The strong Szegő theorem in random matrix theory and number theory

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Historical background: Toeplitz determinants

\[ T_n := \begin{pmatrix} c_0 & c_1 & c_2 & \ldots & \ldots & c_{n-1} \\ c_{-1} & c_0 & c_1 & \ddots & & \vdots \\ c_{-2} & c_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_1 & c_2 \\ \vdots & \ddots & c_{-1} & c_0 & c_1 & c_2 \\ c_{-(n-1)} & \ldots & \ldots & c_{-2} & c_{-1} & c_0 \end{pmatrix} \]

What is \( \text{det}(T_n) \), for large \( n \)?
Historical background: Toeplitz determinants

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What is \( \det(T_n) \), for large \( n \)?
For $w : \mathbb{R}/\mathbb{Z} \to \mathbb{R}_+$, and $\hat{w}_k := \int_0^1 w(\vartheta)e^{-i2\pi \vartheta} d\vartheta$.

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\end{pmatrix}$$

What is $\det(T_n)$, for large $n$?

Has applications to: Orthogonal polynomials, Polynomial approximation, Linear prediction theory, Random matrix theory, etc.
Szegő’s Limit Theorems, for Toeplitz determinants

Theorem (Szegő, 1915)

For $w > 0$ a.e and $w \in L^1(\mathbb{R}/\mathbb{Z}, d\vartheta)$,

$$(\det T_n)^{1/n} \sim \exp \left( \int_0^1 \log w(\vartheta) \, d\vartheta \right).$$
### Theorem (Szegő, 1915)

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\left( \det T_n \right)^{1/n} \sim \exp \left( \int_0^1 \log w(\vartheta) \, d\vartheta \right).
\]

### Theorem (strong Szegő theorem, 1952)

For \( \inf w > 0 \) and \( w \in C^{1+\alpha}(\mathbb{R}/\mathbb{Z}) \),

\[
\det T_n \sim \exp \left( n \int_0^1 \log w(\vartheta) \, d\vartheta + \sum_{k=1}^{\infty} k \left| (\log w)^{\hat{k}} \right|^2 \right).
\]

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**strong Szegő: random matrices and number theory**
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\]

**Theorem (Ibragimov, 1968)**

The strong Szegő theorem is true so long as \( w > 0 \) a.e., \( w \in L^1(d\vartheta) \), and \( \log w \in L^1(d\vartheta) \), with

\[
\sum_{k=-\infty}^{\infty} |k| \left| (\log w)^\wedge_k \right|^2 < +\infty.
\]
$U(M) := \{ g \in \mathbb{C}_{M \times M} : gg^* = I \} \text{ is the group of } M \times M \text{ unitary matrices, endowed with Haar probability measure.}$
$U(M) := \{g \in \mathbb{C}_{M \times M} : gg^* = I\}$ is the group of $M \times M$ unitary matrices, endowed with Haar probability measure.

$\omega_1 = e^{i2\pi \theta_1}, \omega_2 = e^{i2\pi \theta_2}, \ldots, \omega_M = e^{i2\pi \theta_M}$ are the eigenvalues of $g$, with $\theta_i \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. 
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Random matrix theory: what is the distribution of $\{\omega_1, \ldots, \omega_M\}$, for large $M$?
A primer on $U(M)$

Random unitary matrix

Independent uniform points

Figure: The eigenvalues of a randomly generated $25 \times 25$ unitary matrix

Figure: 25 points independently placed on the unit circle with uniform distribution
Recall $\omega_i = e^{i2\pi \theta_i}$ with $\theta_i \in [-\frac{1}{2}, \frac{1}{2})$.

Stretch the eigenangles of an element of $U(M)$ out to have mean unit spacing:

$$\xi_i := M\theta_i.$$  

Microscopic distribution of eigenvalues (stated informally):

- For fixed $x$,

$$P(\text{one } \xi_i \in [x, x + dx]) \sim dx \quad \text{as } M \to \infty.$$
Microscopic statistics of eigenvalues

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- For fixed $x_1$ and $x_2$,
  $$\mathbb{P}\left(\text{one } \xi_i \in [x_1, x_1 + dx_1], \text{ one } \xi_j \in [x_2, x_2 + dx_2]\right) \sim \left(1 - \left(\frac{\sin \pi (x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2\right) dx_1 dx_2.$$  

small when $x_1 \approx x_2$
Recall $\omega_i = e^{i2\pi \theta_i}$ with $\theta_i \in [-\frac{1}{2}, \frac{1}{2})$.

