

CONTROLLABILITY OF COUPLED PARABOLIC SYSTEMS WITH MULTIPLE UNDERACTUATIONS, PART 2: NULL CONTROLLABILITY

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Abstract. This paper is the second of two parts which together study the null controllability of a system of coupled parabolic PDEs. Our work specializes to an important subclass of these control problems which are coupled by first and zero-order couplings and are, additionally, underactuated. In the first part of our work [11], we posed our control problem in a framework which divided the problem into interconnected components: the algebraic control problem, which was the focus of the first part; and the analytic control problem, whose treatment was deferred to this paper. We use slightly non-classical techniques to prove null controllability of the analytic control problem by means of internal controls appearing on every equation. We combine our previous results in [11] with the ones derived below to establish a null controllability result for the original problem.

Key words. Controllability, Parabolic systems, Carleman estimates, Fictitious control method.

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1. Introduction. We begin with defining some notation and conventions.

1.1. Notation and conventions. Throughout this work, we define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and similarly, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $n, k \in \mathbb{N}^*$, we denote the set of $n \times k$ matrices with real-valued entries by $\mathcal{M}_{n \times k}(\mathbb{R})$, and we denote the set of $n \times n$ matrices with real-valued entries by $\mathcal{M}_n(\mathbb{R})$. We denote the set of linear maps from a vector space U to a vector space V by $\mathcal{L}(U; V)$. For (X, \mathcal{T}_X) a topological space and $U \subset X$, we denote the closure of U by \bar{U} . We now recall the coupled parabolic system of interest.

1.2. A system of interest. In this second part of this two-part work, the primary objective is maintained from the first part: that is, for $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$ for some $T > 0$, we wish to study the controllability properties of the system of coupled parabolic PDEs given by

$$(1.1) \quad \begin{cases} \partial_t y = \operatorname{div}(D \nabla y) + G \cdot \nabla y + Ay + r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0, \cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases}$$

where $D := \operatorname{diag}(d_1, \dots, d_m)$, $G := (g_{pk})_{1 \leq p, k \leq m} \in \mathcal{M}_m(\mathbb{R}^n)$ and $A := (a_{pk})_{1 \leq p, k \leq m} \in \mathcal{M}_m(\mathbb{R})$. We associate to this system the differential operator

$$(1.2) \quad \mathcal{L}y = \sum_{p=1}^m \left(-\operatorname{div}(d_p \nabla y_p) - \sum_{k=1}^m g_{pk} \cdot \nabla y_k - \sum_{k=1}^m a_{pk} y_k \right) \mathbf{e}_p,$$

where $g_{pk} := (g_{pk}^1, \dots, g_{pk}^n) \in \mathbb{R}^n$, $d_p \in \mathcal{M}_n(\mathbb{R})$ is symmetric and \mathbf{e}_p is the p -th canonical basis vector in \mathbb{R}^m , for $p \in \{1, \dots, m\}$. We call g_{pk} the *first-order coupling coefficients* and a_{pk} the *zero-order coupling coefficients*, which are constant in space and time.

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In this work, we assume that \mathcal{L} satisfies the uniform ellipticity condition: that is, there exists $C > 0$ such that,

$$(1.3) \quad \sum_{i,j=1}^n d_p^{ij} \xi_i \xi_j \geq C |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. For $u, v \in H_0^1(\Omega)^m$, we define the bilinear form

$$B[u, v] := \int_{\Omega} \sum_{p,k=1}^m \left(\sum_{i,j=1}^n d_p^{ij} (\partial_{x_i} u_p) (\partial_{x_j} v_p) - \sum_{i=1}^n g_{pk}^i (\partial_{x_i} u_k) v_p - a_{pk} u_k v_p \right) \mathbf{e}_p dx.$$

One has the following definition.

DEFINITION 1.1. *Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. A function*

$$\mathbf{y} \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m$$

is said to be a weak solution of system (1.1) provided that for every $v \in H_0^1(\Omega)^m$ and almost every $t \in [0, T]$

- (i) $\langle \frac{d}{dt} \mathbf{y}, v \rangle + B[\mathbf{y}, v] = \int_{\Omega} \mathbf{r}^T v dx$, and;
- (ii) $\mathbf{y}(0) = y^0$,

where $\langle \cdot, \cdot \rangle$ denotes the appropriate duality pairing.

It can be deduced, for example, from [6, Theorems 3 and 4, Section 7.1.2], that for any initial condition $y^0 \in L^2(\Omega)^m$ and $r \in L^2(Q_T)^m$, system (1.1) admits a unique solution. From now on, we mean by “solution to a coupled parabolic system” the weak solution in the sense of Definition 1.1.

1.3. A parabolic regularity result. We state a regularity result for the solution of system (1.1) which is essential in the work to follow.

THEOREM 1.2. [6, Theorem 6, Subsection 7.1.3] *For $d \in \mathbb{N}$, assume $y^0 \in H^{2d+1}(\Omega)^m$, $\mathbf{r} \in L^2((0, T); H^{2d}(\Omega))^m \cap H^d((0, T); L^2(\Omega))^m$, and assume that $\mathbf{y} \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m$ is the solution of system (1.1). Suppose also that the following compatibility conditions hold:*

$$\begin{cases} g^0 := y^0 \in H_0^1(\Omega)^m; \\ g^1 := \mathbf{r}(0) - \mathcal{L}g^0 \in H_0^1(\Omega)^m; \\ \vdots \\ g^d := \frac{d^{d-1}}{dt^{d-1}} \mathbf{r}(0) - \mathcal{L}g^{d-1} \in H_0^1(\Omega)^m. \end{cases}$$

Then $\mathbf{y} \in L^2((0, T); H^{2d+2}(\Omega))^m \cap H^{d+1}((0, T); L^2(\Omega))^m$ and we have the estimate

$$(1.4) \quad \|\mathbf{y}\|_{L^2((0, T); H^{2d+2}(\Omega))^m \cap H^{d+1}((0, T); L^2(\Omega))^m} \leq C \left(\|\mathbf{r}\|_{L^2((0, T); H^{2d}(\Omega))^m \cap H^d((0, T); L^2(\Omega))^m} + \|y^0\|_{H^{2d+1}(\Omega)^m} \right).$$

1.4. The control problem. This work specializes to the case of internal (or distributed) control: that is, for $\omega \subset \Omega$ nonempty and open, we study the case where $r = \mathbb{1}_{\omega} B u$, for $B = (\text{Id}_c \quad 0_{c \times (m-c)})^T \in \mathcal{M}_{m \times c}(\mathbb{R})$ and $1 \leq c \leq m$, and henceforth,

we denote by q_T the set $(0, T) \times \omega$.

An interesting control problem that arises in many engineering applications is underactuation, that is, when $c < m$. Our work will further specialize to this case, where there are currently few results for *first and zero-order couplings*, for arbitrary m and $c < m - 1$ (even for the case of constant coefficients). We focus on a particular type of controllability property, which is defined next.

DEFINITION 1.3. *We say that (1.1) is null controllable in time T if for every initial condition $y^0 \in L^2(\Omega)^m$, there exists $u \in L^2(Q_T)^c$ such that the solution $y \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m$ to (1.1) satisfies $y(T) = 0$ in Ω .*

This work's main objective that we aim to achieve by selecting appropriate control inputs is null controllability of system (1.1). Next, we recall the method which we introduced in [11]; we employ this method to achieve our main objective.

1.5. Fictitious control method. We first described following control problem, referred to as the *analytic control problem*: for any $\tilde{y}^0 \in L^2(\Omega)^m$, proving the existence of (\tilde{y}, \tilde{u}) a solution of

$$(1.5) \quad \begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla \tilde{y}) + G \cdot \nabla \tilde{y} + A\tilde{y} + \mathcal{N}(\mathbb{1}_\omega \tilde{u}), & \text{in } Q_T, \\ \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0, \cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega, \end{cases}$$

such that $\tilde{y}(T, \cdot) = 0$, where \mathcal{N} is a differential operator that was chosen to be the identity in [11], \tilde{u} acts on all equations in (1.5), and we denote by $\mathbb{1}_\omega$ a *smooth version* of the indicator function (this can be constructed via mollification; cf. relation (4.2) for its exact definition).

We next consider a different control problem, referred to as the *algebraic control problem*: proving the existence of a solution (\hat{y}, \hat{u}) of

$$(1.6) \quad \begin{cases} \partial_t \hat{y} = \operatorname{div}(D\nabla \hat{y}) + G \cdot \nabla \hat{y} + A\hat{y} + B\hat{u} + \mathcal{N}(\mathbb{1}_\omega \hat{u}), & \text{in } Q_T, \\ \hat{y} = 0, & \text{on } \Sigma_T, \\ \hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0, & \text{in } \Omega, \end{cases}$$

where \hat{u} acted only on the first c equations.

We defined the notion of *algebraic solvability* of (1.6) in [11], which is a property that enabled us to algebraically “invert” the differential operator associated to (1.6) and recover the solution to this control problem locally. We recovered the following result.

PROPOSITION 1.4. *Given m, n and c in \mathbb{N}^* with $\lfloor \frac{m}{2} \rfloor + 1 \leq c \leq m$, if the matrix $C \in \mathcal{M}_{c \times (m-c)(n+1)}(\mathbb{R})$ given by*

$$C := \begin{pmatrix} a_{(c+1)1} & \cdots & a_{m1} & g_{(c+1)1}^1 & \cdots & g_{m1}^1 & \cdots & g_{(c+1)1}^n & \cdots & g_{m1}^n \\ a_{(c+1)2} & \cdots & a_{m2} & g_{(c+1)1}^1 & \cdots & g_{m1}^1 & \cdots & g_{(c+1)1}^n & \cdots & g_{m1}^n \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{(c+1)c} & \cdots & a_{mc} & g_{(c+1)c}^1 & \cdots & g_{mc}^1 & \cdots & g_{(c+1)c}^n & \cdots & g_{mc}^n \end{pmatrix}$$

has full rank, then (1.6) is algebraically solvable in q_T .

