ABSTRACT. We say that one point process on the line \( \mathbb{R} \) mimics another at a bandwidth \( B \) if for each \( n \geq 1 \) the two point processes have \( n \)-level correlation functions that agree on bandwidth \( [-B, B] \). This paper asks the question of for what values \( a \) and \( B \) can a given point process on the real line be mimicked at bandwidth \( B \) by a point process supported on the lattice \( a \mathbb{Z} \). For Poisson point processes we give a complete answer, and for the sine-kernel process we give a partial answer.

A companion paper gives an application to the Alternative Hypothesis regarding the scaled spacing of zeros of the Riemann zeta function.

1. Introduction

1.1. Objective. In this paper we ask the following question: how well can the statistics of a point process on the real line \( \mathbb{R} \) be mimicked by the statistics of a point process restricted to a lattice \( a \mathbb{Z} = \{aj : j \in \mathbb{Z}\} \)? The statistics we consider are correlation functions, and what we mean by ‘mimicking’ is explained below. We give an analysis of the mimicking problem for two distinct point processes, the Poisson process and the sine-kernel process, and uncover some surprising mismatches between the two.

This problem has its origins in a problem regarding the zeros of the Riemann zeta-function, which we discuss at the end of the introduction and treat more fully in a companion paper [20]. Additional questions raised by this work are discussed in a final section.

1.2. Background and conventions for point processes. We first recall the definition of a point process, fix our notation, and discuss some fundamental properties. More leisurely introductions to the concept of point process with conventions similar to ours include [1, 9, 11, 33].

Informally, a point process is just a recipe to randomly lay down points in some topological space. In more technical terms: we consider a locally compact separable topological space \( \mathfrak{X} \); in fact for us \( \mathfrak{X} \) will always be \( \mathbb{R} \) or \( a \mathbb{Z} \) for some \( a > 0 \), equipped with their usual topology. A point configuration \( u \) in \( \mathfrak{X} \) is a sequence of elements \( u := (u_j)_{j \in \mathbb{Z}} \) with \( u_i \in \mathfrak{X} \) for all \( i \in \mathbb{Z} \). For \( u \) a configuration, and \( V \subset \mathfrak{X} \), we use the notation

\[
\#_V(u) := \#\{i : u_i \in V\}
\]

to denote the number of elements of the configuration \( u \) inside \( V \). We let the configuration space \( \text{Conf}(\mathfrak{X}) \) be the set of locally finite configurations, that is

\[
\text{Conf}(\mathfrak{X}) := \{u : \#_K(u) < +\infty \text{ for all compact } K\}.
\]
Unlike configuration spaces in topology we allow $u_j$ to take repeated values. We let $\mathcal{M}$ be the smallest topology on $\text{Conf}(X)$ that contain all cylinder sets $C^V_m$, where

$$C^V_m := \{ u \in \text{Conf}(X) : \#_V(u) = m \},$$

where $V$ is any bounded Borel set and $m$ is any non-negative integer.

We let $\mathcal{B}(\mathcal{M})$ be the Borel $\sigma$-algebra generated by $\mathcal{M}$. A point process on $\mathcal{X}$ is a random element $u$ taking values in $(\text{Conf}(X), \mathcal{B}(\mathcal{M}))$.

With this definition, the sets

$$\{ u : \#_{B_1}(u) = m_1, \#_{B_2}(u) = m_2, ... , \#_{B_n}(u) = m_n \}$$

are measurable events, for any finite collection of Borel subsets $B_1, B_2, ..., B_n$ of $X$ and for any finite collection of non-negative integers $m_1, ..., m_n$. This definition allows points to coincide; they may have a finite multiplicity. A point process is said to be simple if (with probability one) any configuration has $u_i \neq u_j$ if $i \neq j$.

In this paper we specialize to the case that the space is $X = \mathbb{R}$ or $a\mathbb{Z}$ for some $a > 0$.

For the point processes we will be interested in we impose an additional condition.

**Uniform Local Moments Condition.** For each $n \geq 1$ there exists a constant $C_n < \infty$ such that

$$E[\#_{|L,L+1|}(u)]^n \leq C_n, \quad \text{for all } L \in \mathbb{R}. \quad (1)$$

**Note that $C_n$ does not depend on $L$.**

We say that a point process satisfying (1) has uniform local moments, and refer to it subsequently as a u.m. point process.

For any point process on $\mathbb{R}$, for any $n \geq 1$ and any $\phi \in C_c(\mathbb{R}^n)$, the sum

$$\sum_{j_1, ..., j_n \text{ distinct}} \phi(u_{j_1}, ..., u_{j_n}) \quad (2)$$

defines a random variable (that is, a measurable mapping from $\text{Conf}(X)$ to $\mathbb{C}$).

In the case that our point process has uniform moments, the Riesz representation theorem implies for that measure that for all $n \geq 1$ that there exists a unique measure $\rho_n$ on $\mathbb{R}^n$ such that

$$E \sum_{j_1, ..., j_n \text{ distinct}} \phi(u_{j_1}, ..., u_{j_n}) = \int_{\mathbb{R}^n} \phi(x_1, ..., x_n) d\rho_n(x_1, ..., x_n), \quad (3)$$

for all $\phi \in C_c(\mathbb{R}^n)$. (In the case that $X = a\mathbb{Z}$, the measure $\rho_n$ will be supported on $(a\mathbb{Z})^n$.) The measure $\rho_n$ is called the $n$-level correlation measure of the process $u$. (The name $n$-level joint intensity measure is used interchangeably in some literature.)

We recall the well-known fact that if $V$ is any Borel subset of $\mathcal{M}$ (or $a\mathbb{Z}$), we have

$$\sum_{j_1, ..., j_n \text{ distinct}} \mathbf{1}_V(u_{j_1}) \cdots \mathbf{1}_V(u_{j_n}) = \prod_{i=0}^{n-1} (\#_V(u) - i) \quad (4)$$

\footnote{See Theorem A.1 in Appendix A.1}
where \(1_V\) is the indicator function of the set \(V\). In consequence

\[
E \prod_{i=0}^{n-1} (\#_V(u) - i) = \int_{\Omega^n} d\rho_n(x_1, \ldots, x_n).
\]

From this it follows that if a point process has uniform moments then \(\rho_n([L, L + 1]^n) \leq C_n\).

For a u.m. point process, (2) can be extended to a slightly wider class of functions \(\phi\) than \(C_c(\mathbb{R}^n)\). Let \(S(\mathbb{R}^n)\) be the Schwartz class of functions on \(\mathbb{R}^n\):

**Proposition 1.1.** Let \(u\) be a u.m. point process and let \(\rho_n\) the \(n\)-level correlation measure of the process \(u\) (defined by (3) for all \(\phi \in C_c(\mathbb{R}^n)\)). Then for all \(n \geq 1\) and \(\eta \in S(\mathbb{R}^n)\), the sum

\[
\sum_{j_1, \ldots, j_n \text{ distinct}} \eta(u_{j_1}, \ldots, u_{j_n})
\]

converges almost surely and defines an integrable random variable, with

\[
E \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(u_{j_1}, \ldots, u_{j_n}) = \int_{\mathbb{R}^n} \eta(x_1, \ldots, x_n) d\rho_n(x_1, \ldots, x_n).
\]

This is proved via a simple limiting argument combined with the dominated convergence theorem. Theorem A.3 in Appendix A.2 gives a slightly more general result with a full proof.

**Remark 1.2.** It is possible for two distinct point processes share the same correlation functions for all \(n\). This phenomenon is not the usual situation: if for the two point processes the constants \(C_n\) in (1) do not grow too quickly with \(n\), and if the two point processes have the same correlation measures, then they are identical in distribution. See [11, Remark 1.2.4]. There is a comparison with the classical moment problem for random variables.

**Remark 1.3.** Not all measures \(\rho_n\) are realizable as the correlation measures of some point process. An abstract criterion that correlation measures must satisfy to be a point process was given by Lenard [21].

Distinguishing those measures \(\rho_n\) which are realizable this way from those that are not will play an important role later in this paper.

In this paper we will focus in particular on two point processes: the Poisson process and the sine-kernel process.

