm{pv}(\rho^{-1})(\phi) = -\theta_g(\phi') = &-\int_{\mathbb{R}}\log|t|\phi'(t)\,\mathrm{d}t\\
&= \lim_{\epsilon\to 0}\Big(-\int_{-\infty}^{\epsilon}\log|t|\phi'(t)\,\mathrm{d}t - \int_{\epsilon}^{\infty}\log|t|\phi'(t)\,\mathrm{d}t\Big)\\
&= \lim_{\epsilon\to 0}\Big((\phi(\epsilon)-\phi(-\epsilon))\log\epsilon + \int_{|t|\geq\epsilon}\frac{\phi(t)}{t}\,\mathrm{d}t\Big)\\
&= \lim_{\epsilon\to 0}\Big((2\epsilon\phi'(t_0))\log\epsilon + \int_{|t|\geq\epsilon}\frac{\phi(t)}{t}\,\mathrm{d}t\Big)\\
&= \lim_{\epsilon\to 0}\int_{|t|\geq\epsilon}\frac{\phi(t)}{t}\,\mathrm{d}t = \mathrm{pv}\int_{\mathbb{R}}\frac{\phi(t)}{t}\,\mathrm{d}t,
\end{aligned}$$

where we have used integration by parts in the third step and the Mean Value Theorem in the third step. This then makes it sensible to denote the distribution g' by $\mathrm{pv}(\rho^{-1})$.

5. *Solve* $\rho\theta = \theta_u$ *for* $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{C})$*:* Here, recall, u denotes the signal that takes the value 1 everywhere. We first claim that $\theta = \mathrm{pv}(\rho^{-1})$ solves this equation $\rho\theta = \theta_u$. To verify this we note that if $\phi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ we have

$$(\rho\mathrm{pv}(\rho^{-1}))(\phi) = \mathrm{pv}(\rho^{-1})(\rho\phi) = \mathrm{pv}\int_{\mathbb{R}} \rho(t)\frac{\phi(t)}{t}\,\mathrm{d}t = \int_{\mathbb{R}} \phi(t)\,\mathrm{d}t = \theta_u(\phi).$$

Now we note that *any* solution of the equation $\rho\theta = \theta_u$ will have the form $\theta = \mathrm{pv}(\rho^{-1}) + \sigma$, where $\sigma \in \mathscr{D}'(\mathbb{R};\mathbb{C})$ satisfies $\rho\sigma = 0$. But this means, as we have seen in step 3, that $\sigma = c\delta_0$ for some $c \in \mathbb{C}$. Therefore, a general solution of $\rho\theta = \theta_u$ has the form $\theta = \mathrm{pv}(\rho^{-1}) + c\delta_0$ for $c \in \mathbb{C}$.

Now we proceed with our determination of $\mathscr{F}_{\mathrm{CC}}(\theta_1)$. We have $\rho\mathscr{F}_{\mathrm{CC}}(\theta_1) = \frac{1}{2\pi\mathrm{i}}\theta_u$. This means, by part 5 of our above discussion, that $\mathscr{F}_{\mathrm{CC}}(\theta_1) = \frac{1}{2\pi\mathrm{i}}\mathrm{pv}(\rho^{-1}) + c\delta_0$ for some $c \in \mathbb{C}$. To determine the value of c, note that $\mathrm{pv}(\rho^{-1})$ is an odd distribution and that δ_0 is an even distribution. Therefore

$$\begin{aligned} c\delta_0 &= \tfrac{1}{2}(\mathscr{F}_{\mathrm{CC}}(\theta_1) + \sigma^*\mathscr{F}_{\mathrm{CC}}(\theta_1)) \\ &= \tfrac{1}{2}(\mathscr{F}_{\mathrm{CC}}(\theta_1) + \mathscr{F}_{\mathrm{CC}}(\sigma^*\theta_1)) \\ &= \tfrac{1}{2}(\mathscr{F}_{\mathrm{CC}}(\theta_1 + \sigma^*\theta_1)) \\ &= \tfrac{1}{2}\mathscr{F}_{\mathrm{CC}}(\theta_u) = \delta_0. \end{aligned}$$

Using Example 13.4.3–4. Thus $c = \frac{1}{2}$. Thus, after some not insignificant effort, we have derived the formula

$$\mathscr{F}_{\mathrm{CC}}(\theta_1) = \frac{1}{2\pi\mathrm{i}}\mathrm{pv}(\rho^{-1}) + \frac{1}{2}\delta_0. \qquad \bullet$$

Let us consider another useful example, and one which requires less development.

13.4.8 Example (The CCFT of a polynomial) We let $\rho(t) = t$, noting that $\rho^k \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ for $k \in \mathbb{Z}_{\geq 0}$. By Proposition 13.4.6, noting that the CCFT of the signal $\rho_0\colon t \mapsto 1$ is δ_0, we have

$$\mathscr{F}_{\mathrm{CC}}(\rho^k) = \frac{1}{(-2\pi\mathrm{i})^k}\mathscr{F}_{\mathrm{CC}}(\rho_0)^{(k)} = \frac{1}{(-2\pi\mathrm{i})^k}\delta_0^{(k)} \qquad \bullet$$

13.4.5 Inversion of the CCFT for tempered distributions

The matter of inverting the CCFT on $\mathscr{S}'(\mathbb{R};\mathbb{C})$ mirrors the situation we saw for the inversion of the CDFT on periodic distributions, in that all the complexities of inversion that are present for signals get washed away. Indeed, the main result here is the following. Continuity in the following theorem means that convergent sequences are mapped to convergent sequences.

13.4.9 Theorem (The CCFT is an isomorphism of the tempered distributions) *The map $\mathscr{F}_{CC}\colon \mathscr{S}'(\mathbb{R};\mathbb{C}) \to \mathscr{S}'(\mathbb{R};\mathbb{C})$ is a continuous bijection with a continuous inverse. Furthermore, the inverse is $\overline{\mathscr{F}}_{CC}\colon \mathscr{S}'(\mathbb{R};\mathbb{C}) \to \mathscr{S}'(\mathbb{R};\mathbb{C})$.*

Proof Continuity of $\mathscr{F}_{CC}$ in this case means that if $(\theta_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence converging to zero in $\mathscr{S}'(\mathbb{R};\mathbb{C})$, then the sequence $(\mathscr{F}_{CC}(\theta_j))_{j\in\mathbb{Z}_{>0}}$ converges to zero in $\mathscr{S}'(\mathbb{R};\mathbb{C})$. To see that this relation holds, we let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and compute

$$\lim_{j\to\infty} \mathscr{F}_{CC}(\theta_j)(\phi) = \lim_{j\to\infty} \theta_j(\mathscr{F}_{CC}(\phi)) = 0$$

since $\mathscr{F}_{CC}(\phi) \in \mathscr{S}(\mathbb{R};\mathbb{C})$. This shows continuity of $\mathscr{F}_{CC}$, and continuity of $\overline{\mathscr{F}}_{CC}$ is shown in exactly the same way.

To see that $\mathscr{F}_{CC}$ is a bijection, we shall show that its inverse is $\overline{\mathscr{F}}_{CC}$. For $\theta \in \mathscr{S}'(\mathbb{R};\mathbb{C})$ and $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ we compute

$$(\overline{\mathscr{F}}_{CC} \circ \mathscr{F}_{CC}(\theta))(\phi) = \mathscr{F}_{CC}(\theta)(\overline{\mathscr{F}}_{CC}(\phi)) = \theta(\mathscr{F}_{CC} \circ \overline{\mathscr{F}}_{CC}(\phi)) = \theta(\phi),$$

since $\phi, \overline{\mathscr{F}}_{CC}(\phi) \in \mathsf{L}^{(1)}(\mathbb{R};\mathbb{C})$ and since ϕ is continuous (here we are invoking Theorem 13.2.26). This shows that $\overline{\mathscr{F}}_{CC} \circ \mathscr{F}_{CC}(\theta) = \theta$. That $\mathscr{F}_{CC} \circ \overline{\mathscr{F}}_{CC}(\theta) = \theta$ follows in an entirely similar manner. Thus $\mathscr{F}_{CC}$ is a bijection whose inverse is $\overline{\mathscr{F}}_{CC}$. ■

13.4.6 The CCFT for distributions with compact support

Since distributions with compact support are tempered distributions (by Propositions 10.5.9 and 10.4.9), the basic theory of the CCFT for distributions with compact support follows from the discussion of the preceding sections. However, since distributions with compact support have additional structure over tempered distributions, one can ask how this additional structure is manifested in the CCFT for these distributions. It is this that we dedicate ourselves to in this section.

The reader will recall at this point the discussion in Section 13.3.6 of the Paley–Weiner Theorem in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. The main point in that discussion is that there is a particular class of entire functions that are associated with the CCFT of signals in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ with compact support. The next definition gives the corresponding class of entire functions that arise as the CCFT of distributions with compact support.

13.4.10 Definition (Entire function of exponential type and slow growth) An entire function $F \in \mathsf{H}(\mathbb{C};\mathbb{C})$ is of ***exponential type α and slow growth*** if there exists $M \in \mathbb{R}_{>0}$, α and $N \in \mathbb{Z}_{>0}$ such that

$$|F(z)| \le M(1 + |z|^2)^N e^{\alpha|z|}, \qquad z \in \mathbb{C}.$$

The set of entire functions of exponential type and slow growth is denoted by $\mathsf{P}_{\exp}(\mathbb{C};\mathbb{C})$ and the set of entire functions of exponential type α and slow growth is denoted by $\mathsf{P}_{\exp,\alpha}(\mathbb{C};\mathbb{C})$. •

It is plausible but not obvious that a function of exponential type and slow growth is of exponential type. Precisely, we have the following result.

13.4.11 Lemma (Exponential type and slow growth implies exponential type) *The following statements hold:*

(i) $\mathsf{H}_{\exp,\alpha}(\mathbb{C};\mathbb{C}) \subseteq \mathsf{P}_{\exp,\alpha}(\mathbb{C};\mathbb{C})$;

(ii) *for every* $\epsilon \in \mathbb{R}_{>0}$, $\mathsf{P}_{\exp,\alpha}(\mathbb{C};\mathbb{C}) \subseteq \mathsf{H}_{\exp,\alpha+\epsilon}(\mathbb{C};\mathbb{C})$.

Proof (i) This is obvious since $(1+|z|^2) \geq 1$ for all $z \in \mathbb{C}$.

(ii) Let $F \in \mathsf{P}_{\exp,\alpha}(\mathbb{C};\mathbb{C})$ and let $M, \alpha \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$ be such that

$$|F(z)| \leq M(1+|z|^2)^N e^{\alpha|z|}.$$

Note that $\lim_{x\to\infty}(1+x^2)^N e^{-\epsilon x} = 0$ by L'Hôpital's Rule. Thus let $R \in \mathbb{R}_{>0}$ be large enough that $(1+x^2)e^{-\epsilon x} \leq 1$ for $x \geq R$. Then define $M' = M(1+R^2)^N$. If $|z| < R$ we have

$$|F(z)| \leq M(1+|z|^2)^N e^{\alpha|z|} \leq M' e^{(\alpha+\epsilon)|z|}.$$

If $|z| \geq R$ we have

$$|F(z)| \leq M(1+|z|^2)^N e^{-\epsilon|z|} e^{(\alpha+\epsilon)|z|} \leq M' e^{(\alpha+\epsilon)|z|},$$

giving the lemma. ■

With this definition, we may characterise the CCFT of a distribution with compact support.

13.4.12 Theorem (Paley–Wiener–Schwartz Theorem) *For* $\theta \in \mathscr{D}'(\mathbb{R};\mathbb{C})$ *and for* $\mathrm{T} \in \mathbb{R}_{>0}$, *the following statements are equivalent:*

(i) $\operatorname{supp}(\theta) \subseteq [-\mathrm{T}, \mathrm{T}]$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\theta)$ *is a regular distribution and, moreover, there exists* $\mathrm{F} \in \mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$ *such that* $\mathscr{F}_{\mathrm{CC}}(\theta)(\nu) = \mathrm{F}(\nu + \mathrm{i}0)$ *for all* $\nu \in \mathbb{R}$.

Proof First suppose that θ has support $[-T, T]$. By Theorem 10.5.20 there exists $m \in \mathbb{Z}_{>0}$ and continuous signals $f_1, \dots, f_m$ with support in $[-T, T]$ such that

$$\theta = \sum_{j=1}^{m} \theta_{f_j}^{(j)}.$$

By Proposition 13.4.5 we have

$$\mathscr{F}_{\mathrm{CC}}(\theta) = \sum_{j=1}^{m} \mathscr{F}_{\mathrm{CC}}(\theta_{f_j}^{(j)}) = \sum_{j=1}^{m} (2\pi\mathrm{i}\rho)^j \mathscr{F}_{\mathrm{CC}}(f_j),$$

where $\rho(\nu) = \nu$. Since f_j is a signal with compact support, by Corollary 13.1.13 we have $\mathscr{F}_{\mathrm{CC}}(f_j)$ as a regular, indeed infinitely differentiable, function. More precisely,

$$\mathscr{F}_{\mathrm{CC}}(\theta) = \sum_{j=1}^{m} (2\pi\mathrm{i}\nu)^j \mathscr{F}_{\mathrm{CC}}(f_j)(\nu).$$

Also $\lim_{|\nu|\to\infty} \mathscr{F}_{\mathrm{CC}}(f_j)(\nu) = 0$ by the Riemann–Lebesgue Lemma. which is evidently a function of slow growth, being a linear combination of functions, each of which is the product of a bounded function decaying to zero at infinity with a polynomial function.

To see that $\mathscr{F}_{\mathrm{CC}}(\theta)$ can be extended to a function in $\mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$, define

$$F(z)=\sum_{j=1}^{m}(2\pi \mathrm{i}z)^j\int_{\mathbb{R}} f_j(t)\mathrm{e}^{-2\pi\mathrm{i}zt}\,\mathrm{d}t.$$

Since f_j, $j\in\{1,\dots,m\}$, has compact support, the definition makes sense for each $z\in\mathbb{C}$. Moreover, we clearly have $\mathscr{F}_{\mathrm{CC}}(\theta)(\nu)=F(\nu+\mathrm{i}0)$ for each $\nu\in\mathbb{R}$. It remains to show that $F\in\mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$. Since f_j, $j\in\{1,\dots,m\}$ has support contained in $[-T,T]$ and is continuous, from Theorem 13.3.18 it follows that

$$G_j\colon z\mapsto\int_{\mathbb{R}} f_j(t)\mathrm{e}^{-2\pi\mathrm{i}zt}\,\mathrm{d}t$$

is in $\mathsf{H}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$. Thus there exists $M_j\in\mathbb{R}_{>0}$, $j\in\{1,\dots,m\}$, such that

$$|G_j(z)|\le M_j\mathrm{e}^{\alpha|z|},\qquad z\in\mathbb{C}.$$

If N is such that $2N>m$ then one can readily verify that there exists $M\in\mathbb{R}_{>0}$ such that

$$|F(z)|\le\sum_{j=1}^{m}(2\pi)^j|z|^j|G_j(z)|\le M(1+|z|^2)^N\mathrm{e}^{\alpha|z|},$$

as desired.

For the converse, suppose that $\mathscr{F}_{\mathrm{CC}}(\theta)$ is a regular distribution (we shall, therefore, think of $\mathscr{F}_{\mathrm{CC}}(\theta)$ as being a function, and so evaluate it as a function) and that there exists $F\in\mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$ such that $\mathscr{F}_{\mathrm{CC}}(\theta)(\nu)=F(\nu+\mathrm{i}0)$ for $\nu\in\mathbb{R}$. Define $\phi\in\mathscr{D}(\mathbb{R};\mathbb{C})$ by $\phi(t)=C\wedge(\frac{t}{T})$ where $C\in\mathbb{R}_{>0}$ is such that

$$\int_{\mathbb{R}}\phi(t)\,\mathrm{d}t=1.$$

Then define $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathscr{D}(\mathbb{R};\mathbb{C})$ by $\phi_j(t)=j\phi(jt)$. By Proposition 10.5.24 and Example 11.3.21 we know that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is a delta-sequence. Moreover, $\operatorname{supp}(\phi_j)=[-\frac{T}{j},\frac{T}{j}]$. Note that $\mathscr{F}_{\mathrm{CC}}(\phi_j)\in\mathscr{S}(\mathbb{R};\mathbb{C})$, $j\in\mathbb{Z}_{>0}$, since $\phi_j\in\mathscr{S}(\mathbb{R};\mathbb{C})$. Note that since $F\in\mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$, it follows that θ is a signal of slow growth. Therefore, $\mathscr{F}_{\mathrm{CC}}(\phi_j)\mathscr{F}_{\mathrm{CC}}(\theta)\in\mathscr{S}(\mathbb{R};\mathbb{C})$ since the product of a signal of slow growth with a signal in $\mathscr{S}(\mathbb{R};\mathbb{C})$ is in $\mathscr{S}(\mathbb{R};\mathbb{C})$ (see Example 10.3.5–4). We then compute

$$\mathscr{F}_{\mathrm{CC}}^{-1}(\mathscr{F}_{\mathrm{CC}}(\phi_j)\mathscr{F}_{\mathrm{CC}}(\theta))=\phi_j*\theta$$

by ***missing stuff***. Since $\lim_{j\to\infty}\phi_j=\delta$ in $\mathscr{D}'(\mathbb{R};\mathbb{C})$, from ***missing stuff*** we know that

$$\lim_{j\to\infty}\phi_j*\theta=\delta*\theta=\theta,$$

convergence being in $\mathscr{D}'(\mathbb{R};\mathbb{C})$.

