Anytime Proximity Moving Horizon Estimation: Stability and Regret for Nonlinear Systems*

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***Abstract*— In this paper, we reduce computational burden of moving horizon estimation (MHE) for discrete-time constrained nonlinear systems by providing an iteration scheme that computes a suboptimal state estimate after a limited number of gradient-based optimization algorithm iterations. The optimization algorithm is warm-started by an a priori estimate constructed based on a locally stable, model-based, and recursive state estimation strategy. Due to this implicit stabilizing regularization approach of the a priori estimate, we establish local exponential stability of the underlying estimation error by using Lyapunov arguments. Furthermore, by assuming convexity of the MHE problem, we characterize the performance of the iteration scheme relative to an estimator that knows for instance the optimal solutions in terms of regret upper bounds.**

I. Introduction

Moving horizon estimation (MHE) is an optimization-based approach that estimates the state of a dynamical system by solving a suitable optimization problem at each sampling instant. Hereby, MHE takes a fixed and limited number of the most recent input and measurement data into account and the considered horizon of data is moved forward in time (in a receding horizon manner) when a new measurement becomes available. The main benefits of MHE consist of its capability to handle general nonlinear systems and states constraints, as well as the flexibility it provides for designing performance criteria. During the last decades, MHE has received significant attention from both theoretical and practical viewpoints, see e.g. [3]–[8]. In this paper, we devote our attention to the computational efficiency of MHE by designing an iterative scheme for the state estimation problem of discrete-time nonlinear systems, following our recent results [9] and [10] in the framework of proximity-based MHE (pMHE). In [9], we presented a pMHE scheme for nonlinear systems, where we solve at each time instant an optimization problem that employs a suitable Bregman distance as a proximity measure to a stabilizing a priori estimate constructed based on the extended Kalman filter (EKF), as well as a convex and rather flexible stage cost that allows for instance to handle the case where measurements are affected by outliers. While desirable stability properties can be ensured by a simple design of the Bregman distance and the a priori estimate, the user's freedom in selecting suitable stage costs and keeping the horizon length small is not compromised. In [10], we considered the MHE problem of linear systems and proposed an efficient pMHE iteration scheme for solving the underlying optimization problem. The iteration scheme executes a limited number of optimization algorithm iterations and is shown to possess the *anytime* property, where global exponential stability of the resulting estimation error is guaranteed after any number of optimization iterations.

The contributions of this paper are as follows. Instead of solving the nonlinear pMHE problem presented in [9] at each time instance, we employ an idea similar to [10] to compute a suboptimal estimate after a limited number of optimization algorithm iterations. In particular, a simple convex problem is solved at each iteration, where a suitable Bregman distance is used as a proximity regularizing term to the previous iterate. In the warm start, we employ a stable recursive estimator such as the EKF to generate a stabilizing a priori estimate. As a main result, we show that the anytime pMHE iteration scheme proposed in [10] can be extended to nonlinear systems by proving exponential stability of the underlying estimation error under suitable assumptions and independently of the number of internal iterations. In contrast to the (explicit) stabilizing regularization of the EKF-based a priori estimate in [9], and similarly to [10], stability of the pMHE iteration scheme is induced from the a priori estimate that is only used in the warm start. This is an implicit stabilizing regularization, in the sense that the (suboptimal) bias of the stabilizing a priori estimate is fading away with each iteration and hence an improved performance can be achieved. In addition, we give performance guarantees for the pMHE iteration scheme by presenting an upper bound for the regret, i.e., the difference between the loss incurred by the pMHE algorithm and that of an arbitrary comparator sequence, in the case where the MHE problem is convex.

The paper is structured as follows. In Section II, we introduce the problem setup and present the pMHE iteration scheme for discrete-time nonlinear systems. In Section III, we investigate the nominal stability properties of the resulting estimation error. In Section IV, we address the performance of the proposed iteration scheme in terms of a regret analysis. In Section V, we use a simulation example to illustrate our theoretical results. We conclude the paper in Section VI. All the technical proofs can be found in the Appendix.

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Notation: We denote by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the sets of natural, positive natural, nonnegative real and positive real numbers, respectively. Moreover, we denote by $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ the sets of positive semi-definite and positive definite matrices of dimension $n \in \mathbb{N}_+$, respectively. For a vector v, let $\|v\| = \|v\|_2 = \sqrt{v^\top v}$ and $\|v\|_P \coloneqq \sqrt{v^\top P v}$ for any $P \in \mathbb{S}^n_+$. Let $\|M\| = \sqrt{\lambda_{\max}(M^\top M)}$ refer to the spectral norm of a generic matrix M.

II. Problem setup and the pMHE algorithm

We consider the state estimation problem of nonlinear discrete-time systems of the form

$$x_{k+1} = f(x_k, u_k, w_k), \tag{1a}$$
$$y_k = h(x_k) + v_k, \tag{1b}$$

where $k \in \mathbb{N}$ denotes the discrete time instant, $x_k \in \mathbb{R}^n$ the state vector, $u_k \in \mathbb{R}^m$ the input vector, and $y_k \in \mathbb{R}^p$ the measurement vector. The vectors $w_k \in \mathbb{R}^{m_w}$ and $v_k \in \mathbb{R}^p$ denote unknown process and measurement disturbances. We assume that the functions $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_w} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable in all of their arguments. Moreover, the state x_k is known to lie in a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. We let $x(k; x_0, \mathbf{u}, \mathbf{w})$ refer to the solution of system (1) at time k with initial state x_0 and input and disturbance sequences $\mathbf{u} = \{u_0, \cdots, u_{k-1}\}$ and $\mathbf{w} = \{w_0, \cdots, w_{k-1}\}$. We will simplify the notation to $x(k; x_0, \mathbf{u})$ in the disturbance-free case where $w_k = 0$ for all $k \in \mathbb{N}$. The goal is to find at each time instant k an estimate of the state x_k in a moving horizon fashion given the model (1), the constraint set $\mathcal{X}$, and a constant number $N \in \mathbb{N}_+$ of past inputs $\{u_{k-N}, \ldots, u_{k-1}\}$ and measurements $\{y_{k-N}, \ldots, y_{k-1}\}$. More specifically, at the time instant k, our goal is to find a solution to the following nonlinear MHE problem

$$\min_{\hat{x}_{k-N}, \hat{\mathbf{w}}_k} \quad \sum_{j=k-N}^{k-1} r(\hat{v}_j) + q(\hat{w}_j) \tag{2a}$$
$$\text{s.t.} \quad \hat{x}_{j+1} = f(\hat{x}_j, u_j, \hat{w}_j), \quad j = k-N, \ldots, k-1 \tag{2b}$$
$$y_j = h(\hat{x}_j) + \hat{v}_j, \quad j = k-N, \ldots, k-1 \tag{2c}$$
$$\hat{x}_j \in \mathcal{X}, \quad j = k-N, \tag{2d}$$

where the variables $\hat{x}_j \in \mathbb{R}^n$, $\hat{w}_j \in \mathbb{R}^{m_w}$, $\hat{v}_j \in \mathbb{R}^p$ represent estimates of the states, the model disturbances and the output residuals, respectively. We set the initial state $\hat{x}_{k-N}$ and the model disturbance sequence $\hat{\mathbf{w}}_k \coloneqq \{\hat{w}_{k-N}, \cdots, \hat{w}_{k-1}\}$ as decision variables of problem (2) since the remaining variables $\hat{x}_i$ and $\hat{v}_i$ depend implicitly on $(\hat{x}_{k-N}, \hat{\mathbf{w}}_k)$ through the model (2b), (2c). In the cost function (2a), $r : \mathbb{R}^p \to \mathbb{R}_+$ and $q : \mathbb{R}^{m_w} \to \mathbb{R}_+$ are convex nonnegative functions which penalize the estimated output residual $\hat{v}_j$ and process disturbance $\hat{w}_j$, respectively. For the time instants $k \leq N$, we set $N = k$ in problem (2).

By collecting the decision variables in the vector

$$\hat{\mathbf{z}}_k \coloneqq \begin{bmatrix} \hat{x}_{k-N}^\top & \hat{w}_{k-N}^\top & \ldots & \hat{w}_{k-1}^\top \end{bmatrix} \in \mathbb{R}^{Nm_w+n} \tag{3}$$

and eliminating the system dynamics (2b) and (2c), we can reformulate problem (2) as

$$\min_{\hat{\mathbf{z}}_k \in \mathcal{S}} \quad F_k(\hat{\mathbf{z}}_k). \tag{4a}$$

Here, the set $\mathcal{S} \subseteq \mathbb{R}^{Nm_w+n}$ is convex and given by

$$\mathcal{S} \coloneqq \{\mathbf{z} = \begin{bmatrix} x^\top & \mathbf{w}^\top \end{bmatrix}^\top, x \in \mathbb{R}^n, \mathbf{w} \in \mathbb{R}^{Nm_w} : x \in \mathcal{X}\}, \tag{4b}$$

and the cost function $F_k : \mathbb{R}^{Nm_w+n} \to \mathbb{R}_+$ denotes the sum of stage costs

$$F_k(\hat{\mathbf{z}}_k) = \sum_{j=k-N}^{k-1} r(y_j - h(x(j; \hat{x}_{k-N}, \mathbf{u}, \hat{\mathbf{w}}))) + q(\hat{w}_j), \tag{4c}$$

where, $\mathbf{u} = \{u_{k-N}, \cdots u_{j-1}\}$ and $\hat{\mathbf{w}} = \{\hat{w}_{k-N}, \cdots \hat{w}_{j-1}\}$ for each $j = k-N, \cdots, k-1$. In the recently proposed pMHE scheme for nonlinear systems [9], stability can be guaranteed by solving a regularized form of (4), where a generalized proximity measure to a stabilizing a priori estimate is added to the cost function F_k. In this work, we take the dynamics of the underlying optimization problem into account in the stability analysis. More specifically, we design and analyze a pMHE iteration scheme where a fast online implementation is ensured by executing only a finite number of optimization algorithm iterations at each time instant k.

Anytime pMHE algorithm

In the following, we extend the pMHE iteration scheme proposed in [10] for linear systems to the nonlinear case by computing a suboptimal solution to problem (4) after a limited number $\text{it}(k) \in \mathbb{N}_+$ of optimization algorithm iterations. More specifically, at each time k, a stabilizing a priori estimate, which we refer to as

$$\bar{\mathbf{z}}_k \coloneqq \begin{bmatrix} \bar{x}_{k-N}^\top & \bar{w}_{k-N}^\top & \ldots & \bar{w}_{k-1}^\top \end{bmatrix}^\top \in \mathbb{R}^{Nm_w+n}, \tag{5}$$

is used to generate a warm start $\hat{\mathbf{z}}_k^0$ for the optimization algorithm. Then, a first-order iterative optimization algorithm is employed, from which the sequence of iterates $\{\hat{\mathbf{z}}_k^1, \cdots \hat{\mathbf{z}}_k^{\text{it}(k)}\}$ is calculated. We define the i-th iterate of the optimization algorithm as

$$\hat{\mathbf{z}}_k^i \coloneqq \begin{bmatrix} \hat{x}_{k-N}^{i^\top} & \hat{w}_{k-N}^{i^\top} & \ldots & \hat{w}_{k-1}^{i^\top} \end{bmatrix} \in \mathbb{R}^{Nm_w+n}. \tag{6}$$

The steps of the iteration scheme are given in Algorithm 1. In line 3 of Algorithm 1, the warm start $\hat{\mathbf{z}}_k^0$ is computed as the so-called Bregman projection of the a priori estimate $\bar{\mathbf{z}}_k$ onto the set $\mathcal{S}$ in order to make sure that the optimization algorithm is warm-started by an estimate that satisfies the constraints. Here, $D_k : \mathbb{R}^{Nm_w+n} \times \mathbb{R}^{Nm_w+n} \to \mathbb{R}_+$ denotes the Bregman distance induced from a continuously differentiable and strongly convex function $\psi_k : \mathbb{R}^{Nm_w+n} \to \mathbb{R}$ as

$$D_k(\mathbf{z}_1, \mathbf{z}_2) = \psi_k(\mathbf{z}_1) - \psi_k(\mathbf{z}_2) - (\mathbf{z}_1 - \mathbf{z}_2)^\top \nabla \psi_k(\mathbf{z}_2). \tag{8}$$

Note that since ψ_k is convex, it follows that $D_k(\mathbf{z}_1, \mathbf{z}_2)$ is nonngeative, and that $D_k(\mathbf{z}_1, \mathbf{z}_2) = 0$ if and only if $\mathbf{z}_1 = \mathbf{z}_2$. A brief overview on its useful properties is provided in

