CONTROLLABILITY OF COUPLED PARABOLIC SYSTEMS WITH MULTIPLE UNDERACTUATIONS, PART 1: ALGEBRAIC SOLVABILITY

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Abstract. This paper is the first of two parts which together study the null controllability of a system of coupled parabolic PDEs. This work specializes to an important subclass of these control problems which are coupled by first and zero-order couplings and are, additionally, underactuated. In this paper, we pose our control problem in a fairly new framework which divides the problem into interconnected components: we refer to the first component as the analytic control problem; we refer to the second component as the algebraic control problem, where we use an algebraic method to "algebraically invert" a linear partial differential operator that describes our system; this allows us to recover null controllability by means of internal controls which appear on only a few of the equations. Treatment of the analytic control problem is deferred to the second part of this work [21]. The conclusion of this two-part work is a null controllability result for the original problem.

Key words. Controllability, Parabolic systems, Algebraic solvability, Fictitious control method.

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1. Introduction.

In recent years, problems concerning controllability of coupled parabolic PDEs have received much interest from the mathematical control community, see [3] and references therein. One classification of these numerous control problems is into problems with zero-order couplings (i.e., the reaction term in a usual parabolic PDE is now replaced with terms which couple the evolution of the solution with the solutions to other PDEs in the system) and problems with first-order couplings (i.e., the advection term is now replaced with terms which couple the evolution of the solution with the gradient of the solutions to other PDEs in the system). The applications of such control problems are ubiquitous: zero-order couplings arise in engineering problems modelled by reaction-diffusion equations, such as [6, 11, 20], whereas first-order couplings arise in engineering problems modelled by reaction-advection-diffusion equations, such as [8, 16, 17, 22].

1.1. Literature review.

For systems of several coupled parabolic equations, an important problem is to establish their controllability with reduced number of controls; we refer to such systems with reduced controls as underactuated systems of coupled parabolic PDEs. For the case of zero-order couplings and with internal controls, this control problem has been studied extensively in [1, 2]. In [2], a necessary and sufficient condition for exact controllability is proved for a system of m equations with constant coupling coefficients, which mimics the Kalman rank condition for finite-dimensional systems. In [1], some results similar to the Silverman-Meadows condition are obtained for time-varying coefficients.

General conditions for controllability of systems with first and zero-order couplings and internal controls have proven to be more elusive. In [14], a system of $n+1$ coupled heat equations with constant couplings and with one underactuation is studied, and a sufficient condition for null controllability is given under some restrictions

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on the controls. In [4], a system of three parabolic equations coupled by (time and space) varying coefficients is studied for two underactuations. The authors were able to recover a null controllability condition under some technical restrictions on the control domain and the coupling terms. In [10], a necessary and sufficient condition for null controllability is given for a system of m equations with one underactuation and constant coupling coefficients; furthermore, the authors study the case of (time and space) varying coupling coefficients and prove a sufficient controllability condition for a system of two equations with one underactuation, under some technical conditions.

1.2. Statement of contributions. The first part of this work has one main contribution: it achieves in proving the so-called algebraic solvability of a system of coupled parabolic PDEs under moderate assumptions, where controls appear on more than half of the equations. Algebraic solvability of this underactuated system, which is referred to as the algebraic control problem, allows one to generate its solution locally, and this solution inherits zero as its initial and final conditions from the particular treatment that is employed. This result is a key component of the fictitious control method, which can be used to prove controllability results for underactuated coupled PDE systems and is employed in Section 4. The technique used to prove our result is adapted from [8].

2. Preliminaries.

In this section, we introduce some notational conventions and present some mathematical background that we utilize throughout this work.

2.1. Notation and conventions. Throughout this work, we define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and similarly, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $n, k \in \mathbb{N}^*$, we denote the set of $n \times k$ matrices with real-valued entries by $\mathcal{M}_{n\times k}(\mathbb{R})$, and we denote the set of $n \times n$ matrices with real-valued entries by $\mathcal{M}_n(\mathbb{R})$. We denote the set of linear maps from a vector space U to a vector space V by $\mathscr{L}(U; V)$. For $(X, \mathcal{T}_X)$ a topological space and $U \subset X$, we denote the closure of U by $\bar{U}$.

2.2. A system of interest. In many fields of engineering, equations which describe the conservation of physical quantities are paramount. Among these conservation equations, the general second-order diffusion equation is routinely used to model engineering processes. Let $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$ for some $T > 0$; consider the second-order PDE

$$\begin{cases} \partial_t y + \mathcal{L} y = r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0, \cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $r : Q_T \to \mathbb{R}$ and $y^0 : \Omega \to \mathbb{R}$ are known, $y : \bar{Q}_T \to \mathbb{R}$ is the unknown, and for each $t \in (0, T)$, $\mathcal{L}$ denotes the second-order linear differential operator given by

$$\mathcal{L} y = -\sum_{i,j=1}^{n} \partial_{x_j} \left(d^{ij}(t, x) \partial_{x_i} y \right) + \sum_{i=1}^{n} g^i(t, x) \partial_{x_i} y + a(t, x) y, \tag{2.2}$$

for given coefficients d^{ij}, g^i, a, for $i, j \in \{1, \ldots, n\}$. Equation (2.1) can be used to describe the evolution in time of the distribution of some quantity y (e.g., heat), where the second-order term models diffusion, the first-order term models advection, the zero-order term models linear generation or depletion, and the forcing function accounts for external sources or sinks. We begin with some definitions that help us classify (2.1).

DEFINITION 2.1. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index and denote $\partial_{\alpha_1} \cdots \partial_{\alpha_n} y$ by $\partial_\alpha y$. For $k,\ l \in \mathbb{N}$ and $(d_\alpha)_\alpha$ coefficients, where $d_\alpha : Q_T \to \mathbb{R}$, a linear time-variant differential operator of order $l = 2k$ on Ω given by*

$$\mathcal{L}y = \sum_{|\alpha| \le l} d_\alpha(t,x) \partial_\alpha y$$

satisfies the uniform ellipticity condition if there exists $C > 0$ such that,

$$\sum_{|\alpha| = l} d_\alpha(t,x) \xi^\alpha \ge C|\xi|^l, \qquad \forall\, \xi \in \mathbb{R}^n, \forall (t,x) \in Q_T, \tag{2.3}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

DEFINITION 2.2. *A partial differential operator $\partial_t + \mathcal{L}$ is (uniformly) parabolic if $\mathcal{L}$ satisfies the uniform ellipticity condition.*

Of greater interest in many areas of engineering is the study a *system of second-order parabolic PDEs* (e.g., [18], [23]). We express systems consisting of m equations in vector form as

$$\begin{cases} \partial_t y + \mathcal{L}y = r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.4}$$

where $y^0 := (y_1, \dots, y_m)$ and $r := (r_1, \dots, r_m)$ are known, $y := (y_1, \dots, y_m)$ are the unknowns, and the differential operator $\mathcal{L}$ is now defined as

$$\mathcal{L}y = \sum_{k=1}^{m} \left(-\sum_{i,j=1}^{n} \partial_{x_j} \left(d_k^{ij}(t,x) \partial_{x_i} y_k \right) + \sum_{i=1}^{n} g_k^i(t,x) \partial_{x_i} y_k + a_k(t,x) y_k \right) \mathbf{e_k},$$

where $\mathbf{e_k}$ is the k-th canonical basis vector in $\mathbb{R}^m$. Yet another very practical extension of this system of second-order PDEs is when the *equations within the system are coupled* (e.g., [3, 15, 20]): denoting the p-th entry of $\mathcal{L}y$ as $\mathcal{L}_p y$ for $p \in \{1, \dots, m\}$, we now have

$$\mathcal{L}_p y = \sum_{k=1}^{m} \left(-\sum_{i,j=1}^{n} \partial_{x_j} \left(d_{pk}^{ij}(t,x) \partial_{x_i} y_k \right) + \sum_{i=1}^{n} g_{pk}^i(t,x) \partial_{x_i} y_k + a_{pk}(t,x) y_k \right). \tag{2.5}$$

When $p \ne k$, we call d_{pk}^{ij} the *second-order coupling coefficients*, g_{pk}^i the *first-order coupling coefficients*, and a_{pk} the *zero-order coupling coefficients*. This work studies a particular case of first and zero-order constant coupling coefficients, where for δ_{ij} denoting the Kronecker delta function, $d_{pk}^{ij}(t,x) = d_p^{ij} \delta_{pk} \in \mathbb{R}$, $g_{pk}^i(t,x) = -g_{pk}^i \in \mathbb{R}$ and $a_{pk}(t,x) = -a_{pk} \in \mathbb{R}$, for $i,j \in \{1, \dots, n\}$ and $p \in \{1, \dots, m\}$. Additionally, we study the case where $d_p^{ij} = d_p^{ji}$, for $i,j \in \{1, \dots, n\}$ and $p \in \{1, \dots, m\}$. Hence, we can write $\mathcal{L}y$ as

$$\mathcal{L}y = \sum_{p=1}^{m} \left(-\mathrm{div}(d_p \nabla y_p) - \sum_{k=1}^{m} g_{pk} \cdot \nabla y_k - \sum_{k=1}^{m} a_{pk} y_k \right) \mathbf{e_p}, \tag{2.6}$$

where $g_{pk} := (g_{pk}^1, \dots, g_{pk}^n) \in \mathbb{R}^n$, $d_p \in \mathcal{M}_n(\mathbb{R})$ is symmetric and $\mathbf{e_p}$ is the p-th canonical basis vector in $\mathbb{R}^m$, for $p \in \{1, \dots, m\}$. With these choices of coefficients, system (2.4) becomes

$$\begin{cases} \partial_t y = \mathrm{div}(D\nabla y) + G \cdot \nabla y + Ay + r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0, \cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.7}$$

where $D := \mathrm{diag}(d_1, \dots, d_m)$, $G := (g_{pk})_{1 \le p,k \le m} \in \mathcal{M}_m(\mathbb{R}^n)$ and $A := (a_{pk})_{1 \le p,k \le m} \in \mathcal{M}_m(\mathbb{R})$.