Now, since $\mathscr{S}(\mathbb{R};\mathbb{C})\subseteq\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$, by Theorem 13.3.18 we know that $\mathscr{F}_{\mathrm{CC}}(\phi_j)$ can be extended to a function $\Phi_j\in\mathsf{H}_{\exp,2\pi\frac{T}{j}}(\mathbb{C};\mathbb{C})$. Since $\Phi_j\in\mathsf{H}_{\exp,2\pi\frac{T}{j}}(\mathbb{C};\mathbb{C})$ and since $F\in\mathsf{P}_{\exp,2\pi T}(\mathbb{C};\mathbb{C})$, it follows from Lemma 13.4.11 that $\Phi_jF\in\mathsf{H}_{\exp,2\pi(1+\frac{2}{j})T}(\mathbb{C};\mathbb{C})$. Therefore, by Theorem 13.3.18 and Remark 13.3.19–2 we know that

$$\operatorname{supp}(\phi_j*\theta)\subseteq[-(1+\frac{2}{j})T,(1+\frac{2}{j})T].$$

Let $\psi \in \mathscr{D}(\mathbb{R};\mathbb{C})$ be a test function for which $\mathrm{supp}(\psi) \subseteq \mathbb{R} \setminus [-T, T]$. Then, since $\mathrm{supp}(\psi)$ is closed, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\mathrm{supp}(\phi) \subseteq \mathbb{R} \setminus [-(1 + \frac{2}{N})T, (1 + \frac{2}{N})T].$$

Therefore, $(\phi_j * \theta)(\psi) = 0$ for $j \geq N$ and so $\theta(\psi) = 0$ by definition of convergence in $\mathscr{D}'(\mathbb{R};\mathbb{C})$. Therefore, we conclude that $\mathrm{supp}(\theta) \subseteq [-(1 + \frac{2}{j})T, (1 + \frac{2}{j})T]$ for every $j \in \mathbb{Z}_{>0}$, i.e., $\mathrm{supp}(\theta) \subseteq [-T, T]$. ■

During the course of the first part of the proof of the preceding theorem, we proved the following result, recalling that for $a \in \mathbb{C}$ $\mathsf{E}_a \colon \mathbb{R} \to \mathbb{C}$ is defined by $\mathsf{E}_a(t) = \mathrm{e}^{at}$.

13.4.13 Corollary (The CCFT of a distribution with compact support) *If $\theta \in \mathscr{E}'(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CC}}(\theta)$ is an infinitely differentiable function of slow growth which satisfies $\mathscr{F}_{\mathrm{CC}}(\theta)(\nu) = \theta(\mathsf{E}_{-2\pi\mathrm{i}\nu})$.*

Proof This follows from the computation, using the notation from the proof of the theorem,

$$\begin{aligned}\theta(\mathsf{E}_{-2\pi\mathrm{i}\nu}) &= \sum_{j=1}^{m} \theta_{f_j}^{(j)}(\mathsf{E}_{-2\pi\mathrm{i}\nu}) = \sum_{j=1}^{m} (-1)^j \theta_{f_j}(\mathsf{E}_{-2\pi\mathrm{i}\nu}^{(j)}) \\ &= \sum_{j=1}^{m} (2\pi\mathrm{i}\nu)^j \int_{\mathbb{R}} f_j(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t,\end{aligned}$$

which is exactly the expression we derived in the proof of the theorem for $\mathscr{F}_{\mathrm{CC}}(\theta)(\nu)$. In the above computation we used Proposition 10.2.35. ■

Let us give an example which verifies the Paley–Wiener–Schwartz Theorem.

13.4.14 Example (Paley–Wiener–Schwartz Theorem) We note that for $a \in \mathbb{R}$, δ_a is a distribution with compact support. In Example 13.4.3–3 we computed $\mathscr{F}_{\mathrm{CC}}(\delta_a) = \theta_{\mathsf{E}_{-2\pi\mathrm{i}a}}$. Thus $\mathscr{F}_{\mathrm{CC}}(\delta_a)$ is indeed a regular, indeed infinitely differentiable, distribution. Note that $F(z) = \mathrm{e}^{-2\pi a\mathrm{i}z}$ has the property that $\mathscr{F}_{\mathrm{CC}}(\delta_a)(\nu) = F(\nu + \mathrm{i}0)$. Since $\mathrm{supp}(\delta_a) \subseteq [-a, a]$, we should verify that $\mathscr{F}_{\mathrm{CC}}(\delta_a) \in \mathsf{P}_{\exp,2\pi a}(\mathbb{C};\mathbb{C})$. However, we clearly have $F \in \mathsf{H}_{\exp,2\pi a}(\mathbb{C};\mathbb{C})$ and so our conclusion follows from Lemma 13.4.11, •

13.4.7 The CCFT for periodic distributions

As with distributions of compact support, the CCFT for periodic distributions follows in its generalities from the CCFT for tempered distributions (by Theorem 10.7.18). However, periodic distributions possess particular structure which shows up in an essential way in the theory of the CCFT for these distributions. In this section we investigate this.

Exercises

13.4.1 Let $\phi \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

(a) Show that $\mathscr{F}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(\phi)) = \sigma^*\phi$.

(b) Show that ϕ is the inverse Fourier transform of $\mathscr{F}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(\mathscr{F}_{\mathrm{CC}}(\phi)))$.

13.4.2 Prove Proposition 13.4.4.

13.4.3 The CCFT is sometimes used to solve ordinary differential equations, just like the Laplace transform that you may have encountered in the same rôle. In the following exercise you will be asked to use the CCFT to solve two ordinary inhomogeneous differential equations for the distribution θ with a forcing function given by the delta-signal δ_0. For each differential equation do the following:

1. take the CCFT of the differential equation, using the fact, as demonstrated in Example 13.4.3–3, that $\mathscr{F}_{\mathrm{CC}}(\delta_0)$ is the distribution corresponding to the locally integrable signal $\nu \mapsto 1$;
2. in the frequency domain, solve for $\mathscr{F}_{\mathrm{CC}}(\theta)$;
3. compute the inverse CCFT to show that θ actually corresponds to a locally integrable signal x in the time-domain (i.e., $\theta = \theta_x$), and provide the expression for $x(t)$;

   Hint: *In both examples, one can compute the inverse CCFT by reference to Examples 13.1.3–1 and 2.*
4. sketch the solution to the differential equation;
5. Is the signal x causal?

The differential equations are:

(a) $\theta' + \theta = \delta_0$;

(b) $\theta'' - \theta = \delta_0$.

13.4.4 Let $f \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ and define a generalised signal in the frequency domain by

$$\theta = \sum_{n\in\mathbb{Z}} f(nT^{-1})\delta_{nT^{-1}}.$$

Answer the following questions.

(a) Show that $\theta \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

(b) Compute $\overline{\mathscr{F}}_{\mathrm{CC}}(\theta)$.

(c) Show that $\overline{\mathscr{F}}_{\mathrm{CC}}(\theta)$ is the distribution associated with a T-periodic signal g (i.e., $\theta = \theta_g$) and show that $\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1}) = f(nT^{-1})$.

(d) Carry out part (b) in the case where $f(0) = 0$ and $f(nT^{-1}) = \frac{1}{2n^2}$, $n \in \mathbb{Z}\setminus\{0\}$, and plot $\overline{\mathscr{F}}_{\mathrm{CC}}(\theta)$ in this case (take $T = 1$ for concreteness).

Chapter 14

Discrete-time Fourier transforms

In this chapter we study Fourier transform theory for discrete-time signals, mirroring the developments of Chapters 12 and 13. Things are somewhat easier for the discrete-time theory, and so we are able to combine the discrete-time analogues of these three chapters in a single chapter.

In Sections 14.1 and 14.2 we complete our Fourier transform quadrangle by giving the discrete versions of the Fourier transform for aperiodic and periodic signals, respectively. The development here is much simpler than in the continuous-time case. Indeed, for the DCFT we can make use of much of the machinery already in place from our somewhat thorough study of the inverse of the CDFT in Section 12.2. For the DDFT things are simpler because the signal spaces involved are finite-dimensional.

In Section **??** we summarise the relationships between the various Fourier transforms. Some of these are more or less obvious, but some are a little deep, requiring, for example, comprehension of the transforms for various classes of distributions.

Do I need to read this chapter? If you are interested in understanding the Fourier transforms that may be applied to discrete-time signals, then this is the place to start. •

Contents

missing stuff

Section 14.1

The discrete-continuous Fourier transform

The transform we consider in this section takes as input an aperiodic discrete-time signal and returns a periodic continuous-time signal. This transform is very often known by the name "discrete-time Fourier transform." However, for us to call it anything other than the DCFT would be absurd.

Do I need to read this section? If you want to know about the DCFT, then you will be reading this section. •

14.1.1 Definition of the ℓ^1-DCFT

As with the CDFT and the CCFT, we refer the reader to material in Section 9.6.4 for motivation for the transform we discuss here. Assuming this motivation, we proceed with the definition.

14.1.1 Definition (DCFT) The ***discrete-continuous Fourier transform*** or ***DCFT*** assigns to $f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ the signal $\mathscr{F}_{\mathrm{DC}}(f)\colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n\in\mathbb{Z}} f(n\Delta) \mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu}, \qquad \nu \in \mathbb{R}.$$ •

14.1.2 Remarks (Comments on the definition of the DCFT)

1. As mentioned in the preamble to this section, what we call the DCFT is most often called the "discrete-time Fourier transform." Our decision to use the much less common "DCFT" is based solely on rational concerns.
2. It is important to note the relationship of the DCFT with Fourier series. This relationship, along with our observations in Sections 12.2 and 12.3.2, should make one wonder whether the DCFT is well-defined in that the sum exists. However, the assumption that the DCFT is applied to signals in $\ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ is sufficient to ameliorate any concerns about convergence. We shall make this formal in Theorem 14.1.7.
3. As we have done with our previous Fourier transforms, we shall regard $\ell^1(\mathbb{Z}(\Delta); \mathbb{R})$ as a subspace of $\ell^1(\mathbb{Z}(\Delta; \mathbb{C})$, so $\mathbb{R}$-valued signals are treated most often as special cases of $\mathbb{C}$-valued signals.
4. One often sees the DCFT defined with the domain being a signal on the time-domain $\mathbb{Z}$ rather than on $\mathbb{Z}(\Delta)$ as we have done. Our explicit involvement of Δ makes it clear the rôle of the sampling time.

Let us compute the DCFT for some examples.

14.1.3 Examples (Computing the DCFT)

1. Let us consider the unit pulse $\mathsf{P}\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$\mathsf{P}(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Trivially, the DCFT of P is

$$\mathscr{F}_{\mathrm{DC}}(\mathsf{P})(\nu) = \Delta, \qquad \nu \in \mathbb{R}.$$

2. Let us generalise the preceding example slightly, and consider the shifted pulse $\mathsf{P}_N\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$\mathsf{P}_N(t) = \begin{cases} 1, & t = N\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

In this case,

$$\mathscr{F}_{\mathrm{DC}}(\mathsf{P}_N)(\nu) = \Delta \mathrm{e}^{-2\pi \mathrm{i} N\Delta\nu}$$

3. Next we consider a discrete version of the square wave. Thus we define $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ by

$$f(t) = \begin{cases} 1, & t \in \{-N\Delta, -\Delta, 0, \Delta, \dots, N\Delta\}, \\ 0, & \text{otherwise.} \end{cases}$$

We plot this signal in Figure 14.1. In this case, the sum defining the DCFT of f is finite, and we have

$$\mathscr{F}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n=-N}^{N} \mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu} = \Delta D^{\mathrm{per}}_{\Delta^{-1},N}(\nu),$$

using Lemma **??** from Example **??** and the definition

$$D^{\mathrm{per}}_{\Delta^{-1},N}(\nu) = \begin{cases} \dfrac{\sin\big((2N+1)\pi\Delta\nu\big)}{\sin\big(\pi\Delta\nu\big)}, & t \neq 0, \\ 2N+1, & t = 0 \end{cases}$$

of the discrete Dirichlet kernel; see the discussion in Section 12.2.2.

4. The final example we consider is a signal which is the discrete analogue of triangular wave. Thus we consider $g(\Delta)\colon \mathbb{Z} \to \mathbb{C}$ defined by

$$g(t) = \begin{cases} -\frac{t}{N\Delta} + 1, & t \in \{0, \Delta, \dots, (N-1)\Delta\}, \\ \frac{t}{N\Delta} + 1, & t \in \{-(N-1)\Delta, \dots, -\Delta\}, \\ 0, & \text{otherwise.} \end{cases}$$

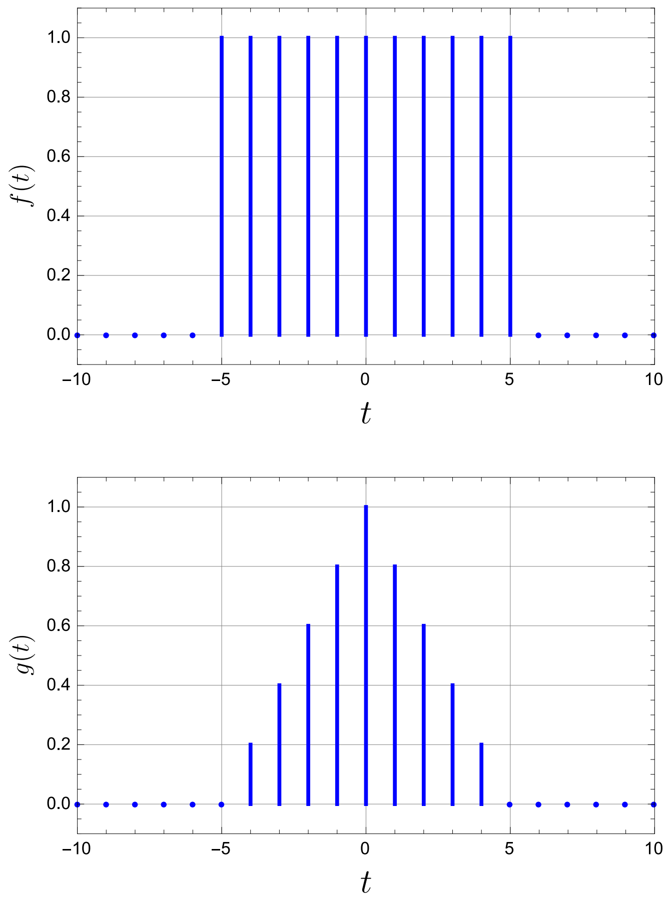


Figure 14.1 A discrete square wave (top) and a discrete triangular wave (bottom) for $\Delta = 1$ and $N = 5$

This signal is plotted in Figure 14.1. The compute the DCFT of g, we take $\theta = 2\pi\nu$ in Lemmata **??** and **??** from Example **??–??** to get

$$\mathscr{F}_{\mathrm{DC}}(g)(\nu) = \Delta F^{\mathrm{per}}_{\Delta^{-1},N}(\nu),$$

where

$$F^{\mathrm{per}}_{\Delta^{-1},N}(t) = \begin{cases} \dfrac{1}{N}\dfrac{\sin^2\!\big(\pi N\Delta\nu\big)}{\sin^2(\pi\Delta\nu)}, & t \neq 0, \\ N, & t = 0 \end{cases}$$

is the Fejér kernel. We refer the discussion following the proof of Theorem 12.2.1 for some properties of the Fejér kernel. •

Recall with the CDFT and the CCFT there are sine and cosine versions of the transform and these are related with the complex exponential version; see Definitions 12.1.4 and 13.1.4, and Propositions 12.1.5 and 13.1.5. We have a similar construction for the DCFT, although this is less frequently presented as it is less frequently useful.

14.1.4 Definition (DCCT and DCST)

(i) The ***discrete-continuous cosine transform*** or ***DCCT*** assigns to $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{C}_{\mathrm{DC}}(f)\colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ by

$$\mathscr{C}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n \in \mathbb{Z}_{\geq 0}} f(n\Delta)\cos(2\pi n\Delta\nu), \qquad \nu \in \mathbb{R}.$$

(ii) The ***discrete-continuous sine transform*** or ***DCST*** assigns to $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{S}_{\mathrm{DC}}(f)\colon \mathbb{R}_{>0} \to \mathbb{C}$ by

$$\mathscr{S}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta)\sin(2\pi n\Delta\nu), \qquad \nu \in \mathbb{R}.$$

The relationships between the DCFT and the DCCT and DCST are given as follows.