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- For fixed $x_1, x_2, x_3$,
  
  $$\mathbb{P}(\text{one } \xi_i \in [x_1, x_1 + dx_1], \text{ one } \xi_j \in [x_2, x_2 + dx_2], \text{ one } \xi_k \in [x_3, x_3 + dx_3]) \sim \det_{2\times2} K(x_i - x_j) dx_1 dx_2 dx_3.$$ 

- And so on, with $K(x) := \sin(\pi x)/\pi x$...
Theorem (Weyl - Gaudin - Dyson)

For fixed \( \eta \in C_c(\mathbb{R}^k) \),

\[
\mathbb{E}_{\mathcal{U}(M)} \sum_{i_1, \ldots, i_k \text{ distinct}} \eta(\xi_{i_1}, \ldots, \xi_{i_k}) \\
= \int \eta(x_1, \ldots, x_k) \det_{k \times k} \left[ \frac{1}{M} K_M(\frac{x_i - x_j}{M}) \right] \, d^k x \quad \text{for } K_M(x) := \frac{\sin(\pi M x)}{\sin(\pi x)} \\
\sim \int_{\mathbb{R}^k} \eta(x_1, \ldots, x_k) \det_{k \times k} \left[ K(x_i - x_j) \right] \, d^k x \quad \text{for } K(x) := \frac{\sin(\pi x)}{\pi x}.
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\sim \int_{\mathbb{R}^k} \eta(x_1, \ldots, x_k) \det_{k \times k} \left[ K(x_i - x_j) \right] d^k x
$$

for $K_M(x) := \frac{\sin(\pi M x) \sin(\pi)}{\sin(\pi x)}$ and $K(x) := \frac{\sin(\pi x)}{\pi x}$.

Remark: The correlation functions give the same information as the moments of linear statistics:

$$
\mathbb{E}_{U(M)} \left( \sum_i \eta(\xi_i) \right)^k
$$
Theorem (Weyl - Gaudin - Dyson)

For fixed $\eta \in C_c(\mathbb{R}^k)$,

$$
\mathbb{E}_{U(M)} \sum_{\substack{i_1, \ldots, i_k \\ \text{distinct}}} \eta(\xi_{i_1}, \ldots, \xi_{i_k}) \\
= \int \eta(x_1, \ldots, x_k) \det_{k \times k} \left[ \frac{1}{M} K_M \left( \frac{x_i - x_j}{M} \right) \right] d^k x \\
\sim \int_{\mathbb{R}^k} \eta(x_1, \ldots, x_k) \det_{k \times k} \left[ K \left( x_i - x_j \right) \right] d^k x
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for $K_M(x) := \frac{\sin(\pi M x)}{\sin(\pi x)}$.

Remark: The correlation functions give the same information as the moments of linear statistics:

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Remark: This is a little more information than knowing the distribution of counts $\#_I(\xi_i)$, for fixed intervals $I$. 

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strong Szegő: random matrices and number theory
In probabilistic language: the point process

\[ \{\xi_i\} = \{M\theta_i\} \]

induced by stretching out the eigenangles of \( U(M) \) tends in
distribution to \( S \), the determinantal point process with
sine-kernel.
In probabilistic language: the point process

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induced by stretching out the eigenangles of \( U(M) \) tends in distribution to \( S \), the determinantal point process with sine-kernel.