2.3. The solution of coupled parabolic systems. To adapt classical existence and uniqueness results to a system of *coupled* parabolic PDEs such as in system (2.7), one can follow the treatment, for example, in [12, Section 7], but write all intermediary results for a system of solutions. From now on, we assume that $\mathcal{L}$ satisfies (2.3). Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. For $u, v \in H_0^1(\Omega)^m$, we define the bilinear form

$$B[u,v] := \int_\Omega \sum_{p,k=1}^{m} \left(\sum_{i,j=1}^{n} d_p^{ij} (\partial_{x_i} u_p)(\partial_{x_j} v_p) - \sum_{i=1}^{n} g_{pk}^i (\partial_{x_i} u_k) v_p - a_{pk} u_k v_p \right) \mathbf{e_p} dx.$$

One has the following definition.

DEFINITION 2.3. *Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. A function*

$$\mathbf{y} \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$$

is said to be a weak solution of system (2.7) provided that for every $v \in H_0^1(\Omega)^m$ and almost every $t \in [0,T]$

(i) $\langle \frac{d}{dt}\mathbf{y}, v \rangle + B[\mathbf{y}, v] = \int_\Omega \mathbf{r}^T v dx$, *and;*

(ii) $\mathbf{y}(0) = y^0$,

where $\langle \cdot, \cdot \rangle$ denotes the appropriate duality pairing.

From now on, we mean by "solution to a coupled parabolic system" the weak solution in the sense of Definition 2.3.

2.4. A parabolic regularity result. We state a regularity result for the solution of system (2.7) which is essential in the work to follow.

THEOREM 2.4. *[12, Theorem 6, Subsection 7.1.3] For $d \in \mathbb{N}$, assume $y^0 \in H^{2d+1}(\Omega)^m$, $\mathbf{r} \in L^2((0,T); H^{2d}(\Omega))^m \cap H^d((0,T); L^2(\Omega))^m$, and assume that $\mathbf{y} \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$ is the solution of system (2.7). Suppose* *also that the following compatibility conditions hold:*

$$\begin{cases} g^0 := y^0 \in H_0^1(\Omega)^m; \\ g^1 := \mathbf{r}(0) - \mathcal{L}g^0 \in H_0^1(\Omega)^m; \\ \vdots \\ g^d := \frac{d^{d-1}\mathbf{r}}{dt^{d-1}}(0) - \mathcal{L}g^{d-1} \in H_0^1(\Omega)^m. \end{cases}$$

Then $\mathbf{y} \in L^2((0,T);H^{2d+2}(\Omega))^m \cap H^{d+1}((0,T);L^2(\Omega))^m$ *and we have the estimate*

$$\begin{aligned} ||\mathbf{y}||_{L^2((0,T);H^{2d+2}(\Omega))^m \cap H^{d+1}((0,T);L^2(\Omega))^m} &\leq C \left(||\mathbf{r}||_{L^2((0,T);H^{2d}(\Omega))^m \cap H^d((0,T);L^2(\Omega))^m} \right. \\ &\left. + ||y^0||_{H^{2d+1}(\Omega)^m} \right). \end{aligned} \tag{2.8}$$

2.5. Some sparse matrix theory preliminaries. When studying the invertibility of certain linear operators of interest, we are faced with studying non-singularity conditions for matrices associated to coupled parabolic PDEs of interest (cf. Subsection 4.3). By nature of their construction, these matrices are *sparse*. We describe an algorithm that can be used to decompose a sparse matrix into block triangular form. Importantly, this algorithm can be applied to matrices with symbolic entries as it only makes use of the placement of zero entries in the matrix.

Given a matrix $P \in \mathcal{M}_{q\times r}(\mathbb{R})$, consider the bipartite graph associated to P given by the triple $G(P) := (R, C, E)$, where $R := \{r_1, \dots, r_q\}$ is the set of row vertices associated to P, $C := \{c_1, \dots, c_r\}$ is the set of column vertices associated to P, and E denotes the set the edges (r_i, c_j) associated to every nonzero entry p_{ij} of P, for $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, r\}$. As in [5], we have the following definitions.

DEFINITION 2.5. *A matching $M \subset E$ in $G(P)$ is such that the edges in M have no common vertices. We define the cardinality of M as the number of edges in M. A maximum matching is a matching with maximum cardinality. Furthermore, M is said to be column-perfect if every column vertex in C is matched; it is said to be row-perfect if every row vertex in R is matched; and it is said to be perfect if it is both column-perfect and row-perfect. A vertex v_i is said to be matched with respect to M if there exists $(v_i, v_j) \in M$ for appropriate indices i, j.*

DEFINITION 2.6. *The structural rank of a matrix $P \in \mathcal{M}_{q\times r}(\mathbb{R})$ is the cardinality of a maximum matching $M \subset E$ in $G(P)$.*

DEFINITION 2.7. *For an appropriate index i, let either $v_i = r_i$ or $v_i = c_i$. Fix a maximum matching M in $G(P)$. For $k \in \mathbb{N}^*$, a walk is a sequence of (possibly repeated) vertices $(v_i)_{i=0}^k$ such that (v_i, v_{i+1}) is an edge for $i \in \{1, \dots, k-1\}$. An alternating walk is a walk with every second edge belonging to M. An alternating path is an alternating walk with no repeated vertices.*

Next, we define some important subsets of R and C.

DEFINITION 2.8. *Let M be a maximum matching in $G(P)$ with row set R and column set C. We define the following sets of vertices with respect to M:*

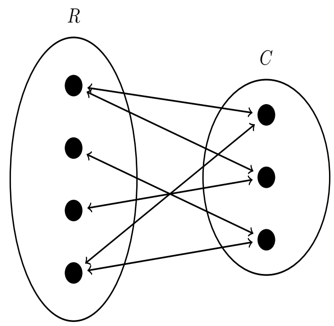
(i) $VR := \{$*row vertices reachable by alternating paths from an unmatched row*$\}$;
(ii) $HR := \{$*row vertices reachable by alternating paths from an unmatched col.*$\}$;
(iii) $VC := \{$*col. vertices reachable by alternating paths from an unmatched row*$\}$;
(iv) $HC := \{$*col. vertices reachable by alternating paths from an unmatched col.*$\}$;
(v) $SR := R \setminus (VR \cup HR)$, *and;*

(vi) $SC := C \setminus (VC \cup HC)$.

It was proven in [9] that VR, HR and SR are pairwise disjoint, and also that VC, HC and SC are pairwise disjoint. We demonstrate these definitions on an example.

EXAMPLE 2.9. *Consider the matrix $P \in \mathcal{M}_{4\times 3}(\mathbb{R})$ and its bipartite graph $G(P)$ given by*

$$P = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & a_{43} \end{pmatrix}$$



Consider two maximum matchings

$$M_1 := \{(r_1, c_1), (r_2, c_3), (r_3, c_2)\} \quad \text{and} \quad M_2 := \{(r_1, c_1), (r_3, c_2), (r_4, c_3)\}$$

in $G(P)$. Note that M_1 and M_2 are column-perfect and the structural rank of A is 3. For matching M_1, an alternating path is given by the sequence r_4, c_1, r_1, c_2, r_3. Furthermore, for matching M_1, we have $VR := \{r_1, r_2, r_3, r_4\}$ and $VC := \{c_1, c_2, c_3\}$.

In the above example, the structural rank of P is equal to the rank of P in general. It is easily deduced that the structural rank of a matrix in $\mathcal{M}_{q,r}(\mathbb{R})$ is an upper bound on the rank of that matrix, and is never greater than $\min\{q, r\}$. We arrive at the following important result, which is identified in literature as the *Dulmage-Mendelsohn decomposition* and can be deduced from [9, 19].