14.1.5 Proposition (The DCFT, and the DCCT and the DCST) *For* $\mathrm{f} \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ *the following statements hold:*

(i) $\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(0) = \mathscr{C}_{\mathrm{DC}}(\mathrm{f})(0)$*;*

(ii) $\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(\nu) = \mathscr{C}_{\mathrm{DC}}(\mathrm{f})(\nu) - \mathrm{i}\mathscr{S}_{\mathrm{DC}}(\mathrm{f})(\nu)$ *and* $\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(-\nu) = \mathscr{C}_{\mathrm{DC}}(\mathrm{f})(\nu) + \mathrm{i}\mathscr{S}_{\mathrm{DC}}(\mathrm{f})(\nu)$ *for every* $\nu \in \mathbb{R}_{>0}$*;*

(iii) $\mathscr{C}_{\mathrm{DC}}(\mathrm{f})(\nu) = \frac{1}{2}(\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(\nu) + \mathscr{F}_{\mathrm{DC}}(\mathrm{f})(-\nu))$ *for every* $\nu \in \mathbb{R}_{\geq 0}$*;*

(iv) $\mathscr{S}_{\mathrm{DC}}(\mathrm{f})(\nu) = \frac{\mathrm{i}}{2}(\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(\nu) - \mathscr{F}_{\mathrm{DC}}(\mathrm{f})(-\nu))$ *for every* $\nu \in \mathbb{R}_{>0}$*.*

Proof This is a direct application of Euler's formula:

$$\mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu} = \cos(2\pi n\Delta\nu) - \mathrm{i}\sin(2\pi n\Delta\nu). \qquad \blacksquare$$

Using evenness of cosine and oddness of sine, we can also write the DCCT and the DCST as

$$\mathscr{C}_{\mathrm{DC}}(f)(\nu) = 2\Delta \sum_{n \in \mathbb{Z}_{\geq 0}} f_{\mathrm{even}}(n\Delta)\cos(2\pi n\Delta\nu),$$
$$\mathscr{S}_{\mathrm{DC}}(f)(\nu) = 2\Delta \sum_{n \in \mathbb{Z}_{>0}} f_{\mathrm{odd}}(n\Delta)\sin(2\pi n\Delta\nu),$$

where

$$f_{\mathrm{even}}(t) = \tfrac{1}{2}(f(t) + f(-t)), \quad f_{\mathrm{odd}}(t) = \tfrac{1}{2}(f(t) - f(-t)).$$

It is not too difficult to reason that one might want to apply the DCCT and the DCST for signals that are zero on $\mathbb{Z}_{<0}$.

14.1.2 Properties of the DCFT

Let us now consider some properties of the DCFT that are analogous to those we have seen for the other Fourier transforms. We begin by looking at elementary properties. For $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ we have $\sigma^* f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ defined by $\sigma^* f(t) = f(-t)$.

If $a \in \mathbb{Z}(\Delta)$ then we define $\tau_a^* f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ by $\tau^* f(t) = f(t-a)$. Finally, we define $\overline{\mathscr{F}}_{\mathrm{DC}}(f) \colon \mathbb{R} \to \mathbb{C}$ by

$$\overline{\mathscr{F}}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta) \mathrm{e}^{2\pi \mathrm{i} n \Delta \nu}.$$

With this notation we have the following result.

14.1.6 Proposition (Elementary properties of the DCFT) *For* $\mathrm{f} \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ *the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{DC}}(\mathrm{f})} = \overline{\mathscr{F}}_{\mathrm{DC}}(\bar{\mathrm{f}})$;

(ii) $\mathscr{F}_{\mathrm{DC}}(\sigma^* \mathrm{f}) = \sigma^*(\mathscr{F}_{\mathrm{DC}}(\mathrm{f})) = \overline{\mathscr{F}}_{\mathrm{DC}}(\mathrm{f})$;

(iii) *if* f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{DC}}(\mathrm{f})$ *is even (resp. odd);*

(iv) *if* f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{DC}}(\mathrm{f})$ *is real and even (resp. imaginary and odd);*

(v) *if* $\mathrm{a} \in \mathbb{Z}(\Delta)$ *then* $\mathscr{F}_{\mathrm{DC}}(\tau_{\mathrm{a}}^* \mathrm{f})(\nu) = \mathrm{e}^{-2\pi \mathrm{i} \mathrm{a} \nu} \mathscr{F}_{\mathrm{DC}}(\mathrm{f})(\nu)$.

Proof The proof consists of direct verifications of all assertions. ■

Next we consider the character of the function $\mathscr{F}_{\mathrm{DC}}(f)$ when $f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$. The following result indicates that this function is actually continuous and periodic with period Δ^{-1}. Moreover, the result also gives an important topological property of the DCFT.

14.1.7 Theorem (Continuity of the DCFT) *If* $\mathrm{f} \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ *then* $\mathscr{F}_{\mathrm{DC}}(\mathrm{f}) \in \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C})$. *Moreover, the mapping* $\mathrm{f} \mapsto \mathscr{F}_{\mathrm{DC}}(\mathrm{f})$ *is a continuous linear mapping from* $(\ell^1(\mathbb{Z}(\Delta); \mathbb{C}), \|\cdot\|_1)$ *to* $(\mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C}), \|\cdot\|_\infty)$.

Proof Since $f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$, the series

$$\mathscr{F}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta) \exp^{-2\pi \mathrm{i} n \Delta \nu}$$

converges uniformly by the Weierstrass *M*-test. Since the uniform limit of a sequence of bounded continuous functions is continuous (this is Theorem 3.5.8), it follows that $\mathscr{F}_{\mathrm{DC}}(f)$ is continuous. Linearity of the DCFT follows from the fact that convergent series are linear. Using this fact we also have

$$\begin{aligned}
\mathscr{F}_{\mathrm{DC}}(f)(\nu + \Delta^{-1}) &= \Delta \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta) \exp^{-2\pi \mathrm{i} n \Delta(\nu + \Delta^{-1})} \\
&= \Delta \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta) \exp^{-2\pi \mathrm{i} n \Delta \nu} = \mathscr{F}_{\mathrm{DC}}(\nu),
\end{aligned}$$

giving the Δ^{-1}-periodicity of $\mathscr{F}_{\mathrm{DC}}(f)$.

To verify the final assertion of the theorem, we compute

$$|\mathscr{F}_{\mathrm{DC}}(f)(\nu)| = \Delta \Big| \sum_{n \in \mathbb{Z}_{>0}} f(n\Delta) \exp^{-2\pi \mathrm{i} n \Delta \nu} \Big| \leq \Delta \sum_{n \in \mathbb{Z}_{>0}} |f(n\Delta)| = \Delta \|f\|_1.$$

Thus we have

$$\|\mathscr{F}_{\mathrm{DC}}(f)\|_\infty = \sup\{|\mathscr{F}_{\mathrm{DC}}(f)(\nu)| \mid \nu \in \mathbb{R}\} \leq \Delta \|f\|_1.$$

missing stuff Thus $\mathscr{F}_{\mathrm{DC}}$ is a bounded linear map, and so continuous by Theorem 6.5.8. ■

14.1.8 Remark (The DCFT and inversion of the CDFT) Note that the first assertion of the previous theorem really follows from Theorem 12.2.33 which deals with uniform convergence of Fourier series. The natural assumption that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ corresponds to the natural hypotheses for uniform convergence of Fourier series. •

We also have a version of the Fourier Reciprocity Relation for the DCFT.

14.1.9 Proposition (Fourier Reciprocity Relation for the DCFT) *If* $\mathsf{f}, \mathsf{g} \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ *then*

$$\Delta \sum_{\mathsf{n}\in\mathbb{Z}} \mathsf{f}(\mathsf{n}\Delta)\mathsf{g}(\mathsf{n}\Delta) = \int_0^{\Delta^{-1}} \mathscr{F}_{\mathrm{DC}}(\mathsf{f})(\nu)\overline{\mathscr{F}}_{\mathrm{DC}}(\mathsf{g})(\nu)\,\mathrm{d}\nu.$$

Proof We leave this to the reader as Exercise 14.1.2. ■

14.1.3 The effect of coefficient decay on the DCFT

In Section 12.1.3 we saw that there were relationships between the differentiability of a periodic signal and the rate of decay of CDFT at large frequencies. In this section we establish a sort of converse of this, understanding that the rôles of time and frequency are reversed for the DCFT.

For smoothness of the DCFT of a signal, the following result is the main one.

14.1.10 Proposition (Differentiability of the DCFT of a signal) *For* $\mathsf{k} \in \mathbb{Z}_{>0}$*, suppose that* $\mathsf{f}\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ *has the property that the signal* $\mathsf{t} \mapsto \mathsf{t}^{\mathsf{k}}\mathsf{f}(\mathsf{t})$ *is in* $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$*. Then* $\mathscr{F}_{\mathrm{DC}}(\mathsf{f}) \in \mathrm{C}^{\mathsf{k}}(\mathbb{R};\mathbb{C})$ *and*

$$\mathscr{F}_{\mathrm{DC}}(\mathsf{f})^{(\mathsf{k})}(\nu) = \Delta \sum_{\mathsf{n}\in\mathbb{Z}} (-2\pi\mathrm{i}\mathsf{n}\Delta)^{\mathsf{k}}\mathsf{f}(\mathsf{n}\Delta)\mathrm{e}^{-2\pi\mathrm{i}\mathsf{n}\Delta\nu}.$$

Proof First of all, note that the hypotheses of the proposition ensure that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. Indeed, consider the signal $g\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$g(t) = \begin{cases} f(t), & t = 0, \\ t^k f(t), & \text{otherwise}. \end{cases}$$

Then, our hypotheses ensure that $g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. Since $|f(t)| \le g(t)$ for all $t \in \mathbb{Z}(\Delta)$ it follows from the Comparison Test that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$.

We now prove the result by induction on k. For $k = 0$ it is true from Theorem 14.1.7. So suppose the result holds for $k \in \{0, 1, \dots, r-1\}$ and let $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ be such that $t \mapsto t^r f(t)$ is in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. It follows as in the first paragraph of the proof that $t \mapsto t^{r-1} f(t)$ is in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ and so, by the induction hypothesis,

$$\mathscr{F}_{\mathrm{DC}}(f)^{(r-1)}(\nu) = \Delta \sum_{n\in\mathbb{Z}} (-2\pi\mathrm{i}n\Delta)^{r-1} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu},$$

with the convergence of the series on the left being uniform by the Weierstrass M-test. By our hypotheses on f, the series

$$\sum_{n\in\mathbb{Z}} n^r f(n\Delta)$$

converges absolutely. By the Weierstrass M-test, this means that the series

$$\sum_{n\in\mathbb{Z}}(-2\pi i n\Delta)^r f(n\Delta)e^{-2\pi i n\Delta\nu}$$

converges uniformly. Therefore, by Theorem 3.5.24, it follows that $\mathscr{F}_{DC}(f)$ is r-times continuously differentiable and that

$$\mathscr{F}_{DC}(f)^r(\nu) = \Delta\sum_{n\in\mathbb{Z}}(-2\pi i n\Delta)^r f(n\Delta)e^{-2\pi i n\Delta\nu},$$

as desired. ■

This results gives rise to the following corollary, which gives an easy class of signals with smooth DCFT's.

14.1.11 Corollary (Differentiability of the DCFT of a signal) *For* $k\in\mathbb{Z}_{>0}$*, suppose that* $f\colon\mathbb{Z}(\Delta)\to\mathbb{C}$ *has the property that* $\lim_{|t|\to\infty}t^{k+1+\epsilon}f(t) = 0$ *for some* $\epsilon\in\mathbb{R}_{>0}$*. Then* $\mathscr{F}_{DC}(f)\in C^k(\mathbb{R};\mathbb{C})$ *and*

$$\mathscr{F}_{DC}(f)^{(k)}(\nu) = \Delta\sum_{n\in\mathbb{Z}}(-2\pi i n\Delta)^k f(n\Delta)e^{-2\pi i n\Delta\nu}.$$

Proof Let $N\in\mathbb{Z}_{>0}$ be sufficiently large that $|t|^{k+1+\epsilon}|f(t)| < 1$ for $|t|\geq N\Delta$. Therefore,

$$|f(t)| < |t|^{-k-1-\epsilon} \quad\Longrightarrow\quad |t|^k|f(t)| < |t|^{-1-\epsilon}.$$

Therefore, $t\mapsto t^k f(t)$ is in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ by Example 2.4.2–??. The result then follows from Proposition 14.1.10. ■

14.1.4 Convolution, multiplication, and the DCFT

As with the CDFT and the CCFT, there are connections between the DCFT and convolution. Here we state these in the ℓ^1-case, referring to Section 14.1.6 for the situation in the ℓ^2-case.***missing stuff*** The first result is that the DCFT of a convolution is the product of the DCFT's.

14.1.12 Proposition (The ℓ^1-DCFT of a convolution is the product of the ℓ^1-DCFT's) *If* $f,g\in\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ *then*

$$\mathscr{F}_{DC}(f*g)(\nu) = \mathscr{F}_{DC}(f)(\nu)\mathscr{F}_{DC}(g)(\nu)$$

for every $t\in\mathbb{R}$.

Proof We have

$$\begin{aligned}\mathscr{F}_{DC}(f*g)(\nu) &= \sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}f((n-k)\Delta)g(k\delta)\Big)e^{-2\pi i n\Delta\nu}\\ &= \sum_{m\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}f((m\Delta)g(k\delta)\Big)e^{-2\pi i(m+k\Delta\nu}\\ &= \Big(\sum_{m\in\mathbb{Z}}f(m\Delta)e^{-2\pi i m\Delta\nu}\Big)\Big(g(k\Delta)e^{-2\pi i k\Delta\nu}\Big)\\ &= \mathscr{F}_{DC}(f)(\nu)\mathscr{F}_{DC}(g)(\nu),\end{aligned}$$

interchanging the sums by Fubini's Theorem. ■

For the DCFT of products, we have the following result.

14.1.13 Proposition (The ℓ^1-DCFT of a product is the convolution of the ℓ^1-DCFT's) *If* $f, g \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ *then*

$$\mathscr{F}_{DC}(fg)(\nu) = \mathscr{F}_{DC}(f) * \mathscr{F}_{DC}(g)(\nu), \qquad \nu \in \mathbb{R}.$$

Proof Note that $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \subseteq \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ by Theorem 8.2.7(v). By Exercise 8.2.5 we then have $fg \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. We can thus compute

$$\begin{aligned}
\mathscr{F}_{DC}(f) * \mathscr{F}_{DC}(g)(\nu) &= \int_0^{\Delta^{-1}} \mathscr{F}_{DC}(f)(\nu-\mu)\mathscr{F}_{DC}(g)(\mu)\,d\mu \\
&= \int_0^{\Delta^{-1}} \Big(\sum_{n\in\mathbb{Z}} f(n\Delta)e^{-2\pi i n\Delta(\nu-\mu)}\Big)\Big(\sum_{m\in\mathbb{Z}} g(m\Delta)e^{-2\pi i m\Delta\mu}\Big)\,d\mu \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f(n\Delta)g(m\Delta)e^{-2\pi i n\Delta\nu}\int_0^{\Delta^{-1}} e^{2\pi i n\Delta\mu}e^{-2\pi i m\Delta\mu}\,d\mu \\
&= \Delta\sum_{n\in\mathbb{Z}} f(n\Delta)g(n\Delta)e^{-2\pi i n\Delta\nu} = \mathscr{F}_{DC}(fg)(\nu),
\end{aligned}$$

where the sums and the integral have been interchanged by Fubini's Theorem. ■

14.1.5 Inversion of the DCFT

Inversion of the DCFT is more easily accomplished than for the continuous-time Fourier transforms, the CDFT and the CCFT. This is partly due to the fact that the transform has a set of discrete-time signals as its domain, and partly because we have done some of the work to invert the DCFT in our discussion of the CDFT.

As with the CDFT and the CCFT, it is first useful to understand the basic properties of $\mathscr{F}_{DC}$ as a map from $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ to $C^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$.

14.1.14 Theorem (The DCFT is injective) *The map* $\mathscr{F}_{DC}\colon \ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \to C^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ *is injective.*

Proof Because $\mathscr{F}_{DC}$ is linear, to show that it is injective it suffices to show that only the zero signal in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ maps to the zero signal in $C^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, cf. Exercise 4.3.23. Thus suppose that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ has the property that $\mathscr{F}_{DC}(f)(\nu) = 0$ for every $\nu \in \mathbb{R}$. Thus

$$\sum_{n\in\mathbb{Z}} f(n\Delta)e^{-2\pi i n\Delta\nu} = 0, \qquad \nu \in \mathbb{R}.$$

As we have already seen, the sum on the left converges uniformly by the Weierstrass M-test. Using Theorem 3.5.23 we have, for $m \in \mathbb{Z}$

$$0 = \sum_{n\in\mathbb{Z}} f(n\Delta)\int_0^{\Delta^{-1}} e^{2\pi i m\Delta\nu}e^{-2\pi i n\Delta\nu}\,d\nu = \frac{f(m\Delta)}{\Delta},$$

using Lemma 12.3.2. This gives the result. ■

One might now hope that the DCFT is an isomorphism. However, if the reader was following along in Chapters 12 and 13, then they will realise that this is too much to hope for.

14.1.15 Proposition (The DCFT is not onto $\mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$) *The map* $\mathscr{F}_{\mathrm{DC}}\colon \ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \to \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ *is not surjective.*

Proof Let $F \in \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ be a signal whose Fourier series does not converge, cf. Example 12.2.10. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be defined by $\sigma(\nu) = -\nu$. Then the Fourier series of σ^*F also does not converge, where $\sigma^*F(\nu) = F(-\nu)$. Suppose that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ is such that $\mathscr{F}_{\mathrm{DC}}(f) = \sigma^*F$. Thus

$$F(\nu) = \Delta \sum_{n\in\mathbb{Z}} f(n\Delta)\mathrm{e}^{2\pi i n\Delta\nu}.$$

Since $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ the series on the left must converge uniformly to F. However, the series on the left is the Fourier series of F which we have assumed is divergent. This is a contradiction, and so there can exist no $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ such that $\mathscr{F}_{\mathrm{DC}}(f) = \sigma^*F$. Since $\sigma^*F \in \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, this gives the result. ■

As we did with the CDFT and the CCFT, we can seek a left-inverse for DCFT. For the CDFT and the CCFT, this required some work. For the DCFT the inverse is easily written down by virtue of our work on convergence of Fourier series in Sections 12.2.4 and 12.2.5.