This is a ‘microscopic’ statement: It has only to do with eigenvalues separated by a distance of \( O(1/M) \).
Theorem (Costin - Lebowitz)

Let \( n(M) \) be a function \( \rightarrow \infty \) as \( M \rightarrow \infty \), but so that \( n(M) = o(M) \). Let \( I_M = [-n(M)/2, n(M)/2] \). Consider the counting function

\[
\Delta_M = \# I_M(\{\xi_i\}).
\]

Then

\[
\mathbb{E}_{U(M)} \Delta_M = n(M)
\]

\[
\text{Var}_{U(M)} \Delta_M \sim \frac{1}{\pi^2} \log n(M)
\]

and in distribution

\[
\frac{\Delta_M - \mathbb{E} \Delta_M}{\sqrt{\text{Var} \Delta_M}} \Rightarrow N(0, 1)
\]
Mesoscopic collections of eigenvalues

**Theorem (Costin - Lebowitz)**

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Here $n(M) = o(M)$ is a natural boundary.
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Here $n(M) = o(M)$ is a natural boundary.
Theorem (The Strong Szegő Theorem)

If \( f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \) is bounded and satisfies

\[
\sum_{k=-\infty}^{\infty} |k| |\hat{f}_k|^2 < \infty
\]

then for

\[
\Delta f := \sum_{j=1}^{M} f\left(\frac{\xi_j}{M}\right) = \sum_{j=1}^{M} f(\theta_j)
\]

we have

\[
\mathbb{E}_{U(M)} \Delta f = M \int_0^1 f(\vartheta) \, d\vartheta
\]

and

\[
\Delta f - \mathbb{E} \Delta f \Rightarrow N(0, 1) \left( \sum_{k=-\infty}^{\infty} |k| |\hat{f}_k|^2 \right)^{1/2}
\]
Connection to Toeplitz determinants

### Proposition (Heine)

$$\det T_M(w) = \mathbb{E}_{U(M)} \prod_{i=1}^{M} w(\theta_i).$$

Heine & Szegő $\Rightarrow$: Setting $w = e^{\lambda f}$,

$$\mathbb{E}_{U(M)} e^{\lambda(f(\theta_1) + \cdots + f(\theta_M))} = \det T_M(e^{\lambda f})$$

$$\sim \exp \left( M\lambda \int_{0}^{1} f(\vartheta) \, d\vartheta + \lambda^2 \sum_{k=1}^{\infty} k \hat{f}_k \hat{f}_{-k} \right)$$
Connection to Toeplitz determinants

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\[ \sim \ exp \left( M \lambda \int_0^1 f(\vartheta) \, d\vartheta + \lambda^2 \sum_{k=1}^{\infty} k \hat{f}_k \hat{f}_{-k} \right) \]

⇒:

\[ \mathbb{E}_{U(M)} \exp \left( \lambda \left( \sum f(\theta_i) - M \int_0^1 f(\vartheta) \, d\vartheta \right) \right) \sim \ exp \left( \frac{\lambda^2}{2} \sum_{k=-\infty}^{\infty} |k| |\hat{f}_k|^2 \right). \]
Corollary (Diaconis - Shahshahani)

Consider a random matrix $g \in U(M)$. For fixed $\lambda \geq 1$,

$$\text{Tr}(g^\lambda) = \sum_{j=1}^{M} e^{i2\pi \lambda \theta_j},$$

satisfies as $M \to \infty$, for any measurable $B \subset \mathbb{C}$

$$\mathbb{P}(\text{Tr}(g^\lambda) \in B) \to \mathbb{P}(\sqrt{\lambda} N_{\mathbb{C}}(0, 1) \in B) = \int_B \frac{1}{\pi \lambda} e^{-|z|^2/\lambda} \, dz.$$

In fact, as $M \to \infty$, the variables $\text{Tr}(g^\lambda)$ as $M \to \infty$ are independent for distinct (but fixed) $\lambda$'s.