1.2.1. **Poisson point process.** The Poisson process for us is in fact a family of point processes indexed by a parameter \(\lambda > 0\) called intensity. The Poisson process of intensity \(\lambda\) may be characterized in the following way [12, Ex. 2.5]: it is the unique point process \(w\) with correlation measures defined by \(d\rho_n(x) = \lambda^n d^n x\):

\[
E \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(w_{j_1}, \ldots, w_{j_n}) = \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) \cdot \lambda^n \, dx_1 \cdots dx_n,
\]

for all \(n \geq 1\) and all \(\phi \in C_r(\mathbb{R}^n)\). From this and (5) it is easy to see for any \(\lambda\) that the Poisson point process of intensity \(\lambda\) has uniform moments.
1.2.2. Sine-kernel process. The sine-kernel process is a shorter name for the determinantal point process with sine-kernel. The sine-kernel process may be characterized in the following way (see [11, Ch. 4]): it is the unique point process \( z \) with correlation measures

\[
E \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(z_{j_1}, \ldots, z_{j_n}) = \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) \cdot \det_{n \times n} [S(x_i - x_j)] \, dx_1 \cdots dx_n,
\]

for all \( n \geq 1 \) and all \( \phi \in C_c(\mathbb{R}^n) \). Here \( \det_{n \times n} [\cdot] \) denotes an \( n \times n \) determinant, and

\[
S(x) = \begin{cases} 
\sin \frac{\pi x}{x} & x \neq 0 \\
1 & x = 0.
\end{cases}
\]

Furthermore, by convention, the right hand side of (8) for \( n = 1 \) has the meaning \( \int_{\mathbb{R}} \phi(x_1) \, dx_1 \). From this correlation measure and (5) it is easy to see that the sine-kernel process also has uniform moments.

1.3. Statement of the problem. We make the following definition.

**Definition 1.4.** Let \( u \) and \( v \) be u.m. point processes in \( \mathbb{R} \), and let \( B > 0 \). Suppose that for each \( n \geq 1 \) and all \( \eta \in \mathcal{S}(\mathbb{R}^n) \) whose Fourier transform \( \hat{\eta} \) is supported in \( [-B, B]^n \), we have

\[
E \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(u_{j_1}, \ldots, u_{j_n}) = E \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(v_{j_1}, \ldots, v_{j_n}).
\]

Then we say that \( v \) mimics \( u \) at the bandwidth \( [-B, B] \), or just \( v \) mimics \( u \) at the bandwidth \( B \).

For Fourier transforms we use the convention that

\[
\hat{\eta}(\xi) = \int \eta(x) e(-x \cdot \xi) \, dx,
\]

where \( e(y) = e^{i2\pi y} \).

The mimicry relation is an equivalence relation: it is reflexive, symmetric and transitive. The symmetry property is if \( u \) mimics \( v \) at bandwidth \( B \), then \( v \) mimics \( u \) at bandwidth \( B \).

We have been motivated to consider this definition by an application to number theory described later.

A point process is said to be supported on \( a\mathbb{Z} \) if all configurations lie in \( a\mathbb{Z} \). We ask the following question in general:

**Question.** For a given u.m. point process \( u \) in \( \mathbb{R} \), for what values \( a \) and \( B \) does there exist a u.m. point process \( u^* \) supported in \( a\mathbb{Z} \) such that \( u^* \) mimics \( u \) at the bandwidth \( B \)?

Note that we do not require the point processes we consider to be simple. (A point process is said to be simple if for any configuration \( u, u_i \neq u_j \) for \( i \neq j \).)

The problem of whether or not a given collection of measures \( \rho_n^* \) are realized as the correlation measures of some point process is referred to as the realizability of point processes. It is a subtle matter. The problem was first studied in an abstract context by Lenard [21] and has been more recently been the subject of considerable work [19, 18, 4].
1.4. **Sampling and interpolation.** There is a certain resemblance between this problem and the classical problems of sampling and interpolating a signal. The sine kernel plays a special role in these problems.

Indeed, the sampling theorem (see [8, Thm. 5.6.9]) tells us that for a function \( \eta \in S(\mathbb{R}^n) \) with \( \text{supp} \, \hat{\eta} \subset (1/2a, 1/2a)^n \) can be reconstructed (by interpolation) from its sample values on the lattice \( a\mathbb{Z} \), by the Whittaker-Shannon interpolation formula

\[
\eta(x) = \sum_{k \in (a\mathbb{Z})^n} \eta(k) \prod_{i=1}^{n} S\left(\frac{x_i - k_i}{a}\right),
\]

where \( S(x) \) is defined in (9).

Hence under mild conditions on a point process’s \( n \)-point correlation measures \( \rho_n \), we should expect to be able to reconstruct the integral of a function \( \eta \in S(\mathbb{R}^n) \) having \( \text{supp} \, \hat{\eta} \subset [1/2a, 1/2a]^n \), against the correlation function from samples on the lattice \( (a\mathbb{Z})^n \) by

\[
\int_{\mathbb{R}^n} \eta(x) d\rho_n(x) = \sum_{k \in (a\mathbb{Z})^n} \eta(k) \int_{\mathbb{R}^n} \prod_{i=1}^{n} S\left(\frac{x_i - k_i}{a}\right) d\rho_n(x)
\]

\[
= \int_{\mathbb{R}^n} \eta(x) d\rho'_n(x), \tag{10}
\]

where \( \rho'_n(x) \) is the atomic measure supported on the lattice \( (a\mathbb{Z})^n \) with

\[
\rho'_n(\{k\}) = \int_{\mathbb{R}^n} \prod_{i=1}^{n} S\left(\frac{x_i - k_i}{a}\right) d\rho_n(x),
\]

for \( k \in (a\mathbb{Z})^n \). Under mild conditions on \( \rho_n \) such an integral will converge in a natural sense. Nonetheless the existence of a measure \( \rho'_n \) supported on \( (a\mathbb{Z})^n \) satisfying (10) is only a necessary condition that there exist a point process having such correlation measures.

The problem of whether or not a given collection of measures \( \rho^*_n \) are realized as the correlation measures of some point process is referred to as the realizability of point processes. It is a subtle matter. The problem was first studied in an abstract context by Lenard [21] and has been more recently been the subject of considerable work [19, 18, 4].

As we will see, in some cases the measures defined this way by (10) are sometimes realized as correlation measures, but sometimes they are not. The bandwidth \( \frac{1}{2a} \) for the lattice \( a\mathbb{Z} \) nonetheless retains a certain importance, and we use the convention that the bandwidth \( B = \frac{1}{2a} \) is called the **Nyquist bandwidth**, following a naming convention in sampling theory.

1.5. **Main results for point processes.** For the Poisson process and the sine-kernel process, we give a complete characterization of those values \( a \) for which they can be mimicked at the Nyquist bandwidth \( [-\frac{1}{2a}, \frac{1}{2a}] \) by a point process supported on \( a\mathbb{Z} \).

**Theorem 1.5** (Poisson process mimicry - Nyquist bandwidth). Let \( \lambda \) be arbitrary. For each \( a > 0 \), the Poisson process with finite intensity \( \lambda \) can be mimicked at the Nyquist bandwidth \( [-\frac{1}{2a}, \frac{1}{2a}] \) by a u.m. point process supported on \( a\mathbb{Z} \).
Theorem 1.6 (Sine-kernel process mimicry - Nyquist bandwidth). The sine-kernel process be mimicked by a u.m. point process supported on aZ at the Nyquist bandwidth $[-\frac{1}{2a}, \frac{1}{2a}]$ if and only if $0 < a \leq \frac{1}{2}$.

For each lattice spacing $a$, one may ask more generally about the full range of bandwidths $B$ for which the point process can be mimicked. For the Poisson process we have a complete characterization.

Theorem 1.7 (Poisson process mimicry - general bandwidth). Let $\lambda > 0$ be arbitrary. We have,

(i) For all $a > 0$, if $B \leq \frac{1}{a}$, then the Poisson process with intensity $\lambda$ can be mimicked at bandwidth $[-B, B]$ by a u.m. point process supported on aZ.

(ii) For all $a > 0$, if $B > \frac{1}{a}$, then the Poisson process with intensity $\lambda$ cannot be mimicked at Bandwidth $[-B, B]$ by a u.m. point process supported on aZ.

For the sine-kernel process we have a partial answer for the full bandwidth.

