Ensemble Controllability of linear control systems

by

Daniel Owusu Adu

A project submitted to the
Department of Mathematics and Statistics
in conformity with the requirements for
the degree of Master of Science

Queen's University
Kingston, Ontario, Canada
September 2017

Abstract

This thesis is concerned about ensemble controllability. We give an overview of the notions of ensemble controllability, in particular, L_2-ensemble controllability, and uniform ensemble controllability. We review the results presented in [7] on uniform ensemble controllability of one-parameter time-invariant linear systems and in [11] on L_2-ensemble controllability. In contrast to the notions in [7] and [11], we investigate on the possibility to steer an ensemble using constrained control signals in the unit interval, which we call uniform *null* ensemble controllability for one-parameter time-invariant linear systems using constrained control signals in the unit interval. We give a necessary as well as a sufficient condition for uniform *null* ensemble controllability of one-parameter time-invariant linear systems using constrained control signals in the unit interval. Using tools from complex approximation theory, we show that in the discrete-time scenario, the problem of uniform *null* ensemble controllability of one-parameter time-invariant linear systems using control signals in the unit interval is equivalent to polynomial approximation problem.

Acknowledgments

First of all, I would like to thank the Almighty God through Jesus Christ for granting me success in my master's program. I will also like to thank my mother for her support and encouragement.

I also wish to express my utmost gratitude to my supervisor Professor Bahman Gharesifard for his unrelenting support, guidance and encouragement throughout my master's study. I am also grateful for Professor Abdol-Reza Mansouri and Professor Andrew Lewis for agreeing to serve on my examination committee. Needless to say, any error that remains is solely mine.

Lastly I like to thank all my lovely friends especially Naomi Naa Merley Botchway and Mohammad Jahvani.

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Chapter 1

Introduction

The task of controlling a continuum of structurally similar systems has received great interest in recent years. In practice, systems exhibit variations in parameter. For example, in the experiment of magnetic resonance, the spin of a continuum of systems is likely to show dispersions in the coupling strength between coupled spins [15]. Also, systems in aircraft, manufacturing processes, spacecraft, communication systems and many more are parameter-dependent. Applications such as the control of platoons in [16], the control of large number of networks of systems, controlling of the flock system in [3], and control of some classes of biological systems motivates the investigation of the possibility of controlling a large number of parameter-dependent systems. Although in practice, there are often a finite number of such structurally similar systems, it is also reasonable, and mathematically interesting, to study a continuum of such parameters which we term *ensemble* of systems. Indeed, our main objective is to control such an ensemble. Given that the characteristics of the parameter that differentiates these systems is often unknown, our main objective is to design a parameter-independent control signal that steers an ensemble of control system from a given initial states to any desired states.

The possibility to control a continuum of systems have being investigated in different frameworks. In [11], ensemble controllability is investigated in the L_2 sense. However the conditions derived in [11] depends heavily on the singular decomposition of the input-to-state operator. In [7], necessary as well as sufficient conditions for uniform ensemble controllability of one-parameter of time-invariant linear systems are derived. It should be pointed out that, although the conditions in [11] do not apply to the framework used in [7], comparing the conditions in [7] to [11], the conditions in [7] are easily checkable, although the framework in [7] is restrictive compared to the framework in [11].

Statement of contribution. The main objective of this current project is to provide a summary of the results on ensemble controllability given in [11] and [7]. We proceed with this objective by investigating notions of ensemble controllability both in uniform and L_2 sense. Moreover, inspired by [18], we study controllability of ensemble control systems for scenarios where the control set is constrained, by introducing the notion of uniform *null* ensemble controllability. In particular, we provide necessary and sufficient conditions for uniform null ensemble controllability.

Organization

The thesis is organized as follows. In Chapter 2, we give a general problem statement of ensemble controllability. In Chapter 3, the problem of L_2-ensemble controllability is investigated. In Chapter 4, the problem of uniform ensemble controllability of one-parameter time-invariant linear systems is investigated. In Chapter 5, we extend Chapter 4 by considering uniform ensemble controllability of one-parameter time-invariant linear systems to the origin using constrained control signals. In Chapter 6, we state the conclusion and future research directions. In Chapter 7, the thesis closes by proving some propositions in the appendix.

Chapter 2

Problem Statement

2.1 Definitions and problem statement

In this thesis, we study a *continuum* of linear systems, which we later call an *ensemble* of linear systems. In order to define these family of systems formally, we consider both continuous-time and discrete-time scenarios. In continuous-time, we consider a family of systems of the form

$$\frac{\partial x}{\partial t}(t,\theta) = A(t,\theta)x(t,\theta) + B(t,\theta)u(t), \tag{2.1}$$

where $x(t,\theta) \in \mathbb{R}^n$, $A(t,\theta) \in \mathbb{R}^{n\times n}$, $B(t,\theta) \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$, with $t \geq 0$ and $\theta \in \mathbb{P} := [\theta^-, \theta^+] \subset \mathbb{R}$.

In discrete-time, we consider a family of control systems of the form

$$x(t+1,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \tag{2.2}$$

where $A(\theta) \in \mathbb{R}^{n\times n}$, $B(\theta) \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$ with $\theta \in \mathbb{P} := [\theta^-, \theta^+] \subset \mathbb{R}$.

We will often identify an ensemble of control systems given by (2.1), or (2.2) with

$\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ and $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$, respectively. Given initial states $x(0, \theta)$, for all $\theta \in \mathbb{P}$ and a finite time $T > 0$, using the variation of constants formula, the general solution for (2.1) is given by,

$$x(T, \theta) = \Phi(T, 0, \theta)x(0, \theta) + \int_0^T \Phi(T, \tau, \theta)B(\tau, \theta)u(\tau)d\tau, \tag{2.3}$$

where $\Phi(T, t, \theta)$ is the transition matrix of the uncontrolled ensemble of the form

$$\frac{\partial x}{\partial t}(t, \theta) = A(t, \theta)x(t, \theta),$$

where $\theta \in \mathbb{P}$. Similarly, in (2.2) we have

$$x(T, \theta) = A^T(\theta)x(0, \theta) + \sum_{r=1}^{T-1} A^r(\theta)B(\theta)u(T - 1 - r). \tag{2.4}$$

Let $x(T, \theta)$ and $x_d(\theta)$ denote the final and desired states, respectively. We say that a control signal u steers the trajectories in (2.1) or (2.2) between $x_1(\theta)$ and $x_2(\theta)$, for all $\theta \in \mathbb{P}$, if and only if its corresponding trajectory $x(., \theta)$ for all $\theta \in \mathbb{P}$ in (2.3) or (2.4) induced by u satisfies the conditions $x(t_1, \theta) = x_1(\theta)$ and $x(t_2, \theta) = x_2(\theta)$, for all $\theta \in \mathbb{P}$, respectively. Let us denote the set of n-tuples entries are real-valued continuous functions defined on $\mathbb{P}$ by $C(\mathbb{P}, \mathbb{R}^n)$ and the set of n-tuple whose entries are real-valued measurable functions defined on $\mathbb{P}$ such that its L_p-norm is bounded by $L_p(\mathbb{P}, \mathbb{R}^n)$, where $n,\ p \in\ \mathbb{N}$. We proceed to give formal definitions of the notions of ensemble controllability.

Definition 2.1.1. *An ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is L_p-ensemble controllable in $L_p(\mathbb{P}; \mathbb{R}^n)$ if and only if, for all x_0 and x_d in $L_p(\mathbb{P}; \mathbb{R}^n)$ and $\epsilon > 0$, there exists a finite time $T > 0$*

and a control signal $u \in L_q([0,T];\mathbb{R}^m)$ *that steers the trajectories of* $\Sigma_{\mathrm{C}}(\mathbb{P},A,B)$*, for all* $\theta \in \mathbb{P}$ *from* $x(0,\theta)$ *to* $x(T,\theta)$*, where* $x(T,\theta)$ *satisfies the relation*

$$\left(\int_{\mathbb{P}} \|x_d(\theta) - x(T,\theta)\|^p d\theta\right)^{\frac{1}{p}} < \epsilon, \tag{2.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

We introduce another notion for ensemble controllability.

Definition 2.1.2. *An ensemble* $\Sigma_{\mathrm{C}}(\mathbb{P},A,B)$ *is uniformly ensemble controllable in* $C(\mathbb{P};\mathbb{R}^n)$ *if and only if, for all* x_0 *and* x_d *in* $C(\mathbb{P};\mathbb{R}^n)$*, and* $\epsilon > 0$*, there exists a finite time* $T > 0$ *and a control signal* $u \in L_1([0,T];\mathbb{R}^m)$ *that steers the trajectories of* $\Sigma_{\mathrm{C}}(\mathbb{P},A,B)$*, for all* $\theta \in \mathbb{P}$ *from* $x(0,\theta)$ *to* $x(T,\theta)$ *,where* $x(T,\theta)$ *satisfies the relation*

$$\sup_{\theta \in \mathbb{P}} \|x_d(\theta) - x(T,\theta)\| < \epsilon. \tag{2.6}$$

We describe later as to why the controls are chosen from $L_q([0,T];\mathbb{R}^m)$ in this definition. It is worth pointing out that in Chapter 5, we will consider a special case of Definition 2.1.2 where the control set is constrained. We also point out that, the definitions hold for discrete-time scenario.

Chapter 3

L_2-Ensemble Controllability of Finite-Dimensional Time-Varying Linear Systems

3.1 L_2-Ensemble Controllability

In this chapter, for reasons that will become clear later, we find it convenient to consider the system parameters over $\mathbb{C}$. We let $M := [0,T] \times \mathbb{P}$ and we consider the ensemble of control systems $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ introduced in (2.1), where $(A, B) \in L_\infty(M; \mathbb{C}^{n\times n}) \times L_2(M; \mathbb{C}^{n\times m})$. That is, $A_{ij} \in L_\infty(M; \mathbb{C})$ and $B_{ij} \in L_2(M; \mathbb{C})$, where A_{ij} and B_{ij} are the ijth entries of A and B, respectively, and $i, j \in \{1, ..., n\}$. In this chapter, we assume that $u \in L_2([0,T]; \mathbb{C}^m)$.