(e.g. $\text{Tr}(g)$ and $\text{Tr}(g^2)$ are asymptotically independent for large $M$.)
Macrosopic collections of eigenvalues

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One may proceed the opposite direction also, from this to the strong Szegő theorem:

$$\sum_j \left( f(\theta_j) - \int_0^1 f(\vartheta) \, d\vartheta \right) = \sum_{\lambda \in \mathbb{Z} \setminus 0} \hat{f}_\lambda \left( \sum_{j=1}^{M} e^{i2\pi \lambda \theta_j} \right).$$
Theorem (Diaconis - Shahshahani)

Let $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ be non-negative integers, and $Z_1, \ldots, Z_\ell$ i.i.d. copies of $N_{\mathbb{C}}(0, 1)$. Then as long as
\[ \sum_{j=1}^{\ell} ja_j + \sum_{j=1}^{\ell} jb_j \leq 2M, \]
we have
\[
\mathbb{E}_{U(M)} \left[ \prod_{j=1}^{\ell} \text{Tr}(g^j)^{a_j} \overline{\text{Tr}(g^j)^{b_j}} \right] = \mathbb{E} \left[ \prod_{j=1}^{\ell} (\sqrt{jZ_j})^{a_j} \overline{\sqrt{jZ_j})^{b_j}} \right]
\]
Theorem

(Spohn-Soshnikov) Let $\eta \in L^1(\mathbb{R})$ be real-valued and of bounded variation, and for $\xi_i = M\theta_i$ as before, define

$$\Delta_{\eta,M} = \sum_{j=1}^{M} \eta \left( \frac{\xi_j}{n(M)} \right).$$

Then

$$\mathbb{E}\Delta_{\eta,M} \sim n(M) \int_{-\infty}^{\infty} \eta(x) \, dx$$

$$\text{Var}\Delta_{\eta,M} \sim \int_{-n(T)}^{n(T)} |x| \hat{\eta}(x) \hat{\eta}(-x) \, dx$$

and in distribution

$$\frac{\Delta_{\eta,M} - \mathbb{E}\Delta_{\eta,M}}{\sqrt{\text{Var}\Delta_{\eta,M}}} \Rightarrow N(0, 1)$$

as $M \to \infty$. 
A primer on the Riemann zeta function

- Non-trivial zeros: those with real part in $(0, 1)$.

- First few:
  \[
  \frac{1}{2} + i14.13, \; \frac{1}{2} + i21.02, \; \frac{1}{2} + i25.01.
  \]

- We assume the Riemann Hypothesis in what follows: all nontrivial zeros have the form \( \frac{1}{2} + i\gamma \), for \( \gamma \in \mathbb{R} \).

**Figure**: \( \zeta(s) \). Hue is argument, brightness is modulus. Made with Jan Homann’s ComplexGraph Mathematica code.
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- We assume the Riemann Hypothesis in what follows: all nontrivial zeros have the form \( \frac{1}{2} + i\gamma \), for \( \gamma \in \mathbb{R} \).

- Around height \( T \), zeros have density roughly \( \log T / 2\pi \). More precisely:

\[
N(T) = \# \{ \gamma \in (0, T), \zeta(\frac{1}{2} + i\gamma) = 0 \} = \frac{T}{2\pi} \log(\frac{T}{2\pi}) - \frac{T}{2\pi} + O(\log T)
\]

Figure: \( \zeta(s) \). Hue is argument, brightness is modulus. Made with Jan Homann's ComplexGraph Mathematica code.

Theorem (Riemann - von Mangoldt)
What is the distribution of $\gamma$ near a large random $s$?

Consider collections of $\gamma$ near a random $s \in [T, 2T]$, stretched out to have mean unit density:

$$\left\{ \frac{\log T}{2\pi} (\gamma - s) \right\} \quad s \in [T, 2T]$$
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1-level density:

$\mathbb{P}\left( \text{one } \frac{\log T}{2\pi} (\gamma - s) \in [x, x + dx] \right) \sim dx$. 

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What is the distribution of \( \gamma \) near a large random \( s \)?