Importantly, the “inverse” differential operator that we recovered in [11], denoted by \mathcal{B} , was of differential order at most $p + 2$ in space. This differential order is of consequence in Section (4), where we require the controls in the analytic system (1.5) to be regular enough to withstand these $p + 2$ spacial derivatives. This regularity

on the controls is necessary for the solution that was constructed for the algebraic problem to be well-defined.

With the algebraic problem resolved, solving the analytic problem is this paper's secondary objective. Achieving this secondary objective will allow us to attain this work's main objective, as will be shown in Section 4.

1.6. Statement of contributions. The first contribution is a partial generalization of [5, Theorem 1]. In particular, our result gives a sufficient condition for the null and approximate controllability of an underactuated system of coupled parabolic PDEs, with constant first and zero-order couplings, when more than half of the equations are actuated. Importantly, this controllability condition applies to systems with multiple underactuators. Furthermore, this condition, which requires the rank of a matrix containing some of the coupling coefficients as entries to be full rank, is generic. The technique used to prove this result is adapted from [4].

Our second contribution is Proposition 3.7, which is an extension of [5, Proposition 2.2]. Specifically, our Carleman estimate contains higher differential order terms on its lefthand side, which allows us to construct very regular controls in Proposition 4.2. Importantly, these highly regular controls are necessary when applying Theorem 2.1 to problems with many underactuators.

2. Main result. The main controllability theorem of this work is stated next, where we assume that more than half of the equations in system (1.1) are actuated.

THEOREM 2.1. *For a fixed m , suppose $\Omega \subset \mathbb{R}^n$ nonempty, open and bounded. Furthermore, suppose Ω is connected and of class C^{p+2} . For $\lfloor \frac{m}{2} \rfloor + 1 \leq c \leq m$, if the matrix $C \in \mathcal{M}_{c \times (m-c)(n+1)}(\mathbb{R})$ given by*

$$C := \begin{pmatrix} a_{(c+1)1} & \cdots & a_{m1} & g_{(c+1)1}^1 & \cdots & g_{m1}^1 & \cdots & g_{(c+1)1}^n & \cdots & g_{m1}^n \\ a_{(c+1)2} & \cdots & a_{m2} & g_{(c+1)2}^1 & \cdots & g_{m2}^1 & \cdots & g_{(c+1)2}^n & \cdots & g_{m2}^n \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{(c+1)c} & \cdots & a_{mc} & g_{(c+1)c}^1 & \cdots & g_{mc}^1 & \cdots & g_{(c+1)c}^n & \cdots & g_{mc}^n \end{pmatrix}$$

has full rank, then the system (1.1) is null controllable in time T .

The rest of this work is devoted to proving the above result. First, we will resolve the analytic control problem in Section 4; next, we will utilize the solutions to the algebraic and analytic control problems to solve our original control problem in Section 3, which is the null controllability of the underacted system (1.1).

3. A Carleman estimate for the analytic problem. In this section, we study the analytic system:

$$(3.1) \quad \begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla \tilde{y}) + G \cdot \nabla \tilde{y} + A\tilde{y} + \mathbb{1}_\omega \tilde{u}, & \text{in } Q_T, \\ \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0, \cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega. \end{cases}$$

The goal of this section is to prove that the solution (\tilde{y}, \tilde{u}) to the analytic control system (3.1) satisfies the following so-called *weighted observability inequality*, which will help us deduce its null controllability. To this end, we consider the adjoint system

to system (3.1) given by

$$(3.2) \quad \begin{cases} -\partial_t \tilde{\psi} = \operatorname{div}(D\nabla \tilde{\psi}) - G^* \cdot \nabla \tilde{\psi} + A^* \tilde{\psi}, & \text{in } Q_T, \\ \tilde{\psi} = 0, & \text{on } \Sigma_T, \\ \tilde{\psi}(T, \cdot) = \tilde{\psi}^0(\cdot), & \text{in } \Omega, \end{cases}$$

where $\tilde{\psi}^0 \in L^2(\Omega)^m$.

We state the weighted observability inequality we aim to establish.

PROPOSITION 3.1. *For every $\tilde{\psi}^0 \in L^2(\Omega)^m$, the solution $\tilde{\psi}$ of system (3.2) satisfies*

$$(3.3) \quad \int_{\Omega} \left\| \tilde{\psi}(0, x) \right\|_1^2 dx \leq C_{obs} \iint_{(0,T) \times \omega_0} e^{-2s_1 \alpha \xi^{2p+7}} \left\| \tilde{\psi}(t, x) \right\|_1^2 dx dt,$$

where $C_{obs} := CT^9 e^{C(1+3T/4+1/T^5)} > 0$ and $\|\cdot\|_1$ denotes the Euclidean norm. We call (3.3) a weighted observability inequality with weight $\rho := e^{-2s_1 \alpha \xi^{2p+7}}$, for α and ξ defined below in (3.5) and (3.6), respectively, where $s_1 := \sigma(T^5 + T^{10})$ for $\sigma > 0$ depending on Ω and ω_0 .

We utilize the *Carleman estimate* technique to develop an estimate which will help us establish the observability inequality stated above; the proof of Proposition 3.1 follows from Proposition 3.7 and is given in the Appendix. This section builds upon the technique developed in [5, Section 2.2]: in particular, it incorporates the higher-order terms found on the lefthand side of (3.12) which allow us to construct highly regular controls for system (3.1). Constructing a solution (\tilde{y}, \tilde{u}) to system (3.1) with highly regular controls and satisfying $\tilde{y}(T, \cdot) = 0$ is treated in Section 4.

3.1. Some notation and technical results. We begin by introducing some notation. For the multi-index β of length l consisting of multi-indices, consider the l^{th} -order tensor given by $C := (C_{\beta})_{\beta}$, where β_i has length n_i , for $n_i \in \mathbb{N}^*$, for $i \in \{1, \dots, l\}$. We associate to C the element-wise norm:

$$\|\cdot\|_l := \left(\sum_{i_1=1, \dots, i_l=1}^{n_1, \dots, n_l} C_{\beta_1(i_1), \dots, \beta_l(i_l)}^2 \right)^{1/2}.$$

An equivalent interpretation of $\|\cdot\|_l$ is the following: given a l^{th} -order tensor C , one *vectorizes* C into a vector of length $\sum_{i=1}^l n_i$ and then applies the Euclidean norm to recover $\|\cdot\|_l$. Fix a sequence $(\omega_i)_{i=0}^{p+2}$ of nonempty open subsets of ω such that

$$\begin{cases} \bar{\omega}_i \subset \omega_{i-1}, & \text{for } i \in \{1, \dots, p+2\}, \\ \bar{\omega}_0 \subset \omega. \end{cases}$$

The next result is an adaptation of [8, Lemma 1.1] (see also [2, Lemma 2.68]).

LEMMA 3.2. *Assume that Ω is of class C^{p+2} and connected. Then there exists $\eta^0 \in C^{p+2}(\bar{\Omega})$ such that*

$$(3.4) \quad \begin{cases} \|\nabla \eta^0\|_1 \geq \kappa, & \text{in } \Omega \setminus \omega_{p+2}, \\ \eta^0 > 0, & \text{in } \Omega, \\ \eta^0 = 0, & \text{on } \partial\Omega, \end{cases}$$

for some $\kappa > 0$.

REMARK 3.3. In (4.17), we require η^0 to be $(p+2)$ -times differentiable; this is why we require spatial domain boundary regularity in Theorem 2.1.

For $(t, x) \in Q_T$ we define

$$(3.5) \quad \alpha(t, x) := \frac{e^{12\lambda\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}}{t^5(T-t)^5}$$

and

$$(3.6) \quad \xi(t, x) := \frac{e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}}{t^5(T-t)^5}.$$

Additionally, for $t \in (0, T)$ we define

$$(3.7) \quad \alpha^*(t) := \max_{x \in \Omega} \alpha(t, x)$$

and

$$(3.8) \quad \xi^*(t) := \min_{x \in \Omega} \xi(t, x).$$

For $s, \lambda > 0$ and $u \in L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$, let us define

$$(3.9) \quad \mathcal{I}(s, \lambda; u) := s^3 \lambda^4 \iint_{Q_T} e^{-2s\alpha} \xi^3 |u|^2 dx dt + s \lambda^2 \iint_{Q_T} e^{-2s\alpha} \xi \|\nabla u\|_1^2 dx dt.$$

In the work to follow, for $u \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m$, we use a slight abuse of notation and define $\mathcal{I}(s, \lambda; u)$ as above but with $|\cdot|$ replaced by $\|\cdot\|_1$, and with $\|\cdot\|_1$ replaced by $\|\cdot\|_2$.

We now state a Carleman estimate result for the heat equation; the proof is quite technical and is omitted here.

LEMMA 3.4. [7, Theorem 1] Assume that $d > 0$, $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$ and $f_2 \in L^2(\Sigma_T)$. Then there exists a constant $C := C(\Omega, \omega_{p+2}) > 0$ such that the solution to

$$\begin{cases} -\partial_t u = \operatorname{div}(d\nabla u) + f_1, & \text{in } Q_T, \\ \frac{\partial u}{\partial n} = f_2, & \text{on } \Sigma_T, \\ u(T, \cdot) = u^0(\cdot), & \text{in } \Omega, \end{cases}$$

satisfies

$$\mathcal{I}(s, \lambda; u) \leq C \left(s^3 \lambda^4 \iint_{(0, T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 |u|^2 dx dt + \iint_{Q_T} e^{-2s\alpha} |f_1|^2 dx dt + s \lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* |f_2|^2 d\sigma dt \right)$$

for all $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

We can adapt the Carleman estimate in Lemma 3.4 to system (3.2) with Neumann boundary condition; this adapted Carleman estimate will be used later (cf. (A.5)).