Note that if it happens that $x(T, .) = x_d$, then, from (2.3) we have that,

$$\vartheta(\theta) = \int_0^T \Phi(0, \tau, \theta) B(\tau, \theta) u(\tau) d\tau, \tag{3.1}$$

where $\vartheta \in L_2[\mathbb{P}; \mathbb{C}^n]$ is defined as

$$\vartheta(\theta) = \Phi(0, T; \theta) x_d(\theta) - x_0(\theta).$$

We need to recall a few mathematical notions. Recall that the space, $L_2([a,b];\mathbb{C}^k), a, b \in \mathbb{R}, k \in \mathbb{N}$ has an inner product defined by

$$\langle f, g \rangle = \int_a^b f^\dagger(t) g(t) dt,$$

for all $f, g \in L_2([a,b];\mathbb{C}^k)$, where $\dagger$ denote the conjugate transpose. Let $H_1 = L_2([0,T];\mathbb{C}^m)$ and $H_2 = L_2(\mathbb{P};\mathbb{C}^n)$. We define an operator $L : H_1 \to H_2$ by

$$(Lu)(\theta) = \int_0^T \Phi(0,\tau,\theta) B(\tau,\theta) u(\tau) d\tau. \tag{3.2}$$

From (3.1) and (3.2) we have that

$$(Lu)(\theta) = \vartheta(\theta), \tag{3.3}$$

for all $\theta \in \mathbb{P}$. With this new formulation at hand, ensemble controllability is equivalent to solving the operator equation (3.3). That is, we wish to find $u \in H_1$ that solves

$$Lu = \vartheta. \tag{3.4}$$

It is shown in [11] that, the operator L defined in (3.2) is bounded and compact. We include a proof of this fact in the appendix for completeness (see Theorem 7.1.2 and Proposition 7.1.3). Hence L is a bounded compact linear operator. Under these conditions on L, it is well-known in [8] that L has an adjoint operator L^* which is also a bounded compact linear operator such that, for all $f \in H_2$ and $u \in H_1$, L^* satisfy the relation

$$\langle f, Lu \rangle_{H_2} = \langle L^* f, u \rangle_{H_1}, \tag{3.5}$$

where $\langle .,. \rangle_{H_1}$ and $\langle .,. \rangle_{H_2}$ are inner products defined on the space H_1 and H_2, respectively. From (3.5) one can show that, for all $f \in H_2$, L^* is given by

$$(L^* f)(t) = \int_{\mathbb{P}} B^\dagger(\tau,\theta)\Phi^\dagger(0,\tau,\theta) f(\theta) d\theta. \tag{3.6}$$

Now, since compact operators are not invertible (see Proposition 7.1.5), the operator equation (3.4) does not have a unique solution. For this, we state a result and refer the reader to [12] for the proof.

Theorem 3.1.1. *[12, Theorem 6.10]: Let H_1 and H_2 be Hilbert space and let $L \in B(H_1, H_2)$ with range space of L denoted by $R(L)$, closed in H_2. Then, for $\vartheta \in R(L)$, the vector of minimum norm u satisfying $Lu = \vartheta$ is given by $u = L^* z$, where z is any solution of $LL^* z = \vartheta$.*

Using (3.2) and (3.6), one can show that the operator $LL^* : H_2 \to H_2$ takes the form

$$(LL^* z)(\theta) = \int_{\mathbb{P}} \int_0^T \Phi(0,\tau,\theta) B(\tau,\theta) B^\dagger(\tau,\theta') \Phi^\dagger(0,\tau,\theta') z(\theta') d\tau d\theta'. \tag{3.7}$$

Before we proceed to state and prove the main results, we give the following definition.

Definition 3.1.2. *[11]: Let H_1 and H_2 be Hilbert spaces and $L : H_1 \to H_2$ be compact operator. If (λ_j^2, ψ_j) is an eigensystem of LL^* and (λ_j^2, ϕ_j) is an eigensystem of $L^* L$, namely $LL^* \psi_j = \lambda_j^2 \psi_j$, $\psi_j \in H_2$ and $L^* L \phi_j = \lambda_j^2 \phi_j$, $\phi_j \in H_1$, where $\lambda_j > 0$ $(j \geq 1)$, then, the two systems are related by the equations*

$$L\phi_j = \lambda_j \psi_j \quad \textit{and} \quad L^* \psi_j = \lambda_j \phi. \tag{3.8}$$

We say that the triple $(\lambda_j, \phi_j, \psi_j)$ is a singular system of L.

Now we proceed to state and prove the main result of this chapter.

Theorem 3.1.3. *[11] An ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is L_2-ensemble controllable in $L_2(\mathbb{P}; \mathbb{R}^n)$ if and only if, for any given initial and desired states x_0 and $x_d \in L_2(\mathbb{P}; \mathbb{R}^n)$ and for $\vartheta(\theta) = \Phi(0, T; \theta)x_d(\theta) - x_0(\theta)$, the conditions*

1. $\sum_{j=1}^{\infty} \frac{|\langle \vartheta, \psi_j \rangle|^2}{\lambda_j^2} < \infty$
2. $\vartheta \in \overline{R(L)}$

hold, where $\overline{R(L)}$ denotes the closure of the range space of L. Furthermore, the control law

$$u = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle \varphi, \psi_j \rangle \phi_j \tag{3.9}$$

satisfies

$$\langle u, u \rangle \leq \langle u_0, u_0 \rangle,$$

for all $u_0 \in \mathbb{F}$ and $u \neq u_0$, where

$$\mathbb{F} = \{u \in L_2([0, T]; \mathbb{R}^m) \mid Lu = \vartheta \textit{ with conditions } 1 \textit{ and } 2 \textit{ of Theorem 3.1.3 satisfied}\}.$$

In addition, for a given $\epsilon > 0$,

$$u_r = \sum_{j=1}^{r} \frac{\langle \vartheta, \psi_j \rangle \phi_j}{\lambda_j}$$

is such that

$$\|\vartheta - Lu_m\| < \epsilon, \tag{3.10}$$

for all $m \geq r$, where $r \in \mathbb{N}$ and depends on ϵ, where

$$u_m = \sum_{j=1}^{m} \frac{\langle \vartheta, \psi_j \rangle \phi_j}{\lambda_j}. \tag{3.11}$$

Proof. We start by proving the necessity. Suppose there exist $u \in H_1$ that satisfies (3.4). Then,

$$\langle \vartheta, \psi_j \rangle = \langle Lu, \psi_j \rangle, \tag{3.12}$$

which implies

$$\frac{1}{\lambda_j} \langle \vartheta, \psi_j \rangle = \langle u, \phi_j \rangle. \tag{3.13}$$

Since LL^* is a self-adjoint compact operator, the sequences $\{\phi_j\}_{j\geq 1} \subset H_1$ and $\{\psi_j\}_{j\geq 1} \subset H_2$ are orthonormal sequences (see [6, pp. 248]). Using Bessel's inequality, we have that,

$$\sum_{j=1}^{\infty} \frac{|\langle \vartheta, \psi_j \rangle|^2}{\lambda_j^2} \leq \|u\|_2^2 < \infty.$$

This ends the proof of the first statement. Also, for any $\alpha \in N(L^*)$, we have that $\alpha \in H_2$ such that

$$L^* \alpha = 0.$$

It follows that,

$$\langle \vartheta, \alpha \rangle = \langle Lu, \alpha \rangle = \langle u, L^* \alpha \rangle = 0.$$

Hence,

$$\vartheta \in N(L^*)^{\perp} = \overline{R(L^*)}.$$

This ends the proof of the second statement. Conversely, suppose the first and second conditions hold. Then, let

$$\beta_j = \frac{\langle \vartheta, \psi_j \rangle}{\lambda_j}. \tag{3.14}$$

From the first condition, we have that

$$\sum_{j=1}^{\infty} |\beta_j|^2 < \infty. \tag{3.15}$$

By Proposition 7.1.6, there exist $u \in H_1$ such that

$$u = \sum_{j=1}^{\infty} \beta_j \phi_j. \tag{3.16}$$

In [11], it has been shown that $\{\phi_j\}_{j\geq 1}$ and $\{\psi_j\}_{j\geq 1}$ are an orthonormal basis for $\overline{R(L^*)}$ and $\overline{R(L)}$, respectively, and since $u \in R(L^*) \subset H_1$, we have that

$$u = \sum_{j=1}^{\infty} \langle u, \phi_j \rangle \phi_j. \tag{3.17}$$

Now, since $\{\phi_j\}_{j\geq 1}$ is an orthonormal basis its coefficients are unique. Hence, from (3.16) and (3.17) we have that,

$$\langle u, \phi_j \rangle = \frac{\langle \vartheta, \psi_j \rangle}{\lambda_j}.$$

We claim that $u \in H_1$ in (3.17) is not in $N(L)$. We prove this by a contradiction argument.

Suppose $u \in N(L)$, then $Lu = 0$. Now, by linearity and continuity of L we obtain,

$$Lu = \sum_{j=1}^{\infty} \beta_j (L\phi_j) = \sum_{j=1}^{\infty} \langle \vartheta, \psi_j \rangle \psi_j = 0. \tag{3.18}$$

Now, since $\{\psi_j\}_{j\geq 1}$ is an orthonormal basis, it follows that $\langle \vartheta, \psi_j \rangle = 0$ for $j \in \{1, 2, ...\}$. This implies that $\vartheta = 0$, which is a contradiction. Hence the assumption $u \in N(L)$ is false.

Now, since $\vartheta \in \overline{R(L)}$ and $\{\psi_j\}_{j\geq 1}$ is an orthonormal basis in $\overline{R(L)}$, we see that the right hand side of equation (3.18) is

$$\sum_{j=1}^{\infty} \langle \vartheta, \psi_j \rangle \psi_j = \vartheta.$$

Hence u in (3.17) solves the operator equation (3.4). Furthermore, let

$$u_N = \sum_{j=1}^{N} \frac{\langle \vartheta, \psi_j \rangle}{\lambda_j} \phi_j, \tag{3.19}$$

where $N \in \mathbb{N}$. Using the fact that $\{\phi_j\}_{j\geq 1}$ is an orthonormal sequence, we obtain

$$\|u - u_N\|_2^2 = \sum_{j=N+1}^{\infty} \frac{1}{\lambda_j^2} |\langle \varphi, \psi_j \rangle|^2 \to 0 \quad \text{as} \quad N \to \infty. \tag{3.20}$$

This implies that

$$\|\vartheta - Lu_N\|_2^2 = \sum_{j=N+1}^{\infty} \lambda_j^2 |\langle u, \phi_j \rangle|^2 \to 0 \quad \text{as} \quad N \to \infty. \tag{3.21}$$

This completes the proof. □

3.1.1 Optimal control of an Ensemble of Harmonic Oscillators

We include an example from [11] to illustrate the construction of the ensemble controller. We consider a fixed endpoint optimal control problem of an ensemble of harmonic oscillators. Consider

$$\frac{\partial x}{\partial t}(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \tag{3.22}$$

where $\theta \in \mathbb{P} \subset \mathbb{R}$, $x(t,\theta) = (x_1(t,\theta), x_2(t,\theta))^{\mathrm{T}} \in \mathbb{R}^2$, $u(t) = (u_1(t), u_2(t))^{\mathrm{T}} \in \mathbb{R}^2$ such that each $u_i \in L_2([0,T];\mathbb{R})$ for $i \in \{1,2\}$,

$$A(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \quad \text{and} \quad B(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Given x_0, $x_d \in L_2([0,T];\mathbb{R}^2)$, we wish to find $u \in L_2([0,T];\mathbb{R}^2)$ that steers the trajectories of (3.22) from $x(0,\theta)$ to $x(T,\theta) \in \mathbb{R}^2$ in the sense of L_2-ensemble controllability such that, u minimizes the cost functional

$$\min_{u \in L_2([0,T];\mathbb{R}^2)} J(u) = \int_0^T \|u(t)\|^2 dt.$$

We use the fact that $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$ and let

$$\begin{aligned} x(t,\theta) =& x_1(t,\theta) + ix_2(t,\theta), \\ u(t) =& u_1(t) + iu_2(t). \end{aligned}$$