Algorithm 1 pMHE iteration scheme

1: **Initialize:** Choose $\hat{x}_0$ and set $\bar{\mathbf{z}}_0 = \hat{x}_0$
2: **for** $k = 0, 1, \cdots$ **do**
3: $\quad \hat{\mathbf{z}}_k^0 = \arg\min_{\mathbf{z}\in\mathcal{S}} \quad D_k(\mathbf{z}, \bar{\mathbf{z}}_k)$ warm start
4: $\quad$ **for** $i = 0, \ldots, \mathrm{it}(k) - 1$ **do** optimizer update

$$\hat{\mathbf{z}}_k^{i+1} = \arg\min_{\mathbf{z}\in\mathcal{S}} \left\{ \eta_k^i \, \nabla F_k \left(\hat{\mathbf{z}}_k^i\right)^\top \mathbf{z} + D_k(\mathbf{z}, \hat{\mathbf{z}}_k^i) \right\} \quad (7)$$

5: $\quad$ **end for**
6: $\quad$ obtain $\hat{x}_k$ based on $\hat{\mathbf{z}}_k^{\mathrm{it}(k)}$
7: $\quad$ compute $\bar{\mathbf{z}}_{k+1} = \Phi_k(\hat{\mathbf{z}}_k^{\mathrm{it}(k)})$
8: **end for**

Appendix A. In line 4 of Algorithm 1, a fixed number $\mathrm{it}(k)$ of optimization algorithm iterations is performed via (7), where the next iterate $\hat{\mathbf{z}}_k^{i+1}$ is obtained by minimizing a linear approximation of the function F_k defined in (4c) and the Bregman distance D_k centered around the previous iterate $\hat{\mathbf{z}}_k^i$ (and not around $\bar{\mathbf{z}}_k$ as in [9]). Here, $\eta_k^i \in \mathbb{R}_{++}$ refers to the step size at the i-th iteration at time k. Notice that for $\mathcal{S} = \mathbb{R}^{Nm_{\mathrm{w}}+n}$ and a quadratic Bregman distance $D_k(\mathbf{z}_1, \mathbf{z}_2) = \frac{1}{2}\|\mathbf{z}_1 - \mathbf{z}_2\|_{P_k}^2$ induced from the function $\psi_k(\mathbf{z}) = \frac{1}{2}\|\mathbf{z}\|_{P_k}^2$, where $P_k \in \mathbb{S}_{++}^{Nm_{\mathrm{w}}+n}$, the optimizer update step (7) can be written as

$$\hat{\mathbf{z}}_k^{i+1} = \hat{\mathbf{z}}_k^i - \eta_k^i P_k^{-1} \nabla F_k(\hat{\mathbf{z}}_k^i). \quad (9)$$

In particular, if P_k is the identity matrix, we recover the iteration step of the classical gradient descent algorithm and hence a fast online implementation can be ensured. In line 6, based on the last iterate $\hat{\mathbf{z}}_k^{\mathrm{it}(k)}$, the state estimate

$$\hat{x}_k = x\left(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k, \hat{\mathbf{w}}_k^{\mathrm{it}(k)}\right) \quad (10)$$

is computed via a forward prediction of the system dynamics (2b) given the initial condition $\hat{x}_{k-N}^{\mathrm{it}(k)}$, input sequence $\mathbf{u}_k = \{u_{k-N}, \cdots, u_{k-1}\}$ and estimated process disturbance sequence $\hat{\mathbf{w}}_k^{\mathrm{it}(k)} = \left\{\hat{w}_{k-N}^{\mathrm{it}(k)}, \cdots, \hat{w}_{k-1}^{\mathrm{it}(k)}\right\}$. We impose the following assumptions on the function F_k and the Bregman distance D_k.

Assumption 1 *The functions r and q in F_k defined in* (4c) *are continuously differentiable, convex, nonnegative, and attain their minimum zero at zero. Additionally, the gradient of r is Lipschitz continuous with Lipschitz constants $L_r \in \mathbb{R}_{++}$. Moreover, for any $k \in \mathbb{N}$, the gradient of F_k is Lipschitz continuous with Lipschitz constant $L_k \in \mathbb{R}_{++}$, and there exists $L_{\mathrm{F}} \in \mathbb{R}_{++}$ such that $L_k \leq L_{\mathrm{F}}$ for all $k \in \mathbb{N}$.*

Assumption 2 *For any $k \in \mathbb{N}$, the function ψ_k is continuously differentiable, strongly convex with constant $\sigma_k \in \mathbb{R}_{++}$, and its gradient is Lipschitz continuous with Lipschitz constant $\gamma_k \in \mathbb{R}_{++}$, which implies the following for the Bregman distance*

$$\frac{\sigma_k}{2}\|\mathbf{z}_1 - \mathbf{z}_2\|^2 \leq D_k(\mathbf{z}_1, \mathbf{z}_2) \leq \frac{\gamma_k}{2}\|\mathbf{z}_1 - \mathbf{z}_2\|^2 \quad (11)$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{Nm_{\mathrm{w}}+n}$. Moreover, there exist $\sigma \in \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}_{++}$ such that $\sigma_k \geq \sigma$ and $\gamma_k \leq \gamma$ for all $k \in \mathbb{N}$.

In line 7 of Algorithm 1, the a priori estimate $\bar{\mathbf{z}}_{k+1}$ for the next time instant is calculated based on the last pMHE iterate $\hat{\mathbf{z}}_k^{\mathrm{it}(k)}$. In the framework of pMHE, the idea is to construct the a priori estimate from a simple, model-based, and recursive state estimator whose dynamics can be described by an operator $\Phi_k : \mathbb{R}^{Nm_{\mathrm{w}}+n} \to \mathbb{R}^{Nm_{\mathrm{w}}+n}$ and from which stability of pMHE can be inherited. Note that Φ_k is time varying since it generates the a priori estimate based on inputs and measurements of the system. We require the following assumption on Φ_k when evaluated at the true system state

$$\mathbf{z}_k := \begin{bmatrix} x_{k-N}^\top & w_{k-N}^\top & \ldots & w_{k-1}^\top \end{bmatrix}^\top \in \mathbb{R}^{Nm_{\mathrm{w}}+n}, \quad (12)$$

i.e., the true state x_{k-N} with the true process disturbance sequence $\{w_{k-N}, \cdots, w_{k-1}\}$.

Assumption 3 *In the disturbance-free case, it holds that $\Phi_k(\mathbf{z}_k) = \mathbf{z}_{k+1}$, where $\mathbf{z}_k = \begin{bmatrix} x_{k-N}^\top & 0 & \ldots & 0 \end{bmatrix}^\top$ and $\mathbf{z}_{k+1} = \begin{bmatrix} x_{k-N+1}^\top & 0 & \ldots & 0 \end{bmatrix}^\top$.*

Assumption 3 states that, in the absence of disturbances, if we substitute the true system state $\mathbf{z}_k$ in the a priori estimate operator Φ_k, we should obtain the next true sytem state $\mathbf{z}_{k+1}$. This reflects the intuitive fact that, in this case, the stabilizing estimator based on which the a priori estimate is constructed should follow the true system dynamics.

Remark 1 We can use Algorithm 1 for the time instants $k \leq N$ by setting all the negative indices to zero. In particular, we have $\hat{\mathbf{z}}_k^i = \begin{bmatrix} \hat{x}_0^{i\top} & \hat{w}_0^{i\top} & \ldots & \hat{w}_{k-1}^{i\top} \end{bmatrix}^\top$ and set $\bar{\mathbf{z}}_k = \begin{bmatrix} \hat{x}_0^\top & \bar{w}_0^\top & \ldots & \bar{w}_{k-1}^\top \end{bmatrix}^\top$, where $\hat{\mathbf{z}}_k^i, \bar{\mathbf{z}}_k \in \mathbb{R}^{km_{\mathrm{w}}+n}$ and $\hat{x}_0$ in the a priori estimate denotes the initial guess. Note that for $k = 0$, we have $\bar{\mathbf{z}}_0 = \hat{x}_0 \in \mathbb{R}^n$. In other words, at each time $k \leq N$, we take all the available measurements into account in the estimation process.

In [10], Algorithm 1 is employed for the state estimation problem of linear systems. The results therein show that if the a priori estimate operator Φ_k incorporates the dynamics of the Luenberger observer and the Bregman distance D_k is chosen suitably, then global exponential stability (GES) of the underlying estimation error can be ensured through a proper choice of the step size η_k^i. Thereby, Algorithm 1 for linear systems can be considered as an anytime MHE algorithm, in which stability is guaranteed after any number of optimization algorithm iterations. In the following section, we extend the stability analysis to nonlinear systems by deriving sufficient stability conditions on the Bregman distance D_k, a priori estimate operator Φ_k and step size η_k^i. Subsequently, a specific design approach for D_k and Φ_k that is based on the extended Kalman filter (EKF) is discussed.

III. Stability analysis

In this section, we analyze the stability properties of the proposed pMHE iteration scheme in Algorithm 1 applied to system (1) with $w_k = 0, v_k = 0, k \in \mathbb{N}$. More specifically, we establish sufficient conditions for the local exponential stability of the resulting estimation error. We define the pMHE error as

$$\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)} = \begin{bmatrix} x_{k-N} \\ \mathbf{w}_k \end{bmatrix} - \begin{bmatrix} \hat{x}_{k-N}^{\mathrm{it}(k)} \\ \hat{\mathbf{w}}_k^{\mathrm{it}(k)} \end{bmatrix} = \begin{bmatrix} e_{k-N} \\ \mathbf{w}_k - \hat{\mathbf{w}}_k^{\mathrm{it}(k)} \end{bmatrix}, \tag{13}$$

and the estimation error as

$$x_k - \hat{x}_k = x\left(k; x_{k-N}, \mathbf{u}_k, \mathbf{w}_k\right) - x\left(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k, \hat{\mathbf{w}}_k^{\mathrm{it}(k)}\right). \tag{14}$$

We first derive in the subsequent two lemmas useful properties of the employed optimization algorithm. The MHE problem (4) can be reformulated as

$$\min_{\hat{\mathbf{z}}_k \in \mathbb{R}^{N m_{\mathrm{w}}+n}} \quad F_k(\hat{\mathbf{z}}_k) + I_{\mathcal{S}}(\hat{\mathbf{z}}_k), \tag{15}$$

where $I_{\mathcal{S}}$ denotes the indicator function of the convex set $\mathcal{S}$ defined as $I_{\mathcal{S}}(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathcal{S}$ and $I_{\mathcal{S}}(\mathbf{z}) = +\infty$ for all $\mathbf{z} \notin \mathcal{S}$. Hence, any local minimum $\mathbf{z}^*$ of (15) must satisfy $-\nabla F_k(\mathbf{z}^*) \in \partial I_{\mathcal{S}}(\mathbf{z}^*)$, where $\partial I_{\mathcal{S}}(\mathbf{z}^*)$ refers to the subdifferential of the convex function $I_{\mathcal{S}}$ at the point $\mathbf{z}^*$ [11]. The optimizer update step (7) can be accordingly reformulated as

$$\begin{aligned} \hat{\mathbf{z}}_k^{i+1} &= \underset{\mathbf{z} \in \mathbb{R}^{N m_{\mathrm{w}}+n}}{\arg\min} \left\{ \eta_k^i \, I_{\mathcal{S}}(\mathbf{z}) + \eta_k^i \, \nabla F_k\left(\hat{\mathbf{z}}_k^i\right)^\top \mathbf{z} + D_k(\mathbf{z}, \hat{\mathbf{z}}_k^i) \right\} \\ &=: T_{\eta_k^i}(\hat{\mathbf{z}}_k^i), \end{aligned} \tag{16}$$

where $T_{\eta_k^i} : \mathbb{R}^{N m_{\mathrm{w}}+n} \to \mathbb{R}^{N m_{\mathrm{w}}+n}$ refers to the optimizer update operator. A fundamental property of this operator is that minimizers of (15) are fixed points of $T_{\eta_k^i}$, i.e.,

$$\begin{aligned} T_{\eta_k^i}(\mathbf{z}^*) = \mathbf{z}^* &\Leftrightarrow 0 \in \eta_k^i \left(\partial I_{\mathcal{S}}(\mathbf{z}^*) + \nabla F_k\left(\mathbf{z}^*\right)\right) + \nabla D_k(\mathbf{z}^*, \mathbf{z}^*) \\ &\Leftrightarrow -\nabla F_k\left(\mathbf{z}^*\right) \in \partial I_{\mathcal{S}}(\mathbf{z}^*) \\ &\Leftrightarrow \mathbf{z}^* \text{ is a local minimum of (15).} \end{aligned} \tag{17}$$

Lemma 1 *Consider Algorithm 1 and suppose Assumptions 1-2 hold. Then, the optimizer update operator* (16) *satisfies*

$$\left\| T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2) \right\| \le \frac{\eta_k^i}{\sigma_k}\left(L_k + \frac{\gamma_k}{\eta_k^i}\right)\left\| \mathbf{z}_1 - \mathbf{z}_2 \right\| \tag{18}$$

for any $k \in \mathbb{N}$ *and* $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{N m_w + n}$.