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\]

1-level density:

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P\left( \text{one } \frac{\log T}{2\pi} (\gamma - s) \in [x, x + dx] \right) \sim dx.
\]

2-level density (Montgomery’s pair correlation conjecture):

\[
P\left( \text{one } \frac{\log T}{2\pi} (\gamma_1 - s) \in [x_1, x_1 + dx_1], \text{ one } \frac{\log T}{2\pi} (\gamma_2 - s) \in [x_2, x_2 + dx_2] \right) \\
\sim \left( 1 - \left( \frac{\sin \pi(x_1-x_2)}{\pi(x_1-x_2)} \right)^2 \right) dx_1 dx_2.
\]
A histogram of the pair correlation conjecture

Figure: A histogram of $\frac{\log T}{2\pi} (\gamma - \gamma')$ for the first 10000 zeros, in intervals of size .05, compared to the appropriately scaled prediction $1 - (\frac{\sin \pi x}{\pi x})^2$. 
Conjecture (GUE)

For fixed \( k \) and fixed \( \eta \) (Schwartz, say)

\[
\frac{1}{T} \int_{T}^{2T} \sum_{\gamma_1, \ldots, \gamma_k \text{ distinct}} \eta\left( \frac{\log T}{2\pi} (\gamma_1 - s), \ldots, \frac{\log T}{2\pi} (\gamma_k - s) \right) ds \sim \int_{\mathbb{R}^k} \eta(x) \det (K(x_i - x_j)) d^k x
\]
Conjecture (GUE)

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$$\frac{1}{T} \int_T^{2T} \sum_{\gamma_1, \ldots, \gamma_k \text{ distinct}} \eta\left(\frac{\log T}{2\pi} (\gamma_1 - s), \ldots, \frac{\log T}{2\pi} (\gamma_k - s)\right) \, ds \sim \int_{\mathbb{R}^k} \eta(x) \det (K(x_i - x_j)) \, d^k x$$

Theorem (Mongtomery, Hejhal, Rudnick-Sarnak)

For fixed $k$ and $\eta$ with $\operatorname{supp} \hat{\eta} \subset \left(-\frac{2}{k}, \frac{2}{k}\right)$

$$\frac{1}{T} \int_T^{2T} \sum_{\gamma_1, \ldots, \gamma_k \text{ distinct}} \eta\left(\frac{\log T}{2\pi} (\gamma_1 - s), \ldots, \frac{\log T}{2\pi} (\gamma_k - s)\right) \, ds \sim \int_{\mathbb{R}^k} \eta(x) \det (K(x_i - x_j)) \, d^k x$$
Conjecture (GUE)

For fixed \( k \) and fixed \( \eta \) (Schwartz, say)

\[
\frac{1}{T} \int_T^{2T} \sum_{\gamma_1, \ldots, \gamma_k \text{ distinct}} \eta \left( \frac{\log T}{2\pi} (\gamma_1 - s), \ldots, \frac{\log T}{2\pi} (\gamma_k - s) \right) ds \sim \int_{\mathbb{R}^k} \eta(x) \det_{k \times k} (K(x_i - x_j)) \, d^k x
\]

Theorem (Mongtomery, Hejhal, Rudnick-Sarnak)

For fixed \( k \) and \( \eta \) with \( \text{supp} \hat{\eta} \in \{ y : |y_1| + \cdots + |y_k| < 2 \} \)

\[
\frac{1}{T} \int_T^{2T} \sum_{\gamma_1, \ldots, \gamma_k \text{ distinct}} \eta \left( \frac{\log T}{2\pi} (\gamma_1 - s), \ldots, \frac{\log T}{2\pi} (\gamma_k - s) \right) ds \sim \int_{\mathbb{R}^k} \eta(x) \det_{k \times k} (K(x_i - x_j)) \, d^k x
\]

Theorem (Hughes-Rudnick)

Fix \( k \) and \( \eta : \mathbb{R} \to \mathbb{R} \), with \( \text{supp} \hat{\eta} \subset (-2/k, 2/k) \),

\[
\frac{1}{T} \int_T^{2T} \left( \sum_{\gamma} \eta \left( \frac{\log T}{2\pi} (\gamma - s) \right) \right)^k ds \sim c_k \left( \int_{-2/k}^{2/k} \left| x \right| \left| \hat{\eta}(x) \right|^2 \, dx \right)^{k/2}
\]
In less formal language:

These results are microscopic: they concern only collections of zeros at height $T$, all separated by $O(1/\log T)$.

