Geometry of Dirac Operators

By

Babak Beheshti Vadeqan

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Abstract

Let M be a compact, oriented, even dimensional Riemannian manifold and let S be a Clifford bundle over M with Dirac operator D. Then

$$\text{Atiyah Singer:}\quad \operatorname{Ind} \mathsf{D} = \int_M \hat{\mathcal{A}}(TM) \wedge \operatorname{ch}(\mathcal{V})$$

where $\mathcal{V} = \operatorname{Hom}_{\mathbb{C}l(TM)}(\$, S)$.

We prove the above statement with the means of the heat kernel of the heat semigroup e^{-tD^2}.

The first outstanding result is the McKean-Singer theorem that describes the index in terms of the supertrace of the heat kernel. The trace of heat kernel is obtained from local geometric information. Moreover, if we use the asymptotic expansion of the kernel we will see that in the computation of the index only one term matters. The Berezin formula tells us that the supertrace is nothing but the coefficient of the Clifford top part, and at the end, Getzler calculus enables us to find the integral of these top parts in terms of characteristic classes.

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To My Sister Azadeh

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Chapter 1

Introduction

October,1910 - Götingen

"In conclusion, there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans. In an enclosure with a perfectly reflecting surface, there can form stand- ing electromagnetic waves analogous to tones over an organ pipe: we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$. To this end, he calculates the number of overtones which lie between frequencies ν and $\nu + d\nu$, and multiplies this number by the energy which belongs to the frequency ν, and which according to a theorem of statistical mechanics, is the same for all frequencies. It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure, and is simply proportional to its volume. For many shapes for which calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures, like membranes, air masses, etc., should also hold."

Hendrik Lorentz [22]

Four months later, Hermann Weyl proved the above statement. In fact, he showed that the spectrum of the Laplacian with zero condition on the boundary of a domain Ω has the following property:

$$N(\lambda) = \#\{\lambda_i : \lambda_i < \lambda\} \sim \frac{\text{area}(\Omega)}{2\pi}\lambda.$$

This is a concrete illustration of a more general phenomenon: an elliptic operator ,to some extent, encodes the geometry and topology of the underlying manifold. It is in fact the main objective of this thesis to show that elliptic operators and specifically Dirac operators contain certain information about the geometry and topology of the base manifold and the related vector bundles. In particular, using the heat kernel, we aim to prove the celebrated Atiyah-Singer index theorem for Dirac operators.

We shall also see that the study of Dirac operators reveals some interconnections between the geometry and the topology of the underlying manifold. Perhaps one of the most well-known results of this type is the Gauss-Bonnet theorem:

$$\chi(M) = \frac{1}{4\pi}\int_M \kappa \, \mathrm{d}x.$$

The hidden point about this equation is that the above quantity is equal to the index of the well-known Euler operator $d + d^*$, considering it as an operator from even forms to odd forms.

In this chapter, we first briefly go through the original thoughts of Atiyah and Singer and their motivation behind the index theorem. We will see that their early approach to the index problem is somehow purely topological and uses different techniques from the theory of characteristic classes and also K-theory. Afterwards we introduce another approach to the index problem, the so called the heat kernel method, the one that we adopt later for the study of the index, which is more geometric in essence.

1.1 Elliptic topology

Around spring 1962 Michael Atiyah and Isadore Singer started their collaboration on the index theory of elliptic operators. Following the works of Hirzebruch on the Riemann-Roch theorem, they were first looking for a similar analytic interpretation of the $\hat{\mathcal{A}}$-genus of a spin manifold. They proved that the $\hat{\mathcal{A}}$-genus is the difference between the dimension of the positive harmonic spinors and the negative harmonic spinors. We shall prove this in the last chapter.

While they were working on this problem Steve Smale drew their attention to Gelfand's conjecture concerning the topology of elliptic operators. We briefly explain Gelfand's main idea.

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. A linear (bounded) operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be a ***Fredholm operator*** if dim ker T and dim coker T are both finite. In this case the index of T is defined to be the difference

$$\text{Ind } T := \dim \ker T - \dim \operatorname{coker} T.$$

Let $\mathfrak{F} = \mathfrak{F}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of Fredholm operators. A notable property of the index is that it is locally constant on $\mathfrak{F}$ and in fact, it provides a bijection

$$\text{Ind } : \pi_0(\mathfrak{F}) \xrightarrow{\approx} \mathbb{Z}.$$

(See [23] Proposition 7.1 for a proof).

Now, if we have a smooth manifold M and E, F are two vector bundles over M, a linear map

$$P : C^\infty(E) \to C^\infty(F)$$

that takes sections of E to sections of F is called a differential operator of order m provided that it can be given locally as

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$$

where the coefficients $a_\alpha(x)$ are matrices of appropriate dimension (depending on the ranks of E and F).

Given a cotangent vector $\xi \in T_x^* M$, the principal symbol of P at $x \in M$ is given by

$$\sigma_{x,\xi}(P) := \sum_{|\alpha|=m} (i)^{|\alpha|} a_\alpha(x) \xi^\alpha.$$

In fact it is obtained from P by replacing $\partial/\partial x^j$ by $i\xi_j$ in the highest order. One can check that the symbol is an intrinsic object (does not depend on a specific local trivialization) and provides a linear map

$$\sigma_{x,\xi}(P) : E_x \to F_x.$$

The differential operator P is called ***elliptic*** if the above mapping is an isomorphism for any non-zero ξ at every point.

Back to our discussion about Fredholm operators, it is well-known that for a compact, oriented, Riemannian manifold M, a differential operator $P : C^\infty(E) \to C^\infty(F)$ extends to a bounded linear map $P : H^s(E) \to H^{s-m}(F)$ on the Sobolev spaces for each s. Also if P is elliptic the extensions are Fredholm and they all share the same index (the index does not depend on s). Another important topological property of index is that the index of an elliptic operator depends only on its homotopy class and hence it only depends on its principal symbol. (for the proofs see [23], Chapter III.) This topological invariance of the index led Gelfand to the conjecture that the index is expressible in terms of its principal symbol. The first verification of this conjecture was given by Atiyah and Singer [3]. Here is a sketch of their results in [3]:

Let $S(M)$ and $B(M)$ denote the unit ball and the unit sphere subbundles of the cotangent bundle T^*M, respectively. As we just saw the principal symbol of P gives a bundle homomorphism

$$\sigma(P) : p^*E \to p^*F$$

between the pull backs of E and F by the projection $p : B(M) \to M$. Ellipticity of P means that $\sigma(P)$ is an isomorphism on the submanifold $S(M) \subset B(M)$. It follows by some results in [2] that one can associate to $(p^*E, p^*F, \sigma(P))$ a difference bundle $d(p^*E, p^*F, \sigma(P)) \in K(B(M)/S(M))$ in the K-group. Recall that for a topological space X, the K-group of X denoted by $K(X)$ is the abelian group associated to the semi group $(Vec(X), \oplus)$ of all complex vector bundles on X equipped with the direct sum. Also recall that the Chern character gives a ring homomorphism

$$\mathrm{ch} \ : K(X) \to H^*(X, \mathbb{Q})$$

of the K-group into the rational cohomology ring. Now taking the Chern character one obtains an element $\mathrm{ch}\, d(p^*E, p^*F, \sigma(P)) \in H^*(B(M)/S(M), \mathbb{Q})$. At last, by means of the Thom isomorphism

$$\phi : H^k(M, \mathbb{Q}) \xrightarrow{\cong} H^{n+k}(B(M)/S(M), \mathbb{Q}) \qquad (n = \dim M),$$

we obtain the element

$$\phi^{-1}\mathrm{ch}\, d(p^*E, p^*F, \sigma(P)) \in H^*(M, \mathbb{Q})$$

in the rational cohomology of M which is denoted by $\mathrm{ch}\, \sigma(P)$. With the above the Atiyah-Singer result reads

Theorem 1.1 (Atiyah-Singer Index Theorem [3]). *For any elliptic operator P on a compact oriented smooth manifold M the index of P is given by the formula*

$$Ind\ P = \{ch\ \sigma(P)\mathfrak{J}(TM)\}[M]$$

where $\mathfrak{J}(TM)$ denotes the Todd class of the tangent bundle, and $\alpha[M]$ denotes the value of the top dimensional component of α on the fundamental homology class of M for any $\alpha \in H^*(M, \mathbb{Q})$.

1.2 Elliptic geometry

There is another approach to the index problem that originated in the analysis of elliptic operators and in particular the study of the heat kernel. In this section we briefly explain this line of thought.

Around the beginning of twentieth century the spectrum of the Laplacian proved to be a critical object that contains some geometric information of Riemannian manifolds, and the question of to what extent the spectrum reflects the shape of the base manifold intrigued the thoughts of many mathematicians then. (cf. M. Kac [22])

Weyl formula is perhaps one of the earliest attack to this problem. To put it in the amusing language of Kac, Weyl showed that one can hear the area of D. It was proved by Kac himself that it is also possible to hear the length of ∂D.

On the other hand it turned out that the study of the behavior of certain functions on the spectrum can be a crucial step for the better understanding of Kac's problem. There are two functions that were mainly used for this purpose:

- The zeta function $\sum_\lambda \lambda^{-s}$.
- and the exponential $\sum_\lambda e^{-\lambda t}$.

We are now trying to take a closer look at the second case.

Let M be an n-dimensional, oriented, Riemannian manifold. Let Δ_p denote the restriction of the Laplacian $(d+d^*)^2$ on p-forms. Note that

$$\Delta_0 = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i} g^{ij}\sqrt{\det g}\frac{\partial}{\partial x^j}$$

is the Laplace-Beltrami operator on functions.

Let $Z_p = \sum_{\text{spec }\Delta_p} e^{-\lambda t}$. It was shown by N. G. de Bruijn, V. Arnold and McKean & Singer [25] that there is an asymptotic expansion

$$Z_0 \sim \frac{1}{(4\pi t)^{n/2}}\left(\operatorname{vol}(M) + \frac{t}{6}\int_M \kappa \,\mathrm{d}x + \frac{t^2}{180}\int_M 10A - B + 2C \,\mathrm{d}x + \dots\right)$$

where κ denotes the scalar curvature and A, B, C are some polynomial of the curvature tensor. For $n=2$ it turns out that $10A - B + 2C = 12\kappa^2$ and it follows by the Gauss-Bonnet theorem that

$$Z_0 \sim \left(\frac{\operatorname{area}(M)}{4\pi t} + \frac{1}{6}\chi(M) + \frac{t}{60\pi}\int_M \kappa^2 \,\mathrm{d}x + \dots\right).$$

In particular the Euler characteristic of M (the number of holes) is audible.

The main idea behind this estimate lies on the relation between the above sum and the heat equation. In fact the sum Z_p is the trace of the heat operator $e^{-t\Delta_p}$ and can be computed via the integral of the heat kernel $k_t^p(x,y)$ over the diagonal. Indeed

$$Z_p = \operatorname{tr} e^{-t\Delta_p} = \int_M \operatorname{tr} k_t^p(x,x)\,\mathrm{d}x. \tag{1.1}$$

Regarding this idea Minakshisundaram [26] proved that there exists an asymptotic expansion for the heat kernel

$$(4\pi t)^{n/2} k_t(x,y) \sim 1 + a_1(x,y)t + a_2(x,y)t^2 + \dots \tag{1.2}$$

where the coefficients a_j are some polynomials in the curvature and its derivatives. He also computed the term $a_1 = \frac{1}{6}\kappa$ as pointed out above. However Minakshisundaram derived the above result only for functions (zero forms) and it was shown later by McKean and Singer [25] that the result holds for arbitrary forms as well.

Now one might consider the alternating sum

$$\begin{aligned} Z =& Z_0 - Z_1 + Z_2 - \cdots \pm Z_n \\ =& \sum_{\lambda} e^{-t\lambda} \left(\dim \mathcal{H}_{\lambda}^{\text{even}} - \dim \mathcal{H}_{\lambda}^{\text{odd}}\right) \end{aligned}$$

where $\mathcal{H}^{\text{even}}$ and $\mathcal{H}^{\text{odd}}$ denote the set of even and odd λ-eigenforms. Since $d + d^*$ provides an isomorphism between $\mathcal{H}^{\text{even}}$ and $\mathcal{H}^{\text{odd}}$ for non-zero λ, the above sum turns to be

$$\dim \mathcal{H}_0^{\text{even}} - \dim \mathcal{H}_0^{\text{odd}}$$

the difference between the dimension of even and odd harmonics. It then follows from the Hodge theorem that

$$Z = \dim \mathcal{H}_0^{\text{even}} - \dim \mathcal{H}_0^{\text{odd}} = \chi(M).$$

On the other hand $\dim \mathcal{H}_0^{\text{even}} - \dim \mathcal{H}_0^{\text{odd}}$ is equal to the index of $d + d^*$ considering it as an operator from even forms to odd forms. Hence one obtains

$$\text{Ind } d + d^* = \chi(M).$$

For several reasons the above observation was a breakthrough in the development of the index theory from the heat equation point of view. For one thing because it relates an analytic object to a topological invariant.

For another thing in view of Chern's discovery

$$\chi(M) = \frac{1}{(2\pi)^{n/2}} \int_M \text{Pf}(\Omega) \tag{1.3}$$

and paying attention to the Equations 1.1, 1.2 McKean and Singer suggested, on an a priori, that a remarkable cancellation might take place in the alternating sum

$$\sum_P (-1)^p \operatorname{tr} a^p_{n/2}(x,x),$$

for n even and the remaining term must coincide with the Pfaffian $\mathrm{Pf}(\Omega)$ which is an algebraic expression -in fact a characteristic form- in terms of the curvature Ω and only the curvature and not its derivatives. They themselves proved that this guess is true for $n = 2$. Note that this gives an alternative proof of the Gauss-Bonnet theorem. It took a few years until Patodi [27] verified that the McKean and Singer's optimism is true for any even n. Later Peter Gilkey [12] and Atiyah-Bott-Patodi [1] improved the result for a more general type of elliptic differential operators which are said to be Dirac type operators.

Let us put these ideas in the context of elliptic operators. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic operator and let P' denote its adjoint. Then P and PP' are both Fredholm and one can show that (see Section 5.2 for more details)

$$\begin{aligned}
\text{Ind } P =& \dim \ker P - \dim \ker P' \\
=& \dim \ker P'P - \dim \ker PP' \\
=& \operatorname{tr} e^{-tP'P} - \operatorname{tr} e^{-tPP'}.
\end{aligned}$$

Let k_t^+ and k_t^- be the heat kernel for the heat operators $e^{-tP'P}$ and $e^{-tPP'}$ respectively. It then follows from the asymptotic expansions

$$k_t^\pm \sim (4\pi t)^{-n/2} \sum_j a_j^\pm t^j$$

that

$$\text{Ind } P = \int_M \operatorname{tr} a^+_{n/2}(x,x) - \operatorname{tr} a^-_{n/2}(x,x)\, \mathrm{d}x \tag{1.4}$$

where all the coefficients $a_j^{\pm}$ as well as $k_t^{\pm}$ are sections of $F \boxtimes E^*$.

For a general elliptic operator P there is no clue how one can compute the coefficients $a_{n/2}^{\pm}(x,x)$ and how it can be related to the characteristic forms appearing in the Atiyah-Singer index theorem. However, there is hope that this can be done for those operator that come from the geometric structures on M. Some instances of these geometric elliptic operators are:

- The Euler operator $E = d + d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$ that takes even forms to odd forms.

- The Hirzebruch signature operator $S = d + d^* : \Omega^+(M) \to \Omega^-(M)$ where $\Omega^{\pm}(M)$ are the ± 1 eigenspaces for $\tau = (i)^{p(p-1)+n/2} \star : \Omega^p(M) \to \Omega^{n-p}(M)$ ($n = 4k$) and $\star$ denotes the Hodge star operator.

- The Dirac operator $\not{D}^+ : C^\infty(\not{S}^+) \to C^\infty(\not{S}^-)$ that acts on the sections of spinors.

As it was mentioned earlier, for the Euler operator following the McKean-Singer conjecture [25] it was shown by Patodi [27] that the integrand in the right hand side of Equation 1.4 coincides with the Pfaffian polynomial of the curvature. Hence this gives another proof of the Chern equation 1.3 since Ind $E = \chi(M)$.

For the signature operator S it was proved by Gilkey [12] and Atiyah-Bott-Patodi [1] that the right hand side integrand of 1.4 is in fact the L-polynomial of the curvature. In this case Equation 1.4 reads

$$\operatorname{sign}(M) = \int_M L$$

since Ind $S = \operatorname{sign}(M)$.

At last, for the case of the Dirac operator it was shown by Gilkey [12] and Atiyah-Bott-Patodi [1] that the characteristic polynomial on the right hand side is $\hat{A}(TM)$ and we have

$$\text{Ind } \not{\partial}^+ = \int_M \hat{\mathcal{A}}(TM)$$

The above operators are all typical of a large class of elliptic operators called Dirac operators. In this thesis we are planning to study this type of operators. We are going in detail through the algebraic and geometric structures required to construct Dirac operators on manifolds. We will start by introducing Clifford algebras and their representation which is the algebraic foundation of Dirac operators structure. In chapter 3 we introduce Clifford bundles and Dirac operators. This is in fact the geometric facet of our constructions. Then we study the analysis of Dirac including the heat equation and the asymptotic expansion of the heat kernel. In the last chapter we go through Getzler′s" idea to prove the index theorem and we will see that Equation 1.4 for Dirac operators becomes

$$\text{Ind } \mathsf{D} = \int_M \hat{\mathcal{A}}(TM) \wedge \mathrm{ch}(\mathcal{V}), \tag{1.5}$$

the Atiyah-Singer index theorem. This is also equivalent to Theorem 1.1 for the case of Dirac operators on Clifford bundles.

We assume as background basics of differential geometry in particular Chern-Weyl theory of characteristic classes.

In our study we borrow many ideas from the references [6], [11], [13], [19], [23], [28]. In particular most of our proofs are a slight variation of the proofs in the references.

Chapter 2

Clifford Algebras and Their Representations

In the introduction we saw the significance of elliptic operators from the geometric and topological point of view. Now we are going to see how Clifford algebras and Clifford modules come in to the stage. Here we shall give two motivations that reveal the role of Clifford structures in our work. The first one is a mathematical idea and the second one comes from physics. Indeed that is the idea that led Dirac to introduce a new equation (known as Dirac equation) that describes the relativistic motion of a particle in the context of quantum theory.

Example 2.1. As we saw in the first chapter the study of $d + d^*$ which is simply the square root of the Laplacian can be beneficial to a better understanding of geometry and topology. Now one might ask what is the square root of Laplacian in the case of a trivial vector bundle on a flat space? For instance We are looking for a first order differential operator

$$P = \sum_{i=1}^{N} \gamma^i \partial_i \tag{2.1}$$

acting on the vector valued functions $f : \mathbb{R}^N \to V$ on the euclidean space, that satisfies the condition

$$P^2 = \Delta = -\sum_{i=1}^{N} \partial_i^2.$$

For the above equation to be satisfied, we get

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij} \qquad 1 \leq i, j \leq N.$$

One can prove that for $N \geq 2$ there are neither real nor complex scalars that satisfy this condition. However the above equation is exactly the algebraic equation that defines the generators of the Clifford algebra associated to $\mathbb{R}^N$. Hence we would be able to find the desired differential operator P if we had a representation of the Clifford algebra. In this case the meaning of γ^i in 2.1 is the Clifford action on V.

Now we see another evidence that shows the importance of the square root of the Laplacian and the Clifford algebra structure. This time from the physics point of view.

Example 2.2 (Dirac's discovery)**.** Dirac's main idea in [10] is to find a Schrödinger type equation $i\partial_t\psi = H\psi$ that describes the relativistic motion of a point-charge electron of mass m moving in the presence of an arbitrary electro-magnetic field. Here $\psi = \psi(t)$ denotes the state of the particle at time t (the wave function) and H is the Hamiltonian. The aim is to find an appropriate Hamiltonian consistent with the special theory of relativity. Let us only consider the case of a free particle.
Let $q^\mu = (E/c, p_1, p_2, p_3)$ be the momentum-energy 4-vector. Dirac's starting point was the fundamental equation of special relativity

$$q_\mu q^\mu = (mc)^2$$

where $q_\mu = g_{\mu\nu} q^\nu$ and $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Lorentz metric. For the sake of simplicity let $c = 1$. Then the above equation, in terms of the components, reads

$$E^2 = p_1^2 + p_2^2 + p_3^2 + m^2. \tag{2.2}$$

Note that in the Schrödinger equation the Hamiltonian H is the infinitesimal generator of the time translation and it represents the total energy E. On the other hand according to the basic axioms of Quantum mechanics, H which is an observable corresponding to the energy has to be a self-adjoint operator. A plausible candidate for the Hamiltonian takes the form

$$H = \frac{1}{i}(\alpha^1 \partial_1 + \alpha^2 \partial_2 + \alpha^3 \partial_3) + m\alpha^4. \tag{2.3}$$

Besides, Equation 2.2 quantized to

$$H^2 = -\partial_1^2 - \partial_2^2 - \partial_3^2 + m^2, \tag{2.4}$$

as the quantization procedure sends p_i to $-i\partial_i$, and m to the point wise multiplication operator by m itself. Now plugging 2.3 into 2.4 yields

$$\alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij} \qquad \text{for } 1 \leq i, j \leq 4.$$

As we shall see later in this chapter the above equation is exactly the condition for the generators of the Clifford algebra of $\mathbb{R}^4$ with the negative definite quadratic form

$$Q(x) = -(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2.$$

In contrast with the prevalent view among physicists in the early days of quantum theory, Dirac's insight was that the wave function ψ does not necessarily need to be complex valued and it can be a vector space valued function. This enables him to find the desired coefficients α^i in 2.3. In the modern terminology he in fact found out the irreducible representation of $\mathbb{C}l_4$ which is $\mathbb{C}^4$ along with the following actions

$$\alpha^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \quad \text{for } i = 1, 2, 3 \quad \text{and,} \quad \alpha^4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where σ_i are Pauli matrices. So in Dirac's theory wave functions are $\mathbb{C}^4$-valued.

As it became evident to us that the study of Clifford algebras and Clifford modules are somehow indispensable for a better understanding of the geometry of the elliptic operators, our aim now is to study these algebraic structures in details. Moreover, these structures are not only an essential part of constructing Dirac operators and the bundle of spinors, but also they play a critical role in the Getzler proof of the index theorem.

2.1 Clifford algebras

Throughout this chapter, $\mathbb{K}$ denotes either the field of real or complex numbers.

Let V be a real or complex vector space with a quadratic form Q. Roughly speaking, the Clifford algebra $Cl(V, Q)$ is the associative $\mathbb{K}$-algebra with unit generated by the elements of V subject to the relations

$$v.v = -Q(v)1 \quad \text{for each } v \in V. \tag{2.5}$$

It follows, from the above equation that

$$u.v = -v.u - 2Q(u, v) \tag{2.6}$$

holds in $Cl(V, Q)$ for any $u, v \in V$ where $Q(.,.)$ is the inner product on V induced by Q.

Here is a more concrete definition of the Clifford algebra given by Chevalley [7], however he uses a different sign convention $v.v = Q(v)1$.

Definition 2.3. Let $\mathscr{T}(V) = \mathbb{K} + V + V \bigotimes V + \dots$ be the tensor algebra of V. The Clifford algebra $Cl(V, Q)$ is the quotient of $\mathscr{T}(V)$ by the two-sided ideal $\mathcal{I}_Q$ generated by all elements of the form $v \otimes v + Q(v)1$ for $v \in V$.

If $j : V \hookrightarrow \mathscr{T}(V)$ is the embedding of V into the tensor algebra and $\pi : \mathscr{T}(V) \to Cl(V, Q)$ is the canonical projection, one can compose π with j to obtain the natural embedding of V into $Cl(V, Q)$:

$$\iota : V \overset{j}{\hookrightarrow} \mathscr{T}(V) \overset{\pi}{\to} Cl(V, Q).$$

We shall show that ι is an injection:
Suppose $\iota(\phi) = 0$ then ϕ can be written in the form $\phi = \sum_{i=1}^{n} a_i \otimes (v_i \otimes v_i + Q(v_i)) \otimes b_i$. Without loss of generality we may assume that a_i and b_i are homogeneous tensors. Since $\phi \in V$ we have $\sum_{i \in J} a_i \otimes (v_i \otimes v_i) \otimes b_i = 0$ where J is the set of indices for which $\deg a_i + \deg b_i$ is maximum. The above equation implies that $\sum_{i \in J} Q(v_i) a_i \otimes b_i$ must be in the ideal $\mathcal{I}_Q$. Therefore $\sum_{i \in J} Q(v_i) a_i \otimes b_i = 0$ and repeating this argument shows that $\phi = 0$.

The following proposition gives a very useful characterization of Clifford algebras.

Proposition 2.4 ([23] Proposition 1.1.)**.** *Let $f : V \to \mathscr{A}$ be a linear map into an associative $\mathbb{K}$-algebra with unit, such that*

$$f(v).f(v) = -Q(v)1, \tag{2.7}$$

for all $v \in V$. Then f extends uniquely to a $\mathbb{K}$-algebra homomorphism $\tilde{f} : Cl(V, Q) \to \mathscr{A}$ Furthermore, $Cl(V, Q)$ is the unique associative $\mathbb{K}$-algebra with this property.

Proof. It is clear that any linear map $f : V \to \mathscr{A}$ extends uniquely to an algebra homomorphism $\bar{f} : \mathscr{T}(V) \to \mathscr{A}$ and by property 2.7 $\bar{f}$ descends to an algebra homomorphism

$\tilde{f} : Cl(V,Q) \to \mathscr{A}$. Now suppose that $\mathscr{C}$ is an associative unital $\mathbb{K}$-algebra and that $i : V \hookrightarrow \mathscr{C}$ is an embedding with the property that for any linear map $f : V \to \mathscr{A}$ we have the following extension

$$\begin{tikzcd} \mathscr{C} \arrow[dr, "\tilde{f}"] & \\ V \arrow[u, "i"] \arrow[r, "f"'] & \mathscr{A} \end{tikzcd}$$

For the uniqueness note that the isomorphism $Cl(V,Q) \supset V \xrightarrow{i} i(V) \subset \mathscr{C}$ can be lifted to the algebra isomorphism $Cl(V,Q) \xrightarrow{\tilde{i}} \mathscr{C}$. Indeed $\widetilde{i^{-1}} \circ \tilde{i} = \mathrm{id}_{Cl(V,Q)}$, and $\tilde{i} \circ \widetilde{i^{-1}} = \mathrm{id}_{\mathscr{C}}$. $\square$

This characterization of Clifford algebras shows that any linear map $f : (V,Q) \to (V',Q')$ that preserves the quadratic forms (e.g. $f^*Q' = Q$) can be extended to an algebra homomorphism $\tilde{f} : Cl(V,Q) \to Cl(V',Q')$. Moreover given another map $g : (V',Q') \to (V'',Q'')$ since both $\widetilde{g \circ f}$ and $\tilde{g} \circ \tilde{f}$ are extensions of $g \circ f$ by the uniqueness property, we have $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$. In particular if $f, g \in O(V,Q)$ this argument shows that there is an embedding

$$\mathrm{O}(V,Q) \hookrightarrow \mathrm{Aut}(Cl(V,Q))$$

where

$$\mathrm{O}(V,Q) = \{L \in GL(V) : L^*Q = Q\}$$

is the orthogonal group of Q.