14.1.16 Theorem (The inverse of the DCFT) *The map* $\mathscr{F}^{-1}_{\mathrm{DC}}\colon \mathsf{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(\Delta);\mathbb{C})$ *defined by*

$$\mathscr{F}^{-1}_{\mathrm{DC}}(\mathrm{F})(\mathrm{n}\Delta) = \int_0^{\Delta^{-1}} \mathrm{F}(\nu)\mathrm{e}^{2\pi\mathrm{i}\mathrm{n}\Delta\nu}\,\mathrm{d}\nu$$

has the property that $\mathscr{F}^{-1}_{\mathrm{DC}} \circ \mathscr{F}_{\mathrm{DC}}(\mathrm{f}) = \mathrm{f}$ *for every* $\mathrm{f} \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. *That is to say,* $\mathscr{F}^{-1}_{\mathrm{DC}}$ *is a left-inverse for* $\mathscr{F}_{\mathrm{DC}}$.

Proof First of all, note that the Riemann–Lebesgue Lemma ensures that $\mathscr{F}^{-1}_{\mathrm{DC}}$ takes values in $\mathsf{c}_0(\mathbb{Z}(\Delta);\mathbb{C})$, as stated. Now let $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$, let $m \in \mathbb{Z}$, and compute, using Theorem 3.5.23 and Lemma 12.3.2,

$$\begin{aligned}(\mathscr{F}^{-1}_{\mathrm{DC}} \circ \mathscr{F}_{\mathrm{DC}}(f))(m\Delta) &= \Delta \int_0^{\Delta^{-1}} \sum_{n\in\mathbb{Z}} f(n\Delta)\mathrm{e}^{-2\pi i n\Delta\nu}\mathrm{e}^{2\pi i m\Delta\nu}\,\mathrm{d}\nu \\ &= \Delta \sum_{n\in\mathbb{Z}} f(n\Delta) \int_0^{\Delta^{-1}} \mathrm{e}^{-2\pi i n\Delta\nu}\mathrm{e}^{2\pi i m\Delta\nu}\,\mathrm{d}\nu = f(m\Delta),\end{aligned}$$

as desired. ■

14.1.6 The ℓ^2-DCFT

Recall from Section 8.2.6 that $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \subseteq \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ and that the inclusion is strict. For example, the signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$f(t) = \begin{cases} \frac{1}{|t|}, & t \neq 0, \\ 0, & t = 0 \end{cases}$$

is in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ but not in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. Therefore, the definition of the DCFT from Section 14.1.1 cannot be directly applied to signals in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$. However, the work done in Section 12.3 can be applied here.

We shall mirror the constructions of Section 13.3 for the L^2-CCFT, although this is not quite necessary since the ℓ^2-DCFT is a little simpler than the L^2-CCFT. However, it is useful to see the two transforms developed in the same manner.

We first state two lemmata for the DCFT that mirror Lemmata 13.3.1 and 13.3.2 for the CCFT.

14.1.17 Lemma ($\mathscr{F}_{\mathrm{DC}}(\ell^1 \cap \ell^2) \subseteq \ell^2$) *If* $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *then* $\|\mathscr{F}_{\mathrm{DC}}(f)\|_2 = \|f\|_2$. *In particular,* $\mathscr{F}_{\mathrm{DC}}(f) \in \mathsf{L}^{(2)}_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$.

Proof We leave this to the reader as Exercise 14.1.3. ■

14.1.18 Lemma ($\ell^1 \cap \ell^2$ is dense in ℓ^2) $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *is dense in* $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$.

Proof We leave this to the reader as Exercise 14.1.4. ■

From this we have the following result, whose proof we encourage the reader to look at, as it makes clear connections with our constructions of Section 12.3.

14.1.19 Theorem (Plancherel's Theorem for the DCFT) *There exists a unique continuous linear map* $\tilde{\mathscr{F}}_{\mathrm{DC}} \colon \ell^2(\mathbb{Z}(\Delta);\mathbb{C}) \to \mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ *with the properties*

(i) $\tilde{\mathscr{F}}_{\mathrm{DC}}(f) = \mathscr{F}_{\mathrm{DC}}(f)$ *for* $f \in \ell^1(\mathbb{Z}(\Delta;\mathbb{C}) \cap \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *and*

(ii) $\|\tilde{\mathscr{F}}_{\mathrm{DC}}(f)\|_2 = \|f\|_2$ *(**Parseval's equality** or **Plancherel's equality**).*

Furthermore, if $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *and if* $(f_j)_{j\in\mathbb{Z}_{>0}}$ *is a sequence in* $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *for which* $\lim_{j\to\infty}\|f - f_j\|_2 = 0$, *then* $\lim_{j\to\infty}\|\tilde{\mathscr{F}}_{\mathrm{DC}}(f) - \mathscr{F}_{\mathrm{DC}}(f_j)\|_2 = 0$.

Proof We could adopt the same proof as Theorem 13.3.3, but give a somewhat more explicit proof, using facts about the L^2-CDFT.

For $j \in \mathbb{Z}_{>0}$ and $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$, define

$$f_j(n\Delta) = \begin{cases} f(n\Delta), & n \in \{-j, \dots, -1, 0, 1, \dots, j\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(f_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ (this is obvious) converging to f in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ (this is easy to prove, and furnishes a solution to Exercise 14.1.4. Note that

$$\mathscr{F}_{\mathrm{DC}}(f_j) = \Delta \sum_{n=-j}^{j} f(n\Delta)\mathrm{e}^{-2\pi i n\Delta\nu}.$$

In the proof we recall from Section 8.2.3 that the inner product on $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ is

$$\langle f, g\rangle_2 = \Delta \sum_{n\in\mathbb{Z}} f(n\Delta)\overline{g(n\Delta)}.$$

Let $\mathsf{E}_{-2\pi i n\Delta} \in \mathsf{L}^{(2)}_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ be defined by $\mathsf{E}_{-2\pi i n\Delta^{-1}}(\nu) = \mathrm{e}^{-2\pi i n\Delta^{-1}\nu}$. As in Theorem 12.3.3, $(\sqrt{\Delta}\mathsf{E}_{-2\pi i n\Delta^{-1}})_{n\in\mathbb{Z}}$ is a Hilbert basis for $(\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}), \langle\cdot,\cdot\rangle_2)$. Therefore, by Theorem 7.3.29 it follows that, if $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$, then the sequence $(\mathscr{F}_{\mathrm{DC}}(f_j))_{j\in\mathbb{Z}_{>0}}$ converges in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$. ■

The map $\tilde{\mathscr{F}}_{\mathrm{DC}}$ from the preceding theorem is an isomorphism of Hilbert spaces.

14.1.20 Theorem (The inverse of the ℓ^2-DCFT) *The map $\tilde{\mathscr{F}}_{\mathrm{DC}}$ is a Hilbert space isomorphism from $(\ell^2(\mathbb{Z}(\Delta), \mathbb{C}), \langle\cdot,\cdot\rangle_2)$ to $(\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}), \langle\cdot,\cdot\rangle_2)$ with inverse*

$$\mathscr{F}^{-1}_{\mathrm{DC}}(\mathrm{F})(\mathrm{n}\Delta) = \int_0^{\Delta^{-1}} \mathrm{F}(\nu)\mathrm{e}^{2\pi\mathrm{in}\Delta\nu}\,\mathrm{d}\nu.$$

Proof The map $\tilde{\mathscr{F}}_{\mathrm{DC}}$ assigning to $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ the resulting limit is an isomorphism by Corollary 7.3.35. That $\mathscr{F}^{-1}_{\mathrm{DC}}$ is as stated in the theorem follows since, by Theorem 7.3.29,

$$\begin{aligned} &\mathscr{F}_{\mathrm{DC}}(f) = \Delta\sum_{n\in\mathbb{Z}} f(n\Delta)\mathsf{E}_{-2\pi in\Delta} \\ \implies\quad & f(n\Delta) = \langle \mathscr{F}_{\mathrm{DC}}(f), \mathsf{E}_{-2\pi in\Delta^{-1}}\rangle_2 \\ \implies\quad & f(n\Delta) = \int_0^{\Delta^{-1}} \mathscr{F}_{\mathrm{DC}}(f)(\nu)\mathrm{e}^{2\pi in\Delta\nu}\,\mathrm{d}\nu. \end{aligned}$$

That $\tilde{\mathscr{F}}_{\mathrm{DC}}$ is a Hilbert space isomorphism follows just as does the proof of the similar statement in Theorem 12.3.8. ■

Of course, we shall not use the symbol $\tilde{\mathscr{F}}_{\mathrm{DC}}$ for the ℓ^2-DCFT, but shall use $\mathscr{F}_{\mathrm{DC}}$ instead, the exact meaning of this symbol being clear from context.

Note that the fact that the DCFT cannot be computed directly for signals in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ is analogous to the fact that the CCFT cannot be computed directly for signals in $\mathsf{L}^{(2)}(\mathbb{R};\mathbb{C})$. For the L^2-CCFT we followed a procedure which led us to the fact that the limit

$$\lim_{T\to\infty}\int_{-T}^{T} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t$$

exists in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$, so defining the L^2-CCFT of $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. This is analogous to the situation in the preceding theorem where we define the ℓ^2-DCFT of $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ as the limit

$$\lim_{N\to\infty}\Delta\sum_{n=-N}^{N} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$. We refer to Section 12.3 for a discussion of the pointwise convergence properties of this limit.

Let us consider some examples illustrating the issues involved with the ℓ^2-DCFT.

14.1.21 Examples (The ℓ^2-DCFT)

1. Let us consider the discrete square wave f from Example 14.1.3–3. Note that $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \subseteq \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$. Thus the ℓ^2-DCFT of f is “the same” as the ℓ^1-CDFT:

$$\mathscr{F}_{\mathrm{DC}}(f)(\nu) = \Delta D^{\mathrm{per}}_{1,N}(t).$$

Note, however, when dealing with the ℓ^2-DCFT, we are really thinking of the DCFT as being the equivalence class in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ containing the ℓ^1-CDFT.

2. Here we consider $f\colon \mathbb{Z} \to \mathbb{C}$ defined by

$$f(n) = \begin{cases} 0, & n = 0, \\ \mathrm{i}\frac{1-(-1)^n}{n\pi}, & \text{otherwise}. \end{cases}$$

Note that $f \in \ell^2(\mathbb{Z};\mathbb{C})$ but that $f \notin \ell^1(\mathbb{Z};\mathbb{C})$. Using our Fourier series computations from Example 12.2.30–1 we have

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \mathrm{i}\frac{1-(-1)^n}{n\pi}\mathrm{e}^{-2\pi\mathrm{i}n\nu} = \lim_{N\to\infty} \sum_{n=-N}^{N} \mathrm{i}\frac{(-1)^n-1}{n\pi}\mathrm{e}^{2\pi\mathrm{i}n\nu} = \begin{cases} 0, & \nu \in \{0, \frac{1}{2}, 1\}, \\ 1, & \nu \in (0, \frac{1}{2}), \\ -1, & \nu \in (\frac{1}{2}, 1), \end{cases}$$

with the value of $\mathscr{F}_{\mathrm{DC}}(f)$ being defined for all frequencies by periodic extension. Thus, in this case, the series defining $\mathscr{F}_{\mathrm{DC}}(f) \in \mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ converges for every $\nu \in \mathbb{R}$. Nonetheless, one should be careful to understand that Theorem 14.1.19 gives convergence in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, not pointwise convergence.

3. The final example we consider is the signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$f(n\Delta) = \begin{cases} 0, & n = 0, \\ \frac{1}{n}, & \text{otherwise}. \end{cases}$$

Note that $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ but that $f \notin \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. Note that at $\nu = 0$ the limit

$$\lim_{N\to\infty} \Delta \sum_{n=-N}^{N} \frac{1}{n}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

does not exist. Nonetheless, Theorem 14.1.19 ensures that the limit

$$\lim_{N\to\infty} \Delta \sum_{n=-N}^{N} \frac{1}{N}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

exists in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$.***missing stuff*** •

Let us provide the results regarding convolution and multiplication for the DCFT in the ℓ^2-case. As with the CCFT, for the ℓ^2-DCFT we have to modify our expectations for what we can say about the DCFT of a convolution. Indeed, by Corollary 11.2.36 we have that $f * g \in \ell^\infty(\mathbb{Z}(\Delta);\mathbb{C})$ and the DCFT of signals in $\ell^\infty(\mathbb{Z}(\Delta);\mathbb{C})$ is not generally defined. The result that we can state is the following.

14.1.22 Proposition (The convolution of ℓ^2 is the inverse DCFT of the product of the ℓ^2-DCFT's) *If* $\mathrm{f}, \mathrm{g} \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ *then*

$$\mathrm{f} * \mathrm{g}(\mathrm{n}\Delta) = \mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(\mathrm{f})\mathscr{F}_{\mathrm{DC}}(\mathrm{g}))(\mathrm{n}\Delta)$$

for every $\mathrm{n} \in \mathbb{Z}$.

Proof Define

$$f_j(n\Delta) = \begin{cases} f(n\Delta), & n \in \{-j,\dots,-1,0,1,\dots,j\}, \\ 0, & \text{otherwise}, \end{cases}$$
$$g_j(n\Delta) = \begin{cases} g(n\Delta), & n \in \{-j,\dots,-1,0,1,\dots,j\}, \\ 0, & \text{otherwise}. \end{cases}$$

As the reader might verify en route to doing Exercise 14.1.4, the sequences $(f_j)_{j\in\mathbb{Z}_{>0}}$ and $(g_j)_{j\in\mathbb{Z}_{>0}}$ are in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell 2(\mathbb{Z}(\Delta);\mathbb{C})$ and converge in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ to f and g, respectively. Moreover, since f_j and g_j have finite support, by Proposition 11.1.25 it follows that $f_j * g_j$ has finite support for each $j \in \mathbb{Z}_{>0}$. Thus $f_j * g_j \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \cap \ell 2(\mathbb{Z}(\Delta);\mathbb{C})$. Then, according to Proposition 14.1.12,

$$\begin{aligned} & \mathscr{F}_{\mathrm{DC}}(f_j * g_j) = \mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j) \\ \Longrightarrow \quad & f_j * g_j = \mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)) \qquad j \in \mathbb{Z}_{>0}, \end{aligned}$$

because $f_j * g_j \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ and since $\mathscr{F}_{\mathrm{DC}}^{-1} \circ \mathscr{F}_{\mathrm{DC}}$ is the identity on $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ by Theorem 14.1.20.

By ***missing stuff*** the sequence $((f_j, g_j))_{j\in\mathbb{Z}_{>0}}$ converges to (f, g) in the product topology on $\ell^2(\mathbb{Z}(\Delta);\mathbb{C}) \times \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$. By Corollary 11.2.36 the sequence $(f_j * g_j)_{j\in\mathbb{Z}_{>0}}$ converges to $f * g$ using the ∞-norm.

We claim that $(\mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)))_{j\in\mathbb{Z}_{>0}}$ converges to $\mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{CC}}(g))$ in the ∞-norm. Indeed, we have

$$\begin{aligned} \|\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g) - \mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)\|_1 &\le \|\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g) - \mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g)\|_2 \\ &\quad + \|\mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g) - \mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)\|_2 \\ &\le \|\mathscr{F}_{\mathrm{DC}}(f) - \mathscr{F}_{\mathrm{DC}}(f_j)\|_2 \|\mathscr{F}_{\mathrm{DC}}(g)\|_2 + \|\mathscr{F}_{\mathrm{DC}}(f_j)\| \|\mathscr{F}_{\mathrm{DC}}(g) - \mathscr{F}_{\mathrm{DC}}(g_j)\|_2 \\ &= \|f - f_j\|_2 \|g\|_2 + \|f_j\| \|g - g_j\|_2 \end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality and Parseval's equality. Thus

$$\lim_{j\to\infty} \|\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g) - \mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)\|_1 = 0.$$

By Corollary 12.1.10 (applied to $\mathscr{F}_{\mathrm{DC}}^{-1}$ rather than $\mathscr{F}_{\mathrm{CD}}$) it then follows that $(\mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)))_{j\in\mathbb{Z}_{>0}}$ to $\mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g))$ in the ∞-norm, as desired.