THEOREM 2.10. *Let $P \in \mathcal{M}_{q\times r}(\mathbb{R})$, and let M be a maximum matching in $G(P)$. Then, one can permute the rows and columns of P to obtain the following block-triangular form (which we refer to as coarse decomposition):*

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & 0 & P_{23} & P_{24} \\ 0 & 0 & 0 & P_{34} \\ 0 & 0 & 0 & P_{44} \end{pmatrix},$$

where

(i) (P_{11}, P_{12}) *is the underdetermined part of the matrix (i.e., more rows than columns), is generated by $(r_i, c_i) \in HR \times HC$, and has row-perfect matching;*

(ii) $\begin{pmatrix} P_{34} \\ P_{44} \end{pmatrix}$ *is the overdetermined part of the matrix (i.e., more columns than rows), is generated by $(r_i, c_i) \in VR \times VC$, and has column-perfect matching;*

(iii) P_{23} *is generated by $(r_i, c_i) \in SR \times SC$, and;*

(iv) P_{12}, P_{23}, P_{34} *are square matrices with nonzero diagonal, and hence have perfect matchings (i.e., they are of maximal structural rank).*

Moreover, P_{12}, P_{23}, P_{34} can be further decomposed into block-triangular form with nonzero diagonal (which we refer to as fine decomposition). The structural rank of P is given by the sum of the structural ranks of P_{12}, P_{23}, P_{34}.

Remark 2.11. *If P is overdetermined, then (P_{11}, P_{12}) will be present only if P does not have a column-perfect matching. Similarly, if P is underdetermined, then (P_{34}, P_{44}) will appear only if P does not have a row-perfect matching. In both of these cases, the presence of P_{23} depends on the nonzero structure of P. If P is square, non-symmetric and has a perfect maximum matching, then its coarse decomposition will consist only of P_{23}.*

Remark 2.12. *It was proven in [9] that the Dulmage-Mendelsohn decomposition is independent of the choice of maximum matching in $G(P)$.*

We are now ready to study system (2.7) under the framework of control systems, in the sense that we "select" the forcing term r to drive the system to a desired final state in some time $T \in \mathbb{R}^*$.

3. Problem statement.

We revisit the system consisting of m coupled second-order parabolic PDEs given by system (2.7), where it can be deduced, for example, from [12, Theorems 3 and 4, Section 7.1.2], that for any initial condition $y^0 \in L^2(\Omega)^m$ and $r \in L^2(Q_T)^m$, system (2.7) admits a unique solution

$$y \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m.$$

We now introduce the problem of interest.

3.1. The control problem.

We recast system (2.7) as a *control system*, where $r = Bu$ with $u \in L^2(Q_T)^c$ being control inputs to be chosen, and $B \in \mathcal{M}_{m\times c}(\mathbb{R})$, with $0 < c \leq m$, yielding

$$\begin{cases} \partial_t y = \operatorname{div}(D\nabla y) + G \cdot \nabla y + Ay + Bu, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega. \end{cases} \tag{3.1}$$

Let us now introduce our objectives that we aim to achieve by selecting appropriate control inputs. We have the following notions of *controllability* for system (3.1).

Definition 3.1. *We say that system* (3.1) *is null controllable in time T if for every initial condition $y^0 \in L^2(\Omega)^m$, there exists a control $u \in L^2(Q_T)^c$ such that the solution $y \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$ to* (3.1) *satisfies*

$$y(T) = 0 \quad in \quad \Omega.$$

Definition 3.2. *We say that system* (3.1) *is approximately controllable in time T if for every $\epsilon > 0$, for every initial condition $y^0 \in L^2(\Omega)^m$ and for every $y_T \in L^2(\Omega)^m$, there exists a control $u \in L^2(Q_T)^c$ such that the solution $y \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$ to* (3.1) *satisfies*

$$||y(T) - y_T||^2_{L^2(\Omega)^m} \leq \epsilon.$$

This work specializes to the case of internal (or distributed) control: that is, for $\omega \subset \Omega$ nonempty and open, we study the case where $r = \mathbb{1}_\omega Bu$, and henceforth, we denote by q_T the set $(0,T) \times \omega$.

An interesting control problem that arises in many engineering applications is underactuation, that is, when $c < m$. Our work will further specialize to this case, where there are currently few results for *first and zero-order couplings*, for arbitrary m and $c < m - 1$ (even for the case of constant coefficients).

Since we treat the particular case of a system of linear parabolic PDEs with constant coefficients (constant in space and *time*), we are easily able to ascertain approximate controllability of system (3.1) from its null controllability.

THEOREM 3.3. *[7, Theorem 2.45] Assume that for every $T > 0$, the control system* (3.1) *is null controllable in time T. Then, for every $T > 0$, system* (3.1) *is approximately controllable in time T.*

We now outline the treatment that we use throughout this work.

4. Fictitious control method. This section presents a technique that can be used to prove the null controllability of the coupled system (3.1) with possibly multiple underactuations (i.e., when $c \leq m - 1$). We first introduce the so-called *fictitious control method*, developed in [8], which allows one to bifurcate the null controllability problem into interconnected components: an analytic control problem, where *fictitious* controls act on every equation in the coupled system (3.1); and an algebraic control problem, where there are possibly many underactuations. For the analytic problem, one can prove a so-called *weighted observability inequality* which will help deduce null controllability of the analytic system. For the algebraic problem, one can pose this underactuated control problem as an underdetermined system involving differential operators, and, under some conditions, "invert" one of these operator algebraically. The first part of this two-part work focuses on the latter treatment.

4.1. Definitions. Recall that we denote our control domain by $q_T := (0, T) \times \omega$. We begin with some definitions.

DEFINITION 4.1. *For $n \in \mathbb{N}^*$, let α be a multi-index of length $n + 1$. For $k, l \in \mathbb{N}^*$, a linear map $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^l$ is called a linear partial differential operator of order $m \in \mathbb{N}$ in q_T if for every α verifying $|\alpha| \leq m$, there exists $A_\alpha \in C^\infty(q_T; \mathscr{L}(\mathbb{R}^k; \mathbb{R}^l))$ such that for all $\phi \in C^\infty(q_T)^k$ and $(t, x) \in q_T$,*

$$(\mathcal{B}\phi)(t,x) = \sum_{|\alpha| \leq m} A_\alpha(t,x)\partial_\alpha \phi(t,x). \tag{4.1}$$

Let $c, m, k \in \mathbb{N}$ and consider the linear partial differential operators

$$\begin{cases} \mathcal{L} : C^\infty(q_T)^{m+c} \to C^\infty(q_T)^m, \\ \mathcal{N} : C^\infty(q_T)^k \to C^\infty(q_T)^m. \end{cases}$$

Suppose that for $(\hat{y} \quad \hat{u})^T \in C^\infty(q_T)^{m+c}$ and $\tilde{u} \in C^\infty(q_T)^k$, the linear equation

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) = \mathcal{N}(\tilde{u}) \tag{4.2}$$

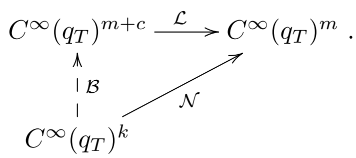
is of interest, where $\tilde{u}$ is given and $(\hat{y} \quad \hat{u})^T$ are the unknowns. We characterize the solvability of (4.2).

DEFINITION 4.2. *We say that the linear equation* (4.2) *is algebraically solvable in q_T if there exists a linear partial differential operator $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^{m+c}$ such that*

$$\mathcal{L} \circ \mathcal{B} = \mathcal{N}, \tag{4.3}$$

that is, $\mathcal{B}(\tilde{u})$ is a solution to (4.2) for every $\tilde{u} \in C^\infty(q_T)^k$. If $k = m$ and $\mathcal{N} = Id_{C^\infty(q_T)^m}$, then we call $\mathcal{B}$ the right inverse of $\mathcal{L}$.

In other words, we wish to find $\mathcal{B}$ such that the following diagram is commutative:



4.2. The fictitious control method. Our goal is to prove null controllability in time T for the control system (3.1), where there are m coupled parabolic equations and less than m controls. To accomplish this for an arbitrary number of controls $c \leq m-1$, our strategy is to divide this control problem into two separate parts as was done in [8, 10].

4.2.1. Analytic control problem. We first consider following control problem: for any $\tilde{y}^0 \in L^2(\Omega)^m$, prove the existence of $(\tilde{y}, \tilde{u})$ a solution of

$$\text{(4.4)} \qquad \begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla\tilde{y}) + G\cdot\nabla\tilde{y} + A\tilde{y} + \mathcal{N}\left(\mathbb{1}_\omega \tilde{u}\right), & \text{in } Q_T, \\ \quad \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0,\cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega, \end{cases}$$

such that $\tilde{y}(T,\cdot) = 0$, where $\mathcal{N}$ is a differential operator that is to be determined (cf. Section 4.3), $\tilde{u}$ *acts on all equations* in (4.4), and we denote by $\mathbb{1}_\omega$ a *smooth version* of the indicator function which will be constructed in [21]. Note that $(\tilde{y}, \tilde{u})$ has to be in a suitable space: in particular, depending on our choice of differential operator $\mathcal{N}$, $\tilde{u}$ has to be regular enough to withstand the derivatives applied by $\mathcal{N}$.