Remark 2.5. Since, later on, we are going to use the theory of Clifford algebras on Riemannian manifolds (in particular we will talk about the Clifford algebra of the tangent spaces and the bundle of Clifford modules) it suffices for our purposes to assume that the underlying vector space is of finite dimension and the quadratic form Q is non-degenerate.

Remark 2.6. If $(e_1, e_2, \ldots, e_n)$ is a $Q-$orthonormal basis of V then the set

$$\{e_I : I \subset \{1, 2, \ldots, n\}\}$$

forms a basis of $Cl(V, Q)$ where

$$e_I = \begin{cases} e_{i_1} \ldots e_{i_k} & \text{If } I = \{i_1 < \cdots < i_k\} \\ 1 & \text{If } I = \varnothing \end{cases}$$

Suppose that V is an inner product space for any $v \in V$ the interior product

$$\iota_v : \bigwedge^k V \to \bigwedge^{k-1} V$$
$$\xi \mapsto \iota_\alpha \xi$$

is defined by

$$\iota_v(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} \langle v, v_i \rangle v_1 \wedge \ldots \hat{v}_i \cdots \wedge v_k$$

where the hat denotes deletion.

There is a canonical vector space isomorphism between the exterior algebra and the Clifford algebra that identifies $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $e_{i_1} \ldots e_{i_k}$ and one can check that this identification is indeed independent of the choice of the orthonormal basis. The following proposition describes the Clifford action of a vector under this identification.

Proposition 2.7. *With respect to the canonical vector space isomorphism $Cl(V, Q) \cong \bigwedge^* V$ the Clifford multiplication of a vector $v \in V$ and any element $\phi \in Cl(V, Q)$ is of the following form in the exterior algebra*

$$v.\phi \cong (v \wedge -\iota_v)\phi$$

Proof. Suppose that $(e_1, \ldots, e_n)$ is an orthonormal basis for V. Let $v = e_r$ and $\phi = e_{i_1} \ldots e_{i_m}$. Then

$$v.\phi = \begin{cases} e_r e_{i_1} \ldots e_{i_m} \cong (v \wedge -\iota_v)\phi & \text{If } r < i_1 \\ (-1)^q e_{i_1} \ldots \hat{e_{i_q}} \ldots e_{i_m} \cong (v \wedge -\iota_v)\phi & \text{If } r = i_q \\ (-1)^q e_{i_1} \ldots e_{i_q} e_r e_{i_{q+1}} \ldots e_{i_m} \cong (v \wedge -\iota_v)\phi & \text{If } i_q < r < i_{q+1} \\ (-1)^m e_{i_1} \ldots e_{i_m} e_r \cong (v \wedge -\iota_v)\phi & \text{If } r > i_m \end{cases}$$

The results immediately follows if one writes any vector and an element of the Clifford algebra in terms of the basis elements. □

The above argument suggests that we can think of the exterior algebra as a $Cl(V, Q)$-module. Using the identification $V \xrightarrow{\flat} V^*$ induced by the inner product on V, we can say that $\bigwedge^* V^*$ is also a $Cl(V, Q)$-module. In this case the Clifford action of a vector $v \in V$ on $\phi \in \bigwedge^* V^*$ turns out to be

$$v.\phi = (v^\flat \wedge -\iota_{v^\flat})\phi. \tag{2.8}$$

Remark 2.8. The interior product defined above is different from the standard interior product in the sense that for a covector $\alpha \in V^*$ the standard interior product

$$\begin{aligned} \iota_\alpha : \bigwedge^k V &\to \bigwedge^{k-1} V \\ \xi &\mapsto \iota_\alpha \xi \end{aligned}$$

is defined by

$$\iota_\alpha \xi(\alpha_1, \ldots, \alpha_{k-1}) = \xi(\alpha, \alpha_1, \ldots, \alpha_{k-1}) \quad \text{for } \alpha_1, \ldots, \alpha_{k-1} \in V^*.$$

Consequently,

$$\iota_\alpha(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} \langle \alpha, v_i \rangle v_1 \wedge \ldots \hat{v}_i \cdots \wedge v_k.$$

With respect to this notation Equation 2.8 becomes

$$v.\phi \cong (v^\flat \wedge - \iota_v)\phi.$$

The Clifford algebra inherits a natural filtration from the tensor algebra. Indeed if we let $\mathscr{F}^r := \bigoplus_{k \leq r} \bigotimes^k V$ then the projections $Cl_r := \pi(\mathscr{F}^r)$ give the filtration

$$Cl_0 \subset Cl_1 \subset Cl_2 \subset \cdots \subset Cl(V, Q)$$

with the property

$$Cl_r.Cl_s \subset Cl_{r+s}$$

and we can show that

Proposition 2.9 ([23] Proposition 1.2.). *The associated graded algebra of the Clifford algebra is the exterior algebra. i.e.*

$$\bigoplus_r Cl_r / Cl_{r-1} \cong \bigwedge^* V$$

Proof. Consider the family of maps

$$\lambda_r : \bigwedge^r V \to Cl_r / Cl_{r-1}$$

defined by

$$[v_1 \otimes \cdots \otimes v_r] \mapsto [v_1 \ldots v_r].$$

These are clearly well-defined surjective homomorphisms that give rise to the map

$$\lambda : \bigwedge^{*} V \to \bigoplus_{r=1}^{n} Cl_r / Cl_{r-1}.$$

We need to check that this is an injective mapping.

Any zero element in Cl_r/Cl_{r-1} is a finite sum of terms of the form

$$[a \otimes (v \otimes v + Q(v)) \otimes b] = [a \otimes v \otimes v \otimes b]$$

where we can assume that a and b are homogeneous elements with $\deg(a) + \deg(b) \leq r - 2$. But the above expression is obviously zero in the exterior algebra $\bigwedge^r V$, thus the kernel of λ is trivial and it must be injective. □

Another fact of high importance about Clifford algebras is that they have the structure of a $\mathbb{Z}_2$ **graded algebra**. Indeed consider the automorphism $\tilde{\alpha} : Cl(V, Q) \to Cl(V, Q)$ which extends the mapping

$$\begin{aligned} \alpha : & V \to V \\ & v \mapsto -v. \end{aligned}$$

Since $\tilde{\alpha}^2 = \mathrm{id}_{Cl(V,Q)}$ one can define $Cl^0(V, Q)$, and $Cl^1(V, Q)$ to be the eigenspaces corresponding to the ± 1 eigenvalues respectively. Consequently we get the decomposition

$$Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$$

along with the relations

$$Cl^i(V, Q).Cl^j(V, Q) \subset Cl^{i+j}(V, Q) \qquad i, j \in \{0, 1\}$$

that provide a $\mathbb{Z}_2$ grading for a Clifford algebra.

2.2 The groups Pin and Spin

In this section we shall construct double coverings for the classical groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$. To this end we first need to study the multiplicative group of units in $Cl(V,Q)$ and its representation into the algebra automorphisms of the Clifford algebra.

As we mentioned before, we are assuming from now on that (V,Q) is a finite dimensional vector space equipped with a non-degenerate quadratic form. This assumption allows us to use some significant results from linear algebra.

Let $Cl^{\times}(V,Q)$ denotes **The multiplicative group of units** in the Clifford algebra

$$Cl^{\times}(V,Q) = \{\phi \in Cl(V,Q) : \phi \text{ is invertible}\}.$$

If $Q(v) \neq 0$ for $v \in V$ then $v.\dfrac{-v}{Q(v)} = 1$ and

$$v^{-1} = \frac{-v}{Q(v)}. \tag{2.9}$$

Therefore, $Cl^{\times}(V,Q)$ contains of all vectors $v \in V$ with $Q(v) \neq 0$.

There is a natural representation of $Cl^{\times}(V,Q)$ into the group of algebra automorphisms of $Cl(V,Q)$

$$\mathrm{Ad} : Cl^{\times}(V,Q) \to \mathrm{Aut}(Cl(V,Q))$$
$$\phi \mapsto \mathrm{Ad}_{\phi}$$

given by

$$\mathrm{Ad}_{\phi}(x) = \phi x \phi^{-1}.$$

We shall see that this representation has certain geometric properties. If $v, w \in V$ are non-zero vectors using 2.6 and 2.9 we have

$$\begin{aligned}\mathrm{Ad}_v(w) &= v^{-1}wv = \frac{-v}{Q(v)}wv = \frac{-1}{Q(v)}(-wv - 2Q(v,w))v \\ &= -(w - 2Q(v,w)v) \\ &= -\rho_v(w)\end{aligned}$$

where $\rho_v : V \to V$ denotes the reflection across the hyperplane $v^{\perp}$.

To get rid of the negative sign above, we consider **the twisted adjoint representation**

$$\widetilde{\mathrm{Ad}} : Cl^{\times}(V,Q) \to \mathrm{GL}(Cl(V,Q))$$

defined by

$$\widetilde{\mathrm{Ad}}_{\phi}(x) = \tilde{\alpha}(\phi)x\phi^{-1}$$

where $\tilde{\alpha}$ denotes the grading operator and $\mathrm{GL}(Cl(V,Q))$ denotes the group of the vector space isomorphisms of $Cl(V,Q)$.

With this definition we see that for $v_1, v_2, \ldots, v_r \in V$

$$\widetilde{\mathrm{Ad}}_{v_1 v_2 \ldots v_r} = \rho_{v_1} \circ \rho_{v_2} \circ \cdots \circ \rho_{v_r}. \tag{2.10}$$

The above observation leads us to consider the subgroup $\mathrm{P}(V,Q) \subset Cl^{\times}(V,Q)$ consisting of all elements of the form

$$\mathrm{P}(V,Q) = \{v_1 v_2 \ldots v_r \in Cl^{\times}(V,Q) : v_i \in V, Q(v_i) \neq 0\}.$$

In fact $\mathrm{P}(V,Q)$ is the subgroup generated by all vectors $v \in V$ with $Q(v) \neq 0$. Moreover it is clear from 2.10 that the twisted adjoint representation gives rise to the homomorphism

$$\mathrm{P}(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V,Q).$$

It is clear from 2.10 and the definition of $P(V,Q)$ that the image of $\mathrm{P}(V,Q)$ under the mapping $\widetilde{\mathrm{Ad}}$ is the subgroup of $\mathrm{O}(V,Q)$ generated by all reflections. However the Cartan-Dieudonné Theorem says that this subgroup is exactly the entire group $\mathrm{O}(V,Q)$.

Theorem 2.10 (Cartan-Dieudonné)**.** *[[23] Theorem 2.7.] Let Q be a non-degenerate form on a finite dimensional vector space V. Then any element $g \in O(V,Q)$ can be written as a product of a finite number of reflections*

$$g = \rho_{v_1} \circ \rho_{v_2} \circ \cdots \circ \rho_{v_r}.$$

In order to achieve the double coverings of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ we shall examine certain subgroups of of $\mathrm{P}(V,Q)$ as follows.

Definition 2.11. The **Pin group** of (V,Q) is the subgroup of $\mathrm{P}(V,Q)$ which is generated by all vectors $v \in V$ with $Q(v)) = \pm 1$ and denoted by $\mathrm{Pin}(V,Q)$.
The **spin group** of (V,Q) denoted by $\mathrm{Spin}(V,Q)$ is the even part of $\mathrm{Pin}(V,Q)$

$$\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap Cl^0(V,Q).$$

We shall also consider the subgroup $\mathrm{SP}(V,Q) \subset \mathrm{P}(V,Q)$ which is the even part of $\mathrm{P}(V,Q)$

$$\mathrm{SP}(V,Q) = \mathrm{P}(V,Q) \cap Cl^0(V,Q).$$

Using Theorem 2.10, one can show that the restriction of $\widetilde{\mathrm{Ad}}$ to the subgroup $\mathrm{SP}(V,Q)$ gives the surjective homomorphism

$$\mathrm{SP}(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,Q) \tag{2.11}$$

where

$$\mathrm{SO}(V,Q) = \{L \in \mathrm{O}(V,Q) : \det(L) = 1\}.$$

To see this, first note that $\det(\rho_v) = -1$ for each $v \in V$. Indeed if we choose a Q-orthogonal basis $(v, v_2, \dots, v_n)$ of V then $\rho_v(v) = -v$ and $\rho_v(v_j) = v_j$ for $j \geq 2$ therefore $\det(\rho_v) = -1$.

Also if $L \in \mathrm{SO}(V,Q)$ by Theorem 2.10 there exist $v_1, \dots, v_r \in V$ such that

$$L = \rho_{v_1} \circ \cdots \circ \rho_{v_r}.$$

Since $\det(L) = 1$ we conclude that r must be even and the mapping 2.11 is surjective.

At this point we have two surjective homomorphisms

$$\mathrm{P}(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,Q) \qquad \mathrm{SP}(V,Q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,Q);$$

One might ask whether the twisted adjoint representation $\widetilde{\mathrm{Ad}}$ maps the groups $\mathrm{Pin}(V,Q)$ and $\mathrm{Spin}(V,Q)$ onto $\mathrm{O}(V,Q)$ and $\mathrm{SO}(V,Q)$ respectively. Since $\rho_{tv} = \rho_v$ and one can always normalize any vector $v \in V$, it is clear that the answer to this question is positive. However the fact that we are working with real or complex scalars is of crucial importance here. Indeed in order to normalize a vector we get

$$t^2 Q(v) = Q(tv) = 1$$

and we need to solve the equation $t^2 = 1$ which is solvable both in $\mathbb{R}$ and $\mathbb{C}$.

Therefore we have two homomorphisms:

$$\mathrm{Pin}(V, Q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V, Q) \qquad \mathrm{Spin}(V, Q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V, Q).$$

Having in mind that the main objective of this discussion is to find the covering of the Lie group $\mathrm{SO}(n)$, what we need to do at this point is to look for the kernel of the above mappings. The following theorem gives the main result:

Theorem 2.12 ([23] Theorem 2.9.). *Let V be a finite dimensional vector space over $\mathbb{K}$ with a non-degenerate quadratic form Q. Then there are short exact sequences*

$$0 \to F \to Spin(V, Q) \xrightarrow{\widetilde{Ad}} SO(V, Q) \to 1$$

$$0 \to F \to Pin(V, Q) \xrightarrow{\widetilde{Ad}} O(V, Q) \to 1$$

where

$$F = \begin{cases} \mathbb{Z}_2 & \textit{if } \sqrt{-1} \notin \mathbb{K} \\ \mathbb{Z}_4 & \textit{otherwise} \end{cases}$$

In particular when $V = \mathbb{R}^n$ with the standard quadratic form we have

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\widetilde{Ad}} SO(n) \to 1.$$

Moreover, the map $Spin(n) \xrightarrow{\widetilde{Ad}} SO(n)$ is the universal double covering of $SO(n)$ for $n \geq 3$.

Proof. Let $(e_1, e_2, \ldots, e_n)$ be a Q-orthogonal basis for V. It follows from the definition of Clifford algebra and Equation 2.6 that

$$e_i e_j = -e_i e_j \quad \text{for } i \neq j \quad \text{and} \quad e_i^2 = -Q(e_i).$$

Suppose that $\phi \in \mathrm{Pin}(V, Q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$. We can decompose ϕ into its even and odd parts $\phi = \phi_0 + \phi_1$. Since $\widetilde{\mathrm{Ad}}(\phi) = 1$ we have $\tilde{\alpha}(\phi)v\phi = v$ and the following equations must be satisfied for any $v \in V$:

$$\phi_0 v = v\phi_0 \quad \text{and} \quad \phi_1 v = -v\phi_1. \tag{2.12}$$

We are going to show that both ϕ_0 and ϕ_1 do not involve any basis element. This implies that $\phi_1 = 0$ and $\phi = \phi_0 = k$ for some scalar $k \in \mathbb{K}$. To this end we first consider ϕ_0. We can write $\phi_0 = \theta_0 + e_1\theta_1$ where none of θ_0 and θ_1 involve e_1. Since θ_1 is an odd element which does not have e_1 we have $e_1\theta_1 = -\theta_1 e_1$. Considering this equation and the first equation in 2.12 we conclude that $\theta_1 = 0$ and that ϕ_0 does not involve e_1. Repeating this argument with other basis elements shows that ϕ_0 does not involve any basis element. Since ϕ_0 is even it must be a scalar. With the same logic one can verify that neither ϕ_1 involves any basis element and because it's odd it has to be zero. Hence $\phi = \phi_0 = k$ can only be scalar. Furthermore considering the fact that $Q(\phi) = 1$ we get $k^2 = Q(k) = 1$ which determines the kernel both in real and complex cases.
Now we have to show that $\pi_1\mathrm{Spin}(n)$ is trivial for $n \geq 3$. We have the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(n) \to 1$$

which gives us the long exact sequence in homotopy

$$\pi_1\mathbb{Z}_2 \to \pi_1\mathrm{Spin}(n) \to \pi_1\mathrm{SO}(n) \to \pi_0\mathbb{Z}_2 \to \pi_0\mathrm{Spin}(n) \to \pi_0\mathrm{SO}(n).$$

We know that $\pi_1\mathbb{Z}_2$ and $\pi_0\mathrm{SO}(n)$ are trivial, $\pi_0\mathbb{Z}_2$ is $\mathbb{Z}_2$ and $\pi_1\mathrm{SO}(n)$ is $\mathbb{Z}_2$. Therefore from the above sequence it suffices to show that $\pi_0\mathrm{Spin}(n)$ is trivial. In other words, $\mathrm{Spin}(n)$ is path connected.

If $x \in \mathrm{Spin}(n)$ then

$$x = u_1 u_2 \dots u_{2m} \qquad \text{where } \; u_i \in \mathbb{S}^{n-1}.$$

Since $\mathbb{S}^{n-1}$ is path connected we can connect every u_i to E_1 through a path γ_i. So we can connect x to $E_1 \dots E_1$ ($2m$ times) in $\mathrm{Spin}(n)$ via the path $\gamma_1 \dots \gamma_{2m}$.
Since $E_1 \dots E_1$ ($2m$ times) is either $+1$ or -1 it is enough to find a path in $\mathrm{Spin}(n)$ connecting ± 1. This can be done using

$$\gamma(t) = (E_1 \cos \frac{\pi}{2} t + E_2 \sin \frac{\pi}{2} t)(E_1 \cos \frac{\pi}{2} t - E_2 \sin \frac{\pi}{2} t).$$

$\square$

2.3 The Algebras $Cl_{r,s}$ and $\mathbb{C}l_n$

Let us consider now the Clifford algebras $Cl_{r,s} := Cl(\mathbb{R}^{r+s}, Q_{r+s})$ of euclidean spaces with non-degenerate quadratic forms of different signatures

$$Q_{r+s}(x) = (x^1)^2 + \dots + (x^r)^2 - (x^{r+1})^2 - \dots - (x^{r+s})^2.$$

In order to simplify the notations we often write Cl_n instead of $Cl_{n,0}$ for the Clifford algebra of $\mathbb{R}^n$ with the standard positive definite form.
Recall from Equation 2.6 that given any $Q_{r,s}$-orthonormal basis of $\mathbb{R}^{r+s}$ we have the relations

$$e_i e_j + e_j e_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \\ 2\delta_{ij} & \text{if } i > r \end{cases} \tag{2.13}$$

Also recall that for any Clifford algebra we have the decomposition $Cl = Cl^0 + Cl^1$ into the even and odd parts.

Theorem 2.13 ([23] Theorem 3.7.). *There is an algebra isomorphism*

$$Cl_{r,s} \cong Cl^0_{r+1,s}.$$

In particular

$$Cl_n \cong Cl^0_{n+1}.$$

Proof. Let $(e_1, \ldots, e_{r+s+1})$ be a basis for $\mathbb{R}^{r+s+1}$ such that

$$Q(e_i) = \begin{cases} 1 & \text{if } i \leq r+1 \\ -1 & \text{if } i > r+1 \end{cases}$$

Also let $\mathbb{R}^{r+s} = \text{span}\{e_i : i \neq r+1\}$. We can define the linear mapping

$$f : \mathbb{R}^{r+s} \to Cl^0_{r+1,s}$$
$$e_i \mapsto e_{r+1}e_i.$$

Now, for $x = \sum_{i \neq r+1} x^i e_i$ we observe that

$$\begin{aligned} f(x)^2 =& \sum_{i,j} x^i x^j e_{r+1} e_i e_{r+1} e_j \\ =& \sum_{i,j} x^i x^j e_i e_j \\ =& x.x = -Q(x). \end{aligned}$$

Hence by the universal property, f extends to an algebra homomorphism

$$\tilde{f} : Cl_{r,s} \to Cl^0_{r+1,s}.$$

It is clear, from the definition that $\tilde{f}$ is surjective. Moreover, one can see that

$$\dim(Cl_{r,s}) = \dim(Cl^0_{r+1,s}).$$

In fact

$$Cl_{r,s} = \operatorname{span}\{e_I : I \subset \{1, 2, \ldots, r+s\}\}$$

where $e_I = e_{i_1} e_{i_2} \ldots e_{i_k}$ for $I = \{i_1 < i_2 < \cdots < i_k\}$ hence $\dim(Cl_{r,s}) = 2^{r+s}$.
On the other hand, $\dim(Cl^0_{r+1,s}) = 2^{r+s} = \dim(Cl_{r,s})$ since $\sum_{k=1}^{r+s+1} \binom{r+s+1}{k}(-1)^k = 0$.
Therefore, $\tilde{f}$ is an isomorphism. $\square$

It is quite useful, both for the classification and the representation theory of Clifford algebras, to identify $Cl_{r,s}$ algebras as a matrix algebra over some well-known field. The following example is the first step towards this goal.

Example 2.14.

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R} \qquad Cl_1 \cong \mathbb{C} \qquad Cl_2 \cong \mathbb{H}.$$

The first identity is pretty obvious and the rest follows from the fact that $Cl_1 = \mathbb{R} \oplus \operatorname{span}\{e_1\}$ and $Cl_2 = \mathbb{R} \oplus \operatorname{span}\{e_1, e_2, e_1e_2\}$ subject to the relations

$$(e_1)^2 = (e_2)^2 = -1 \quad \text{and} \quad e_1e_2 = -e_2e_1.$$

Also one can show that

$$Cl_{0,2} \cong \mathbb{R}(2),$$

considering the linear mapping $\phi : Cl_{0,2} \to \mathbb{R}(2)$ defined by

$$e_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $\mathbb{R}(2)$ denotes the algebra of real 2×2 matrices.

Theorem 2.15 ([23] Theorem 4.1.). *There exist isomorphisms*

$$\begin{aligned} Cl_{n,0} \otimes Cl_{0,2} &\cong Cl_{0,n+2} \\ Cl_{0,n} \otimes Cl_{2,0} &\cong Cl_{n+2,0} \\ Cl_{r,s} \otimes Cl_{1,1} &\cong Cl_{r+1,s+1}. \end{aligned} \tag{2.14}$$

Proof. In order to prove the first identity let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\mathbb{R}^{n+2}$ with respect to the standard inner product. Also let $(e'_1, \ldots, e'_n)$ and (e''_1, e''_2) be standard generators for $Cl_{n,0}$ and $Cl_{0,2}$ respectively (in the sense of Equation 2.13).
Let $f : \mathbb{R}^{n+2} \to Cl_{n,0} \otimes Cl_{0,2}$ be the map defined by

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & 1 \leq i \leq n \\ 1 \otimes e''_{i-n} & i = n+1, n+2 \end{cases}$$

One can easily check that

$$f(e_i)f(e_j) + f(e_j)f(e_i) = 2\delta_{ij} 1 \otimes 1.$$

It then follows that

$$f(x)f(x) = -Q_{0+(n+2)}(x) 1 \otimes 1.$$

Therefore, by the universal property, f extends to the algebra homomorphism $\tilde{f}$: $Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}$. It also follows from the construction of $\tilde{f}$ that it is surjective. Counting the dimensions shows that $\tilde{f}$ must be a bijection. Similar arguments establish the next two identities as well.

□

Let $\mathbb{C}l_{r,s}$ denotes the complexification of $Cl_{r,s}$ i.e.

$$\mathbb{C}l_{r,s} = Cl_{r,s} \otimes \mathbb{C}.$$

We will show that $\mathbb{C}l_{r,s}$ is isomorphic to $Cl(\mathbb{C}^{r+s}, Q^{\mathbb{C}}_{r,s})$ where $Q^{\mathbb{C}}_{r,s}$ denotes the extension of the real quadratic form $Q_{r,s}$ over complex scalars. (e.g. $Q^{\mathbb{C}}_{r,s}(z \otimes u) = z^2 Q_{r,s}(u)$ for any vector $u \in \mathbb{R}^{r+s}$ and any complex number z.)
To see this, first consider the inclusion

$$f : \mathbb{C}^{r+s} \hookrightarrow \mathbb{C}l_{r,s}.$$

One can easily verify that $f(w)^2 = Q^{\mathbb{C}}_{r,s}(w).1$. Hence by Proposition 2.4 f extends to a homomorphism

$$\tilde{f} : Cl(\mathbb{C}^{r+s}, Q^{\mathbb{C}}_{r,s}) \to \mathbb{C}l_{r,s}.$$

The result follows by dimension counting and the fact that $\tilde{f}$ is surjective.
In particular we get the isomorphism

$$\mathbb{C}l_n \cong Cl(\mathbb{C}^n, Q^{\mathbb{C}})$$

where

$$Q^{\mathbb{C}}(z) = \sum_{j=1}^{n} z_i^2.$$

But we should note that the complexification of the quadratic form $Q_{r,s}$ ignores the signature. Hence we will have the following identities

$$Cl(\mathbb{C}^n, Q^{\mathbb{C}}) \cong \mathbb{C}l_n \cong \mathbb{C}l_{n-1,1} \cong \ldots \cong \mathbb{C}l_{0,n}. \tag{2.15}$$

Example 2.16. Recall from Example 2.14 that

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R} \quad \text{and} \quad Cl_{0,2} \cong \mathbb{R}(2)$$

hence

$$\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C} \quad \text{and} \quad \mathbb{C}l_2 \cong \mathbb{C}(2)$$

The following periodicity theorem is the key to the classification of complex Clifford algebras.