Lemma 2 *Consider the optimizer update step* (7) *in Algorithm 1 and suppose Assumptions 1-2 hold. Then,*

$$F_k(\mathbf{z}) - F_k\left(T_{\eta_k^i}(\mathbf{z})\right) \ge \left(\frac{\sigma_k}{\eta_k^i} - L_k\right)\left\| \mathbf{z} - T_{\eta_k^i}(\mathbf{z}) \right\|^2 \tag{19}$$

for any $k \in \mathbb{N}$ *and* $\mathbf{z} \in \mathcal{S}$.

Lemma 1 establishes the Lipschitz continuity property of the optimizer update operator $T_{\eta_k^i}$. Lemma 2 indicates that by choosing the step size such that it verifies $\eta_k^i \le \frac{\sigma_k}{L_k}$ for all $i = 0, \ldots, \mathrm{it}(k) - 1$ and $k \in \mathbb{N}_+$, we can make sure that the sequence $\{F_k(\hat{\mathbf{z}}_k^0), \cdots, F_k(\hat{\mathbf{z}}_k^{\mathrm{it}(k)})\}$ is nonincreasing.
In addition, we require the following two preparatory lemmas before presenting the main stability result.

Lemma 3 *Consider system* (1) *with* $w_k = 0, v_k = 0, k \in \mathbb{N}$ *and Algorithm 1. Suppose Assumptions 1-2 hold true. Then, there exist constants* $\delta, \kappa \in \mathbb{R}_{++}$ *such that*

$$\begin{aligned} D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^{i+1}\right) \le D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^i\right) + \frac{\eta_k^i}{2}\left(L_k - \frac{\sigma_k}{\eta_k^i}\right)\left\| \hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i \right\|^2 \\ + \eta_k^i \, \kappa \| \mathbf{z}_k - \hat{\mathbf{z}}_k^i \|^3 \end{aligned} \tag{20}$$

holds for all $i = 0, \ldots, \mathrm{it}(k) - 1$, $k \in \mathbb{N}$, *and* $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ *with* $\| \mathbf{z}_k - \hat{\mathbf{z}}_k^i \| \le \delta$, *where* $\mathbf{z}_k$ *is defined in* (12).

Lemma 4 *Consider system* (1) *with* $w_k = 0, v_k = 0, k \in \mathbb{N}$ *and Algorithm 1. Suppose that Assumptions 1-2 hold and that* $\mathrm{it}(k+1) \le \mathrm{it}(k)$. *If we choose the step sizes in* (7) *such that* $\eta_k^i \le \frac{\sigma_k}{L_k}$, *then there exists* $\alpha \in \mathbb{R}_{++}$ *such that*

$$\| \mathbf{z}_k - \hat{\mathbf{z}}_k^i \| \le \alpha \, \| \mathbf{z}_k - \bar{\mathbf{z}}_k \| \tag{21}$$

for all $i = 0, \ldots, \mathrm{it}(k) - 1$ *and* $k \in \mathbb{N}_+$.

Theorem 1 *Consider system* (1) *with* $w_k = 0, v_k = 0, k \in \mathbb{N}$ *and Algorithm 1. Let Assumptions 1-3 hold and suppose that* $\mathrm{it}(k+1) \le \mathrm{it}(k)$. *If we choose the step sizes in* (7) *such that*

$$\eta_k^i \le \frac{\sigma_k}{L_k}, \tag{22}$$

for all $i = 0, \ldots, \mathrm{it}(k) - 1$ *and* $k \in \mathbb{N}_+$, *and if there exist* $\epsilon, c \in \mathbb{R}_{++}$ *such that the Bregman distance* D_k *and the a priori estimate operator* Φ_k *satisfy*

$$D_k(\Phi_{k-1}(\mathbf{z}), \Phi_{k-1}(\hat{\mathbf{z}})) - D_{k-1}(\mathbf{z}, \hat{\mathbf{z}}) \le -c\| \mathbf{z} - \hat{\mathbf{z}} \|^2 \tag{23}$$

for any $k \in \mathbb{N}_+$ *and* $\mathbf{z}, \hat{\mathbf{z}}$ *with* $\| \mathbf{z} - \hat{\mathbf{z}} \| \le \epsilon$, *then the estimation error* (14) *is (locally) exponentially stable, i.e., there exist* $\tilde{\epsilon}, \tilde{\alpha} \in \mathbb{R}_{++}$ *and* $\beta \in (0, 1)$ *such that*

$$\| x_k - \hat{x}_k \| \le \tilde{\alpha} \beta^k \| x_0 - \hat{x}_0 \| \tag{24}$$

for any $k \in \mathbb{N}_+$ *and* $x_0, \hat{x}_0 \in \mathbb{R}^n$ *with* $\| x_0 - \hat{x}_0 \| \le \tilde{\epsilon}$.

Theorem 1 implies that Algorithm 1 possesses the anytime property, in the sense that (local) exponential stability of the estimation error can be ensured independently of the number of optimization algorithm iterations $\mathrm{it}(k)$. A crucial issue for this property to hold is the validity of condition (23). It states that, given two vectors $\mathbf{z}, \hat{\mathbf{z}}$ with $\| \mathbf{z} - \hat{\mathbf{z}} \| \le \epsilon$ and the Bregman distance $D_{k-1}(\mathbf{z}, \hat{\mathbf{z}})$, a prediction from the a priori estimate operator Φ_{k-1} will yield a contracting Bregman distance D_k. Hence, condition (23) can be fulfilled by designing the a priori estimate operator from an estimator with locally (exponentially) stable dynamics and selecting the Bregman distance as the Lyapunov function with which the local stability of this estimator can be verified. This includes for instance the computationally efficient discrete-time EKF (see below).

Remark 2 To make sure that the estimate of the state x_k satisfies the constraints, we can perform a projection of the obtained pMHE state $\hat{x}_k$ onto the convex set $\mathcal{X}$ as

$$\hat{x}_k^{\mathrm{proj}} = \underset{x \in \mathcal{X}}{\arg\min} \| x - \hat{x}_k \|. \tag{25}$$

In fact, this does not jeopardize the stability properties of the resulting estimation error, which we briefly discuss in the following. The proximal operator $\text{prox}_{I_{\mathcal{X}}}$ of the indicator function $I_{\mathcal{X}}$ is the Euclidean projection onto $\mathcal{X}$ [11]. Due to the fact that $\text{prox}_{I_{\mathcal{X}}}$ is nonexpansive, i.e. Lipschitz continuous with Lipschitz constant one, it holds that

$$\|\text{prox}_{I_{\mathcal{X}}}(x_k) - \text{prox}_{I_{\mathcal{X}}}(\hat{x}_k)\| \leq \|x_k - \hat{x}_k\|. \tag{26}$$

Since the true state satisfies the constraints, $\text{prox}_{I_{\mathcal{X}}}(x_k) = x_k$ and we obtain by (26) that $\|x_k - \hat{x}_k^{\text{proj}}\| \leq \|x_k - \hat{x}_k\|$. Substituting this inequality in (24) yields

$$\|x_k - \hat{x}_k^{\text{proj}}\| \leq \tilde{\alpha}\beta^k \|x_0 - \hat{x}_0\| \tag{27}$$

for any $k \in \mathbb{N}_+$ and $x_0, \hat{x}_0 \in \mathbb{R}^n$ with $\|x_0 - \hat{x}_0\| \leq \tilde{\epsilon}$, which proves the local exponential stability of the estimation error computed based on the projected estimate $\hat{x}_k^{\text{proj}}$.

Φ_k and D_k based on the extended Kalman filter

We present in the following a design approach for the a priori estimate operator and the associated Bregman distance, which will be demonstrated to satisfy condition (23). Recall that the key idea in the pMHE framework is to use in the warm start a stabilizing a priori estimate in order to implicitly inherit its stability properties. From the various exponential observer design methods in the literature [12], we use the EKF to construct the a priori estimate operator Φ_k and choose the Bregman distance D_k as a weighted quadratic function, where the weight matrix consists of the inverse of the EKF covariance matrix.
Let $\mathbf{z} = \begin{bmatrix} x^\top & \mathbf{w}^\top \end{bmatrix}^\top$, $\hat{\mathbf{z}} = \begin{bmatrix} \hat{x}^\top & \hat{\mathbf{w}}^\top \end{bmatrix}^\top$, $x, \hat{x} \in \mathbb{R}^n$, $\mathbf{w}, \hat{\mathbf{w}} \in \mathbb{R}^{Nm_w}$. After retrieving the stored EKF covariance matrix $P_{k-N} \in \mathbb{S}_{++}^n$, the input u_{k-N} and the measurement y_{k-N}, we construct the a priori estimate operator at time instant k as follows:

$$C_{k-N} = \frac{\partial h}{\partial x}\bigg|_x \tag{28a}$$
$$K_{k-N} = P_{k-N} C_{k-N}^\top \left(C_{k-N} P_{k-N} C_{k-N}^\top + R\right)^{-1} \tag{28b}$$
$$x^+ = x + K_{k-N}\left(y_{k-N} - h(x)\right) \tag{28c}$$
$$\Phi_k(\mathbf{z}) = \begin{bmatrix} f\left(x^+, u_{k-N}, 0\right) \\ \mathbf{0} \end{bmatrix}, \tag{28d}$$

where $\mathbf{0} \in \mathbb{R}^{Nm_w}$, $R \in \mathbb{S}_{++}^p$, and the matrix $K_{k-N} \in \mathbb{R}^{n \times p}$ refers to the EKF gain. The associated Bregman distance is

$$D_k(\mathbf{z}, \hat{\mathbf{z}}) = \frac{1}{2}\|x - \hat{x}\|^2_{P_{k-N}^{-1}} + \frac{1}{2}\|\mathbf{w} - \hat{\mathbf{w}}\|^2_W, \tag{29}$$

where $W \in \mathbb{S}_{++}^{Nm_w}$ is an arbitrary weight matrix. The EKF covariance matrix P_{k-N+1} for the time instant $k+1$ is computed via

$$A_{k-N} = \frac{\partial f}{\partial x}\bigg|_{(x^+, u_{k-N}, 0)} \tag{30a}$$
$$P_{k-N}^+ = (I - K_{k-N} C_{k-N}) P_{k-N} \tag{30b}$$
$$P_{k-N+1} = A_{k-N} P_{k-N}^+ A_{k-N}^\top + Q, \tag{30c}$$

where $Q \in \mathbb{S}_{++}^n$. In particular, consider the last MHE iterate $\hat{\mathbf{z}}_k^{\text{it}(k)}$ computed at time k. In view of line 7 of Algorithm 1, the a priori estimate, whose notation is given in (5), is calculated for the next time instant as

$$\bar{\mathbf{z}}_{k+1} = \Phi_k(\hat{\mathbf{z}}_k^{\text{it}(k)}) = \begin{bmatrix} \bar{x}_{k-N+1}^\top & 0 & \dots & 0 \end{bmatrix}^\top, \tag{31a}$$

where we have zero a priori process disturbances, i.e., $\bar{w}_j = 0$ for $j = k-N+1, \dots, k$, and

$$\bar{x}_{k-N+1} = f\left(\bar{x}_{k-N}^+, u_{k-N}, 0\right) \tag{31b}$$
$$\bar{x}_{k-N}^+ = \hat{x}_{k-N}^{\text{it}(k)} + K_{k-N}\Big(y_{k-N} - h\left(\hat{x}_{k-N}^{\text{it}(k)}\right)\Big). \tag{31c}$$

In order to establish that the proposed design based on the EKF satisfies condition (23), we require the following assumption.

Assumption 4 *For any $k \in \mathbb{N}$, A_k is nonsingular, $\|A_k\| \leq \bar{a}$, $\|C_k\| \leq \bar{c}$, $\|K_k\| \leq \bar{k}$ for some $\bar{a}, \bar{c}, \bar{k} \in \mathbb{R}_{++}$, and the matrices P_k, P_k^+ are uniformly bounded by*

$$\underline{p} I \preceq P_k \preceq \bar{p} I, \qquad \underline{p} I \preceq P_k^+ \preceq \bar{p} I, \tag{32}$$

where $\underline{p}, \bar{p} \in \mathbb{R}_{++}$.

Proposition 1 *Consider system* (1) *with $w_k = 0$, $v_k = 0$, $k \in \mathbb{N}$ and let Assumption 4 hold. Then, the Bregman distance* (29) *and the a priori estimate operator* (28) *verify Assumptions 2 and 3, respectively. Moreover, there exist constants $\epsilon, c \in \mathbb{R}_{++}$ such that they satisfy* (23) *for any $k \in \mathbb{N}_+$ and $\mathbf{z}, \hat{\mathbf{z}}$ with $\|\mathbf{z} - \hat{\mathbf{z}}\| \leq \epsilon$.*

Overall, we obtain the following (inherent) stability result for the pMHE iteration scheme based on the EKF, which is a direct consequence of Theorem 1 and Proposition 1.