Hence (3.22) can be written as

$$\frac{\partial x}{\partial t}(t,\theta) = i\theta x(t,\theta) + u(t).$$

From the variation of constants formula we obtain

$$x(T,\theta) = e^{i\theta}x(0,\theta) + \int_0^T e^{i\theta(t-s)}u(s)ds.$$

As a result,

$$\vartheta(\theta) = \int_0^T e^{-i\theta s}u(s)ds, \tag{3.23}$$

where

$$\vartheta(\theta) = e^{-i\theta T}x(T,\theta) - x(0,\theta).$$

Let $H_1 = L_2([0,T];\mathbb{C})$ and $H_2 = L_2(\mathbb{P};\mathbb{C})$. We define an operator $L : H_1 \to H_2$ by

$$(Lu)(\theta) = \int_0^T e^{-i\theta s}u(s)ds. \tag{3.24}$$

From (3.23) and (3.24) we obtain

$$(Lu)(\theta) = \vartheta(\theta), \tag{3.25}$$

for all $\theta \in \mathbb{P}$. Now, since $u \in H_1$ and the kernel $k(t,\theta) = e^{-i\theta T}$ is bounded, it implies that the operator L defined in (3.24) is a bounded compact linear operator and hence it has an adjoint. Note that, for all $f \in H_2$ we have

$$\langle f, Lu\rangle_{H_2} = \int_0^T \int_{\theta^-}^{\theta^+} e^{-i\theta s}f(\theta)^\dagger d(\theta)u(s)ds. \tag{3.26}$$

Hence the adjoint operator satisfies

$$(L^*f)(s) = \int_{\theta^-}^{\theta^+} e^{i\theta s}f(\theta)d\theta. \tag{3.27}$$

From Theorem 3.1.1, we have that

$$L^* z = u,$$

where z satisfies

$$LL^* z = \vartheta.$$

Substituting (3.27) into (3.24), the operator $LL^* : H_2 \to H_2$ is of the form

$$(LL^* z)(\theta_1) = \int_0^T \int_{\theta^-}^{\theta^+} e^{i(\theta' - \theta_1)s} z(\theta') d\theta' ds. \tag{3.28}$$

Using Fubini's Theorem, we obtain

$$(LL^* z)(\theta_1) = \int_{\theta^-}^{\theta^+} \left(\int_0^T e^{i(\theta' - \theta_1)s} ds \right) z(\theta') d\theta'. \tag{3.29}$$

By direct calculation, we have

$$(LL^* z)(\theta_1) = \int_{\theta^-}^{\theta^+} \left(\frac{e^{i(\theta' - \theta_1)T} - 1}{i(\theta' - \theta_1)} \right) z(\theta') d\theta'. \tag{3.30}$$

We have that:

$$
\begin{aligned}
\frac{e^{i(\theta'-\theta_1)T}-1}{i(\theta'-\theta_1)} =& \frac{\cos((\theta'-\theta_1)T)-1+i\sin((\theta'-\theta_1)T)}{i(\theta'-\theta_1)},\\
=& \frac{\left(\cos^2((\theta'-\theta_1)\frac{T}{2})-1\right)-\sin^2((\theta'-\theta_1)\frac{T}{2})+i(2\sin((\theta'-\theta_1)\frac{T}{2})\cos((\theta'-\theta_1)\frac{T}{2}))}{i(\theta'-\theta_1)},\\
=& \frac{-2\sin^2((\theta'-\theta_1)\frac{T}{2})+i(2\sin((\theta'-\theta_1)\frac{T}{2})\cos((\theta'-\theta_1)\frac{T}{2}))}{i(\theta'-\theta_1)},\\
=& \frac{2\pi\sin((\theta'-\theta_1)\frac{T}{2})}{\pi(\theta'-\theta_1)}\left(\cos\left((\theta'-\theta_1)\frac{T}{2}\right)+i\sin\left((\theta'-\theta_1)\frac{T}{2}\right)\right),\\
=& 2\pi e^{i(\theta'-\theta_1)\frac{T}{2}}\left(\frac{\sin((\theta'-\theta_1)\frac{T}{2})}{\pi(\theta'-\theta_1)}\right).
\end{aligned}
$$

Let $\omega' = \frac{\theta'}{\theta}$, $\omega = \frac{\theta_1}{\theta}$ and $\alpha = \frac{T\theta}{2}$, then $\omega', \omega \in [-1,1]$. Using this observation, equation (3.30) can be rewritten as

$$
(LL^*z)(\omega) = \int_{-1}^{1} 2\pi e^{i(\omega'-\omega)\alpha}\left(\frac{\sin((\omega'-\omega)\alpha)}{\pi(\omega'-\omega)}\right)z(\omega')d\omega'. \tag{3.31}
$$

We consider the equation

$$
\int_{-1}^{1}\left(\frac{\sin((\omega'-\omega)\alpha)}{\pi(\omega'-\omega)}\right)\beta_j(\omega',\alpha)d\omega' = \upsilon_j(\alpha)\beta_j(\omega,\alpha), \tag{3.32}
$$

where $\beta_j(\omega,\alpha)$ is the jth eigenfunction and υ_j is its corresponding eigenvalue of a well-known prolate spheroidal wave function [14], [4], [19], [9] and [10]. Similarly, consider

$$
(LL^*\psi_j)(\omega,\alpha) = \int_{-1}^{1} 2\pi e^{i(\omega'-\omega)\alpha}\left(\frac{\sin((\omega'-\omega)\alpha)}{\pi(\omega'-\omega)}\right)\psi_j(\omega',\alpha)d\omega' = \rho_j(\alpha)\psi_j(\omega,\alpha). \tag{3.33}
$$

Rearranging (3.33) we have that

$$\int_{-1}^{1} e^{i\omega'\alpha}\left(\frac{\sin((\omega'-\omega)\alpha)}{\pi(\omega'-\omega)}\right)\psi_j(\omega',\alpha)d\omega' = \frac{1}{2\pi}e^{i\omega\alpha}\rho_j(\alpha)\psi_j(\omega,\alpha). \tag{3.34}$$

Let

$$e^{i\omega'\alpha}\psi_j(\omega',\alpha) = \beta_j(\omega',\alpha). \tag{3.35}$$

Then,

$$\upsilon_j(\alpha)\beta_j(\omega,\alpha) = \frac{1}{2\pi}e^{i\omega\alpha}\rho_j(\alpha)\psi_j(\omega,\alpha). \tag{3.36}$$

By evaluating (3.36) at ω', we obtain

$$\upsilon_j(\alpha)\beta_j(\omega',\alpha) = \frac{1}{2\pi}e^{i\omega'\alpha}\rho_j(\alpha)\psi_j(\omega',\alpha). \tag{3.37}$$

By comparing equations (3.35) and (3.37), we have

$$\rho_j = 2\pi\upsilon_j. \tag{3.38}$$

Therefore from (3.35) and (3.38) the eigenvectors and eigenvalues of the operator LL^* can be represented in terms of υ_j and β_j, respectively. It is well-known (see for example in [14]) that β_j's are orthogonal and complete on $L_2[-1,1]$. Now, let

$$z = \sum_{j=1}^{\infty}\frac{1}{\rho_j}\langle\vartheta,\tilde{\psi}_j\rangle\tilde{\psi}_j, \tag{3.39}$$

where

$$\tilde{\psi}_j = e^{-i\omega\alpha}\frac{\beta_j}{\|\beta_j\|}. \tag{3.40}$$

Then, we have that

$$LL^*z = \sum_{j=1}^{\infty} \langle \vartheta, \tilde{\psi}_j \rangle \tilde{\psi} j = \vartheta.$$

By applying Theorem 3.1.1 to LL^* with respect to the orthonormal basis $\{\tilde{\psi}_j\}_{j\geq 1}$ in $\overline{R(L)}$. We can easily observe that

$$u = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle \vartheta, \tilde{\psi}_j \rangle \tilde{\phi}_j, \tag{3.41}$$

where

$$\lambda_j = \sqrt{\rho_j}. \tag{3.42}$$

We can also express the control signal only in terms on $\tilde{\psi}_j$, noting that $\tilde{\phi}_j$ can be obtained by the same reasoning using the operator L^*L. The control signal can also be written as

$$u(t) = \int_{-\theta}^{\theta} e^{i\tilde{\theta}t} \sum_{j=1}^{\infty} \frac{1}{\rho_j} \langle \vartheta(\tilde{\theta}), \tilde{\psi}_j(\tilde{\theta}) \rangle \tilde{\psi}_j(\tilde{\theta}) d(\tilde{\theta}). \tag{3.43}$$

Let

$$z_N = \sum_{j=1}^{N} \frac{1}{\rho_j} \langle \vartheta, \tilde{\psi}_j \rangle \tilde{\psi}_j,$$

where $N \in \mathbb{N}$. As a result,

$$LL^*z_N = \sum_{j=1}^{N} \rho_j \langle z, \tilde{\psi}_j \rangle \tilde{\psi}_j.$$

We have that

$$\|\vartheta - LL^*z_N\|_2^2 = \sum_{j=N+1}^{\infty} |\langle \vartheta, \tilde{\psi}_j \rangle|^2 \tag{3.44}$$

goes to zero as $N \to \infty$. Hence, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $N \in \mathbb{N}$ such that for $N > n$, we have

$$\|\vartheta - LL^* z_N\|_2 < \epsilon. \tag{3.45}$$

Now, since $LL^* z_N$ approximates ϑ in this sense, it follows that

$$\begin{aligned}
\|LL^* z - LL^* z_N\|_2 =& \|L(L^* z - L^* z_N)\|_2, \\
\leq& \|L\|_2 \|L^* z - L^* z_N\|_2, \\
\leq& \|L\|_2^2 \|z - z_N\|_2 < \epsilon.
\end{aligned}$$

Now since $z_N \to z$ as $N \to \infty$, it implies that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $N \in \mathbb{N}$ such that $N > n$, we have

$$\|L^* z - L^* z_N\| < \epsilon.$$

Hence, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $N \in \mathbb{N}$ such that for $N > n$, we have

$$\|u - u_N\|_2 < \epsilon,$$

where $u_N = L^* z$. Therefore, the best approximation of the control signal u that achieves a minimum norm is given by the sequence of control inputs

$$u_N = L^* z_N.$$

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Chapter 4

Uniform Ensemble Controllability of One-parameter Time-invariant Linear Systems

4.1 Uniform Ensemble Controllability

In this Chapter, we let $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ be an ensemble of continuous time-invariant linear systems, where the pair $(A, B) \in C(\mathbb{P}; \mathbb{R}^{n\times n}) \times C(\mathbb{P}; \mathbb{R}^{n\times m})$. We also assume that $u \in L_1([0, T]; \mathbb{R}^m)$. Let $x(0, \theta) = 0$, for all $\theta \in \mathbb{P}$. Let us define the reachable set $\mathcal{R}(T) \subset \mathbb{R}^n$ at time $t = T$ for the time-invariant scenario of (2.1), with the constraint control set $L_1([0, T]; \mathbb{R}^m)$, to be the set

$$\begin{aligned}\mathcal{R}(T) =& \{x(T, \theta) \in \mathbb{R}^n \mid \ x \text{ is the solution to the time-invariant scenario of (2.1)} \\ & \text{for some } u \in L_1([0, T]; \mathbb{R}^m)\};\end{aligned}$$

we denote by $\mathcal{R}$ the union of all the reachable sets over time $T \geq 0$. As usual, we define the spectra of $A(\theta)$ to be the set $\sigma(A(\theta))$ of all eigenvalues of $A(\theta)$, where