LEMMA 3.5. Assume that $\tilde{\psi}^0 \in L^2(\Omega)^m$ and $u \in L^2(\Sigma_T)^m$. Then there exists a constant $C := C(\Omega, \omega_{p+2}) > 0$ such that the solution to

$$(3.10) \quad \begin{cases} -\partial_t \tilde{\psi} = \operatorname{div}(D\nabla \tilde{\psi}) - G^* \cdot \nabla \tilde{\psi} + A^* \tilde{\psi}, & \text{in } Q_T, \\ \frac{\partial \tilde{\psi}}{\partial n} = u, & \text{on } \Sigma_T, \\ \tilde{\psi}(T, \cdot) = \tilde{\psi}^0(\cdot), & \text{in } \Omega, \end{cases}$$

satisfies

$$(3.11) \quad \mathcal{I}(s, \lambda; \tilde{\psi}) \leq C \left(s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 \|\tilde{\psi}\|_1^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|u\|_1^2 d\sigma dt \right)$$

for all $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

The proof of Lemma 3.5 can be deduced from Lemma 3.4 and the definitions of ξ and α (one can absorb the integral with coupling terms appearing as the integrand into $\mathcal{I}(s, \lambda; \tilde{\psi})$ on the lefthand side).

We will also use the following estimate in the ensuing treatment (cf. (A.18) and (A.19)).

LEMMA 3.6. [3, Lemma 3] Let $r \in \mathbb{R}$. There exists a $C := C(\Omega, \omega_{p+2}, r) > 0$ such that for every $T > 0$ and every $u \in L^2((0, T); H^1(\Omega))$,

$$\begin{aligned} s^{r+2} \lambda^{r+3} \iint_{Q_T} e^{-2s\alpha} \xi^{r+2} |u|^2 dxdt &\leq C \left(s^r \lambda^{r+1} \iint_{Q_T} e^{-2s\alpha} \xi^r \|\nabla u\|_1^2 dxdt \right. \\ &\quad \left. + s^{r+2} \lambda^{r+3} \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^{r+2} |u|^2 dxdt \right) \end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Finally, one can establish the following Carleman estimate for system (3.2), which is an extension of [5, Proposition 1].

PROPOSITION 3.7. There exists a constant $C := C(\Omega, \omega_0) > 0$ such that for every $\tilde{\psi}^0 \in L^2(\Omega)^m$, the solution $\tilde{\psi}$ to system (3.2) satisfies

$$(3.12) \quad \begin{aligned} \iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \|\nabla^{p+4-k} \tilde{\psi}\|_{p+5-k}^2 dxdt \\ \leq C s^{2p+7} \lambda^{2p+8} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{2p+7} \|\tilde{\psi}\|_1^2 dxdt \end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

REMARK 3.8. It is worth pointing out to the fact that (3.12) contains spatial derivatives past order one, since $\tilde{\psi}^0$ is assumed to be in $L^2(\Omega)^m$, and hence $\tilde{\psi} \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); L^2(\Omega))^m$. However, due to inequalities (A.15) and (A.16) and since the weight $e^{-2s\alpha}$ absorbs the singularity of ξ at $t = 0$, one can deduce that these integrals exist.

4. Proof of main theorem. Recall from Section 2 that our principal goal was to prove null controllability of system (1.1) with multiple underactuators. To this end, we studied an algebraic system and an analytic system both related to system (1.1). In [11], we developed an algebraic method which allowed us to solve the algebraic control problem under the assumption that the source term $\mathbb{1}_\omega \tilde{u}$ be regular enough so that our algebraic solution $\mathcal{B}(\mathbb{1}_\omega \tilde{u})$ be well-defined, where \mathcal{B} is a differential operator of order zero in time and at most $p + 2$ in space. In Section 3, we established the weighted observability inequality (3.3) for the analytic system (3.1).

The goal of this section is to solve the analytic control problem (1.5) with *regular enough* controls $\mathbb{1}_\omega \tilde{u}$ so that the algebraic control problem (1.6) also be solved. The treatment presented in this section is an extension of that used in [5, Section 2.3]. In particular, since the right inverse \mathcal{B} of \mathcal{L} derived implicitly in [11] is in general of order at most $p + 2$ in space, we require higher regularity of controls in the analytic problem than in [5].

4.1. An optimal control result. We do not use the weighted observability inequality to directly deduce null controllability of system (3.1). Instead, we use a method developed in [8] to construct controls with high regularity which will help us deduce controllability results; to do this, we rely on the following unconstrained optimal control result.

THEOREM 4.1. [9, Section 3, Theorem 2.2] *Let $y^0 \in L^2(\Omega)^m$, $u \in L^2(Q_T)^m$, $B \in \mathcal{L}(L^2(Q_T)^m; L^2(Q_T)^m)$, and suppose \mathcal{L} given in (1.2) satisfies the uniform ellipticity condition (1.3). Let $N \in \mathcal{L}(L^2(Q_T)^m; L^2(Q_T)^m)$ such that $(Nu, u)_{L^2(Q_T)^m} \geq \nu \|u\|_{L^2(Q_T)^m}^2$ for $\nu > 0$ and for all $u \in L^2(Q_T)^m$, and let $D \in \mathcal{L}(H_0^1(\Omega))^m; H_0^1(\Omega))^m$. Consider the optimal control problem associated to system (1.1) with cost functional $J(u) : L^2(Q_T)^m \rightarrow \mathbb{R}^+$ given by*

$$(4.1) \quad J(u) := (Nu, u)_{L^2(Q_T)^m} + (Dy_u(T, \cdot) - z_d)_{L^2(\Omega)^m}^2,$$

for some $z_d \in H_0^1(\Omega)^m$. This problem has a unique solution, and the optimal control is characterized by the following relations:

$$\begin{cases} \partial_t y_u = \operatorname{div}(D\nabla y_u) + G \cdot \nabla y_u + Ay_y + Bu, & \text{in } Q_T, \\ y_u = 0, & \text{on } \Sigma_T, \\ y_u(0, \cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\partial_t \psi_u = \operatorname{div}(D\nabla \psi_u) - G^* \cdot \nabla \psi_u + A^* \psi_u, & \text{in } Q_T, \\ \psi_u = 0, & \text{on } \Sigma_T, \\ \psi_u(T, \cdot) = D^* (Dy_u(T, \cdot) - z_d), & \text{in } \Omega, \end{cases}$$

and

$$B^* \psi_u + Nu = 0.$$

Hence, for this unconstrained optimal control problem, the second term in (4.1) has no dependence on u (nor do the primal/adjoint systems).

4.2. Null controllability of the analytic problem. Recall that in [11], we fixed a p large enough such that we recovered algebraic solvability of (1.6). In this section, we establish the following proposition.

PROPOSITION 4.2. Consider $\theta \in C^{p+2}(\bar{\Omega})$ such that

$$(4.2) \quad \begin{cases} \text{Supp}(\theta) \subseteq \omega, \\ \theta = 1, & \text{in } \omega_0, \\ 0 \leq \theta \leq 1, & \text{in } \Omega. \end{cases}$$

Then there exists $v \in L^2(Q_T)^m$ such that

$$(\tilde{y}, \theta v) \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m \times L^2(Q_T)^m$$

is a solution to the analytic control problem (1.5) satisfying $\tilde{y}(T, \cdot) = 0$ in Ω . Moreover, for every $K \in (0, 1)$, we have

$$(4.3) \quad \begin{aligned} e^{Ks_1\alpha^*} v &\in L^2((0, T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0, T); L^2(\Omega))^m, \text{ and} \\ \|e^{Ks_1\alpha^*} v\|_{L^2((0, T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0, T); L^2(\Omega))^m} &\leq C \|\tilde{y}^0\|_{L^2(\Omega)^m}. \end{aligned}$$

Proof. Let $\tilde{y}^0 \in L^2(\Omega)^m$, $\rho := e^{-2s_1\alpha}\xi^{2p+7}$ and $C := C(\Omega, \omega_0, T) > 0$. Let $k \in \mathbb{N}^*$ and denote by $L^2(Q_T, \rho^{-1/2})^m$ the space of functions which, when multiplied by $\rho^{-1/2}$, are L^2 -integrable (i.e., for $u \in L^2(Q_T, \rho^{-1/2})^m$, we require $\iint_{Q_T} \rho^{-1} \|u\|_1^2 dxdt < \infty$). Consider the following optimal control problem:

$$(4.4) \quad \begin{cases} \text{minimize} & J_k(v) := \frac{1}{2} \iint_{Q_T} \rho^{-1} \|v\|_1^2 dxdt + \frac{k}{2} \int_{\Omega} \|\tilde{y}(T, \cdot)\|_1^2 dx, \\ \text{subject to} & v \in L^2(Q_T, \rho^{-1/2})^m, \end{cases}$$

where $\tilde{y} \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m$. The functional J_k is differentiable, coercive and strictly convex on $L^2(Q_T, \rho^{-1/2})^m$. By Theorem 4.1 (for $D = \sqrt{k}$, $N = \rho^{-1}$ and $z_d = 0$ in Q_T), there exists a unique solution to this problem, and the optimal control is characterized by the solution \tilde{y}_k to the analytic system

$$(4.5) \quad \begin{cases} \partial_t \tilde{y}_k = \text{div}(D\nabla \tilde{y}_k) + G \cdot \nabla \tilde{y}_k + A\tilde{y}_k + \theta v_k, & \text{in } Q_T, \\ \tilde{y}_k = 0, & \text{on } \Sigma_T, \\ \tilde{y}_k(0, \cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega, \end{cases}$$

the solution $\tilde{\psi}_k$ to its adjoint system

$$(4.6) \quad \begin{cases} -\partial_t \tilde{\psi}_k = \text{div}(D\nabla \tilde{\psi}_k) - G^* \cdot \nabla \tilde{\psi}_k + A^* \tilde{\psi}_k, & \text{in } Q_T, \\ \tilde{\psi}_k = 0, & \text{on } \Sigma_T, \\ \tilde{\psi}_k(T, \cdot) = k\tilde{y}(T, \cdot), & \text{in } \Omega, \end{cases}$$

and the relation

$$(4.7) \quad \begin{cases} v_k = -\rho\theta\tilde{\psi}_k, & \text{in } Q_T, \\ v_k \in L^2(Q_T, \rho^{-1/2})^m. \end{cases}$$