Theorem 2.17 ([23] Theorem 4.3.)**.**

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2.$$

Proof. Using the identities 2.15 and the Theorem 2.15 we see that

$$\begin{aligned} \mathbb{C}l_{n+2} \cong \mathbb{C}l_{0,n+2} &\cong (Cl_n \otimes Cl_{0,2}) \otimes \mathbb{C} \\ &\cong Cl_n \otimes \mathbb{C}l_2 \\ &\cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2. \end{aligned}$$

□

The above periodicity theorem, along with Example 2.16, completely determine the classification of complex Clifford algebras:

Theorem 2.18.

$$\mathbb{C}l_n = \begin{cases} \mathbb{C}(2^m) & \textit{for } \; n = 2m \\ \mathbb{C}(2^m) \oplus \mathbb{C}(2^m) & \textit{for } \; n = 2m+1 \end{cases}$$

2.4 Representations of $\mathbb{C}l_n$

In this section we study (complex) representations of $\mathbb{C}l_n$ and $\mathrm{Spin}(n)$ group. At the end we will discover the irreducible graded representation of $\mathbb{C}l_{2m}$ which becomes crucial in our study of the index theorem.

Definition 2.19. A (complex) representation of the Clifford algebra $\mathbb{C}l_n$ is a $\mathbb{C}$-linear algebra homomorphism

$$\rho : \mathbb{C}l_n \to \mathrm{Hom}(W, W)$$

where W is a complex finite-dimensional vector space. The space W is called a **$\mathbb{C}l_n$-module** over $\mathbb{C}$.

We often use the notation $\phi.w$ for the action $\rho(\phi)(w)$ of the Clifford algebra on W.

Definition 2.20. Two representations $\rho_i : \mathbb{C}l_n \to \mathrm{Hom}(W_i, W_i)$ $(i = 1, 2)$ are said to be **equivalent** if there exists a $\mathbb{C}$-linear isomorphism $F : W_1 \to W_2$ such that the following diagram commutes for any $\phi \in \mathbb{C}l_n$

$$\begin{tikzcd} W_1 \arrow[r, "\rho_1(\phi)"] \arrow[d, "F"'] & W_1 \arrow[d, "F"] \\ W_2 \arrow[r, "\rho_2(\phi)"'] & W_2 \end{tikzcd}$$

Recall from the classification of complex Clifford algebras that there are isomorphisms

$$\mathbb{C}l_{2m} \cong \mathbb{C}(2^m) \quad \text{and} \quad \mathbb{C}l_{2m+1} \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m).$$

Now we may borrow a classical fact from the representation theory of matrix algebras which states:

Theorem 2.21 ([23] Theorem 5.6.). *The natural representation of $\mathbb{C}(n)$ on $\mathbb{C}^n$ is -up to isomorphism- the only irreducible representation of $\mathbb{C}(n)$. Moreover the algebra $\mathbb{C}(n) \oplus \mathbb{C}(n)$ has exactly two equivalence classes of irreducible representations given by*

$$\rho_1(\phi_1, \phi_2) = \rho(\phi_1) \quad \textit{and} \quad \rho_2(\phi_1, \phi_2) = \rho(\phi_2)$$

where ρ is the representation of $\mathbb{C}(n)$ acting on $\mathbb{C}^n$.

Therefore using the above theorem we conclude that

Theorem 2.22 ([23] Theorem 5.7.). *Let $\nu_n^{\mathbb{C}}$ be the number of inequivalent irreducible complex representations of $\mathbb{C}l_n$. Then*

$$\nu_n^{\mathbb{C}} = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Furthermore, for both $n = 2m$ and $n = 2m + 1$ the dimension of any irreducible representation of $\mathbb{C}l_n$ is 2^m.

There is an element of Cl_n namely the **volume element** which is denoted by ω and is defined by

$$\omega = e_1 \dots e_n.$$

If $(e'_1, \dots, e'_n)$ is another orthonormal basis then using Equation 2.13 it follows that

$$e_1 \dots e_n = \det(g) e'_1 \dots e'_n$$

where g is the change of basis matrix. Therefore, ω does not depend on the choice of basis. For the complex case the volume element of $\mathbb{C}l_n$ is given by

$$\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \dots e_n.$$

We may immediately observe that $\omega_{\mathbb{C}}^2 = 1$ and that $\omega_{\mathbb{C}}$ is central when n is odd. We also obtain the decomposition

$$\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$$

where

$$\mathbb{C}l_n^{\pm} = (1 \pm \omega_{\mathbb{C}}) \mathbb{C}l_n.$$

Proposition 2.23 ([23] Proposition 3.6.). *Suppose n is even and let W be any representation of $\mathbb{C}l_n$. Then there is a decomposition*

$$W = W^+ \oplus W^-$$

into the ± 1 eigenspaces of $\rho(\omega_{\mathbb{C}})$. In fact

$$W^+ = (1 + \rho(\omega_{\mathbb{C}}))W \quad \textit{and} \quad W^- = (1 - \rho(\omega_{\mathbb{C}}))W.$$

Furthermore, the action of any vector $v \in \mathbb{C}^n$ gives the isomorphisms

$$\rho(v) : W^+ \to W^- \quad \textit{and} \quad \rho(v) : W^- \to W^+.$$

Proof. The decomposition follows from the fact that $\omega_{\mathbb{C}}^2 = 1$ and that

$$\omega_{\mathbb{C}}(1 \pm \omega_{\mathbb{C}}) = \pm(1 \pm \omega_{\mathbb{C}}).$$

The isomorphisms follows by observing that

$$v(1 \pm \omega_{\mathbb{C}}) = (1 \mp \omega_{\mathbb{C}})v$$

and

$$\rho(v) \circ \rho(-v) = \mathrm{id}.$$

□

Proposition 2.24 ([23] Proposition 5.9.). *Let $\rho : \mathbb{C}l_n \to Hom(W, W)$ be any irreducible representation where n is odd. Then either*

$$\rho(\omega_{\mathbb{C}}) = Id \quad \textit{or} \quad \rho(\omega_{\mathbb{C}}) = -Id.$$

Both possibilities can occur and the corresponding representations are inequivalent.

Proof. Since $\omega_{\mathbb{C}}$ is central, the possibilities of the two cases follow from the fact that both $\mathbb{C}l_n^{\pm}$ are $\mathbb{C}l_n$-modules.

To see that $\rho(\omega_{\mathbb{C}}) = id$ or $\rho(\omega_{\mathbb{C}}) = -id$, first note that $\rho^2(\omega_{\mathbb{C}}) = 1$. Therefore we can let $W^{\pm}$ to be the ± 1 eigenspaces for $\rho(\omega_{\mathbb{C}})$. Since $\omega_{\mathbb{C}}$ is central both $W^{\pm}$ are $\mathbb{C}l_n$-invariant and it follows from the irreducibility of W that $W^+ = W$ or $W^- = W$.

Let $\rho_{\pm}$ be the two representations above with $\rho_{\pm}(\omega_{\mathbb{C}}) = \pm Id$; since for any isomorphism $F : W \to W$ we have $F \circ \rho_{\pm}(\omega_{\mathbb{C}}) \circ F^{-1} = \rho_{\pm}$, the two representations can not be equivalent. $\square$

Proposition 2.25 ([23] Proposition 5.10.). *Let $\rho : \mathbb{C}l_n \to Hom(W, W)$ be any irreducible representation where n is even. And consider the decomposition*

$$W = W^+ \oplus W^-$$

where $W^{\pm} = (1 \pm \rho(\omega_{\mathbb{C}}))W$ as in Proposition 2.23 above. Then each W^+ and W^- is invariant under the even subalgebra $\mathbb{C}l_n^0$. Under the isomorphism $\mathbb{C}l_n^0 \cong \mathbb{C}l_{n-1}$ (Theorem 2.13), these spaces are the two irreducible representations of $\mathbb{C}l_{n-1}$ discussed in the previous proposition.

Proof. Since any even element of the Clifford algebra commutes with the volume element $\omega_{\mathbb{C}}$ both $W^{\pm}$ are invariant under the action of the even subalgebra $\mathbb{C}l_n^0$.

Furthermore by Theorem 2.22 and Proposition 2.23 $W^{\pm}$ are irreducible representations of $\mathbb{C}l_n^0 \cong \mathbb{C}l_{n-1}$.

Under the isomorphism $\mathbb{C}l_{n-1} \xrightarrow{\simeq} \mathbb{C}l_n^0$ the volume element $\omega'_{\mathbb{C}} \in \mathbb{C}l_{n-1}$ is sent to the volume element $\omega_{\mathbb{C}} \in \mathbb{C}l_n^0$. Hence $\rho(\omega'_{\mathbb{C}}) = \text{id}$ on W^+ and $\rho(\omega'_{\mathbb{C}}) = -\text{id}$ on W^- and it follows by the previous proposition that these two representations are inequivalent. $\square$

Now we get to the main concept of this section, that of Spin representation.

Definition 2.26. Suppose that Δ is an irreducible representation of $\mathbb{C}l_n$. The **Spin Representation** of the group $\text{Spin}(n)$ is the homomorphism $\text{Spin}(n) \to \text{GL}(\Delta)$ given by restricting the representation $\mathbb{C}l_n \to \text{Hom}(\Delta, \Delta)$ to $\text{Spin}(n) \subset Cl_n^0 \subset \mathbb{C}l_n$.

Since $\mathbb{C}l_n$ has two irreducible representations when n is odd one has to check that there is no ambiguity in the above definition. In fact we have

Proposition 2.27 ([23] Proposition 5.15.)**.** *When n is odd, the above definition is independent of which irreducible Representation of $\mathbb{C}l_n$ is used. Furthermore when n is odd the Spin representation is irreducible and when n is even there is a decomposition*

$$\Delta = \Delta^+ \oplus \Delta^-$$

into a direct sum of two inequivalent irreducible representations of Spin(n).

Proof. Suppose that n is odd we saw that the two irreducible representations of $\mathbb{C}l_n$ are basically the inner action of the Clifford algebra on $\mathbb{C}l_n^{\pm}$. First note that both $\mathbb{C}l_n^{\pm}$ are invariant under the action of $\mathbb{C}l_n^0$. Moreover we have the automorphism $\alpha : \mathbb{C}l_n \to \mathbb{C}l_n$ that interchanges $\mathbb{C}l_n^+$ with $\mathbb{C}l_n^-$ and vise versa. One can see that any even element $x \in \mathbb{C}l_n^0$ can be written as $x = \phi + \alpha(\phi)$ with respect to the decomposition $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$.
Now let $c(x)$ denotes the Clifford action of x on $\mathbb{C}l_n^+$ or on $\mathbb{C}l_n^+$ since $c(x)$ commutes with α the actions of $\mathbb{C}l_n^0$ on $\mathbb{C}l_n^{\pm}$ are equivalent.
On the other hand if n is even, by the previous proposition, restricting the action of the Clifford algebra to the even subalgebra $\mathbb{C}l_n^0$ splits the irreducible representation Δ into two parts

$$\Delta = \Delta^+ \oplus \Delta^-$$

each of which are inequivalent irreducible representations of $\mathbb{C}l_n^0 \cong \mathbb{C}l_{n-1}$.
Finally, since any element of Cl_n^0 can be written as a finite sum of elements in Spin(n), restricting an irreducible representation of Cl_n^0 to Spin(n) remains irreducible. $\square$

At the end of this section we will prove that any representation of $\mathbb{C}l_n$ has a natural inner product with respect to which the Clifford action of any vector is skew-adjoint.

Proposition 2.28 ([23] Proposition 5.16.). *Let W be a representation of $\mathbb{C}l_n$. Then there exists an inner product on W such that the Clifford action of any unit vector $e \in \mathbb{R}^n$ is unitary. i.e.*

$$\langle e.w, e.w' \rangle = \langle w, w' \rangle.$$

Proof. Let $F_n \subset Cl_n^\times$ be the finite group generated by an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. Choose a Hermitian inner product on W and average it over the action of F_n. Now if $e = \sum_i a_i e_i$ with $\sum_i a_i^2 = 1$, then

$$\langle ew, ew' \rangle = \sum_i a_i^2 \langle e_i w, e_i w' \rangle + \sum_{i \neq j} a_i a_j \langle e_i w, e_j w' \rangle = \langle w, w' \rangle$$

since $\langle e_i w, e_i w' \rangle = \langle w, w' \rangle$, and $\langle e_i w, e_j w' \rangle = \langle e_j e_i w, -w' \rangle = \langle e_i e_j w, w' \rangle = -\langle e_j w, e_i w' \rangle$ for $i \neq j$. □

Corollary 2.29. *Let $\langle ., . \rangle$ be the inner product defined in the above proposition. Then the Clifford action of any $v \in \mathbb{R}^n$ is skew-adjoint with respect to $\langle ., . \rangle$.*

Proof. Assume that $v \neq 0$. Then

$$\langle v.w, w' \rangle = \langle \frac{v}{\|v\|}.v.w, \frac{v}{\|v\|}.w' \rangle = \frac{1}{\|v\|^2} \langle v^2.w, v.w' \rangle = -\langle w, v.w' \rangle.$$

□

2.5 The Lie algebra of Spin(n)

As the universal covering of SO(n) the Lie group Spin(n) inherits its Lie algebra from the Lie algebra of SO(n) which is the algebra of skew-symmetric matrices. However it is worthwhile, for some computional purposes, to see how this Lie algebra structure relates to the Clifford structure.

The group of invertibles

$$Cl_n^\times = \{\phi \in Cl_n : \phi \text{ is invertible}\}$$

is an open subset of the Clifford algebra and consequently a Lie group of dimension 2^n. Its Lie algebra is the Clifford algebra $\mathfrak{cl}_n^\times = Cl_n$ with the Lie bracket

$$[x, y] = xy - yx.$$

The adjoint representation is given by

$$\begin{aligned} &\mathrm{Ad} : Cl_n^\times \to \mathrm{Aut}(Cl_n) \\ &\mathrm{Ad}_\phi(x) = \phi x \phi^{-1} \end{aligned}$$

and the corresponding adjoint representation at the level of Lie algebras is

$$\begin{aligned} &\mathrm{ad} : \mathfrak{cl}_n^\times \to \mathrm{End}(Cl_n) \\ &\mathrm{ad}_y(x) = [y, x]. \end{aligned}$$

Proposition 2.30 ([23] Proposition 6.1.). *The Lie subalgebra of $(Cl_n, [.,.])$ corresponding to the subgroup $Spin(n) \subset Cl_n^\times$ is*

$$\mathfrak{spin}(n) = \bigwedge^2 \mathbb{R}^n.$$

Proof. Let $(E_1, \ldots, E_n)$ be the standard basis of $\mathbb{R}^n$ and consider the family of curves

$$\{(E_i \cos t + E_j \sin t)(-E_i \cos t + E_j \sin t)\}_{i<j}.$$

These curves lie in $\mathrm{Spin}(n)$ and are tangent to 1 at $t = 0$.
Let $\gamma(t) = (E_i \cos t + E_j \sin t)(-E_i \cos t + E_j \sin t) = \cos 2t + E_i E_j \sin 2t$. Since $\frac{d}{dt}|_{t=0}\gamma(t) = 2E_iE_j$, the Lie algebra $\mathfrak{spin}(n)$ contains $\mathrm{span}\{E_iE_j\}_{i<j} = \bigwedge^2 \mathbb{R}^n$.

On the other hand as $\dim \mathfrak{spin}(n) = \dim \bigwedge^2 \mathbb{R}^n = n(n-1)/2$ we get the desired equality. $\square$

There is natural correspondence between $\mathfrak{so}(n)$ and $\bigwedge^2 \mathbb{R}^n$ that sends the element $v \wedge w$ to the orthogonal transformation $v \curlywedge w$ defined by

$$v \curlywedge w(x) = \langle v, x\rangle w - \langle w, x\rangle v.$$

The set $\{E_i \curlywedge E_j\}_{i<j}$ constitutes a natural basis for $\mathfrak{so}(n)$.

Proposition 2.31 ([23] Proposition 6.2.). *let $\xi : Spin(n) \to SO(n)$ be the double covering. Then the corresponding isomorphism at the level of the Lie algebras is given by*

$$\begin{aligned} \Xi := \xi_* : &\mathfrak{spin}(n) \to \mathfrak{so}(n) \\ &E_i E_j \mapsto 2E_i \curlywedge E_j. \end{aligned}$$

Consequently for $v, w \in \mathbb{R}^n$

$$\Xi^{-1}(v \curlywedge w) = \frac{1}{4}[v, w].$$

Proof. Let $\gamma(t) = \cos t + E_i E_j \sin t = \exp(t E_i E_j)$ as in the proof of the previous proposition. Since $\xi(\phi)(x) = \mathrm{Ad}_\phi(x)$

$$\Xi(E_i E_j)(x) = \frac{d}{dt}|_{t=0}\xi(\gamma(t))(x) = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{\gamma(t)}(x) = \mathrm{ad}_{E_i E_j}(x) = [E_i E_j, x].$$

$$\begin{aligned} \Xi(E_i E_j)(x) =& E_i E_j x - x E_i E_j \\ =& E_i E_j x + (E_i x + 2\langle E_i, x\rangle) E_j \\ =& E_i E_j x - E_i E_j x - 2\langle E_j, x\rangle E_i + 2\langle E_i, x\rangle E_j \\ =& 2E_i \curlywedge E_j(x). \end{aligned}$$

$\square$

Chapter 3

Dirac Operators and Spin Structures on Manifolds

3.1 Clifford Bundles and Dirac Operators

In this chapter, we start our study of Dirac operators on vector bundles. In order to define the Dirac operator, the vector bundle under consideration needs to carry some special structure namely the Clifford module structure. For our purposes, we always assume that the base manifold is a closed oriented Riemannian manifold.
Before we start our study of Clifford bundles we briefly recall some notions from the theory of vector bundles and connections.

Let $S \to M$ be a vector bundle. A covariant derivative on S is a linear map

$$\nabla : C^\infty(S) \to C^\infty(T^*M \otimes S)$$

that satisfies the following equation:

$$\nabla(fs) = df \otimes s + f\nabla(s) \qquad \text{for all } f \in C^\infty(M), s \in C^\infty(S).$$

Remark 3.1. Since any covariant derivative is induced from a connection on the corresponding principal bundle the words "covariant derivative" and "connection" are used interchangeably for the above mapping when the context is clear.

Using the contraction between TM and T^*M, for any $X \in C^\infty(TM)$, we obtain a linear operator

$$\nabla_X : C^\infty(S) \to C^\infty(S)$$

which is basically the directional derivative along X.
A connection can be extended to all types of tensor fields by the following properties:

- It acts on functions (e.g. $(0,0)$ tensor fields) as the exterior derivative $\nabla f = df$ for any $f \in C^\infty(M)$.
- It satisfies the Leibniz rule $\nabla(a \otimes b) = \nabla(a) \otimes b + a \otimes \nabla(b)$. i.e. it defines a derivation

$$\nabla : C^\infty(\mathcal{T}(S) \otimes \mathcal{T}(S^*)) \to C^\infty(T^*M \otimes \mathcal{T}(S) \otimes \mathcal{T}(S^*))$$

- It commutes with any contraction.

In particular, if $\langle , \rangle \in C^\infty(S^* \otimes S^*)$ is a metric on S, we say that the connection is metric (or compatible with the metric) if $\nabla\langle , \rangle = 0$. This is equivalent to saying that

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle,$$

for any sections $s_1, s_2 \in C^\infty(S)$.
Another equivalent characterization of a metric connection is that the induced parallel transports along any curve preserve the inner product on the corresponding fibers.

Note that a connection is a linear map

$$\nabla : \Omega^0(M;S) \to \Omega^1(M;S)$$

and it can be extended to the family of maps

$$\ldots \xrightarrow{\nabla} \Omega^r(M;S) \xrightarrow{\nabla} \Omega^{r+1}(M;S) \xrightarrow{\nabla} \ldots$$

by enforcing the Leibniz rule

$$\nabla(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^{\deg \alpha} \wedge \nabla\sigma.$$

Also recall that an operator of high significance is

$$K := \nabla^2 : \Omega^0(M;S) \to \Omega^2(M;S)$$

which is called the **curvature tensor operator**. In fact, since

$$K(fs) = fKs \qquad \text{for any } f \in C^\infty(M)\,, s \in C^\infty(S)$$

the value of Ks at any point $p \in M$ depends only on $s(p)$. Therefore K is a two form with values in the algebra $\mathrm{End}(S)$. One can show that for any $X, Y \in C^\infty(TM)$ and $s \in C^\infty(S)$ we have

$$K(X,Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s.$$

Furthermore, if the connection is metric, K is an element of $\Omega^2(M;\mathrm{Ad}S)$, where $\mathrm{Ad}S$ denotes the set of skew-adjoint endomorphisms of S.

Suppose that M is an oriented Riemannian Manifold. Since at each point the tangent space T_xM is endowed with a nondegenerate (in fact positive definite) inner product it makes sense to talk about the bundle of Clifford algebras over M. Let $Cl(TM)$ denotes the bundle of Clifford algebras. i.e. a complex vector bundle whose fiber over $x \in M$ is the Clifford algebra $\mathbb{C}l(T_xM)$.
Let S be a bundle of Clifford modules. This means each fiber S_x is a $\mathbb{C}l(T_xM)$-module.

Definition 3.2. We call S a **Clifford module bundle** (or briefly a Clifford bundle) if it is equipped with a Hermitian metric and a metric connection such that

1. The Clifford action of any tangent vector is skew-adjoint. i.e. For each $v \in T_xM$ and any $s_1, s_2 \in S_x$
$$\langle vs_1, s_2 \rangle = -\langle s_1, vs_2 \rangle$$
2. The connection on S is compatible with the Levi-Civita connection on TM in the following sense:
$$\nabla_X(Ys) = (\nabla_X Y)s + Y\nabla_X s$$
for any $X, Y \in C^\infty(TM)$, and $s \in C^\infty(S)$.

Remark 3.3. We can think of the Clifford action $c : X \mapsto c(X)$ as an element of $\Omega^1(M; \mathrm{End}(S)$. In this respect the second property above is saying that $\nabla c = 0$.

Remark 3.4. Using the second property in the definition of a Clifford bundle above, one can show that the connection on S is a derivation with respect to the $Cl(TM)$-module structure. i.e. in a Clifford bundle we have

$$\nabla(\phi s) = (\nabla\phi)s + \phi\nabla s \qquad \text{for } \phi \in C^\infty(Cl(TM))\,,\, s \in C^\infty(S)$$

To verify this we first observe that the exterior bundle $\bigwedge^* TM$ inherits a connection from the Levi-Civita connection and the covariant derivative is an even derivation of the graded algebra $C^\infty(\bigwedge^* TM)$. The above result follows once we use the canonical identification $Cl(TM) \cong \bigwedge^* TM$.

Definition 3.5. The **Dirac operator** of a Clifford bundle S which is denoted by D is defined by the following compositions

$$C^\infty(S) \to C^\infty(T^*M \otimes S) \to C^\infty(TM \otimes S) \to C^\infty(S)$$

where the first map is given by the connection of S, the second map is just identification of the cotangent bundle with the tangent bundle, and the third map is the Clifford action of TM on S.

Given a local orthonormal frame $(e_1, \ldots, e_n)$ of TM, the Dirac operator can be written locally in the following form

$$D = \sum_i e_i \nabla_{e_i}.$$

Remark 3.6. In a local coordinate $(x^1, \ldots, x^n)$ the Dirac operator has the following expression:

$$D = g^{ji}\partial_j \nabla_{\partial_i} = \partial^i \nabla_{\partial_i}.$$

Now we shall show that D^2 is equal-modulo a zero order differential operator- to the Laplacian. Before going through the study of the general case it is worthwhile to see this fact in a trivial example.

Example 3.7. Let M be the flat euclidean space $\mathbb{R}^n$ and let S be the trivial flat bundle $\mathbb{R}^n \times V$, where V is a $\mathbb{C}l_n$-module given by the map

$$\gamma : \mathbb{C}l_n \to \mathrm{Hom}(V, V)$$

We first observe that γ commutes with the partial derivatives. Moreover, by Corollary 2.29, we can always choose a Hermitian inner product on V so that the Clifford action of vectors becomes skew-adjoint therefore S is Clifford bundle.If $(E_1, \ldots, E_n)$ is the standard orthonormal basis of $\mathbb{R}^n$, the Dirac operator can be written as

$$D = \sum_{i=1}^{n} \gamma_i \partial_i$$

where $\gamma_i = \gamma(E_i)$. We obtain:

$$\begin{aligned} D^2 = \sum_{i,j} \gamma_j \partial_j (\gamma_i \partial_i) = \sum_{i,j} \gamma_j \gamma_i \partial_j \partial_i &= \sum_i \gamma_i^2 \partial_i^2 + \sum_{i<j} (\gamma_j \gamma_i + \gamma_i \gamma_j) \partial_j \partial_i \\ &= -\sum_i \partial_i^2, \end{aligned}$$

which is the standard Laplacian.

Definition 3.8. Let K be the the curvature tensor. The Clifford contraction of the curvature denoted by $\mathcal{K}$ is the element of $\mathrm{End}(S)$ given by

$$\mathcal{K} = \sum_{i<j} e_i e_j K(e_i, e_j)$$

where $(e_1, \ldots, e_n)$ is a local orthonormal frame for the tangent bundle.

Note that since $K(Y, X) = -K(X, Y)$ for any $X, Y \in TM$ the right hand side of the above equation is equal to

$$\frac{1}{2} \sum_{i,j} e_i e_j K(e_i, e_j).$$

Also one has to verify that this definition is independent of the choice of an orthonormal frame. If $(e'_1, \ldots, e'_n)$ is another orthonormal frame then $e'_i = \sum_r g_i^r e_r$ and

$$\frac{1}{2} \sum_{i,j} e'_i e'_j K(e'_i, e'_j) = \frac{1}{2} \sum_{i,j,r,s} (g_i^r)^2 (g_j^s)^2 e_r e_s K(e_r, e_s) = \frac{1}{2} \sum_{r,s} e_r e_s K(e_r, e_s)$$

since $(g_j^i) \in \mathrm{O}(n)$.

We now proceed to compute D^2, to this aim we prefer to work in a synchronous frame at some point $x \in M$. i.e. we pick an orthonormal basis of T_xM and then we extend it to a neighborhood via parallel transports along geodesics. The frame field obtained is the so called synchronous frame field for TM centered at x. Then by definition it has the property that $(\nabla_{e_i} e_j) = 0$ at x. The pairwise Lie bracket of the fields also vanishes at x. Therefore, at x we have

$$\begin{aligned}
D^2 s &= \sum_{i,j} e_j \nabla_j (e_i \nabla_i) s \\
&= \sum_{i,j} e_j e_i \nabla_j \nabla_i s \\
&= -\sum_i \nabla_i^2 s + \sum_{i<j} e_j e_i (\nabla_j \nabla_i - \nabla_i \nabla_j) s \\
&= -\sum_i \nabla_i^2 s + \mathcal{K} s.
\end{aligned}$$

We will show that the term $-\sum_i \nabla_i^2$ in the above formula is equal to $\nabla^* \nabla$, where ∇^* is the formal adjoint of ∇.

Lemma 3.9 ([28] Lemma 3.9.). *In a local coordinate the operator $\nabla^* : C^\infty(T^*M \otimes S) \to C^\infty(S)$ can be given by the fomula*

$$\nabla^*(dx^j \otimes s_j) = -g^{jk}(\nabla_j s_k - \Gamma^i_{jk} s_i).$$

Therefore, if we choose normal coordinates,

$$\nabla^*(dx^j \otimes s_j) = -\nabla_j s_j$$

at the origin.(We have used the Einstein summation convention.)