$\theta \in \mathbb{P}$. For a given controllable pair $(A(\theta), B(\theta))$, we let

$$\mathcal{C}_{(A(\theta),B(\theta))} = (B(\theta), A(\theta)B(\theta), ..., A^{n-1}(\theta)B(\theta)) \tag{4.1}$$

be the corresponding controllability matrix for the pair $(A(\theta), B(\theta))$, where $\theta \in \mathbb{P}$. We define Hermite indices as in [20]: we let the matrix $H_{\mathcal{C}_{(A(\theta),B(\theta))}}$ be

$$H_{\mathcal{C}_{(A(\theta),B(\theta))}} = (b_1(\theta), A(\theta)b_1(\theta), ..., A^{n-1}(\theta)b_1(\theta), ..., b_m(\theta), ..., A^{n-1}(\theta)b_m(\theta)), \tag{4.2}$$

where $b_i(\theta)$ is the ith column vector of $B(\theta)$; we then choose from the left to the right of (4.2) the first linearly independent columns to obtain a list of basis vectors

$$b_1(\theta), ..., A^{K_1-1}(\theta)b_1(\theta), ..., b_m(\theta), ..., A^{K_m-1}(\theta)b_m(\theta). \tag{4.3}$$

We call the set $\{K_1, ..., K_m\}$, which may depend on θ and, where $K_i = 0$ when b_i is not selected, the Hermite indices. One can easily check that if the system is controllable, then, the sum of the Hermite indices will be equal to the rank of the controllability matrix. Following [5, pp. 508], we consider

$$\frac{\partial x_j}{\partial t}(t, \theta) = A_j(\theta)x_j(t, \theta) + B_j(\theta)u(t), \quad j \in \{1, 2, ..., r\} \tag{4.4}$$

with $(A_j(\theta), B_j(\theta)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. As usual, we denote the set of all real-valued polynomial with parameter y by $\mathbb{R}[y]$, the set of $n \times m$ polynomial matrices whose entries are univariate polynomials in $\lambda \in \mathbb{R}$ by $\mathbb{R}[\lambda]^{n\times m}$, and the set of $n \times m$ polynomial matrices whose entries are rational function in $\lambda \in \mathbb{R}$ by $\mathbb{R}(\lambda)^{n\times m}$. As usual, we will define the state space of the system to be the Euclidean space whose axes are the variables of the system.

We proceed to state the main result.

Theorem 4.1.1. *[7]: An ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is uniformly ensemble controllable provided the following are satisfied:*

1. *The pair $(A(\theta), B(\theta))$ is controllable, for all $\theta \in \mathbb{P}$.*
2. *The input Hermite indices $K_1(\theta), ..., K_m(\theta)$ of $(A(\theta), B(\theta))$ are independent of $\theta \in \mathbb{P}$.*
3. *We have $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$, for any pair of distinct parameters θ, $\theta' \in \mathbb{P}$.*
4. *The eigenvalues of $A(\theta)$ have algebraic multiplicity of one, for all $\theta \in \mathbb{P}$.*

Before we proceed with the proof of this sufficient condition, we also state a necessary condition for uniform ensemble controllability. For this, we need to introduce some notions.

Definition 4.1.2. *[5, Definition 2.26]: The polynomial matrices $P_i(\lambda, \theta) \in \mathbb{R}[\lambda]^{n\times m_i}$, where $\theta \in \mathbb{P}$ and $i \in \{1, ...r\}$ are* left coprime *if there exists a matrix $X(\lambda, \theta) \in \mathbb{R}[\lambda]^{k\times n}$ such that $D(\lambda, \theta) = \mathrm{gcld}(P_1(\lambda, \theta), ..., P_r(\lambda, \theta))$ satisfies $D(\lambda, \theta)X(\lambda, \theta) = I_n$, for all $\theta \in \mathbb{P}$ where* gcld *abbreviates greatest common left divisor. The right coprime polynomial matrices are defined similarly.*

Theorem 4.1.3. *[5, Theorem 2.29]: Let $G(\lambda, \theta) \in \mathbb{R}(\lambda)^{n\times m}$, where $\theta \in \mathbb{P}$. There exist left coprime polynomial matrices $N_1(\lambda, \theta) \in \mathbb{R}[\lambda]^{n\times m}$, $D_1(\lambda, \theta) \in \mathbb{R}[\lambda]^{n\times n}$, with $\det(D_1(\lambda, \theta)) \neq 0$, such that $G(\lambda, \theta) = D(\lambda, \theta)^{-1}N(\lambda, \theta)$.*

Using the right coprime factorization given by Theorem 4.1.3, one can conclude that

$$(\lambda I - A(\theta))^{-1}B(\theta) = K(\lambda, \theta)R(\lambda, \theta)^{-1} \tag{4.5}$$

where $R(\lambda, \theta) \in \mathbb{R}[\lambda]^{m\times m}$ is a non-singular polynomial matrix and $K(\lambda, \theta) \in \mathbb{R}[\lambda]^{n\times m}$ is a polynomial matrix. We proceed to state and prove a necessary condition for uniform ensemble controllability.

Theorem 4.1.4. *[7]: Assume $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is uniformly ensemble controllable. Then,*

1. *For each $\theta \in \mathbb{P}$, the pair $(A(\theta), B(\theta))$ is controllable.*
2. *For any finite number of parameters $\theta_1, \theta_2, \ldots, \theta_r \in \mathbb{P}$, the $m \times m$ polynomial matrices $R(\lambda, \theta_1), R(\lambda, \theta_2), \ldots, R(\lambda, \theta_r)$ are mutually left coprime.*
3. *For any finite number $r \geq\ m+1$ of distinct parameters $\theta_1, \theta_2, \ldots, \theta_r$, the spectra of $A(\theta)$ satisfies $\sigma(A(\theta_1)) \cap \sigma(A(\theta_2)) \cap \cdots \cap \sigma(A(\theta_r)) = \emptyset$.*
4. *For the case where $m = 1$, and for any pair of distinct parameter θ, $\theta' \in \mathbb{P}$, we have $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$.*

Proof. Let θ be fixed in $\mathbb{P}$, $x_d(\theta) \in \mathbb{R}^n$. For $\epsilon > 0$, by assumption of uniform ensemble controllability, there exists a finite time $t = T$ and a control signal $u \in L_1([0, T]; \mathbb{R}^m)$ such that

$$\sup_{\theta\in\mathbb{P}} \|x_d(\theta) - x(T, \theta)\| < \epsilon. \tag{4.6}$$

This implies

$$\|x_d(\theta) - x_T(\theta)\| < \epsilon. \tag{4.7}$$

Thus, $x_d(\theta)$ is in a neighborhood of $x_T(\theta)$. Now, since $\mathcal{R}(T)$ is a closed set, we have that $x_d(\theta) \in \mathcal{R}(T)$. Hence, the pair $(A(\theta), B(\theta))$ is controllable.

Secondly, we observe that uniform ensemble controllability of $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ implies uniform ensemble controllability of $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ for finitely many $\theta \in \mathbb{P}$. Consider (4.4), where $A_j(\theta) = A(\theta_j)$ and $B_j(\theta) = B(\theta_j)$, for all $i \in \{1, ..., r\}$. Then, we have that

$$\mathbb{A} = \begin{pmatrix} A(\theta_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A(\theta_r) \end{pmatrix} \text{and} \quad \mathbb{B} = \begin{pmatrix} B(\theta_1) \\ \vdots \\ B(\theta_r) \end{pmatrix}. \tag{4.8}$$

We observe that, each controllable pair $(A(\theta_i), B(\theta_i))$ can be associated to a non-singular $m \times m$ polynomial matrix $R(\lambda, \theta_i)$, for $i \in \{1, ..., r\}$. With this observation, we let $(\lambda I - A(\theta_i))^{-1}B(\theta_i) = K(\lambda, \theta_i)R(\lambda, \theta_i)^{-1}$, for each $i \in \{1, ..., r\}$. Then, using Theorem 10.2 in [5], we have that (4.8) is controllable if and only if the $m \times m$ polynomial matrices $R(\lambda, \theta_1), ..., R(\lambda, \theta_r)$ are mutually left coprime. This proves the second statement.

The controllability of (4.8) implies that there is at most m Jordan blocks in $\mathbb{A}$ for each eigenvalues of $\mathbb{A}$. This means, there exists no eigenvalues of $\mathbb{A}$ after m. Hence, for $r \geq m + 1$, we have $\sigma(A(\theta_1)) \cap \sigma(A(\theta_2)) \cap ... \cap \sigma(A(\theta_r)) = \emptyset$.

Taking the case where $m = 1$, it follows that the spectra of $A(\theta_1)$ and $A(\theta_2)$ are disjoint. $\square$

We now focus on discrete-time single-input scenarios.

Proposition 4.1.5. *A family $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ of discrete-time single-input systems is uniformly ensemble controllable if and only if, for all $\epsilon > 0$, and $x_d \in C(\mathbb{P}; \mathbb{R}^n)$, there exists a real-valued polynomial $\chi(y) \in \mathbb{R}[y]$ such that*

$$\sup_{\theta \in \mathbb{P}} \|x_d(\theta) - \chi(A(\theta))B(\theta)\| < \epsilon, \tag{4.9}$$

where $\chi(A(\theta))$ is an $n \times n$ matrix-valued continuous function in $\theta \in \mathbb{P}$ induced by the real-valued polynomial $\chi(y) \in \mathbb{R}[y]$ evaluated at $A(\theta)$ for all $\theta \in \mathbb{P}$.

Proof. Note that (2.4) can be written as

$$x(T, \theta) = \chi(A(\theta))B(\theta), \tag{4.10}$$

where

$$\chi(y) = \sum_{r=1}^{N} z_{N-r} y^r.$$

Thus, by assumption of uniformly ensemble controllability of discrete-time single-input systems, we have that

$$\sup_{\theta \in \mathbb{P}} \|x_d(\theta) - \chi(A(\theta))B(\theta)\| < \epsilon. \tag{4.11}$$

By the definition of uniform ensemble controllability, the converse is true. □

Assume

$$\mathcal{C}_{(A(\theta),B(\theta))} = (B(\theta), A(\theta)B(\theta), ..., A^{n-1}(\theta)B(\theta)) \tag{4.12}$$

is an $n \times n$ invertible controllability matrix, for all $\theta \in \mathbb{P}$. Then, for any $x_d \in C(\mathbb{P}; \mathbb{R}^n)$, we define a polynomial $\chi_\theta(y) \in \mathbb{R}[y]$ by

$$\chi_\theta(y) := (1, y, ..., y^{n-1})\mathcal{C}^{-1}_{(A(\theta),B(\theta))} x_d(\theta). \tag{4.13}$$

The next result focuses on the case where all the systems in $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ are controllable.

Proposition 4.1.6. *[7]: Assume the pair $(A(\theta), B(\theta))$ is controllable for all $\theta \in \mathbb{P}$.*

Then the following are equivalent:

1. *The ensemble* $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ *is uniformly ensemble controllable.*

2. *For any* $x_d \in C(\mathbb{P}; \mathbb{R}^n)$ *there exists a real-valued polynomial* $\chi(y) \in \mathbb{R}[y]$ *such that, for all* $\epsilon > 0$ *the relation*

$$\|(\chi_\theta - \chi)(A(\theta))B(\theta)\| < \epsilon \tag{4.14}$$

   holds, for all $\theta \in \mathbb{P}$.

3. *For any* $x_d \in C(\mathbb{P}; \mathbb{R}^n)$ *there exists a real-valued polynomial* $\chi(y) \in \mathbb{R}[y]$ *such that, for all* $\epsilon > 0$ *the relation*

$$\|\chi_\theta(A(\theta)) - \chi(A(\theta))\| < \epsilon \tag{4.15}$$

   holds, for all $\theta \in \mathbb{P}$.