Theorem 1.8 (Sine-kernel process mimicry - general bandwidth). We have,

(i) For all $0 < a \leq \frac{1}{2}$, if $B \leq \frac{1-a}{a}$, then the sine-kernel process can be mimicked at bandwidth $[-B, B]$ by a u.m. point process supported on aZ.

(ii) For all $0 < a \leq \frac{1}{2}$, if $B > \frac{1-a}{a}$, then the sine-kernel process cannot be mimicked at bandwidth $[-B, B]$ by a u.m. point process supported on aZ.

(iii) If $a > \frac{1}{2}$ and $B \geq \frac{1}{2a}$, then the sine-kernel process cannot be mimicked at bandwidth $[-B, B]$ by a u.m. point process supported on aZ.

Figure 1. A plot of the regions $(a, B)$ for which the Poisson process and sine-kernel process can be mimicked at bandwidth $B$ by a point process with uniform moments supported on aZ. In the green region these point processes can be mimicked, while in the red region they cannot. In the white region of the second plot we currently have no information.
The regions of $a, B$ spelled out by these theorems are plotted in Figure 1. It would be very interesting to better understand those $a, B$ not described by Theorem 1.8, left white in Figure 1.

The reader should take a moment to verify that Theorems 1.7 and 1.8 imply Theorems 1.5 and 1.6.

1.6. **An application to the Alternative Hypothesis.** The questions treated in this paper were motivated by a problem originating in number theory regarding zeros of the Riemann zeta function. We treat this problem more fully in a companion paper [20], and give a brief description here. Recently Tao has independently treated much the same questions (using slightly different methods) in a blog post [35].

We assume the Riemann Hypothesis holds and let $\{1/2 + i\gamma_k\}_{k \in \mathbb{Z}}$ be the non-trivial zeros of the Riemann zeta function. We define the rescaled zeta zero ordiates

$$\tilde{\gamma}_k := \frac{1}{\pi} \gamma_k \log \gamma_k.$$  

The $\tilde{\gamma}_k$ have have on average a spacing of 1 between consecutive values. The *Alternative Hypothesis* refers to the (seemingly outlandish) supposition that the spacings $\tilde{\gamma}_{k+1} - \tilde{\gamma}_k$ always lie approximately in the set $1/2 \mathbb{Z}$.

The Alternative Hypothesis is of special interest because of known connections between spacings of zeros of the zeta function and the existence of Landau-Siegel zeros. The Alternative Hypothesis is expected to be false, and indeed it is contradicted by the well-known GUE Hypothesis, that the spacing between zeros of the zeta function follow a distribution coming from random matrix theory. On the other hand, the GUE Hypothesis remains a conjecture, even on the Riemann Hypothesis, and it is natural to ask whether the Alternative Hypothesis can be ruled out just by *what is known about the statistical distribution of zeros of the zeta function*. By this we mean the known information about $n$-level correlation functions of zeros that was first proved by Rudnick and Sarnak [30] for all $n \geq 1$. Rudnick and Sarnak characterized the correlation functions of zeros against certain band-limited test functions; their result amounts to knowing just a bit less than the assertion that the renormalized zeros mimic the sine-kernel process at a bandwidth $B = 1$.

In the companion paper [20], using ideas related to those in this paper, we show that the Alternative Hypothesis cannot be ruled out by what is known about the statistical distribution of zeros of the zeta function. This is done via the construction of a counterexample collection of points with statistics mimicking what is known about the zeros of the zeta function, but with renormalized distance between points always a multiple of 1/2. Very recently, T. Tao in [35] independently proved this fact, using slightly different techniques.

The present paper treats a more general question in the context of point processes, with the purpose of getting a deeper understanding. Theorems 1.6 and 1.8 in particular show that the bandwidth $B = 1$ and lattice spacing $1/2 \mathbb{Z}$ lie at something of a boundary for the sine-kernel process – mimicry at any larger bandwidth $B > 1$ entails a more stringent restriction on the lattices on which the point process could be supported.
2. On the Nyquist Bandwidth

In this section we prove that past the Nyquist bandwidth the correlation measures of any mimicking point process are uniquely determined. This result does not address the existence of an underlying point process. The ideas involved are more or less standard ones from sampling theory.

In what follows for $0 < \varepsilon < 1/2$, we let $\beta_\varepsilon(\xi)$ be an even bump function with the following four properties:

\begin{align}
0 \leq \beta_\varepsilon(\xi) &\leq 1, \quad \text{for all } \xi \in \mathbb{R}, \\
\beta_\varepsilon(\xi) & = 1, \quad \text{for } |\xi| \leq 1/2 - \varepsilon, \\
\beta_\varepsilon(\xi) & = 0, \quad \text{for } |\xi| \geq 1/2 + \varepsilon, \\
\beta_\varepsilon(\frac{1}{2} + x) & = 1 - \beta_\varepsilon(\frac{1}{2} - x), \quad \text{for all } 0 \leq x < 1/2.
\end{align}

A drawing will convince the reader that such a function exists.

Lemma 2.1. Let $u$ be a u.m. point process. If $B > 1/2a$ and $u$ can be mimicked at bandwidth $B$ by a u.m. point process $u'$ supported on $a\mathbb{Z}$, then the correlation measures $\rho_n'$ of $u'$ are uniquely determined and satisfy

\[
\rho_n'(\{k\}) = \int_{\mathbb{R}^n} \prod_{i=1}^n \beta_\varepsilon\left(\frac{x_i - k_i}{a}\right) d\rho_n(x) \quad \text{for all } k \in (a\mathbb{Z})^n,
\]

for all sufficiently small $\varepsilon$.

Remark 2.2. For $\varepsilon > 0$, the function $\hat{\beta}_\varepsilon$ is a Schwartz function and the integral above converges due to the uniform moments of the point process $u$.

Proof. We begin by showing that for $x \in (a\mathbb{Z})^n$,

\[
\prod_{i=1}^n \beta_\varepsilon\left(\frac{x_i - k_i}{a}\right) = \mathbf{1}_k(x).
\] (15)

For note that

\[
\prod_{i=1}^n \beta_\varepsilon\left(\frac{x_i - k_i}{a}\right) = \prod_{i=1}^n \int_{\mathbb{R}} \beta_\varepsilon(\xi) e(\xi(x_i - k_i)/a) d\xi.
\] (16)

As $(x_i - k_i)/a \in \mathbb{Z}$, note that $\xi \mapsto \xi(x_i - k_i)/a$ has period 1 and so using the properties (12), (13), and (14),

\[
\int_{\mathbb{R}} \beta_\varepsilon(\xi) e(\xi(x_i - k_i)/a) d\xi = \int_{-1/2}^{1/2} [\beta_\varepsilon(\xi) + \beta_\varepsilon(-1 + \xi) + \beta_\varepsilon(1 + \xi)] e(\xi(x_i - k_i)/a) d\xi
\]

\[
= \int_{-1/2}^{1/2} 1 \cdot e(\xi(x_i - k_i)/a) d\xi
\]

\[
= \mathbf{1}_{k_\varepsilon}(x_i).
\]

Applying this in (16) yields (15).

Let

\[
\eta(x) = \prod_{i=1}^n \beta_\varepsilon\left(\frac{x_i - k_i}{a}\right).
\] (17)

We have

\[
\hat{\eta}(\xi) = a^n e(-k \cdot \xi) \prod_{i=1}^n \beta_\varepsilon(a\xi_i).
\]
If $B > 1/2a$, then for sufficiently small $\varepsilon$ we have $\text{supp } \hat{\eta} \subset [-B, B]^n$.

If $u$ is mimicked at bandwidth $B$ by $u'$ supported on $a\mathbb{Z}$ as in the Lemma, then

$$\rho_n'({\{k\}}) = \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(u'_{j_1}, \ldots, u'_{j_n}) = \int_{\mathbb{R}^n} \eta(x) d\rho_n(x),$$

with the first equality above following from (15) and the support of $u'$ and the second equality from band-limited mimicry. From the definition (17) of $\eta$ this establishes the lemma.

3. Mimicry of the Poisson process

3.1. The discrete Poisson process. In this section we prove Theorem 1.7 describing when the Poisson process can be mimicked.

It ends up that when the process can be mimicked, it is mimicked just by the discrete Poisson process.