Thus we have

$$\lim_{j\to\infty} f_j g_j = fg, \quad \lim_{j\to\infty} \mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f_j)\mathscr{F}_{\mathrm{DC}}(g_j)) = \mathscr{F}_{\mathrm{DC}}^{-1}(\mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g)),$$

with both limits being with respect to the ∞-norm. From this the result follows. ■

14.1.23 Proposition (The DCFT of a product is the convolution of the DCFT's) *If* $\mathsf{f}, \mathsf{g} \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ *and if* $\mathscr{F}_{\mathrm{CD}}(\mathsf{f}), \mathscr{F}_{\mathrm{CD}}(\mathsf{g}) \in \ell^1(\mathbb{Z}(\mathrm{T}^{-1};\mathbb{C})$*, then*

$$\mathscr{F}_{\mathrm{DC}}(\mathsf{fg})(\nu) = \mathscr{F}_{\mathrm{DC}}(\mathsf{f}) * \mathscr{F}_{\mathrm{DC}}(\mathsf{g})(\nu), \qquad \nu \in \mathbb{R}.$$

Proof As in the proof of Proposition 14.1.22, let $(f_j)_{j\in\mathbb{Z}_{>0}}$ and $(g_j)_{j\in\mathbb{Z}_{>0}}$ be sequences of finitely supported signals in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})\cap\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ converging in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ to f and g, respectively. By Proposition 14.1.13 we have

$$\mathscr{F}_{\mathrm{DC}}(f_jg_j)(\nu) = \mathscr{F}_{\mathrm{DC}}(f_j) * \mathscr{F}_{\mathrm{DC}}(g_j)(\nu)$$

for every $j\in\mathbb{Z}_{>0}$ and for every $\nu\in\mathbb{R}$.

By continuity of the ℓ^2-DCFT it follows that the sequences $(\mathscr{F}_{\mathrm{DC}}(f_j))_{j\in\mathbb{Z}_{>0}}$ and $(\mathscr{F}_{\mathrm{DC}}(g_j))_{j\in\mathbb{Z}_{>0}}$ converge in $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ to $\mathscr{F}_{\mathrm{DC}}(f)$ and $\mathscr{F}_{\mathrm{DC}}(g)$, respectively. Thus, by ***missing stuff***, the sequence $((\mathscr{F}_{\mathrm{DC}}(f_j),\mathscr{F}_{\mathrm{DC}}(f_j)))_{j\in\mathbb{Z}_{>0}}$ converges in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})\times\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ to $(\mathscr{F}_{\mathrm{CC}}(f),\mathscr{F}_{\mathrm{CC}}(g))$ with the product topology. By Corollary 11.2.29 it follows that the sequence $(\mathscr{F}_{\mathrm{DC}}(f_j) * \mathscr{F}_{\mathrm{DC}}(g_j))_{j\in\mathbb{Z}_{>0}}$ converges to $\mathscr{F}_{\mathrm{DC}}(f) * \mathscr{F}_{\mathrm{DC}}(g)$ uniformly.

We claim that the sequence $(\mathscr{F}_{\mathrm{DC}}(f_jg_j))$ converges uniformly to $\mathscr{F}_{\mathrm{DC}}(fg)$. Indeed, we have

$$\|fg-f_jg_j\|_1 \le \|fg-f_jg\|_2 + \|f_jg-f_jg_j\| \le \|f-f_j\|_2\|g\|_2 + \|f_j\|\|g-g_j\|_2,$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Thus

$$\lim_{j\to\infty}\|fg-f_jg_j\|_1 = 0.$$

By Theorem 14.1.7 it then follows that $(\mathscr{F}_{\mathrm{DC}}(f_jg_j))_{j\in\mathbb{Z}_{>0}}$ converges uniformly to $\mathscr{F}_{\mathrm{CC}}(fg)$, as desired.

Thus

$$\lim_{j\to\infty}\mathscr{F}_{\mathrm{DC}}(f_jg_j) = \mathscr{F}_{\mathrm{DC}}(fg),\quad \lim_{j\to\infty}\mathscr{F}_{\mathrm{DC}}(f_j) * \mathscr{F}_{\mathrm{DC}}(g_j) = \mathscr{F}_{\mathrm{DC}}(f)\mathscr{F}_{\mathrm{DC}}(g),$$

with convergence being uniform in each case. This gives the result. ■

14.1.7 The DCFT for signals of slow growth

Having now considered the DCFT for signals in $\ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ and $\ell^2(\mathbb{Z}(\Delta);\mathbb{C})$, classes of signals that decay to zero at infinity, we now turn to more general classes of signals that do not necessarily decay to zero at infinity. Of course, in such cases the direct definition of Definition 14.1.1 is hopeless. For those who have been following along carefully, it will come as no surprise to see that we circumvent the difficulties with a direct definition using distribution theory.

We begin by restricting ourselves to a class of signals that do not grow too quickly at infinity. The reader will notice obvious similarities between the presentation here and the presentation in Section **??** of the CDFT for periodic distributions.

14.1.24 Definition (Signal of slow growth) A signal $f\colon\mathbb{Z}(\Delta)\to\mathbb{C}$ is a ***signal of slow growth*** if there exists $M\in\mathbb{R}_{>0}$ and $k\in\mathbb{Z}_{\ge0}$ such that $|f(n\Delta)|\le M|n|^k$ for every $n\in\mathbb{Z}$. Let us denote by $\mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ the set of signals of slow growth. •

14.1.25 Remark (Signals of slow growth form a vector space) It is easily verified that $\mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ is a $\mathbb{C}$-vector space; the reader may verify this as Exercise 14.1.6. •

The fundamental result in this section is then the following. We use the notation that, if $f \in \mathsf{L}^{(1)}_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, then $\theta_f \in \mathscr{D}'_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ denotes the Δ^{-1}-periodic distribution as in Example 10.7.12–1.

14.1.26 Theorem (DCFT for signals of slow growth) *The map $\mathscr{F}_{\mathrm{DC}}$ which assigns to $\mathrm{f} \in \mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ the periodic distribution*

$$\mathscr{F}_{\mathrm{DC}}(\mathrm{f}) = \Delta \sum_{\mathrm{n}\in\mathbb{Z}} \mathrm{f}(\mathrm{n}\Delta)\theta_{\mathsf{E}_{-2\pi\mathrm{in}\Delta}}$$

is an isomorphism of $\mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ with $\mathscr{D}'_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$. Moreover, the inverse of $\mathscr{F}_{\mathrm{DC}}$ is

$$\mathscr{F}^{-1}_{\mathrm{DC}}(\Theta)(\mathrm{n}\Delta) = \Theta(\mathsf{E}_{2\pi\mathrm{in}\Delta}).$$

Proof Let $\sigma^* f(t) = f(-t)$; it is evident that $\sigma^* f \in \mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$. By Theorem **??** we have

$$\mathscr{F}_{\mathrm{DC}}(f) = \Delta \sum_{n\in\mathbb{Z}} f(n\Delta)\theta_{\mathsf{E}_{-2\pi in\Delta}} = \Delta \sum_{n\in\mathbb{Z}} \sigma^* f(n\Delta)\theta_{\mathsf{E}_{2\pi in\Delta}} \in \mathscr{D}'_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}).$$

Thus the $\mathscr{F}_{\mathrm{DC}}$ as written indeed has $\mathscr{D}'_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ as its codomain. By Corollary **??** it also follows that $\mathscr{F}_{\mathrm{DC}}$ is an isomorphism.

Let us define $\mathscr{F}^{-1}_{\mathrm{DC}}$ as stated in the theorem, and show that this is indeed the inverse of $\mathscr{F}_{\mathrm{DC}}$ (thereby introducing an abuse of notation whose temporal support is so small as to be negligible). By Corollary **??**

$$\begin{aligned}\mathscr{F}^{-1}_{\mathrm{DC}} \circ \mathscr{F}_{\mathrm{DC}}(f)(m\Delta) &= \Big(\Delta \sum_{n\in\mathbb{Z}} f(-n\Delta)\theta_{\mathsf{E}_{2\pi in\Delta}}\Big)(\mathsf{E}_{2\pi im\Delta}) \\ &= \Big(\Delta \sum_{n\in\mathbb{Z}} f(n\Delta)\theta_{\mathsf{E}_{-2\pi in\Delta}}\Big)(\mathsf{E}_{2\pi im\Delta}) = f(m\Delta),\end{aligned}$$

using the fact that

$$\theta_{\mathsf{E}_{-2\pi in\Delta}}(\mathsf{E}_{2\pi im\Delta}) = \int_0^{\Delta^{-1}} \mathrm{e}^{-2\pi in\Delta\nu}\mathrm{e}^{2\pi im\Delta\nu}\,\mathrm{d}\nu = \Delta^{-1}$$

by Lemma 12.3.2. Thus $\mathscr{F}^{-1}_{\mathrm{DC}}$ is a left-inverse for $\mathscr{F}_{\mathrm{DC}}$. Moreover, by Corollary **??** we also compute

$$\mathscr{F}_{\mathrm{DC}} \circ \mathscr{F}^{-1}_{\mathrm{DC}}(\Theta) = \Delta \sum_{n\in\mathbb{Z}} \Theta(\mathsf{E}_{2\pi in\Delta})\theta_{\mathsf{E}_{-2\pi in\Delta}} = \Delta \sum_{n\in\mathbb{Z}} \Theta(\mathsf{E}_{-2\pi in\Delta})\theta_{\mathsf{E}_{2\pi in\Delta}} = \Theta$$

for $\Theta \in \mathscr{D}'_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$. Thus $\mathscr{F}^{-1}_{\mathrm{DC}}$ is also a right-inverse for $\mathscr{F}_{\mathrm{DC}}$, and so an inverse. ■

Let us give some examples of the DCFT for signals of slow growth.

14.1.27 Examples (DCFT for signals of slow growth)

1. Let us take $f \in \mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ to be defined by $f(n\Delta) = 1$ for every $n \in \mathbb{Z}$. We have

$$\mathscr{F}_{\mathrm{DC}}(f) = \Delta \sum_{n\in\mathbb{Z}} \theta_{\mathsf{E}_{-2\pi i n\Delta}} = \sum_{n\in\mathbb{Z}} \delta_{n\Delta^{-1}},$$

using Example **??–??**. To verify the validity of this expression to oneself, it suffices to verify its sensibility when evaluated on functions $\psi \in \mathscr{D}_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$. Such functions can be written as

$$\psi(\nu) = \Delta \sum_{m\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(\psi)(m\Delta)\mathrm{e}^{2\pi \mathrm{i} m\Delta\nu}.$$

We then have

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}} \theta_{\mathsf{E}_{-2\pi i n\Delta}}\Big)(\psi) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \Delta\mathscr{F}_{\mathrm{CD}}(\psi)(n\Delta)\theta_{\mathsf{E}_{-2\pi i n\Delta}}(\mathsf{E}_{2\pi \mathrm{i} m\Delta}) \\ &= \sum_{n\in\mathbb{Z}_{>0}} \mathscr{F}_{\mathrm{CD}}(\psi)(n\Delta) \\ &= \sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(\psi)(n\Delta)\Big(\sum_{m\in\mathbb{Z}} \delta_{m\Delta^{-1}}(\theta_{\mathsf{E}_{2\pi \mathrm{i} n\Delta\nu}}\Big) \\ &= \Big(\sum_{m\in\mathbb{Z}} \delta_{m\Delta^{-1}}\Big)(\psi),\end{aligned}$$

verifying the desired equality.

2.

14.1.8 The DCFT for general signals

We now carry the results of the preceding section one step further to allow the taking of the DCFT for an arbitrary signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$.

Exercises

14.1.1 In Table 14.1.1 are given plots of three discrete-time signals defined on $\mathbb{Z}$ and their DCFT's. You are not told which signal goes with which DCFT. Without doing any computations, indicate which signal in the left column goes with which DCFT in the right column.

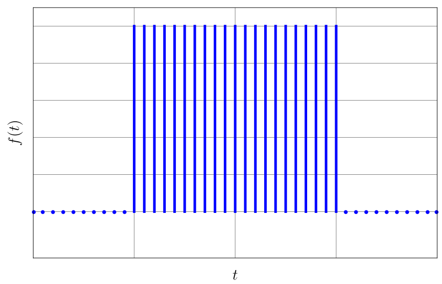
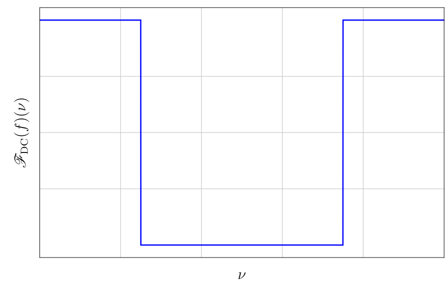
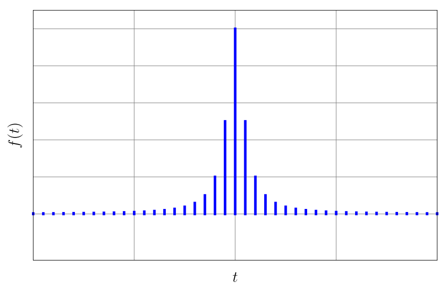
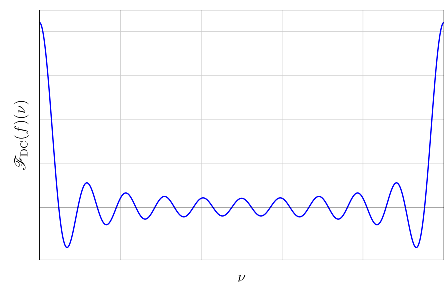
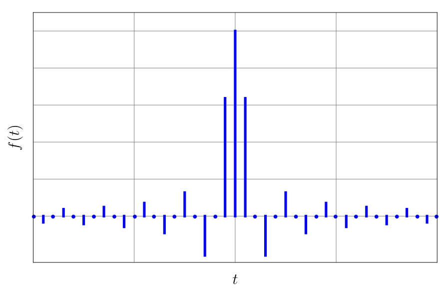
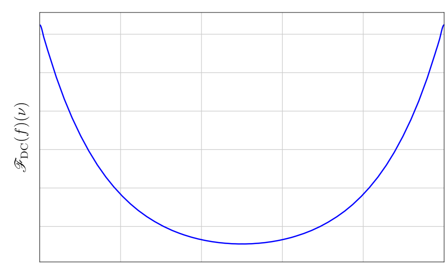
14.1.2 Prove Proposition 14.1.9.

14.1.3 Prove Lemma 14.1.17.

14.1.4 Prove Lemma 14.1.18.

14.1.5 Find a signal $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ such that the signal $\mathscr{F}_{\mathrm{DC}}(f)$ is not continuous.

14.1.6 Show that $\mathsf{S}(\mathbb{Z}(\Delta);\mathbb{C})$ is a $\mathbb{C}$-vector space.

	Graphs of signals		Graphs of DCFT's
1.		**1.**	
2.		**2.**	
3.		**3.**	

Section 14.2

The discrete-discrete Fourier transform

The last of the four Fourier transforms we discuss is the simplest, the most easily computed, and the most widely used in practice. Indeed, were one to adopt an excessively simplistic point of view, one might assert that the *only* Fourier transform worth learning in the DDFT. However, the fact remains that, even though one might use the DDFT in practice, one often uses it as an approximation of what one really wishes to compute, typically the CCFT. Thus, the flipside of the usefulness of the DDFT is the fact that it is often a mere computational approximation to what one wants to do. In this way, perhaps the DDFT is the one transform that one should *not* learn.

The above polemical remarks aside, the DDFT is best understood as an extremely useful computational tool which is best appreciated in relation to the other Fourier transforms. In this section we study the DDFT outright, leaving to Section **??** the comparisons with the other transforms.

Do I need to read this section? If you are learning Fourier transform theory, and particularly if you ever plan to use the Fourier transform in a non-textbook setting, then you must read this section. •

14.2.1 The DDFT signal spaces

The signals we consider in this section are the discrete-time periodic signals as described in Section 8.2.4. The two spaces of interest are

$$\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C}) = \{f\colon \mathbb{Z}(\Delta)\to\mathbb{C} \mid\ f((n+N)\Delta) = f(n\Delta) \text{ for all } n\in\mathbb{Z}\}$$

in the time-domain and

$$\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C}) = \{f\colon \mathbb{Z}((N\Delta^{-1}))\to\mathbb{C} \mid\ f(\tfrac{n+N}{N\Delta}) = f(\tfrac{n}{N\Delta}) \text{ for all } n\in\mathbb{Z}\}$$

in the frequency-domain, both defined for some $N\in\mathbb{Z}_{>0}$. In particular, we saw in Section 8.2.4 that the spaces we are dealing with above are finite-dimensional. Therefore, unlike for all of our other Fourier transforms, one does not have to discriminate between spaces with various summability properties, i.e., the ℓ^p spaces.

We shall make use in this section of inner products on these two finite-dimensional vector spaces. The inner products we use are those derived from the 2-norms from Section 8.2.3. Let us recall these here. On $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta;\mathbb{C})$ we use the inner product

$$\langle f,g\rangle_{\mathrm{time}} = \Delta\sum_{n=0}^{N-1} f(n\Delta)\overline{g(n\Delta)}$$

and on $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ we use the inner product

$$\langle F, G\rangle_{\mathrm{freq}} = (N\Delta)^{-1}\sum_{n=0}^{N-1} F(n(N\Delta)^{-1})\overline{G(n(N\Delta)^{-1})}.$$

Let us record orthonormal bases for these spaces with these inner products.