   Assume that for all $\theta \in \mathbb{P}$*, the eigenvalues of* $A(\theta)$ *are distinct. Let*

$$W := \{(\lambda, \theta) \in \mathbb{C} \times \mathbb{P} \mid \det(\lambda I - A(\theta)) = 0\}.$$

   Then, any of the first three statements of the Proposition is equivalent to:

4. *For any* $x_d \in C(\mathbb{P}; \mathbb{R}^n)$ *there exists a real-valued polynomial* $\chi(y) \in \mathbb{R}[y]$ *such that, for all* $\epsilon > 0$

$$|\chi_\theta(\lambda) - \chi(\lambda)| < \epsilon, \tag{4.16}$$

for all $(\lambda, \theta) \in W$ *and for any* $x_d \in C(\mathbb{P}; \mathbb{R}^n)$.

Proof. Suppose the first statement of holds. Then, from equation (4.13), we have that $\chi_\theta(A(\theta))B(\theta) = x_d(\theta)$. Thus, by Proposition 4.1.5, we conclude that

$$\|(\chi_\theta - \chi)(A(\theta))B(\theta)\| < \epsilon. \tag{4.17}$$

The converse is true. Therefore, first and second statement are equivalent. Suppose third statement holds. Then, by Cauchy-Schwarz inequality, we have that

$$\|(\chi_\theta - \chi)(A(\theta))B(\theta)\| \leq \|(\chi_\theta - \chi)(A(\theta))\|\|B(\theta)\|.$$

Thus, since $B \in C(\mathbb{P}; \mathbb{R}^n)$, there exists a real number g such that $\|B(\theta)\| \leq g$. Therefore,

$$\|(\chi_\theta - \chi)(A(\theta))\| < \epsilon. \tag{4.18}$$

Hence, the third statement implies the second statement. Suppose the second statement holds. Let $\chi' = \chi_\theta - \chi$. Then,

$$\|\chi'(A(\theta))B(\theta)\| < \epsilon.$$

Let

$$\chi'(A(\theta)) = \mathcal{C}^{-1}_{(A(\theta),B(\theta))}\mathcal{C}_{(A(\theta),B(\theta))}\chi'(A(\theta)).$$

Then,

$$\begin{aligned}
\|\chi'(A(\theta))\| =&\|\mathcal{C}^{-1}_{(A(\theta),B(\theta))}\mathcal{C}_{(A(\theta),B(\theta))}\chi'(A(\theta))\|,\\
=&\|\mathcal{C}^{-1}_{(A(\theta),B(\theta))}(I, A(\theta), ..., A(\theta)^{n-1})(\chi'(A(\theta))B(\theta))\|,\\
\leq&\|\mathcal{C}^{-1}_{(A(\theta),B(\theta))}(I, A(\theta), ..., A(\theta)^{n-1})\|\|\chi'(A(\theta))B(\theta)\| < c\epsilon,
\end{aligned}$$

where

$$c = n\sup_{\theta\in\mathbb{P}}\|\mathcal{C}^{-1}_{(A(\theta),B(\theta))}\|\sup_{0\leq k\leq n-1}\|A(\theta)\|^k.$$

Thus,

$$\|\chi'(A(\theta))\| < \epsilon.$$

By substituting back $\chi' = \chi_\theta - \chi$, we have

$$\|(\chi_\theta - \chi)A(\theta)\| < \epsilon.$$

Finally, the equivalence between third and forth statement follows from Cayley-Hamilton Theorem. □

The proof of Theorem 4.1.1 depends on the so-called Mergelyan Theorem, which we state next; we refer the reader to [17] for a proof.

Theorem 4.1.7. *[17, Theorem 20.5] Suppose K is a compact set in $\mathbb{C}$ and complement of K is connected. Suppose further that f is continuous on K and analytic in the interior of K. Then, for all $\epsilon > 0$, there exists a polynomial h such that for all $z \in K$,*

$$|f(z) - h(z)| < \epsilon. \tag{4.19}$$

We are now in a position to give a proof of Theorem 4.1.1. We start with single-input systems and then generalize to multi-input scenarios.

Proof of Theorem 4.1.1. We begin with the discrete-time scenario.

Case 1 (single-input systems)

Given $\Sigma_{\mathrm{D}} = (\mathbb{P}, A, B)$, by the fourth statement of Theorem 4.1.4, we consider the set

$$W := \{(\lambda, \theta) \in \mathbb{C} \times \mathbb{P} \mid \det(\lambda I - A(\theta)) = 0\}.$$

We define a map $\pi : W \to \mathbb{C}$ to be

$$\pi(\lambda, \theta) = \lambda. \tag{4.20}$$

Clearly, the function (4.20) is continuous . Let $\{\lambda_1, ..., \lambda_n\}$ be the spectra of $A(\theta)$, where $\theta \in \mathbb{P}$. Then, each mapping $\lambda_i : \mathbb{P} \to \mathbb{C}$, for all $i \in \{1, ..., n\}$ is a continuous mapping and injective, by the third statement of Theorem 4.1.4. Hence, λ_i is a homeomorphism, for all $i \in \{1, ..., n\}$. Let $M = M_1 \cup ... \cup M_n$, where $M_i = \lambda_i(\mathbb{P})$, for all $i \in \{1, ..., n\}$. Then, M_i is homeomorphic to $\mathbb{P}$, for all $i \in \{1, ..., n\}$. Hence, the union M is homeomorphic to $\mathbb{P}$. Therefore, M is simply connected in $\mathbb{C}$ since $\mathbb{P}$ is simply connected in $\mathbb{R}$. This implies that, the complement of M is connected. Now, for all $x_d \in C(\mathbb{P}, \mathbb{R}^n)$, we consider (4.13) and define a map $f : M \to \mathbb{C}$ by

$$f(\lambda) = \chi_\theta(\lambda). \tag{4.21}$$

Clearly, f is analytic in the interior of M. Hence, by Theorem 4.1.7, for all

$\epsilon > 0$, there exists a polynomial $p(\lambda)$ such that

$$|f(\lambda) - p(\lambda)| < \epsilon \tag{4.22}$$

for all $z \in M$. Equivalently, we have that

$$|\chi_\theta(\lambda) - p(\lambda)| < \epsilon. \tag{4.23}$$

Now, since $p(\lambda)$ is complex-valued polynomial, its conjugate $p^\dagger(\lambda)$ will also satisfy Theorem 4.1.7. Thus, since $\chi_\theta(\lambda)$ is real-valued, we replace $p(\lambda)$ in (4.23) by $s(\lambda) = \frac{1}{2}(p(\lambda) + p^\dagger(\lambda))$. Hence, we have that, for all $\epsilon > 0$, there exists a real-valued polynomial $s(\lambda) \in \mathbb{R}[\lambda]$ such that

$$|\chi_\theta(\lambda) - s(\lambda)| < \epsilon. \tag{4.24}$$

From Proposition 4.1.6, we conclude that, $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ is uniform ensemble controllable. This ends the proof in discrete-time single-input case.

In the proceeding case, we will identify $\Sigma_{\mathrm{D}} = (\mathbb{P}, A, B)$ as multiple-input

- Case 2 (multi-input systems)

Given $\Sigma_{\mathrm{D}} = (\mathbb{P}, A, B)$, by the first and second statements of Theorem 4.1.1 and, since the Hermite indices are independent of $\theta \in \mathbb{P}$, without loss of generality, we assume they are constants. Under this assumption, it is well-known (for example in [1]) that, there exists $D \in C(\mathbb{P}; Gl_n(\mathbb{C}))$, where $Gl_n(\mathbb{C})$ is a general linear group of degree n whose entries are complex numbers, such that

$$(D(\theta)A(\theta)D^{-1}(\theta), D(\theta)B(\theta)) = (A_D(\theta), B_D),$$

where

$$A_D(\theta) = \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) & \dots & A_{1m}(\theta) \\ 0 & A_{22}(\theta) & \dots & A_{2m}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{mm}(\theta) \end{pmatrix} \quad \text{and} \quad B_D = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_m \end{pmatrix}, \tag{4.25}$$

with each $(A_{ii}(\theta), b_i) \in \mathbb{R}^{n_i \times n_i} \times \mathbb{R}^{n_i}$. Let $x_d = (x_{d_1}, ..., x_{d_m})^{\mathrm{T}}$, where $x_{d_i} \in C(\mathbb{P}; \mathbb{R}^{n_i})$. For any integer N, let $\mathcal{C}^N_{i(A_{ii}(\theta), b_i)}$ denote the $n_i \times N$ controllability matrix of the pair $(A_{ii}(\theta), b_i)$ and $\mathcal{C}^{Nm}_{(A_D(\theta), B_D)}$ be the controllability matrix of size $n \times Nm$ of the pair $(A_D(\theta), B_D)$, then $\mathcal{C}^{Nm}_{(A_D(\theta), B_D)}$ is a block-upper triangular matrix of the form

$$\mathcal{C}^{Nm}_{(A(\theta), B_D)} = \begin{pmatrix} \mathcal{C}^N_{1(A_{11}(\theta), b_1)} & \dots & \mathcal{C}^N_{1m(A_{1m}(\theta), b_1)} \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{C}^N_{m(A_{mm}(\theta), b_m)} \end{pmatrix}. \tag{4.26}$$

We observe that, for $u(T) = (\gamma(T), ..., \gamma(1))^{\mathrm{T}} \in \mathbb{R}^T$, the solution to a system induced by $\Sigma_{\mathrm{D}}(\mathbb{P}, A_{ii}, b_i)$ at time T will be of the form

$$x_i(T, \theta) = \sum_{r=0}^{T-1} b_i A^r_{ii}(\theta) \gamma(T - r).$$

This sum can be rewritten as

$$\sum_{r=0}^{T-1} b_i A_{ii}^r(\theta) u_{T-r} = [b_i \quad b_i A_{ii}(\theta) \quad \ldots \quad b_i A_{ii}^{T-1}(\theta)] \begin{bmatrix} \gamma(T) \\ \vdots \\ \gamma(1) \end{bmatrix},$$
$$= \mathcal{C}^T_{i(A_{ii}(\theta),b_i)} u(T).$$

Hence, we have that

$$x_i(T,\theta) = \mathcal{C}^T_{i(A_{ii}(\theta),b_i)} u(T). \tag{4.27}$$

By Proposition 4.1.5, we see that (4.27) can also be expressed as

$$\chi(A_{ii}(\theta))b_i = \mathcal{C}^T_{i(A_{ii}(\theta),b_i)} u(T) \tag{4.28}$$

where $\theta \in \mathbb{P}$, for all $i \in \{1, ..., m\}$. Now, since m is finite, we consider the case where $m = 2$. The general case follows from the same argument. We consider the equation

$$x_2(t+1,\theta) = A_{22}(\theta)x_2(t,\theta) + b_2 u(t). \tag{4.29}$$

We have already proved, in case 1, that systems of the form (4.29) is uniform ensemble controllable . This means that, for all $x_d \in C(\mathbb{P};\mathbb{R}^n)$ and $\epsilon > 0$, there exists a sequence of control signals $u(T)$ and a real-valued polynomial $\chi(z) \in \mathbb{R}[z]$ be such that,

$$\sup_{\theta\in\mathbb{P}} \|\mathcal{C}^T_{2(A_{22}(\theta),b_2)} u(T) - x_{d_2}(\theta)\| < \epsilon, \tag{4.30}$$

where

$$\chi(A_{22}(\theta))b_2 = \mathcal{C}^T_{2(A_{22}(\theta),b_2)}u(T). \tag{4.31}$$