From (4.5) and (4.6), we calculate

$$\begin{aligned}
& \int_0^T \left((\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \right) dt \\
&= \frac{d}{dt} \int_0^T (\tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} dt \\
(4.8) \quad &= (\tilde{y}_k(T, \cdot), k\tilde{y}_k(T, \cdot))_{L^2(\Omega)^m} - (\tilde{y}^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m},
\end{aligned}$$

and

$$\begin{aligned}
& (\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \\
&= (\tilde{y}_k, -\operatorname{div}(D\nabla \tilde{\psi}_k) + G^* \cdot \nabla \tilde{\psi}_k - A^* \tilde{\psi}_k)_{L^2(\Omega)^m} \\
&+ (\operatorname{div}(D\nabla \tilde{y}_k) + G \cdot \nabla \tilde{y}_k + A\tilde{y}_k + \theta v_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \\
(4.9) \quad &= (\theta v_k, \tilde{\psi}_k)_{L^2(\Omega)^m}.
\end{aligned}$$

It follows from (4.7), (4.8) and (4.9) that

$$\begin{aligned}
J_k(v_k) &= -\frac{1}{2} \int_0^T (\theta \tilde{\psi}_k, v_k)_{L^2(\Omega)^m} dt + \frac{1}{2} (\tilde{y}_k(T, \cdot), \tilde{\psi}_k(T, \cdot))_{L^2(\Omega)^m} \\
&= -\frac{1}{2} \int_0^T (\tilde{\psi}_k, \theta v_k)_{L^2(\Omega)^m} dt + \frac{1}{2} \int_0^T \left((\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \right) dt \\
(4.10) \quad &+ \frac{1}{2} (y^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m} = \frac{1}{2} (y^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m}.
\end{aligned}$$

Moreover, employing the weighted observability inequality (3.3) along with (4.2), (4.7), (4.4), (4.10) and the Cauchy-Schwarz inequality successively, we have

$$\begin{aligned}
\|\tilde{\psi}_k(0, \cdot)\|_{L^2(\Omega)^m}^2 &\leq C_{obs} \iint_{(0,T) \times \omega_0} \rho \theta^2 \|\tilde{\psi}_k\|_1^2 dx dt \\
&\leq C_{obs} \iint_{Q_T} \rho \theta^2 \|\tilde{\psi}_k\|_1^2 dx dt \\
&= C_{obs} \iint_{Q_T} \rho^{-1} \|v_k\|_1^2 dx dt \\
&\leq 2C_{obs} J_k(v_k) \\
&\leq 2C_{obs} \|\tilde{\psi}_k(0, \cdot)\|_{L^2(\Omega)^m} \|y^0\|_{L^2(\Omega)^m},
\end{aligned}$$

from which we deduce

$$(4.11) \quad \|\tilde{\psi}_k(0, \cdot)\|_{L^2(\Omega)^m} \leq 2C_{obs} \|y^0\|_{L^2(\Omega)^m}.$$

Furthermore, by (4.10), (4.11) and the Cauchy-Schwarz inequality, we obtain

$$(4.12) \quad J_k(v_k) \leq C_{obs} \|y^0\|_{L^2(\Omega)^m}^2.$$

One can deduce from parabolic regularity, (4.2) and (4.12) that

$$\begin{aligned}
\|\tilde{y}_k\|_{L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m} &\leq C \left(\|\theta v_k\|_{L^2(Q_T)^m} + \|\tilde{y}^0\|_{L^2(\Omega)^m} \right) \\
&\leq C \left(\|v_k\|_{L^2(Q_T)^m} + \|\tilde{y}^0\|_{L^2(\Omega)^m} \right) \\
(4.13) \qquad \qquad \qquad &\leq C(1 + \sqrt{2C_{obs}})\|\tilde{y}^0\|_{L^2(\Omega)^m},
\end{aligned}$$

since for our choice of s_1 (which depends on p ; see (A.33)) and by (3.5) and (3.6), $\rho \leq 1$ in Q_T . Owing to the well-known result that in Hilbert spaces, bounded sequences have weakly convergent subsequences (see, for example, [1]), along with (4.4) (4.12), and (4.13), one can extract subsequences of $(v_k)_k$ and $(\tilde{y}_k)_k$ (which we still denote by v_k and \tilde{y}_k) such that

$$\begin{cases} v_k \rightharpoonup v & \text{in } L^2(Q_T, \rho^{-1/2})^m, \\ \tilde{y}_k \rightharpoonup \tilde{y} & \text{in } L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m, \\ \tilde{y}_k(T, \cdot) \rightarrow 0 & \text{in } L^2(\Omega)^m. \end{cases}$$

Hence, $(\tilde{y}, \theta v)$ is the solution to the analytic control problem (1.5) with $\theta v \in L^2(Q_T, \rho^{-1/2})$. Furthermore, we deduce from (4.4) by taking $k \rightarrow \infty$ that $\tilde{y}(T, \cdot) = 0$ (in the sense of Definition 1.1). In addition, by (4.12) and since $\rho \leq 1$ in Q_T for our choice of s_1 ,

$$\|v\|_{L^2(Q_T)}^2 \leq \sqrt{2C_{obs}}\|y^0\|_{L^2(\Omega)^m}^2,$$

as claimed. It is left to show that (4.3) is verified. Note that for every $K \in (0, 1)$, there exists a $C_K := C_K(\Omega)$ such that

$$(4.14) \qquad e^{2Ks_1\alpha^*} \leq C_K \xi^{-2p-7} e^{2s_1\alpha},$$

for all $(t, x) \in Q_T$. Hence, utilizing (4.14), (4.4) and then (4.12), we obtain

$$\begin{aligned}
\|e^{2Ks_1\alpha^*} v_k\|_{L^2(Q_T)^m}^2 &\leq C_K \iint_{Q_T} \rho^{-1} \|v_k\|_1^2 dx dt \\
(4.15) \qquad \qquad \qquad &\leq C_K \|\tilde{y}^0\|_{L^2(\Omega)^m}^2.
\end{aligned}$$

For $a > 0$, one has (see (A.9))

$$(4.16) \qquad |\partial_t(\xi^a e^{-2s_1\alpha})| \leq CT \xi^{a+6/5} e^{-2s_1\alpha}.$$

Furthermore, for $r = \{0, \dots, p+2\}$ one has

$$(4.17) \qquad \|\nabla^r(\xi^a e^{-2s_1\alpha})\|_r \leq C \xi^{a+r} e^{-2s_1\alpha}.$$

Indeed,

$$\begin{aligned}
\nabla(\xi^a e^{-2s_1\alpha}) &= a\xi^{a-1} \lambda \nabla \eta^0 \xi e^{-2s_1\alpha} - 2s_1 \xi^a e^{-2s_1\alpha} (-\lambda \nabla \eta^0 \xi) \\
&= \lambda \nabla \eta^0 \left(\frac{a}{\xi} + 2s_1 \right) \xi^{a+1} e^{-2s_1\alpha},
\end{aligned}$$

and since $C := C(\Omega, \omega_0, T)$, (4.17) is verified for $r = 1$. The same reasoning can be used for the r -th derivative, where we have fixed $\eta^0 \in C^{p+2}(\bar{\Omega})$. Hence, by (4.7), the

triangle inequality and then (4.17) for $a = 2p + 7$, we obtain

$$\begin{aligned}
& \|e^{Ks_1\alpha^*} \nabla v_k\|_{L^2(Q_T)^m}^2 \\
&= \iint_{Q_T} e^{2Ks_1\alpha^*} \|\nabla v_k\|_2^2 dxdt \\
&= \iint_{Q_T} e^{2Ks_1\alpha^*} \|\nabla(-\xi^{2p+7} e^{-2s_1\alpha} \tilde{\psi}_k)\|_2^2 dxdt \\
&\leq C \iint_{Q_T} e^{2Ks_1\alpha^*} \left(\|\nabla(\xi^{2p+7} e^{-2s_1\alpha})\|_1^2 \|\tilde{\psi}_k\|_1^2 + \|\xi^{2p+7} e^{-2s_1\alpha} \nabla \tilde{\psi}_k\|_2^2 \right) dxdt \\
(4.18) \quad &\leq C \iint_{Q_T} e^{2Ks_1\alpha^* - 4s_1\alpha} \left(\xi^{4p+16} \|\tilde{\psi}_k\|_1^2 + \xi^{4p+14} \|\nabla \tilde{\psi}_k\|_2^2 \right) dxdt,
\end{aligned}$$

and similarly, for $r \in \{1, \dots, p+2\}$, we obtain

$$(4.19) \quad \|e^{Ks_1\alpha^*} \nabla^r v_k\|_{L^2(Q_T)^m}^2 \leq C \iint_{Q_T} e^{2Ks_1\alpha^* - 4s_1\alpha} \left(\sum_{l=0}^r \xi^{4p+14+2l} \|\nabla^{r-l} \tilde{\psi}_k\|_{r-l+1}^2 \right) dxdt.$$

By (4.16) and since $\tilde{\psi}_k$ satisfies system (4.6), we obtain

$$\begin{aligned}
(4.20) \quad & \|\partial_t(e^{Ks_1\alpha^*} v_k)\|_{L^2(Q_T)^m}^2 \\
&\leq C \iint_{Q_T} e^{2Ks_1\alpha^* - 4s_1\alpha} \left(\xi^{(20p+82)/5} \|\tilde{\psi}_k\|_1^2 + \xi^{2p+14} \|\partial_t \tilde{\psi}_k\|_1^2 \right) dxdt \\
&\leq C \iint_{Q_T} e^{2Ks_1\alpha^* - 4s_1\alpha} \left(\xi^{(20p+82)/5} \|\tilde{\psi}_k\|_1^2 \right. \\
(4.21) \quad & \left. + \xi^{2p+14} \left(\|\nabla \tilde{\psi}_k\|_3^2 + \|\nabla \tilde{\psi}_k\|_2^2 + \|\tilde{\psi}_k\|_1^2 \right) \right) dxdt.
\end{aligned}$$