4.2.2. Algebraic control problem. We next consider a different control problem: prove the existence of a solution $(\hat{y}, \hat{u})$ of

$$\text{(4.5)} \qquad \begin{cases} \partial_t \hat{y} = \operatorname{div}(D\nabla\hat{y}) + G\cdot\nabla\hat{y} + A\hat{y} + B\hat{u} + \mathcal{N}\left(\mathbb{1}_\omega \tilde{u}\right), & \text{in } Q_T, \\ \quad \hat{y} = 0, & \text{on } \Sigma_T, \\ \hat{y}(0,\cdot) = \hat{y}(T,\cdot) = 0, & \text{in } \Omega, \end{cases}$$

where $\hat{u}$ acts only on the first c equations and $B = \begin{pmatrix} \mathrm{Id}_c & 0_{c\times(m-c)} \end{pmatrix}^T \in \mathcal{M}_{m\times c}(\mathbb{R})$. The notions of algebraic solvability, as described in Section 4.1, will be used to resolve this control problem in the next subsection. The analytic and algebraic control problems differ in the following ways: in the analytic problem, the controls are $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$, whereas in the algebraic problem, the controls are $\hat{u}$, and furthermore, $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$ appears but is considered to be a source term; and in the analytic problem, one has to prove that $\tilde{y}(T,\cdot) = 0$ (we will accomplish this in [21] by means of an observability inequality), whereas in the algebraic problem, $\hat{y}(T,\cdot) = 0$ is inherited from the construction of the solution $(\hat{y}, \hat{u})$ (cf. Remark 4.3).

Solving both the analytic and algebraic problems will prove the null controllability of system (3.1). Indeed, defining

$$(y, u) := (\tilde{y} - \hat{y}, -\hat{u}),$$

one notices that (y, u) is the solution to (3.1) in a suitable space with $y(T, \cdot) = 0$. Note that the controls in the analytic system, $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$, are eliminated via the subtraction $\tilde{y} - \hat{y}$; this gives meaning to the name of the method we employ.

4.3. Algebraic solvability. In this section, we study the algebraic solvability of differential operators corresponding system (4.5) which contains m equations and c controls, for $c \in \{1, \ldots, m-1\}$. To this end, we consider the linear partial differential operator defined by

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) := \partial_t \hat{y} - \mathrm{div}(D\nabla\hat{y}) - G \cdot \nabla\hat{y} - A\hat{y} - B\hat{u}, \tag{4.6}$$

which is an underdetermined operator, and we consider $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$ as a source term, where $\mathcal{N}$ is to be chosen later. One can write system (4.5) as

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) = \mathcal{N}(\mathbb{1}_\omega \tilde{u}); \tag{4.7}$$

we study the algebraic solvability of (4.7) in q_T. Recall from Definition 4.2 that this is equivalent to proving the existence of a linear partial differential operator $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^m$ such that $(\hat{y} \quad \hat{u})^T = \mathcal{B}(\mathbb{1}_\omega \tilde{u})$ for any $\mathbb{1}_\omega \tilde{u} \in C^\infty(q_T)^m$, and hence by reason of $\mathcal{B}$ being a differential operator, $(\hat{y}, \hat{u})$ will have support in q_T. With a slight abuse of notation, from now on we denote the extension by zero of $(\hat{y}, \hat{u})$ to Q_T also by $(\hat{y}, \hat{u})$, so that $\hat{y} = 0$ on Σ_T and $\hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0$ in Ω.

REMARK 4.3. *For simplicity, we formulated the notion of algebraic solvability for controls in the analytic problem $\mathbb{1}_\omega \tilde{u} \in C^\infty(q_T)$, which dictates the regularity of $(\hat{y}, \hat{u})$; however, in [21] we will need to expand the space of controls that we may access to recover null controllability results for system* (4.4)*. For controls with weaker regularity, we must additionally show that these controls vanish at times $t = 0$ and $t = T$. This treatment is deferred to [21]. For the time being, assume $(\hat{y}, \hat{u})$ are regular enough such that $\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right)$ is well-defined.*

We study the adjoint system associated to system (4.5):

$$\begin{cases} -\partial_t \hat{\psi} = \mathrm{div}(D\nabla\hat{\psi}) - G^* \cdot \nabla\hat{\psi} + A^*\hat{\psi}, & \text{in } Q_T, \\ \hat{\psi} = 0, & \text{on } \Sigma_T, \\ \hat{\psi}(T, \cdot) = \hat{\psi}^0(\cdot), & \text{in } \Omega, \end{cases} \tag{4.8}$$

for $\hat{\psi}^0 \in L^2(\Omega)^m$.

4.3.1. One underactuation. This section follows the treatment in [10, Subsection 2.1] and is presented here to contrast the existing technique to treat the null controllability of system (4.5) with one underactuation and the proposed technique in Subsection 4.3.2, which treats the case of multiple underactuations. The method presented here succeeds in algebraically solving (4.7) by utilizing the first and zero-order couplings to isolate for the unknown, and is henceforth referred to as the *direct isolation technique*.

Choose $k = m$; we wish to find a linear partial differential operator $\mathcal{B}$ such that

$$\mathcal{L} \circ \mathcal{B} = \mathcal{N}, \tag{4.9}$$

where $\mathcal{L}$ is given in (4.6) and $\mathcal{N}$ is to be chosen. Note that this is equivalent to solving the adjoint problem: that is, finding a linear partial differential operator $\mathcal{B}^*$ such that

$$\mathcal{B}^* \circ \mathcal{L}^* = \mathcal{N}^*. \tag{4.10}$$

We calculate the (formal) adjoint of differential operator $\mathcal{L}$: for $\hat{\psi} \in L^2(Q_T)^m$, we have

$$\begin{aligned}
&\left(\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right), \hat{\psi}\right) \\
&= \left(\iint_{Q_T} \sum_{k=1}^{m} \left(\partial_t \hat{y} \hat{\psi}_k - \operatorname{div}(d_k \nabla \hat{y}_k) \hat{\psi}_k - \sum_{i=1}^{m} (g_{ki} \cdot \nabla \hat{y}_k + a_{ki} \hat{y}_k) \hat{\psi}_k\right)\right. \\
&\left. + \sum_{l=1}^{c} \hat{u}_l \hat{\psi}_l dxdt\right);
\end{aligned}$$

as a result,

$$\begin{aligned}
&= \iint_{Q_T} \sum_{k=1}^{m} \hat{y}_k \mathcal{L}_k^* \hat{\psi} + \sum_{l=1}^{c} \hat{u}_l \mathcal{L}_{m+l}^* \hat{\psi} \\
&= \left((\hat{y} \quad \hat{u})^T, \mathcal{L}^* \hat{\psi}\right),
\end{aligned}$$

and hence

$$\mathcal{L}^* \hat{\psi} = \begin{pmatrix}
-\left(\partial_t + \operatorname{div}(d_1 \nabla)\right) \hat{\psi}_1 + \sum_{j=1}^{m} \left(g_{j1} \cdot \nabla - a_{j1}\right) \hat{\psi}_j \\
-\left(\partial_t + \operatorname{div}(d_2 \nabla)\right) \hat{\psi}_2 + \sum_{j=1}^{m} \left(g_{j2} \cdot \nabla - a_{j2}\right) \hat{\psi}_j \\
\vdots \\
-\left(\partial_t + \operatorname{div}(d_m \nabla)\right) \hat{\psi}_m + \sum_{j=1}^{m} \left(g_{jm} \cdot \nabla - a_{jm}\right) \hat{\psi}_j \\
\hat{\psi}_1 \\
\vdots \\
\hat{\psi}_{m-1}
\end{pmatrix}. \tag{4.11}$$

We state the following lemma, which is a reformulation of [10, Theorem 1].