We repeat similar argument to the system

$$x_1(t+1,\theta) = A_{11}(\theta)x_1(t,\theta) + b_1 u(t) \tag{4.32}$$

but consider family of desired states of the form $x_{d_1}(\theta) - \mathcal{C}^T_{12(A_{12}(\theta),b_1)}u(T)$. Then, by assumption there exists a sequence of control signals $v(T) = (\eta(T), ..., \eta(1))^{\mathrm{T}}$ such that

$$\sup_{\theta\in\mathbb{P}} \|\mathcal{C}^T_{1(A_{11}(\theta),b_1)}v(T) - x_{d_1}(\theta) + \mathcal{C}^T_{12(A_{12}(\theta),b_1)}u(T)\| < \epsilon.$$

Thus,

$$\sup_{\theta\in\mathbb{P}} \|\mathcal{C}^T_{1(A_{11}(\theta),b_1)}v(T) + \mathcal{C}^T_{12(A_{12}(\theta),b_1)}u(T) - x_{d_1}(\theta)\| < \epsilon. \tag{4.33}$$

From equations (4.30) and (4.34), we conclude that

$$\sup_{\theta\in\mathbb{P}} \|\mathcal{C}^{T2}_{(A(\theta),B(\theta))}\xi(T) - x_d(\theta)\| < \epsilon, \tag{4.34}$$

where

$$\mathcal{C}^{T2}_{(A(\theta),B(\theta))} = \begin{pmatrix} \mathcal{C}^T_{1(A_{11}(\theta),b_1)} & \mathcal{C}^T_{12(A_{12}(\theta),b_1)} \\ 0 & \mathcal{C}^T_{2(A_{22}(\theta),b_2)} \end{pmatrix}, \quad \xi(T) = (v(T), u(T))^{\mathrm{T}},$$

and

$$x_d(\theta) = (x_{d_1}(\theta), x_{d_2}(\theta))^{\mathrm{T}}.$$

The general case m follows by the same reasoning. This means the ensemble $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ is uniform ensemble controllable.

We now give a proof for the continuous-time case.

- Case 3 (continuous-time system)

Given $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$, one can obtain the discrete-time ensemble,

$$x(t+1, \theta) = G(\theta)x(t, \theta) + H(\theta)u(t), \tag{4.35}$$

where

$$G(\theta) = e^{\omega A(\theta)} \quad \text{and} \quad H(\theta) = \Big(\int_0^\omega e^{sA(\theta)} ds \Big) B(\theta), \tag{4.36}$$

by sampling with a small rate $\omega > 0$. We will show that, this sampled ensemble satisfies the conditions in Theorem 4.1.1.

For sufficiently small sampling rate ω, if a continuous-time linear system induced by $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is controllable, by sampling the control signal, we obtain a sequence of control signals that will steer the trajectories of the discrete-time system that agrees with the sampled points of the continuous-time system induced by $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$. With this observation, we can conclude that, controllability of continuous-time systems implies controllability of discrete-time systems; therefore, for all $\theta \in \mathbb{P}$, the pair $(G(\theta), H(\theta))$ is controllable.

Let $D(\theta) = \int_0^\omega e^{sA(\theta)} ds$. Then, we observe that, $\det(e^{sA(\theta)}) = e^{\mathrm{Tr}(sA(\theta))} \neq 0$ Therefore, we have that, $D \in C(\mathbb{P}; Gl_n)$. This implies that

$$D^{-1}(\theta)G(\theta)D(\theta) = \big(\int_0^\omega e^{sA(\theta)} ds \big)^{-1} e^{\omega A(\theta)} \big(\int_0^\omega e^{sA(\theta)} ds \big) = e^{\omega A(\theta)}. \tag{4.37}$$

We also see that $D^{-1}(\theta)H(\theta) = B(\theta)$. The second statement of Theorem 4.1.1 implies that the Hermite indices are constants. Hence, they are invariant under non-singular transformation [20]. This implies that, the Hermite indices for the pair $(e^{\omega A(\theta)}, B(\theta))$ and $(G(\theta), H(\theta))$ are the same. We also have that the pair $(e^{\omega A(\theta)}, B(\theta))$ and $(A(\theta), B(\theta))$ are the same. Therefore, the Hermite indices will be the same for $(A(\theta), B(\theta))$ and $(G(\theta), H(\theta))$.

Now, since the exponential function is one-to-one, it follows that, $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$ if and only if $\sigma(e^{\omega A(\theta)}) \cap \sigma(e^{\omega A(\theta')}) = \emptyset$, for all $\theta \neq \theta' \in \mathbb{P}$. Also, $A(\theta)$ has algebraic multiplicity of one if and only if $e^{\omega A(\theta)}$ has algebraic multiplicity of one has for all $\theta \in \mathbb{P}$. Therefore, statement three and four of Theorem 4.1.1 follow.

Under sampling, discrete-time and continuous-time system agree on the same sampled points. Hence by interpolating over the sampled points, we obtain a piecewise-continuous control signal that steers the trajectories to the family of continuous-time linear systems. This complete the proof. $\square$

Chapter 5

Uniform Null Ensemble Controllability for One-parameter Linear Systems Using Constrained Control Signals in the Unit Interval.

5.1 Uniform Null Ensemble Controllability.

In this chapter, we study a special scenario of uniform ensemble controllability studied in the previous chapter. We investigate the possibility to steer the trajectories of an ensemble to the origin using constrained control signals. We call this notion *uniform null ensemble controllability*. We focus our attention on single-input systems with constrained control set $[0, 1]$ throughout. Let $\partial\mathcal{R}$ denote the boundary of $\mathcal{R}$. We first recall the notion of null controllability for classical linear control systems in [2]. To this end, let $\Sigma = (A, B)$ be linear control system in continuous-time. Then, Σ is null-controllable if there exists an open set $\mathbb{V} \subset \mathbb{R}^n$ containing the origin such that, for any $x_0 \neq 0 \in \mathbb{V}$, there exists a control signal u that steers $x(0) = x_0$ to the origin in finite time. We proceed to give a definition of uniform null ensemble controllability using control signals in $[0, 1]$.

Definition 5.1.1. *Let $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ be an ensemble of continuous-time single-input systems. Then, $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is uniformly null ensemble controllable if and only if there exists an open set $\mathbb{V} \subset \mathbb{R}^n$ containing the origin, a finite time $T > 0$, and a control signal $u \in L_1([0,T];[0,1])$ such that, for all $x(0,\theta) \neq 0 \in \mathbb{V}$, u steers $x(0,\theta)$ to $x(T,\theta) \in \mathbb{V}$, where*

$$\sup_{\theta \in \mathbb{P}} \|x(T,\theta)\| < \epsilon. \tag{5.1}$$

The same definition holds for the discrete-time single-input scenario. We proceed to state the main result of this chapter.

Theorem 5.1.2. *An ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ of continuous-time single-input systems is uniformly null ensemble controllable if the following condition holds:*

1. *The eigenvalues of $A(\theta)$ has nonzero imaginary part, for all $\theta \in \mathbb{P}$.*

2. *The pair $(A(\theta), B(\theta))$ is null controllable, for all $\theta \in \mathbb{P}$.*

3. *$\sigma(A(\theta)) \cap \sigma(A(\theta^{'})) = \emptyset$, for any pair of distinct parameter $\theta, \theta^{'} \in \mathbb{P}$.*

4. *The eigenvalues of $A(\theta)$ have algebraic multiplicity of one, for each θ.*

Before we proceed with the proof of Theorem 5.1.2, we also state a necessary condition for uniform null ensemble controllability. For this, we need the next result.

Lemma 5.1.3. *Let $x(0,\theta) = 0$, for all $\theta \in \mathbb{P}$. Assume the parameter-dependent matrix $A(\theta) \in \mathbb{R}^{n \times n}$ induced by the ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ has a real eigenvalue λ for some $\theta \in \mathbb{P}$. Then, $0 \in \partial\mathcal{R}$.*

Proof. Suppose the system matrix $A(\theta)$ has a real eigenvalue λ, for some $\theta \in \mathbb{P}$. Then, λ will also be an eigenvalue of the matrix $A(\theta)^{\mathrm{T}}$. This means, there exists a vector

$v \in \mathbb{R}^n$ such that $A(\theta)^{\mathrm{T}} v = \lambda v$. Let $\phi(t,\theta) = v^{\mathrm{T}} x(t,\theta)$, where $x(t,\theta)$ is the trajectory of the system at time t induced by $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$. Then,

$$\frac{\partial \phi}{\partial t}(t,\theta) = v^{\mathrm{T}}(A(\theta)x(t,\theta) + B(\theta)u(t)), \tag{5.2}$$

$$\frac{\partial \phi}{\partial t}(t,\theta) = v^{\mathrm{T}}A(\theta)x(t,\theta) + v^{\mathrm{T}}B(\theta)u(t), \tag{5.3}$$

and

$$\frac{\partial \phi}{\partial t}(t,\theta) = \lambda\phi(t,\theta) + v^{\mathrm{T}}B(\theta)u(t). \tag{5.4}$$

Now, since $\phi(0,\theta) = 0$, by applying the variation of constants formula on (5.4), we get

$$\phi(T,\theta) = \int_0^T e^{\lambda(t-s)} v^{\mathrm{T}} B(\theta)u(s)ds. \tag{5.5}$$

One can observe that $v^{\mathrm{T}}B(\theta) \in \mathbb{R}$. Without loss of generality, we let $v^{\mathrm{T}}B(\theta) \geq 0$. Then since $u(T) \geq 0$, for all $T \geq 0$, we obtain the following relations:

$$\text{if} \quad v^{\mathrm{T}}B(\theta) > 0 \quad \text{then} \quad \phi(T,\theta) \geq 0,$$

and

$$\text{if} \quad v^{\mathrm{T}}B(\theta) = 0 \quad \text{then} \quad \phi(T,\theta) = 0.$$

This means that, if $x(T,\theta) \in \mathcal{R}(T)$ then, for all $T \geq 0$, we have $v^{\mathrm{T}}x(T,\theta) \geq 0$, whenever $v^{\mathrm{T}}B(\theta) > 0$, or if $x(T,\theta) \in \mathcal{R}(T)$ then, for all $T \geq 0$, we have $v^{\mathrm{T}}x(T,\theta) = 0$, whenever $v^{\mathrm{T}}B(\theta) = 0$. By choosing $u := 0$, this implies that $x(T,\theta) = 0 \in \mathcal{R}(T)$ for all $T > 0$ and $x(T, \theta) = 0$ satisfies both relations, for all $T \geq 0$. Now, since $\mathcal{R}(T)$ is compact and convex [13] it implies that $\mathcal{R}(T)$ is closed, convex, and bounded, for all $T \geq 0$. Hence there exists a supporting hyperplane to $\mathcal{R}(T)$. Now using the relation between the normal vector of the supporting hyperplane and the states in the reachable set, a supporting hyperplane will pass through 0. Hence, $0 \in \partial\mathcal{R}(T)$, for all $T \geq 0$. Hence, $0 \in \partial\mathcal{R}$. □

We proceed to give a necessary condition for uniform null ensemble controllability of single-input systems.