Definition 3.1. The discrete Poisson process on $a\mathbb{Z}$ of intensity $\lambda$ is the point process $w^*$ such that for each $k \in a\mathbb{Z}$, the number of points at each site $\#_k(w^*)$ are independent and identically distributed random variables, with each variable a Poisson random variable with mean $a\lambda$.

The correlation functions of this process are as follows.

Proposition 3.2. Letting $w^*$ be the discrete Poisson process on $a\mathbb{Z}$ of intensity $\lambda$, we have for all $n \geq 1$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$E \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(w^*_{j_1}, \ldots, w^*_{j_n}) = \sum_{k \in (a\mathbb{Z})^n} (a\lambda)^n \phi(k).$$

Proof. This follows from the independence of the random variables $\#_k(w^*)$ for different $k$, and the fact that the factorial moments of Poisson random variables satisfy

$$E \#_k(w^*) (\#_k(w^*) - 1) \cdots (\#_k(w^*) - (m - 1)) = (a\lambda)^m.$$

Details are left to the reader. □

3.2. Mimicry for $B \leq \frac{1}{a}$. We show that the Poisson process can be mimicked by the discrete Poisson process; this is the first part of Theorem 1.7. The proof depends on the Poisson summation formula, which we recall for the reader:

Theorem 3.3 (Poisson summation formula). For all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$a^n \sum_{k \in (a\mathbb{Z})^n} \phi(k) = \sum_{j \in (a^{-1}\mathbb{Z})^n} \hat{\phi}(j).$$

Proof. The usual formulation of Poisson summation is this for $a = 1$ (see [8, Theorem 3.1.17]): $\sum_{k \in \mathbb{Z}} \phi(k) = \sum_{j \in \mathbb{Z}} \hat{\phi}(j)$. Replacing $\phi(x)$ with $a^n \phi(ax)$ yields the result for general $a$. □

Proof of Theorem 1.7 part (i). We show that for $B \leq 1/a$, the Poisson process with intensity $\lambda$ is mimicked at bandwidth $[-B, B]$ by the discrete Poisson process on $a\mathbb{Z}$ with intensity $\lambda$. Using Proposition 3.2 for the discrete Poisson process and
(7) for the Poisson process, this is just a matter of showing that for \( \eta \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp} \hat{\eta} \subset [-B, B]^n \),

\[
\int_{\mathbb{R}^n} \eta(x) \lambda^n \, dx = \sum_{k \in (a\mathbb{Z})^n} (a\lambda)^n \eta(k). \tag{18}
\]

But by Poisson summation we indeed have

\[
(a\lambda)^n \sum_{k \in (a\mathbb{Z})^n} \eta(k) = \lambda^n \sum_{j \in (a^{-1}\mathbb{Z})^n} \hat{\eta}(j) = \lambda^n \hat{\eta}(0),
\]

by the support of \( \hat{\eta} \), and this is just (18). □

3.3. No mimicry for \( B > 1/a \). We now show the other half of Theorem 1.7. We suppose the Poisson process can be mimicked on \( a\mathbb{Z} \) for bandwidth \( B > 1/a \) and obtain a contradiction. Our main tool is Lemma 2.1 and we will obtain a contradiction even for 1-level correlation measures.

Proof of Theorem 1.7, part (ii). Let \( w \) be the Poisson process on \( \mathbb{R} \) with intensity \( \lambda \), and suppose there exists a u.m. point process \( w' \) supported on \( a\mathbb{Z} \) which mimics \( w \) at bandwidth \( B > 1/a \). Since then we certainly have \( B > 1/2a \), Lemma 2.1 (for \( n = 1 \)) implies for any \( k \in a\mathbb{Z} \),

\[
\rho'_1(k) = \int_{\mathbb{R}} \hat{\beta}_x \left( \frac{x-k}{a} \right) \lambda \, dx = a\lambda \hat{\beta}_x(0) = a\lambda,
\]

where we have used \( \int \hat{f}(x) \, dx = f(0) \) for arbitrary \( f \in \mathcal{S}(\mathbb{R}) \). Thus for any \( \eta \in \mathcal{S}(\mathbb{R}) \),

\[
\mathbb{E} \sum_j \eta(w'_j) = \sum_{k \in a\mathbb{Z}} a\lambda \eta(k) = \lambda \sum_{\ell \in a^{-1}\mathbb{Z}} \hat{\eta}(\ell). \tag{19}
\]

Yet if \( w' \) mimics the Poisson process at bandwidth \( B \), for supp \( \hat{\eta} \subset [-B, B] \), we have

\[
\mathbb{E} \sum_j \eta(w'_j) = \lambda \int_{\mathbb{R}} \eta(x) \, dx = \lambda \hat{\eta}(0).
\]

This contradicts (19) if \( \eta \) is chosen such that \( \hat{\eta}(\xi) \geq 0 \) for all \( \xi \) and \( \hat{\eta}(\frac{1}{a}) \neq 0 \). □

4. Mimicry of the sine-kernel process

4.1. The discrete sine-kernel process. In this section we prove Theorem 1.8. A key tool will be the discrete sine-kernel process.

Theorem 4.1. For each \( 0 < a \leq 1 \), there exists a unique point process \( z^* \) on \( a\mathbb{Z} \) such that for all \( n \geq 1 \) and all \( \phi \in \mathcal{S}(\mathbb{R}^n) \),

\[
\mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(z^*_j, \ldots, z^*_n) = \sum_{k \in (a\mathbb{Z})^n} a^n \det_{n \times n} [S(k_i - k_j)] \phi(k).
\]

Moreover \( z^* \) has uniform moments.

Definition 4.2. The point process \( z^* \) described by Theorem 4.1 is called the discrete sine-kernel process on \( a\mathbb{Z} \).
The discrete sine-kernel process is not new; in various guises it has appeared in [2, 12, 36, 37] and a proof of its existence follows the same ideas as for the (continuous) sine-kernel process, coming from the theory of determinantal point processes. The details of this proof however do not seem to be in the literature, and we include a proof of Theorem 4.1 in the appendix of a companion paper [20].

4.2. Mimicry for \( B \leq \frac{1-a}{a} \). We show that the sine-kernel process can be mimicked by the discrete sine-kernel process; this is the first part of Theorem 1.8. As in the previous section, our proof depends on Poisson summation.

Proof of Theorem 1.8 part (i). We show for \( B \leq \frac{1-a}{a} = 1/a - 1 \), the sine-kernel process is mimicked by the discrete sine-kernel process on \( a\mathbb{Z} \). By Theorem 4.1 and [8] this is just a matter of showing that for \( \eta \in \mathcal{S}(\mathbb{R}^n) \) with supp \( \tilde{\eta} \subset [-B, B]^n \),

\[
\int_{\mathbb{R}^n} \eta(x) \det \left[ S(x_i - x_j) \right] d^n x = a^n \sum_{k \in (a\mathbb{Z})^n} \eta(k) \det \left[ S(k_i - k_j) \right].
\]

(20)

Let \( g(x) = \eta(x) \det_{n \times n}[S(x_i - x_j)] \). Then (20) is just the claim that

\[
\int_{\mathbb{R}^n} g(x) d^n x = a^n \sum_{k \in (a\mathbb{Z})^n} g(k),
\]

and as the left hand side is \( \hat{g}(0) \), this identity will be verified by Poisson summation if we show \( \hat{g}(y) = 0 \) whenever \( y \notin (-1/a, 1/a)^n \).