Corollary 1 *Consider system* (1) *with $w_k = 0$, $v_k = 0$, $k \in \mathbb{N}$ and Algorithm 1 with a priori estimate operator* (28) *and Bregman distance* (29)*. Let Assumptions 1 and 4 hold and suppose that $\text{it}(k+1) \leq \text{it}(k)$. If we choose the step sizes in* (7) *such that $\eta_k^i \leq \frac{\sigma_k}{L_k}$ for all $k \in \mathbb{N}_+$ and $i = 0, \dots, \text{it}(k) - 1$, then the estimation error* (14) *is (locally) exponentially stable.*

IV. Regret Analysis

In this section, we characterize the overall performance of the pMHE iteration scheme in Algorithm 1 in terms of the performance criterion F_k of the original problem (4). More specifically, we describe the total loss of Algorithm 1 by

$$\sum_{k=1}^{T} F_k(\hat{\mathbf{z}}_k^{\text{it}(k)}). \tag{33}$$

Note that in view of Lemma 2, by choosing the step size η_k^i such that $\eta_k^i \leq \frac{\sigma_k}{L_k}$ for all $i = 0, \cdots, \text{it}(k) - 1$ and $k \in \mathbb{N}_+$, we can make sure that the sequence $\{F_k(\hat{\mathbf{z}}_k^0), \cdots, F_k(\hat{\mathbf{z}}_k^{\text{it}(k)})\}$ generated by the optimizer update (7) at each time instant k is monotonically nonincreasing, i.e., $F_k(\hat{\mathbf{z}}_k^0) \geq \cdots \geq F_k(\hat{\mathbf{z}}_k^{\text{it}(k)})$.
We aim to attain a low regret relative to a comparator sequence $\{\mathbf{z}_1^c, \mathbf{z}_2^c, \dots, \mathbf{z}_T^c\}$ satisfying $\mathbf{z}_k^c \in \mathcal{S}$ at each time k.

The regret of $\{\hat{\mathbf{z}}_1^{\mathrm{it}(1)}, \cdots, \hat{\mathbf{z}}_T^{\mathrm{it}(T)}\}$ with respect to the comparator $\{\mathbf{z}_1^c, \ldots, \mathbf{z}_T^c\}$ is defined as

$$R(T) \coloneqq \sum_{k=1}^{T} F_k(\hat{\mathbf{z}}_k^{\mathrm{it}(k)}) - \sum_{k=1}^{T} F_k(\mathbf{z}_k^c). \tag{34}$$

Establishing regret bounds can be useful in characterizing how well our algorithm, which delivers suboptimal estimates after limited numbers of optimization algorithm iterations, performs relative to an estimator with access to the optimal solutions $\{\mathbf{z}_1^c, \ldots, \mathbf{z}_T^c\}$. Similar to [10], we present regret bounds in terms of

$$C_T(\mathbf{z}_1^c, \cdots, \mathbf{z}_T^c) \coloneqq \sum_{k=1}^{T} \left\| \mathbf{z}_{k+1}^c - \Phi_k\left(\mathbf{z}_k^c\right) \right\|, \tag{35}$$

i.e. the deviation of the comparator sequence from the dynamics of the estimator used to construct the a priori estimate and characterized by the operator Φ_k. We impose the following assumption.

Assumption 5 *For any $k \in \mathbb{N}$, the sum of stage costs F_k defined in* (4c) *is convex.*

Since the functions r, q are convex by Assumption 1, Assumption 5 can be satisfied if the system dynamics (1) are linear, i.e.,

$$f(x, u, w) \coloneqq Ax + Bu + Gw, \qquad h(x) \coloneqq Cx, \tag{36}$$

where the matrices A, B, G and C are of compatible dimensions. Note that the resulting MHE problem (4) is a convex optimization problem due to the convexity of the constraint set $\mathcal{S}$. We introduce the following notations:

$$M_1 \coloneqq \max_{\mathbf{z} \in \mathcal{S}, k \in \mathbb{N}} \|\nabla \psi_k(\mathbf{z})\|, \quad M_2 \coloneqq \max_{\mathbf{z} \in \mathcal{S}, k \in \mathbb{N}} \|\nabla \psi_k(\Phi(\mathbf{z}))\|$$
$$M \coloneqq M_1 + M_2, \qquad D_{\max} \coloneqq \max_{\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{S}, k \in \mathbb{N}} D_k(\mathbf{z}_1, \mathbf{z}_2),$$

where we assume that the maximum in each definition exists and is finite. The following result establishes conditions under which Algorithm 1 achieves GES of the underlying estimation error as well as a regret bound of the form $\mathcal{O}(\sqrt{T}(1 + C_T))$, similar to [10, Theorem 3] as well as the established regret bound for the online convex optimization algorithm involving a dynamical model and proposed in [13].

Theorem 2 *Consider Algorithm 1 and any comparator sequence $\{\mathbf{z}_1^c, \mathbf{z}_2^c, \ldots, \mathbf{z}_T^c\}$ with $\mathbf{z}_k^c \in \mathcal{S}$. Suppose that Assumptions 1-3 and 5 hold and that* $\mathrm{it}(k+1) \le \mathrm{it}(k)$*. If we choose the step sizes in* (7) *such that*

$$\eta_k^i = \frac{\sigma}{L_{\mathrm{F}}} \frac{1}{\sqrt{k}}, \tag{37}$$

for all $i = 0, \ldots, \mathrm{it}(k) - 1$ and $k \in \mathbb{N}_+$, and if there exists $c \in \mathbb{R}_{++}$ such that the Bregman distance D_k and the a priori estimate operator Φ_k satisfy

$$D_k(\Phi_{k-1}(\mathbf{z}), \Phi_{k-1}(\hat{\mathbf{z}})) - D_{k-1}(\mathbf{z}, \hat{\mathbf{z}}) \le -c \|\mathbf{z} - \hat{\mathbf{z}}\|^2 \tag{38}$$

for any $k \in \mathbb{N}_+$ and $\mathbf{z}, \hat{\mathbf{z}} \in \mathbb{R}^{Nm_{\mathrm{w}}+n}$, then the estimation error is GES and we have that

$$R(T) \le \frac{\sqrt{T}}{\mathrm{it}(T)} \frac{L_{\mathrm{F}}}{\sigma} \Big(D_{max} + M \sum_{k=1}^{T} \left\| \mathbf{z}_{k+1}^c - \Phi_k\left(\mathbf{z}_k^c\right) \right\| \Big). \tag{39}$$

Some remarks on Theorem 2 are in order. First, given the linear dynamics (36), condition (38) can be globally satisfied if we construct the a priori estimate operator Φ_k based on the Kalman filter for linear time-invariant systems and choose the Bregman distance D_k in (29). Alternatively, a design approach that is based on the Luenberger observer can be employed [10]. Second, in (39), we can see that for fixed $T \in \mathbb{N}_+$, the regret upper bound becomes smaller by increasing the number of optimization algorithm iterations $\mathrm{it}(T)$, and that for fixed $\mathrm{it}(T) \in \mathbb{N}_+$, the average regret $R(T)/T$ tends to zero when T goes to infinity and if the comparator sequence follows the dynamics Φ_k.

V. Simulation Example

In order to demonstrate the stability and performance properties of Algorithm 1, we consider the example of a constant volume batch reactor in which the reaction $2A \rightleftarrows B$ takes place [3]. The state consists of the partial pressures $x = \begin{bmatrix} p_A & p_B \end{bmatrix}^\top$ and the system is modeled by

$$\dot{x}_1 = -2k_1 x_1^2 + 2k_2 x_2, \tag{40a}$$
$$\dot{x}_2 = k_1 x_1^2 - k_2 x_2, \tag{40b}$$
$$y = x_1 + x_2, \tag{40c}$$

where $k_1 = 0.16\,\mathrm{min}^{-1}\,\mathrm{atm}^{-1}$ and $k_2 = 0.0064\,\mathrm{min}^{-1}$. The system is discretized using the Euler method with sample time $h = 0.1\,\mathrm{min}$, which yields a system of the form (1) without inputs u_k. Since the states represent pressures, the physical state constraints are given by $x_k \ge 0$. The initial condition of the system is $x_0 = [3\ 1]^\top$.

The goal is to estimate the state x_k by using the pMHE iteration scheme based on the EKF operator (28) and Bregman distance (29), where we select $r(v) = \frac{1}{2}\|v\|_{R^{-1}}^2$ in the cost function (4c), consider only $\hat{x}_{k-N}$ as decision variable by setting $q = 0$ in (4c) and ignoring the process disturbances, and select for the optimizer update (7) the Bregman distance $D_k\left(x_1, x_2\right) = \frac{1}{2}\|x_1 - x_2\|_{P_{k-N}^{-1}}^2$. The corresponding design parameters are $R = 0.01$, $Q = \mathrm{diag}(10^{-4}, 0.01)$, and $P_0 = I_2$. Furthermore, we set the horizon length $N = 3$ and fix the number of optimization algorithm iterations by setting $\mathrm{it} \coloneqq \mathrm{it}(k) = \mathrm{it}(k+1)$ for all time instants k.

We compare the obtained results with the EKF designed accordingly and the nonlinear pMHE scheme from [9]. We initialize the considered estimators with a poor initial guess $\hat{x}_0 = \begin{bmatrix} 0.1 & 4.5 \end{bmatrix}^\top$. The resulting state estimates and estimation errors are depicted in Figures 1 and 2, respectively. Figure 1 shows that the EKF estimates of the partial pressure p_A violate the inequality constraints and are negative. In contrast, the employed pMHE approaches yield positive estimates and exponentially stable estimation errors

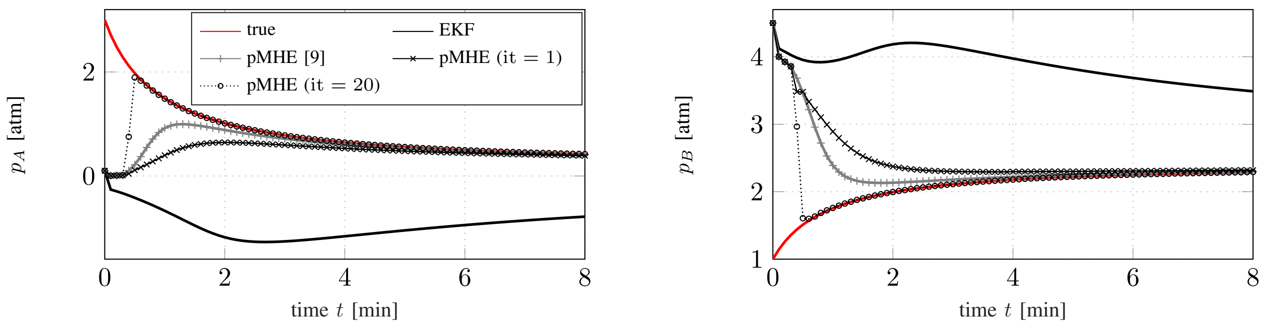

Fig. 1: Resulting estimates of p_A, p_B generated by the EKF, pMHE [9], and the pMHE iteration scheme in Algorithm 1.

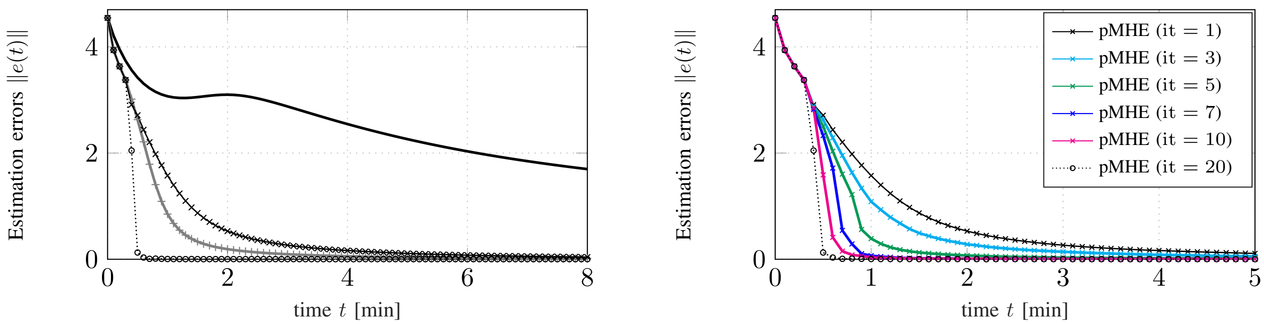

Fig. 2: Resulting estimation errors generated by the EKF, pMHE [9], and the pMHE iteration scheme in Algorithm 1.

as depicted in Figure 2. Moreover, on the left of the latter figure, we can see that using Algorithm 1 with $\mathrm{it} = 20$ yields estimation errors smaller than those generated by the pMHE scheme [9], which are in turn smaller that those of Algorithm 1 with a single iteration $\mathrm{it} = 1$. On the right of Figure 2, we can observe that increasing the number of optimization algorithms iterations in the pMHE iteration scheme yields smaller estimation errors. This observation is validated by the computed root mean square error (RMSE)

$$RMSE = \sqrt{\sum_{k=N}^{T_{\text{sim}}} \frac{\|e_k\|^2}{T_{\text{sim}} - N + 1}} \tag{41}$$

for different values of optimization iterations, where $T_{\text{sim}} = 100$ denotes the simulation time for the discretized system. These are reported in Table I, along with the obtained RMSE with other values of the horizon length N. We can see that, as expected, the RMSE decreases with increasing values of optimization algorithm iterations as well as with increasing values of N.