Note that for every $a, b > 0$ and $K \in (0, 1)$, there exists $C_{a,b,K} := C_{a,b,K}(\Omega) > 0$ such that

$$(4.22) \quad \left| \xi^a e^{2Ks_1\alpha^* - 4s_1\alpha} \right| \leq C_{a,b,K} \xi^b e^{2s_1\alpha}.$$

From (4.15), (4.18), (4.19), (4.20) and utilizing (4.22) for appropriate a and b ,

$$\begin{aligned}
& \|e^{Ks_1\alpha^*} v_k\|_{L^2((0,T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0,T); L^2(\Omega))^m} \\
&\leq C_{max,K} \iint_{Q_T} e^{-2s_1\alpha} \sum_{k=2}^{p+4} \xi^{2k-1} \|\nabla^{p+4-k} \tilde{\psi}_k\|_{p+5-k}^2 dxdt,
\end{aligned}$$

where $C_{max,K} := \max\{\max_{a,b}\{C_{a,b,K}\}, C_K\}$. Owing to (4.2), Proposition 3.7 and (4.7), we deduce

$$\begin{aligned}
& \|e^{Ks_1\alpha^*} v_k\|_{L^2((0,T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0,T); L^2(\Omega))^m} \\
&\leq C_{max,K} C_{obs} \iint_{Q_T} e^{-2s_1\alpha} \xi^{2p+7} \|\theta \tilde{\psi}_k\|_1^2 dxdt = C_{max,K} C_{obs} \|v_k\|_{L^2(Q_T)^m}^2.
\end{aligned}$$

Lastly, for $\bar{C}_K := \bar{C}_K(\Omega, \omega_0, T)$, (4.12) yields the inequality

$$\|e^{Ks_1\alpha^*} v_k\|_{L^2((0,T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0,T); L^2(\Omega))^m} \leq \bar{C}_K \|\tilde{y}^0\|_{L^2(\Omega)^m},$$

from which (4.3) is verified by taking a convergent subsequence and $k \rightarrow \infty$. \square

With algebraic solvability of the algebraic control problem (1.6) and null controllability of the analytic control problem (1.5) established for highly regular controls, we can now prove null controllability of the system (1.1) with internal controls $\hat{u} \in L^2(Q_T)^c$, where $c < m - 1$.

In Proposition 4.2, we showed the existence of $(\tilde{y}, \theta v) \in L^2((0, T); H_0^1(\Omega))^m \cap H^1((0, T); H^{-1}(\Omega))^m \times L^2(Q_T)^m$ satisfying

$$(4.23) \quad \begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla \tilde{y}) + G \cdot \nabla \tilde{y} + A\tilde{y} + \theta v, & \text{in } Q_T, \\ \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0, \cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases}$$

such that $\tilde{y}(T, \cdot) = 0$ in Ω . Furthermore, we established the following higher regularity for v :

$$(4.24) \quad e^{Ks_1\alpha^*} v \in L^2((0, T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0, T); L^2(\Omega))^m,$$

for all $k \in (0, 1)$. Notice that (4.24) implies that v is exponentially decaying as $t \rightarrow 0$ and $t \rightarrow T$. For the linear partial differential operator \mathcal{B} (of order zero in time and at most $p + 2$ in space) constructed implicitly in [11], let us define

$$\begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix} := \mathcal{B}(\theta v),$$

which is well-defined by (4.24). By virtue of \mathcal{B} being a linear partial differential operator of the stated orders with constant coefficients, we conclude that

$$(4.25) \quad (\hat{y}, \hat{u}) \in L^2(Q_T) \times L^2(Q_T)^c;$$

we then extend (\hat{y}, \hat{u}) by zero to Q_T . Since v decays exponentially as $t \rightarrow 0$ and $t \rightarrow T$, $\hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0$ in Ω . Furthermore, it follows from the discussions in Subsection 1.5 that (\hat{y}, \hat{u}) is the solution to

$$(4.26) \quad \begin{cases} \partial_t \hat{y} = \operatorname{div}(D\nabla \hat{y}) + G \cdot \nabla \hat{y} + A\hat{y} + B\hat{u} + \theta v, & \text{in } Q_T, \\ \hat{y} = 0, & \text{on } \Sigma_T, \\ \hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0, & \text{in } \Omega, \end{cases}$$

where, by (4.25) and by parabolic regularity, (\hat{y}, \hat{u}) satisfies Definition 1.1. Defining $(y, u) := (\tilde{y} - \hat{y}, -\hat{u})$, it is immediate that (y, u) is the solution to (1.1) with $y(T, \cdot) = 0$ in Ω . This finishes the proof of Theorem 2.1.

5. Conclusion. Using the powerful fictitious control technique, which has allowed us to pose our controllability problem as two interconnected problems, we have derived a sufficient condition for the null controllability of a system of coupled parabolic PDEs, where the couplings were constant in space and time and of first and zero-order and more than half of the equations in the system were actuated. This controllability condition is generic.

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Appendix. In a proof to follow, we rely on the so-called Gagliardo-Nirenberg interpolation inequality, which is stated next.

THEOREM A.1. [10] *For $\Omega \subset \mathbb{R}^n$ open, for $q, r \in \mathbb{R}$ such that $1 \leq q, r \leq \infty$ and for $m \in \mathbb{N}$, let $u : \Omega \rightarrow \mathbb{R}$ such that $u \in L^q(\Omega) \cap W^{m,r}(\Omega)$. For $0 \leq j \leq m$, we have*

$$(A.1) \quad \|u\|_{W^{j,p}(\Omega)} \leq C \|u\|_{W^{m,r}(\Omega)}^\alpha \|u\|_{L^q(\Omega)}^{1-\alpha},$$

where p satisfies

$$\frac{1}{p} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n} \right) + \frac{1-\alpha}{q}$$

for all α in the interval $\frac{j}{m} \leq \alpha \leq 1$, where $C := C(n, m, j, q, r, \alpha)$, with the following exceptional assumptions:

- (i) if $j = 0$, $rm < n$, $q = \infty$, then we require $u \rightarrow 0$ at infinity, and;
- (ii) if $1 < r < \infty$ and $m - j - \frac{n}{r}$ a nonnegative integer, then (A.1) only holds for α satisfying $\frac{j}{m} \leq \alpha < 1$.

Proof. (Proof of Proposition 3.7): We denote by C various positive constants which depend on Ω and ω_0 . We define the operator

$$(A.2) \quad \mathcal{L}^* := (-\operatorname{div}(D\nabla) + G^* \cdot \nabla - A^*).$$

By density of $H^k(\Omega)^m \cap H_0^1(\Omega)^m$ in $L^2(\Omega)^m$ for $k \in \mathbb{N}$ (this follows from the inclusion $C_c^\infty(\Omega)^m \subset H^k(\Omega)^m \cap H_0^1(\Omega)^m \subset L^2(\Omega)^m$ and since $C_c^\infty(\Omega)^m$ dense in $L^2(\Omega)^m$), we assume without loss of generality that $\tilde{\psi}^0 \in H^{2p+5}(\Omega)^m$ and $\left((\mathcal{L}^*)^k \tilde{\psi}^0 \right)_{k=0}^{p+2} \subset H_0^1(\Omega)$.

Hence by Theorem 1.2, the solution $\tilde{\psi}$ to system (3.2) is an element of

$$(A.3) \quad L^2((0, T); H^{2p+6}(\Omega))^m \cap H^{p+3}((0, T); L^2(\Omega))^m.$$

We apply the differential operator ∇^{p+2} to system (3.2) and, for β a multi-index with $|\beta| = p + 2$, we denote $\partial_\beta \psi$ by ϕ_β so that ϕ_β satisfies

$$(A.4) \quad \begin{cases} -\partial_t \phi_\beta = \operatorname{div}(D\nabla \phi_\beta) - G^* \cdot \nabla \phi_\beta + A^* \phi_\beta, & \text{in } Q_T, \\ \frac{\partial \phi_\beta}{\partial n} = \nabla \phi_\beta \cdot \mathbf{n}, & \text{on } \Sigma_T, \\ \phi_\beta(T, \cdot) = \partial_\beta \tilde{\psi}^0(\cdot), & \text{in } \Omega. \end{cases}$$

Indeed, since D , G^* and A^* are constant, ∇^{p+2} commutes with all the terms in system (3.2). We define the $(p+3)$ -th order tensor $\phi := (\phi_\beta)_{1 \leq \beta_1, \dots, \beta_{p+2} \leq n}$; applying Lemma 3.5 to system (A.4), we have a Carleman inequality for ϕ :

$$(A.5) \quad \begin{aligned} & \mathcal{I}(s, \lambda; \phi) \\ & \leq C \left(s^3 \lambda^4 \iint_{(0, T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 \|\phi\|_{p+3}^2 dx dt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla \phi \cdot n\|_{p+3}^2 d\sigma dt \right) \end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. The rest of this proof follows three steps:

- (i) We will estimate the boundary term on the righthand side of (A.5) with a global interior term involving $\tilde{\psi}$, which will be absorbed into the lefthand side;
- (ii) we will relate $\mathcal{I}(s, \lambda; \phi)$ with the lefthand side of (3.12);
- (iii) we will estimate the local term on the righthand side of (A.5) with a local term of zero differential order (as appearing in (3.12)) and some other local terms which will be absorbed into the lefthand side.