Proof. Using the given expression for ∇^* we have to show that

$$(s, \phi) = (s, \nabla^* \phi) \quad \text{for any } s \in C^\infty(S),\, \phi \in C^\infty(T^*M \otimes S).$$

Note that the above inner products are actually the global L^2-inner products on $C^\infty(S)$, and $C^\infty(T^*M \otimes S)$.

By the divergence theorem it suffices though to show that the difference between the fiber-wise inner products $\langle \nabla s, \phi \rangle - \langle s, \nabla^* \phi \rangle$ is equal to $\operatorname{div}(Z)$ for some vector field Z.

Let $s \in C^\infty(S)$ and $\phi = dx^j \otimes s_j \in C^\infty(T^*M \otimes S)$. Since $\nabla s = dx^i \otimes \nabla_i s$ and $\nabla^* \phi = -g^{jk}(\nabla_j s_k - \Gamma^i_{jk} s_i)$ we have

$$\begin{aligned}
\langle \nabla s, \phi \rangle - \langle s, \nabla^* \phi \rangle &= -g^{ij} \langle \nabla_i s, s_j \rangle - g^{jk} \langle s, \nabla_j s_k \rangle + g^{jk} \Gamma^i_{jk} \langle s, s_i \rangle \\
&= -g^{jk} \partial_j \langle s, s_k \rangle + g^{jk} \Gamma^i_{jk} \langle s, s_i \rangle \\
&= -\operatorname{div}(Z)
\end{aligned}$$

where $Z = g^{ij} \langle s, s_j \rangle \partial_i$. More precisely the global expression for the vector field Z is $Z^\flat(Y) = \langle Y \otimes s, \phi \rangle$. □

The above discussion leads to the following formula:

Theorem 3.10 (Weitzenbock Formula)**.**

$$D^2 = \nabla^* \nabla + \mathcal{K}.$$

Remark 3.11. We saw that $\nabla^* \nabla$ is a positive operator:

$$\langle \nabla^* \nabla s, s \rangle = \| \nabla s \|^2 \geq 0.$$

Remark 3.12. Since the curvature tensor is a two form with value in AdS and the Clifford action of vectors is also skew-adjoint, the curvature contraction operator is a self-adjoint endomorphism of S.

Now we are ready to state and prove a theorem that shows how the geometry of the Clifford bundle can affect the existence of harmonic sections.

Theorem 3.13 (Bochner)**.** *([28] Theorem 3.10.) If the least eigenvalue of $\mathcal{K}$ at each point of M is positive, then the equation $D^2 s = 0$ does not have any non-trivial solution.*

Proof. Let Λ_x be the least eigenvalue of $\mathcal{K}$ at point x. Since $\mathcal{K} \in C^\infty(\mathrm{End})$, it follows from Rouche's theorem that Λ_x continuously depends on x. Hence we may define

$$\Lambda := \min_{x \in M} \Lambda_x.$$

By the assumption, Λ_x is positive everywhere thus Λ must be positive as well. Using the Weitzenbock formula we get

$$\begin{aligned} \langle D^2 s, s \rangle &= \langle \nabla^* \nabla s, s \rangle + \langle \mathcal{K} s, s \rangle \\ &= \|\nabla s\|^2 + \langle \mathcal{K} s, s \rangle \\ &\geq \langle \mathcal{K} s, s \rangle \\ &\geq \Lambda \|s\|^2. \end{aligned}$$

The result immediately follows as $\Lambda > 0$. □

The next proposition shows that the Dirac operator is formally self-adjoint.

Proposition 3.14 ([28] Proposition 3.11.)**.** *Let s_1 and s_2 be two sections of S, then*

$$\langle D s_1, s_2 \rangle = \langle s_1, D s_2 \rangle.$$

Proof. Similar to the argument in 3.9 we shall show that

$$\langle D s_1, s_2 \rangle - \langle s_1, D s_2 \rangle = \mathrm{div}(V),$$

for some vector field V.

Choose an orthonormal frame $(e_1, \dots, e_n)$ synchronous at x. We have

$$\begin{aligned}\langle Ds_1(x), s_2(x)\rangle &= \langle e_i \nabla_i s_1(x), s_2(x)\rangle \\ &= -\langle \nabla_i s_1(x), e_i s_2(x)\rangle \\ &= -(e_i\langle s_1, e_i s_2\rangle)_x + \langle s_1(x), \nabla_i(e_i s_2)(x)\rangle \\ &= -(e_i\langle s_1, e_i s_2\rangle)_x + \langle s_1(x), e_i \nabla_i s_2(x)\rangle \\ &= -(e_i\langle s_1, e_i s_2\rangle)_x + \langle s_1(x), Ds_2(x)\rangle \\ &= -\mathrm{div}(V)(x) + \langle s_1(x), Ds_2(x)\rangle\end{aligned}$$

where V is the vector field whose components are $\langle s_1(x), e_i s_2(x)\rangle$. i.e

$$V^\flat(W) = \langle s_1, W s_2\rangle.$$

To verify this we notice that

$$\begin{aligned}\mathrm{div}(V)(x) = \sum_j \langle \nabla_j V, e_j\rangle_x = \sum_{j,i}\langle \nabla_j \langle s_1, e_i s_2\rangle e_i, e_j\rangle_x &= \sum_{j,i}\langle e_j\langle s_1, e_i s_2\rangle e_i, e_j\rangle_x \\ &= \sum_i (e_i\langle s_1, e_i s_2\rangle)_x.\end{aligned}$$

□

We saw that the connection does not commute with the Clifford action. e.g.

$$[\nabla_Y, c(X)] = c(\nabla_Y X).$$

Now we might ask whether the curvature tensor commutes with the Clifford action or not? Although (as we will see shortly) the answer to this question is negative in general, it is always possible to decompose the curvature into two parts $K = R^S + F^S$ so that F^S always commutes with the Clifford action.

Lemma 3.15 ([28] Lemma 3.13.)**.** *Let R denote the (Riemannian) curvature corresponding to the Levi-Civita connection on TM. And let K be the curvature operator of S. Then*

$$[K(X,Y), c(Z)] = c(R(X,Y)Z),$$

for any tangent vectors X, Y, Z.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal frame for TM synchronous at a point $p \in M$. We can assume that $X = e_i|_p, Y = e_j|_p, Z = e_k|_p$. The result follows by noticing that at point p

$$\nabla_i \nabla_j (e_k s) = (\nabla_i \nabla_j e_k)s + e_k \nabla_i \nabla_j s.$$

□

Definition 3.16. For a Clifford bundle S, the Riemann endomorphism of S denoted by R^S is an $\mathrm{End}(S)$-valued two form defined by

$$R^S(X,Y) = \frac{1}{4} \sum_{k,l} c(e_k)c(e_l)\langle R(X,Y)e_k, e_l\rangle.$$

It is easy to check that this definition is independent of the choice of orthonormal basis.

Lemma 3.17 ([28] Lemma 3.15.)**.**

$$[R^S(X,Y), c(Z)] = c(R(X,Y)Z)$$

Proof. The Riemannian curvature R is a 3-covariant, 1-contravariant tensor. Recall that the corresponding 4-covariant tensor is given by

$$R_{lkij} = \langle R(e_i, e_j)e_k, e_l\rangle.$$

This means

$$R(e_i, e_j)e_k = \sum_k R_{lkij} e_l.$$

We can assume that $X = e_i$, $Y = e_j$, $Z = e_a$, then

$$R^S(e_i, e_j) = \frac{1}{4} \sum_{k,l} c(e_k)c(e_l)R_{lkij}.$$

Therefore,

$$R^S(e_i, e_j)c(e_a) = \frac{1}{4} \sum_{k,l} R_{lkij}\{c(e_k)c(e_l)c(e_a) - c(e_a)c(e_k)c(e_l)\}.$$

The above expression vanishes when $k = l$, or when k, l, a are all distinct. So the only remaining terms are those where $a = k \neq l$ or $a = l \neq k$ and the above expression reduces to

$$\begin{aligned} R^S(e_i, e_j)c(e_a) &= \frac{1}{2} \sum_l R_{laij} c(e_l) - \frac{1}{2} \sum_k R_{akij} c(e_k) \\ &= \sum_l R_{laij} c(e_l) \\ &= c(R(e_i, e_j)e_a) \end{aligned}$$

since $R_{la**} = -R_{al**}$. □

From Lemma 3.15 and Lemma 3.17 we can conclude that

Proposition 3.18 ([28] Proposition 3.16.)**.** *The curvature of a Clifford bundle can be written as*

$$K = R^S + F^S$$

where R^S is the Riemann endomorphism of S and F^S is an an $End(S)$-valued two form that commutes with the Clifford action.

Lemma 3.19.

$$\sum_{i,j,k} R_{lkij} e_i e_j e_k = -2 \sum_j \boldsymbol{Ric}_{lj} e_j$$

where

$$\boldsymbol{Ric}_{lj} = \sum_i R_{ilij}$$

is the Ricci tensor.

Proof. Suppose that i, j, k are distinct then

$$e_i e_j e_k = e_k e_i e_j = e_j e_k e_i.$$

Using the first Bianchi identity

$$R_{lkij} + R_{lijk} + R_{ljki} = 0,$$

we get

$$R_{lkij} e_i e_j e_k + R_{lijk} e_j e_k e_i + R_{ljki} e_k e_i e_j$$
$$+ R_{lkji} e_j e_i e_k + R_{ljik} e_i e_k e_j + R_{likj} e_k e_j e_i = 0.$$

Also the terms with $i = j$ vanish since $R_{**ij} = -R_{**ji}$. Therefore,

$$\begin{aligned}\sum_{i,j,k} R_{lkij} e_i e_j e_k &= \sum_{i,j} R_{liij} e_j + \sum_{i,j} R_{ljij} e_i \\ &= -2 \sum_{i,j} R_{ilij} e_j \\ &= -2 \sum_j \mathbf{Ric}_{lj} e_j.\end{aligned}$$

Now we can obtain a more refined version of Weitzenbock formula 3.10

□

Theorem 3.20 (Lichnerowicz-Schrödinger). *([28] Proposition 3.18.)*

$$D^2 = \nabla^* \nabla + \mathcal{F}^S + \frac{1}{4}\kappa$$

where $\mathcal{F}^S = \sum_{i<j} e_i e_j F^S(e_i, e_j)$ is the Clifford contraction of the twisting curvature, and κ is the scalar curvature of the Riemannian manifold.

Proof. Comparing with the Weitzenbock formula it is enough to show that

$$\sum_{i<j} e_i e_j R^S(e_i, e_j) = \frac{1}{4}\kappa.$$

Using the definition of R^S (Definition 3.16), we have

$$\begin{aligned}\sum_{i<j} e_i e_j R^S(e_i, e_j) = \frac{1}{2}\sum_{ij} e_i e_j R^S(e_i, e_j) = \frac{1}{8}\sum_{ijkl} e_i e_j e_k e_l \langle R(e_i, e_j)e_k, e_l\rangle &= \frac{1}{8}\sum_{ijkl} e_i e_j e_k e_l R_{lkij} \\ &= -\frac{1}{4}\sum_{lj} \mathbf{Ric}_{lj} e_j e_l \\ &= +\frac{1}{4}\sum_{l} \mathbf{Ric}_{ll} \\ &= \frac{1}{4}\kappa.\end{aligned}$$

□

3.2 $\bigwedge^* TM^* \otimes \mathbb{C}$ as a Clifford bundle

In this section we get to a well-known example of a Clifford bundle namely the exterior bundle. In fact Proposition 2.7 suggests that $\bigwedge^* TM^* \otimes \mathbb{C}$ is a Clifford bundle. Recall that $\bigwedge^* V^*$ is a $Cl(V, Q)$- module and the action of a vector $v \in V$ on $\phi \in \bigwedge^* V^*$ is given by

$$v.\phi = (v^\flat \wedge -\iota_v)\phi.$$

Proposition 3.21 ([28] Lemma 3.21.)**.** *The exterior bundle $\bigwedge^* TM^* \otimes \mathbb{C}$ equipped with its natural metric, the Levi-Civita connection, and the above Clifford action is a Clifford bundle.*

Proof. We have to check two things:

i. The Clifford action of any tangent vector is skew-adjoint.

ii. The Clifford action is compatible with the connection.

Suppose $\dim(M) = n$. For part (i) let ω_1 be a $k-1$-form, ω_2 be a k-form, and $X \in TM$. Since $\langle \xi, \eta \rangle = *(\xi \wedge *\eta)$ and if ω is a p -form $\iota_v \omega = (-1)^{n(p+1)} * (v^\flat \wedge *\omega)$ we have

$$\begin{aligned}\langle \omega_1, \iota_X \omega_2 \rangle = &* (\omega_1 \wedge * \iota_X \omega_2) = (-1)^{n(k+1)} * (\omega_1 \wedge * * (X^\flat \wedge *\omega_2)) \\ =&(-1)^{n(k+1)+(n-k+1)(k-1)} * (\omega_1 \wedge X^\flat \wedge *\omega_2) \\ =&(-1)^{n(k+1)+(n-k+1)(k-1)+(k-1)} * (X^\flat \wedge \omega_1 \wedge *\omega_2) = \langle X^\flat \wedge \omega_1, \omega_2 \rangle.\end{aligned}$$

The above observation shows that the interior product is the adjoint of the wedge product and accordingly the Clifford action is skew adjoint.
Part (ii) is obvious since both wedge product and interior product are compatible with the connection. □

Considering $\bigwedge^* TM^* \otimes \mathbb{C}$ as a Clifford bundle one might ask what is the corresponding Dirac operator. The following proposition answers this question.

Proposition 3.22 ([21] Lemma 4.3.4.). *The Dirac operator of the Clifford bundle $\bigwedge^* TM^* \otimes \mathbb{C}$ is equal to*

$$D = d + d^*.$$

Consequently D^2 is the Laplace-Beltrami operator $dd^* + d^*d$.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal frame and $(\eta^1, \ldots, \eta^n)$ be the dual coframe. We will prove that

$$d = \eta^i \wedge \nabla_{e_i} \quad \text{and,} \quad d^* = -\iota_{e_i} \nabla_{e_i}.$$

First let

$$\tilde{d} := \eta^i \wedge \nabla_{e_i} \quad \text{and,} \quad \tilde{d}^* := -\iota_{e_i} \nabla_{e_i} \tag{3.1}$$

We are going to show that $d = \tilde{d}$ and $d^* = \tilde{d}^*$ in two main steps:

1- The definition of $\tilde{d}$ and $\tilde{d}^*$ is independent of the choice of frame.

2- The desired equalities hold at the centre of a normal coordinate.

1- Suppose that $(f_1, \ldots, f_n)$ is another orthonormal frame with dual $(\zeta^1, \ldots, \zeta^n)$ then

$$f_i = a_i^k e_k \quad , \quad \zeta^i = b_k^i \eta^k$$

with

$$a_i^k b_l^i = \delta_l^k \quad , \quad a_i^k a_i^l = \delta^{kl}$$

We have

$$\zeta^i \wedge \nabla_{f_i} = a_i^k b_l^i \eta^l \wedge \nabla_{e_k} = \eta^k \wedge \nabla_{e_k}$$

and

$$-\iota_{f_i} \nabla_{f_i} = -a_i^k a_i^l \iota_{e_k} \nabla_{e_l} = -\iota_{e_k} \nabla_{e_k}$$

2 Choose a normal coordinated $(x^1, \ldots, x^n)$ centred at x_0. Let $e_i = \frac{\partial}{\partial x^i}$ and $\eta^i = dx^i$. These frames are orthonormal at the centre. Also at the centre we have

$$\nabla_{\frac{\partial}{\partial x^i}} dx^j = 0$$

Therefore we get, at the point x_0,

$$\begin{aligned}
\tilde{d}(\phi(x) dx^1 \wedge \cdots \wedge dx^p) =& dx^i \wedge (\nabla_{\frac{\partial}{\partial x^i}} \phi) dx^1 \wedge \cdots \wedge dx^p \\
=& \frac{\partial \phi}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^p \\
=& d(\phi(x) dx^1 \wedge \cdots \wedge dx^p)
\end{aligned}$$

and

$$\begin{aligned}\tilde{d}^*(\phi(x)dx^1 \wedge \cdots \wedge dx^p) =& -\iota_{\frac{\partial}{\partial x^i}}(\frac{\partial \phi}{\partial x^i})dx^1 \wedge \cdots \wedge dx^p \\ =&(-1)^i(\frac{\partial \phi}{\partial x^i})dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^p \\ =&d^*(\phi(x)dx^1 \wedge \cdots \wedge dx^p)\end{aligned}$$

□

3.3 Spin Structures on Manifolds

Now we are going to study a significant example of Clifford bundle namely the bundle of spinors, or spinor bundle. In order to prepare the setting we first recall some basic facts and ideas about the theory of principal bundles and their associated bundles briefly.
Let $P \xrightarrow{\pi} M$ be a $G-$principal bundle. For each element $A \in \mathfrak{g}$ of the Lie algebra the infinitesimal action of the one-parameter subgroup $\{\exp(tA)\}_{t\in\mathbb{R}}$ defines a vector field $\tilde{A} \in C^\infty(TP)$ in some literature called **the fundamental vector field** of A. For instance

$$\tilde{A}_p f = \frac{d}{dt}|_{t=0} f(p.\exp(tA)) \quad \text{for any } f \in C^\infty(P)$$

The mapping $A \mapsto \tilde{A}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $C^\infty(TP)$.

Definition 3.23 (Vertical subspace). For each $p \in P$ the vertical subspace $V_pP \subset T_pP$ of the tangent space at point p is defined to be the kernel of the map

$$\pi_* : T_pP \to T_{\pi(p)}M$$

In fact it consists of all vectors $X \in T_pP$ such that

$$X(\pi^* f) = 0 \quad \text{for every } f \in C^\infty(M)$$

Lemma 3.24. *Let $A \in \mathfrak{g}$ and $\tilde{A}$ be the fundamental vector field of A. Then*

$$\text{For each } p \in P \quad \tilde{A}_p \in V_pP$$

It follows by the above lemma that we have a family of isomorphisms

$$\imath_p : \mathfrak{g} \to V_pP$$

Definition 3.25. Let $P \to M$ be a $G-$principal bundle a connection on P is a smooth distribution that assigns to each tangent spaceT_pP a subspace $H_pP \subset T_pP$ called the horizontal subspace in a way that

- H_pP is complement to V_pP : $H_pP \oplus V_pP = T_pP$.
- It is $G-$ invariant: $H_{p.g}P = g_*H_pP$.

Such a distribution defines for any vector field $X \in C^\infty(TP)$ two smooth vector fields X^V and X^H which are the vertical and the horizontal parts of X.

Definition 3.26 (connection form). Given a connection on P the connection form ω is constructed as follows

$$\omega_p : T_pP \to \mathfrak{g} \quad X \mapsto \imath_p^{-1}(X^V)$$

where $\imath_p$ is the isomorphism defined by Lemma 3.24. In fact ω takes any tangent vector to the corresponding Lie algebra element of its vertical part.

One can check that the connection form is smooth and hence defines a $\mathfrak{g}$-valued one form $\omega \in \Omega^1(P; \mathfrak{g})$. Furthermore the connection form adequately encodes the information about the connection. For instance given ω one can recover H_pP by

$$H_pP = \ker \omega_p$$

Proposition 3.27. *The connection form ω has the following properties*

- $\omega(\tilde{A}) = A$ *for any* $A \in \mathfrak{g}$.

$$\begin{tikzcd} V_p P \arrow[dr, "\omega_p"] & \\ \mathfrak{g} \arrow[u, "\imath_p"] \arrow[r, "id"'] & \mathfrak{g} \end{tikzcd}$$

- $g^*\omega = Ad_{g^{-1}}\omega$

where $Ad : G \to Aut(\mathfrak{g})$ *is the adjoint representation of the Lie group* G.

Definition 3.28. Let $\phi \in \Omega^k(P; W)$ be a differential k-form with values in a vector space W. The **exterior covariant derivative** of ϕ is a W-valued $k+1$-form given by

$$D\phi(X_1, \dots, X_{k+1}) := d\phi(X_1^H, \dots, X_{k+1}^H)$$

Since the connection form ω is a $\mathfrak{g}$-valued one form one might consider $D\omega$ which is an important geometric object called the curvature form of the connection.

Definition 3.29 (Curvature form). Let P equipped with a connection. The curvature form of the connection denoted by Ω is defined to be the exterior covariant derivative of the connection form

$$\Omega := D\omega$$

So Ω is an element of $\Omega^2(P; \mathfrak{g})$.

Proposition 3.30. *The following identities hold:*

1. $\Omega = d\omega + [\omega, \omega]$.

2. ***Bianchi identity*** $D\Omega = 0$.

where $[\omega_1, \omega_2](X, Y) := [\omega_1(X), \omega_2(Y)]$.

In practice one wishes to pull these global objects down to an open subset $U \subset M$ of the base manifold. If we choose a local section $\sigma : U \to P$ this induces two local objects:

1- a local connection form $A := \sigma^*\omega \in \Omega^1(U; \mathfrak{g})$ (sometimes called the gauge field)

2- a local curvature form $F := \sigma^*\Omega \in \Omega^2(U; \mathfrak{g})$ (sometimes called the gauge field strength)

It follows then by Proposition 3.30 that

Proposition 3.31.

$$F = dA + [A, A]$$

Now if we have two local sections $\sigma : U \to P$ and $\sigma' : U' \to P$ one might ask how the corresponding local connection and curvature forms (A, F), and (A', F') relate to one another. Given the transition functions

$$g : U \cap U' \to G$$

we will have

Proposition 3.32.

$$\begin{aligned} A' &= Ad_{g^{-1}}\omega + g^*\Theta^{M\text{-}C} \\ F' &= Ad_{g^{-1}}F \end{aligned}$$

where $\Theta^{M\text{-}C}$ is the Maurer-Cartan form of the Lie group G. In particular if G is a group of matrices the above equations take the form

$$\begin{aligned} A' &= g^{-1}Ag + g^{-1}dg \\ F' &= g^{-1}Fg \end{aligned}$$

We will see later that the family of local connection forms determines a covariant derivative on any associated vector bundle.

Now let V be a linear representation of G given by

$$\rho : G \to \mathrm{GL}(V)$$

and let $E = P \times_G V$ be the associated vector bundle. We shall proceed to define a covariant derivative on E using the connection form of P. This can be done globally if one considers a section $s \in C^\infty(E)$ as a G-equivariant vector valued function

$$f_s : P \to V$$

i.e.

$$f_s(p.g) = g^{-1} f(p)$$

In general let $\overline{\Omega^k}(P; W)$ denotes the set of W-valued k-forms on P that satisfy the following properties:

(H) $\iota_X \alpha = 0$ for any vertical vector X.

(E) $g^* \alpha = g^{-1} \alpha$.

Note that for a zero-form the property (E) automatically implies the property (H). With this notation, $\overline{\Omega^0}(P; W)$ is the set of G-equivariant W-valued functions on P.
We call $\theta \in \overline{\Omega^k}(P; W)$ a **basic** k-form. An important result says that the set of basic forms is invariant under the exterior covariant derivative and we have the map below well-defined

$$D : \overline{\Omega^k}(P; W) \to \overline{\Omega^{k+1}}(P; W)$$

Back to our discussion about an associated vector bundle. In this case we have a linear representation of G say V and the operator D takes the following expression

Proposition 3.33. *For a representation $\rho : G \to GL(V)$ and $\theta \in \overline{\Omega^k}(P;V)$ we have*

$$D\theta = d\theta + \rho_*(\omega) \wedge \theta$$

Also recall that there is a natural correspondence

$$\overline{\Omega^k}(P;V) \cong \Omega^k(M;E)$$

in particular

$$\overline{\Omega^0}(P;V) \cong \Omega^0(M;E)$$

as we mentioned before.

With this setting if the principal bundle P carries a connection one can define a covariant derivative ∇ on E as follows

$$\begin{tikzcd} \overline{\Omega^0}(P;V) \arrow[r, "D = d + \rho_*(\omega)"] & \overline{\Omega^1}(P;V) \arrow[d] \\ \Omega^0(M;E) \arrow[u] \arrow[r, "\nabla"] & \Omega^1(M;E) \end{tikzcd}$$

We wish to give a local expression for ∇:

Choose a local section $\sigma : U \to P$. Then any section $s \in C^\infty(E)$ takes a local expression

$$s = [\sigma, f] \quad \text{where } f : U \to V$$

and

$$\nabla s = [\sigma, df + \rho_*(\sigma^*\omega) f]$$

since $\sigma^*(\omega) = A$ we can say that locally

$$\nabla = d + A$$

Moreover if we take a basis $(f_1, \ldots, f_N)$ of V to get the local sections $\psi_\alpha := [\sigma, f_\alpha]$ then the action of ∇ is determined by its action on each ψ_α.

Remark 3.34. It follows by Proposition 3.32 (the transformation rule for curvature) that the curvature two form on the principal bundle (globally) defines an $\mathrm{End}(E)$-valued two form on any associated vector bundle E which is called the curvature operator of the vector bundle as we introduced before.

Definition 3.35 (Spin Structure). Let M be an n-dimensional oriented Riemannian manifold. Let $\mathrm{SO}(M)$ denotes the $\mathrm{SO}(n)$-principal bundle of the orthonormal frames of TM. A **spin structure** on M denoted by $\mathrm{Spin}(M)$ is a double covering of $\mathrm{SO}(M)$ and also a $\mathrm{Spin}(n)$-principal bundle over M so that the diagram below commutes

$$\begin{tikzcd}
\mathrm{Spin}(M) \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\eta \times \xi"] & \mathrm{Spin}(M) \arrow[d, "\eta"'] \\
\mathrm{SO}(M) \times \mathrm{SO}(n) \arrow[r] & \mathrm{SO}(M)
\end{tikzcd}$$

where $\eta : \mathrm{SO}(M) \to \mathrm{Spin}(M)$ is a double cover of $\mathrm{SO}(M)$ and $\xi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double cover of $\mathrm{SO}(n)$ discussed in Section 2.2.

Remark 3.36. The existence of a spin structure on M relates to the question that whether the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n)$ have a lift $\tilde{g}_{\alpha\beta} \in \mathrm{Spin}(n)$

$$\begin{tikzcd}
 & \mathrm{Spin}(n) \arrow[d, "\xi"'] \\
U_\alpha \cap U_\beta \arrow[ur, "\tilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & \mathrm{SO}(n)
\end{tikzcd}$$

so that they satisfy the cocycle condition

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1$$

This is a standard problem in algebraic topology and we just state a theorem that provides us with an answer to this question.