TABLE I: RMSE of the pMHE iteration scheme for different numbers of optimization algorithm iterations it and values of the horizon length N.

RMSE	$N = 2$	$N = 3$	$N = 5$	$N = 10$
it = 1	1.0301	0.7747	0.1724	0.0254
it = 5	0.8903	0.5985	0.1560	0.0203
it = 10	0.7583	0.4790	0.1453	0.0193
it = 20	0.5698	0.4011	0.1410	0.0193

Furthermore, we investigate for Algorithm 1 as well as for the pMHE scheme [9] the resulting regret (34) with respect to the comparator sequence given by the true system state which yields zero cost F_k. More specifically, after each simulation time T, we compute and plot the resulting regrets $R(T)$ and average regrets $R(T)/T$ in Figure 3. Although the underlying MHE problem is nonconvex and hence the results of Theorem 2 cannot be applied in this case, we can see that the regret decreases with an increasing number of iterations and that the average regret goes to zero for $T \to \infty$.

VI. Conclusions

In this work, we extended the results developed in [10] to the case of nonlinear systems based on results from [9]. In particular, we proved that (local) exponential stability of the underlying estimation error can be ensured for any number of optimization algorithm iterations. Moreover, we showed that by considering a convex MHE problem as a special case, both global exponential stability and a sublinear regret can be guaranteed, where a smaller regret bound can be achieved by increasing the number of optimization algorithm iterations.

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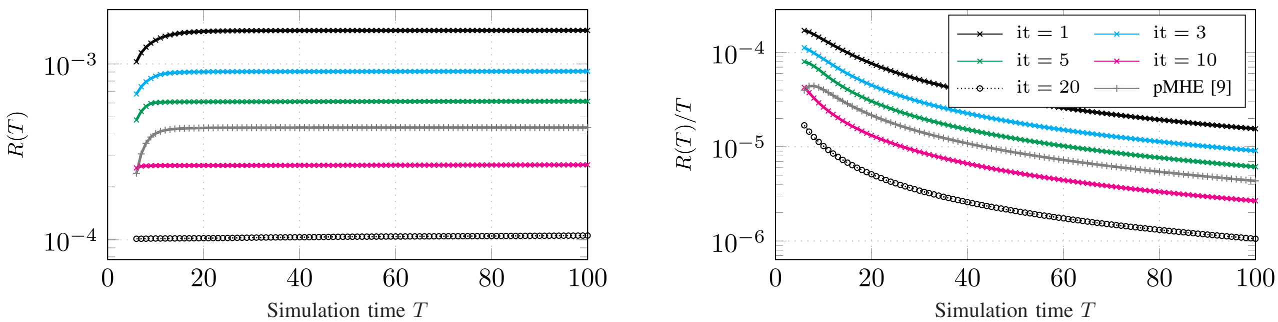

Fig. 3: Resulting regret and average regret of Algorithm 1 with different number of optimization algorithms iterations.

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APPENDIX

A. Bregman distances

We shortly present central properties of Bregman distances defined in (8). In analogy with the classical projection, the Bregman projection $\Pi_{k,\mathcal{S}}(\bar{\mathbf{z}})$ onto a convex set $\mathcal{S}$ is defined as the closest point in $\mathcal{S}$ to $\bar{\mathbf{z}}$ with respect to the Bregman distance D_k:

$$\Pi_{k,\mathcal{S}}(\bar{\mathbf{z}}) = \operatorname*{arg\,min}_{\mathbf{z}\in\mathcal{S}} \quad D_k(\mathbf{z},\bar{\mathbf{z}}). \tag{42}$$

The next key identity can be proven by directly using the definition of D_k in (8).

Lemma 5 *Let the function D_k denote a Bregman distance induced from ψ_k. Then for any $a, b, c \in \mathbb{R}^{Nm_{\mathrm{w}}+n}$, the following three-points identity holds*

$$\begin{aligned} D_\psi(c,a)+&D_\psi(a,b) - D_\psi(c,b) \\ &= \left(\nabla\psi(b) - \nabla\psi(a)\right)^\top (c-a). \end{aligned} \tag{43}$$

We also require the next result from [14, Proposition 3.5].

Lemma 6 *Let the set $\mathcal{S} \subset \mathbb{R}^{Nm_{\mathrm{w}}+n}$ be nonempty, closed and convex. Suppose $\bar{\mathbf{z}} \notin \mathcal{S}$ and $\mathbf{z} \in \mathcal{S}$. Then,*

$$D_k\big(\Pi_{k,\mathcal{S}}(\bar{\mathbf{z}}),\bar{\mathbf{z}}\big) \le D_k(\mathbf{z},\bar{\mathbf{z}}) - D_k\big(\mathbf{z},\Pi_{k,\mathcal{S}}(\bar{\mathbf{z}})\big). \tag{44}$$

More details on Bregman distances can be found in [14].

B. Proof of Lemma 1

Proof. By optimality of $T_{\eta_k^i}(\mathbf{z}_1)$ and $T_{\eta_k^i}(\mathbf{z}_2)$, we have

$$u \in \partial I_{\mathcal{S}}(T_{\eta_k^i}(\mathbf{z}_1)), \qquad v \in \partial I_{\mathcal{S}}(T_{\eta_k^i}(\mathbf{z}_2)) \tag{45}$$

where

$$u = -\nabla F_k\left(\mathbf{z}_1\right) - 1/\eta_k^i \nabla D_k(T_{\eta_k^i}(\mathbf{z}_1),\mathbf{z}_1) \tag{46}$$

$$v = -\nabla F_k\left(\mathbf{z}_2\right) - 1/\eta_k^i \nabla D_k(T_{\eta_k^i}(\mathbf{z}_2),\mathbf{z}_2). \tag{47}$$

By the monotonicity of the subdifferential of the convex function $I_{\mathcal{S}}$, we have

$$(u-v)^\top (T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)) \ge 0, \tag{48}$$

which implies that

$$\begin{aligned} &\frac{1}{\eta_k^i}\left(\nabla\psi_k(T_{\eta_k^i}(\mathbf{z}_1)) - \nabla\psi_k(T_{\eta_k^i}(\mathbf{z}_2))\right)^\top \left(T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right) \\ &\le \left(\nabla F_k\left(\mathbf{z}_2\right) - \nabla F_k\left(\mathbf{z}_1\right)\right)^\top \left(T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right) \\ &\quad + \frac{1}{\eta_k^i}\left(\nabla\psi_k(\mathbf{z}_1) - \nabla\psi_k(\mathbf{z}_2)\right)^\top \left(T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right). \end{aligned} \tag{49}$$

In view of Assumption 2, ψ_k is strongly convex with constant σ_k. We therefore get in (49) that

$$\begin{aligned} &\frac{\sigma_k}{\eta_k^i}\left\|T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right\|^2 \\ &\le \left\|\nabla F_k\left(\mathbf{z}_2\right) - \nabla F_k\left(\mathbf{z}_1\right)\right\| \left\|T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right\| \\ &\quad + \frac{1}{\eta_k^i}\left\|\nabla\psi_k(\mathbf{z}_1) - \nabla\psi_k(\mathbf{z}_2)\right\| \left\|T_{\eta_k^i}(\mathbf{z}_1) - T_{\eta_k^i}(\mathbf{z}_2)\right\|. \end{aligned} \tag{50}$$

By Assumptions 1 and 2, the gradients of F_k and ψ_k are Lipschitz continuous with constants L_k, γ_k, respectively. Hence, we obtain

$$\frac{\sigma_k}{\eta_k^i}\left\|T_{\eta_k^i}(\mathbf{z}_1)-T_{\eta_k^i}(\mathbf{z}_2)\right\|^2 \leq \left(L_k+\frac{\gamma_k}{\eta_k^i}\right)\left\|\mathbf{z}_1-\mathbf{z}_2\right\|\left\|T_{\eta_k^i}(\mathbf{z}_1)-T_{\eta_k^i}(\mathbf{z}_2)\right\|. \tag{51}$$

Dividing the last inequality by $\|T_{\eta_k^i}(\mathbf{z}_1)-T_{\eta_k^i}(\mathbf{z}_2)\|$ yields the desired result of the Lemma. □

C. Proof of Lemma 2

Proof. By optimality of $T_{\eta_k^i}(\mathbf{z})$ in the convex optimization problem (16), we have for any $\tilde{\mathbf{z}} \in \mathcal{S}$

$$\left(\eta_k^i \nabla F_k(\mathbf{z})+\nabla D_k\left(T_{\eta_k^i}(\mathbf{z}),\mathbf{z}\right)\right)^\top\left(\tilde{\mathbf{z}}-T_{\eta_k^i}(\mathbf{z})\right) \geq 0, \tag{52}$$

Since $\mathbf{z} \in \mathcal{S}$, setting $\tilde{\mathbf{z}}=\mathbf{z}$ in (52) yields

$$\eta_k^i \nabla F_k(\mathbf{z})^\top\left(\mathbf{z}-T_{\eta_k^i}(\mathbf{z})\right) \geq \nabla D_k\left(T_{\eta_k^i}(\mathbf{z}),\mathbf{z}\right)^\top\left(T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right). \tag{53}$$

Since the gradient of F_k is Lipschitz continuous by Assumption 1, we have that

$$F_k\left(T_{\eta_k^i}(\mathbf{z})\right) \leq F_k(\mathbf{z})+\nabla F_k(\mathbf{z})^\top\left(T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right) + \frac{L_k}{2}\left\|T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right\|^2, \tag{54}$$

which we substitute in (53) to obtain for all $\mathbf{z} \in \mathcal{S}$

$$\eta_k^i\left(F_k(\mathbf{z})-F_k\left(T_{\eta_k^i}(\mathbf{z})\right)\right) \geq -\frac{L_k\,\eta_k^i}{2}\left\|T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right\|^2 + \nabla D_k\left(T_{\eta_k^i}(\mathbf{z}),\mathbf{z}\right)^\top\left(T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right). \tag{55}$$

Due to the fact that $\nabla D_k(\mathbf{z}_1,\mathbf{z}_2)=\nabla\psi_k(\mathbf{z}_1)-\nabla\psi_k(\mathbf{z}_2)$ as well as the strong convexity of ψ_k in Assumption 2, it holds for the last term in the right-hand side of (55) that

$$\left(\nabla\psi_k\left(T_{\eta_k^i}(\mathbf{z})\right)-\nabla\psi_k(\mathbf{z})\right)^\top\left(T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right) \geq \sigma_k\left\|T_{\eta_k^i}(\mathbf{z})-\mathbf{z}\right\|^2. \tag{56}$$

Substituting the last inequality in (55) and dividing by $\eta_k^i \in \mathbb{R}_{++}$ yields the desired result of the lemma. □

D. Proof of Lemma 3

Proof. By optimality of $\hat{\mathbf{z}}_k^{i+1}$ in the convex optimization problem (7), we have for any $\mathbf{z} \in \mathcal{S}$

$$\left(\eta_k^i \nabla F_k\left(\hat{\mathbf{z}}_k^i\right)+\nabla D_k\left(\hat{\mathbf{z}}_k^{i+1},\hat{\mathbf{z}}_k^i\right)\right)^\top\left(\mathbf{z}-\hat{\mathbf{z}}_k^{i+1}\right) \geq 0, \tag{57}$$

We equivalently reformulate the last inequality for all $\mathbf{z} \in \mathcal{S}$

$$\nabla F_k\left(\hat{\mathbf{z}}_k^i\right)^\top\left(\mathbf{z}-\hat{\mathbf{z}}_k^i\right) \geq \nabla F_k\left(\hat{\mathbf{z}}_k^i\right)^\top\left(\hat{\mathbf{z}}_k^{i+1}-\hat{\mathbf{z}}_k^i\right) - \frac{1}{\eta_k^i}\nabla D_k\left(\hat{\mathbf{z}}_k^{i+1},\hat{\mathbf{z}}_k^i\right)^\top\left(\mathbf{z}-\hat{\mathbf{z}}_k^{i+1}\right). \tag{58}$$