Step (i): Consider a function $\theta \in C^2(\bar{\Omega})$ such that $\nabla \theta \cdot \mathbf{n} = \theta = 1$ in $\bar{\Omega}$, where \mathbf{n} is the outward pointing normal of $\partial\Omega$. With this construction, $\nabla \theta = \mathbf{n}$. Indeed, for any $q \in \partial\Omega$ and for any parametrized curve $\gamma : \mathbb{R} \rightarrow \Omega$ passing through point q at time 0, we have

$$\frac{d}{dt} \theta(\gamma(t)) \Big|_{t=0} = \nabla \theta \Big|_q \frac{d\gamma(t)}{dt} \Big|_{t=0} = 0,$$

since $\theta = 1$ in $\bar{\Omega}$. Hence, since $\nabla \theta$ is orthogonal to the tangent of any curve passing through any arbitrary point $q \in \partial\Omega$ at $t = 0$, it must be equal to \mathbf{n} . Let β and γ be

multi-indices of length n ; we integrate the boundary term by parts to obtain

$$\begin{aligned}
& s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla\phi \cdot \mathbf{n}\|_{p+3}^2 d\sigma dt \\
&= s\lambda \sum_{|\beta|=p+3} \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* (\partial_\beta\psi \cdot \nabla\theta) (\partial_\beta\psi \cdot \mathbf{n}) d\sigma dt \\
&= \sum_{\substack{|\beta|=p+3 \\ |\gamma|=p+4}} \left(s\lambda \iint_{Q_T} e^{-2s\alpha^*} \xi^* (\partial_\gamma\psi) (\partial_\beta\psi \cdot \nabla\theta) dxdt \right. \\
&\quad \left. + s\lambda \iint_{Q_T} e^{-2s\alpha^*} \xi^* \nabla(\partial_\beta\psi \cdot \nabla\theta) \cdot \partial_\beta\psi dxdt \right).
\end{aligned}$$

Next, we employ Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned}
& s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla\phi \cdot \mathbf{n}\|_{p+3}^2 d\sigma dt \\
\text{(A.6)} \quad & \leq C\lambda \left(\int_0^T e^{-2s\alpha^*} (s\xi^*)^{2k} \|\tilde{\psi}\|_{H^{p+4}(\Omega)^m}^2 dt + \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} \|\tilde{\psi}\|_{H^{p+3}(\Omega)^m}^2 dt \right),
\end{aligned}$$

for $k \in (0, 1)$ to be chosen later. We define $\hat{\psi} := \rho\tilde{\psi}$, with $\rho \in C^\infty([0, T])$ defined by $\rho := (s\xi^*)^a e^{-s\alpha^*}$ for some $a \in \mathbb{R}$ to be chosen later. Note that $\hat{\psi}(T, \cdot) = 0$ in Ω , since ρ decays exponentially to zero as $t \rightarrow T$. Similarly, $\frac{d^i}{dt^i} \rho(0) = 0$, for all $i \in \mathbb{N}$. Furthermore, $\hat{\psi}$ is the solution to

$$\text{(A.7)} \quad \begin{cases} -\partial_t \hat{\psi} = \operatorname{div}(D\nabla \hat{\psi}) - G^* \cdot \nabla \hat{\psi} + A^* \hat{\psi} - \frac{d}{dt} \rho \tilde{\psi}, & \text{in } Q_T, \\ \hat{\psi} = 0, & \text{on } \Sigma_T, \\ \hat{\psi}(T, \cdot) = 0, & \text{in } \Omega. \end{cases}$$

Hence, by (A.3), one can utilize Theorem 1.2 to get the estimate

$$\begin{aligned}
& \|\hat{\psi}\|_{L^2((0, T); H^{2d+2}(\Omega))^m \cap H^{d+1}((0, T); L^2(\Omega))^m} \\
\text{(A.8)} \quad & \leq C \left\| \frac{d}{dt} \rho \tilde{\psi} \right\|_{L^2((0, T); H^{2d}(\Omega))^m \cap H^d((0, T); L^2(\Omega))^m}
\end{aligned}$$

for $d \in \{0, \dots, p+2\}$. Owing to (3.5) and (3.6), we have the bound

$$\text{(A.9)} \quad \left| \frac{d}{dt} \rho \right| \leq CT (s\xi^*)^{a+6/5} e^{-s\alpha^*}.$$

Indeed, for $\bar{c} := \min_{x \in \bar{\Omega}} \{e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}\}$ and $\tilde{c} := \max_{x \in \bar{\Omega}} \{e^{12\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}\}$, we have

$$\begin{aligned}
\left| \frac{d}{dt} \rho \right| &= \left| a s (s\xi^*)^{a-1} e^{-s\alpha^*} \frac{d}{dt} \xi^* - s (s\xi^*)^a e^{-s\alpha^*} \frac{d}{dt} \alpha^* \right| \\
&= e^{-s\alpha^*} \left| s (s\xi^*)^{a-1} \frac{5(2t-T)}{t^6(T-t)^6} (a\bar{c} - (s\xi^*)\tilde{c}) \right| \\
&= (s\xi^*)^a e^{-s\alpha^*} \left| \frac{10t-5T}{t(T-t)} \left(a - \frac{(s\xi^*)\tilde{c}}{\bar{c}} \right) \right| \\
&= (s\xi^*)^{a+6/5} e^{-s\alpha^*} \left| \frac{(10t-5T)}{\bar{c}^{6/5}} \left(\frac{at^5(T-t)^5}{s^{6/5}} - \frac{\tilde{c}}{s^{1/5}} \right) \right|,
\end{aligned}$$

and since $s \geq C(T^5 + T^{10})$, one can obtain (A.9). Similarly, we have

$$(A.10) \quad \left| \frac{d^r}{dt^r} \rho \right| \leq CT^r (s\xi^*)^{a+6r/5} e^{-s\alpha^*},$$

for $r \in \mathbb{N}$. We apply (A.8) to $\hat{\psi}$ for $a = 1 - k$ and $d = \lfloor \frac{p+1}{2} \rfloor$ to obtain

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} \|\tilde{\psi}\|_{H^2 \lfloor \frac{p+3}{2} \rfloor (\Omega)^m}^2 dt \\
&\leq C \left(\int_0^T \left\| \frac{d}{dt} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right\|_{H^2 \lfloor \frac{p+1}{2} \rfloor (\Omega)^m}^2 dt \right. \\
(A.11) \quad &\left. + \sum_{r=1}^{\lfloor \frac{p+1}{2} \rfloor} \int_0^T \left\| \frac{d^r}{dt^r} \left(\frac{d}{dt} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right) \right\|_{L^2(\Omega)^m}^2 dt \right).
\end{aligned}$$

We now apply (A.8) to $\hat{\psi} = \frac{d}{dt} \rho \tilde{\psi}$ (which satisfies a system similar to (A.7) and verifies the compatibility conditions in Theorem 1.2) for $a = 1 - k$ and $d = \lfloor \frac{p+1}{2} \rfloor - 1$ to obtain

$$\begin{aligned}
&\int_0^T \left\| \frac{d}{dt} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right\|_{H^2 \lfloor \frac{p+1}{2} \rfloor (\Omega)^m}^2 dt \\
&+ \sum_{r=1}^{\lfloor \frac{p+1}{2} \rfloor} \int_0^T \left\| \frac{d^r}{dt^r} \left(\frac{d}{dt} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right) \right\|_{L^2(\Omega)^m}^2 dt \\
&\leq C \int_0^T \left\| \frac{d^2}{dt^2} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right\|_{H^2 \lfloor \frac{p+1}{2} \rfloor - 2 (\Omega)^m}^2 dt \\
(A.12) \quad &+ \sum_{r=1}^{\lfloor \frac{p+1}{2} \rfloor - 1} \int_0^T \left\| \frac{d^r}{dt^r} \left(\frac{d^2}{dt^2} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right) \right\|_{L^2(\Omega)^m}^2 dt.
\end{aligned}$$

Repeating this way $\lfloor \frac{p+1}{2} \rfloor - 1$ more times and utilizing (A.10) yields the inequality

$$\begin{aligned}
& \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} \|\tilde{\psi}\|_{H^2\lfloor \frac{p+3}{2} \rfloor(\Omega)^m}^2 dt \\
& \leq C \int_0^T \left\| \frac{d\lfloor \frac{p+1}{2} \rfloor + 1}{dt\lfloor \frac{p+1}{2} \rfloor + 1} \left(e^{-s\alpha^*} (s\xi^*)^{1-k} \right) \tilde{\psi} \right\|_{L^2(\Omega)^m}^2 dt \\
\text{(A.13)} \quad & \leq CT^{2\lfloor \frac{p+1}{2} \rfloor + 2} \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k + \frac{12}{5}(\lfloor \frac{p+1}{2} \rfloor + 1)} \|\tilde{\psi}\|_{L^2(\Omega)^m}^2 dt.
\end{aligned}$$

We can get very similar estimates (A.11) and (A.12) for $a = 3k - 1$, $d = \lceil \frac{p+2}{2} \rceil$, and by using (A.10), we obtain

$$\begin{aligned}
& \int_0^T e^{-2s\alpha^*} (s\xi^*)^{6k-2} \|\tilde{\psi}\|_{H^2\lceil \frac{p+4}{2} \rceil(\Omega)^m}^2 dt \\
& \leq C \int_0^T \left\| \frac{d\lceil \frac{p+2}{2} \rceil + 1}{dt\lceil \frac{p+2}{2} \rceil + 1} \left(e^{-s\alpha^*} (s\xi^*)^{3k-1} \right) \tilde{\psi} \right\|_{L^2(\Omega)^m}^2 dt \\
\text{(A.14)} \quad & \leq CT^{2\lceil \frac{p+2}{2} \rceil + 2} \int_0^T e^{-2s\alpha^*} (s\xi^*)^{6k-2 + \frac{12}{5}(\lceil \frac{p+2}{2} \rceil + 1)} \|\tilde{\psi}\|_{L^2(\Omega)^m}^2 dt.
\end{aligned}$$