Theorem 3.37 ([6] Proposition 3.34.). *An oriented manifold M has a spin structure if and only if its second Stieffel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. If this is the case, then the different spin structures are parametrized by elements of $H^1(M, \mathbb{Z}_2)$.*

Definition 3.38. The **Spinor bundle** (or **bundle of spinors**) which we denote by $\not{S}$ is the associated vector bundle of $\mathrm{Spin}(M)$ with respect to the spin representation $\mathrm{Spin}(n) \to \mathrm{GL}(\Delta)$.

$$\not{S} = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \Delta$$

Any section $s \in C^\infty(\not{S})$ can be written locally as

$$s = [\tilde{\sigma}, f]$$

where

$$\tilde{\sigma} : U \to \mathrm{Spin}(M) \quad \text{and} \quad f : U \to \Delta$$

We shall show that a spinor bundle is a Clifford bundle.
The action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ induces an action on Cl_n since the ideal generated by the elements $v \otimes v + \langle v, v \rangle$ is invariant under the orthogonal transformations.
Hence we have the representation

$$\mathrm{SO}(n) \to \mathrm{Aut}(Cl_n) \tag{3.2}$$

This observation leads us to the view that the bundle of Clifford algebras $Cl(TM)$ can be described as the associated bundle of $\mathrm{SO}(M)$ with the above representation.

$$Cl(TM) = \mathrm{SO}(M) \times_{\mathrm{SO}(n)} Cl_n$$

On the other hand we have also the adjoint representation of $\mathrm{Spin}(n)$:

$$\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{Aut}(Cl_n)$$
$$g \mapsto \mathrm{Ad}_g$$
$$\mathrm{Ad}_g(\phi) = g\phi g^{-1}$$

Since $\mathrm{Ad}_1 = \mathrm{Ad}_{-1} = 1$ this representation descends to a representation of $\mathrm{SO}(n)$

$$\mathrm{Ad}' : \mathrm{SO}(n) \to \mathrm{Aut}(Cl_n)$$

which is the same as the representation in 3.2.
Indeed we can see this if we think of elements of $\mathrm{Spin}(n)$ as even numbers of reflections that generate the group of rotations as we discussed before.
Put all these together we obtain that $Cl(TM)$ is the associated bundle of $\mathrm{Spin}(M)$ with the adjoint representation

$$Cl(TM) = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} Cl_n.$$

In particular the tangent space itself is also an associated bundle of $\mathrm{Spin}(M)$ if one restrict the adjoint representation to $\mathbb{R}^n$.

$$TM = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{R}^n$$

The above discussion prepares the adequate setting to introduce the Clifford bundle structure of the bundle of spinors.

In fact given $s = [\tilde{\sigma}, f] \in \$$ and $X = [\tilde{\sigma}, v] \in TM$ the Clifford action of X on s is

$$X.s = [\tilde{\sigma}, v.f]$$

If we equipped $\$$ with the natural Hermitian metric discussed in 2.28 the above action becomes skew-adjoint as we wish.

Definition 3.39. The **Spinorial connection form** on $\mathrm{Spin}(M)$ is the unique one-form $\tilde{\omega}$ that lifts the Levi-Civita connection form ω of $\mathrm{SO}(M)$ so that the following diagram commutes.

$$\begin{tikzcd} T\mathrm{Spin}(M) \arrow[r, "\tilde{\omega}"] \arrow[d, "\eta_*"] & \mathfrak{spin}(n) \arrow[d, "\xi_*"] \\ T\mathrm{SO}(M) \arrow[r, "\omega"] & \mathfrak{so}(n) \end{tikzcd}$$

Choose a local section (frame) $\mathcal{E} = (e_1, \ldots, e_n) : U \to \mathrm{SO}(M)$ and let $A = \mathcal{E}^*\omega$ and $F = \mathcal{E}^*\Omega$ be the local expressions for the connection and curvature forms of $\mathrm{SO}(M)$.

$$A = -\sum_{i<j} A_{ij} E_i \curlywedge E_j \quad , \quad F = -\sum_{i<j} F_{ij} E_i \curlywedge E_j$$

The local section $\mathcal{E}$ lifts to the section $\tilde{\mathcal{E}} : U \to \mathrm{Spin}(M)$

$$\begin{tikzcd} & \mathrm{Spin}(M) \arrow[d, "\eta"] \\ U \arrow[ur, "\tilde{\mathcal{E}}"] \arrow[r, "\mathcal{E}"] & \mathrm{SO}(M) \end{tikzcd}$$

and it is clear from the above diagrams and Proposition 2.31 that

Proposition 3.40 ([19] Proposition 4.3.). ***i.*** *Let $\tilde{A} = \tilde{\mathcal{E}}^*\tilde{\omega}$ be the gauge field of $\tilde{\omega}$, then*

$$\tilde{A} = -\frac{1}{2}\sum_{i<j} A_{ij}E_iE_j$$

ii. *Choose an orthonormal basis $(f_1, \ldots, f_N)$ of Δ and set $\psi_\alpha := [\tilde{\mathcal{E}}, f_\alpha]$, then*

$$\nabla\psi_\alpha = \frac{1}{4}\sum_{i,j} A_{ji}e_ie_j\psi_\alpha$$

iii. *Let $\tilde{F} = \tilde{\mathcal{E}}^*\tilde{\Omega}$ then*

$$\tilde{F} = \frac{1}{4}\sum_{i,j} F_{ji}E_iE_j$$

Consequently the curvature operator K of $\$$ is

$$K = \frac{1}{4}\sum_{i,j} F_{ji}e_ie_j = \frac{1}{4}\sum_{i,j}\langle Re_i, e_j\rangle e_ie_j$$

Where R is the (Riemannian) curvature operator of TM.

Proposition 3.41 ([19] Proposition 4.4.). *The spinorial connection is compatible with the Hermitian metric of $\$$ and the Clifford action. For instance*

$$X\langle s_1, s_2\rangle = \langle\nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle \tag{3.3}$$

$$\nabla_X(Y.s) = (\nabla_X Y).s + Y.\nabla_X s \tag{3.4}$$

Proof. To verify 3.3 first suppose that $s_1 = \psi_\alpha$ and $s_2 = \psi_\beta$, then

$$\begin{aligned}
\langle \nabla_X \psi_\alpha, \psi_\beta \rangle =& \frac{1}{4} \sum_{i,j} A_{ji} \langle e_i e_j \psi_\alpha, \psi_\beta \rangle \\
=& \frac{1}{4} \sum_{i,j} A_{ji} \langle \psi_\alpha, e_j e_i \psi_\beta \rangle \\
=& -\frac{1}{4} \sum_{i,j} A_{ji} \langle \psi_\alpha, e_i e_j \psi_\beta \rangle \\
=& - \langle \psi_\alpha, \nabla_X \psi_\beta \rangle + X \langle \psi_\alpha, \psi_\beta \rangle \quad (\text{since } \langle \psi_\alpha, \psi_\beta \rangle = \delta_{\alpha\beta})
\end{aligned}$$

Since

$$X\langle f s_1, s_2 \rangle = X(f)\langle s_1, s_2 \rangle + f X \langle s_1, s_2 \rangle$$

Equation 3.3 holds for arbitrary sections as well □

The above arguments shows that the spinor bundle with its natural Hermitian metric and the spinorial connection is a Clifford bundle.

At the end, it follows from Definition 3.16 , Proposition 3.18, and Proposition 3.40 (iii) that

Theorem 3.42. *The twisting curvature of the spinor bundle is zero.*

Chapter 4

Analysis of the Dirac Operator and the Heat Equation

4.1 Sobolev Spaces on Vector Bundles

There are several approaches to define Sobolev spaces of sections. We take advantage of the assumption that the base manifold is compact and we define the Sobolev spaces without any direct appeal to the local trivialization. However our definition is local in essence and one can go back and forth to the vector valued functions on $\mathbb{R}^n$ in order to carry out some calculations.

Let $S \to M$ be a Hermitian vector bundle equipped with a metric connection.

Definition 4.1. We define the space $L^2(S)$ to be the completion of $C^\infty(S)$ with respect to the L^2 inner product

$$\langle s_1, s_2 \rangle = \int_M \langle s_1(x), s_2(x) \rangle \, \mathrm{d}x$$

We are using the Riemann volume measure for integration on M.

Definition 4.2. Since the bundle $\bigotimes^p T^*M \otimes S$ inherits an inner product from the Riemannian metric of M and the Hermitian metric of S we can similarly define, for a positive integer k, the H^k inner product on $C^\infty(S)$

$$\begin{aligned}\langle s_1, s_2\rangle_k =& \int_M \langle s_1(x), s_2(x)\rangle + \langle \nabla s_1(x), \nabla s_2(x)\rangle + \cdots + \langle \nabla^k s_1(x), \nabla^k s_2(x)\rangle \,\mathrm{d}x \\ =& \langle s_1, s_2\rangle + \langle \nabla s_1, \nabla s_2\rangle + \cdots + \langle \nabla^k s_1, \nabla^k s_2\rangle\end{aligned}$$

The Sobolev space $H^k(S)$ is the completion of $C^\infty(S)$ with respect to the above norm.

We denote the H^k norm or inner product by the subscript k. The absence of subscript refers to L^2.

It is clear from the definitions that

Proposition 4.3. *There is a bounded inclusion*

$$H^{k'}(E) \hookrightarrow H^k(E) \quad \text{for } k' > k$$

More precisely if $\phi \in H^{k'}(E)$ then $\phi \in H^k(E)$ and $\|\phi\|_k \leq \|\phi\|_{k'}$.

Proposition 4.4 ([11] Proposition 3.1.9.)**.** *If $\| \|_k'$ is the Sobolev norm for different choices of metric on M, metric on S, and connection on S then $\| \|_k'$ is equivalent to $\| \|_k$. i.e. there are constants c_1 and c_2 such that for any $s \in C^\infty(S)$*

$$c_1\|s\|_k \leq \|s\|_k' \leq c_2\|s\|_k$$

Proof. One can see that the different choices of Riemannian and Hermitian metrics yield an equivalent norm since

- $\sqrt{\det g'}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \sqrt{\det(g'g^{-1})}\sqrt{\det g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$
- Any two norms on a finite dimensional vector space are equivalent.
- The base space M is compact.

Now we work out the case that we are given a different connection ∇' on S. This can also resembles somehow the nature of the precise proof for the above argument.
We know that the difference $\nabla' - \nabla$ is an $\mathrm{End}(S)$-valued one form on M. For instance

$$\nabla' = \nabla + A$$

where $A \in \Omega^1(M; \mathrm{End}(S))$.
Since M is compact there exists a positive number c such that $\|A(x)s(x)\| \leq c\|s(x)\|$ for every $s \in C^\infty(S)$ and $x \in M$. Indeed we can let $c = \max_{x \in M} \|A(x)\|$.
Now it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\langle \nabla' s, \nabla' s \rangle =& \langle \nabla s, \nabla s \rangle + 2\mathrm{Re}\langle \nabla s, As \rangle + \langle As, As \rangle \\ \leq& \|\nabla s\|^2 + c\|\nabla s\|\|s\| + c^2\|s\|^2\end{aligned}$$

Consequently

$$\begin{aligned}\|s\|_1' =& \|s\|^2 + \|\nabla' s\|^2 \\ \leq& \|s\|^2 + \|\nabla s\|^2 + c\|\nabla s\|\|s\| + c^2\|s\|^2 \\ \leq& \|s\|_1^2 + c\|s\|_1^2 + c^2\|s\|_1^2 \\ =& (1 + c + c^2)\|s\|_1^2 \\ =& c_1\|s\|_1^2\end{aligned}$$

The proof for the higher Sobolev norms follows by a similar logic. □

Proposition 4.5 ([11] Proposition 3.1.13.)**.** *a. The covariant derivative extends to a bounded mapping*

$$\nabla : H^k(S) \to H^{k-1}(T^*M \otimes S) \quad \textit{for any } k \geq 1$$

b. Any vector bundle map $L : S \to S'$ extends to a bounded map

$$L : H^k(S) \to H^k(S') \quad \textit{for any } k \geq 0$$

Proof. The first assertion immediately follows from the definition.

To prove (b) we will show that there exists a constant c such that for any $s \in C^\infty(S)$

$$\|Ls\|'_k \leq c\|s\|_k \tag{4.1}$$

We proceed to prove the above statement by induction on k. At the same time we show that for $k \geq 1$

$$\|[L, \nabla^k]s\|' \leq c\|s\|_{k-1} \tag{4.2}$$

For $k = 0$, Equation 4.1 is obvious since M is compact. Similarly, since the commutator $[\nabla, L]$ is a bundle map, Equation 4.2 holds for $k = 1$.

Now, assume that both 4.1 and 4.2 hold for $m < k$. We can write

$$[L, \nabla^k] = [L, \nabla]\nabla^{k-1} + \nabla[L, \nabla]\nabla^{k-2} + \dots \nabla^{k-1}[L, \nabla] \tag{4.3}$$

Since $[L, \nabla]$ is a bundle map, $[L, \nabla] : H^m \to H^m$ is bounded for $m < k$ by the induction hypothesis. Hence the above equation along with part (a) imply that $[L, \nabla^k] : H^k \to L^2$ is bounded.

Furthermore

$$\begin{aligned}
\|\nabla^k Ls\| &\leq \|L\nabla^k s\| + \|[L, \nabla^k]s\| \\
&\leq c(\|\nabla^k s\| + \|s\|_{k-1}) \\
&\leq c\|s\|_k
\end{aligned}$$

and using the induction hypothesis, we obtain

$$
\begin{aligned}
\|Ls\|_k &\leq \|\nabla^k Ls\| + \|Ls\|_{k-1} \\
&\leq C(\|s\|_k + \|s\|_{k-1}) \\
&\leq 2C\|s\|_k
\end{aligned}
$$

□

Remark 4.6. One needs to be careful through the above computations. For instance the commutators in Equation 4.3 are all different in terms of their domains and codomains, or the commutator $[L, \nabla^k]$ is including the tensor product of L with the identity map if it is required.

Theorem 4.7 (Rellich Lemma). *([13] Lemma 1.1.5.) The inclusion $H^{k'} \hookrightarrow H^k$ is compact for $k' > k$.*

Proof. As we mentioned before because the Sobolev norms are local objects, in practice one can imply local reduction to the trivial bundle. So cover M by finite number of charts $(U_\alpha, x_\alpha)_\alpha$ with the trivialization of the vector bundle over each U_α. We can choose the coordinate charts so that $x_\alpha(U_\alpha) = B^n(1)$ the unit ball centred at the origin. With the means of a partition of unity subordinate to our cover any section can be written as $s = \sum_1^m s_\alpha$ where each s_α is represented by a vector valued function

$$
s_\alpha : \mathbb{R}^n \to \mathbb{C}^N
$$

with compact support in $B^n(1)$.

With this preparation the study of sections of S is reduced to the study of sections of the trivial bundle with compact support.

In fact since the Sobolev norms do not depend on a particular choice of connection one can use the trivial (flat) connection $\nabla = d$ on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N$ and it is clear that in this case the H^k norm of our compactly supported sections become

$$\|s\|_k^2 = \sum_{|\alpha \leq k} \|D^\alpha s\|^2$$

It is straightforward to show that the above norm is equivalent to the standard H^k norm which is defined with the means of Fourier transform on the Schwartz class sections as follows

$$\|s\|_k^2 = \int_{\mathbb{R}^n} (1+|\xi|)^{2k} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi$$

Now we get back to the original problem. We have to show that for a bounded sequence $(s_n) \subset H^{k'}(\mathbb{R}^n)$ (e.g. $\|s_n\|_{k'} \leq C$) with compact support in $B^n(1)$, there is a subsequence that converges in $H^k(\mathbb{R}^n)$ for $k' > k$. Since $H^{k'}$ is complete it suffices to find a Cauchy subsequence.

If ϕ is an bump function such that $\phi|_{B^n(1)} = 1$, then $s_n = \phi s_n$ and so $\hat{s}_n = \hat{\phi} * \hat{s}_n$

$$\hat{s}_n(\xi) = \int_{\mathbb{R}^n} \hat{\phi}(\xi - \eta) \hat{s}_n(\eta) \,\mathrm{d}\eta$$

It follows by Cauchy-Schwarz that

$$\begin{aligned} |\hat{s}_n(\xi)|^2 &\leq \int_{\mathbb{R}^n} (1+|\eta|)^{2k'} |\hat{s}_n(\eta)|^2 \,\mathrm{d}\eta \int_{\mathbb{R}^n} (1+|\eta|)^{-2k'} |\hat{\phi}(\xi-\eta)|^2 \,\mathrm{d}\eta \\ &= \|s_n\|_{k'}^2 K(\xi) \leq C^2 K(\xi) \end{aligned}$$

where K is a continuous function defined by the second integral.

The above trick shows that the sequence $(\hat{s}_n)$ is equibounded and equicontinuous on compact sets. Hence by Arzela-Ascoli theorem there is a subsequence, we still denote it by $(\hat{s}_n)$, that converges uniformly on compact sets.

Then for any positive r

$$\|s_n - s_m\|_k^2 = \int_{|\xi|>r} (1+|\xi|)^{2k} |\hat{s}_n(\xi) - \hat{s}_m(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|\le r} (1+|\xi|)^{2k} |\hat{s}_n(\xi) - \hat{s}_m(\xi)|^2 \,\mathrm{d}\xi$$

Since

$$(1+|\xi|)^{2k} \le (1+r)^{-2(k'-k)} (1+|\xi|)^{2k'} \quad \text{for } |\xi| > r,$$

the first integral is bounded by

$$\frac{\|s_n - s_m\|_{k'}^2}{(1+r)^{2(k'-k)}} \le \frac{2C}{(1+r)^{2(k'-k)}}$$

which in turn can be made small enough by choosing r large enough.
On the other hand since $(\hat{s}_n)$ is uniformly Cauchy on the closed ball $\overline{B^n(r)}$, the second integral can also be made small enough by choosing n, m large enough. □

Theorem 4.8 (The Sobolev Embedding Theorem)**.** *([13] Lemma 1.1.4.) If $k' > k + \frac{n}{2}$, then there is a continuous embedding*

$$H_{k'} \hookrightarrow C^k$$

Proof. Using the same line of reasoning as we did at the beginning of the previous proof, it is enough to show that there exists a constant C such that for any $s \in \mathcal{S}(\mathbb{R}^n)$ belonging to the Schwartz class

$$\|s\|_{C^k} \le C\|s\|_{k'} \quad \text{for } k' > k + \frac{n}{2}$$

where $\|.\|_{C^k}$ denotes the sup norm on $C^k(\mathbb{R}^n)$ defined by

$$\|s\|_{C^k} = \|s\|_{\infty,k} = \sup_{\mathbb{R}^n} \sum_{\alpha \le k} |D^\alpha s(x)| \quad \text{for } s \in \mathcal{S}$$

First assume that $k = 0$, we have then for $s \in \mathcal{S}$

$$
\begin{aligned}
s(x) &= \int_{\mathbb{R}^n} e^{ix.\xi}\, \hat{s}(\xi)\, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} e^{ix.\xi}\, \hat{s}(\xi)(1+|\xi|)^{k'}(1+|\xi|)^{-k'}\, \mathrm{d}\xi
\end{aligned}
$$

Using Cauchy-Schwarz and the assumption that $k' > \frac{n}{2}$, we obtain

$$
|s(x)| \leq C\|s\|_{k'}
$$

which implies the desired result for $k = 0$.

For $k > 0$, it follows by the above result that

$$
\|D^\alpha s\|_{\infty,o} \leq C\|D^\alpha s\|_{k'-|\alpha|} \quad \text{for } |\alpha| \leq k \text{ and } k' > k + \frac{n}{2}
$$

Moreover

$$
\|D^\alpha s\|_{k'-|\alpha|} \leq \|s\|_{k'}
$$

Putting these two inequalities together and summing over α gives the required result. □

4.2 Analysis of the Dirac operator

Our goal in this section is to prove that the Dirac operator admits spectral decomposition on $L^2(S)$ and the eigenvectors are smooth (elliptic regularity). We begin by proving two crucial estimates known as Garding and Elliptic estimates. These are somehow backwards to the Sobolev and Rellich estimates in the sense that they provide us with some H^k regularity information of ϕ in terms of L^2 regularities of ϕ and $D\phi$.

First recall that the Dirac operator is defined by the composition of three maps

$$
C^\infty(S) \to C^\infty(T^*M \otimes S) \to C^\infty(TM \otimes S) \to C^\infty(S)
$$

It then follows from 4.5 that the Dirac operator extends to a bounded operator

$$D : H^k(S) \to H^{k-1}(S)$$

for any $k \geq 1$.

Theorem 4.9 (Garding). *([11] Proposition 3.2.4.) There exists a positive constant C such that for any $u \in H^1(S)$*

$$\langle D^2 u, u \rangle + C\langle u, u \rangle \geq \|u\|_1^2$$

Proof. It suffices to establish the above statement for an arbitrary $s \in C^\infty(S)$.
It follows by the Weitzenbock formula

$$D^2 s = \nabla^* \nabla s + \mathcal{K} s$$

that

$$s + \nabla^* \nabla s = D^2 s - (\mathcal{K} - 1)s$$

Since $\mathcal{K}$ is a bundle map , we have

$$\|s\|_1^2 = \langle s, s \rangle + \langle \nabla s, \nabla s \rangle \leq \langle D^2 s, s \rangle + C\langle s, s \rangle$$

as we want. □

Theorem 4.10 (Elliptic Estimate). *([11] Proposition 3.2.15.) For any $k \geq 0$ there exits a positive constant C_k such that for any $u \in H^{k+1}(S)$*

$$\|u\|_{k+1} \leq C_k(\|u\|_k + \|Du\|_k) \tag{4.4}$$

Proof. The case $k = 0$ follows immediately by the Garding's inequality. We proceed to prove the general case by induction on k.

We first claim that the commutator $[D, \nabla]$ is a bundle map. To see this choose a synchronous frame $(e_1, \ldots, e_n)$ with the dual $(\eta^1, \ldots, \eta^n)$. Then at the centre we have

$$\begin{aligned}[D, \nabla] =& \eta^l \otimes c(e_k)\nabla_k \nabla_l - \eta^l \otimes c(e_k)\nabla_l \nabla_k \\ =& \eta^l \otimes c(e_k) K(e_k, e_l)\end{aligned}$$

Now we assume that Equation 4.4 holds for $m < k$ and let $u \in H^{k+1}(S)$. By the induction hypothesis there is $C_{k-1} > 0$ such that

$$\begin{aligned}\|\nabla u\|_k \leq& C_{k-1}(\|\nabla u\|_{k-1} + \|D\nabla u\|_{k-1}) \\ \leq& C_{k-1}(\|\nabla u\|_{k-1} + \|[D, \nabla]u\|_{k-1} + \|\nabla D u\|_{k-1}) \\ \leq& C'_{k-1}(\|u\|_k + \|u\|_{k-1} + \|Du\|_k) \\ \leq& C''_{k-1}(\|u\|_k + \|Du\|_k)\end{aligned}$$

The result will follow if one adds $\|u\|_k$ to the both sides. □

Definition 4.11. Let A be an operator on a Hilbert space $\mathcal{H}$. The graph of A is the subspace of $\mathcal{H} \oplus \mathcal{H}$ defined by

$$G_A := \{(x, Ax) : x \in \mathrm{Dom}(A)\}$$

Lemma 4.12 ([28] Lemma 5.18.)**.** *Let G denote the graph of the Dirac operator then $\overline{G}$ is also a graph.*

Proof. Suppose that $\overline{G}$ is not a graph. This means that there is an obstruction to define $A_{\overline{G}}x = y$ for some $(x, y) \in \overline{G}$. For instance there exists $y' \in \mathcal{H} = L^2(S)$ such that both $(x, y), (x, y')$ lie in $\overline{G}$.

Since $\overline{G}$ is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$ we have got an element $(0, z) \in \overline{G}$ where z is non-zero. So there must be a sequence $(s_n) \subset C^\infty(S)$ of smooth sections with

$$s_n \to 0 \quad \text{and} \quad Ds_n \to z \quad \text{in } L^2(S)$$

then for any smooth s,

$$\langle s_n, D^\dagger s \rangle \to 0 \quad \text{and} \quad \langle Ds_n, s \rangle \to \langle z, s \rangle \quad \text{in } L^2(S)$$

We can conclude that

$$\langle z, s \rangle = 0 \quad \text{for any } s \in C^\infty(S)$$

Hence $z = 0$, as $C^\infty(S)$ is dense in $L^2(S)$, which is a contradiction. □

We denote by $\overline{D}$ the operator that $\overline{G}$ defines. The next lemma shows that $\overline{D}$ is actually the extension of D to $H^1(S)$.

Lemma 4.13. *The domain of $\overline{D}$ is $H^1(S)$.*

Proof. Let $x \in \mathrm{Dom}(\overline{D})$, then there must be a sequence $(s_n) \subset C^\infty(S)$ such that

$$s_n \to x \quad \text{and} \quad Ds_n \to y \quad \text{in } L^2(S)$$

for some $y \in L^2(S)$.
By Garding's inequality 4.9,

$$\|s_n\|_1 \leq C(\|s_n\| + \|Ds_n\|)$$

which implies that (s_n) is a Cauchy sequence with respect to the H^1 norm and so defines an element $x' = [(s_n)] \in H^1(S)$. Since $H^1(S) \subset L^2(S)$, x' lies also in $L^2(S)$. But $x' = x$ since they have a common element, say (s_n). We have shown that $x \in H^1(S)$.
On the other hand if $x \in H^1(S)$, then x is the equivalence class of some Cauchy sequence

$(s_n) \subset C^\infty(S)$. To put it another way

$$s_n \to x \quad \text{in } H^1(S)$$

therefore

$$s_n \to x \quad \text{in } L^2(S)$$

Moreover, $\|Ds_n\| \leq C\|s_n\|_1$ hence there is some $y \in L^2(S)$ so that

$$Ds_n \to y \quad \text{in } L^2(S)$$

Thus $x \in \mathrm{Dom}(\overline{D})$. $\square$

Since we do not make any distinction between D and its extensions, we occasionally drop the bar and we write simply D instead of $\overline{D}$.

Definition 4.14 (Weak Solution of a PDE)**.** For $x, y \in L^2(S)$ we say that $Dx = y$ weakly if

$$\langle x, D^\dagger s\rangle = \langle y, s\rangle \quad \text{for all } s \in C^\infty(S)$$

Definition 4.15 (Smoothing Operators)**.** A bounded operator $A \in \mathcal{L}(L^2(S))$ is called a smoothing operator if there exists a smooth kernel $k \in C^\infty(S \boxtimes S^*)$ such that for any $u \in L^2(S)$

$$Au(p) = \int_M k(p,q)u(q)\,\mathrm{d}q$$

where $S \boxtimes S^* := \pi_1^* S \otimes \pi_2^* S^*$ is the bundle over $M \times M$ defined by using the pull backs of the projections

$$M \times M \xrightarrow{\pi_1, \pi_2} M$$

Definition 4.16 (Friedrich Mollifier)**.** A Friedrich mollifier for S is a family of self-adjoint, bounded operators $\{F_\epsilon\}_{\epsilon \in (0,1)}$ on $L^2(S)$ such that

(i) $[B, F_\epsilon]$ extends to a bounded family of operators on $L^2(S)$, for any first order differential operator B on S.