Moral: Provided we 'count' zeros with test functions that aren't too oscillatory (too narrowly concentrated, that is, by the uncertainty principle) at the microscopic level, we can rigorously determine their distribution. Said another way: We can see the distribution of the zeros even at a microscopic scale, but at a blurred resolution.
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Said another way: We can see the distribution of the zeros even at a microscopic scale, but at a blurred resolution.
Idea of proof: zeros to primes

Ex: $\Lambda(n) \approx 1 - \sum_{\gamma} n^{-1/2+i\gamma} + \text{lower order}$
Idea of proof: zeros to primes

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**Theorem**

*For nice \( g \)*

\[
\int_{\mathbb{R}} \hat{g} \left( \frac{\xi}{2\pi} \right) dS(\xi) = \int_{-\infty}^{\infty} [g(x) + g(-x)] e^{-x/2} d(e^x - \psi(e^x)),
\]

where

\[
dS(\xi) \approx \sum_{\gamma} \delta_{\gamma}(\xi) - \frac{\log \xi}{2\pi}.
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\]

After some analysis: for \( \text{supp } \hat{\eta} \subset [-2/k, 2/k] \) (and \( \eta \) even),

\[
\frac{1}{T} \int_{-T}^{T} \left( \sum_{\gamma} \eta\left(\frac{\log T}{2\pi} (\gamma - s)\right) - \int \eta(x) \, dx \right)^k \, ds
\]

\[
\approx \frac{1}{T} \int_{-T}^{T} \left( -\frac{1}{\log T} \sum_p \hat{\eta}\left(\frac{\log p}{\log T}\right) \log p \left(p^{it} + p^{-it}\right) \right)^k \, dt.
\]

Brad Rodgers

strong Szegő: random matrices and number theory
Idea of proof: primes

Theorem (Bohr)

If $\nu_1, ..., \nu_k$ (fixed) are linearly independent, and $a_1, ..., a_k$ (fixed) are integers,

\[
\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} \prod_{\ell=1}^k (e^{i\nu_\ell t})^{a_\ell} \, dt = \mathbb{E} \prod_{\ell=1}^k S_{a_\ell} = \delta_{a=0}.
\]
Idea of proof: primes

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If $\nu_1, \ldots, \nu_k$ (fixed) are linearly independent, and $a_1, \ldots, a_k$ (fixed) are integers,

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$$

**Example:** $\nu_1 = \log p_1, \ldots, \nu_k = \log p_k$, for distinct primes $p_\ell$. 

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Idea of proof: primes

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**Example:** $\nu_1 = \log p_1, \ldots, \nu_k = \log p_k$, for distinct primes $p_\ell$.

**Proposition**

As long as $|a_1| \log p_1 + \cdots + |a_k| \log p_k \leq 2 \log T$,

$$
\frac{1}{T} \int_T^{2T} \prod_{\ell=1}^k \left( e^{i \nu_\ell t} \right)^{a_\ell} \, dt \approx \mathbb{E} \prod_{\ell=1}^k S_{a_\ell} = \delta_{a=0}.
$$
Idea of proof

\[ \frac{1}{T} \int_{T}^{2T} \left( \sum_{\gamma} \eta \left( \frac{\log T}{2\pi} (\gamma - s) \right) - \int \eta(x) \, dx \right)^{k} \, ds \]

By prop.

\[ \approx \mathbb{E} \left( \frac{-1}{\log T} \sum_{p} \hat{\eta} \left( \frac{\log p}{\log T} \right) \frac{\log p}{\sqrt{p}} (S_{p} + \overline{S_{p}}) \right)^{k} \]
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CLT

\[
= c_k \left( \frac{1}{\log^2 T} \sum_p \hat{\eta} \left( \frac{\log p}{\log T} \right)^2 \frac{\log^2 p}{p} \right)^{k/2}
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\]

PNT

\[
\approx c_k \left(\int \hat{\eta}(x)^2 |x| \, dx \right)^{k/2}
\]
Idea of proof