For notational reasons we let \( E = [-1/2, 1/2] \). One has the well-known computation

\[
S(x) = \int_{\mathbb{R}} 1_E(\xi) d\xi
\]

so, where \( S_n \) is the symmetric group,

\[
\det_{n \times n}[S(x_i - x_j)] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n S(x_i - x_j)
\]

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int_{E^n} e \left( \sum_{i=1}^n (\xi_i - x_{\sigma(i)}) \right) d^n \xi
\]

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int_{E^n} e \left( \sum_{i=1}^n (x_i - \xi_{\sigma^{-1}(i)}) \right) d^n \xi.
\]

Hence for \( y \in \mathbb{R}^n \),

\[
\hat{g}(y) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int_{E^n} \int_{\mathbb{R}^n} e(-x \cdot y) e \left( \sum_{i=1}^n (x_i - \xi_{\sigma^{-1}(i)}) \right) \eta(x) d^n x d^n \xi
\]

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int_{E^n} \hat{\eta}(y_1 - (\xi_1 - \xi_{\sigma^{-1}(1)}), \ldots, y_n - (\xi_n - \xi_{\sigma^{-1}(n)})) d^n \xi.
\]

(21)

But for \( y \notin (-1/a, 1/a)^n \), we must have \( |y_i| \geq 1/a \) for some \( i \), and hence for \( \xi \in E^n \), we have \( |y_i - (\xi_1 - \xi_{\sigma^{-1}(1)})| \geq 1/a - 1 \). If \( \hat{\eta} \) is supported in \([-B, B]^n \) with \( B \leq 1/a - 1 \), we therefore have that the integrand in (21) vanishes for all \( y \notin (-1/a, 1/a)^n \), and \( \hat{g}(y) = 0 \) as we wanted. This therefore verifies (20) and proves the claim. \( \square \)
4.3. **No mimicry for** $a \leq 1/2$ and $B > \frac{1-a}{a}$. By an argument analogous to that of section 3.3 we prove part (ii) of Theorem 1.8. For $a \leq 1/2$, our strategy will be to suppose the sine-kernel process can be mimicked for bandwidth $B > 1/a - 1$ and obtain a contradiction. Our main tool, as before, is Lemma 2.1 but now we use 2-level correlations.

**Proof of Theorem 1.8, part (ii).** Let $a \leq 1/2$ and let $z$ be the sine-kernel process. Suppose there exists a u.m. point process $z'$ supported on $a\mathbb{Z}$ which mimics $z$ at bandwidth $B > \frac{1-a}{a} = 1/a - 1$; we will obtain a contradiction.

For $a \leq 1/2$, this implies $B \geq 1/2a$ and so Lemma 2.1 applies. Thus for any $k \in (a\mathbb{Z})^2$ and all sufficiently small $\varepsilon > 0,$

$$
\rho'_2(k) = \int_{\mathbb{R}^2} \hat{\beta}_\varepsilon\left(\frac{x_1 - k_1}{a}\right) \hat{\beta}_\varepsilon\left(\frac{x_2 - k_2}{a}\right) (1 - S(x_1 - x_2)^2) \, dx_1 dx_2
$$

$$
= \int_{\mathbb{R}^2} \beta_\varepsilon(\xi_1) \beta_\varepsilon(\xi_2) e\left(-\frac{k_1 \xi_1 + k_2 \xi_2}{a}\right) \left[\delta\left(\frac{\xi_1}{a}\right) \delta\left(\frac{\xi_2}{a}\right) - \delta\left(\frac{\xi_1 + \xi_2}{a}\right) (1 - |\frac{\xi_1}{a}|_+ + |\frac{\xi_2}{a}|_+)ight] \, d\xi_1 d\xi_2
$$

$$
= a^2 \left[1 - \int_{\mathbb{R}} \beta_\varepsilon(\nu)^2 e((k_1 - k_2)\nu)(1 - |\nu|_+) \, d\nu\right],
$$

where the computation in the second line uses the Fourier pair $f(x) = S(x)^2,$ $\hat{f}(\xi) = (1 - |\xi|)_+,$ and the computation in the third line uses of the fact that $\beta_\varepsilon$ is even to simplify the resulting expression. As this is true for all sufficiently small $\varepsilon$, we can take the limit as $\varepsilon \to 0$, and see that

$$
\rho'_2(k) = a^2 \left[1 - \int_{-1/2}^{1/2} e((k_1 - k_2)\nu)(1 - |\nu|_+) \, d\nu\right] = a^2 (1 - S(k_1 - k_2)^2),
$$

with the last identity following because $(1 - |\nu|_+)_{+}$ is supported in $[-1/2a, 1/2a]$ for $a \leq 1/2$.

Hence for any $\eta \in \mathcal{S}(\mathbb{R})$, we must have for the point process $z'$,

$$
\mathbb{E} \sum_{j_1, j_2 \text{ distinct}} \eta(z'_{j_1}, z'_{j_2}) = a^2 \sum_{k \in (a\mathbb{Z})^2} \eta(k)(1 - S(k_1 - k_2)^2). \quad (22)
$$

Yet if $z'$ mimics the sine-kernel process at bandwidth $B$ for supp $\hat{\eta} \subset [-B, B]^2$,

$$
\mathbb{E} \sum_{j_1, j_2 \text{ distinct}} \eta(z'_{j_1}, z'_{j_2}) = \int_{\mathbb{R}^2} \eta(x)(1 - S(x_1 - x_2)^2) \, dx_1 dx_2. \quad (23)
$$

Let $g(x) = \eta(x)(1 - S(x_1 - x_2)^2)$, so that as a consequence of (21) for $n = 2$,

$$
\hat{g}(y_1, y_2) = \hat{\eta}(y_1, y_2) - \int_{\mathbb{R}} \hat{\eta}(y_1 - \xi, y_2 + \xi)(1 - |\xi|)_{+} \, d\xi. \quad (24)
$$

By Poisson summation the expression on the right hand side of (22) is

$$
\sum_{j \in (a^{-1}\mathbb{Z})^2} \hat{g}(j),
$$

while the expression in (23) is

$$
\hat{g}(0).
$$
These expressions are not equal if $\eta$ is chosen such that $\hat{\eta}(\xi) \geq 0$ for all $\xi$ and $\hat{\eta}$ is supported in a sufficiently small neighborhood of the point $(1/a - 1, -(1/a - 1))$ with $\hat{\eta}(1/a - 1, -(1/a - 1)) \neq 0$, since in this case
\begin{equation}
\sum_{j \in (a^{-1}Z)^2} \hat{g}(j) = \hat{g}(0) - \int_{\mathbb{R}} \hat{\eta}(1/a - \xi, -1/a + \xi)(1 - |\xi|) d\xi \tag{25}
\end{equation}
due to (24) and the facts that $\hat{\eta}(j) = 0$ for any $j \in (a^{-1}Z)^2$ and $\hat{\eta}(j_1 - \xi, j_2 + \xi) = 0$ for all $\xi \in (-1, 1)$ if $j \in (a^{-1}Z)^2$ unless $j = (1/a, -1/a)$ (or possibly $j = 0$ if $a = 1/2$). But then (25) is not equal to $\hat{g}(0)$ since $\hat{\eta}(1/a - 1, -(1/a - 1)) \neq 0$.

This shows that (22) cannot equal (23), a contradiction. □

4.4. No mimicry for $a > \frac{1}{2}$ and $B \geq \frac{1}{2}a$. Finally we prove part (ii) of Theorem 1.8. This proof is rather more involved than the other proofs in this paper, and we break it into three steps:

(i) in step 1, we show that band-limited mimicry can be extended to a slightly more general class of test-functions $\eta$ than Schwartz-class;
(ii) in step 2 we develop some computations involving the sine-determinant;
(iii) in step 3 we suppose the sine-kernel process can be mimicked for the relevant $a$ and $B$ and obtain a contradiction.

**Step 1**: We extend the class of test functions outside the Schwartz class.

**Lemma 4.3.** If $u$ and $v$ are u.m. point processes and $u$ mimics $v$ at bandwidth $[-B, B]$, then for all $n \geq 1$ if $\eta \in C(\mathbb{R}^n)$ is a function that can be written as
$$\eta(x_1, ..., x_n) = h(x_1) \cdot ... \cdot h(x_n)$$
with
(i) $\hat{h}(\xi) = \int_{-\infty}^{\infty} \sigma(t) dt$ where $\sigma$ is of bounded variation, and
(ii) $\sigma$ and $\hat{h}$ are supported in $[-B, B]$
then we have
$$\mathbb{E} \sum_{j_1, ..., j_n \text{ distinct}} \eta(u_{j_1}, ..., u_{j_n}) = \mathbb{E} \sum_{j_1, ..., j_n \text{ distinct}} \eta(v_{j_1}, ..., v_{j_n}).$$

The proof of Lemma 4.3 naturally follows the proof of Theorem A.3 in Appendix A, and we give the proof there.

The point of Lemma 4.3 is that $\eta$ is just slightly out of the Schwartz class, but expectations of these statistics can still be taken.

**Step 2**: This step consists of a series of technical computations to be used in the last step. The reader may want to skim these computations on first read and then come back to check them more carefully once they are called upon.