(ii) $F_\epsilon \to 1$ in the weak topology of $\mathcal{L}(L^2(S))$. More precisely for any $x, y \in L^2(S)$,

$$\langle F_\epsilon x, y \rangle \to \langle x, y \rangle \quad \text{as } \epsilon \to 0$$

One can show that a Friedrich mollifier exists for S.

Proposition 4.17 ([28] Proposition 5.22.). *Suppose that $Dx = y$ weakly. Then $x \in H^1(S)$ and $\overline{D}(x) = y$.*

Proof. Let F_ϵ be a Friedrich mollifier, and let $x_\epsilon = F_\epsilon x$. Then x_ϵ is smooth and for any smooth section s we have

$$\begin{aligned}
|\langle Dx_\epsilon, s\rangle| =& |\langle x_\epsilon, D^\dagger s\rangle| \\
=& |\langle x, F_\epsilon D^\dagger s\rangle| \\
\leq& |\langle x, D^\dagger F_\epsilon s\rangle| + |\langle x, [F_\epsilon, D^\dagger] s\rangle \\
\leq& |\langle y, F_\epsilon s\rangle| + |\langle x, [F_\epsilon, D^\dagger] s\rangle \\
\leq& c_1 \|y\| \|s\| + c_2 \|x\| \|s\|
\end{aligned}$$

This shows that there exists a constant C such that

$$|\langle Dx_\epsilon, s\rangle| \leq C\|s\| \quad \text{for any } \epsilon \in (0,1)$$

Since for any element $x \in E$ of a Banach space

$$\|x\| = \sup_{l \in E', \|l\| \leq 1} |\langle l, x\rangle|$$

and $C^\infty(S)$ is dense is $L^(S)$, we can conclude that

$$\|Dx_\epsilon\| \leq C \quad \text{for any } \epsilon \in (0,1)$$

Now it follows by Garding's inequality 4.9 that $\{x_\epsilon\}_{\epsilon\in(0,1)}$ is a bounded subset of $H^1(S)$. Since any closed ball is compact in the weak topology of $H^1(S)$, there must be a subsequence $x_n = x_{\epsilon_n}$ with $\epsilon_n \to 0$ so that $x_n \to x_0$ weakly in $H^1(S)$. Weak convergence in H^1 implies weak convergence in L^2 since the map $u \mapsto \langle f, u\rangle$ defines a bounded functional on H^1 for any $f \in L^2$. So we can deduce that

$$x_n \to x_0 \quad \text{weakly in } L^2(S).$$

On the other hand, for any $y \in L^2(S)$

$$\langle x_n, y\rangle = \langle F_{\epsilon_n} x, x\rangle \to \langle x, y\rangle$$

hence $x = x_0 \in H^1(S)$. □

Remark 4.18. For A an unbounded operator on a Banach space X, Let A' denote the adjoint of A. Recall that $\phi \in \mathrm{Dom}(A')$ if there exists $\psi \in X'$ such that

$$\psi(x) = \phi(Ax) \quad \text{for every } x \in \mathrm{Dom}(A)$$

In Section 3.1 we saw that the Dirac operator is symmetric. Now the above lemma implies that the Dirac operator is in fact self adjoint in the sense of unbounded operators on a Hilbert space.

Indeed as we are working on the Hilbert space $\mathcal{H} = L^2(S)$, if $x \in \mathrm{Dom}(D')$ there must be

$y \in \mathcal{H}$ such that

$$\langle x, Ds \rangle = \langle y, s \rangle \quad \text{for any } s \in C^{\infty}(S)$$

that is $Dx = y$ weakly. Hence by the above lemma $x \in H^1(S)$ and so $\mathrm{Dom}(D') \subset H^1(S)$. Similarly one can also show that $H^1(S) \subset \mathrm{Dom}(D')$.

Let $\ker(D) = \{x \in H^1(S) : Dx = 0\}$, Then we have an important regularity result:

Theorem 4.19 (Elliptic Regularity)**.** *([28] Proposition 5.24.) The kernel of Dirac operator consists of smooth sections.*

Proof. Let $u \in \ker(D)$. We use induction to show that

$$u \in H^n(S) \quad \text{for all } n$$

which implies - by the Sobolev embedding- that u is smooth.
We already know that $u \in H^1$. Now assume that $u \in H^{n-1}$. One can show (by induction and using the elliptic estimate) that F_ϵ and $[D, F_\epsilon]$ both define bounded operators on each Sobolev space $H^k(S)$. If $u = [(s_n)]$ We can use the elliptic estimate to obtain

$$\begin{aligned} \|F_\epsilon s_n\|_k \leq & C_k(\|F_\epsilon s_n\|_{k-1} + \|DF_\epsilon s_n\|_{k-1}) \\ = & C_k(\|F_\epsilon s_n\|_{k-1} + \|[D, F_\epsilon] s_n\|_{k-1}) \quad \text{since } Ds_n = 0 \end{aligned}$$

which shows that $F_\epsilon u \in H^k$ for any $\epsilon \in (0, 1)$.
With the same logic as in the proof of Proposition 4.17 the sequence $u_n = F_{\epsilon_n} u$ -for some appropriate $\epsilon_n \to 0$- weakly converges to some element of H^k both in H^k and in L^2. But $u_n \to u$ weakly in L^2 by the second property of a mollifier. Hence $u \in H^k$ as desired. □

Remark 4.20*.* One can modify the proof of the elliptic estimate for the Dirac operator so that the statement holds for any generalized Dirac operator as well. By a generalized

Dirac operator we mean a first order differential operator D such that

$$D^2 = \nabla^* \nabla + B$$

where B is a first order differential operator. Consequently elliptic regularity also holds for a generalized Dirac operator. For more details see [28] chapter 5.

Lemma 4.21 ([28] Lemma 5.25.). *Let $\mathcal{H} = L^2(S)$ and $J : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ denotes the map $(x, y) \mapsto (y, -x)$. Then there exists an orthogonal decomposition*

$$\mathcal{H} \oplus \mathcal{H} = \overline{G} \oplus J\overline{G} \tag{4.5}$$

where G is the graph of the Dirac operator.

Proof. If $(x, y) \in \overline{G}^{\perp}$, then

$$\langle (x, y), (s, Ds) \rangle = 0 \quad \text{for every } s \in C^{\infty}(S)$$

In other words,

$$\langle x, s \rangle + \langle y, Ds \rangle = 0 \quad \text{for every } s \in C^{\infty}(S)$$

But this means that $Dy = -x$ weakly since the Dirac operator is self adjoint. Hence $\overline{D}y = -x$ and $(x, y) \in J\overline{G}$. □

Now using the above lemma, we define the operator

$$Q : \mathcal{H} \to \mathcal{H}$$
$$x \mapsto Qx$$

where Qx is determined so that $(Qx, \overline{D}Qx)$ is the orthogonal projection of $(x, 0)$ onto $\overline{G}$ in the decomposition 4.5.

First we note that Q is a bounded operator. In fact by the above decomposition $\|Qx\| \leq \|x\|$ and so $\|Q\|_{\mathcal{L}(\mathcal{H})} \leq 1$.

Furthermore, it is clear that Q maps $\mathcal{H} = L^2(S)$ into $H^1(S)$ and it follows by Garding's inequality that

$$\|Qx\|_1 \leq C(\|Qx\| + \|DQx\|) \leq C\|x\|$$

Therefore, Q is a bounded map from $L^2(S)$ to $H^1(S)$ and by Rellich's lemma it must be a compact operator on $L^2(S)$.

This is a fantastic result for us towards obtaining the spectral decomposition of D.

One can also verify that Q is an injection. Indeed if $Qx = Qy$, then by the above decomposition

$$(x, 0) = (Qx, DQx) + (D^2Qx, -DQx) = (y, 0)$$

which shows that $y = x$.

Moreover since $x = Qx + D^2Qx$,

$$\begin{aligned}\langle Qx, x\rangle = \langle Qx, Qx + D^2Qx\rangle =& \langle Qx, Qx\rangle + \langle Qx + D^2Qx\rangle \\ =& \|Qx\|^2 + \|DQx\|^2 \geq 0\end{aligned}$$

which shows that Q is positive and self-adjoint.

The above argument enables us to get into the main result of this section:

Theorem 4.22 (Spectral Theorem). *([28] Theorem 5.27.) There is a direct sum decomposition of $\mathcal{H}$ into a countable number of orthogonal subspaces $\mathcal{H}_\lambda$. Each $\mathcal{H}_\lambda$ is a finite dimensional subspace of smooth sections, and is an eigenspace for D with eigenvalue λ. The eigenvalues form a discrete subset of real numbers.*

Proof. It follows by the spectral theorem for compact, self-adjoint operators that

$$\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_{\lambda_i}$$

where each $\mathcal{H}_{\lambda_i}$ is a finite dimensional eigenspace of Q and the set of eigenvalues is a discrete subset of $\mathbb{R}$ with the accumulation point zero. The eigenvalues are in fact positive since Q is a positive injection.

If x is eigenvector for Q, with eigenvalue $\rho > 0$, by lemma 4.21 there exists $y \in H^1(S)$ such that

$$(x, 0) = (Qx, DQx) + (-Dy, y)$$

In fact $y = -DQx = -\rho Dx \in H^1$ and we get

$$(x, 0) = \rho^2(x, Dx) + \rho^2(D^2x, -Dx).$$

Now, let $\lambda = \sqrt{\frac{1-\rho}{\rho}}$ and $z = \frac{1}{\lambda}Dx$, then we will have

$$Dx = \lambda z \quad \text{and} \quad Dz = \lambda x.$$

Therefore, $x + z$ and $x - z$ are eigenvectors of D with eigenvalues λ and $-\lambda$ respectively. Thus we can write $\mathcal{H}$ as an orthogonal direct sum of eigenspaces of D.

Furthermore each eigenvector must be smooth since it belongs to the kernel of the generalized Dirac operator $D - \lambda$. (see Remark 4.20.) □

Once we obtain the above spectral decomposition for the Dirac operator we can write any section $s \in L^2(S)$ as a Fourier expansion

$$s = \sum_n c_n \phi_n = \sum_n \langle s, \phi_n \rangle \phi_n$$

or simply

$$s = \sum_{\lambda \in \sigma(D)} s_\lambda$$

where s_λ is the orthogonal projection of s into the span$\{\phi_\lambda\}$.

We proceed to develop an appropriate functional calculus using the Fourier expansion.

Proposition 4.23 ([28] Proposition 5.29.). *A section $s \in L^2(S)$ is smooth if and only if $\|s_\lambda\| = O(|\lambda|^{-k})$ for any k.*

Proof. Suppose that $\|s_\lambda\| = O(|\lambda|^{-k})$ for each k. It follows by the elliptic estimate that

$$\|s_\lambda\|_p \leq C(1+\lambda)^p \|s_\lambda\| \leq C'(1+\lambda^p)\|s_\lambda\|$$

Since $\|s_\lambda\|$ is a rapidly decreasing function of λ the Fourier expansion converges in each H^p space which implies that s is smooth. $\square$

Now for a function f which is bounded on $\sigma(D)$ we can define the operator $f(D)$ by setting

$$f(D)s = \sum_{\lambda \in \sigma(D)} f(\lambda) s_\lambda$$

It is clear that $f(D)$ is a bounded operator on $L^2(S)$, and by the above proposition it maps smooth sections to smooth sections.

A crucial question for us to study is that whether and under what conditions $f(D)$ is a smoothing operator.

We first note that we can write $f(D)$ in terms of projections

$$f(D) = \sum_\lambda f(\lambda) P_\lambda$$

Moreover each projection P_λ is a smoothing operator. In fact

$$\begin{aligned} P_\lambda s(x) =& \langle \phi_\lambda, s\rangle \phi_\lambda(x) \\ &= \int_M \phi_\lambda(x)\langle \phi_\lambda(y), s(y)\rangle \,\mathrm{d}y \\ &= \int_M K_\lambda(x,y)s(y)\,\mathrm{d}y \end{aligned}$$

where $K_\lambda(x,y) = \phi_\lambda(x) \otimes \phi_\lambda^\flat(y)$ is the kernel of P_λ.

We claim that if f is rapidly decreasing then $f(D)$ will be a smoothing operator. To justify this we first note that we can formally write

$$f(D)s(x) = \int_M K_f(x,y)s(y)\,\mathrm{d}y$$

where $K_f = \sum_\lambda f(\lambda)K_\lambda$.

Since $\|K_\lambda\|_p \leq \|\phi_\lambda\|_p^2 \leq C(1+|\lambda|^{2p})$ and f is rapidly decreasing the above sum converges in each H^p space. So K_f is smooth as we asserted. We can now summarize the above discussion in the following theorem:

Theorem 4.24 ([28] Proposition 5.30.)**.** *(i) The map $f \mapsto f(D)$ is a homomorphism from the ring of the bounded functions on $\sigma(D)$ to $\mathcal{B}(\mathcal{H})$. Moreover $f(D)$ maps $C^\infty(S)$ to $C^\infty(S)$.*

(ii) If f is rapidly decreasing the operator $f(D)$ is a smoothing operator and the mapping $f \mapsto K_f$ is continuous from $\mathcal{S}(\mathbb{R})$ to $C^\infty(S \boxtimes S^*)$.

4.3 Hodge Theorem and Bochner Theorem

In this section we will show that how the study of Dirac operator reveals some interconnections between the geometry and topology of the base manifold. In fact Bochner's theorem says that if the first Betti number of a compact Riemannian manifold is not zero, then the manifold has no metric of positive Ricci curvature. The main ingredients to prove this are Weitzenbock's formula and the Hodge theorem.

We begin first by defining Dirac complexes which are a generalization of the de Rham complex.

Definition 4.25 (Dirac complex)**.** Let M be a compact oriented Riemannian manifold, and let $S_0, S_1, \ldots, S_k$ be a sequence fn Hermitian vector bundles over M equipped with metric connections. Suppose given differential operators $d_j : C^\infty(S_j) \to C^\infty(S_{j+1})$ such that $d_{j+1}d_j = 0$. This is called a Dirac complex if $S = \bigoplus S_j$ is a Clifford bundle whose Dirac operator is $d + d^*$.

As we saw in Section 3.2 the de Rham complex is an example of a Dirac complex.

Theorem 4.26 (Hodge Theorem)**.** *([28] Theorem 6.2.) Each cohomology class of a Dirac complex contains a unique harmonic representative. (e.g. a section $s \in C^\infty(S_j)$ thats satisfies $D^2 s = 0$.)*

In fact the cohomology class H^j is isomorphic as a vector space to the space of harmonic sections of S_j.

Proof. Let $\mathcal{H}^j$ be the subspace of harmonic sections of $C^\infty(S_j)$. Then the $\mathcal{H}$'s form a subcomplex of the Dirac complex with trivial differential.

Since any harmonic section is closed the inclusion map $\imath : \mathcal{H}^j \to C^\infty(S_j)$ defines a chain map. We shall show that it is also a chain equivalence between these two complexes.

$$\begin{tikzcd}
\mathcal{H}^{j-1} \arrow[r, "0"] \arrow[d, "\imath"'] & \mathcal{H}^{j} \arrow[r, "0"] \arrow[d, "\imath"'] & \mathcal{H}^{j+1} \arrow[r] \arrow[d, "\imath"'] & {} \\
C^\infty(S_{j-1}) \arrow[r, "d_{j-1}"] & C^\infty(S_j) \arrow[r, "d_j"] & C^\infty(S_{j+1}) \arrow[r] & {}
\end{tikzcd}$$

Let $P : C^\infty(S_j) \to \mathcal{H}^j$ be the restriction to $C^\infty(S_j)$ of the orthogonal projection $L^2(S_j) \to \mathcal{H}^j$. Then $P\imath = 1$, and $\imath P = 1 - f(D)$. Where

$$f(\lambda) = \begin{cases} 1 & \text{If } \lambda \neq 0 \\ 0 & \text{If } \lambda = 0 \end{cases}$$

Although $\imath P$ is not equal to identity, it is enough to find a chain homotopy between $\imath P$ and 1.

To this end we define $G = g(D)$ where

$$g(\lambda) = \begin{cases} \lambda^{-2} & \text{If } \lambda \neq 0 \\ 0 & \text{If } \lambda = 0 \end{cases}$$

Then $D^2G = f(D) = 1 - \imath P$. We note that G commutes with d since D commutes with d. So $D^2G = (dd^* + d^*d)G = d\delta + \delta d$, where $\delta = d^*G$. This shows that δ defines a chain homotopy between $\imath P$ and 1.

$$\begin{tikzcd}
C^\infty(S_{j-1}) \arrow[r, "d_{j-1}"] \arrow[d, "\imath P"', "1"] & C^\infty(S_j) \arrow[r, "d_j"] \arrow[d, "\imath P"', "1"] \arrow[dl, "\delta"'] & C^\infty(S_{j-1}) \arrow[r] \arrow[d, "\imath P"', "1"] \arrow[dl, "\delta"'] & {} \\
C^\infty(S_{j-1}) \arrow[r, "d_{j-1}"] & C^\infty(S_j) \arrow[r, "d_j"] & C^\infty(S_{j+1}) \arrow[r] & {}
\end{tikzcd}$$

□

Now we have everything we need to prove Bochner theorem. We first state and prove a helpful lemma.

Lemma 4.27 ([28] Lemma 6.8.). *Consider the bundle $\bigwedge^* T^*M$ as a Clifford bundle. Then the restriction of the Clifford contraction of the curvature (see 3.8 for the definition) to one-forms is the Ricci curvature operator.*

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal frame for the tangent bundle with the dual $(\eta^1, \ldots \eta^n)$.

Note that we have the natural isomorphisms

$$Cl(TM) \xrightarrow{\theta} \bigwedge^* TM \xrightarrow{\flat} \bigwedge^* T^*M$$

In fact these three vector bundles are all the same Clifford bundle but with different Clifford actions. For clarity let us denote the Clifford actions of a vector e on $\bigwedge^* TM$, and $\bigwedge^* T^*M$ by $c(e)$ and $c'(e)$ respectively. Also let $\mathcal{K}$ and $\mathcal{K}'$ denote the Clifford contraction of the curvature for these two bundles. Using the above identifications it follows by Lemma 3.19 that

$$\begin{aligned}
\mathcal{K}'\eta_k = (\mathcal{K}(e_k))^\flat =& \{\frac{1}{2}\sum_{i,j} c(e_i)c(e_j)R(e_i, e_j)e_k\}^\flat \\
=& \{\frac{1}{2}\sum_{i,j,l} c(e_i)c(e_j)R_{lkij}e_l\}^\flat \\
=& \{\frac{1}{2}\sum_{i,j,l} R_{lkij}\theta(e_ie_je_l)\}^\flat \\
& \{\sum_j \mathbf{Ric}_{kj}e_j\}^\flat \\
& \sum_j \mathbf{Ric}_{kj}\eta_j
\end{aligned}$$

□

Theorem 4.28 (Bochner)**.** *([28] Theorem 6.9.) Let M be a compact oriented manifold whose first Betti number is nonzero. Then M does not have any metric of positive Ricci curvature.*

Proof. Since H^1_{dR} is not trivial, by Hodge theorem there should exist a harmonic one form ω. It then follows by Weitzenbock formula that

$$\langle \mathcal{K}\omega, \omega \rangle = -\|\nabla\omega\|^2 \leq 0$$

We can conclude, using the above lemma, that there is no metric of positive Ricci curvature. □

4.4 The Heat Kernel and Its Asymptotic Expansion

Definition 4.29. Let S be a Clifford bundle. The heat equation for D is the partial differential equation

$$\frac{\partial}{\partial t}s + D^2 s = 0$$

Proposition 4.30. *For any given intial data $\psi \in L^2(S)$ the Cauchy problem for the heat equation has a unique smooth solution s_t. The solution exists for all $t > 0$ and it satisfies $\|s_t\| \leq \|\psi\|$. Moreover*

$$\lim_{t \mapsto 0^+} s_t = \psi \quad \text{in } L^2$$

and if ψ is smooth, then

$$\lim_{t \to 0^+} s_t = \psi$$

in the C^∞ topology.

Proof. Using the operational calculus we developed at the end of previous section, let

$$s_t = e^{-tD^2}\psi = \sum_\lambda e^{-t\lambda^2}\psi_\lambda \tag{4.6}$$

for $t > 0$. Then $s_t \in C^\infty(S)$ since the function $\lambda \mapsto \exp(-t\lambda^2)$ is rapidly decreasing for $t > 0$.

We have to first verify that one can differentiate Equation 4.6 with respect to t to find that s_t depends smoothly on t and satisfies the heat equation.

Using the Elliptic estimate we get

$$\begin{aligned}
&\|[\frac{e^{-(t+h)D^2}-e^{-tD^2}}{h}-D^2e^{-tD^2}]\psi\|_k \\
&\leq C(\|[\frac{e^{-(t+h)D^2}-e^{-tD^2}}{h}-D^2e^{-tD^2}]\psi\|+\|D^k[\frac{e^{-(t+h)D^2}-e^{-tD^2}}{h}-D^2e^{-tD^2}]\psi\|
\end{aligned}$$

But

$$\|D^k[\frac{e^{-(t+h)D^2}-e^{-tD^2}}{h}-D^2e^{-tD^2}]\psi\|^2=\sum_\lambda[\lambda^{k+2}e^{-t\lambda^2}(\frac{e^{-h\lambda^2}-1}{h\lambda^2}-1)]^2\|\psi\|^2$$

Since $[\lambda^{k+2}e^{-t\lambda^2}(\frac{e^{-h\lambda^2}-1}{h\lambda^2}-1)]^2$ is bounded and also approaches zero uniformly in λ as h goes to zero, and $\sum_\lambda\|\psi_\lambda\|^2<\infty$ we can use Lebesgue's dominated convergence theorem to conclude that the above sum goes to zero as $h\to 0$ for any $k\in\mathbb{N}$. This argument establishes the convergence of the heat equation in every H^k and Consequently in C^∞ topology.

For the uniqueness part we note that

$$\|s_t\|^2=\sum_\lambda e^{-2t\lambda^2}\|\psi_\lambda\|\leq\|\psi\|^2$$

which actually implies the uniqueness.

Now we study the behavior of the solution s_t when t approaches zero.

$$\|s_t-\psi\|^2=\|(e^{-tD^2}-1)\psi\|^2=\sum_\lambda(e^{-t\lambda^2}-1)^2\|\psi_\lambda\|^2\leq\sum_\lambda\|\psi_\lambda\|^2<\infty$$

Therefore by the Lebesgue's dominated theorem we can conclude that

$$\lim_{t\to 0^+}s_t=\psi$$

in L^2.

If ψ is smooth, using the same techniques, we can show that

$$\lim_{t\to 0^+} s_t = \psi$$

in each H^k and therefore in C^∞ topology. $\square$

From Theorem 4.24 we know that e^{-tD^2} is a smoothing operator and there is a time-dependent kernel $k_t \in C^\infty(S \boxtimes S^*)$ called the ***heat kernel*** such that

$$e^{-tD^2}\psi(p) = \int_M k_t(p,q)\psi(q)\,\mathrm{d}q$$

for all $t > 0$.

Proposition 4.31 ([28] Proposition 7.6.)**.** *The heat kernel k_t has the following properties.*

(i) For each fixed q the section $p \mapsto k_t(p,q)$ of $S \bigotimes S_q^*$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + D^2\right) k_t(.,q) = 0$$

(ii) For each smooth section s,

$$\int_M k_t(p,q)s(q)\,dq \to s(p)$$

uniformly in p as $t \to 0^+$.

Moreover, the heat kernel is the unique time-dependent section of $S \boxtimes S^*$ which is C^2 in the spatial variables, C^1 in time, and has the above properties.

Proof. Since

$$K_t(p,q) = \sum_\lambda e^{-t\lambda^2}\phi_\lambda(p) \otimes \phi_\lambda^\flat(q)$$

applying the same logic as we used in the previous proof shows that the heat kernel satisfies the heat equation

$$[\frac{\partial}{\partial t} + D^2]k_t(.,q) = 0$$

Furthermore if $s \in C^\infty(S)$, then

$$\begin{aligned}\lim_{t\to 0^+}\int_M k_t(p,q)s(q)\,\mathrm{d}q &= \lim_{t\to 0^+}\int_M \sum_\lambda e^{-t\lambda^2}\phi_\lambda(p)\otimes\phi_\lambda^\flat(q)s(q)\,\mathrm{d}q \\ &= \sum_\lambda \langle\phi_\lambda, s\rangle\phi_\lambda(p) = s(p)\end{aligned}$$

In fact

$$\sum_\lambda \langle\phi_\lambda, s\rangle\phi_\lambda \to s$$

in C^∞ topology. In particular

$$\|\sum_\lambda \langle\phi_\lambda, s\rangle\phi_\lambda(p)(e^{-t\lambda^2}-1)\| \to 0 \quad \text{as } t \to 0^+$$

uniformly in p.

For the uniqueness, suppose that k_t has theses properties and let $\{K_t\}_{t>0}$ be the family of smoothing operators with kernels k_t. Then it follows by the uniqueness for solutions of the heat equation that

$$K_t s = e^{-(t-\epsilon)D^2}K_\epsilon s$$

for every positive ϵ.

By property (ii) above, for any smooth section s, $K_\epsilon s \to s$ uniformly as $\epsilon \to 0$ also $e^{-(t-\epsilon)D^2} \to e^{-tD^2}$ in the norm topology of $\mathcal{B}(L^2(S))$ as $\epsilon \to 0$, So we have

$$K_t s = e^{-(t-\epsilon)D^2}K_\epsilon s \to e^{-tD^2}s$$

for any smooth s.

Since smooth sections are dense in $L^2(S)$, we have proved that $K_t = e^{-tD^2}$. $\square$

Definition 4.32. Let m be a positive integer. An ***approximate heat kernel*** of order m is a time-dependent section k'_t of $S \boxtimes S^*$ which is C^1 in time and C^2 in spatial variables and satisfies property (ii) in the previous proposition, and also approximately satisfies the heat equation in the sense that

$$\left(\frac{\partial}{\partial t} + D^2\right) k'_t(.,q) = t^m r_t(.,q)$$

where r_t is a C^m section of $S \boxtimes S^*$ and depends continuously on t for $t \geq 0$.

Proposition 4.33 (Duhamel's Principle)**.** *[[28] Proposition 7.9.] Let s_t be a continuously varying C^2 section of S. Then there is a unique smooth section $\tilde{s}_t$ of S, differentiable in t and with $\tilde{s}_0 = 0$, satisfying the inhomogeneous heat equation*

$$\left(\frac{\partial}{\partial t} + D^2\right) \tilde{s}_t = s_t$$

In fact $\tilde{s}_t$ is given by the integral formula:

$$\tilde{s}_t = \int_0^t e^{-(t-t')D^2} s_{t'}\, dt'$$

Proof. uniqueness follows from the uniqueness of the homogeneous heat equation. For the existence we can differentiate the integral formula to get

$$\frac{\partial}{\partial t}\tilde{s}_t = s_t + \int_0^t \big(- D^2 e^{-(t-t')D^2} s_{t'}\big)\, \mathrm{d}t' = s_t - D^2 \tilde{s}_t$$

$\square$

Corollary 4.34 ([28] Corollary 7.10.). *For each $k \in \mathbb{N}$ there is an estimate in Sobolev norms for the solution of the inhomogeneous heat equation of the form*

$$\|\tilde{s}_t\|_k \leq tC_k \sup_{t' \in [0,t]} \|s_{t'}\|_k$$

Proof. This follows once we know that the operators e^{-tD^2} are uniformly bounded in each H^k for $t \geq 0$.