Suppose for the moment that p is odd. By applying Theorem A.1 to the appropriate spatial derivative of $\tilde{\psi}$ with $j = 1$, $m = q = p = r = 2$ and $\alpha = 1/2$, and then employing the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
& \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2k} \|\tilde{\psi}\|_{H^{p+4}(\Omega)^m}^2 dt \\
& \leq C \int_0^T \|e^{-s\alpha^*} (s\xi^*)^{3k-1} \tilde{\psi}\|_{H^2\lceil \frac{p+4}{2} \rceil(\Omega)^m} \|e^{-s\alpha^*} (s\xi^*)^{1-k} \tilde{\psi}\|_{H^2\lfloor \frac{p+3}{2} \rfloor(\Omega)^m} dt \\
& \leq C \left(\int_0^T e^{-2s\alpha^*} (s\xi^*)^{6k-2} \|\tilde{\psi}\|_{H^2\lceil \frac{p+4}{2} \rceil(\Omega)^m}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} \|\tilde{\psi}\|_{H^2\lfloor \frac{p+3}{2} \rfloor(\Omega)^m}^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Choosing $k = \frac{1}{2} + \frac{3}{10} (\lfloor \frac{p+1}{2} \rfloor - \lceil \frac{p+2}{2} \rceil)$ verifies

$$2 - 2k + \frac{12}{5} \left(\left\lfloor \frac{p+1}{2} \right\rfloor + 1 \right) = 6k - 2 + \frac{12}{5} \left(\left\lceil \frac{p+2}{2} \right\rceil + 1 \right),$$

and hence by utilizing (A.13) and (A.14), we obtain

$$\begin{aligned}
& \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2k} \|\tilde{\psi}\|_{H^{p+4}(\Omega)^m}^2 dt \\
\text{(A.15)} \quad & \leq CT^{\lceil \frac{p+2}{2} \rceil + \lfloor \frac{p+1}{2} \rfloor + 2} \int_0^T e^{-2s\alpha^*} (s\xi^*)^{\frac{17}{5} + \frac{9}{5}\lfloor \frac{p+1}{2} \rfloor + \frac{3}{5}\lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_{L^2(\Omega)^m}^2 dt.
\end{aligned}$$

Identical steps can be followed for the case when p is even to obtain

$$(A.16) \quad \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} \|\tilde{\psi}\|_{H^{p+3}(\Omega)_m}^2 dt \\ \leq CT^{\lceil \frac{p+2}{2} \rceil + \lceil \frac{p+1}{2} \rceil + 2} \int_0^T e^{-2s\alpha^*} (s\xi^*)^{\frac{17}{5} + \frac{3}{5} \lceil \frac{p+1}{2} \rceil + \frac{9}{5} \lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_{L^2(\Omega)_m}^2 dt.$$

It follows from (A.6), (A.13) and (A.15) that

$$s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla\phi \cdot n\|_{p+3}^2 d\sigma dt \\ \leq C\lambda \left(T^{2\lceil \frac{p+1}{2} \rceil + 2} + T^{\lceil \frac{p+2}{2} \rceil + \lceil \frac{p+1}{2} \rceil + 2} \right) \int_0^T e^{-2s\alpha^*} (s\xi^*)^{\frac{17}{5} + \frac{9}{5} \lceil \frac{p+1}{2} \rceil + \frac{3}{5} \lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_{L^2(\Omega)_m}^2 dt,$$

for p odd, and it follows from (A.6), (A.14) and (A.16)

$$s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla\phi \cdot n\|_{p+3}^2 d\sigma dt \\ \leq C\lambda \left(T^{2\lceil \frac{p+2}{2} \rceil + 2} + T^{\lceil \frac{p+2}{2} \rceil + \lceil \frac{p+1}{2} \rceil + 2} \right) \int_0^T e^{-2s\alpha^*} (s\xi^*)^{\frac{17}{5} + \frac{3}{5} \lceil \frac{p+1}{2} \rceil + \frac{9}{5} \lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_{L^2(\Omega)_m}^2 dt,$$

for p even. In what follows, we choose p even without loss of generality (the exact same technique can be used for p odd), and since

$$\left(T^{2\lceil \frac{p+2}{2} \rceil + 2} + T^{\lceil \frac{p+2}{2} \rceil + \lceil \frac{p+1}{2} \rceil + 2} \right) \leq Cs^{2p - \frac{3}{5} \lceil \frac{p+1}{2} \rceil - \frac{9}{5} \lceil \frac{p+2}{2} \rceil + \frac{17}{5}},$$

for $s \geq C(T^5 + T^{10})$, we use (3.7) and (3.8) to obtain

$$s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi^* \|\nabla\phi \cdot n\|_{p+3}^2 d\sigma dt \\ \leq Cs^{2p+34/5} \lambda \int_0^T e^{-2s\alpha^*} (\xi^*)^{\frac{17}{5} + \frac{9}{5} \lceil \frac{p+1}{2} \rceil + \frac{3}{5} \lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_{L^2(\Omega)_m}^2 dt \\ \leq Cs^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{\frac{17}{5} + \frac{9}{5} \lceil \frac{p+1}{2} \rceil + \frac{3}{5} \lceil \frac{p+2}{2} \rceil} \|\tilde{\psi}\|_1^2 dx dt.$$

Denoting by $l(p)$ the exponent $\frac{17}{5} + \frac{9}{5} \lceil \frac{p+1}{2} \rceil + \frac{3}{5} \lceil \frac{p+2}{2} \rceil$, we arrive at the end of Step (i) to conclude that

$$(A.17) \quad \mathcal{I}(s, \lambda; \phi) \\ \leq C \left(s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 \|\phi\|_{p+3}^2 dx dt + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \|\tilde{\psi}\|_1^2 dx dt \right)$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Step (ii): In this step, we relate $\mathcal{I}(s, \lambda; \phi)$ to the lefthand side of (3.12). We apply

Lemma 3.6 to $\tilde{\psi}$ for $r = 2p + 5$ to obtain

$$\begin{aligned}
& s^{2p+7} \lambda^{2p+8} \iint_{Q_T} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt \\
& \leq C \left(s^{2p+5} \lambda^{2p+6} \iint_{Q_T} e^{-2s\alpha} \xi^{2p+5} \left\| \nabla \tilde{\psi} \right\|_2^2 dxdt \right. \\
\text{(A.18)} \quad & \left. + s^{2p+7} \lambda^{2p+8} \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Similarly, for $k \in \{0, \dots, p\}$, we apply Lemma 3.6 to $\nabla^{p+1-k} \tilde{\psi}$ for $r = 2k + 3$ to obtain

$$\begin{aligned}
& s^{2k+5} \lambda^{2k+6} \iint_{Q_T} e^{-2s\alpha} \xi^{2k+5} \left\| \nabla^{p+1-k} \tilde{\psi} \right\|_{p+2-k}^2 dxdt \\
& \leq C \left(s^{2k+3} \lambda^{2k+4} \iint_{Q_T} e^{-2s\alpha} \xi^{2k+3} \left\| \nabla^{p+2-k} \tilde{\psi} \right\|_{p+3-k}^2 dxdt \right. \\
\text{(A.19)} \quad & \left. + s^{2k+5} \lambda^{2k+6} \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^{2k+5} \left\| \nabla^{p+1-k} \tilde{\psi} \right\|_{p+2-k}^2 dxdt \right),
\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. One can upper bound the first term in the righthand side of (A.18) by (A.19) for $k = p$ and continue this way by backwards iteration on k . The global terms on the righthand side of (A.19) can be absorbed in the exact same way. Hence, a combination of (A.17), (A.18) and (A.19) gives

$$\begin{aligned}
& \iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \\
& \leq C \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \left. + s^3 \lambda^4 \iint_{Q_T} e^{-2s\alpha} \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. By utilizing (A.17) once more, we arrive at the inequality

$$\begin{aligned}
& \iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \\
& \leq C \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
\text{(A.20)} \quad & \left. + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

which is verified for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Step (iii): In this final step, we absorb the higher-order local terms in the righthand

side of (A.20). Consider the function $\theta_{p+1} \in C^2(\bar{\Omega})$ satisfying

$$(A.21) \quad \begin{cases} \text{Supp}(\theta_{p+1}) \subseteq \omega_{p+1}, \\ \theta_{p+1} = 1, & \text{in } \omega_{p+2}, \\ 0 \leq \theta_{p+1} \leq 1 & \text{in } \Omega. \end{cases}$$

Let β be a multi-index of length n . Since $\bar{\omega}_{p+2} \subset \omega_{p+1}$, where ω_{p+1} is an open subset of Ω , we integrate the rightmost term in (A.20) by parts and employ the Cauchy-Schwarz inequality to obtain

$$(A.22) \quad \begin{aligned} & s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha \xi^3} \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt \\ & \leq s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+1}} \theta_{p+1} e^{-2s\alpha \xi^3} \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt \\ & = -s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+1}} \sum_{\substack{i=1 \\ |\beta|=p+1}}^n \left(\partial_i (\theta_{p+1} e^{-2s\alpha \xi^3}) \partial_i \partial_\beta \tilde{\psi} + \theta_{p+1} e^{-2s\alpha \xi^3} \partial_i^2 \partial_\beta \tilde{\psi} \right) \left(\partial_\beta \tilde{\psi} \right) dxdt \\ & \leq s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+1}} \left(\left\| \nabla (\theta_{p+1} e^{-2s\alpha \xi^3}) \right\|_1 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3} \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2} \right. \\ & \quad \left. + \theta_{p+1} e^{-2s\alpha \xi^3} \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4} \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2} \right) dxdt. \end{aligned}$$

By (3.5) and (3.6), we have that

$$(A.23) \quad \left\| \nabla (\theta_{p+1} e^{-2s\alpha \xi^3}) \right\|_1 \leq Cs\lambda e^{-2s\alpha \xi^4}.$$

Indeed,

$$\begin{aligned} \left\| \nabla (\theta_{p+1} e^{-2s\alpha \xi^3}) \right\|_1 &= \left\| e^{-2s\alpha \xi^3} (\nabla \theta_{p+1} + 2s\lambda \theta_{p+1} \xi \nabla \eta^0 + 3\lambda \theta_{p+1} \nabla \eta^0) \right\|_1 \\ &= s\lambda e^{-2s\alpha \xi^4} \left\| \frac{\nabla \theta_{p+1}}{s\lambda \xi} + 2\theta_{p+1} \nabla \eta^0 + \frac{3\theta_{p+1} \nabla \eta^0}{s\xi} \right\|_1, \end{aligned}$$

and since $s \geq C(T^5 + T^{10})$, (A.23) is verified. Hence, by (A.21), (A.23) and using Young's inequality with $\epsilon > 0$, we have

$$(A.24) \quad \begin{aligned} & s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha \xi^3} \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt \\ & \leq Cs^3 \lambda^4 \iint_{(0,T) \times \omega_{p+1}} \left(s\lambda e^{-2s\alpha \xi^4} \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3} \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2} \right. \\ & \quad \left. + e^{-2s\alpha \xi^3} \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4} \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2} \right) dxdt \\ & \leq C \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha \xi^3} \left(\epsilon s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + \epsilon s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right. \\ & \quad \left. + \frac{2}{\epsilon} s^5 \lambda^6 \xi^5 \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2}^2 \right) dxdt. \end{aligned}$$