LEMMA 4.4. *The linear partial differential equation* (4.10) *is algebraically solvable in* q_T *if there exists an index* $i_0 \in \{1, \ldots, m-1\}$ *such that*

$$g_{mi_0} \neq 0 \quad \text{or} \quad a_{mi_0} \neq 0. \tag{4.12}$$

Proof. One need only look at the i_0-th entry of $\mathcal{L}^*$ to verify this assertion:

$$\begin{aligned}
\mathcal{L}_{i_0}^* \hat{\psi} &= -\left(\partial_t + \operatorname{div}(d_{i_0} \nabla)\right) \hat{\psi}_{i_0} + \sum_{j=1}^{m} \left(g_{ji_0} \cdot \nabla - a_{ji_0}\right) \hat{\psi}_j \\
&= -\left(\partial_t + \operatorname{div}(d_{i_0} \nabla)\right) \mathcal{L}_{m+i_0}^* \hat{\psi} + \sum_{j=1}^{m-1} \left(g_{ji_0} \cdot \nabla - a_{ji_0}\right) \mathcal{L}_{m+j}^* \hat{\psi} \\
&+ \left(g_{mi_0} \cdot \nabla - a_{mi_0}\right) \hat{\psi}_m,
\end{aligned}$$

which one can use to isolate for the unknown $\hat{\psi}_m$ and its spatial derivative:
(4.13)

$$(g_{mi_0} \cdot \nabla - a_{mi_0})\,\hat{\psi}_m = \mathcal{L}^*_{i_0}\hat{\psi} + (\partial_t + \operatorname{div}(d_{i_0}\nabla))\,\mathcal{L}^*_{m+i_0}\hat{\psi} - \sum_{j=1}^{m-1} (g_{ji_0} \cdot \nabla - a_{ji_0})\,\mathcal{L}^*_{m+j}\hat{\psi}.$$

Hence, a careful choice of $\mathcal{N}^*$ yields the desired result: choosing

$$\mathcal{N}^*\hat{\psi} := \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \vdots \\ \hat{\psi}_{m-1} \\ (g_{mi_0} \cdot \nabla - a_{mi_0})\,\hat{\psi}_m \end{pmatrix},$$

one can define for $\phi \in C^\infty(Q_T)^{2m-1}$

$$\mathcal{B}^*\phi := \begin{pmatrix} \phi_{m+1} \\ \phi_{m+2} \\ \vdots \\ \phi_{2m-1} \\ \phi_{i_0} + (\partial_t + \operatorname{div}(d_{i_0}\nabla))\,\phi_{m+i_0} - \sum_{j=1}^{m-1} (g_{ji_0} \cdot \nabla - a_{ji_0})\,\phi_{m+j} \end{pmatrix},$$

so that

$$(\mathcal{B}^* \circ \mathcal{L}^*)\,\hat{\psi} = \mathcal{N}^*\hat{\psi}$$

is verified for every $\hat{\psi} \in C^\infty(Q_T)^m$. □

REMARK 4.5. *One notices that condition* (4.12) *is also necessary for algebraic solvability of* (4.10).

4.3.2. Multiple underactuations. We specialize to the case where system (4.5) has more than one underactuation (i.e., when $c < m-1$).

Direct isolation technique: we begin by employing the technique presented in Subsection 4.3.1. For the moment, we focus on the simplest case, when $c = m-2$. We have

$$\mathcal{L}^*\hat{\psi} = \begin{pmatrix} -(\partial_t + \operatorname{div}(d_1\nabla))\,\hat{\psi}_1 + \sum_{j=1}^m (g_{j1} \cdot \nabla - a_{j1})\,\hat{\psi}_j \\ -(\partial_t + \operatorname{div}(d_2\nabla))\,\hat{\psi}_2 + \sum_{j=1}^m (g_{j2} \cdot \nabla - a_{j2})\,\hat{\psi}_j \\ \vdots \\ -(\partial_t + \operatorname{div}(d_m\nabla))\,\hat{\psi}_m + \sum_{j=1}^m (g_{jm} \cdot \nabla - a_{jm})\,\hat{\psi}_j \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{m-2} \end{pmatrix}.$$

A natural necessary condition for algebraic solvability of (4.5) as in Lemma (4.4) is the following: without loss of generality, suppose there exists indices $i_0, i_1 \in \{1,\dots,m-2\}$

such that

$$\begin{cases} g_{(m-1)i_0} \neq 0 & \text{or} \quad a_{(m-1)i_0} \neq 0, \\ g_{mi_1} \neq 0 & \text{or} \quad a_{mi_1} \neq 0. \end{cases}$$

One immediately encounters the issue that none of the entries of $\mathcal{L}^*$ can be used to isolate for the individual unknowns $\hat{\psi}_{m-1}$ and $\hat{\psi}_m$ (and their spatial derivatives). Instead, we recover the system of equations

$$\begin{cases} \left(g_{(m-1)i_0} \cdot \nabla - a_{(m-1)i_0}\right) \hat{\psi}_{m-1} + \left(g_{mi_0} \cdot \nabla - a_{mi_0}\right) \hat{\psi}_m \\ \qquad = \mathcal{L}^*_{i_0} \hat{\psi} + \left(\partial_t + \operatorname{div}(d_{i_0} \nabla)\right) \mathcal{L}^*_{m+i_0} \hat{\psi} \\ \qquad\quad - \sum_{j=1}^{m-2} \left(g_{ji_0} \cdot \nabla - a_{ji_0}\right) \mathcal{L}^*_{m+j} \hat{\psi}, \\ \left(g_{(m-1)i_1} \cdot \nabla - a_{(m-1)i_1}\right) \hat{\psi}_{m-1} + \left(g_{mi_1} \cdot \nabla - a_{mi_1}\right) \hat{\psi}_m \\ \qquad = \mathcal{L}^*_{i_1} \hat{\psi} + \left(\partial_t + \operatorname{div}(d_{i_1} \nabla)\right) \mathcal{L}^*_{m+i_1} \hat{\psi} \\ \qquad\quad - \sum_{j=1}^{m-2} \left(g_{ji_1} \cdot \nabla - a_{ji_1}\right) \mathcal{L}^*_{m+j} \hat{\psi}. \end{cases} \tag{4.14}$$

While one can define an appropriate $\mathcal{N}^*$ using (4.14) such that (4.5) is algebraically solvable, in general this $\mathcal{N}^*$ will have entries involving both $\hat{\psi}_{m-1}$ and $\hat{\psi}_m$ (and their spatial derivatives). Such an $\mathcal{N}^*$ introduces an unresolvable issue when attempting to solve the analytic control problem. Alas, we are not aware of a procedure through which one can hope to recover a general necessary and sufficient condition for algebraic solvability of (4.5) using this technique.

Prolongation technique: inspired by [8, Section 3], we present a method to prove the algebraic solvability of (4.10) by means of *prolongation*: that is, since $\mathcal{L}^*\hat{\psi} = \mathcal{N}^*\hat{\psi}$ is an overdetermined system (i.e., there are $m+c$ equations and only m unknowns), we can expect to differentiate each equation a sufficient amount of times with respect to all of the spatial variables in order to gain more equations than "algebraic unknowns", which we make more precise in what follows. An inversion technique, which is inspired by the results in [13, Section 2.3.8], is then used to recover the unknowns from the overdetermined system.

We consider system (4.5) for an arbitrary $c \in \{1, \dots, m-2\}$ and define the linear partial differential operator

$$\mathcal{N}\zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{pmatrix},$$

for $\zeta \in C^\infty(Q_T)^m$. With this choice of $\mathcal{N}$, it suffices to consider differential operators

$\bar{\mathcal{L}} : C^\infty(Q_T)^m \to C^\infty(Q_T)^{m-c}$ and $\bar{\mathcal{N}} : C^\infty(Q_T)^{m-c} \to C^\infty(Q_T)^{m-c}$ defined by

$$\bar{\mathcal{L}}\zeta := \begin{pmatrix} (\partial_t - \mathrm{div}(d_{c+1}\nabla))\,\zeta_{c+1} - \sum_{i=1}^m \left(g_{(c+1)i}\cdot\nabla + a_{(c+1)i}\right)\zeta_i \\ (\partial_t - \mathrm{div}(d_{c+2}\nabla))\,\zeta_{c+2} - \sum_{i=1}^m \left(g_{(c+2)i}\cdot\nabla + a_{(c+2)i}\right)\zeta_i \\ \vdots \\ (\partial_t - \mathrm{div}(d_m\nabla))\,\zeta_m - \sum_{i=1}^m \left(g_{mi}\cdot\nabla + a_{mi}\right)\zeta_i \end{pmatrix}$$

and

$$\bar{\mathcal{N}}\zeta := \begin{pmatrix} \zeta_{c+1} \\ \vdots \\ \zeta_m \end{pmatrix}$$

to prove algebraic solvability of (4.9). Indeed, with our choice of $\mathcal{N}$ we can write system (4.5) as

$$\mathcal{L}(\hat{y}, \hat{u}) = \mathbb{1}_\omega \tilde{u}, \tag{4.15}$$

where $\hat{u}$ acts on the first c equations; also, finding a linear partial differential operator $\mathcal{B}$ satisfying (4.9) is equivalent to finding $\mathcal{B}$ such that

$$\begin{cases} \hat{y}_1 = \mathcal{B}_1(\mathbb{1}_\omega \tilde{u}), \\ \quad\vdots \\ \hat{y}_m = \mathcal{B}_m(\mathbb{1}_\omega \tilde{u}), \\ \hat{u}_1 = \mathcal{B}_{m+1}(\mathbb{1}_\omega \tilde{u}), \\ \quad\vdots \\ \hat{u}_c = \mathcal{B}_{m+c}(\mathbb{1}_\omega \tilde{u}). \end{cases} \tag{4.16}$$