Theorem 5.1.4. *Assume a family of continuous-time single-input systems* $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ *is uniformly null ensemble controllable. Then, the following holds:*

1. *The pair* $(A(\theta), B(\theta))$ *is null controllable, for all* $\theta \in \mathbb{P}$.

2. *We have* $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$, *for any pair of distinct parameter* $\theta, \theta' \in \mathbb{P}$.

3. *All eigenvalues of* $A(\theta)$ *have nonzero imaginary part, for all* $\theta \in \mathbb{P}$.

Proof. By assumption of uniform null ensemble controllability, for all $\epsilon > 0$ and $x(0, \theta) \neq 0 \in \mathbb{V}$, we have that $x(T, \theta) \in \mathbb{V}$, where

$$\sup_{\theta \in \mathbb{P}} \|x(T, \theta)\| < \epsilon. \tag{5.6}$$

This implies

$$\|x(T, \theta)\| < \epsilon, \tag{5.7}$$

for all $\theta \in \mathbb{P}$. Hence, $x(T, \theta) = 0$, for all $\theta \in \mathbb{P}$. Therefore, the first statement holds. The proof of the second statement follows from a similar argument as the one in the proof of the forth statement in Theorem 4.1.4. Lastly, suppose there exists $\theta \in \mathbb{P}$ such that $A(\theta)$ has real eigenvalue. Then by lemma 5.1.3, we have that $0 \in \partial\mathcal{R}(T)$, for all time $T \geq 0$. This implies that, there exists no open set $\mathbb{V} \subset \mathbb{R}^n$ containing the origin such that $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$ is uniformly null ensemble controllable. This is a contradiction. $\square$

We next give a proposition about uniform ensemble controllability of interconnection of systems.

Let $(A_{i,j}, B_{i,j}) \in C(\mathbb{P}; \mathbb{R}^{n_i \times n_j}) \times C(\mathbb{P}; \mathbb{R}^{n_i})$, where $1 \leq i, j \leq N$, $\bar{n} = \sum_{i=1}^{N} n_i$ and $M = \sum_{i=1}^{M} 1$. Let now $(\mathcal{A}, \mathcal{B}) \in C(\mathbb{P}; \mathbb{R}^{\bar{n} \times \bar{n}}) \times C(\mathbb{P}; \mathbb{R}^{\bar{n} \times M})$ be given by

$$\mathcal{A}(\theta) = \begin{pmatrix} A_{1,1}(\theta) & \dots & A_{1,N}(\theta) \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{N,N}(\theta) \end{pmatrix} \quad \text{and} \quad \mathcal{B}(\theta) = \begin{pmatrix} B_{1,1}(\theta) & \dots & B_{1,M}(\theta) \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{N,M}(\theta) \end{pmatrix}, \tag{5.8}$$

We have the following result.

Proposition 5.1.5. *The ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, \mathcal{A}, \mathcal{B})$ is uniformly null ensemble controllable if and only if each $\Sigma_{\mathrm{C}}(\mathbb{P}, A_{i,j}, B_{i,j})$, $i, j \in \{1, ...N\}$, is uniformly null ensemble controllable.*

Proof. Suppose $\Sigma_{\mathrm{C}}(\mathbb{P}, \mathcal{A}, \mathcal{B})$ is uniform null ensemble controllable. Then, each ensemble $\Sigma_{\mathrm{C}}(\mathbb{P}, A_{i,j}, B_{i,j})$, $i, j \in \{1, ...N\}$ is uniform null ensemble controllable. We prove the converse. For simplicity, we focus on the case when $N = 2$; the general case follows by the exact same reasoning. By construction, we have

$$\frac{\partial x_1}{\partial t}(t, \theta) = A_{1,1}(\theta)x_1(t, \theta) + A_{1,2}(\theta)x_2(t, \theta) + B_{1,1}(\theta)u_1(t) + B_{1,2}(\theta)u_2(t), \tag{5.9}$$

and

$$\frac{\partial x}{\partial t}(t,\theta) = A_{2,2,}(\theta)x_2(t,\theta) + B_{2,2}(\theta)u_2(t). \tag{5.10}$$

By assumption of uniform null ensemble controllability of $\Sigma_{\mathrm{C}}(\mathbb{P}, A_{2,2}, B_{2,2})$, it follows that, there exists an open set $\mathbb{V}_2$ containing the origin such that, for all $\epsilon > 0$, and $x_2(0,\theta) \neq 0 \in \mathbb{V}_2$, we have that $x_2(T,\theta) \in \mathbb{V}_2$ satisfies the relation

$$\sup_{\theta \in \mathbb{P}} \|x_2(T,\theta)\| < \epsilon, \tag{5.11}$$

where

$$x_2(T,\theta) = e^{A_{22}(\theta)T}x_2(0,\theta) + \int_0^T e^{(T-s)A_{2,2}(\theta)}B_{2,2}(\theta)u_2(s)ds.$$

By substituting $x_2(t,\theta)$ for time $t \leq T$ into (5.9), we obtain

$$\begin{aligned}\frac{\partial x_1}{\partial t}(t,\theta) =& A_{1,1}(\theta)x_1(t,\theta) + A_{1,2}(\theta)\Big(e^{A_{22}(\theta)T}x_2(0,\theta) \\ &+ \int_0^t e^{(t-\nu)A_{2,2}(\theta)}B_{2,2}(\theta)u_2(\nu)d\nu\Big) + B_{1,1}(\theta)u_1(t)B_{1,2}(\theta)u_2(t).\end{aligned}$$

Thus, by applying the constants of variation formula, we have that

$$x_1(T,\theta) = e^{A_{1,1}(\theta)T}x_1(0,\theta) + \int_0^T e^{(T-s)A_{1,1}(\theta)}B_{1,1}(\theta)u_1(s)ds + z(T,\theta), \tag{5.12}$$

where

$$\begin{aligned}z(T,\theta) =& \int_0^T e^{(T-s)A_{1,1}(\theta)}\big(A_{1,2}(\theta)e^{A_{22}(\theta)T}x_2(0,\theta) \\ &+ A_{1,2}(\theta)\int_0^s e^{(s-\nu)A_{2,2}(\theta)}B_{2,2}(\theta)u_2(\nu)d\nu + B_{1,2}(\theta)u_2(s)\Big)ds.\end{aligned}$$

We observe that, by rearranging (5.12) we obtain

$$x_1(T,\theta) - z(T,\theta) = e^{A_{11}(\theta)T}x_1(0,\theta) + \int_0^T e^{(T-s)A_{1,1}(\theta)}B_{1,1}(\theta)u_1(s)ds. \tag{5.13}$$

Let

$$\bar{x}_1(T,\theta) = e^{A_{11}(\theta)T}x_1(0,\theta) + \int_0^T e^{(T-s)A_{1,1}(\theta)}B_{1,1}(\theta)u_2(s)ds.$$

Then, $\bar{x}_1(.,\theta)$ is the solution to a system induced by $\Sigma_{\mathrm{C}}(\mathbb{P}, A_{1,1}, B_{1,1})$. By assumption of uniform null ensemble controllability on $\Sigma_{\mathrm{C}}(\mathbb{P}, A_{1,1}, B_{1,1})$, there exists an open set $\mathbb{V}_1$ containing the origin such that, for all $\epsilon > 0$ and $\bar{x}_1(0,\theta) + z_0(\theta) = x_1(0,\theta) \in \mathbb{V}_1$, we have that $x_1(T,\theta) = \bar{x}_1(T,\theta) + z(T,\theta) \in \mathbb{V}_1$ satisfies the relation

$$\sup_{\theta\in\mathbb{P}} \|x_1(T,\theta)\| < \epsilon. \tag{5.14}$$

Let $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$. Then, $\mathbb{V}$ is an open set containing the origin such that, for all $\epsilon > 0$ and $x(0,\theta) = (x_1(0,\theta), x_2(0,\theta))^{\mathrm{T}} \in \mathbb{V}$, we have that $x(T,\theta) = (x_1(T,\theta), x_2(T,\theta))^{\mathrm{T}} \in \mathbb{V}$ satisfies the relation

$$\sup_{\theta\in\mathbb{P}} \|x(T,\theta)\| < \epsilon. \tag{5.15}$$

We conclude that $\Sigma_{\mathrm{C}}(\mathbb{P}, \mathcal{A}, \mathcal{B})$ given by (5.8) is uniformly null ensemble controllable for $N = 2$. This ends the proof. $\square$

We now go back to proving Theorem 5.1.2. Our proof relies on a corresponding discrete-time scenario, which we present next.

Proposition 5.1.6. *An ensemble $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ of discrete-time single-input systems is uniformly null ensemble controllable if and only if there exists an open set $\mathbb{V} \subset \mathbb{R}^n$ containing the origin, a finite time $T > 0$ and a real-valued polynomial $\varphi(z) \in \mathbb{R}[z]$*

such that, for all $x(0,\theta) \neq 0 \in \mathbb{V}$, we have that $x(T,\theta) \in \mathbb{V}$ satisfies that relation

$$\sup_{\theta \in \mathbb{P}} \|\varphi(A(\theta))b(\theta)\| < \epsilon. \tag{5.16}$$

The proof is the same as the proof of Proposition 4.1.5.

We proceed to next proposition.

Proposition 5.1.7. *Assume $(A(\theta), B(\theta))$ is null controllable, for all $\theta \in \mathbb{P}$. Then, the following are equivalent.*

1. *The ensemble $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ is uniform null ensemble controllable.*
2. *For all $\epsilon > 0$, there exists a polynomial $\chi(y) \in \mathbb{R}[y]$ such that, $\|\chi(A(\theta))B(\theta)\| < \epsilon$.*
3. *For all $\epsilon > 0$, there exists a polynomial $\chi(y) \in \mathbb{R}[y]$ such that, $\|\chi(A(\theta))\| < \epsilon$. Moreover, if for each $\theta \in \mathbb{P}$, the eigenvalues of $A(\theta)$ are distinct.*

$$W := \{(z,\theta) \in C(\mathbb{P};\mathbb{C}) \times \mathbb{P} \mid \det(zI - A(\theta)) = 0\} \tag{5.17}$$

Then, any of the first to third statements are all equivalent to:

4. *For all $\epsilon > 0$, there exists a polynomial $\chi(y) \in \mathbb{R}[y]$ such that $|\chi(z)| < \epsilon$, for all $(z,\theta) \in W$.*

The proof follows similarly to Proposition 4.1.6.

We now give a prove of Theorem 5.1.2.

Proof of Theorem 5.1.2. We give the proof in two cases. We first prove for the case of discrete-time single-input and the continuous-time case is reduce to the discrete-time case by sampling over a sufficiently small rate.