14.2.1 Lemma (Orthonormal bases for the DDFT time-dimain) *For $\Delta \in \mathbb{R}_{>0}$ and $\mathrm{N} \in \mathbb{Z}_{>0}$, define $\mathrm{e}_0, \dots, \mathrm{e}_{\mathrm{N}-1}, \mathrm{E}_0, \dots, \mathrm{E}_{\mathrm{N}-1} \in \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta;\mathbb{C})$ by defining*

$$\mathrm{e}_{\mathrm{j}}(\mathrm{n}\Delta) = \begin{cases} 1, & \mathrm{j} = \mathrm{n}, \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\mathrm{E}_{\mathrm{j}}(\mathrm{n}\Delta) = \mathrm{e}^{2\pi\mathrm{i}\frac{\mathrm{n}}{\mathrm{N}}\mathrm{j}}$$

for $\mathrm{j}, \mathrm{n} \in \{0, 1, \dots, \mathrm{N}-1\}$, and then defining then on all of $\mathbb{Z}(\Delta)$ so that they are $\mathrm{N}\Delta$-periodic. Then the sets $\{\Delta^{-1/2}\mathrm{e}_0, \dots, \Delta^{-1/2}\mathrm{e}_{\mathrm{N}-1}\}$ and $\{(\mathrm{N}\Delta)^{-1/2}\mathrm{E}_0, \dots, (\mathrm{N}\Delta)^{-1/2}\mathrm{E}_{\mathrm{N}-1}\}$ are orthonormal bases for $(\ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C}), \langle\cdot,\cdot\rangle_{\mathrm{time}})$.

Proof We have

$$\langle e_j, e_k\rangle_{\mathrm{time}} = \Delta\sum_{n=0}^{N-1} e_j(n)\overline{e_k(n)} = \begin{cases} \Delta, & j = k, \\ 0, & \text{otherwise}, \end{cases}$$

which gives the result for $\{\Delta^{-1/2}e_0, \dots, \Delta^{-1/2}e_{N-1}\}$. For $j, k \in \{0, \dots, N-1\}$ with $j > k$ we have

$$\Delta^{-1}\langle E_j, E_k\rangle_{\mathrm{time}} = \sum_{n=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\frac{n}{N}j}\mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}k} = \sum_{n=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\frac{n}{N}(j-k)} = \sum_{n=0}^{N-1} (\mathrm{e}^{2\pi\mathrm{i}\frac{1}{N}})^{n(j-k)}.$$

Note that $\mathrm{e}^{2\pi\mathrm{i}\frac{1}{N}}$ is a primitive Nth root of unity. By Proposition **??**, since $j - k \in \{1, \dots, N-1\}$,

$$\langle E_j, E_k\rangle_{\mathrm{time}} = \Delta\sum_{n=0}^{N-1} (\mathrm{e}^{2\pi\mathrm{i}\frac{1}{N}})^{n(j-k)} = 0.$$

Similarly, for $k > j$ we have $\langle E_j, E_k\rangle_{\mathrm{time}} = 0$. Finally, if $j = k$ we immediately have $\langle E_j, E_k\rangle_{\mathrm{time}} = N\Delta$, and the result now follows. ■

The following lemma records the corresponding result for the frequency-domain of the DDFT.

14.2.2 Lemma (Orthonormal bases for the DDFT frequency-dimain) *For $\Delta \in \mathbb{R}_{>0}$ and $\mathrm{N} \in \mathbb{Z}_{>0}$, define $\mathrm{f}_0, \dots, \mathrm{f}_{\mathrm{N}-1}, \mathrm{F}_0, \dots, \mathrm{F}_{\mathrm{N}-1} \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((\mathrm{N}\Delta)^{-1};\mathbb{C})$ by defining*

$$\mathrm{f}_{\mathrm{j}}(\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \begin{cases} 1, & \mathrm{j} = \mathrm{n}, \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\mathrm{F}_{\mathrm{j}}(\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \mathrm{e}^{2\pi\mathrm{i}\frac{\mathrm{n}}{\mathrm{N}}\mathrm{j}}$$

for $\mathrm{j}, \mathrm{n} \in \{0, 1, \dots, \mathrm{N}-1\}$, and then defining then on all of $\mathbb{Z}((\mathrm{N}\Delta)^{-1})$ so that they are Δ^{-1}-periodic. Then the sets $\{(\mathrm{N}\Delta)^{1/2}\mathrm{f}_0, \dots, (\mathrm{N}\Delta)^{1/2}\mathrm{f}_{\mathrm{N}-1}\}$ and $\{\Delta^{1/2}\mathrm{F}_0, \dots, \Delta^{1/2}\mathrm{F}_{\mathrm{N}-1}\}$ are orthonormal bases for $(\ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C}), \langle\cdot,\cdot\rangle_{\mathrm{freq}})$.

14.2.2 Definition of the DDFT

We refer to Section 9.6.3 for motivational remarks concerning the following definition.

14.2.3 Definition (DDFT) The ***discrete-discrete Fourier transform*** or ***DDFT*** assigns to $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{F}_{\mathrm{DD}}(f) \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\mathscr{F}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}) = \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n}, \qquad k \in \mathbb{Z}. \qquad \bullet$$

14.2.4 Remarks (Comments on the definition of the DDFT)

1. It is perhaps not completely trivial that the DDFT takes values in $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{R})$. This is proved in Proposition 14.2.8 below.
2. What we above call the DDFT is most commonly referred to as the "discrete Fourier transform," for more or less obvious reasons. However, we shall keep to "DDFT" to preserve the rationale of our naming conventions.
3. As there is no discrimination, as with the other Fourier transforms, with summability properties with spaces of periodic discrete-time signals, we do not have an "ℓ^1-DDFT," nor shall we have an "ℓ^2-DDFT." This results because the domain and codomain of the DDFT are both finite-dimensional, indeed having the same dimension. This is one of the results of the comparative simplification of the DDFT relative to the other Fourier transforms.
4. As we have always done in dealing with the Fourier transform, we shall regard $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{R})$ and $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{R})$ as subspaces of $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ and $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$.
5. Sometimes the domain and codomain of the DDFT is taken to be $\{0, 1, \dots, N-1\}$. This is reasonable, given that the signal spaces forming the domain and codomain have dimension N. $\bullet$

Let us give some examples where we can explicitly compute the DDFT. Unlike the other Fourier transforms, the simplicity of the DDFT allows us to give "closed form" expressions (in the form of finite sums) for the DDFT of any signal.

14.2.5 Examples (Computing the DDFT)

1. We first consider the periodic extension of a pulse. Thus we take $\mathsf{P}_{\mathrm{per}} \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ to be defined by

$$\mathsf{P}_{\mathrm{per}}(t) = \begin{cases} 1, & t = kN\Delta,\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In this case we simply have

$$\mathscr{F}_{\mathrm{DD}}(\mathsf{P}_{\mathrm{per}})(k(N\Delta)^{-1}) = \Delta\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}}$$

for each $k \in \mathbb{Z}$.

2. Here we consider the discrete periodic square wave. We thus define $f \in \ell_{\mathrm{per},N}(\mathbb{Z};\mathbb{C})$ by defining it on $\{-\lceil\frac{N}{2}\rceil\Delta,\dots,-\Delta,0,\Delta,\dots,\lceil\frac{N}{2}\rceil\Delta\}$ (recall from the text following Definition 2.2.8 the definition of the ceiling function $\lceil\cdot\rceil$) by

$$f(t)=\begin{cases}1, & t\in\{-M\Delta,\dots,-\Delta,0,\Delta,\dots,M\Delta\},\\ 0, & \text{otherwise},\end{cases}$$

for some $M\in\{1,\dots,\lfloor\frac{N}{2}\rfloor\}$. We depict this signal in Figure 14.2. Here we have

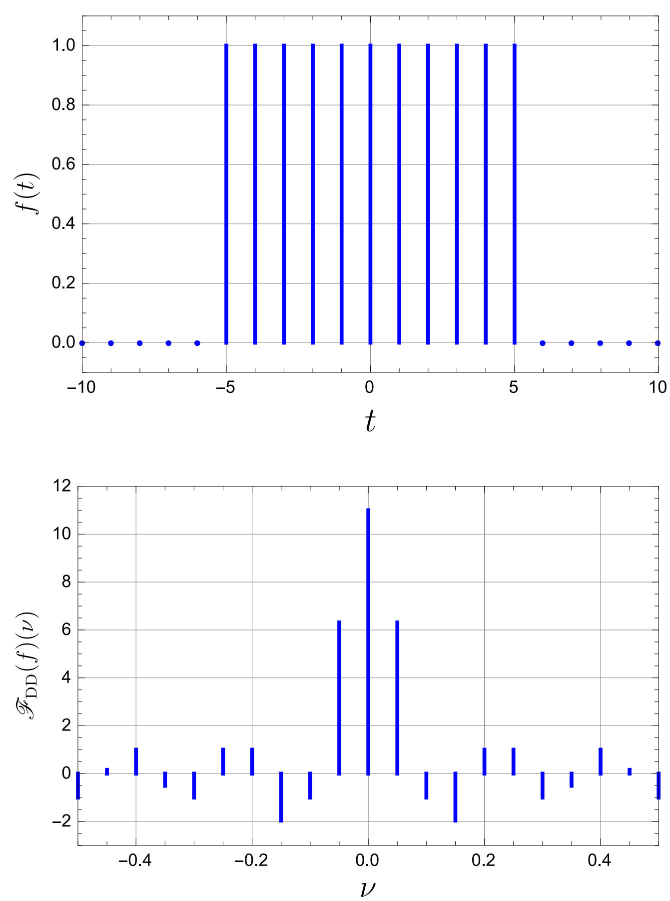


Figure 14.2 A discrete periodic square wave (top) and its DDFT (bottom) for $\Delta=1$, $N=20$, and $M=5$

$$\mathscr{F}_{\mathrm{DD}}(f)(k(N\Delta)^{-1})=\Delta\sum_{n=-M}^{M}\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n}=\Delta D^{\mathrm{per}}_{\Delta^{-1},M}(k(N\Delta)^{-1}),$$

where, by Lemma **??** from Example **??**, we have

$$D^{\mathrm{per}}_{N,M}(k(N\Delta)^{-1})=\begin{cases}\dfrac{\sin\!\Big((2M+1)\pi\frac{k}{N}\Big)}{\sin\!\Big(\pi\frac{k}{N}\Big)}, & t\neq 0,\\ 2M+1, & t=0.\end{cases}$$

We plot $\mathscr{F}_{\mathrm{DD}}(f)$ in Figure 14.2.

3. Finally we consider a discrete periodic triangular. Similarly with the preceding example, we define the function on $\{-\lceil\frac{N}{2}\rceil\Delta,\dots,-\Delta,0,\Delta,\dots,\lceil\frac{N}{2}\rceil\Delta\}$. Here we have

$$g(t)=\begin{cases}-\frac{t}{M\Delta}+1, & t\in\{0,\Delta,\dots,(M-1)\Delta\},\\ \frac{t}{M\Delta}+1, & t\in\{-(M-1)\Delta,\dots,-\Delta\},\\ 0, & \text{otherwise},\end{cases}$$

which we show in Figure 14.3. Here we take $M\in\{1,\dots,\lfloor\frac{N}{2}\rfloor\}$. We then compute,

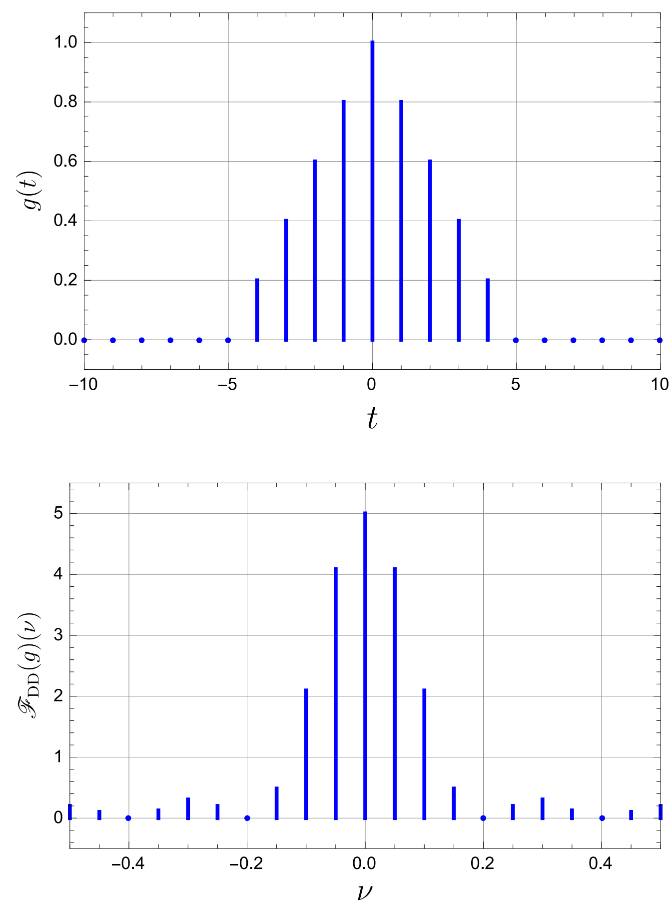


Figure 14.3 A discrete periodic triangular wave (top) and its DDFT (bottom) for $\Delta=1$, $N=20$, and $M=5$

using Lemma **??** from Example **??–??**,

$$\mathscr{F}_{\mathrm{DD}}(g)(k(N\Delta)^{-1})=\Delta\sum_{n=-M}^{M}g(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n}=\Delta F^{\mathrm{per}}_{\Delta^{-1},M}(\tfrac{k}{N})$$

where

$$F^{\mathrm{per}}_{\Delta^{-1},M}(k(N\Delta)^{-1}) = \begin{cases} \dfrac{1}{M}\dfrac{\sin^2\left(\pi M\frac{k}{N}\right)}{\sin^2(\pi\frac{k}{N})}, & t \neq 0, \\ M, & t = 0. \end{cases}$$

We plot this DDFT in Figure 14.3. •

As with our other Fourier transforms, there are sine and cosine versions of the DDFT.

14.2.6 Definition (DDCT and DDST)

(i) The ***discrete-discrete cosine transform*** or ***DDCT*** assigns to $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{C}_{\mathrm{DD}}(f) \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\mathscr{C}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}) = 2\Delta\sum_{n=0}^{N-1} f(n\Delta)\cos\left(2\pi\tfrac{k}{N}n\right), \qquad k \in \mathbb{Z}((N\Delta)^{-1}).$$

(ii) The ***discrete-discrete sine transform*** or ***DDST*** assigns to $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{C}_{\mathrm{DD}}(f) \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\mathscr{S}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}) = 2\Delta\sum_{n=0}^{N-1} f(n\Delta)\sin\left(2\pi\tfrac{k}{N}n\right), \qquad k \in \mathbb{Z}((N\Delta)^{-1}).$$

The DDFT is related to the DDCT and the DDST as follows.

14.2.7 Proposition (The DDFT, and the DDCT and the DDST) *For* $\mathrm{f} \in \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ *the following statements hold:*

(i) $\mathscr{F}_{\mathrm{DD}}(0) = \frac{1}{2}\mathscr{C}_{\mathrm{DD}}(\mathrm{f})(0)$*;*

(ii) $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \frac{1}{2}(\mathscr{C}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) - \mathrm{i}\mathscr{S}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}))$ *and* $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})(-\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \frac{1}{2}(\mathscr{C}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) + \mathrm{i}\mathscr{S}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}))$ *for every* $\mathrm{n} \in \mathbb{Z}_{>0}$*;*

(iii) $\mathscr{C}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) + \mathscr{F}_{\mathrm{DD}}(\mathrm{f})(-\mathrm{n}(\mathrm{N}\Delta)^{-1})$ *for every* $\mathrm{n} \in \mathbb{Z}_{\geq 0}$*;*

(iv) $\mathscr{S}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) = \mathrm{i}(\mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\mathrm{n}(\mathrm{N}\Delta)^{-1}) - \mathscr{F}_{\mathrm{DD}}(\mathrm{f})(-\mathrm{n}(\mathrm{N}\Delta)^{-1}))$ *for every* $\mathrm{n} \in \mathbb{Z}_{>0}$.

Proof This follows by direct computation using Euler's formula

$$\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n} = \cos\left(2\pi\tfrac{k}{N}n\right) + \mathrm{i}\sin\left(2\pi\tfrac{k}{N}n\right). \qquad \blacksquare$$

14.2.3 Properties of the DDFT

In this section we present the basic properties of the DDFT. We begin by verifying that the DDFT is a signal is indeed periodic with the period stated in Definition 14.2.3.

14.2.8 Proposition (The image of the DDFT consists of periodic signals) *If* $\mathrm{f} \in \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ *and if* $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})\colon \mathbb{Z}((\mathrm{N}\Delta)^{-1}) \to \mathbb{C}$ *is defined by*

$$\mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\tfrac{\mathrm{k}}{\mathrm{N}\Delta}) = \Delta \sum_{\mathrm{n}=0}^{\mathrm{N}-1} \mathrm{f}(\mathrm{n}\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{\mathrm{k}}{\mathrm{N}}\mathrm{n}}, \qquad \mathrm{k} \in \mathbb{Z},$$

then $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})$ *is periodic with period* Δ^{-1}*.*

Proof This follows simply because

$$\begin{aligned}\mathscr{F}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta} + \Delta^{-1}) = \Delta\mathscr{F}_{\mathrm{DD}}(f)(\tfrac{k+N}{N\Delta}) &= \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{k+N}{N}n} \\ &= \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n}\mathrm{e}^{-2\pi\mathrm{i}n} = \Delta\mathscr{F}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}),\end{aligned}$$

as desired. ■

The DDFT has the familiar properties with respect to transformations of the domain and codomain. To state these, we let $\sigma, \tau_a\colon \mathbb{Z}(\Delta) \to \mathbb{Z}(\Delta)$ be defined by $\sigma(t) = \sigma(-t)$ and $\tau_a(t) = t - a$ for $a \in \mathbb{Z}(\Delta)$. This then gives, in the usual way, $\sigma^* f(t) = f(-t)$ and $\tau_a^* f(t) = f(t-a)$ for $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$. We also define $\overline{\mathscr{F}}_{\mathrm{DD}}\colon \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C}) \to \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\overline{\mathscr{F}}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}) = \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n},$$

With all this notation, we have the following result whose proof consists of computations.