Since the gradient of F_k is Lipschitz continuous by Assumption 1, we have that

$$F_k\left(\hat{\mathbf{z}}_k^{i+1}\right) \leq F_k\left(\hat{\mathbf{z}}_k^i\right)+\nabla F_k\left(\hat{\mathbf{z}}_k^i\right)^\top\left(\hat{\mathbf{z}}_k^{i+1}-\hat{\mathbf{z}}_k^i\right) + \frac{L_k}{2}\left\|\hat{\mathbf{z}}_k^{i+1}-\hat{\mathbf{z}}_k^i\right\|^2, \tag{59}$$

which we substitute in (58) to obtain for all $\mathbf{z} \in \mathcal{S}$

$$\nabla F_k\left(\hat{\mathbf{z}}_k^i\right)^\top\left(\mathbf{z}-\hat{\mathbf{z}}_k^i\right) \geq F_k\left(\hat{\mathbf{z}}_k^{i+1}\right)-F_k\left(\hat{\mathbf{z}}_k^i\right)-\frac{L_k}{2}\left\|\hat{\mathbf{z}}_k^{i+1}-\hat{\mathbf{z}}_k^i\right\|^2 - \frac{1}{\eta_k^i}\nabla D_k\left(\hat{\mathbf{z}}_k^{i+1},\hat{\mathbf{z}}_k^i\right)^\top\left(\mathbf{z}-\hat{\mathbf{z}}_k^{i+1}\right). \tag{60}$$

Let us now introduce the following vectors defined in terms of the output residuals and estimated process disturbances at the i-th iteration and for a given time instant k

$$D_{r,k}^i=\left[\nabla r^\top(\hat{v}_{k-N}^i)\ \ \dots\ \ \nabla r^\top(\hat{v}_{k-1}^i)\right]^\top \in \mathbb{R}^{Np}, \tag{61a}$$

$$D_{q,k}^i=\left[\nabla q^\top(\hat{w}_{k-N}^i)\ \ \dots\ \ \nabla q^\top(\hat{w}_{k-1}^i)\right]^\top \in \mathbb{R}^{Nm_{\rm w}}, \tag{61b}$$

$$\hat{\mathbf{v}}_k^i=\left[\hat{v}_{k-N}^{i\top}\ \ \dots\ \ \hat{v}_{k-1}^{i\top}\right]^\top \in \mathbb{R}^{Np}. \tag{61c}$$

Moreover, we introduce the function $H_k:\mathbb{R}^{Nm_{\rm w}+n} \to \mathbb{R}^{Np}$

$$H_k(\hat{\mathbf{z}}_k)=\begin{bmatrix} h\left(\hat{x}_{k-N}\right) \\ h\left(x\left(k-N+1;\hat{x}_{k-N},u_{k-N},\hat{w}_{k-N}\right)\right) \\ \vdots \\ h\left(x\left(k-1;\hat{x}_{k-N},\mathbf{u},\hat{\mathbf{w}}\right)\right) \end{bmatrix} \tag{62}$$

and the matrix

$$\tilde{H}=\left[0_{[Nm_{\rm w}\times n]}\quad I_{[Nm_{\rm w}]}\right] \in \mathbb{R}^{Nm_{\rm w}\times(Nm_{\rm w}+n)}. \tag{63}$$

In view of (4c), (62) and the above definitions, we have

$$\nabla F_k(\hat{\mathbf{z}}_k^i)=-\nabla H_k(\hat{\mathbf{z}}_k^i)D_{r,k}^i+\tilde{H}^\top D_{q,k}^i. \tag{64}$$

By (2c), and since the true disturbances are $w_k=0, v_k=0, k\in\mathbb{N}$, it holds that $y_j=h\left(x\left(j;x_{k-N},\mathbf{u}\right)\right)$ where $j=k-N,\cdots,k-1$, and hence, using (61c) and (62),

$$\hat{\mathbf{v}}_k^i=H_k(\mathbf{z}_k)-H_k(\hat{\mathbf{z}}_k^i). \tag{65}$$

By Taylor's theorem [15], we have for the $l-$th component of the vector H_k defined in (62) that

$$H_k^l(\mathbf{z}_k)=H_k^l(\hat{\mathbf{z}}_k^i)+\nabla H_k^l(\hat{\mathbf{z}}_k^i)^\top(\mathbf{z}_k-\hat{\mathbf{z}}_k^i)+d_k^l(\mathbf{z}_k,\hat{\mathbf{z}}_k^i) \tag{66a}$$

where

$$d_k^l(\mathbf{z}_k,\hat{\mathbf{z}}_k^i)=\frac{1}{2}(\mathbf{z}_k-\hat{\mathbf{z}}_k^i)^\top\nabla^2 H_k^l(\xi_j)(\mathbf{z}_k-\hat{\mathbf{z}}_k^i). \tag{66b}$$

Here, $\xi_l := \hat{\mathbf{z}}_k^i+t_l\left(\mathbf{z}_k-\hat{\mathbf{z}}_k^i\right)$ for some $t_l\in(0,1)$. Hence, it holds that

$$H_k(\mathbf{z}_k)=H_k(\hat{\mathbf{z}}_k^i)+\nabla H_k(\hat{\mathbf{z}}_k^i)^\top(\mathbf{z}_k-\hat{\mathbf{z}}_k^i)+\mathbf{d}_k(\mathbf{z}_k,\hat{\mathbf{z}}_k^i), \tag{67}$$

where $\mathbf{d}_k$ is the column vector of the stacked d_k^l defined in (66b) and with $l=1,\cdots,Np$. In view of (67), we therefore get

$$\hat{\mathbf{v}}_k^i=\nabla H_k(\hat{\mathbf{z}}_k^i)^\top(\mathbf{z}_k-\hat{\mathbf{z}}_k^i)+\mathbf{d}_k(\mathbf{z}_k,\hat{\mathbf{z}}_k^i). \tag{68}$$

According to Assumption 1, the functions r and q are smooth, convex, and achieve zero at zero. Hence,

$$r(\hat{v}_j^i)=r(\hat{v}_j^i)-r(0)\leq\nabla r(\hat{v}_j^i)^\top(\hat{v}_j^i-0) \tag{69a}$$

$$q(\hat{w}_j^i)=q(\hat{w}_j^i)-q(0)\leq\nabla q(\hat{w}_j^i)^\top(\hat{w}_j^i-0). \tag{69b}$$

In view of (69) and the definitions in (61), we have

$$\begin{aligned} F_k(\hat{\mathbf{z}}_k^i) &= \sum_{j=k-N}^{k-1} r(\hat{v}_j^i) + q(\hat{w}_j^i) \\ &\leq \sum_{j=k-N}^{k-1} \nabla r(\hat{v}_j^i)^\top \hat{v}_j^i + \nabla q(\hat{w}_j^i)^\top \hat{w}_j^i \\ &= (D_{r,k}^i)^\top \hat{\mathbf{v}}_k^i + (D_{q,k}^i)^\top \hat{\mathbf{w}}_k^i. \end{aligned} \tag{70}$$

By (68) and (64), and since $\hat{\mathbf{w}}_k^i = \tilde{H}(\hat{\mathbf{z}}_k^i - \mathbf{z}_k)$, we obtain

$$\begin{aligned} F_k(\hat{\mathbf{z}}_k^i) &\leq (D_{q,k}^i)^\top \hat{\mathbf{w}}_k^i \\ &\quad + (D_{r,k}^i)^\top (\nabla H_k(\hat{\mathbf{z}}_k^i)^\top (\mathbf{z}_k - \hat{\mathbf{z}}_k^i) + \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)) \\ &= \nabla F_k(\hat{\mathbf{z}}_k^i)^\top \left(\hat{\mathbf{z}}_k^i - \mathbf{z}_k\right) + (D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i). \end{aligned} \tag{71}$$

Substituting (60) evaluated at $\mathbf{z}_k \in \mathcal{S}$ in the last inequality and using the fact that $F_k\left(\hat{\mathbf{z}}_k^{i+1}\right) > 0$ yields

$$\begin{aligned} (D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i) &\geq -\frac{L_k}{2} \left\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\right\|^2 \\ &\quad - \frac{1}{\eta_k^i} \nabla D_k\left(\hat{\mathbf{z}}_k^{i+1}, \hat{\mathbf{z}}_k^i\right)^\top \left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{i+1}\right). \end{aligned} \tag{72}$$

In view of the three points identity of Bregman distances given in (43) and (11) in Assumption 2, we have

$$\begin{aligned} &\left(\nabla\psi_k\left(\hat{\mathbf{z}}_k^i\right) - \nabla\psi_k\left(\hat{\mathbf{z}}_k^{i+1}\right)\right)^\top \left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{i+1}\right) \\ &\quad = D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^{i+1}\right) + D_k\left(\hat{\mathbf{z}}_k^{i+1}, \hat{\mathbf{z}}_k^i\right) - D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^i\right) \\ &\quad \geq D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^{i+1}\right) + \frac{\sigma_k}{2}\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\|^2 - D_\psi\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^i\right). \end{aligned} \tag{73}$$

Therefore, by employing (73) in (72), we get

$$\begin{aligned} D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^{i+1}\right) &\leq D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^i\right) + \frac{\eta_k^i}{2}\left(L_k - \frac{\sigma_k}{\eta_k^i}\right) \left\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\right\|^2 \\ &\quad + \eta_k^i (D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i). \end{aligned} \tag{74}$$

Let us now derive an upper bound for the last term in the right-hand side of (74). By the Lipschitz continuity of ∇r in Assumption 1,

$$\begin{aligned} (D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i) &\leq \|D_{r,k}^i\| \|\mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)\| \\ &\leq L_r \|\hat{\mathbf{v}}_k^i\| \|\mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)\|. \end{aligned} \tag{75}$$

Since f, h are $\mathcal{C}^2$ functions, the Taylor approximation remainders of their compositions are of second order. Hence, in (67), there exist $\kappa_d, \delta \in \mathbb{R}_{++}$ such that

$$\|\mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)\| \leq \kappa_d \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^2 \tag{76}$$

holds for all $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ with $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \leq \delta$. Moreover, the fact that f, h are $\mathcal{C}^2$ functions implies that they are globally Lipschitz continuous, and hence their compositions in H_k defined in (62) is Lipschitz continuous. Let $c_H \in \mathbb{R}_{++}$ refer to the uniform Lipschitz constant of the function H_k. Thus, in view of (65) and (76), we get

$$\begin{aligned} &(D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i) \\ &\quad \leq L_r \|H_k(\mathbf{z}_k) - H_k(\hat{\mathbf{z}}_k^i)\| \|\mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)\| \\ &\quad \leq L_r\, c_H\, \kappa_d\, \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^3 \end{aligned} \tag{77}$$

for all $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ with $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \leq \delta$. Substituting the last inequality in (74) yields the desired result with $\kappa := L_r\, c_H\, \kappa_d$. □

E. Proof of Lemma 4

Proof. We consider (18) in Lemma 1 and set $\mathbf{z}_1 = \mathbf{z}_k$ and $\mathbf{z}_2 = \hat{\mathbf{z}}_k^i$, i.e.,

$$\left\|T_{\eta_k^i}(\mathbf{z}_k) - T_{\eta_k^i}(\hat{\mathbf{z}}_k^i)\right\| \leq \frac{\eta_k^i}{\sigma_k}\left(L_k + \frac{\gamma_k}{\eta_k^i}\right)\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\|. \tag{78}$$

Since the true system state $\mathbf{z}_k$ satisfies the constraints, it holds that $I_{\mathcal{S}}(\mathbf{z}_k) = 0$ in problem (15). Moreover, because we have zero true disturbances and due to Assumption 1, evaluating F_k defined in (4c) at $\mathbf{z}_k$ leads to

$$F_k(\mathbf{z}_k) = \sum_{j=k-N}^{k-1} r(0) + q(0) = 0. \tag{79}$$

Thus, $\mathbf{z}_k$ is a minimum of (15), which yields by (17) that $T_{\eta_k^i}(\mathbf{z}_k) = \mathbf{z}_k$. Hence, (78) becomes

$$\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{i+1}\right\| \leq \left(\frac{\eta_k^i}{\sigma_k} L_k + \frac{\gamma_k}{\sigma_k}\right)\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\|. \tag{80}$$

Given that the step size satisfies $\eta_k^i \leq \frac{\sigma_k}{L_k}$, we get

$$\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{i+1}\right\| \leq \left(1 + \frac{\gamma_k}{\sigma_k}\right)\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\| \tag{81}$$

and therefore

$$\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\| \leq \left(1 + \frac{\gamma_k}{\sigma_k}\right)^i\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^0\right\|. \tag{82}$$