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\[ \approx \mathbb{E} \left( \frac{-1}{\log T} \sum_p \hat{\eta} \left( \frac{\log p}{\log T} \right) \log p \left( S_p + S_p \right) \right)^k \]

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What about larger collections of zeros?
Theorem (Fujii)

Let \( n(T) \) be a function \( \to \infty \) as \( T \to \infty \) but so that \( n(T) = o(\log T) \), and let \( s \) be random and uniformly distributed on \( [T, 2T] \). Let \( J_T = [-n(T)/2, n(T)/2] \), and define

\[
\Delta_T = \# J_T \left( \left\{ \frac{\log T}{2\pi} (\gamma - s) \right\} \right).
\]

Then

\[
E \Delta_T = n(T) + o(1)
\]

\[
\text{Var} \Delta_T = E (\Delta_T - E \Delta_T)^2 \sim \frac{\pi}{2} \log n(T).
\]

And in distribution

\[
\Delta_T - E \Delta_T \sqrt{\text{Var} \Delta_T} \Rightarrow N(0,1)
\]

as \( T \to \infty \).

Heuristic conjecture of Berry (1989): The zeros look like eigenvalues not only microscopically, but also mesoscopically.
Theorem (Fujii)

Let $n(T)$ be a function $\to \infty$ as $T \to \infty$ but so that $n(T) = o(\log T)$, and let $s$ be random and uniformly distributed on $[T, 2T]$. Let $J_T = [-n(T)/2, n(T)/2]$, and define

$$\Delta_T = \#J_T\left(\{\log T\frac{T}{2\pi}(\gamma - s)\}\right).$$

Then

$$E\Delta_T = n(T) + o(1)$$

$$\text{Var}\Delta_T := E(\Delta - E\Delta)^2 \sim \frac{1}{\pi^2} \log n(T)$$

and in distribution

$$\frac{\Delta_T - E\Delta_T}{\sqrt{\text{Var}\Delta_T}} \Rightarrow N(0, 1)$$

as $T \to \infty$. 

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Heuristic conjecture of Berry (1989): The zeros look like eigenvalues not only microscopically, but also mesoscopically.
Theorem (R., Bourgade-Kuan)

Let \( n(T) \to \infty \), but \( n(T) = o(\log T) \). For a fixed \( \eta : \mathbb{R} \to \mathbb{R} \) define

\[
\Delta_{\eta, T} = \sum_{\gamma} \eta \left( \frac{\log T}{2\pi n(T)} (\gamma - s) \right),
\]

For all \( \eta \) with compact support and bounded variation when \( \int |x||\hat{\eta}(x)|^2 \, dx \) diverges, and nearly all such \( \eta \) when the integral converges, we have

\[
\mathbb{E} \Delta_{\eta, T} = n(T) \int_{\mathbb{R}} \eta(\xi) \, d\xi + o(1),
\]

\[
\text{Var} \Delta_{\eta, T} \sim \int_{-n(T)}^{n(T)} |x||\hat{\eta}(x)|^2 \, dx
\]

and in distribution

\[
\frac{\Delta_{\eta, T} - \mathbb{E} \Delta_{\eta, T}}{\sqrt{\text{Var} \Delta_{\eta, T}}} \Rightarrow N(0, 1)
\]

as \( T \to \infty \).
Before, for microscopic collections of zeros: We can say something about the counts as long as the counting functions are sufficiently smooth (at the microscopic level).
**Idea of proof**

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*For mesoscopic collections:* The same remains true.
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*For mesoscopic collections:* The same remains true. Any question that can be asked about zeta zeros, provided answering it does not require counting with functions that are “too oscillatory” in the microscopic regime, can be rigorous answered.
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In particular: for any function \( \eta \) (that isn’t pathological), as \( n(M) \to \infty \), the function

\[
x \mapsto \eta \left( \frac{x}{n(M)} \right)
\]

becomes sufficiently smooth.
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In particular: for any function $\eta$ (that isn’t pathological), as $n(M) \to \infty$, the function

$$x \mapsto \eta\left(\frac{x}{n(M)}\right)$$

becomes sufficiently smooth.

...Or can be closely approximated by such a function.
Thanks:

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