To justify the latter we can use the elliptic estimate and we get

$$\begin{aligned}
\|e^{-tD^2}\phi\|_k &\leq C\big(\|e^{-tD^2}\phi\| + \|D^k e^{-tD^2}\phi\|\big) \\
&\leq C\big(|\lambda \mapsto e^{-t\lambda^2}|_\infty + |\lambda \mapsto \lambda^k e^{-t\lambda^2}|_\infty\big)\|\phi\| \\
&\leq C'\|\phi\| \\
&\leq C'\|\phi\|_k
\end{aligned}$$

$\square$

Proposition 4.35 ([28] Proposition 7.11.). *Let k_t denote the true heat kernel. For every m there exists $m' \geq m$ such that, if k'_t is an approximate heat kernel of order m', then*

$$k_t - k'_t = t^m e_t$$

where $e_t \in C^m(S \boxtimes S^*)$, and it depends continuously on $t \geq 0$.

Proof. Take $m' > m + \frac{1}{2}\dim M$. By definition, the approximate heat kernel k'_t satisfies the equation

$$\left(\frac{\partial}{\partial t} + D^2\right)k'_t(.,q) = t^m r_t(.,q)$$

where t_t is a C^m error term.

Let $s_t(.,q)$ be the unique solution (depending on q) to the inhomogeneous heat equation

$$\left(\frac{\partial}{\partial t} + D^2\right) s_t(.,q) = -t^m r_t(.,q)$$

with $s_0 = 0$. It follows by the uniqueness of the heat kernel that

$$k'_t(p,q) + s_t(p,q) = k_t(p,q)$$

as $k'_t(p,q) + s_t(p,q)$ satisfies both properties of Proposition 4.31.

It suffices to define

$$e_t = \begin{cases} \frac{s_t}{t^m} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

The Sobolev embedding completes the proof since $\|s_t\|_{m'} \leq C t^{m'+1}$ by the above corollary.

□

We are going to build an asymptotic expansion for the heat kernel from the local data. Before that we need to do some helpful calculations.

Let us fix a point q, and take the normal coordinate $\big(U, x = (x^1, \ldots, x^n)\big)$ around q. Let $r^2 = \sum (x^i)^2$, and $h_t(p,q) = (4\pi t)^{-n/2} e^{-r^2(p)/4t}$. One can show that:

Lemma 4.36 ([28] Lemma 7.12.). *We have the following expressions for the derivatives of h.*

(a) $grad(h) = -\frac{h}{2t} r \frac{\partial}{\partial r}$

(b) $\frac{\partial}{\partial t} h + \Delta h = \frac{rh}{4gt} \frac{\partial}{\partial r} g$, *where* $g = \det(g_{ij})$ *is the determinant of the metric.*

We need also some calculations about the commutator of the Dirac operator with the (pointwise) multiplication operator by smooth functions.

Lemma 4.37 ([28] Lemma 7.13.). *For any smooth function* $f \in C^\infty(M)$ *we have*

(a) $[D, f] = c(grad(f))$, *where* c *denotes the Clifford action.*

(b) $[D^2, f] = \Delta f - 2\nabla_{grad(f)}$.

Proof. Choose an orthonormal frame synchronous at a point $p \in M$, then for any smooth section s, we have

$$D(fs) = \sum e_i \nabla_i (fs) = f \sum_i e_i \nabla_i s + \sum_i e_i(f) e_i.s = fDs + c(\mathrm{grad}(f))s$$

at the point p.

For part (b), a similar computation gives

$$\begin{aligned} D^2(fs) =& f \sum_{i,j} e_i e_j \nabla_i \nabla_j s + \sum_{i,j} e_i(e_j(f)) e_i e_j s + \sum_{i,j} e_i e_j \big(e_i(f) \nabla_j s + e_j(f) \nabla_i s\big) \\ =& fD^2 s + \Delta f s - 2\nabla_{\mathrm{grad}(f)} s \end{aligned}$$

since $e_i e_j + e_j e_i = -2\delta_{ij}$. □

Definition 4.38. Let f be a function on $\mathbb{R}^+$ with values in a Banach space E. A formal series

$$f(t) \sim \sum_{k=0}^{\infty} a_k(t)$$

where $a_k : \mathbb{R}^+ \to E$, is called an asymptotic expansion for f near $t = 0$ if for each positive integer n there exists an l_n such that for all $l \geq l_n$ there is a constant $C_{l,n}$ such that

$$\|f(t) - \sum_{k=0}^{l} a_k(t)\| \leq C_{l,n} t^n$$

for sufficiently small t.

Now one can show how to build an asymptotic expansion for the heat kernel.

Theorem 4.39 ([28] Theorem 7.15.)**.** *Let* M *be a compact Riemannian manifold equipped with a Clifford bundle* S *and Dirac operator* D*. Let* k_t *denote the heat kernel of* M*. Then* *(i) There is an asymptotic expansion for k_t, of the form*

$$k_t(p,q) \sim h_t(p,q)\big(\Theta_0(p,q) + t\Theta_1(p,q) + t^2\Theta_2(p,q) + \dots\big)$$

where Θ_j are smooth sections of $S \boxtimes S^*$.

(ii) The expansion is valid in the Banach space $C^r(S \boxtimes S^*)$ for any $r \in \mathbb{N}$.

(iii) The values $\Theta_j(p,p)$ of the sections Θ_j along the diagonal can be computed by algebraic expressions involving the metric and connection coefficients, and their derivatives, of which the first is $\Theta_0(p,q)$.

Chapter 5

Getzler Calculus and the Index Problem

5.1 Super Structure on Clifford Modules

In this section we pay careful attention to the $\mathbb{Z}/2\mathbb{Z}$ grading of Clifford modules. This enables us to define the supertrace of both the global heat semi-group operator e^{-tD^2} and the local heat kernel operator k_t and eventually yields huge cancellations in the asymptotic terms of the heat kernel. The remaining term turns out to give important topological invariants namely the $\hat{\mathcal{A}}$ genus of M coupled with its Chern character.

From now on we assume that the dimension of the base manifold is even. We first recall that when n is even the $\mathrm{Spin}(n)$ representation Δ admits a super grading

$$\Delta = \Delta^+ \oplus \Delta^-$$

The grading operator is given by

$$\omega = i^{\frac{n}{2}} e_1 \dots e_n$$

an involution of Δ with $\Delta^\pm$ its ± 1 eigenspaces.

The super structure of Δ naturally induces a super structure on any Clifford module.

Indeed if W is a Clifford module we can decompose it as

$$W = \Delta \otimes V$$

where V is an axillary vector space that can obtained via $V = \mathrm{Hom}_{Cl_n}(\Delta, W)$.

Now we can build a super structure on W as follows

$$W = W^+ \oplus W^- = (\Delta^+ \otimes V) \oplus (\Delta^- \otimes V)$$

Accordingly for any $a \in \mathrm{End}(W)$ the ***super trace*** of a, denoted by $\mathrm{Str}(a)$, is defined to be

$$\mathrm{Str}(a) = \mathrm{tr}(\epsilon a)$$

where ϵ denotes the grading operator of W. In terms of matrix notation we have

$$\mathrm{Str}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathrm{tr}(a_{11}) - \mathrm{tr}(a_{22})$$

If a, b are homogeneous elements of $\mathrm{End}(W)$ the ***super commutator*** $[a, b]_s$ is defined by

$$[a, b]_s = ab - (-1)^{|a||b|} ba$$

where $|a| = 0$ if $a \in \mathrm{End}(W)^+$, and $|a| = 1$ if $a \in \mathrm{End}(W)^-$.

Proposition 5.1 ([11] Lemma 5.1.7.). *The super trace vanishes on super commutators.*

Proof.

$$\begin{aligned}
\mathrm{Str}[a, b]_s =& tr\big(\epsilon ab - (-1)^{|a||b|} \epsilon ba\big) \\
=& \mathrm{tr}\big(\epsilon ab - b\epsilon a + b\epsilon a - (-1)^{|a||b|} \epsilon ba\big) \\
=& \big((-1)^{|b|} - (-1)^{|a||b|}\big) \mathrm{tr}(\epsilon ba)
\end{aligned}$$

The first term above vanishes unless b is odd and a is even, in which case $\mathrm{tr}(\epsilon ba)$ is zero. $\square$

The parallel constructions go on on the level of Clifford bundles and their sections. For instance, if S is a Clifford bundle

$$S_x = \Delta \otimes V = (\Delta^+ \otimes V) \oplus (\Delta^- \otimes V)$$

and

$$S = \$ \otimes \mathcal{V} \quad \text{where } \mathcal{V} = \mathrm{Hom}_{\mathbb{C}l(TM)}(\$, S)$$

Also

$$\mathrm{End}(S_x) = \mathbb{C}l(T_xM) \otimes \mathrm{End}(V), \quad \mathrm{End}(V) = \mathrm{End}_{\mathbb{C}l(T_xM)}(S_x)$$

It is clear from this setting that

Proposition 5.2. *If $a = c \otimes F \in End(S_x)$, where $c \in \mathbb{C}l(T_xM)$ and $F \in End(V)$, then*

$$Str(a) = Str(c)tr(F)$$

The next result gives the super trace of an element of a Clifford algebra. This is a crucial part of the index problem as it is revealed later.

Proposition 5.3 (Berezin Formula). *([28] Lemma 11.5.) Let $c = \sum_{I \subset \{1,\dots,n\}} c^I e_I$ be an element of $\mathbb{C}l_n$. If we consider c as an element of $End(\Delta)$, then*

$$Str(c) = (-2i)^{\frac{n}{2}} c_{\{1,2,\dots,n\}}$$

Proof. Since n is even we let $n = 2k$. We will show that

$$\mathrm{Str}(e_I) = \begin{cases} (-2i)^k & \text{if } I = \{1, 2, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

When $I = \{1, 2, \ldots, n\}$ by the definition of the supertrace we have

$$\mathrm{Str}(e_I) = \mathrm{tr}(\omega e_I) = \mathrm{tr}(i^k e_I e_I) = (-i)^k \mathrm{tr}(1) = (-i)^k \dim \Delta = (-2i)^k$$

Now if $I \neq \{1, 2, \ldots, n\}$ choose $i \in \{1, 2, \ldots, n\} \setminus I$. One can show that

$$e_I = -\frac{1}{2}[e_i, e_i e_I]_s$$

It then follows by Proposition 5.1 that $\mathrm{Str}(e_I) = 0$. $\square$

5.2 McKean-Singer Formula

Now we are going to investigate how the Dirac operator adjusts itself to this new superstructure. If S is a Clifford bundle we saw that S has a $\mathbb{Z}/2\mathbb{Z}$ grading $S = S^+ \oplus S^-$, where $S^\pm = S = \$ \otimes \mathcal{V}$ and $\mathcal{V} = \mathrm{Hom}_{Cl(TM)}(\$)$. In accordance with this decomposition we get a decomposition for the sections

$$C^\infty(S) = C^\infty(S^+) \oplus C^\infty(S^-)$$

and

$$L^2(S) = \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Since the Clifford action of any vector switches the grading, the Dirac operator is an odd operator and takes the form

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

where $D^\pm$ are the restrictions of Dirac to $C^\infty(S^\pm)$ respectively. Moreover $D^- = D^\dagger$ as Dirac is self adjoint. For the sake of simplicity we use the notation $\mathsf{D} = D^+$. Then we have

$$D = \begin{bmatrix} 0 & \mathsf{D}^\dagger \\ \mathsf{D} & 0 \end{bmatrix} \quad \text{and} \quad D^2 = \begin{bmatrix} \mathsf{D}^\dagger\mathsf{D} & 0 \\ 0 & \mathsf{D}\mathsf{D}^\dagger \end{bmatrix}$$

It follows by the ellipticity of $\mathsf{D}^\dagger\mathsf{D}$ and $\mathsf{D}\mathsf{D}^\dagger$ that D is a *Fredholm operator* and

$$\text{Ind } \mathsf{D} = \dim\ker \mathsf{D} - \dim\ker \mathsf{D}^\dagger \tag{5.1}$$

Indeed one can show that $\mathcal{H}^+$ and $\mathcal{H}^-$ enjoy spectral decompositions with respect to the elliptic operators $\mathsf{D}^\dagger\mathsf{D} : \mathcal{H}^+ \to \mathcal{H}^+$ and $\mathsf{D}\mathsf{D}^\dagger : \mathcal{H}^- \to \mathcal{H}^-$. Also the eigenspaces consist of smooth sections and have finite dimension. Since $\ker \mathsf{D} = \ker \mathsf{D}^\dagger\mathsf{D}$ and $\ker \mathsf{D}^\dagger = \ker \mathsf{D}\mathsf{D}^\dagger$, they are both finite.

The operator D provides an isomorphism between the non-zero eigenspaces of $\mathsf{D}^\dagger\mathsf{D}$ and the non-zero eigenspaces of $\mathsf{D}\mathsf{D}^\dagger$. In fact if we let $\mathcal{H}^\pm_\lambda$ denote the λ- eigenspaces of $\mathcal{H}^\pm$, for $\lambda \neq 0$. Then $s \in \mathcal{H}^\pm_\lambda$ implies that $\mathsf{D}s \in \mathcal{H}^\mp_\lambda$. So we get the maps $\mathsf{D} : \mathcal{H}^\pm_\lambda \to \mathcal{H}^\mp_\lambda$ which are clearly injective and therefore must be isomorphisms.
Intuitively

$$\begin{aligned} \dim \mathcal{H}^+_0 - \dim \mathcal{H}^-_0 =& (\dim\ker \mathsf{D} + \sum_{\lambda\neq 0} \dim \mathcal{H}^+_\lambda) - (\dim\ker \mathsf{D}^\dagger + \sum_{\lambda\neq 0} \dim \mathcal{H}^-_\lambda) \\ =& \dim\ker \mathsf{D} - \dim\ker \mathsf{D}^\dagger \\ =& \text{Ind } \mathsf{D} \end{aligned}$$

We provide a more rigorous proof of this idea:

Proposition 5.4 (McKean-Singer). *([28] Proposition 11.9.) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function that belongs to the Schwartz class with $f(0) = 1$. Then Ind $\mathsf{D} = Str(f(D^2))$, in particular*

$$Ind\ \mathsf{D} = Str(e^{-tD^2})$$

for $t > 0$.

Proof. Since the spectrum of D is discrete we can write the projection P onto the kernel of D as $P = \alpha(D)$ where $\alpha \in C_0^\infty(\mathbb{R})$ and is even, for instance $\alpha(x) = \beta(x^2)$ for some $\beta \in C_0^\infty(\mathbb{R})$ with $\beta(0) = 1$. Now for f explained above we have

$$\begin{aligned} \mathrm{Str}(f(D^2)) = \mathrm{Str}(f(D^2) - \beta(D^2) + \beta(D^2)) &= \mathrm{Str}(f(D^2) - \beta(D^2)) + \mathrm{Str}(\beta(D^2)) \\ &= \mathrm{Str}(\beta(D^2)) \\ &= \mathrm{Ind}\ \mathsf{D} \end{aligned}$$

provide that $\mathrm{Str}(g(D^2)) = 0$ for g rapidly decreasing with $g(0) = 0$.
To prove the latter we can write $g(x) = xh(x) = xh_1(x)h_2(x)$ where h, h_1, h_2 are rapidly decreasing. Then

$$g(D^2) = D^2 h(D^2) = \frac{1}{2}[Dh_1(D), Dh_2(D)]_s$$

and it follows by Proposition 5.1 that $\mathrm{Str}(g(D^2)) = 0$. □

The significance of McKean-Singer formula lies in the idea that it describes the index which is a purely analytic object in terms of the trace of the heat equation smoothing operator which is still analytic. However once we can relate the trace of a smoothing operator to the trace of its heat kernel we then can see the interconnection between analysis of D with the geometry of the base manifold and the Clifford bundle; as the heat kernel is computed from local geometric data.

In the remaining part of this section, our aim is to study the relation between the trace of a smoothing operator and the trace of its heat kernel.

Proposition 5.5 ([11] Proposition 5.2.5.). *Let $k_t^+(q,q) : S_q^+ \to S_q^+$ be the restriction of the heat kernel $k_t(q,q)$ to S_q^+. Then*

$$tr\, e^{-t\mathsf{D}^\dagger\mathsf{D}} = \int_M tr\, k_t^+(q,q)\, dq$$

Proof. As we mentioned before $\mathcal{H}^+$ has an orthonormal basis $\{s_\lambda^+\}_{\lambda\in\mathrm{spec}(\mathsf{D}^\dagger\mathsf{D})}$ consisting of the (smooth) eigensections of $\mathsf{D}^\dagger\mathsf{D}$.
Since for any $e \in S_q^+$

$$k_t^+(p,q)e = \sum_\lambda e^{-t\lambda} s_\lambda^+(p)\langle s_\lambda^+(q), e\rangle$$

we can compute the local trace

$$\mathrm{tr}\, k_t^+(q,q) = \sum_{i=1}^{n} \langle k_t^+(q,q)e_i, e_i\rangle = \sum_\lambda e^{-t\lambda}|s_\lambda^+(q)|^2$$

Integrating the above expression over M and switching the sum with the integral leads to

$$\begin{aligned}\int_M \mathrm{tr}\, k_t^+(q,q)\,\mathrm{d}q =& \sum_\lambda e^{-t\lambda} \int_M |s_\lambda^+(q)|^2\,\mathrm{d}q \\ =& \sum_\lambda e^{-t\lambda} \\ =& \mathrm{tr}\, e^{-t\mathsf{D}^\dagger\mathsf{D}}\end{aligned}$$

□

Doing the same type of computation for the negative part and subtracting the results, we obtain

Theorem 5.6 (McKean-Singer). *([11] Corollary 5.2.11.)*

$$\begin{aligned} Ind\, \mathsf{D} = Str(e^{-tD^2}) = tr\, e^{-t\mathsf{D}^\dagger \mathsf{D}} - tr\, e^{-t\mathsf{D}\mathsf{D}^\dagger} &= \int_M tr\, k_t^+(q,q) - tr\, k_t^-(q,q)\, dq \\ &= \int_M Str\, k_t(q,q)\, dq \end{aligned}$$

Using the heat kernel asymptotic 4.39 gives us

$$\text{Ind } \mathsf{D} = \text{Str}(e^{-tD^2}) \sim \frac{1}{(4\pi t)^{n/2}} \left(\int_M \text{Str } \Theta_0(q,q)\, \mathrm{d}q + t \int_M \text{Str } \Theta_1(q,q)\, \mathrm{d}q + \dots \right)$$

At this point something quite fascinating happens: The index is obviously a constant integer; as a result , there is a huge amount of cancellations in the above integral and only one term will remain.

Proposition 5.7. *The index of the operator* $\mathsf{D} = D^+$ *can be computed by the means of local geometric data:*

$$Ind\, \mathsf{D} = \frac{1}{(4\pi)^{n/2}} \int_M Str\ \Theta_{n/2}(q,q)\, dq$$

5.3 Filtered Algebras and Symbols

So far we know that Ind D can be computed from the local data once we have the Clifford top part of the asymptotic term $\Theta_{n/2}(q,q) \in \text{End}(S_q)$. The Getzler calculus provides us with an organized way of this crucial computation with the means of defining symbols for both differential and integral operators. The basic ideas lie on the notions of filtered and graded algebras and the symbol maps between them.

Definition 5.8. A ***graded algebra*** is an algebra A with a direct sum decomposition $A = \bigoplus_{m \in \mathbb{Z}} A^m$ such that $A^m A^{m'} \subset A^{m+m'}$.

Example 5.9. The exterior algebra $\bigwedge^* V$ and the polynomial algebra $\mathbb{C}[t]$ are two examples of graded algebras.

Definition 5.10. An algebra A is said to be a ***filtered algebra*** if there is a nested family of subspaces $\{A_m\}_{m\in\mathbb{Z}}$ with $A_m \subset A_{m+1}$, and such that $A_m A_{m'} \subset A_{m+m'}$.

Example 5.11. In Section 2.1 we saw that a Clifford algebra is a filtered algebra.

Example 5.12. The algebra $\mathfrak{D}(S)$ of differential operators on a vector bundle S is another example of a filtered algebra. This algebra is generated by the elements of $\Omega^0(M, \mathrm{End}(S))$ as zero ordered terms, and the covariant derivatives as first ordered terms. So we have the filtration

$$\mathfrak{D}_m = \Omega^0(M, \mathrm{End}(S))\,\mathrm{span}\{\nabla_{X_1}, \ldots, \nabla_{X_j}\}_{j\leq m}$$

Example 5.13 (Weyl Algebra)**.** The Weyl algebra is another important example of a graded algebra. It is denoted by $\mathfrak{P}(V)$ and it consists of all differential operators on the vector space V with polynomial coefficients. The degree of the operator $x^\alpha \dfrac{\partial}{\partial x^\beta}$ is $|\beta| - |\alpha|$. To put in another way the Weyl algebra is the algebra generated by

$$\{\partial_1, \ldots, \partial_n, x^1, \ldots, x^n\}$$

subject to the relations

$$\partial_i\partial_j = \partial_j\partial_i, \quad \partial_i x^i - x^i\partial_i = 1, \quad x^i x^j = x^j x^i$$

Example 5.14. Recall the Riemann curvature R is an $\mathrm{Ad}(TM)$-valued two form on M. Given $X \in C^\infty(TM)$ the map

$$T_pM \to \bigwedge^2 T_p^*(M) \quad v \mapsto \langle R_p X_p, v\rangle$$

defines a section of $\mathfrak{P}(TM) \otimes \bigwedge^* TM$ which we denote it by $\langle RX, .\rangle$. Indeed identifying TM with T^*M the above map is a one form with values in $\bigwedge^2 T_p(M)$ so it is an element of $\mathfrak{P}(TM) \otimes \bigwedge^* TM$.

Remark 5.15. If A and B are two filtered algebras we can define a filtration on $A \otimes B$ by letting

$$(A \otimes B)_m = \sum_{k+l=m} A_k B_l$$

Remark 5.16. Any graded algebra A has a filtration. In fact we can define

$$A_m = \bigoplus_{k \leq m} A^k$$

Definition 5.17 (Symbol Map). Let A be a filtered algebra and G be a graded algebra. A symbol map $\sigma_\bullet : A \to G$ is a family of linear maps $\sigma_m : A_m \to G^m$ with the following properties:

(i) If $a \in A_{m-1}$, then $\sigma_m(a) = 0$.

(ii) If $a \in A_m$ and $a' \in A_{m'}$, then $\sigma_{m+m'}(aa') = \sigma_m(a)\sigma_{m'}(a')$.

Definition 5.18. For a filtered algebra A, the ***associated graded algebra*** $G(A)$ is defined by

$$G(A) = \bigoplus_m A_m / A_{m-1}$$

Moreover the projections $A_m \xrightarrow{\pi_m} A_m / A_{m-1}$ give rise to a natural symbol map.

Example 5.19. In section 2.1 we showed (Proposition 2.9) that the exterior algebra is the associated graded algebra for the Clifford algebra. In this case the canonical symbol map (the projection map) gives the top part of Clifford algebra elements.

5.4 Getzler Symbols

Now we have the required background to introduce the Getzler symbols. It is a way of defining symbols for the differential operators of a Clifford bundle and the heat kernels. As it was mentioned before we assume that the base manifold is of even dimension. Let

S be a Clifford bundle. Recall that

$$\mathrm{End}(S) = \mathbb{C}l(TM) \otimes \mathrm{End}(\mathcal{V}) = \mathbb{C}l(TM) \otimes \mathrm{End}_{\mathbb{C}l(TM)}(S)$$

It follows from the above isomorphism that $\mathrm{End}(S)$ is a bundle of filtered algebras. The filtration on each $\mathrm{End}(S_x)$ comes from the filtration of $\mathbb{C}l(T_xM)$, and we assign degree zero to the elements of $\mathrm{End}_{\mathbb{C}l(TM)}(S)$. The algebra $\mathfrak{D}(S)$ of differential operators on S is then generated by

(i) $\mathbb{C}l(TM)$-Endomorphisms of S.

(ii) Clifford actions of vector fields.

(iii) Covariant derivatives.

Definition 5.20 (Getzler Filtration)**.** The ***Getzler filtration*** on $\mathfrak{D}(S)$ is determined by the following assignments of degrees to the generators:

(i) A Clifford module endomorphism of S has degree zero.

(ii) Clifford action of a vector field has degree one.

(iii) Covariant derivative along a vector field has degree one.

In fact the above idea comes from a general situation in which one has an algebra A generated by $B \cup V$, where B is a subalgebra of A and V is a vector subspace of A. Then one can give degree zero to the elements of B and degree one to the elements of V. If we let

$$\bigotimes_B V = B \oplus (B \otimes V \otimes B) \oplus (B \otimes V \otimes B \otimes V \otimes B) \oplus \ldots$$

It is clear that there is a surjective map

$$\bigotimes_B V \to A$$

and therefore the algebra A inherits a filtration from $\bigotimes_B V$.

For the algebra $\mathfrak{D}(S)$ of differential operators we can let B to be $\Omega^0(M; \mathrm{End}_{\mathbb{C}l(TM)}(S))$ and $V = C^\infty(TM) \oplus C^\infty(TM)$ where the first component represents the Clifford action $c(X)$ of a vector field , and the second component represents the covariant derivative ∇_Y. We proceed to define the Getzler symbols on $\mathfrak{D}(S)$.

Proposition 5.21 ([28] Proposition 12.13.). *There is a unique symbol map*

$$\sigma_\bullet : \mathfrak{D}(S) \to C^\infty(\mathfrak{P}(TM) \otimes \bigwedge^* TM \otimes End_{\mathbb{C}l(TM)}(S))$$

called the Getzler symbol such that

(i) $\sigma_0(F) = F$ *for* $F \in \Omega^0(M; End_{\mathbb{C}l(TM)}(S))$.

(ii) $\sigma_1(c(X)) = e(X)$*, that is the exterior multiplication by* $X \in C^\infty(TM)$.

(iii) $\sigma_1(\nabla_X) = \partial_X + \frac{1}{4}\langle RX, .\rangle$.

Since we have specified the action of symbol map on the generators, one just has to check that the symbol map factors through the quotient $\bigotimes_B V \to \mathfrak{D}(S)$ explained above.

$$\begin{tikzcd}
\mathfrak{D}(S) \arrow[dr, "\exists ?\sigma_\bullet"] & \\
\bigotimes_B V \arrow[u] \arrow[r, "\sigma_\bullet"] & C^\infty(\mathfrak{P}(TM) \otimes \wedge^* TM \otimes \mathrm{End}_{\mathbb{C}l(TM)}(S))
\end{tikzcd}$$

Put it another way we have to make sure that the symbol is compatible with the interrelations of the algebra elements. We will address this problem later. (See Proposition 5.28.)