Hence, from our choice of B in (4.5), (4.6), (4.15) and (4.16), we have for $l \in \{1, \ldots, c\}$ that the last c entries of $\mathcal{B}$ must satisfy

$$\begin{aligned} \mathcal{B}_{m+l}(\mathbb{1}_\omega \tilde{u}) &= (\partial_t - \mathrm{div}(d_l\nabla))\,\hat{y}_l - \sum_{i=1}^m (g_{li}\cdot\nabla + a_{li})\,\hat{y}_i - \mathbb{1}_\omega \tilde{u}_l \\ &= (\partial_t - \mathrm{div}(d_l\nabla))\,\mathcal{B}_l(\mathbb{1}_\omega \tilde{u}) - \sum_{i=1}^m (g_{li}\cdot\nabla + a_{li})\,\mathcal{B}_i(\mathbb{1}_\omega \tilde{u}) - \mathbb{1}_\omega \tilde{u}_l, \end{aligned}$$

if (4.9) is to be verified. Thus, one need only to find a $\bar{\mathcal{B}} : C^\infty(Q_T)^{m-c} \to C^\infty(Q_T)^m$ to satisfy the first m lines of (4.16), as the last c lines of (4.16) are completely determined by its first m lines and the respective entry of $\tilde{u}$; consequentially, for our choice of $\mathcal{N}$, the algebraic solvability of (4.9) is equivalent to the algebraic solvability of

$$\bar{\mathcal{L}} \circ \bar{\mathcal{B}} = \bar{\mathcal{N}}. \tag{4.17}$$

We study the adjoint equation of (4.17),

$$\bar{\mathcal{B}}^* \circ \bar{\mathcal{L}}^* = \bar{\mathcal{N}}^*, \tag{4.18}$$

and we call $\bar{\mathcal{B}}^*$ the *left inverse* of $\bar{\mathcal{L}}^*$. Similar to (4.11), we have for $\hat{\psi} \in C^\infty(Q_T)^{m-c}$ that

$$\bar{\mathcal{L}}^*\hat{\psi} = \begin{pmatrix} \sum_{j=c+1}^m (g_{j1} \cdot \nabla - a_{j1})\,\hat{\psi}_j \\ \vdots \\ \sum_{j=c+1}^m (g_{jc} \cdot \nabla - a_{jc})\,\hat{\psi}_j \\ (-\partial_t - \operatorname{div}(d_{c+1}\nabla))\,\hat{\psi}_{c+1} + \sum_{j=c+1}^m \left(g_{j(c+1)} \cdot \nabla - a_{j(c+1)}\right)\hat{\psi}_j \\ \vdots \\ (-\partial_t - \operatorname{div}(d_m\nabla))\,\hat{\psi}_m + \sum_{j=c+1}^m (g_{jm} \cdot \nabla - a_{jm})\,\hat{\psi}_j \end{pmatrix}$$

and

$$\bar{\mathcal{N}}^*\hat{\psi} = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}.$$

Hence, the algebraic solvability of (4.17) is equivalent to proving the existence of a differential operator $\bar{\mathcal{B}}^* : C^\infty(Q_T)^m \to C^\infty(Q_T)^{m-c}$ such that for every $\phi \in C^\infty(Q_T)^m$, if $\hat{\psi} \in C^\infty(Q_T)^{m-c}$ is a solution of

$$\begin{cases} \sum_{j=c+1}^{m} (g_{j1} \cdot \nabla - a_{j1})\,\hat{\psi}_j = \phi_1, \\ \quad\vdots \\ \sum_{j=c+1}^{m} (g_{jc} \cdot \nabla - a_{jc})\,\hat{\psi}_j = \phi_c, \\ (-\partial_t - \operatorname{div}(d_{c+1}\nabla))\,\hat{\psi}_{c+1} + \sum_{j=c+1}^{m} \left(g_{j(c+1)} \cdot \nabla - a_{j(c+1)}\right)\hat{\psi}_j = \phi_{c+1}, \\ \quad\vdots \\ (-\partial_t - \operatorname{div}(d_m\nabla))\,\hat{\psi}_m + \sum_{j=c+1}^{m} (g_{jm} \cdot \nabla - a_{jm})\,\hat{\psi}_j = \phi_m, \end{cases} \tag{4.19}$$

then

$$\bar{\mathcal{B}}^*\phi = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}. \tag{4.20}$$

We encode systems of equations related to system (4.19) using matrices: we utilize a matrix containing the coefficients of D, G, A and -1 (to account for the time derivative terms) as entries to describe system (4.19); this encoding is made precise in the work to follow. Throughout this work, we make the following assumption.

Assumption 4.6. *Assume that the equations in system* (4.19) *are distinct, i.e., that the matrix associated to system* (4.19) *is of full rank.*

An examination of (4.19) reveals that there are m distinct equations and only $m-c$ unknowns, them being $\hat{\psi}_{c+1},\dots,\hat{\psi}_m$. Let us call $\hat{\psi}_{c+1},\dots,\hat{\psi}_m$ the *analytic unknowns*. If we view (4.19) as a linear algebraic system by treating every (time and spatial) derivative of $\hat{\psi}_l$ as an *independent algebraic unknown*, for $l \in \{c+1,\dots,m\}$, then there are many more algebraic unknowns than distinct equations. Under this algebraic viewpoint, one can hope to prolong (or differentiate with respect to every spatial variable) each equation of (4.19) to introduce many new equations and a few new algebraic unknowns (owing to the symmetry property of mixed partial derivatives). Repeating this process a sufficient amount of times, one can hope that the linear algebraic system eventually becomes *overdetermined*, that is, the number of distinct equations eventually exceeds the number of algebraic unknowns. Proceeding this way, we begin by counting the number of derivatives up to the highest order contained in a prolonged version of system (4.19), which is an adaptation of the method used in [8, Subection 3.2.2].

LEMMA 4.7. *Let $p \in \mathbb{N}$ denote the number of prolongations of* (4.19), *and let $F(p)$ denote the distinct number of derivatives of order less than or equal to p for* smooth enough *functions having n variables. Then*

$$F(p) = \binom{p+n}{n}. \tag{4.21}$$

Furthermore, denoting by $U(p)$ and by $E(p)$ the number of algebraic unknowns and the number of equation contained in the prolonged version of system (4.19), *respectively, we have*

$$U(p) = (m-c)\left(F(p+2) + F(p)\right), \tag{4.22}$$

and

$$E(p) = mF(p). \tag{4.23}$$

Proof. Let α be a multi-index of length n such that $|\alpha| \le p$: that is, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, where $\sum_{i=1}^n \alpha_i \le p$. Note that

$$(\alpha_1,\dots,\alpha_n) \mapsto \left\{\alpha_1+1, \alpha_1+\alpha_2+2, \alpha_1+\alpha_2+\alpha_3+3, \dots, \sum_{i=1}^n \alpha_i + n\right\}$$

defines a bijection between the set of tuples $(\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ such that $|\alpha| \le p$ and the set of subsets of $\{1,2,\dots,p+n\}$ having n elements. Furthermore, attributing the multi-index α to the partial derivative operator $\partial_\alpha = \partial_{\alpha_1}\cdots\partial_{\alpha_n}$ takes into account the symmetry of mixed partial derivatives, and thus only counts the *distinct* number of derivatives of order less than or equal to p. Since the cardinality of the set of subsets of $\{1,2,\dots,p+n\}$ having n elements is $\binom{p+n}{n}$, we have (4.21).

Since each analytic unknown contained in system (4.19) has corresponding algebraic unknowns of order up to two in space and one time derivative unknown, and there are $m-c$ analytic unknowns, (4.22) follows.

Since there are m equations, each of which is prolonged p times, and $F(p)$ can be used to represent the number of distinct equations differentiated with respect to the multi-index α, (4.23) follows. □

Concerning our system (4.19), we have the following lemma.