Observe that the first two terms in the righthand side of (A.24) can be bounded above by employing (A.20) and (A.24) recursively: indeed, by positivity of the integrand in Q_T and by (A.20), we obtain

$$\begin{aligned}
& \epsilon \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \left(s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right) dxdt \\
& \leq C \epsilon \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \quad \left. + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right) \\
& = C \epsilon \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \quad \left. + s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right), \tag{A.25}
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Combining (A.25) and (A.24) yields

$$\begin{aligned}
& \epsilon \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \left(s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right) dxdt \\
& \leq C \left(\epsilon \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \quad + \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \epsilon^2 \left(s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right) \\
& \quad + \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} 2s^5 \lambda^6 \xi^5 \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2}^2 dxdt \\
& \quad \left. + \epsilon s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right), \tag{A.26}
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Using the same treatment by adapting (A.24), one can bound the terms with ϵ^2 in (A.26); after r of these recursions,

$$\begin{aligned}
& \epsilon \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \left(s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right) dxdt \\
& \leq C \sum_{j=1}^r \left(\epsilon^j \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \quad + \epsilon^{2(r+1)} \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \left(s^3 \lambda^4 \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 + s \lambda^2 \xi \left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}^2 \right. \\
& \quad \left. \left. + 2j s^5 \lambda^6 \xi^5 \left\| \nabla^{p+1} \tilde{\psi} \right\|_{p+2}^2 \right) dxdt + \epsilon^j s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Taking ϵ sufficiently small and using (A.24),

$$\begin{aligned}
& s^3 \lambda^4 \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \xi^3 \left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3}^2 dxdt \\
& \leq C \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
\text{(A.27)} \quad & \left. + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$, since by (A.22), if $\left\| \nabla^{p+2} \tilde{\psi} \right\|_{p+3} = 0$, then so does $\left\| \nabla^{p+3} \tilde{\psi} \right\|_{p+4}$. Hence from (A.27), we obtain

$$\begin{aligned}
& \iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \\
\text{(A.28)} \quad & \leq C \iint_{(0,T) \times \omega_{p+1}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt,
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. For $r \in \{1, \dots, p+1\}$, consider the functions $\theta_r \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases} \text{Supp}(\theta_{p+1-r}) \subseteq \omega_{p+1-r}, \\ \theta_{p+1-r} = 1, & \text{in } \omega_{p+2-r}, \\ 0 \leq \theta_{p+1-k} \leq 1, & \text{in } \Omega. \end{cases}$$

Using the exact same approach as was used for $r = 0$, one obtains the estimate

$$\begin{aligned}
& s^{2r+3} \lambda^{2r+4} \iint_{(0,T) \times \omega_{p+2-r}} e^{-2s\alpha} \xi^{2r+3} \left\| \nabla^{p+2-r} \tilde{\psi} \right\|_{p+3-r}^2 dxdt \\
& \leq C \left(\iint_{(0,T) \times \omega_{p+2}} e^{-2s\alpha} \sum_{k=3+r}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \right. \\
& \left. + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Hence, it follows that

$$\begin{aligned}
& \iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1} \lambda^{2k} \xi^{2k-1} \left\| \nabla^{p+4-k} \tilde{\psi} \right\|_{p+5-k}^2 dxdt \\
& \leq C \left(s^{2p+7} \lambda^{2p+8} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt \right. \\
\text{(A.29)} \quad & \left. + s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \right),
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Finally, by (3.6) we have the estimate

$$s^{2p+34/5} \lambda \iint_{Q_T} e^{-2s\alpha} \xi^{l(p)} \left\| \tilde{\psi} \right\|_1^2 dxdt \leq C s^{2p+7} \lambda^{2p+8} \iint_{Q_T} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt,$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$ large enough; from now on, we denote this choice of s by s_0 . Hence, one can absorb the global term in the righthand side of (A.29) into its lefthand side, and thus (3.12) is verified. \square

Proof. (Proof of Proposition 3.1): We denote by C various positive constant depending on Ω and ω_0 . From (3.12), we deduce

$$(A.30) \quad \iint_{Q_T} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt \leq C \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{2p+7} \left\| \tilde{\psi} \right\|_1^2 dxdt,$$

for $\lambda \geq C$ and $s \geq s_0$. Note that for $t \in [\frac{T}{4}, \frac{3T}{4}]$, we have

$$(A.31) \quad \begin{aligned} & \min_{t \in [\frac{T}{4}, \frac{3T}{4}]} \{e^{-2s\alpha} \xi^{2p+7}\} \\ &= (e^{-2s\alpha} \xi^{2p+7}) \left(\frac{T}{4}, \cdot \right) = (e^{-2s\alpha} \xi^{2p+7}) \left(\frac{3T}{4}, \cdot \right) \\ &= \left(e^{-2s \frac{4^{10}}{3^5} \left(\frac{e^{12\lambda \|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}}{T^{10}} \right)} \right) \left(\frac{4^{10} e^{(2p+7)\lambda(10\|\eta^0\|_\infty + \eta^0(x))}}{3^5 T^{10}} \right). \end{aligned}$$

We can choose s sufficiently large such that

$$(A.32) \quad \frac{4^{10}}{3^5 T^{10}} e^{-\frac{s}{T^{10}}} \leq e^{-2s\alpha} \xi^{2p+7},$$

for all $t \in [\frac{T}{4}, \frac{3T}{4}]$. Indeed, choosing

$$(A.33) \quad s \geq s_1 := \max \left\{ s_0, \left(\frac{3^5(2p+7)\lambda}{4^{10}} \right) \max_{x \in \Omega} \left\{ \frac{10\|\eta^0\|_\infty + \eta^0(x)}{e^{12\lambda\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}} \right\} \right\}$$

in (A.31) will ensure that (A.32) is verified. Note that we can write s_1 as $s_1 = \sigma(T^5 + T^{10})$, where $\sigma > 0$ depends only on Ω and ω_0 . Fixing $s = s_1$ from now on, we deduce from (A.30) and (A.32) that

$$\iint_{(\frac{T}{4}, \frac{3T}{4}) \times \Omega} \left\| \tilde{\psi} \right\|_1^2 dxdt \leq C T^{10} e^{C(1+1/T^5)} \iint_{(0,T) \times \omega_0} e^{-2s_1\alpha} \xi^7 \left\| \tilde{\psi} \right\|_1^2 dxdt$$

for every $\lambda \geq C$ and $s \geq s_1$. We claim that

$$(A.34) \quad \int_{\Omega} \left\| \tilde{\psi}(T/4, \cdot) \right\|_1^2 dx \leq \frac{C}{T} e^{CT/2} \iint_{(\frac{T}{4}, \frac{3T}{4}) \times \Omega} \left\| \tilde{\psi} \right\|_1^2 dxdt$$

and

$$(A.35) \quad \int_{\Omega} \left\| \tilde{\psi}(0, \cdot) \right\|_1^2 dx \leq e^{CT/4} \int_{\Omega} \left\| \tilde{\psi}(T/4, \cdot) \right\|_1 dx,$$

from which we can deduce (3.3). Indeed, we can multiply system (3.2) by $\tilde{\psi}$, integrate the resulting equation by parts over Ω and use the Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \|\tilde{\psi}\|_1^2 dx + D \int_{\Omega} \|\nabla \tilde{\psi}\|_2^2 dx &= - \int_{\Omega} (\partial_t \tilde{\psi}) \tilde{\psi} dx + \int_{\Omega} \operatorname{div}(D \nabla \tilde{\psi}) \tilde{\psi} dx \\ &= - \int_{\Omega} (G^* \cdot \nabla \tilde{\psi}) \tilde{\psi} dx + \int_{\Omega} (A^* \tilde{\psi}) \tilde{\psi} dx \\ &\leq \frac{1}{2} \int_{\Omega} \|G^* \cdot \nabla \tilde{\psi}\|_1^2 dx + \left(1 + \frac{\|A^*\|_{\infty}}{2}\right) \int_{\Omega} \|\tilde{\psi}\|_1^2 dx. \end{aligned}$$

Hence, since (1.2) satisfies the uniform ellipticity condition (see (1.3)), we obtain

$$-\frac{d}{dt} \int_{\Omega} \|\tilde{\psi}\|_1^2 dx + \int_{\Omega} \|\nabla \tilde{\psi}\|_2^2 dx \leq C \int_{\Omega} \|\tilde{\psi}\|_1^2 dx,$$

from which we deduce

$$(A.36) \quad \frac{d}{dt} \left(e^{Ct} \int_{\Omega} \|\tilde{\psi}\|_1^2 dx \right) = e^{Ct} \left(C \int_{\Omega} \|\tilde{\psi}\|_1^2 dx + \frac{d}{dt} \int_{\Omega} \|\tilde{\psi}\|_1^2 dx \right) \geq e^{Ct} \int_{\Omega} \|\nabla \tilde{\psi}\|_2^2 dx \geq 0,$$

for all $t > 0$. We integrate (A.36) over $[\frac{T}{4}, t]$ to obtain

$$(A.37) \quad \int_{\Omega} \|\tilde{\psi}\|_1^2 dx \geq e^{C(T/4-t)} \int_{\Omega} \|\tilde{\psi}(T/4, \cdot)\|_1^2 dx \geq e^{-CT/2} \int_{\Omega} \|\tilde{\psi}(T/4, \cdot)\|_1 dx,$$

for every $t \in [\frac{T}{4}, \frac{3T}{4}]$. Integrating (A.37) once more over $[\frac{T}{4}, \frac{3T}{4}]$ now yields (A.34). Finally, to show (A.35), we integrate (A.36) over $t \in [0, \frac{T}{4}]$. \square