We fixed $a > 0$ and let $\ell$ be an odd multiple of $a$. Define the functions
$$h_{a, \ell}(x) = S\left(\frac{x}{a}\right) + S\left(\frac{x - \ell}{a}\right),$$
and
$$H_{a, \ell}(x_1, ..., x_n) = h_{a, \ell}(x_1) \cdot ... \cdot h_{a, \ell}(x_n).$$
The reader should verify $h_{a, \ell}(x) = O(\frac{1}{1+x^2})$, so that $H_{a, \ell}(x_1, \ldots, x_n) = O(\frac{1}{1+x_1^2} \cdots \frac{1}{1+x_n^2})$, with implicit constants depending on $a, \ell, n$. Furthermore we define

$$\Phi_n(a) = \lim_{\ell \to \infty, \text{odd}} \int_{\mathbb{R}^n} H_{a, \ell}(x_1, \ldots, x_n) \det [S(x_i - x_j)] d^n x.$$ 

(The limit is over odd multiples of $a$, as $\ell \to \infty$.) Because of the decay of $H_{a, \ell}$ this integral is well-defined, though it is not yet obvious that the limit here exists.

**Lemma 4.4.** The limit defining $\Phi_n(a)$ exists for all $n \geq 1$ and $a > 0$, and

$$\Phi_1(a) = 2a$$

$$\Phi_2(a) = \begin{cases} 2a^2, & \text{if } a \in (0, 1/2] \\ 1/2 - 2a + 4a^2, & \text{if } a \in (1/2, \infty), \end{cases}$$

$$\Phi_3(a) = \begin{cases} 0, & \text{if } a \in (0, 1/2] \\ (2a - 1)^3, & \text{if } a \in (1/2, \infty) \end{cases}$$

$$\Phi_4(a) = \begin{cases} 0, & \text{if } a \in (0, 1/2] \\ (a - 1/2)^2(1 - 20a + 12a^2), & \text{if } a \in (1/2, 1] \\ 17/4 - 22a + 48a^2 - 48a^3 + 16a^4, & \text{if } a \in (1, \infty). \end{cases}$$

The proof of this Lemma really is just a computation. In the proof below we give one reasonably efficient method for this computation, but the reader may find it preferable simply to compute in their own way.

**Proof.** In the first place, note

$$h_{a, \ell}(\xi) = a \cdot (1 + e(-\ell \xi)) I_a(\xi),$$

(26)

where for notational reasons we write $I_a(\xi) = 1_{[-1/2a, 1/2a]}(\xi)$. Fix $n$ and $a$, and for $x \in \mathbb{R}^n$, let

$$g_\ell(x) = H_{a, \ell}(x) \det [S(x_i - x_j)].$$

Using (21), and recalling the notational convention $E = [-1/2, 1/2]$, we see

$$\int_{\mathbb{R}^n} H_{a, \ell}(x) \det [S(x_i - x_j)] d^n x$$

$$= \mathcal{g}_\ell(0) = \sum_{\sigma \in S_n} \sgn(\sigma) \int_{E^n} a^n \prod_{j=1}^n \left(1 + e(\ell(\xi_j - \xi_{\sigma^{-1}(j)}))\right) I_a(\xi_j - \xi_{\sigma^{-1}(j)}) d^n \xi.$$ 

(27)

We will take the limit of this expression as $\ell \to \infty$. By multiplying cross terms of (27), using the Riemann-Lebesgue Lemma to eliminate any terms in which an exponential remains, we see the limit as $\ell \to \infty$ exists and

$$\Phi_n(a) = \sum_{\sigma \in S_n} \sgn(\sigma) N(\sigma) a^n \int_{E^n} \prod_{j=1}^n I_a(\xi_j - \xi_{\sigma^{-1}(j)}) d^n \xi,$$

(28)

where

$$N(\sigma) = \# \{ T \subseteq \{1, \ldots, n\} : \sigma(T) = T \} = 2^\omega(\sigma),$$

for $\omega(\sigma)$ the number of cycles of $\sigma$, and

$$\Phi_n(a) = \lim_{\ell \to \infty, \text{odd}} \int_{\mathbb{R}^n} H_{a, \ell}(x_1, \ldots, x_n) \det [S(x_i - x_j)] d^n x.$$ 

(The limit is over odd multiples of $a$, as $\ell \to \infty$.) Because of the decay of $H_{a, \ell}$ this integral is well-defined, though it is not yet obvious that the limit here exists.
with \( \omega(\sigma) \) the number of cycles in the permutation \( \sigma \). From this point the easiest way to deduce the remainder of the Lemma is simply to compute, noting that the integral in (25) breaks into separate parts for each cycle of \( \sigma \).

The following computations are helpful to have at hand, where for \( \nu \geq 2 \) we define

\[
f_\nu(r) = \int_{E^n} \mathbf{1}_{[-r,r]}(\xi_1 - \xi_2) \cdots \mathbf{1}_{[-r,r]}(\xi_{n-1} - \xi_n) \mathbf{1}_{[-r,r]}(\xi_n - \xi_1) \, d^n \xi.
\]

One can verify

\[
\begin{align*}
f_2(r) &= \begin{cases} 2r - r^2, & \text{if } r \in (0, 1) \\ 1, & \text{if } r \in [1, \infty) \end{cases} \\
f_3(r) &= \begin{cases} 3r^2 - 2r^3, & \text{if } r \in (0, 1) \\ 1, & \text{if } r \in [1, \infty) \end{cases} \\
f_4(r) &= \begin{cases} (16r^3 - 14r^4)/3, & \text{if } r \in (0, 1/2) \\ (1 - 8r + 24r^2 - 16r^3 + 2r^4)/3, & \text{if } r \in [1/2, 1) \\ 1, & \text{if } r \in [1, \infty). \end{cases}
\end{align*}
\]

(A computer algebra system is helpful here.) Painstakingly\(^2\) inserting these into (28) yields the computations of \( \Phi_1, \ldots, \Phi_4 \) that have been claimed.

\( \square \)

**Step 3:** We can now complete the last part of the proof of Theorem 1.8.

**Proof of Theorem 1.8, part (iii).** Take \( a > 1/2 \). We now suppose that the sine-kernel process \( z \) can be mimicked at a bandwidth \( B \geq 1/2a \) by a u.m. point process \( z' \) supported on \( a\mathbb{Z} \), and we will obtain a contradiction. For \( \ell \) always an odd multiple of \( a \), consider the random variable

\[
X_\ell = \sum_j h_{a,\ell}(z'_j) = \#(0,\ell)(z'),
\]

with the second identity dependent on the assumption that \( z' \) is supported on \( a\mathbb{Z} \).

\( X_\ell \) is thus an **integer-valued** random variable, and so clearly

\[
\mathbb{E} X_\ell(X_\ell - 1)(X_\ell - 2)(X_\ell - 3) \geq 0,
\]

and furthermore by a consequence of the Hamburger moment criterion (see e.g. [34, Theorem 1.2]),

\[
D_\ell = \det \begin{pmatrix} m_0^\ell & m_1^\ell & m_2^\ell \\ m_1^\ell & m_2^\ell & m_3^\ell \\ m_2^\ell & m_3^\ell & m_4^\ell \end{pmatrix} \geq 0,
\]

where \( m_\nu^\ell = \mathbb{E} X_\ell^\nu \) are the moments of \( X_\ell \).

Yet we claim that at least one of (31) or (32) will not be the case for sufficiently large \( \ell \), for any choice of \( a > 1/2 \).