LEMMA 4.8. *For all $m \in \mathbb{N}_{>1}$, $n \in \mathbb{N}^*$ and $c \in \{1, \dots, m-2\}$ such that $c > \frac{m}{2}$, there exists $p \in \mathbb{N}^*$ such that*

$$E(p) > U(p).$$

Proof. We claim that there exists $p \in \mathbb{N}^*$ such that

$$c\binom{p+n}{n} > (m-c)\binom{p+n+2}{n}.$$

Indeed, we have

$$(m-c)\binom{p+n+2}{n} = (m-c)\frac{(p+n+2)(p+n+1)}{(p+2)(p+1)}\frac{(p+n)!}{p!n!}$$

and

$$c\binom{p+n}{n} = c\frac{(p+n)!}{p!n!}.$$

First, we show that for fixed m and n, there exist p and c such that

$$\frac{(p+n+2)(p+n+1)}{(p+2)(p+1)} < \frac{c}{(m-c)}; \tag{4.24}$$

Indeed, since $m \in \mathbb{N}_{>1}$, we can choose $c > \frac{m}{\frac{(p+2)(p+1)}{(p+n+2)(p+n+1)}+1}$ to verify (4.24). Note that $\frac{(p+2)(p+1)}{(p+n+2)(p+n+1)} \to 1$ from below as $p \to \infty$, and thus $c > \frac{m}{2}$ is necessary for $E(p) > U(p)$. Since $m \in \mathbb{N}^*$ and $c \in \{1, \dots, m-2\}$, $c > \frac{m}{2}$ is also sufficient since one can always choose $p \in \mathbb{N}$ large enough to verify (4.24) when $c = \lfloor \frac{m}{2} \rfloor + 1$. □

REMARK 4.9. *Lemma 4.8 shows that for a sufficiently regular solution $\hat{\psi}$ to system (4.8), if $c \geq \lfloor \frac{m}{2} \rfloor + 1$, then there exists $p \in \mathbb{N}$ such that we can prolong system (4.19) p times and study the resulting overdetermined linear algebraic system. One can argue the appropriate regularity of $\hat{\psi}$ as follows: without loss of generality, we can take $\hat{\psi}^0 \in H^{p+1}(\Omega)^m$ by a classical density argument; then, ones applies Theorem 2.4. As we will see, under certain conditions, one may hope to extract the analytic unknowns $\hat{\psi}_{c+1}, \dots, \hat{\psi}_m$ from the overdetermined algebraic system. Hence, one can expect the left inverse of the differential operator associated to the prolonged version of system (4.19) to be of maximum differential order $p+2$ in space and 1 in time. Thus, by (4.16) we require the analytic system's controls, $\mathbb{1}_\omega \tilde{u}$, to accommodate $p+2$ spatial differentiations. These highly regular $\mathbb{1}_\omega \tilde{u}$ are constructed in [21].*

We finish this work by proving the main result.

PROPOSITION 4.10. *Given m, n and c in $\mathbb{N}^*$ with $\lfloor \frac{m}{2} \rfloor + 1 \leq c \leq m$, if the matrix*

$C \in \mathcal{M}_{c\times(m-c)(n+1)}(\mathbb{R})$ *given by*

$$C := \begin{pmatrix} a_{(c+1)1} & \cdots & a_{m1} & g^1_{(c+1)1} & \cdots & g^1_{m1} & \cdots & g^n_{(c+1)1} & \cdots & g^n_{m1} \\ a_{(c+1)2} & \cdots & a_{m2} & g^1_{(c+1)1} & \cdots & g^1_{m1} & \cdots & g^n_{(c+1)1} & \cdots & g^n_{m1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{(c+1)c} & \cdots & a_{mc} & g^1_{(c+1)c} & \cdots & g^1_{mc} & \cdots & g^n_{(c+1)c} & \cdots & g^n_{mc} \end{pmatrix}$$

has full rank, then (4.9) *is algebraically solvable in* q_T.

Proof. Without loss of generality, for a given m, n and c, we fix a p large enough such that $E(p) > U(p)$. Consider the overdetermined matrix $\bar{L}^* \in \mathcal{M}_{E(p)\times U(p)}(\mathbb{R})$ with entries equal to the coefficients multiplying the algebraic unknowns generated by prolonging system (4.19) p times. We denote the vector containing the p-times prolonged unknowns by $\hat{z} \in \mathcal{M}_{U(p)\times 1}(L^2(Q_T))$, where the necessary regularity of $\hat{\psi}$ is discussed in Remark 4.9. Similarly, we denote the p-times prolonged version of ϕ by $\Phi \in \mathcal{M}_{E(p)\times 1}(C^\infty(Q_T))$. Hence, we can write the *prolonged algebraic version* of the system (4.19) as

$$\bar{L}^*\hat{z} = \Phi. \tag{4.25}$$

The counterpart of solving (4.19) and (4.20) simultaneously for (4.25) is to find a $P \in \mathcal{M}_{(m-c)\times E(p)}$ such that

$$P\bar{L}^*\hat{z} = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}, \tag{4.26}$$

with P being the algebraic version of $\bar{\mathcal{B}}^*$. We apply Theorem 2.10 to $\bar{L}^*$ so that for $S_{\tilde{\sigma}}$ and S_σ the left and right permutation matrices generated by the Dulmage-Mendelsohn decomposition, respectively, we have

$$S_{\tilde{\sigma}}\bar{L}^*S_\sigma = \begin{pmatrix} \bar{L}^*_{11} & \bar{L}^*_{12} & \bar{L}^*_{13} & \bar{L}^*_{14} \\ 0 & 0 & \bar{L}^*_{23} & \bar{L}^*_{24} \\ 0 & 0 & 0 & \bar{L}^*_{34} \\ 0 & 0 & 0 & \bar{L}^*_{44} \end{pmatrix}, \tag{4.27}$$

where $\bar{L}^*_{34}$ is square and perfectly matched (i.e., it is of maximal structural rank). We permute $\hat{z}$ accordingly by S_σ^{-1}.

Our next steps are as follows. First, we study the structure of $\bar{L}^*$ and argue that under $S_{\tilde{\sigma}}$ and S_σ, every row of C (which appear in $\bar{L}^*$) is permuted to block $\bar{L}^*_{34}$ (possibly with some zero entries to the right). Then, we argue that the unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ contained in $\hat{z}$ are being multiplied by the block $\bar{L}^*_{34}$ (and in particular, the rows of C). Finally, we deduce from the full rank of C that $\bar{L}^*_{34}$ is non-singular (possibly after some row permutations of $\begin{pmatrix}\bar{L}^*_{34} & \bar{L}^*_{44}\end{pmatrix}^T$), which yields a P satisfying (4.26). Immediately following the end of this proof, we supplement our explanations with Example 4.28.

By construction of $\bar{L}^*$, we have that the columns of $\bar{L}^*$ corresponding to any algebraic unknown involving a time derivative are very sparse. Indeed, each of these columns has only one nonzero entry (which is -1). This occurs since we do not prolong system (4.19) with respect to time, and hence each time derivative term appears in one (and only one) equation within the prolonged version of system (4.19). Furthermore, the row associated to any one of these -1 column entries must correspond to the j-th equation (or its prolonged version) in system (4.19), for $j \in \{c+1, \ldots, m\}$. Hence, the coefficients corresponding to the j-th equation (or its prolonged version) in system (4.19) lie in this row, for $j \in \{c+1, \ldots, m\}$.

We claim that there exists a maximum matching M in $G(\bar{L}^*)$ that contains all of the edges (r_i, c_i) corresponding to these -1 entries. Indeed, for any matrix P, a *matching* in $G(P)$ is a subset of nonzero entries of P such that no two of which belong to the same row or column. Hence, since the columns of $\bar{L}^*$ corresponding to any algebraic unknown involving a time derivative contain only one nonzero entry, it is easy to deduce that there exists a maximum matching M in $G(\bar{L}^*)$ chosen to include all of these -1 entries. Importantly, this choice will omit any other edges associated to coefficients corresponding to the j-th equation (or its prolonged version) in system (4.19), for $j \in \{c+1, \ldots, m\}$, from M, and the rows containing these coefficients will be matched (see Example 4.11). Furthermore, we can choose at random enough edges which make M a maximum matching; it follows that all of these edges will correspond to coupling coefficients of the j-th equation (or its prolonged version) in system (4.19), for $j \in \{1, \ldots, c\}$. Without loss of generality, we associate $S_{\tilde{\sigma}}$ and S_σ to this choice of maximum matching.

With our choice of M, we now study vertex sets VR and VC. Recall from Section 2.5 that

$$VR := \{\text{row vertices reachable by alternating paths from some unmatched row}\},$$
$$VC := \{\text{column vertices reachable by alternating paths from some unmatched row}\},$$

where an alternating path is a sequence of (row or column) vertices $(v_i)_{i=0}^k$ such that $(v_{2i}, v_{2i+1}) \in E$ and, additionally, $(v_{2i+1}, v_{2(i+1)}) \in M$ and no vertices are repeated, for $k \in \mathbb{N}^*$. By our choice of M and since $\bar{L}^*$ is overdetermined, there exists unmatched rows, and any unmatched row must correspond to the j-th equation (or its prolonged version) in system (4.19), for $j \in \{1, \ldots, c\}$. One deduces from the structure of $\bar{L}^*$ that these unmatched rows have nonzero entries which lie in matched columns, and hence VR and VC are not empty. Furthermore, these matched columns cannot be those corresponding to algebraic unknowns involving a time derivative. By the structure of $\bar{L}^*$, all row vertices corresponding to the j-th equation in system (4.19) are reachable by an alternating path from some unmatched row, for all $j \in \{1, \ldots, c\}$. This is a consequence of equations in system (4.19) having *first and zero-order coupling coefficients* and since $\bar{L}^*$ is generated by prolongations with respect to spatial variables only. Hence, rows corresponding to the j-th equation (or its prolonged version) in system (4.19) have corresponding row vertices contained in VR, for $j \in \{1, \ldots, c\}$. It follows that columns containing coupling coefficients have corresponding column vertices contained in VC (the same alternating paths yield the column vertices in VC). Hence by Theorem 2.10, only the coefficients that appear in the j-th equation (or its prolonged version) in system (4.19) are permuted to the blocks $\bar{L}^*_{34}$ and $\bar{L}^*_{44}$, for $j \in \{1, \ldots, c\}$.