Since we assumed that $\mathrm{it}(k+1) \leq \mathrm{it}(k)$, it holds for every i-th iteration that $i \leq \mathrm{it}(k) \leq \mathrm{it}(0)$ which yields

$$\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\| \leq \left(1 + \frac{\gamma_k}{\sigma_k}\right)^{\mathrm{it}(0)}\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^0\right\|. \tag{83}$$

Given that $\hat{\mathbf{z}}_k^0 = \Pi_{k,\mathcal{S}}\left(\bar{\mathbf{z}}_k\right)$ as defined in (42) and $\mathbf{z}_k \in \mathcal{S}$, we have in view of (44) in Lemma 6 that

$$0 \leq D_k\left(\hat{\mathbf{z}}_k^0, \bar{\mathbf{z}}_k\right) \leq D_k\left(\mathbf{z}_k, \bar{\mathbf{z}}_k\right) - D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^0\right). \tag{84}$$

Thus, $D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^0\right) \leq D_k\left(\mathbf{z}_k, \bar{\mathbf{z}}_k\right)$. By the lower and upper bounds in (11) in Assumption 2, we get

$$\|\mathbf{z}_k - \hat{\mathbf{z}}_k^0\| \leq \sqrt{\gamma_k/\sigma_k}\|\mathbf{z}_k - \bar{\mathbf{z}}_k\|. \tag{85}$$

Applying the last inequality in (83) yields

$$\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\right\| \leq \left(1 + \frac{\gamma_k}{\sigma_k}\right)^{\mathrm{it}(0)}\sqrt{\frac{\gamma_k}{\sigma_k}}\left\|\mathbf{z}_k - \bar{\mathbf{z}}_k\right\|, \tag{86}$$

which, by the uniform bounds in Assumption 2, proves the result of the Lemma with $\alpha := \left(1 + \gamma/\sigma\right)^{\mathrm{it}(0)}\sqrt{\gamma/\sigma}$. □

F. Proof of Theorem 1

Proof. First, we prove that the pMHE error (13) is exponentially stable. Let V_k be a time-varying candidate Lyapunov function chosen as $V_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) = D_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)})$. In the following, we show that there exist positive constants α_1, α_2, α_3 and $\bar{\epsilon}$ such that V_k satisfies the following conditions

$$\alpha_1\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\right\|^2 \leq V_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\right) \leq \alpha_2\left\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\right\|^2 \tag{87a}$$

and

$$\Delta V_k := V_k\big(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\big) - V_{k-1}\big(\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big) \le -\alpha_3 \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\|^2 \tag{87b}$$

for any $k \in \mathbb{N}_+$ and $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^i$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^i\| \le \bar{\epsilon}$. Note that by Assumption 2, (87a) follows with $\alpha_1 = \frac{\sigma}{2}$ and $\alpha_2 = \frac{\gamma}{2}$.

(i): We start by deriving an upper bound on $V_k\big(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\big)$ in (87b). Applying (20) of Lemma 3 to each two subsequent iterations i and $i-1$ (starting from $i = \mathrm{it}(k)$ until $i = 0$) yields

$$\begin{aligned} &D_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) \\ &\le D_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^0) + \frac{1}{2} \sum_{i=0}^{\mathrm{it}(k)-1} \left(\eta_k^i L_k - \sigma_k\right) \big\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\big\|^2 \\ &\quad + \sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \, \kappa \, \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^3 \end{aligned} \tag{88}$$

for all $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ with $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \le \delta$. Condition (22) on the step sizes implies that $\eta_k^i L_k - \sigma_k \le 0$ and hence,

$$\frac{1}{2} \sum_{i=0}^{\mathrm{it}(k)-1} \left(\eta_k^i L_k - \sigma_k\right) \big\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\big\|^2 \le 0. \tag{89}$$

Moreover, in view of (84), $D_k\left(\mathbf{z}_k, \hat{\mathbf{z}}_k^0\right) \le D_k\left(\mathbf{z}_k, \bar{\mathbf{z}}_k\right)$ and we obtain in (88)

$$D_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) \le D_k(\mathbf{z}_k, \bar{\mathbf{z}}_k) + \sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \, \kappa \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^3 \tag{90}$$

for all $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ with $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \le \delta$.

(ii): We now show that the Lyapunov difference (87b) is locally negative definite. We obtain due to (90) that

$$\begin{aligned} \Delta V_k &= D_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) - D_{k-1}(\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}) \\ &\le D_k(\mathbf{z}_k, \bar{\mathbf{z}}_k) + \sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \, \kappa \, \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^3 \\ &\quad - D_{k-1}(\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}) \end{aligned} \tag{91}$$

for all $\mathbf{z}_k, \hat{\mathbf{z}}_k^i$ with $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \le \delta$. Note that since $\mathbf{z}_k = \Phi_{k-1}\left(\mathbf{z}_{k-1}\right)$ by Assumption 3,

$$D_k(\mathbf{z}_k, \bar{\mathbf{z}}_k) = D_k(\Phi_{k-1}(\mathbf{z}_{k-1}), \Phi_{k-1}(\hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)})). \tag{92}$$

Let us now derive an upper bound for the second term (the sum) in the right-hand side of (91). By using (21) in Lemma 4 and $\mathbf{z}_k = \Phi_{k-1}\left(\mathbf{z}_{k-1}\right)$, we have that

$$\begin{aligned} \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| &\le \alpha \|\mathbf{z}_k - \bar{\mathbf{z}}_k\| \\ &= \alpha \big\|\Phi_{k-1}(\mathbf{z}_{k-1}) - \Phi_{k-1}(\hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)})\big\|. \end{aligned} \tag{93}$$

In view of (23) and the uniform bounds in Assumption 2,

$$\begin{aligned} 0 &\le \sigma \big\|\Phi_{k-1}(\mathbf{z}_{k-1}) - \Phi_{k-1}(\hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)})\big\|^2 \\ &\le (\gamma - 2c) \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\|^2 \end{aligned} \tag{94}$$

for all $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \epsilon$. Using (94) in (93) yields

$$\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \le \alpha \sqrt{(\gamma - 2c)/\sigma} \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\| \tag{95}$$

for all $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \epsilon'$, where

$$\epsilon' := \min\left(\epsilon, \frac{\delta}{\alpha\sqrt{(\gamma - 2c)/\sigma}}\right). \tag{96}$$

Note that this choice of ϵ' ensures that $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \epsilon$ and $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\| \le \delta$ in (95). Moreover, by (22), the condition $\mathrm{it}(k+1) \le \mathrm{it}(k)$, as well as Assumptions 1 and 2,

$$\sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \le \frac{\sigma_k}{L_k} \mathrm{it}(k) \le \frac{\sigma}{L_{\mathrm{F}}} \mathrm{it}(0). \tag{97}$$

Thus, by (95) and the last inequality, we arrive at

$$\begin{aligned} &\sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \, \kappa \, \|\mathbf{z}_k - \hat{\mathbf{z}}_k^i\|^3 \\ &\quad \le \kappa \, \alpha^3 \sqrt{(\gamma - 2c)/\sigma}^3 \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\|^3 \sum_{i=0}^{\mathrm{it}(k)-1} \eta_k^i \\ &\quad \le \tilde{\kappa} \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\|^3, \end{aligned} \tag{98}$$

where $\tilde{\kappa} := \frac{\sigma}{L_{\mathrm{F}}} \mathrm{it}(0) \, \kappa \, \alpha^3 \sqrt{(\gamma - 2c)/\sigma}^3$. By applying (92), (98), and condition (23) in (91), we obtain

$$\Delta V_k \le -c \|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\|^2 + \tilde{\kappa} \big\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\big\|^3 \tag{99}$$

for any $k \in \mathbb{N}_+$ and $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \epsilon'$. By introducing $\bar{\epsilon} := \min(\epsilon', c/(2\tilde{\kappa}))$, we have that ΔV_k is locally negative definite, i.e., it satisfies (87b) with $\alpha_3 = \frac{c}{2}$ for all $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \bar{\epsilon}$.

More specifically, in view of (99) and the uniform lower bound in Assumption 2, we have

$$\begin{aligned} V_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) &\le V_{k-1}(\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}) - \frac{c}{2} \|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\|^2 \\ &\le \tilde{\beta} \, V_{k-1}(\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}) \end{aligned} \tag{100}$$

for all $k \in \mathbb{N}_+$ and $\mathbf{z}_{k-1}, \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}$ with $\|\mathbf{z}_{k-1} - \hat{\mathbf{z}}_{k-1}^{\mathrm{it}(k-1)}\| \le \bar{\epsilon}$, where $\tilde{\beta} := 1 - c/\sigma$. Note that $\tilde{\beta} \in (0, 1)$ since V_k is strictly negative for all $\mathbf{z}_k \ne \hat{\mathbf{z}}_k^{\mathrm{it}(k)}$ and $c/\sigma \in \mathbb{R}_{++}$. By performing a geometric series and in view of Assumption 2, we get

$$\begin{aligned} \frac{\sigma}{2} \|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\|^2 &\le V_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^{\mathrm{it}(k)}) \\ &\le \tilde{\beta}^k \, V_0(\mathbf{z}_0, \hat{\mathbf{z}}_0^{\mathrm{it}(0)}) \le \tilde{\beta}^k \, \frac{\gamma}{2} \|\mathbf{z}_0 - \hat{\mathbf{z}}_0^{\mathrm{it}(0)}\|^2 \end{aligned} \tag{101}$$

and thus

$$\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\| \le \alpha \beta^k \|\mathbf{z}_0 - \hat{\mathbf{z}}_0^{\mathrm{it}(0)}\| \tag{102}$$

for all $\mathbf{z}_0, \hat{\mathbf{z}}_0^{\mathrm{it}(0)}$ with $\|\mathbf{z}_0 - \hat{\mathbf{z}}_0^{\mathrm{it}(0)}\| \le \tilde{\epsilon}$, where $\alpha := \sqrt{\gamma/\sigma}$, $\beta := \sqrt{\tilde{\beta}} \in (0, 1)$, and $\tilde{\epsilon} := \min(\bar{\epsilon}, \frac{\bar{\epsilon}}{\alpha})$. Note that with this choice of $\tilde{\epsilon}$ we can make sure that $\|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\| \le \bar{\epsilon}$ in (102) and hence can recursively apply (100) to get (101). Thus,

the pMHE error (13) is locally exponentially stable.
(iii): In the following, we will derive an upper bound on the estimation error $x_k - \hat{x}_k$ in terms of the pMHE error. For the estimation error (14), we have by the triangular inequality

$$\begin{aligned}\|x_k - \hat{x}_k\| \le \|x(k; x_{k-N}, \mathbf{u}_k) - x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k)\| \quad (103)\\ + \|x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k) - x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k, \hat{\mathbf{w}}_k^{\mathrm{it}(k)})\|.\end{aligned}$$

Since f is a $\mathcal{C}^2$ function, it is Lipschitz continuous in all of its arguments with some Lipschitz constant $c_f \in \mathbb{R}_{++}$. Recalling (13) therefore yields

$$\|x(k; x_{k-N}, \mathbf{u}_k) - x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k)\| \le c_f^N \|e_{k-N}\|, \quad (104a)$$

$$\begin{aligned}\|x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k) - x(k; \hat{x}_{k-N}^{\mathrm{it}(k)}, \mathbf{u}_k, \hat{\mathbf{w}}_k^{\mathrm{it}(k)})\| \quad (104b)\\ \le \sum_{j=k-N}^{k-1} c_f^{k-j} \|\hat{w}_j^{\mathrm{it}(k)}\|.\end{aligned}$$

In view of (104), we obtain in (103) that

$$\|x_k - \hat{x}_k\| \le \bar{c}\, \|\mathbf{z}_k - \hat{\mathbf{z}}_k^{\mathrm{it}(k)}\|, \quad (105)$$

where $\bar{c} := c_f^N + \sum_{j=1}^N c_f^j$. By (102) and the fact that $\hat{\mathbf{z}}_0^{\mathrm{it}(0)} = \bar{\mathbf{z}}_0 = \hat{x}_0$, we get in (105)

$$\|x_k - \hat{x}_k\| \le \bar{c}\, \alpha \beta^k \|\mathbf{z}_0 - \bar{\mathbf{z}}_0\| = \bar{c}\, \alpha \beta^k \|x_0 - \hat{x}_0\|, \quad (106)$$

for all $x_0, \hat{x}_0$ with $\|x_0 - \hat{x}_0\| \le \tilde{\epsilon}$, which proves the local exponential stability of the estimation error, i.e., (24) with $\tilde{\alpha} := \bar{c}\, \alpha$. □