14.2.9 Proposition (Elementary properties of the DDFT) *For* $\mathrm{f} \in \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ *the following statements hold:*

(i) $\overline{\mathscr{F}_{\mathrm{DD}}(\mathrm{f})} = \overline{\mathscr{F}}_{\mathrm{DD}}(\bar{\mathrm{f}})$;

(ii) $\mathscr{F}_{\mathrm{DD}}(\sigma^*\mathrm{f}) = \sigma^*(\mathscr{F}_{\mathrm{DD}}(\mathrm{f})) = \overline{\mathscr{F}}_{\mathrm{DD}}(\mathrm{f})$;

(iii) *if* f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})$ *is even (resp. odd);*

(iv) *if* f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{DD}}(\mathrm{f})$ *is real and even (resp. imaginary and odd);*

(v) *if* $\mathrm{m} \in \mathbb{Z}$ *then* $\mathscr{F}_{\mathrm{DD}}(\tau_{\mathrm{m}\Delta}^*\mathrm{f})(\frac{\mathrm{k}}{\mathrm{N}\Delta}) = \mathrm{e}^{-2\pi\mathrm{i}\frac{\mathrm{k}}{\mathrm{N}}\mathrm{m}}\mathscr{F}_{\mathrm{DC}}(\mathrm{f})(\frac{\mathrm{k}}{\mathrm{N}\Delta})$.

Finally, we state the basic structure of the DDFT as a transformation of signal spaces.

14.2.10 Proposition (The DDFT is a linear map) *$\mathscr{F}_{\mathrm{DD}}$ is an isomorphism of the finite-dimensional $\mathbb{C}$-vector spaces $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ and $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$. Moreover, if we use the inner products from Section 14.2.1, then $\mathscr{F}_{\mathrm{DD}}$ is a mapping of the corresponding Hilbert spaces.*

Proof Linearity of $\mathscr{F}_{\mathrm{DD}}$ is a consequence of finite sums commuting with addition and scalar multiplication of terms in the sum. To verify that $\mathscr{F}_{\mathrm{DD}}$ is a mapping of the stated Hilbert spaces, we compute

$$\begin{aligned}\langle \mathscr{F}_{\mathrm{DD}}(f), \mathscr{F}_{\mathrm{DD}}(g)\rangle_{\mathrm{freq}} &= \frac{1}{N\Delta}\sum_{n=0}^{N-1} \mathscr{F}_{\mathrm{DD}}(f)(\tfrac{n}{N\Delta})\overline{\mathscr{F}_{\mathrm{DD}}(g)(\tfrac{n}{N\Delta})} \\ &= \frac{1}{N\Delta}\sum_{n=0}^{N-1}\Big(\Delta\sum_{j=0}^{N-1} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}j}\Big)\Big(\Delta\sum_{k=0}^{N-1}\overline{g(k\Delta)}\mathrm{e}^{2\pi\mathrm{i}\frac{n}{N}k}\Big) \\ &= \frac{\Delta}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} f(j\Delta)\overline{g(k\Delta)}\sum_{n=0}^{N-1}(\mathrm{e}^{2\pi\mathrm{i}\frac{1}{N}})^{n(k-j)}.\end{aligned}$$

By Proposition **??**, the inner sum is zero unless $j = k$, in which case the sum is equal to N. This then gives

$$\langle \mathscr{F}_{\mathrm{DD}}(f), \mathscr{F}_{\mathrm{DD}}(g)\rangle_{\mathrm{freq}} = \Delta\sum_{n=0}^{N-1} f(n\Delta)\overline{g(n\Delta)} = \langle f, g\rangle_{\mathrm{time}},$$

as desired. ■

As one might expect, there is a version of the Fourier Reciprocity Relation for the DDFT.

14.2.11 Proposition (Fourier Reciprocity Relation for the DDFT) *For $\mathrm{f},\mathrm{g} \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ we have*

$$\Delta\sum_{\mathrm{n}=0}^{\mathrm{N}-1} \mathrm{f}(\mathrm{n}\Delta)\mathrm{g}(\mathrm{n}\Delta) = \frac{1}{\mathrm{N}\Delta}\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta})\overline{\mathscr{F}}_{\mathrm{DD}}(\mathrm{g})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta}).$$

Proof This is left for the reader to prove in Exercise 14.2.4. ■

14.2.4 Convolution, multiplication, and the DDFT

Here we present the formulae relating the DDFT with convolution and multiplication.

14.2.12 Proposition (The DDFT of a convolution is the product of the DDFT's) *If $\mathrm{f},\mathrm{g} \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ then*

$$\mathscr{F}_{\mathrm{DD}}(\mathrm{f} * \mathrm{g})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta}) = \mathscr{F}_{\mathrm{DD}}(\mathrm{f})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta})\mathscr{F}_{\mathrm{DD}}(\mathrm{g})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta})$$

for every $\mathrm{n} \in \mathbb{Z}$.

Proof We compute

$$\begin{aligned}
\mathscr{F}_{\mathrm{DD}}(f * g)(\tfrac{n}{N\Delta}) &= \Delta \sum_{k=0}^{N-1} f * g(k\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} k} \\
&= \Delta \sum_{k=0}^{N-1} \Big(\Delta \sum_{j=0}^{N-1} f((k-j)\Delta) g(j\Delta) \Big) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} k} \\
&= \Delta^2 \sum_{m=0}^{N-1} \sum_{j=0}^{N-1} f(m\Delta) g(j\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N}(m+j)} \\
&= \Big(\Delta \sum_{m=0}^{N-1} f(m\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} m} \Big) \Big(\Delta \sum_{j=0}^{N-1} g(j\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} j} \Big) \\
&= \mathscr{F}_{\mathrm{DD}}(f)(\tfrac{n}{N\Delta}) \mathscr{F}_{\mathrm{DD}}(g)(\tfrac{n}{N\Delta}),
\end{aligned}$$

as desired. ■

14.2.13 Proposition (The DDFT of a product is the convolution of the DDFT's) *If* $\mathrm{f}, \mathrm{g} \in \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta); \mathbb{C})$ *then*

$$\mathscr{F}_{\mathrm{DD}}(\mathrm{fg})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta}) = \mathscr{F}_{\mathrm{DD}}(\mathrm{f}) * \mathscr{F}_{\mathrm{DD}}(\mathrm{g})(\tfrac{\mathrm{n}}{\mathrm{N}\Delta})$$

for every $\mathrm{n} \in \mathbb{Z}$.

Proof We compute

$$\begin{aligned}
\mathscr{F}_{\mathrm{DD}}(f) * \mathscr{F}_{\mathrm{DD}}(g)(\tfrac{n}{N\Delta}) &= \frac{1}{N\Delta} \sum_{k=0}^{N-1} \mathscr{F}_{\mathrm{DD}}(f)(\tfrac{n-k}{N\Delta}) \mathscr{F}_{\mathrm{DD}}(\tfrac{k}{N\Delta}) \\
&= \frac{1}{N\Delta} \sum_{k=0}^{N-1} \Big(\Delta \sum_{j=0}^{N-1} f(j\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n-k}{N} j} \Big) \Big(\Delta \sum_{m=0}^{N-1} g(m\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{k}{N} m} \Big) \\
&= \frac{\Delta}{N} \sum_{j=0}^{N-1} \sum_{m=0}^{N-1} f(j\Delta) g(m\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} j} \Big(\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \frac{k}{N} j} \mathrm{e}^{-2\pi \mathrm{i} \frac{k}{N} m} \Big) \\
&= \Delta \sum_{m=0}^{N-1} f(m\Delta) g(m\Delta) \mathrm{e}^{-2\pi \mathrm{i} \frac{n}{N} m} = \mathscr{F}_{\mathrm{DD}}(fg)(\tfrac{n}{N\Delta}),
\end{aligned}$$

using the fact that, since $\mathrm{e}^{2\pi \mathrm{i} \frac{1}{N}}$ is a primitive Nth root of unity,

$$\sum_{n=0}^{N-1} (\mathrm{e}^{2\pi \mathrm{i} \frac{1}{N}})^{n(k-j)} = \begin{cases} N, & j = k, \\ 0, & j \neq k, \end{cases}$$

cf. the proof of Lemma 14.2.1. ■

14.2.5 Inversion of the DDFT

The inversion of the DDFT is far less complicated than the other transforms. For example, it is immediate that $\mathscr{F}_{\mathrm{DD}}$ is injective since it is a mapping of inner product spaces cf. ***missing stuff***. Therefore, since it is also a mapping of finite-dimensional vector spaces of the same dimension, it is also an isomorphism by Corollary **??**. Thus $\mathscr{F}_{\mathrm{DD}}$ has an inverse, and it only remains to demonstrate it.

14.2.14 Theorem (Inverse of the DDFT) *The map* $\mathscr{F}_{\mathrm{DD}}\colon \ell_{\mathrm{per},\mathrm{N}\Delta}(\mathbb{Z}(\Delta);\mathbb{C}) \to \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((\mathrm{N}\Delta)^{-1});\mathbb{C})$ *is an isomorphism with inverse defined by*

$$\mathscr{F}_{\mathrm{DD}}^{-1}(\mathrm{F})(\mathrm{k}\Delta) = \frac{1}{\mathrm{N}\Delta}\sum_{\mathrm{n}=0}^{\mathrm{N}-1} \mathrm{F}(\mathrm{n}(\mathrm{N}\Delta)^{-1})\mathrm{e}^{2\pi\mathrm{i}\frac{\mathrm{k}}{\mathrm{N}}\mathrm{n}}.$$

Proof We compute

$$\begin{aligned}
\mathscr{F}_{\mathrm{DD}}^{-1}\circ\mathscr{F}_{\mathrm{DD}}(f)(k\Delta) &= \frac{1}{N\Delta}\sum_{n=0}^{N-1}\mathscr{F}_{\mathrm{DD}}(f)(n(N\Delta)^{-1})\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}\\
&= \frac{1}{N\Delta}\sum_{n=0}^{N-1}\Big(\Delta\sum_{j=0}^{N-1} f(j\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}j}\Big)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}\\
&= \frac{1}{N}\sum_{j=0}^{N-1} f(j\Delta)\sum_{n=0}^{N-1}(\mathrm{e}^{-2\pi\mathrm{i}\frac{1}{N}})^{n(k-j)}.
\end{aligned}$$

By Proposition **??**, the inner sum is zero unless $j = k$, in which case the sum is equal to N. This then gives

$$\mathscr{F}_{\mathrm{DD}}^{-1}\circ\mathscr{F}_{\mathrm{DD}}(f)(k\Delta) = f(k\Delta),$$

or $\mathscr{F}_{\mathrm{DD}}^{-1}\circ\mathscr{F}_{\mathrm{DD}}(f) = f$. Thus $\mathscr{F}_{\mathrm{DD}}^{-1}$ is a left-inverse for $\mathscr{F}_{\mathrm{DD}}$, and so an inverse by Corollary **??**. ■

14.2.6 The fast Fourier transform

In this section is provided the reason for the utility of the DDFT in practice. We provide a computational algorithm for computing the DDFT when the discrete time-domain has a cardinality that is a power of 2. In order to simplify notation, we toss out some of the "physical" structure of the DDFT resulting from the sampling time. Thus we think of the DDFT as being a map from $\mathbb{C}^N$ to $\mathbb{C}^N$, labelling points in $\mathbb{C}^N$ by $z = (z(0), z(1), \dots, z(N-1))$. The DDFT in this simplified framework is defined by a map $F_{\mathrm{DD}}\colon \mathbb{C}^N \to \mathbb{C}^N$ given by

$$F_{\mathrm{DD}}(z)(k) = \sum_{n=0}^{N-1} z(n)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}, \qquad k \in \{0, 1, \dots, N-1\}.$$

The relationship between this map and the DDFT is obviously trivial.

The first and crucial observation is the following lemma which states that, when N is even, F_{DD} can be computed via two similar computations, but on $\mathbb{C}^{N/2}$.

14.2.15 Lemma (Decimation step for the DDFT) *Let* $\mathrm{N} \in \mathbb{Z}_{>0}$ *be even and, for* $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^{\mathrm{N}/2}$, *define*

$$\mathbf{z} = (\mathbf{z}_1(0), \mathbf{z}_2(0), \mathbf{z}_1(1), \mathbf{z}_2(1), \dots, \mathbf{z}_1(\tfrac{\mathrm{N}}{2} - 1), \mathbf{z}_2(\tfrac{\mathrm{N}}{2} - 1)).$$

Then

$$\begin{aligned} \mathrm{F}_{\mathrm{DD}}(\mathbf{z})(\mathrm{k}) &= \tfrac{1}{2}(\mathrm{F}_{\mathrm{DD}}(\mathbf{z}_1)(\mathrm{k}) + \mathrm{e}^{-2\pi\mathrm{i}\frac{\mathrm{k}}{\mathrm{N}}}\mathrm{F}_{\mathrm{DD}}(\mathbf{z}_2)(\mathrm{k})), \\ \mathrm{F}_{\mathrm{DD}}(\mathbf{z})(\mathrm{k} + \tfrac{\mathrm{N}}{2}) &= \tfrac{1}{2}(\mathrm{F}_{\mathrm{DD}}(\mathbf{z}_1)(\mathrm{k}) - \mathrm{e}^{2\pi\mathrm{i}\frac{\mathrm{k}}{\mathrm{N}}}\mathrm{F}_{\mathrm{DD}}(\mathbf{z}_2)(\mathrm{k})) \end{aligned}$$

for $\mathrm{k} \in \{0, 1, \dots, \frac{\mathrm{N}}{2} - 1\}$.

Proof For $k \in \{0, 1, \dots, \frac{N}{2}\}$ we compute

$$\begin{aligned} F_{\mathrm{DD}}(z)(k) &= \sum_{n=0}^{N-1} z(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} z(2n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}(2n)} + \sum_{n=0}^{\frac{N}{2}-1} z(2n+1)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}(2n+1)} \\ &= \sum_{n=0}^{\frac{N}{2}-1} z_1(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N/2}n} + \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}} \sum_{n=0}^{\frac{N}{2}-1} z_2(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N/2}n} \\ &= F_{\mathrm{DD}}(z_1)(k) + \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}}F_{\mathrm{DD}}(z_2)(k) \end{aligned}$$

and

$$\begin{aligned} F_{\mathrm{DD}}(z)(k + \tfrac{N}{2}) &= \sum_{n=0}^{N-1} z(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k+\frac{N}{2}}{N}n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} z(2n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k+\frac{N}{2}}{N}(2n)} + \sum_{n=0}^{\frac{N}{2}-1} z(2n+1)\mathrm{e}^{-2\pi\mathrm{i}\frac{k+\frac{N}{2}}{N}(2n+1)} \\ &= \sum_{n=0}^{\frac{N}{2}-1} z_1(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N/2}n} - \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}} \sum_{n=0}^{\frac{N}{2}-1} z_2(n)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N/2}n} \\ &= F_{\mathrm{DD}}(z_1)(k) - \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}}F_{\mathrm{DD}}(z_2)(k), \end{aligned}$$

as desired. ■

Now, if $N = 2^r$ for some $r \in \mathbb{Z}_{>0}$, then we can repeat the above decimation process r times to compute the DDFT of $z \in \mathbb{C}^N$ by computing N DDFT's for $\mathbb{C}^1$, $\frac{N}{2}$ DDFT's for $\mathbb{C}^2$, $\frac{N}{4}$ DDFT's for $\mathbb{C}^4$, $\frac{N}{8}$ DDFT's for $\mathbb{C}^8$, and so on to the computation of two DDFT's for $\mathbb{C}^{N/2}$. Let us make this explicit.

14.2.16 Sequential construction for fast Fourier transform *To illustrate this, given*

$$\zeta_0^0, \zeta_1^0, \dots, \zeta_{2^{\mathrm{r}}-1}^0 \in \mathbb{C}$$

and given $\mathrm{j} \in \{1, \dots, \mathrm{r}\}$, *we define*

$$\zeta_0^{\mathrm{j}}, \zeta_1^{\mathrm{j}}, \dots, \zeta_{2^{\mathrm{r}-\mathrm{j}}-1}^{\mathrm{j}} \in \mathbb{C}^{2^{\mathrm{j}}}$$

iteratively according to the rule

$$\begin{aligned}\zeta_{\mathrm{m}}^{\mathrm{j}+1}(\mathrm{k}) &= \tfrac{1}{2}(\zeta_{2\mathrm{m}}^{\mathrm{j}}(\mathrm{k}) + \mathrm{e}^{-\pi\mathrm{i}\frac{\mathrm{k}}{2^{\mathrm{j}}}}\zeta_{2\mathrm{m}+1}^{\mathrm{j}}(\mathrm{k})),\\ \zeta_{\mathrm{m}}^{\mathrm{j}+1}(2^{\mathrm{j}}+\mathrm{k}) &= \tfrac{1}{2}(\zeta_{2\mathrm{m}}^{\mathrm{j}}(\mathrm{k}) - \mathrm{e}^{-\pi\mathrm{i}\frac{\mathrm{k}}{2^{\mathrm{j}}}}\zeta_{2\mathrm{m}+1}^{\mathrm{j}}(\mathrm{k})),\end{aligned}$$

for $\mathrm{k} \in \{0, \dots, 2^{\mathrm{j}} - 1\}$.