\( \text{footnote} \)

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\( \text{footnote} \)

The reader familiar with cycle index polynomials can make the computation slightly more efficient by noting that if \( Z(S_n; a_1, \ldots, a_n) \) is the cycle index polynomial of \( S_n \) in the variables \( a_1, \ldots, a_n \), the formula \( \Phi_n(a) = (-1)^n n! a^n Z(S_n; -2f_1(1/2a), \ldots, -2f_n(1/2a)) \) simplifies to

\[ \Phi_n(a) = (-1)^n n! a^n Z(S_n; -2f_1(1/2a), \ldots, -2f_n(1/2a)) \]

where we adopt the convention \( f_1(r) = 1 \) for all \( r \).
For consider first \( \alpha \in (1/2, 1] \). Note that from (29), (4) we have
\[
\mathbb{E} X_\ell(X_\ell - 1)(X_\ell - 2)(X_\ell - 3) = \mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} H_{a,\ell}(z'_{j_1}, z'_{j_2}, z'_{j_3}, z'_{j_4}).
\]
(33)

The computation (26) reveals \( \hat{h}_{a,\ell}(x) = \int_{-\infty}^{x} -a e^{-\ell t} I_a(t) \, dt \), with the integrand of bounded variation and supported in \([-1/2a, 1/2a] \subset [-B, B]\), so (4.3) may be applied; if \( z' \) mimics \( z \), then (33) is equal to
\[
\int_{\mathbb{R}^4} H_{a,\ell}(x) \det \begin{bmatrix} S(x_i - x_j) \end{bmatrix} \, d^4x.
\]
Taking the limit of this expression as \( \ell \to \infty \) along odd multiples of \( a \), Lemma 4.4 yields
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell(X_\ell - 1)(X_\ell - 2)(X_\ell - 3) = \Phi_4(a) = (a - 1/2)^2(1 - 20a + 12a^2).
\]
For \( a \in (1/2, 1] \), it can be checked that this number is strictly negative, but this contradicts (31).

Now consider \( \alpha > 1 \). As above we have
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell(X_\ell - 1) \cdots (X_\ell - (n - 1)) = \Phi_n(a),
\]
and from this, using Lemma 4.4 one may extract
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell = 2a, \quad \text{(for } a > 0)\]
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell^2 = \frac{1}{2} + 4a^2, \quad \text{(for } a > 1/2)\]
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell^3 = \frac{1}{2} + 2a + 8a^3, \quad \text{(for } a > 1/2)\]
\[
\lim_{\ell \to \infty, \text{odd}} \mathbb{E} X_\ell^4 = \frac{7}{4} + 2a + 4a^2 + 16a^4, \quad \text{(for } a > 1)\]
and further, using the notation in (32), one may compute
\[
\lim_{\ell \to \infty, \text{odd}} D_\ell = \frac{1}{2} - a^2, \quad \text{(for } a > 1)\]
(A computer algebra system is helpful here.) But this is strictly negative for any choice of \( a \in (1, \infty) \), and this contradicts (32).

Thus we have obtained a contradiction for all \( a > 1/2 \), so in this range such a u.m. point process \( z' \) does not exist.

\[\square\]

5. Further Questions

We draw attention to a few questions that naturally present themselves:

**Question 5.1.** In the white region of Figure 7 can the sine-kernel process be mimicked or not? That is, what happens for those \( a, B \) not described by Theorem 1.8?

**Question 5.2.** What are the analogues of Theorems 1.7 and 1.8 for point processes other than the Poisson and sine-kernel? Is there a general and effective criteria encompassing both of these results?
Question 5.3. What restrictions does band-limited mimicry entail for point processes not necessarily supported on a lattice. For instance, let $\mathcal{T}_1$ be the class of all u.m. point processes $u$ which mimic the sine-kernel process at a bandwidth $B = 1$, and let

$$
\mu = \sup \{ m : \text{there exists } u \in \mathcal{T}_1 \text{ such that almost surely } |u_i - u_j| \geq m \text{ for all } i \neq j \}.
$$

Theorem 1.8 shows that $\mu \geq 1/2$. The method of proof in [5], which makes use only of pair correlation, should be able to be straightforwardly modified to show that $\mu \leq .606894$. It may be that $\mu = 1/2$.

Likewise let

$$
\lambda = \inf \{ \ell : \text{there exists } u \in \mathcal{T}_1 \text{ such that almost surely } |u_{j+1} - u_j| \leq \ell \text{ for all } j \in \mathbb{Z} \}.
$$

What is the value of $\lambda$? Is it finite?

It may be that a reinterpretation of methods from number theory (see e.g. [7]) can yield further upper bounds for $\mu$ and lower bounds for $\lambda$. Questions about both $\mu$ and $\lambda$ are closely connected to classical questions about gaps between zeros of the Riemann zeta function.

Question 5.4. Although Definition 1.4 of bandlimited mimicry is a natural one for our purposes, it is also interesting to investigate questions of the sort we have considered where instead of supposing a point process $u$ mimics a point process $v$ at a bandwidth $B$, one supposes that

$$
E \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(u_{j_1}, \ldots, u_{j_n}) = E \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(v_{j_1}, \ldots, v_{j_n}),
$$

for some other class of test functions $\eta$, for instance those with Fourier transform supported in some region of $\mathbb{R}^n$ other than $[-B, B]^n$. Are there nice analogues of Theorems 1.7 and 1.8 in this more general context? Such questions mirror those arising in number theory.

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Appendix A. Some general results on correlation measures

In this appendix we collect some results regarding the correlation functions of point processes which we have made reference to earlier, with standard proofs, but which we could not find easily in the literature.

A.1. Existence of correlation measures. The following result essentially is [21 Prop 3.2]. We include the simple proof here for completeness.

**Theorem A.1.** If $u$ is a point process on $\mathbb{R}$ such that for any compact set $K$ the random variable $\#_K(u)$ has finite moments of all orders, then for all $n \geq 1$ there exists a unique measure $\rho_n$ on $\mathbb{R}^n$ such that

$$
E \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(u_{j_1}, \ldots, u_{j_n}) = \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) d\rho_n(x_1, \ldots, x_n),
$$

for all $\phi \in C_c(\mathbb{R}^n)$. 

Remark A.2. A point process with uniform moments will satisfy the hypothesis of this theorem.

Proof. The fact that \( \#_K(u) \) has finite \( n \)-th moment for any compact \( K \) implies that for \( \phi \in C_c(\mathbb{R}^n) \), the random variables \( \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(u_{j_1}, \ldots, u_{j_n}) \) are integrable, and thus the mapping \( \Lambda \) defined by

\[
\Lambda \phi = \mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(u_{j_1}, \ldots, u_{j_n}),
\]

is a positive linear functional on \( C_c(\mathbb{R}^n) \). The Riesz representation theorem \cite{29, Ch. 2} thus implies the existence of the measure \( \rho_n \).

A.2. Bootstrapping test functions from \( C_c(\mathbb{R}^n) \) to \( S(\mathbb{R}^n) \). We show that for u.m. point processes the correlation measures make sense with respect to not only \( C_c(\mathbb{R}^n) \) test functions, but also Schwartz class test functions. Actually we show a bit more.

Theorem A.3. If \( u \) is a u.m. point process on \( \mathbb{R} \) and \( \rho_n \) is the measure on \( \mathbb{R}^n \) defined by (34) for \( \phi \in C_c(\mathbb{R}^n) \), then (34) also holds for all \( n \geq 1 \) and all \( \phi \in C(\mathbb{R}^n) \) such that

\[
\phi(x_1, \ldots, x_n) = O\left(\frac{1}{(1 + x_1^2) \cdots (1 + x_n^2)}\right).
\]

Remark A.4. Hence in particular for a point process with uniform moments, (34) holds for all \( \phi \in S(\mathbb{R}^n) \), for all \( n \geq 1 \).

Proof. Let

\[
Q(x_1, \ldots, x_n) = \frac{1}{1 + x_1^2} \cdots \frac{1}{1 + x_n^2}.
\]

We first establish for the point process \( u \) that

\[
\mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} Q(u_{j_1}, \ldots, u_{j_n}) < +\infty.
\]

(36)

For

\[
\mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} Q(u_{j_1}, \ldots, u_{j_n}) \leq \mathbb{E} \sum_{L \in \mathcal{Z}^n} Q(L_1, \ldots, L_n) \#_{|L_1, L_1+1|}(u) \cdots \#_{|L_n, L_n+1|}(u)
\]

\[
\leq \sum_{L \in \mathcal{Z}^n} Q(L_1, \ldots, L_n) \prod_{i=1}^n (\mathbb{E} \#_{|L_i, L_i+1|}(u))^{1/n}
\]

\[
\leq C_n \sum_{L \in \mathcal{Z}^n} Q(L_1, \ldots, L_n) < +\infty,
\]

using Fatou’s lemma and Hölder’s inequality in the second line.