Case 1 (discrete-time systems): Following the same argument in case 1 of the proof of Theorem 4.1.1, we let

$$f(z) = u_\theta(z) \tag{5.18}$$

for all $z \in M$, where

$$u_\theta(z) = (1, z, \dots, z^{n-1}) C^{-1}_{(A(\theta),B(\theta))} v, \tag{5.19}$$

with $v \in \mathbb{R}^n$ such that $\|v\| < \epsilon$. Then, $f(z) \in \mathbb{R}[z]$, for all $z \in M$. Hence, f is continuous and analytic on M. Also, for all $\epsilon > 0$ we have that

$$|f(z)| = |u_\theta(z)| < \epsilon \tag{5.20}$$

for all $z \in M$, since $\|v\| < \epsilon$ for all $\theta \in \mathbb{P}$. By Theorem 4.1.7, we have that $p(z) = 0$, for all $z \in M$. Now, since f is real-valued, we replace $p(z)$ by its real part $R(p(z))$. Hence, for all $\epsilon > 0$, there exists a real-valued polynomial $R(p(z)) \in \mathbb{R}[z]$ such that

$$|R(p(z))| < \epsilon, \tag{5.21}$$

for all $z \in M$ and for all $\theta \in \mathbb{P}$. This implies that (5.21) holds for all $(z, \theta) \in W$. Hence, by Proposition 5.1.7, we have that $\Sigma_{\mathrm{D}}(\mathbb{P}, A, B)$ is uniformly null ensemble controllable. This completes the proof for the discrete-time case.

We now proceed to the countinuous-time scenario

Case 2 (continuous-time): Given $\Sigma_{\mathrm{C}}(\mathbb{P}, A, B)$, we can easily verify for $G(\theta) = e^{\omega A(\theta)}$, if $A(\theta)$ has nonzero imaginary part then $G(\theta)$ has nonzero imaginary part. The rest follows exactly in the proof of Theorem 4.1.1. This ends the proof. $\square$

Chapter 6

Conclusions and future work

6.1 Summary

In this thesis, we have investigated the problem of controlling a continuum of linear systems in continuous-time, and discrete-time scenarios. In particular, we reviewed the notion of L_2-ensemble controllability of family of time-varying systems introduced in [11]. We stated and proved the necessary and sufficient condition for L_2-ensemble controllability presented originally in [11]. We also reviewed the notion of uniform ensemble controllability of family of one-parameter time-invariant linear systems in [7]. We also proved a necessary, as well as a sufficient condition, for uniform ensemble controllability [7]. In contrast to [7], we looked at a different problem of uniform ensemble controllability for the case where the control set is constrained to be $[0, 1]$. We called this notion uniform *null* ensemble controllability for one-parameter time-invariant linear single-input systems using constrained control signals in the unit interval. We gave a necessary as well as a sufficient condition for uniform *null* ensemble controllability for one-parameter family of time-invariant of linear systems. Finally, we showed that in discrete-time scenarios, uniform *null* ensemble controllability of family of one-parameter time-invariant linear systems is equivalent to a polynomial approximation problem.

6.2 Future research directions

In the future, one can try to examine the constrained control situations in the case of L_2-ensemble controllability. Furthermore, one can also look at ensemble controllability using a more general constrained control set than the unit interval.

Chapter 7

Appendix

7.1 Background material.

Definition 7.1.1. *Let $H_0 = L_2(M; \mathbb{R}^{n\times m})$ be a vector space of all those matrix-valued functions f whose ijth entries $f_{ij}(t,\theta)$, $i = \{1,...,n\}$, $j = \{1,...,m\}$, are complex-valued measurable function defined on M. We define an inner product $\langle , \rangle : H_0 \times H_0 \to \mathbb{R}$ to be*

$$\langle f, g\rangle = tr \int_{\mathbb{P}} \int_0^T f(t,\theta) g^{\dagger}(t,\theta) dt d\theta,$$

for all $f,\ g \in H_0$ and its corresponding norm

$$\|f\|^2 = \int_{\mathbb{P}} \int_0^T \|f(t,\theta)\|^2 dt d\theta. \tag{7.1}$$

Then, H_0 is a Hilbert space.

Theorem 7.1.2. *Let $\Sigma_{\mathrm{C}}(M, A, B)$ be an ensemble of continuous-time varying linear systems and suppose $(A, B) \in L_\infty(M; \mathbb{R}^{n\times n}) \times L_2(M; \mathbb{R}^{n\times m})$. Let $\Phi(t, 0, \theta)$ be the* *transition matrix induced by* $\Sigma_{\mathrm{C}}(M, A, B)$ *such that, for all* $\theta \in \mathbb{P}$, $\Phi(t, 0, \theta)$ *satisfies*

$$\frac{\partial \Phi}{\partial t}(t, 0, \theta) = A(t, \theta)\Phi(t, 0, \theta), \quad \Phi(0, 0, \theta) = I.$$

Then, the operator $L : H_1 \to H_2$ *defined by*

$$(Lu)(\theta) = \int_0^T \Phi(0, \tau, \theta)B(\tau, \theta)u(\tau)d\tau \tag{7.2}$$

is compact.

To be able to proof this Theorem, we will state and prove two propositions. We begin by proving that L defined in (7.2) is a bounded linear operator.

Proposition 7.1.3. *Assume the operator* L *defined in* (7.2). *Then,* $L \in B(H_1; H_2)$.

Proof. Clearly L is linear. We proceed to show that it is bounded.

$$\begin{aligned}
\|Lu\|_{H_2}^2 =& \int_{\mathbb{P}} \|(Lu)(\theta)\|^2 d\theta, \\
=& \int_{\mathbb{P}} \| \int_0^T f(t, \theta)u(t)dt\|^2 d\theta, \\
\leq& \int_{\mathbb{P}} \big(\int_0^T \|f(t, \theta)\|^2 dt\big)\big(\int_0^T \|u(t)\|^2 dt\big) d\theta, \\
=& \int_{\mathbb{P}} \big(\int_0^T \|f(t, \theta)\|^2 dt\big) d\theta \|u\|_{H_1}^2, \\
=& \|f\|_{H_0}^2 \|u\|_{H_1}^2.
\end{aligned}$$

This implies that,

$$\|L\| \leq \|f\|_{H_0} < \infty. \tag{7.3}$$

□

Next, we will show that the kernel of L is bounded in H_0.

Proposition 7.1.4. *Assume the operator L defined in (7.2). Let $k(t,\theta) = \Phi(0,t,\theta)B(t,\theta)$. Then, $k \in H_0$.*

Proof.

$$\begin{aligned}
\|k\|^2 =& \int_{\mathbb{P}} \int_0^T \|\Phi(0,t,\theta)B(t,\theta)\|^2 dt d\theta, \\
\leq& \int_{\mathbb{P}} \big(\int_0^T \|\Phi(t,\theta)\|^2 dt\big) \times \big(\int_0^T \|B(t,\theta)\|^2 dt\big) d\theta, \\
\leq& TK^2 \int_{\mathbb{P}} \int_0^T \|B(t,\theta)\|^2 dt d\theta, \\
=& TK^2 \|B(t,\theta)\|^2_{H_0} < \infty.
\end{aligned}$$

Therefore, $k \in H_0$. □

We now give a proof of Theorem 7.1.2.

Proof of Theorem 7.1.2. Now, since H_0 is non-empty Hilbert space, it has an orthonormal basis [8, pp. 168]. Let $\{\varepsilon_j\}_{j\geq 1}$ be an orthonormal basis in H_0. Then, it is well-known for example in [8] that $k \in H_0$ can be written as

$$k(t,\theta) = \sum_{j=1}^{\infty} \langle k(t,\theta), \varepsilon_j(t,\theta)\rangle \varepsilon_j(t,\theta). \tag{7.4}$$

Let

$$k_n(t,\theta) = \sum_{j=1}^{n} \langle k(t,\theta), \varepsilon_j(t,\theta)\rangle \varepsilon_j(t,\theta). \tag{7.5}$$

Then, $k_n \in H_0$ and has finite range. For $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that for $n > N$

we have

$$\|k_n - k\|_{H_0} < \epsilon. \tag{7.6}$$

We define the operator $L_n : H_1 \to H_2$ by

$$(L_n u)(\theta) = \int_0^T k_n(\tau, \theta) u(\tau) d\tau. \tag{7.7}$$

Then, L_n is a bounded linear operator with finite range hence L_n is compact for all $n \in \mathbb{N}$ [8]. From (7.3), we conclude that

$$\|L - L_n\| \leq \|k - k_n\|_{H_0} < \epsilon. \tag{7.8}$$

Therefore, since L is the limit of sequence of compact operators the L is compact [8, pp. 408], . □

Proposition 7.1.5. *Compact operators are not invertible on an infinite dimensional space X.*

We will proof this using contradiction argument.

Proof. Let L be a compact operator and suppose L is invertible on X, then there exist a unique compact operator S such that $LS = I$ where I is the identity operator. This implies that I is compact on X which is a contradiction. Therefore, our assumption of L is invertible is false. □

Theorem 7.1.6. *Riesz-Fischer Theorem: Let $\{\phi_n\}$ be an orthonormal system in $L_2([a, b])$ and let $\{c_n\}$ be a sequence of complex number such that $\sum_{n=0}^{\infty} |c_n|^2 < \infty$, then there exist a function $f \in L^2([a, b])$ with the following:*

1. $\sum_{n=0}^{\infty} |c_n|^2 = \|f(x)\|^2$.

2. $c_k = \langle f, \phi_n \rangle$ *for all* $k \in \{0, 1, 2, 3...\}$.

Proof. 1. We define a sequence $(s_n)_{n=0}^{\infty}$ as follows:

$$s_n(x) = \sum_{k=0}^{n} c_k \phi_k(x). \tag{7.9}$$

We show that $\{s_n\}$ is a cauchy sequence in $L_2([a, b])$. For all $\epsilon > 0$, there exist an $N \in \mathbb{N}$ such that for $m, n \geq N$, $m > n$, we have that:

$$\begin{aligned}
\|s_n(x) - s_m(x)\| &= \| \sum_{k=0}^{m} c_k \phi_k(x) - \sum_{k=0}^{n} c_k \phi_k(x) \|, \\
&= \| \sum_{k=n+1}^{m} c_k \phi_k(x) \|, \\
&= \sum_{k=n+1}^{m} \|c_k \phi_k(x)\|, \\
&= \sum_{k=n+1}^{m} |c_k| < \epsilon.
\end{aligned}$$

Now, since $L^2([a, b])$ is a complete space, there exist an $f \in L^2([a, b])$ such that

$$\lim_{n \to \infty} \|f(x) - s_n(x)\| = 0 \tag{7.10}$$

Now, from the proof of Bessel identity in [8] , we have

$$\sum_{k=0}^{n} |c_k|^2 + \|f(x) - s_n(x)\|^2 = \|f(x)\|^2,$$

thus, as $n \to \infty$, we get that

$$\sum_{n=0}^{\infty} |c_n|^2 = \|f(x)\|^2. \tag{7.11}$$

2. Now for $n \in \{0, 1, 2...\}$, we observe that

$$\langle s_n(x), \phi_k(x) \rangle = \sum_{j=0}^{n} c_j \langle \phi_j(x), \phi_k(x) \rangle = c_k. \tag{7.12}$$

This means that

$$\begin{aligned}
|c_k - \langle f(x), \phi_k(x) \rangle| =& |\langle s_n(x), \phi_k(x) \rangle - \langle f(x), \phi_k(x) \rangle| \\
=& |\langle s_n(x) - f(x), \phi_k(x) \rangle| \\
\leq& \|s_n(x) - f(x)\| \|\phi_k(x)\| \\
=& \|s_n(x) - f(x)\|.
\end{aligned}$$

Taking the limit as $n \to \infty$ implies that $\|s_n(x) - f(x)\| < \epsilon$, hence

$$|c_k - \langle f(x), \phi_k(x) \rangle| < \epsilon, \tag{7.13}$$

thus (2) follows. $\square$

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