Next we relate the above construction to the DDFT. One of the difficulties, evident in Lemma 14.2.15, is that the order of the z gets rearranged at each step. In order to account for this in a systematic way, we define a map $\rho_r\colon \{0, 1, \dots, 2^r - 1\} \to 0, 1, \dots, 2^r - 1\}$ as follows. Given $k \in \{0, 1, \dots, 2^r - 1\}$ write

$$k = \sum_{j=0}^{r-1} d_j 2^j$$

for some uniquely defined $d_j \in \{0, 1\}$, $j \in \{0, 1, \dots, r-1\}$. This is simply the binary decimal expansion of k. We then define

$$\rho_r(k) = \sum_{j=0}^{r-1} d_{r-j-1} 2^j.$$

Thus $\rho_r(k)$ is the number whose binary decimal expansion is the reverse of that for k. Let us record some elementary properties of the bit reversal mapping.

14.2.17 Lemma (Properties of bit reversal) *For* $\mathrm{r} \in \mathbb{Z}_{>0}$ *the following statements hold:*

(i) ρ_{r} *is a bijection satisfying* $\rho_{\mathrm{r}} \circ \rho_{\mathrm{r}}(\mathrm{k}) = \mathrm{k}$ *for every* $\mathrm{k} \in \{0, 1, \dots, 2^{\mathrm{r}} - 1\}$*;*

(ii) $\rho_{\mathrm{r}}(\sum_{\mathrm{j}=0}^{\mathrm{r}-1} \mathrm{d}_{\mathrm{j}} 2^{\mathrm{j}}) = \sum_{\mathrm{j}=0}^{\mathrm{r}-1} \mathrm{b}_{\mathrm{j}} 2^{\mathrm{r}-\mathrm{j}-1}$*;*

(iii) it holds that

$$\begin{aligned}\rho_{\mathrm{r}+1}(2\mathrm{k}) &= \rho_{\mathrm{r}}(\mathrm{k}),\\ \rho_{\mathrm{r}+1}(2\mathrm{k}+1) &= 2^{\mathrm{r}} + \rho_{\mathrm{r}}(\mathrm{k})\end{aligned}$$

for $\mathrm{k} \in \{0, 1, \dots, 2^{\mathrm{r}} - 1\}$*;*

(iv) $\rho_{\mathrm{r}}(2^{\mathrm{j}} + \mathrm{k}) = \rho_{\mathrm{r}}(\mathrm{k}) + 2^{\mathrm{r}-\mathrm{j}-1}$ *for* $\mathrm{j} \in \{0, 1, \dots, \mathrm{r}-1\}$ *and* $\mathrm{k} \in \{0, 1, \dots, 2^{\mathrm{j}} - 1\}$.

Proof (i) It is clear that $\rho_r \circ \rho_r(k) = k$ for each $k \in \{0, 1, \dots, 2^r - 1\}$. From this it follows that ρ_r is a left- and right-inverse for itself, and so ρ_r is invertible.

(ii) This is just the definition of ρ_r with a change of index in the sum.

(iii) Let us write $k = \sum_{j=0}^{r-1} d_j 2^j$. Then

$$2k = \sum_{j=0}^{r-1} d_j 2^{j+1} = \sum_{j=0}^{r} d'_j 2^j,$$

where $d'_j = d_{j-1}$ for $j \in \{1, \dots, r\}$ and $d'_0 = 0$. Then

$$\rho_{r+1}(2k) = \sum_{j=0}^{r} d'_{r-j} 2^j = \sum_{j=0}^{r-1} d'_{r-j} 2^j = \sum_{j=0}^{r-1} d_{r-j-1} 2^j = \rho_r(k).$$

We also have

$$2k+1=\sum_{j=0}^{r-1} d_j 2^{j+1}+1^0=\sum_{j=0}^{r} d''_j 2^j,$$

where $d''_j=d_{j-1}$ for $j\in\{1,\dots,r\}$ and $d'_0=1$. Therefore,

$$\rho_{r+1}(2k+1)=\sum_{j=0}^{r} d''_{r-j}2^j=\sum_{j=0}^{r-1} d'_{r-j}+2^r=\sum_{j=0}^{r-1} d_{r-j-1}2^j+2^r=\rho_r(k)+2^r,$$

as desired.

(iv) For $j\in\{0,1,\dots,r-1\}$ and $k\in\{0,1,\dots,2^j-1\}$ write

$$k=\sum_{m=0}^{j-1} d_m 2^m$$

for unique $d_m\in\{0,1\}$, $m\in\{0,1,\dots,j-1\}$. Thus

$$k+2^j=\sum_{m=0}^{j-1} d_m 2^m+2^j=\sum_{m=0}^{r-1} d'_m 2^m,$$

where $d'_m=d_m$, $m\in\{0,1,\dots,j-1\}$, $d'_j=1$, and $d'_m=0$, $m\in\{j+1,\dots,r-1\}$. Therefore, using part (ii),

$$\rho_r(k+2^j)=\sum_{m=0}^{r-1} d'_m 2^{r-m-1}=\sum_{m=0}^{j-1} d_m 2^{r-m-1}+2^{r-j-1}=\rho_r(k)+2^{r-j-1},$$

as desired. ■

The point of the lemma is that one can compute the numbers $\rho_r(0),\rho_r(1),\dots,\rho_r(2^r-1)$ as follows:

$$\rho_r(0)=0,\ \rho_r(1)=\rho_r(0)+2^{r-1},\ \rho_r(2)=\rho_r(0)+2^{r-2},\\ \rho_r(3)=\rho_r(1)+2^{r-2},\rho_r(4)=\rho_r(0)+2^{r-3},\dots$$

Thus ρ can be computed using no floating point multiplications.

The following lemma clarifies the rôle of bit reversal in the computation of the DDFT.

14.2.18 Lemma (Use of bit reversal in DDFT decimation) *For* $\mathrm{j}\in\{0,1,\dots,\mathrm{r}\}$ *and for* $\mathrm{m}\in\{0,1,\dots,2^{\mathrm{r}-\mathrm{j}}-1\}$ *we have*

$$\zeta^{\mathrm{j}}_{\mathrm{m}}=\mathrm{F}_{\mathrm{DD}}(\zeta^0_{\mathrm{m}2^{\mathrm{j}}+\rho_{\mathrm{j}}(0)},\zeta^0_{\mathrm{m}2^{\mathrm{j}}+\rho_{\mathrm{j}}(1)},\dots,\zeta^0_{\mathrm{m}2^{\mathrm{j}}+\rho_{\mathrm{j}}(2^{\mathrm{j}}-1)}).$$

Proof We prove the lemma by induction on j. For $j=0$ we have

$$\zeta^0_m=F_{\mathrm{DD}}(z^0_m),$$

which indicates that the lemma holds in this case. Now suppose that the lemma holds for $j \in \{0, 1, \dots, k\}$. Then we calculate

$$\begin{aligned}\zeta_m^{k+1}(n) &= \tfrac{1}{2}(\zeta_{2m}^k(n) + \mathrm{e}^{-\pi\mathrm{i}\frac{n}{2^k}}\zeta_{2m+1}^k(n)) \\ &= \tfrac{1}{2}(F_{\mathrm{DD}}(\zeta^0_{2m2^k+\rho_k(0)}, \dots, \zeta^0_{2m2^k+\rho_k(2^k-1)}) \\ &\quad + \mathrm{e}^{-\pi\mathrm{i}\frac{n}{2^k}}F_{\mathrm{DD}}(\zeta^0_{(2m+1)2^k+\rho_k(0)}, \dots, \zeta^0_{(2m+1)2^k+\rho_k(2^k-1)}),\end{aligned}$$

using the definition ζ_m^{k+1} and the induction hypothesis. Now we use Lemma 14.2.15 to write

$$\zeta_m^{k+1}(n) = F_{\mathrm{DD}}(\zeta^0_{s_0}, \dots, \zeta^0_{s_{2^{k+1}-1}}),$$

where

$$\begin{aligned}s_{2n} &= m2^{k+1} + \rho_k(n), \\ s_{2n+1} &= m2^{k+1} + 2^k + \rho_k(n)\end{aligned}$$

for $n \in \{0, 1, \dots, 2^k - 1\}$. By Lemma 14.2.17(iii) we have

$$\begin{aligned}s_{2n} &= m2^{k+1} + \rho_{k+1}(2n), \\ s_{2n+1} &= m2^{k+1} + \rho_{k+1}(2n+1)\end{aligned}$$

for $n \in \{0, 1, \dots, 2^k - 1\}$. This gives $\zeta_m^{k+1}(n)$ is the form asserted by the lemma in the case that $n \in \{0, 1, \dots, 2^k - 1\}$. For $n \in \{2^k, \dots, 2^{k+1} - 1\}$ we use the definition of ζ_m^{k+1}, the induction hypothesis, Lemma 14.2.15, and Lemma 14.2.17 as above, but with a change of sign from $\mathrm{e}^{-\pi\mathrm{i}\frac{n}{2^k}}$ to $-\mathrm{e}^{-\pi\mathrm{i}\frac{n}{2^k}}$. ■

This immediately gives the following characterisation of the DDFT.

14.2.19 Theorem (The fast Fourier transform) *For* $\mathrm{r} \in \mathbb{Z}_{>0}$ *and* $\mathbf{z} \in \mathbb{C}^{2^\mathrm{r}}$ *define* $\zeta^0_\mathrm{j} \in \mathbb{C}^{2^\mathrm{r}}$, $\mathrm{j} \in \{0, 1, \dots, 2^\mathrm{r} - 1\}$*, by*

$$\zeta^0_\mathrm{j} = \mathbf{z}(\rho_\mathrm{r}(\mathrm{j})), \qquad \mathrm{j} \in \{0, 1, \dots, 2^\mathrm{r} - 1\}.$$

Then the procedure in 14.2.16 is such that $\mathrm{F}_{\mathrm{DD}}(\mathbf{z}) = \zeta^\mathrm{r}_0$.

Proof We apply Lemma 14.2.18 in the case of $j = r$ and $m = 0$, noting that $\rho_r \circ \rho_r(k) = k$, $k \in \{0, 1, \dots, 2^r - 1\}$, by Lemma 14.2.17(i). ■

The previous procedure for computing F_{DD} is known as the ***fast Fourier transform*** of ***FFT***. It is now reasonable to ask why this procedure has acquired the name “fast.” Let us consider this a little carefully. First of all, note that the direct computation of each component of $F_{\mathrm{DD}}(z)$ requires N complex multiplications and N complex additions. Thus the direct computation of $F_{\mathrm{DD}}(z)$ requires N^2 complex multiplications and N^2 complex additions. Addition is computationally relatively quick. Therefore, the speed of an algorithm is often measured by the number of multiplications that must be performed. The next result bounds the number of complex multiplications involved in the computation of the FFT.

14.2.20 Theorem (Computational complexity of the FFT) *The number of complex multiplications needed to compute* F_{DD} *on* $\mathbb{C}^{\mathrm{N}}$, $\mathrm{N} = 2^{\mathrm{r}}$, *using the FFT procedure is bounded above by* $\frac{1}{2}\mathrm{N}\log_2\mathrm{N} + 2\mathrm{N}$.

Proof In going from step j to step $j+1$ in the procedure 14.2.16 one must perform the following complex multiplications.

1. The numbers $(\mathrm{e}^{-\pi\mathrm{i}\frac{1}{2^j}})^2, \dots, (\mathrm{e}^{-\pi\mathrm{i}\frac{1}{2^j}})^{2^j-1}$ must be computed. This necessitates $2^j - 2 \leq 2^j$ complex multiplications.
2. Computing each of the vectors ζ_m^{j+1}, $m \in \{0, 1, \dots, 2^{r-j-1} - 1\}$, from the vectors ζ_m^j, $m \in \{0, 1, \dots, 2^{r-j}\}$, involves 2^j complex multiplications. Thus the total number of complex multiplications to be done to compute all vectors ζ_m^{j+1}, $m \in \{0, 1, \dots, 2^{r-j-1}\}$, is $2^{r-j-1} \cdot 2^j = 2^{r-1}$.

Therefore, in going from step j to step $j+1$ in the procedure 14.2.16 one must perform no more than $2^{r-1} + 2^j$ complex multiplications. One must also compute the numbers $\mathrm{e}^{-\pi\mathrm{i}\frac{1}{2}}, \dots \mathrm{e}^{-\pi\mathrm{i}\frac{1}{2^{r-1}}}$. This can be done by computing the last of the numbers, then compute the rest using the rule $\mathrm{e}^{-\pi\mathrm{i}\frac{1}{2^j}} = (\mathrm{e}^{-\pi\mathrm{i}\frac{1}{2^{j+1}}})^2$. This then constitutes $r - 2 \leq r$ complex multiplications.

In summary, we have shown that the FFT on $N = 2^r$ data points requires a number of complex multiplications bounded above by

$$\sum_{j=0}^{r-1}(2^{r-1} + 2^j) + r \leq r2^{r-1} + 2^r + r,$$

using the fact that

$$\sum_{j=0}^{r-1} 2^j = 2^r - 1,$$

an identity which is easily proved by induction on r. Now, since $r = \log_2 N$, we have

$$r2^{r-1} + 2^r + r = \tfrac{1}{2}r2^r + 2^r + r = \tfrac{1}{2}N\log_2 N + N + \log_2 N.$$

The theorem follows since $\log_2 N \leq N$. ■

For large N, there is a significant savings in computing the DDFT using the fast Fourier transform. In Table 14.1 we show the estimates on the number of complex multiplications for the direct and fast computation of the DDFT. One can see from this table the extent of the computational savings by using the FFT for large N.

14.2.7 Notes

The fast Fourier transform has many forms, the one we present perhaps being the most elementary.

Exercises

14.2.1 If $f \in \ell_{\mathrm{per},2\Delta}(\mathbb{Z}(\Delta); \mathbb{C})$, show that

$$\mathscr{F}_{\mathrm{DD}}(f)(k(2\Delta)^{-1}) = \Delta f(0) + (-1)^k \Delta f(\Delta).$$

Table 14.1 Bounds on complex multiplications needed for the direct computation of the DDFT (left) and computation using the FFT (right)

N	N^2	$\frac{1}{2}N\log_2 N + 2N$
$2^1 = 2$	4	5
$2^2 = 4$	16	12
$2^3 = 8$	64	28
$2^4 = 16$	256	64
$2^5 = 32$	1024	144
$2^6 = 64$	4096	320
$2^7 = 128$	16384	704
$2^8 = 256$	65536	1536
$2^9 = 512$	262144	3328
$2^{10} = 1024$	1048576	7168

14.2.2 Let $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ be defined by $f(n\Delta) = (-1)^n$.

(a) Show that f is 2Δ-periodic and compute the DDFT of f in this case.

(b) Show that f is 4Δ-periodic and compute the DDFT of f in this case.

14.2.3 Recall from Section 14.2.1 the orthonormal bases $\{\Delta^{-1/2}e_0, \dots, \Delta^{-1/2}e_{N-1}\}$ and $\{(N\Delta)^{-1/2}E_0, \dots, (N\Delta)^{-1/2}E_{N-1}\}$ for $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ and the orthonormal bases $\{(N\Delta)^{1/2}f_0, \dots, (N\Delta)^{1/2}f_{N-1}\}$ and $\{\Delta^{1/2}F_0, \dots, \Delta^{1/2}F_{N-1}\}$ for $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1};\mathbb{C})$.

(a) Determine the matrix representative of $\mathscr{F}_{\mathrm{DD}}$ relative to the bases $\{\Delta^{-1/2}e_0, \dots, \Delta^{-1/2}e_{N-1}\}$ and $\{(N\Delta)^{1/2}f_0, \dots, (N\Delta)^{1/2}f_{N-1}\}$.

(b) Determine the matrix representative of $\mathscr{F}_{\mathrm{DD}}$ relative to the bases $\{\Delta^{-1/2}e_0, \dots, \Delta^{-1/2}e_{N-1}\}$ and $\{\Delta^{1/2}\overline{F}_0, \dots, \Delta^{1/2}\overline{F}_{N-1}\}$.

(c) Determine the matrix representative of $\mathscr{F}_{\mathrm{DD}}$ relative to the bases $\{(N\Delta)^{-1/2}E_0, \dots, (N\Delta)^{-1/2}E_{N-1}\}$ and $\{(N\Delta)^{1/2}f_0, \dots, (N\Delta)^{1/2}f_{N-1}\}$.

(d) Determine the matrix representative of $\mathscr{F}_{\mathrm{DD}}$ relative to the bases $\{(N\Delta)^{-1/2}E_0, \dots, (N\Delta)^{-1/2}E_{N-1}\}$ and $\{\Delta^{1/2}F_0, \dots, \Delta^{1/2}F_{N-1}\}$.

14.2.4 Prove Proposition 14.2.11.

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Symbol Index