Example 5.22. For instance we are going to justify that the symbol respects the relation

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = K(X,Y) = R^S(X,Y) + F^S(X,Y)$$

proved in Proposition 3.18.

We shall compute the Getzler symbols of the two sides to see if they agree.

For the left hand side, we can ignore $\nabla_{[X,Y]}$ since it is of first degree and we are computing σ_2. We choose an orthonormal basis $(e_1, \ldots, e_n)$ of T_qM with the coordinate functions $x^i = e_i^*$. Then at the point q we have

$$\begin{aligned}
\sigma_1(\nabla_i) =& \frac{\partial}{\partial x^i} + \frac{1}{4}\langle Re_i, .\rangle \\
=& \frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{j,k,l}\langle R(e_k, e_l)e_i, e_j\rangle x^j e_k \wedge e_l \\
=& \frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{j,k,l}\langle R(e_i, e_j)e_k, e_l\rangle x^j e_k \wedge e_l
\end{aligned}$$

and some straightforward computations shows that

$$\begin{aligned}
\sigma_2(\nabla_i\nabla_r - \nabla_r\nabla_i) =& [\sigma_1(\nabla_i), \sigma_1(\nabla_r)] \\
=& \frac{1}{8}\sum_{k,l}\langle R(e_i, e_r)e_k, e_l\rangle e_k \wedge e_l - \frac{1}{8}\sum_{a,b}\langle R(e_r, e_i)e_a, e_b\rangle e_a \wedge e_b \\
=& \frac{1}{4}\sum_{k,l}\langle R(e_i, e_r)e_k, e_l\rangle e_k \wedge e_l
\end{aligned}$$

But this is exactly $\sigma_2(R^S(e_i, e_r))$ (see Definition 3.16). The desired result follows as $\sigma_2(F^S) = 0$.

Example 5.23. The Getzler degree of the Dirac operator is 2 since $D = c(e_i)\nabla_i$. We shall show that $\sigma_2(D)$ is the exterior derivative at any vector space T_qM. Indeed

$$\begin{aligned}
\sigma_2(D) =& \sum_i \sigma_1(c(e_i))\sigma_1(\nabla_i) \\
=& \sum_i e_i \frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{ijkl}\langle R(e_i, e_j)e_k, e_l\rangle x^j e_i \wedge e_k \wedge e_l \\
=& \sum_i e_i \frac{\partial}{\partial x^i} \qquad \text{by Bianchi identity} \\
=& d_{T_qM}
\end{aligned}$$

Example 5.24. Since $d^2 = 0$ it is obvious that D^2 must be of Getzler degree ≤ 3. Surprisingly the Getzler degree of D^2 is 2 and we have, at point q,

$$\sigma_2(D^2) = -\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4} \sum_j R_{ij} x^j \right)^2 + F^S$$

where R_{ij} and F^S are the Riemann curvature and the twisting curvature two-froms at point q.

To verify this we first recall Lichnerowicz-Schrodinger equation 3.20

$$D^2 = \nabla^* \nabla + \mathcal{F}^S + \frac{1}{4}\kappa$$

where $\mathcal{F}^S = \sum_{i<j} c(e_i)c(e_j)F^S(e_i, e_j)$. Also recall from Lemma 3.9 that

$$\begin{aligned} \nabla^* \nabla =& \sum_{ijk} -g^{jk}(\nabla_j \nabla_k - \Gamma^i_{jk} \nabla_i) \\ =& -\sum_i \nabla_i \nabla_i \qquad \text{at the origin} \end{aligned}$$

Hence we can conclude that

$$\begin{aligned} \sigma_2(D^2) =& \sigma_2(\nabla^* \nabla) + \sigma_2(\mathcal{F}^S) \\ =& \sigma_2(-\sum_i \nabla_i \nabla_i) + \sigma_2(\mathcal{F}^S) \\ =& -\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4} \sum_j R_{ij} x^j \right)^2 + F^S \end{aligned}$$

We are going to extend our study of differential operators to the smoothing operators on S. Our aim is to define the Getzler symbol

$$\sigma_\bullet : C^\infty(S \boxtimes S^*) \to C^\infty(\mathbb{C}[[TM]] \otimes \bigwedge^* TM \otimes \mathrm{End}_{Cl(TM)}(S))$$

Although there is an appropriate multiplication on $C^\infty(S \boxtimes S^*)$ with respect to which the Getzler symbol will satisfy the homomorphism-like property of symbols, our study does not concern this problem. What our study rather concerns is the $\mathfrak{D}(S)$-module structure of $C^\infty(S \boxtimes S^*)$ and we want the symbol map respects this module structure in the sense of Proposition 5.27.

Definition 5.25. For a vector space V, we denote by $\mathbb{C}[[V]]$ the algebra of formal power series on V. This is in fact a graded $\mathfrak{P}(V)$-module if we give a monomial x^α the degree $-|\alpha|$.

Now we are going to define the symbol of a section $s \in C^\infty(S \boxtimes S^*)$. For this purpose we fix a point $q \in M$ and we choose a normal coordinate $x = (x^1, \dots, x^n)$ centered at q. Then there is a local Taylor expansion for the section $p \mapsto s(p,q)$ in the form

$$\sum_\alpha x^\alpha s_\alpha$$

Where s_α are parallel transportations of $s_\alpha(0) \in \mathrm{End}(S_q)$ to $s_\alpha(x(p)) \in S_p \otimes S_q^*$. Since each s_α is determined by its value $s_\alpha(0)$ at the origin, the above Taylor series defines an element in $\mathbb{C}[[T_qM]] \otimes \mathrm{End}(S_q)$. If we vary q we then obtain a section $\Sigma s \in C^\infty(\mathbb{C}[[TM]] \otimes \mathrm{End}(S))$. Since both $\mathbb{C}[[T_qM]]$ and $\mathrm{End}(S_q)$ are filtered, the algebra $\mathbb{C}[[T_qM]] \otimes \mathrm{End}(S_q)$ has a tensor product filtration mentioned in Remark 5.15.

Definition 5.26. We say that $s \in C^\infty(S \boxtimes S^*)$ has degree $\leq m$ if the degree of $\Sigma s \in C^\infty(\mathbb{C}[[TM]] \otimes \mathrm{End}(S))$ is $\leq m$. We call this degree the Getzler degree of s.

Also if we compose the Taylor series map Σ with the Clifford symbol map $\mathrm{End}(S_q) = \mathbb{C}l(T_qM) \otimes \mathrm{End}_{\mathbb{C}l(T_qM)}(S_q) \to \bigwedge^* T_qM \otimes \mathrm{End}_{\mathbb{C}l(T_qM)}(S_q)$, we obtain a symbol map

$$\sigma_\bullet : C^\infty(S \boxtimes S^*) \xrightarrow{\Sigma} C^\infty(\mathbb{C}[[TM]] \otimes \mathrm{End}(S)) \to C^\infty(\mathbb{C}[[TM]] \otimes \bigwedge^* TM \otimes \mathrm{End}_{\mathbb{C}l(TM)}(S))$$

which is called the Getzler symbol. We also let $\sigma_m^0(s)$ denote the constant term in the Taylor series $\sigma_m(s)$.

Proposition 5.27 ([28] Proposition 12.22.). *Let $T \in \mathfrak{D}(S)$ be one of the generators we listed in 5.20. Let $m \in \{0, 1\}$ be the Getzler degree of T. Then for any smoothing operator Q with Getzler degree $\leq k$, the smoothing operator TQ has Getzler degree $\leq m + k$, and the relation*

$$\sigma_{m+k}(TQ) = \sigma_m(T)\sigma_k(Q) \tag{5.2}$$

holds between the symbols.

Proof. Let Q be an smoothing operator with kernel $s \in C_k^\infty(S \boxtimes S^*)$. We fix a point $q \in M$ and choose a normal coordinate centered at q. Also let $s_q(x) \sim \sum_\alpha x^\alpha s_\alpha(x)$ be the Taylor expansion of $p \mapsto s(p, q)$. We split the proof into three cases for the three types of generators of $\mathfrak{D}(S)$.

Case 1. When $T = F \in C^\infty(\mathrm{End}_{Cl(TM)}(S))$.

Suppose that F is synchronous at q and let $s' = Fs$. Since $\nabla_Y(Fs) = F\nabla_Y s$ we get

$$s'_\alpha(0) = Fs_\alpha(0)$$

which implies equation $\sigma_k(Fs) = \sigma_0(F)\sigma_k(s)$ as desired.

If F does not happen to be synchronous we can parallel transport it to the near points to obtain a synchronous section F_0. But $\sigma_0(F) = \sigma_0(F_0)$ since the constant term in the Taylor expansion of $F - F_0$ is zero. Therefore,

$$\sigma_k(Fs) = \sigma_k(F_0 s) = \sigma_0(F_0)\sigma_k(s) = \sigma_0(F)\sigma_k(s).$$

Case 2. The same logic shows that Equation 5.2 holds when $T = c(X)$ the Clifford action of some vector field $X \in C^\infty(TM)$.

Case 3. When $T = \nabla_X$.

Let $(\partial_1, \ldots, \partial_n)$ be the local frame associated to the normal coordinate $x = (x^1, \ldots, x^n)$, and $Y = \sum_j x^j \partial_j$. Without loss of generality, we can assume that $X = \partial_i$. Suppose that s is synchronous and let

$$\sum_\alpha x^\alpha t_\alpha$$

be the Taylor series of $\nabla_X s$. Then

$$\begin{aligned} K(X,Y)s =& \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ =& - \nabla_Y \nabla_X s - \nabla_X s \qquad (\text{since } \nabla_Y s = 0, \text{ and } [X,Y] = X) \end{aligned}$$

The above equation implies that the section $K(X,Y)s$ has the Taylor expansion

$$-\sum_\alpha (|\alpha| + 1) x^\alpha t_\alpha$$

and that the degree of $\nabla_X s$ and its Taylor coefficients are determined by those of $K(X,Y)s$.

On the other hand it follows from Proposition 3.18 that

$$K(X,Y)s = \sum_j x^j K(\partial_i, \partial_j)s = \sum_j x^j R^S(\partial_i, \partial_j)s + \sum_j x^j F^S(\partial_i, \partial_j)s$$

Notice that the Getzler degree of $x^j R^S(\partial_i, \partial_j)$ is 1 and $x^j F^S(\partial_i, \partial_j)$ has the Getzler degree ≤ -1. This shows that $\nabla_X s$ has the Getzler degree $\leq k+1$ as we want. Furthermore, it follows from the Taylor expansion of $K(X,Y)s$ that

$$\nabla_X s = -\frac{1}{2} \sum_j x^j R^S(\partial_i, \partial_j)s + \text{ lower degree terms.}$$

Therefore, using the results of case (2) and Example 5.22 we can compute the symbol

$$\begin{aligned}\sigma_{k+1}(\nabla_X s) =&\sigma_{k+1}(-\frac{1}{2}\sum_j x^j R^S(\partial_i,\partial_j)s)\\ =&\sigma_1(-\frac{1}{2}\sum_j x^j R^S(\partial_i,\partial_j))\sigma_k(s)\\ =&-\frac{1}{8}\sum_{j,k,l}\langle R(e_i,e_r)e_k,e_l\rangle x^j e_k\wedge e_l\sigma_k(s)\\ =&\sigma_1(\nabla_X)\sigma_k(s)\end{aligned}$$

□

Now we can prove Proposition 5.21.

Proposition 5.28 ([28] Proposition 12.23.). *The Getzler symbol is well-defined on $\mathfrak{D}(S)$, and satisfies the equation*

$$\sigma_{m+k}(TT') = \sigma_m(T)\sigma_{m'}(T')$$

for any $T \in \mathfrak{D}_m(S)$, and any $T' \in \mathfrak{D}_{m'}(S)$.

Proof. It suffices to show that the symbol is well-defined. The multiplicative property follows then from the definition of symbol.
Suppose that the operator T has two representatives $T_{\in}\mathfrak{D}_m(S)$ and $T'_{\in}\mathfrak{D}_{m'}(S)$. Iterating the previous proposition for T, T', and an arbitrary smoothing operator Q yields

$$\sigma_m(T)\sigma_k(Q) = \sigma_{m+k}(TQ) = \sigma_{m'+k}(T'Q) = \sigma_{m'}(T')\sigma_k(Q)$$

Since Q is arbitrary, multiplying the exterior algebra parts of $\sigma_m(T)$ and $\sigma_{m'}(T')$ with the same type part of $\sigma_k(Q)$ in the above equation implies that $\sigma_m(T)$ and $\sigma_{m'}(T')$ must agree on the exterior algebra part. Hence we left with the Weyl algebra parts which are differential operators with polynomial coefficients. Again since Q is arbitrary this part can be uniquely identified from its action on appropriately enough power series. □

Recall from Theorem 4.39 that the heat kernel has the asymptotic expansion

$$k_t(p,q) \sim h_t(p,q)\big(\Theta_0(p,q) + t\Theta_1(p,q) + t^2\Theta_2(p,q) + \dots\big)$$

It also satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + D^2\right)k_t(.,q) = 0$$

It follows from Proposition 4.39 that

Proposition 5.29 ([28] Proposition 12.24.). *For each j the term Θ_j belongs to $C^\infty_{2j}(S \boxtimes S^*)$. Furthermore the **heat symbol***

$$W_t = h_t(\sigma_0\Theta_0 + t\sigma_2\Theta_1 + \dots + t^{n/2}\sigma_n\Theta_{n/2})$$

solves the differential equation $\dfrac{\partial}{\partial t}W_t + \sigma_2(D^2)W_t = 0$, and it is the unique solution of the form $h_t(v_0 + tv_1 + \dots + t^{n/2}v_{n/2})$ where each v_j is a symbol of Getzler degree $2j$ and $v_0 = 1$.

5.5 Atiyah-Singer Index theorem

In this section we are going to use Mehler's formula to find an explicit solution to the differential equation

$$\frac{\partial}{\partial t}W_t + \sigma_2(D^2)W_t = 0$$

We will see surprisingly then the characteristic class terms come out of this solution. We first consider the above equation for the algebra of reals.

Proposition 5.30 ([28] Proposition 12.25.). *Suppose that $R = (R_{ij})$ is a skew symmetric matrix of real entries, and that F is a real scalar. Then the partial differential equation*

$$\frac{\partial}{\partial t}w - \sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4} \sum_j R_{ij} x^j \right)^2 w + Fw = 0$$

has a solution for $t > 0$. The solution is analytic with respect to F and the entries of R and is asymptotic to $(4\pi t)^{-n/2} \exp(-|x|^2/4t)$ as $t \to 0^+$. Explicitly the solution is equal to

$$(4\pi t)^{-n/2} \det \left(\frac{tR/2}{\sinh tR/2} \right)^{1/2} \exp \left(-\frac{1}{4t} \langle \frac{tR}{2} \coth \frac{tR}{2} x, x \rangle \right) \exp \left(-tF \right) \tag{5.3}$$

Proof. If we multiply the equation by the integrating factor $\mu = \exp(Ft)$ and then use the substitution $w' = \exp(Ft)w$, the term Fw will be eliminated. However we have to keep this factor and multiply the final answer by $\exp(-Ft)$ at the end.

It is helpful first to write the coefficients of the above PDE in matrix notation:

$$\frac{\partial}{\partial t}w - \left(\frac{\partial}{\partial \mathbf{x}}^T \frac{\partial}{\partial \mathbf{x}} + \mathrm{tr}(R) + \frac{\partial}{\partial \mathbf{x}}^T R\mathbf{x} + \mathbf{x}^T R^T R \mathbf{x} \right) w = 0$$

where $\mathbf{x} = [x^1, \dots, x^n]^T$. Then we can apply an orthogonal transformation $\mathbf{x} = P\mathbf{y}$ so that

$$P^T R P = Q = \begin{bmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \theta_k \\ -\theta_k & 0 \end{bmatrix} \qquad 2k = n$$

and that our differential equation takes the form

$$\frac{\partial}{\partial t}w - \left(\frac{\partial}{\partial \mathbf{y}}^T \frac{\partial}{\partial \mathbf{y}} + \mathrm{tr}(Q) + \frac{\partial}{\partial \mathbf{y}}^T Q\mathbf{y} + \mathbf{y}^T Q^T Q \mathbf{y} \right) w = 0$$

Now if we put

$$w = w_1(y^1, z^1) w_2(y^2, z^2) \dots w_k(y^k, z^k)$$

the above PDE splits into k PDEs of the same form each of which takes place in $\mathbb{R}^2$. Thus it is enough to consider the two dimensional case with

$$Q = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$

and multiply the (similar) solutions later.

So we shall study the differential equation

$$\frac{\partial}{\partial t}w - \left(\frac{\partial}{\partial y} - \frac{1}{4}\theta z\right)^2 w - \left(\frac{\partial}{\partial z} + \frac{1}{4}\theta y\right)^2 w = 0$$

Equivalently we can write

$$\frac{\partial}{\partial t}w + (L_0 + L_1)w = 0$$

where

$$\begin{aligned} L_0 =& -\Delta - \frac{1}{16}\theta^2(y^2+z^2) \\ L_1 =& \frac{1}{2}\theta\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \end{aligned}$$

Since L_0 is invariant under the action of $\mathrm{SO}(2)$ and L_1 is a one of the generators of the Lie algebra $\mathfrak{so}(2)$, the fundamental solution of $L_0 + L_1$ with pole at origin is equal to the fundamental solution of L_0 with pole at the origin. Therefore, our study reduce to the equation

$$\frac{\partial}{\partial t}w + L_0 w = 0.$$

Using separation of variables $w(y,z) = Y(y)Z(z)$ again, we are left with the equation

$$\frac{\partial}{\partial t}Y - \frac{\partial^2}{\partial y^2}Y - \frac{1}{16}\theta^2 y^2 Y = 0$$

By Mehler's formula for the heat kernel of the harmonic oscillator, the fundamental solution of the above equation is

$$Y = (4\pi t)^{-\frac{1}{2}} \left(\frac{it\theta/2}{\sinh it\theta/2} \right)^{\frac{1}{2}} \exp\big(- \frac{1}{8} i\theta y^2 \coth(it\theta/2) \big)$$

Taking the product of the above solution with the corresponding solution for Z and also accordingly plugging

$$w = w_1(y^1, z^1) w_2(y^2, z^2) \dots w_k(y^k, z^k)$$

gives the desired solution of the original PDE.
Notice that the determinant and the exponential terms in Equation 5.3 dose not depend on a particular representation of the matrix R as they are characteristic forms. □

Now recall that

$$\sigma_2(D^2) = -\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4} \sum_j R_{ij} x^j \right)^2 + F^S$$

In which $R = (R_{ij})$ -the Riemann curvature- is a skew symmetric matrix of two forms, and F^S is the twisting curvature. Since the entries of R commutes with each other and with F^S, it follows from Proposition 5.29 that the above formula gives a formal ,finite power series expression for the heat symbol

$$W_t = (4\pi t)^{-n/2} \det \left(\frac{tR/2}{\sinh tR/2} \right)^{1/2} \exp\left(- \frac{1}{4t} \langle \frac{tR}{2} \coth \frac{tR}{2} x, x \rangle \right) \exp\big(- tF^S \big)$$

Now the uniqueness part of Proposition 5.29 tells us that

Proposition 5.31 ([28] Proposition 12.26.). *The sum of the constant parts of the Getzler symbols of asymptotic kernels are given by*

$$\sum_0^{n/2} \sigma_{2j}(\theta_j) = \det \left(\frac{R/2}{\sinh R/2} \right)^{1/2} \exp\big(- F^S \big)$$

After a quite long journey passing through several deep canyons and observing many beauties of the realm of Dirac operators we get to the summit the main promise of this thesis.

January,1962 - Oxford

Atiyah: *Why is the $\hat{\mathcal{A}}$-genus an integer for spin manifolds?*

Singer: *You know the answer better than I- why do you ask?*

Atiyah: *There must be a deeper reason.*

Atiyah-Singer Index Theorem. *Let M be a compact, even dimensional Riemannian manifold and let S be a Clifford bundle over M with Dirac operator D. Then*

$$Ind\ D = \int_M \hat{\mathcal{A}}(TM) \wedge ch(\mathcal{V})$$

where $\mathcal{V} = Hom_{\mathbb{C}l(TM)}(\$, S)$, and ch denotes the Chern character.
In particular for the bundle of spinors $\$$ with the Dirac operator $\not{D}$ we get

$$Ind\ \not{D} = \hat{\mathcal{A}}(M)$$

Proof. Recall from Proposition 5.7 that

$$\text{Ind D} = \frac{1}{(4\pi)^{n/2}} \int_M \text{Str}\ \Theta_{n/2}(q,q)\, \mathrm{d}q$$

It then follows from Berezin formula 5.3 and the previous proposition that

$$
\begin{aligned}
\operatorname{Ind} \mathsf{D} =& \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{Str} \Theta_{n/2}(q,q)\, \mathrm{d}q \\
=& \frac{1}{(2\pi i)^{n/2}} \int_M \operatorname{tr} \sigma_0(\Theta_{n/2}) \\
=& \frac{1}{(2\pi i)^{n/2}} \int_M det \left(\frac{R/2}{\sinh R/2} \right)^{1/2} \operatorname{tr}\big(\exp(-F) \big) \\
=& \int_M \hat{\mathcal{A}}(TM) \wedge \operatorname{ch}(\mathcal{V})
\end{aligned}
$$

□

Now let us reflect upon two outstanding breakthrough in this journey. Our first victory was passing through the Mc-Kean Singer defile that connects the index as a purely algebraic-analytic object to local geometric data.

And at the end we swam over the Getzler calculus river at night that connects the local geometric data to globals topological invariants. Our polar star was the Berezin formula, and the Bianchi identity was our help in the peril.

5.6 Conclusion

There are many areas of research in geometry, topology, and theoretical physics that emerges from the study of Dirac operators and index theory. The study of spaces with positive scalar curvature is perhaps one of the most flourishing fields of research that has gained many benefits from study of Dirac operators. André Lichnerowicz [24] obtained one of the earliest results in this direction. With the help of Atiyah-Singer index theorem he showed that there are many examples of manifolds that can not admit any metric of positive scalar curvature.

Theorem 5.32 (Lichnerowicz)**.** *Let M be a compact, Riemannian, spin manifold of positive scalar curvature. Then the $\hat{A}$-genus of M vanishes.*

Proof. It follows from Lichnerowicz-Schrödinger Theorem 3.20, and Theorem 3.42 that

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa.$$

Therefore,

$$\|Ds\|^2 = \|\nabla s\|^2 + \frac{1}{4}\kappa \|s\|^2$$

and the kernel of the Dirac operator is trivial since the scalar curvature is positive. Hence it follows by Atiyah-Singer index theorem that $\hat{\mathcal{A}}(M) = 0$. □

Consequently, for 4-manifolds, as $\hat{\mathcal{A}}(M) = \frac{1}{8}\mathrm{sign}(M)$ the above theorem shows that manifolds of non-zero signature can not carry positive scalar curvature.

Later Nigel Hitchin [20] extended the above result to the exotic spheres and proved that half of the exotic spheres in dimension 1 or 2 (mod 8) can not carry metrics of positive scalar curvature.

Gromov and Lawson added the fundamental group to the ingredients and introduced the notion of *enlargeability* of manifolds. They proved that

Theorem 5.33 ([16] Theorem A)**.** *Let X be an enlargeable manifold. Then X carries no Riemannian metric of positive scalar curvature.*

Theorem 5.34 ([17] Corollary C)**.** *Every compact simply-connected n-manifold, $n > 5$, which is not spin, carries a metric of positive scalar curvature.*

In [18] they extended their studies to non-compact manifolds as well. They also imposed the following conjecture which was proved later by Stephen Stolz [30].

Theorem 5.35 ([30] Theorem A)**.** *Let M be a simply connected, closed, spin manifold of dimension $n > 5$. Then M carries a metric with positive scalar curvature if and only if $\alpha(M) = 0$. Where $\alpha(M)$ denotes the KO-characteristic number of M.*

The relation between the spectrum of the Dirac operator and the geometry of the manifold is another field of intense researches. For instance, in [4] Christian Bär studies the spectrum of the Dirac operator on the spheres and their quotients. More recently she has obtained some eigenvalue estimates for Dirac operators on non-compact manifolds [5].

In [14] Nicolas Ginoux initiates a study of the Dirac operator on Lagrangian submanifolds of Kähler manifolds and obtains some spectral estimates for these type of spaces. Also, Ginoux and Habib [15] has studied the spectral properties of the Dirac operator on Riemannian flows.

Besides, the index problem for elliptic operators on manifolds with singularities has attracted some attention. For instance Nazaikinskii, etal. [31] developed an index formula for elliptic operators on manifolds with edges.

The study of Dirac operators and index theory is also of great importance for theoretical physicists. As an interesting example, Deguchi and Kitsukawa [29] showed that the quantization conditions of Dirac and Schwinger can be derived from the

Atiyah-Singer index theorem in two dimensions.

At the end, it is worthwhile to mention that the development of non-commutative geometry has one of its routs in the theory of Dirac operators. Index theorem and some other results about geometry of Dirac operators is another evidence for the Algebra-Geometry duality.

It is well-known that a locally compact Hausdorff topological space X can be recovered from the algebra of its observables $C(X)$. In fact the Gelfand-Naimark theorem provides two contravariant functors between the category of locally compact topological spaces with the category of C^*-algebras. The first functor sends a space X to the algebra $C(X)$, and the second one sends a C^* algebra A to the space of characters, that is, the set of non-zero homomorphisms $\rho : A \to \mathbb{C}$. In this correspondence the compactness translates to being unital, open subsets corresponds to ideals, closed subset turns to quotient algebra and etc.

So far there is no mention of the geometry of X. Now if we have a Dirac operator on X using 4.4 we can show that

$$d(p,q) = \sup\{|f(p) - f(q)| : f \in C^\infty(M), \|[D,f]\| \leq 1\}.$$

This is a strong evidence that one can replace a space with the set of observables. In fact it was Alain Connes who took these ideas and introduced the concept of an spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consisting of an algebra $\mathcal{A}$ a Hilbert space $\mathcal{H}$ which is also a representation of $\mathcal{A}$. And a self adjoint operator $\mathcal{D} : \mathcal{H} \to \mathcal{H}$.

In the commutative case $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$ and $\mathcal{D}$ is the Dirac operator of the Clifford bundle S.

Theses tools enables us to study more sophisticated spaces that might suffer from the lack of natural measure theoretic, geometric, or topological structures. An important example of such spaces is the space of leaves of a foliation for which Alain Connes extended the Atiyah-Singer index theorem. (See [8] and [9] for more details.) Erik van Erp [32] has used the Connes' tangent groupoid technique to derive an index formula for hypoelliptic operators on contact manifolds.

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