G. Proof of Proposition 1

Proof. Given the choice of D_k in (29), D_k satisfies Assumption 2 with $\sigma_k = \min(\frac{1}{\lambda_{\max}(P_k)}, \lambda_{\min}(W))$ and $\gamma_k = \max(\frac{1}{\lambda_{\min}(P_k)}, \lambda_{\max}(W))$. Moreover, due to the uniform bounds (32) in Assumption 4, $\sigma = \min\{\frac{1}{\bar{p}}, \lambda_{\min}(W)\}$ and $\gamma = \max\{\frac{1}{\underline{p}}, \lambda_{\max}(W)\}$.
Evaluating the a priori estimate operator (28) at the true system state $\mathbf{z}_k = \begin{bmatrix} x_{k-N}^\top & 0 & \dots & 0 \end{bmatrix}^\top$ yields

$$\begin{aligned}\Phi_k(\mathbf{z}_k) &= \begin{bmatrix} f(x_{k-N} + K_{k-N}(y_{k-N} - h(x_{k-N})), u_{k-N}, 0) \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} f(x_{k-N}, u_{k-N}, 0) \\ \mathbf{0} \end{bmatrix} = \mathbf{z}_{k+1} \quad (107)\end{aligned}$$

and hence Assumption 3 is satisfied.
Concerning the last statement of the proposition, we proceed with similar steps to the convergence proof of the EKF in [16]. Let $\mathbf{z} = \begin{bmatrix} x^\top & \mathbf{w}^\top \end{bmatrix}^\top$, $\hat{\mathbf{z}} = \begin{bmatrix} \hat{x}^\top & \hat{\mathbf{w}}^\top \end{bmatrix}^\top$, $x, \hat{x} \in \mathbb{R}^n$, $\mathbf{w}, \hat{\mathbf{w}} \in \mathbb{R}^{N m_{\mathrm{w}}}$. Moreover, let

$$x^+ = x + K_{k-N-1}(y_{k-N-1} - h(x)) \quad (108a)$$

$$\hat{x}^+ = \hat{x} + K_{k-N-1}(y_{k-N-1} - h(\hat{x})). \quad (108b)$$

In view of the EKF-based a priori estimate (28) and Bregman distance (29), and given the above defintions, we have

$$\begin{aligned}\Delta D_k &:= D_k(\Phi_{k-1}(\mathbf{z}), \Phi_{k-1}(\hat{\mathbf{z}})) - D_{k-1}(\mathbf{z}, \hat{\mathbf{z}}) \quad (109a)\\ &= \frac{1}{2}\|f(x^+, u_{k-N-1}, 0) - f(\hat{x}^+, u_{k-N-1}, 0)\|^2_{P_{k-N}^{-1}} \\ &\quad - \frac{1}{2}\|x - \hat{x}\|^2_{P_{k-N-1}^{-1}} - \frac{1}{2}\|\mathbf{w} - \hat{\mathbf{w}}\|^2_W.\end{aligned}$$

Since f and h are $\mathcal{C}^2$ functions, they can be expanded as

$$\begin{aligned}f(x^+, u_{k-N-1}, 0) &= f(\hat{x}^+, u_{k-N-1}, 0) \quad (109b)\\ &\quad + A_{k-N-1}(x^+ - \hat{x}^+) + \varphi(x^+, \hat{x}^+)\\ h(x) &= h(\hat{x}) + C_{k-N-1}(x - \hat{x}) + \chi(x, \hat{x}), \quad (109c)\end{aligned}$$

where

$$A_{k-N-1} = \left.\frac{\partial f}{\partial x}\right|_{(\hat{x}^+, u_{k-N-1}, 0)}, \quad C_{k-N-1} = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}}, \quad (110)$$

and φ, χ are higher order terms. In addition that there exist constants $\epsilon_\varphi, \epsilon_\chi, \kappa_\varphi, \kappa_\chi \in \mathbb{R}_{++}$ such that

$$\|\varphi(x^+, \hat{x}^+)\| \le \kappa_\varphi \|x^+ - \hat{x}^+\|^2 \quad (111a)$$

$$\|\chi(x, \hat{x})\| \le \kappa_\chi \|x - \hat{x}\|^2 \quad (111b)$$

holds for $\|x^+ - \hat{x}^+\| \le \epsilon_\varphi$ and for $\|x - \hat{x}\| \le \epsilon_\chi$, respectively. Given (109c) and (108), we can compute

$$\begin{aligned}x^+ - x &= K_{k-N-1} y_{k-N-1} - K_{k-N-1} h(x) \quad (112)\\ &= \hat{x}^+ - \hat{x} + K_{k-N-1}(h(\hat{x}) - h(x))\\ &= \hat{x}^+ - \hat{x} - K_{k-N-1}(C_{k-N-1}(x - \hat{x}) + \chi(x, \hat{x})).\end{aligned}$$

Hence,

$$\begin{aligned}x^+ - \hat{x}^+ &= (I - K_{k-N-1} C_{k-N-1})(x - \hat{x}) \quad (113)\\ &\quad - K_{k-N-1} \chi(x, \hat{x}).\end{aligned}$$

We therefore get in (109b)

$$\begin{aligned}f(x^+, 0) - (\hat{x}^+, 0) &= A_{k-N-1}(x^+ - \hat{x}^+) + \varphi(x^+, \hat{x}^+)\\ &= \tilde{A}_{k-N-1}(x - \hat{x}) + r_{k-N-1}, \quad (114a)\end{aligned}$$

where $\tilde{A}_{k-N-1} = A_{k-N-1}(I - K_{k-N-1} C)$ and

$$r_{k-N-1} = \varphi(x^+, \hat{x}^+) - A_{k-N-1} K_{k-N-1} \chi(x, \hat{x}). \quad (114b)$$

By substituting (114a) in (109a), we obtain

$$\begin{aligned}\Delta D_k &= \frac{1}{2}\|\tilde{A}_{k-N-1}(x - \hat{x}) + r_{k-N-1}\|^2_{P_{k-N}^{-1}} \quad (115)\\ &\quad - \frac{1}{2}\|x - \hat{x}\|^2_{P_{k-N-1}^{-1}} - \frac{1}{2}\|\mathbf{w} - \hat{\mathbf{w}}\|^2_W.\end{aligned}$$

The result in [16, Lemma 6] implies that

$$\tilde{A}_{k-N-1}^\top P_{k-N}^{-1} \tilde{A}_{k-N-1} - P_{k-N-1}^{-1} \preceq -U_{k-N-1}, \quad (116)$$

where, with $j = k - N + 1$, and by Assumption 4,

$$U_j = P_j^{-1}\left((P_j^+)^{-1} + A_j^\top Q^{-1} A_j\right)^{-1} P_j^{-1} \succ 0. \quad (117)$$

Hence, we get in (115) that

$$\Delta D_k \le -\frac{1}{2}\|x - \hat{x}\|^2_{U_{k-N-1}} + T_{k-N-1} - \frac{1}{2}\|\mathbf{w} - \hat{\mathbf{w}}\|^2_W, \quad (118a)$$

where

$$\begin{aligned}T_{k-N-1} &:= r_{k-N-1}^\top P_{k-N}^{-1} \tilde{A}_{k-N-1}(x - \hat{x}) \quad (118b)\\ &\quad + \frac{1}{2}\|r_{k-N-1}\|^2_{P_{k-N}^{-1}}.\end{aligned}$$

We compute an upper bound for each of the first two summands in (118a). By (117) and Assumption 4, we obtain

$$\frac{1}{2}\|x-\hat{x}\|^2_{U_{k-N-1}} \geq \frac{1}{2\bar{p}^2(\bar{p}+\bar{a}^2/\underline{q})}\|x-\hat{x}\|^2, \tag{119}$$

where $\underline{q} \in \mathbb{R}_{++}$ is the smallest eigenvalue of Q. Furthermore,

$$T_{k-N-1} \leq \frac{\bar{a}}{\underline{p}}\big(1+\bar{k}\bar{c}\big)\|r_{k-N-1}\|\|x-\hat{x}\|+\frac{1}{2\underline{p}}\|r_{k-N-1}\|^2. \tag{120}$$

Let us compute and upper bound for $r_{k-N+1} = r_j$. By (114b) and (111), it holds that

$$\|r_j\| \leq \kappa_\varphi\|x^+-\hat{x}^+\|^2 + \bar{a}\bar{k}\,\kappa_\chi\|x-\hat{x}\|^2 \tag{121}$$

for $\|x^+-\hat{x}^+\| \leq \epsilon_\varphi$ and for $\|x-\hat{x}\| \leq \epsilon_\chi$. By Assumption 4, (113), (111) and the triangular inequality,

$$\|x^+-\hat{x}^+\| \leq (1+\bar{k}\bar{c}+\bar{k}\kappa_\chi\epsilon_\chi)\|x-\hat{x}\|, \tag{122}$$

for $\|x-\hat{x}\| \leq \epsilon_\chi$. In the following, we consider all $x, \hat{x}$ with $\|x-\hat{x}\| \leq \epsilon'$, where

$$\epsilon' := \min\big(\epsilon_\chi, \epsilon_\varphi/(1+\bar{k}\bar{c}+\bar{k}\kappa_\chi\epsilon_\chi)\big)\,. \tag{123}$$

Therefore, we satisfy $\|x^+-\hat{x}^+\| \leq \epsilon_\varphi$ in (122) as well as $\|x-\hat{x}\| \leq \epsilon_\chi$. We obtain for $\|x-\hat{x}\| \leq \epsilon'$

$$\|r_j\| \leq \kappa_r\|x-\hat{x}\|^2, \tag{124}$$

where $\kappa_r = \kappa_\varphi(1+\bar{k}\bar{c}+\bar{k}\kappa_\chi\epsilon_\chi)^2 + \bar{a}\bar{k}\,\kappa_\chi$. Substituting the last inequality in (120) yields

$$\begin{aligned} T_{k-N-1} &\leq \frac{\kappa_r}{\underline{p}}\bar{a}\left(1+\bar{k}\bar{c}\right)\|x-\hat{x}\|^3+\frac{\kappa_r^2}{\underline{p}}\|x-\hat{x}\|^4 \\ &\leq \frac{\kappa_r}{\underline{p}}\left(\bar{a}\left(1+\bar{k}\bar{c}\right)+\kappa_r\epsilon\right)\|x-\hat{x}\|^3 \end{aligned} \tag{125}$$

for all $x, \hat{x}$ with $\|x-\hat{x}\| \leq \epsilon'$. In view of (119) and the last inequality, we get in (118a)

$$\Delta D_k \leq (-\tilde{q}+\tilde{p}\|x-\hat{x}\|)\,\|x-\hat{x}\|^2 - \frac{1}{2}\|\mathbf{w}-\hat{\mathbf{w}}\|^2_W. \tag{126}$$

for all $x, \hat{x}$ with $\|x-\hat{x}\| \leq \epsilon'$, where $\tilde{q} := 1/(2\bar{p}^2(\bar{p}+\bar{a}^2/\underline{q}))$ and $\tilde{p} := (\kappa_r/\underline{p})\left(1+\bar{k}\bar{c}\right)\bar{a} + (\kappa_r^2/\underline{p})\epsilon$. Letting $\epsilon := \min\left(\epsilon', \frac{\tilde{q}}{2\tilde{p}}\right)$ proves the result with $c = \frac{1}{2}(\tilde{q}, \lambda_{\min}(W))$. □

H. Proof of Corollary 2

Proof. GES of the estimation error can be shown as follows. First, by Assumptions 1 and 2, the step size satisfies $\eta_k^i = \frac{\sigma}{L_{\mathrm{F}}}\frac{1}{\sqrt{k}} \leq \frac{\sigma_k}{L_k}$. Second, due to Assumption 5, we can show that instead of (20) in Lemma 3, we have

$$D_k\big(\mathbf{z}_k, \hat{\mathbf{z}}_k^{i+1}\big) \leq D_k\big(\mathbf{z}_k, \hat{\mathbf{z}}_k^i\big) + \frac{\eta_k^i}{2}\big(L_k - \frac{\sigma_k}{\eta_k^i}\big)\left\|\hat{\mathbf{z}}_k^{i+1} - \hat{\mathbf{z}}_k^i\right\|^2 \tag{127}$$

for all $k \in \mathbb{N}$. More specifically, we can skip in the proof of Lemma 3 all the steps from (61) to (70) and directly use convexity of F_k to get instead of (71)

$$F_k(\hat{\mathbf{z}}_k^i) - \underbrace{F_k(\mathbf{z}_k)}_{=0} \leq \nabla F_k(\hat{\mathbf{z}}_k^i)^\top\left(\hat{\mathbf{z}}_k^i - \mathbf{z}_k\right) \tag{128}$$

This means that all the remaining steps of the proof of the lemma hold without the remainder term $\eta_k^i(D_{r,k}^i)^\top \mathbf{d}_k(\mathbf{z}_k, \hat{\mathbf{z}}_k^i)$. In the proof of Theorem 1, using (127) and (38) instead of (20) and (23), which holds only locally, yields GES of the estimation error.

Concerning the regret analysis, thanks to Assumption 5, we can directly use the proof of [10, Theorem 3] in order to establish the regret bound (39). □