For the same reasons, we have

\[
\int_{\mathbb{R}^n} Q(x_1, \ldots, x_n) d\rho_n(x_1, \ldots, x_n) < +\infty.
\]

(37)

Note also that (36) implies that almost surely

\[
\sum_{j_1, \ldots, j_n \text{ distinct}} Q(u_{j_1}, \ldots, u_{j_n}) \text{ converges}.
\]

(38)
Let \( \beta \in C_c(\mathbb{R}^n) \) be a bump function takes the value 1 in some neighborhood of \( 0 \in \mathbb{R}^n \) and which satisfies \( 0 \leq \beta(x) \leq 1 \) for all \( x \in \mathbb{R}^n \). For \( R > 0 \) define \( \phi_R(x) = \phi(x)\beta(x/R) \), and note for all \( x \in \mathbb{R}^n \),

\[
\lim_{R \to \infty} \phi_R(x) = \phi(x).
\]

Moreover \( \phi_R \in C_c(\mathbb{R}^n) \) for all \( R \), and by assumption there is a constant \( C > 0 \) such that

\[
|\phi_R(x_1, \ldots, x_n)| \leq C Q(x_1, \ldots, x_n)
\]

for all \( x \in \mathbb{R}^n \).

Now from (38), it is easy to see that almost surely

\[
\lim_{R \to \infty} \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_R(u_{j_1}, \ldots, u_{j_n}) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(u_{j_1}, \ldots, u_{j_n}).
\]

Hence using (36), (37) and dominated convergence,

\[
\mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \phi(u_{j_1}, \ldots, u_{j_n}) = \lim_{R \to \infty} \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_R(u_{j_1}, \ldots, u_{j_n})
\]

\[
= \lim_{R \to \infty} \int_{\mathbb{R}^n} \phi_R(x_1, \ldots, x_n) \, d\rho_n(x_1, \ldots, x_n)
\]

\[
= \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) \, d\rho_n(x_1, \ldots, x_n),
\]

as claimed. \( \square \)

### A.3. Bootstrapping band-limited test functions.

We prove Lemma 4.3. The proof will involve similar ideas to that of Theorem A.3. We require two lemmas from analysis first.

**Lemma A.5.** Suppose \( s(\xi) = \int_{-\infty}^{\xi} \sigma(t) \, dt \) where \( \sigma \) and \( s \) are integrable and \( \sigma \) is of bounded variation. Then

\[
\hat{s}(x) = O(\min(\|s\|_{L^1(\mathbb{R})}, \text{var}(\sigma)/x^2)).
\]

**Proof.** This is a combination of two standard results. The bound \( \hat{s}(x) \leq \|s\|_{L^1} \) is obvious, and the bound \( \text{var}(\sigma)/x^2 \) comes from integrating by parts twice in computing the Fourier transform:

\[
|\hat{s}(x)| = \left| \int_{-\infty}^{\infty} e^{-i2\pi x \xi} \sigma(\xi) \, d\sigma(\xi) \right| \leq \frac{1}{4\pi^2 x^2} \int_{-\infty}^{\infty} |\sigma(\xi)|.
\]

Combining these bounds proves the lemma. \( \square \)

**Lemma A.6.** If \( \sigma(t) \) is supported on the interval \([A, B]\) and of bounded variation, then for any \( \epsilon > 0 \) there exists a Schwartz function \( \tilde{\sigma}(t) \) supported on \([A, B]\) such that

\[
\text{var}(\tilde{\sigma}) \leq \text{var}(\sigma),
\]

and

\[
\|\tilde{\sigma} - \sigma\|_{L^1(\mathbb{R})}.\]
Proof. As $\sigma$ is of bounded variation, the Jordan decomposition (see [28, Sec. 5.2]) tells us there exists monotonic nondecreasing functions $\sigma_+$ and $\sigma_-$ such that $\sigma = \sigma_+ - \sigma_-$ and $\sigma_+$ and $\sigma_-$ are constant for $t \notin [A, B]$, that is

$$\sigma_+(t) = \sigma_+(A), \quad \text{for all } t \leq A,$$

$$\sigma_-(t) = \sigma_-(B), \quad \text{for all } t \geq B,$$

and moreover $\text{var}(\sigma) = \text{var}(\sigma_+) + \text{var}(\sigma_-)$. It is a straightforward exercise to construct monotonically nondecreasing Schwartz functions $\tilde{\sigma}_+$ and $\tilde{\sigma}_-$ such that for either $+$ or $-$,

$$\|\tilde{\sigma}_+ - \sigma_+\|_{L^1} < \epsilon/2,$$

$$\tilde{\sigma}_+(t) = \sigma_+(A), \quad \text{for all } t \leq A,$$

$$\tilde{\sigma}_-(t) = \sigma_-(B), \quad \text{for all } t \geq B.$$

Note $\text{var}(\tilde{\sigma}_+) = \text{var}(\sigma_+) = \sigma_+(B) - \sigma_-(A)$ and $\text{var}(\tilde{\sigma}_-) = \text{var}(\sigma_-) = \sigma_-(B) - \sigma_-(A)$, so if $\tilde{\sigma} = \tilde{\sigma}_+ - \tilde{\sigma}_-$,

$$\text{var}(\tilde{\sigma}) \leq \text{var}(\tilde{\sigma}_+) + \text{var}(\tilde{\sigma}_-) = \text{var}(\sigma_+) + \text{var}(\sigma_-) = \text{var}(\sigma),$$

verifying the first claim of the lemma. Moreover from the triangle inequality,

$$\|\tilde{\sigma} - \sigma\|_{L^1} \leq \|\tilde{\sigma}_+ - \sigma_+\|_{L^1} + \|\tilde{\sigma}_- - \sigma_-\|_{L^1} < \epsilon,$$

verifying the second.

We finally turn to Lemma 4.3.

Proof of Lemma 4.3. The proof follows that of Theorem A.3. We show for all $R \geq 1$ there exists $\eta_R \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\lim_{R \to \infty} \eta_R(x) = \eta(x), \quad \text{for all } x \in \mathbb{R}^n,$$

$$\text{supp } \hat{\eta_R} \subset [-B, B]^n \quad \text{for all } R \geq 1,$$

$$\eta_R(x) = O(Q(x)), \quad \text{for all } x \in \mathbb{R}^n, \ R \geq 1,$$

where $Q$ is the quadratically decaying function defined in (35). Then exactly by the argument in the proof of Theorem A.3 we have

$$\mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \eta_\ell(u_{j_1}, \ldots, u_{j_n}) = \lim_{R \to \infty} \mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \eta_R(u_{j_1}, \ldots, u_{j_n})$$

$$= \lim_{R \to \infty} \mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \eta_R(v_{j_1}, \ldots, v_{j_n})$$

$$= \mathbb{E} \sum_{j_1, \ldots, j_n \text{ distinct}} \eta(v_{j_1}, \ldots, v_{j_n}).$$

The functions $\eta_R$ are constructed in the following way. For $\sigma$ as in statement of the theorem, let $\tilde{\sigma}_R$ be a function described by Lemma A.6 such that $\text{supp } \tilde{\sigma}_R \subset [-B, B], \text{var}(\tilde{\sigma}_R) \leq \text{var}(\sigma)$ and $\|\tilde{\sigma}_R - \sigma\|_{L^1} \leq 1/R$. Define $h_R$ by

$$\hat{h}_R(\xi) = \int_{-\infty}^{\xi} \tilde{\sigma}_R(t) \, dt,$$

and note that

$$\text{supp } \hat{h}_R \subset [-B, B], \quad (42)$$
and for all $\xi$, $\hat{h}_R(\xi) - \hat{h}(\xi) \leq 1/R$ so that from the support of both functions $\hat{h}, \hat{h}_R$,\[ h_R(x) - h(x) \leq \|\hat{h}_R - \hat{h}\|_{L^1} \leq 2B/R. \]Finally from Lemma A.5 we have\[ h_R(x) = O\left(\min(\|\hat{h}_R\|_{L^1}, \text{var}(\hat{\sigma}_R))/x^2\right) \]
\[ = O\left(\min(2B\|\hat{\sigma}_R\|_{L^1}, \text{var}(\sigma)/x^2)\right) \]
\[ = O\left(\frac{1}{1 + x^2}\right). \]Letting $\eta_R(x_1, \ldots, x_n) = h_1(x_1) \cdots h_n(x_n)$, we see that (39), (40), (41) are satisfied, using (43), (42), (44) respectively. This completes the proof. \[ \square \]

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