By examining system (4.19), one easily deduces that the unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$

are being multiplied by $\bar{L}_{34}^*$ and $\bar{L}_{44}^*$. Let $\bar{A} \in \mathcal{M}_{c\times(m-c)}$ be defined by

$$\bar{A} := \begin{pmatrix} a_{(c+1)1} & \cdots & a_{m1} \\ a_{(c+1)2} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{(c+1)c} & \cdots & a_{mc} \end{pmatrix},$$

and let $\bar{G} \in \mathcal{M}_{c\times n(m-c)}$ be defined by

$$\bar{G} := \begin{pmatrix} g^1_{(c+1)1} & \cdots & g^1_{m1} & \cdots & g^n_{(c+1)1} & \cdots & g^n_{m1} \\ g^1_{(c+1)1} & \cdots & g^1_{m1} & \cdots & g^n_{(c+1)1} & \cdots & g^n_{m1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g^1_{(c+1)c} & \cdots & g^1_{mc} & \cdots & g^n_{(c+1)c} & \cdots & g^n_{mc} \end{pmatrix},$$

so that $C = \begin{pmatrix} \bar{A} & \bar{G} \end{pmatrix}$. One can permute the bottom rows of $S_{\tilde{\sigma}}\bar{L}^*S_\sigma$ such that if one were to partition $\begin{pmatrix} \bar{L}_{34}^* & \bar{L}_{44}^* \end{pmatrix}^T$ into subsequent groups of c rows, these groups would contain nothing but $\bar{A}$ and $\bar{G}$ (with matrices of zeros possibly on either side of $\bar{A}$ and $\bar{G}$). We denote this row permutation on $S_{\tilde{\sigma}}\bar{L}^*S_\sigma$ by $S_{\tilde{\sigma}^0}$. Since C has full rank, one deduces that the row-permuted version of $\bar{L}_{34}^*$, extracted from $S_{\tilde{\sigma}^0}S_{\tilde{\sigma}}\bar{L}^*S_\sigma$, is non-singular. Finally, with a slight abuse of notation, we denote by I various identity matrices with appropriate dimensions; using the row permutation

$$S_{\tilde{\sigma}^1} := \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

the column permutation

$$S_{\sigma^1} := \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{pmatrix},$$

we permute $S_{\tilde{\sigma}^0}S_{\tilde{\sigma}}\bar{L}S_\sigma$ into lower-block triangular form with the row-permuted version of $\bar{L}_{34}^*$ being the top leftmost block, and we define

$$P := \begin{pmatrix} \mathrm{Id}_{m-c} & 0_{(m-c)\times(h-m+c)} \end{pmatrix} \begin{pmatrix} (\bar{L}_{34}^*)^{-1} & 0_{h\times(E(p)-h)} \end{pmatrix} S_{\tilde{\sigma}^1}S_{\tilde{\sigma}^0}S_{\tilde{\sigma}},$$

which verifies (4.26). Hence, by the non-singularity of the row-permuted version of $\bar{L}_{34}^*$, there exists a linear combination of differentiated lines of $\bar{\mathcal{L}}^*$ that allow us to recover the analytic unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$. We denote by $\mathcal{P}$ the differential operator associated to matrix P; it follows that $\bar{\mathcal{B}}^* := \mathcal{P}$ verifies (4.18), and hence $\bar{\mathcal{B}} = \mathcal{P}^*$ verifies (4.17). $\square$

EXAMPLE 4.11. *In this example, we consider the algebraic control system given by* (4.5)*, where we choose* $m = 5$*,* $c = 3$*, and for simplicity,* $n = 1$*. In solving the algebraic version of* (4.18)*, which is given by* (4.26)*, we study the linear algebraic*

operator obtained by prolonging system (4.19) *3 times given by*
(4.28)

$$\bar{L}^* = \begin{pmatrix}
-a_{41} & -a_{51} & 0 & 0 & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{42} & -a_{52} & 0 & 0 & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{43} & -a_{53} & 0 & 0 & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{44} & -a_{54} & 0 & -1 & g_{44} & g_{54} & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{45} & -a_{55} & -1 & 0 & g_{45} & g_{55} & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{44} & -a_{54} & g_{44} & g_{54} & 0 & -1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{45} & -a_{55} & g_{45} & g_{55} & -1 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & 0 & 0 & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & 0 & 0 & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & 0 & 0 & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{44} & -a_{54} & 0 & 0 & g_{44} & g_{54} & 0 & -1 & d_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{45} & -a_{55} & 0 & 0 & g_{45} & g_{55} & -1 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & 0 & 0 & g_{41} & g_{51} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & 0 & 0 & g_{42} & g_{52} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & 0 & 0 & g_{43} & g_{53} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{44} & -a_{54} & 0 & 0 & g_{44} & g_{54} & 0 & -1 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{45} & -a_{55} & 0 & 0 & g_{45} & g_{55} & -1 & 0 & 0 & d_2
\end{pmatrix}.$$

In (4.28)*, we have circled a subset of nonzero entries of* $\bar{L}^*$ *whose corresponding edges make up a maximum matching* M *in* $G(\bar{L}^*)$*. We have chosen* M *such that it contains every edge corresponding to a* -1 *entry of* $\bar{L}^*$*.*

We now populate the set of edges VR *and* VC*. Note that* r_{13} *and* r_{18} *are the only unmatched row vertices; hence, we search for row vertices that are reachable from* r_{13} *and* r_{18} *via an alternating path. A crucial observation is that there exists no alternating paths from these row vertices to the row vertices corresponding to rows containing* -1 *entries: indeed, for a walk starting from row* 13*, since* $(r_{13}, c_i) \notin M$*, for* $i \in \{7, 8, 11, 12\}$*, the next row vertex* r_k *in the walk must be such that* $(c_i, r_k) \in M$*, for* $k \in \{1, \ldots, 12, 14, \ldots, 20\}$*, hence* $r_k \neq r_j$ *for* $j \in \{4, 5, 9, 10, 14, 15, 19, 20\}$*; the exact same argument holds for a walk starting from row* 18*. One can easily deduce by the same reasoning that* r_j *will never be reachable by a (longer) alternating path, for* $j \in \{4, 5, 9, 10, 14, 15, 19, 20\}$*. Furthermore, every other row vertex is reachable by an alternating path from either* r_{13} *or* r_{18}*! Hence,*

$$VR = \{r_1, r_2, r_3, r_6, r_7, r_8, r_{11}, r_{12}, r_{13}, r_{16}, r_{17}, r_{18}\},$$

and it follows that

$$VC = \{c_1, c_2, c_5, c_6, c_7, c_8, c_{11}, c_{12}, c_{15}, c_{16}\}.$$

Hence, we arrive at (possibly after a row permutation)

$$\begin{pmatrix} \bar{L}_{34}^* \\ \bar{L}_{44}^* \end{pmatrix} = \begin{pmatrix} -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} \end{pmatrix},$$

from which we deduce the algebraic solvability condition stated in Proposition 4.10.

REMARK 4.12. *Let* $h := (m-c)(n+1)$. *For* $c \geq h$, *one can take* $p = 0$ *and still deduce algebraic solvability of* (4.9) *even if* $E(p) < U(p)$. *Indeed, by writing the matrix associated to system* (4.19) *with* $p = 0$, *one can immediately extract a matrix* $\bar{C} \in \mathcal{M}_h(\mathbb{R})$ *(which contains nothing but some of the rows of* C*) with zeros to the right which multiplies the analytic unknowns* $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$. *Since* C *has full rank,* $\bar{C}$ *is non-singular and algebraic solvability follows. Hence, for* $c \geq h$, *we have shown that* $\mathcal{B}$ *will be of differential order* 1 *in space and* 0 *in time, so higher regularity of* $\mathbb{1}_\omega \tilde{u}$ *than was proved in [10] is no longer needed for this case.*

5. Conclusion and possible extension. In the first part of this two-part work, we used a powerful technique, the so-called fictitious control method, which allowed us to pose our controllability problem as two interconnected problems. We derived a sufficient condition for the algebraic solvability of a system of coupled parabolic PDEs, where the couplings were constant in space and time and of first and zero-order, when more than half of the equations in the system were actuated. With algebraic solvability established, we can now study the analytic system (4.4); proving its null controllability will help us recover null controllability of the original control system (3.1).

One could explore different choices of differential operator $\mathcal{N}$, which may yield a milder controllability condition that close the gap between sufficiency and necessity (as in Lemma 4.4 for one underactuation).

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