Tight Error Bounds for Space-Time Orthogonal Block Codes under Slow Rayleigh Flat Fading*

Firouz Behnamfar[†], *Fady Alajaji*[†‡] *and Tamás Linder*[†‡]

[†] Department of Mathematics and Statistics
[‡] Department of Electrical and Computer Engineering
Queen's University, Kingston, Ontario K7L 3N6, Canada.

E-mail: {firouz, fady, linder}@mast.queensu.ca

Abstract

The performance of space-time orthogonal block (STOB) codes over slow Rayleigh fading channels and maximum likelihood (ML) decoding is investigated. Two Bonferroni-type bounds (one upper bound and one lower bound) for the symbol error rate (SER) and bit error rate (BER) of the system are obtained. The bounds are expressed in terms of the pairwise error probabilities (PEPs) and the two-dimensional pairwise error probabilities (2-D PEPs) of the transmitted symbols. Furthermore, the bounds can be efficiently evaluated and they hold for arbitrary (non-standard) signaling schemes and mappings. Numerical results demonstrate that the bounds are very accurate in estimating the performance of STOB codes. In particular, the upper and lower bounds often coincide even at low channel signal-to-noise ratios, large constellation sizes, and large diversity orders.

Keywords: Wireless communications, slow Rayleigh fading, diversity, multiple antennas, space-time coding, pairwise error probability, bit and symbol error rates, maximum likelihood decoding.

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1 Introduction

The original papers on space-time trellis (STT) codes [15] and space-time orthogonal block codes [14] adopt the Chernoff upper bound to estimate the pairwise error probability (PEP) of codewords and to establish code design criteria. Although the Chernoff bound yields successful code constructions, it is quite loose even at high values of channel signal-to-noise ratio (CSNR). Furthermore, it is common practice to use the union bound to approximate the symbol error rate or bit error rate. However, the union bound is intrinsically loose, particularly at low CSNRs. Therefore, using the Chernoff bound together with the union bound may result in poor approximations to system performance (see, e.g., Section 4).

The main challenge in determining the codeword PEP of space-time codes under maximum likelihood detection is to average the Gaussian Q-function $Q(X)$, where X is a non-negative random variable. This problem has been mainly studied in [16, 13, 11]. In [16], a method to derive the exact PEP in terms of the residues of the moment generating function (MGF) is presented. The result is used to find an estimate of the BER of STT codes. In [13] a form of the MGF is derived which is simpler to evaluate for block fading channels, and the PEPs are found numerically. The study in [11] presents another exact form of the PEP, which can be evaluated in closed form when the codewords corresponding to a pair of trellis paths have orthogonal rows. Recently, Lu et al. derived exact closed-form expressions for the PEP of general space-time codes [7]. Their derivation of the STOB codeword PEP is based on averaging Craig's formula [5] using the results in [12].

With respect to the SER/BER analysis of space-time orthogonal block codes, one can use the closed-form conditional SER formulas in [12] and average them over the fading coefficients distribution to derive exact SER formulas for standard PSK and square 2^{2p}-ary QAM, where p is a positive integer; this approach is taken in [9]. However, the resulting SER formulas do not hold for arbitrary constellations such as star-QAM and it is not clear how to extend this approach to evaluate the BER, specially if Gray mapping is not used. For PSK signaling, tight bounds on the symbol PEP were found in [4]. Also, in [1], the symbol PEP was found in integral form for diversity reception systems, which can be easily tailored to STOB coding. The symbol PEPs can be used to derive an upper bound on the BER; however, as will be shown in the sequel, the resulting BER upper bound is very loose even at medium CSNRs.

In this work, we show how to use the symbol PEP results to derive bounds for the SER and BER of STOB codes via establishing two algorithmic Bonferroni-type upper and lower bounds. The bounds are general (upper/lower) bounds on the probability of a union of a finite number of events $P(\cup_{i=1}^{n} A_i)$ which are expressed in terms of the individual probabilities $P(A_i), i = 1, ..., n$ as well as the probabilities of intersections of pairs of events $P(A_i \cap A_j), i, j = 1, ..., n, i < j$ [6]. When specialized to the STOB codes setting, the overall error event of the system can be represented by such a finite union of events, where $P(A_i)$ is the PEP, and $P(A_i \cap A_j)$ is the two-dimensional (2-D) pairwise error probability which we determine in closed form. One important feature of the bounds is that they hold for arbitrary constellations and signal mappings since they do not depend on the geometry of the system at hand. Numerical results indicate that the bounds for STOB codes often *coincide* with the true error probabilities obtained via simulations even at low CSNRs. Furthermore, the computational complexity of these algorithmic bounds is very modest even for large constellations and larger number of antennas.

The rest of this letter is organized as follows. The channel model and a review of space-time orthogonal block codes are presented in Section 2. Section 3 briefly describes the algorithmic lower and upper bounds of [6] and derives the exact probability expressions needed for applying the bounds to STOB coding systems. Numerical and simulation results are given in Section 4. Section 5 concludes the paper.

2 Symbol Pairwise Error Probabilities of Space-Time Orthogonal Codes

The multi-antenna communication system considered here employs K transmit and L receive antennas. The input to the system is an independent and identically distributed (i.i.d.) uniform bit-stream. The complex baseband constellation points are denoted by $\{c_k\}_{k=1}^{2^p}$ where p is a positive integer. We assume that the signal constellation has an average energy of one. In order to have an average CSNR of γ_s at each receive antenna, the modulators weight the symbols by $\sqrt{\frac{\gamma_s}{K}}$ (see (1)). The channel is assumed to be Rayleigh flat fading, so that the complex path gain from transmit antenna i to receive antenna j, H_{ji}, has a zero-mean unit-variance complex Gaussian distribution, denoted by $\mathcal{CN}(0, 1)$, with i.i.d. real and imaginary parts. We assume that the receiver, but not the transmitter, has perfect knowledge of the path gains. Moreover, we assume that the channel is quasi-static, meaning that the path gains remain constant during the transmission of a codeword.

The additive noise at receiver j at symbol interval t, N_t^j, is assumed to be $\mathcal{CN}(0,1)$ distributed with i.i.d. real and imaginary parts. The signal at receive antenna j can be written as $R_t^j = \sqrt{\frac{\gamma_s}{K}} \sum_{i=1}^{K} H_{ji} s_t^i + N_t^j$, where $\sqrt{\frac{\gamma_s}{K}} s_t^i$ is the signal sent from antenna i, or in matrix form,

$$\boldsymbol{r}_t = \sqrt{\frac{\gamma_s}{K}} \boldsymbol{H} \mathbf{s}_t + \boldsymbol{n}_t, \tag{1}$$

where $\boldsymbol{r}_t = (R_t^1, ..., R_t^L)^T$, $\mathbf{s}_t = (s_t^1, ..., s_t^K)^T$, $\boldsymbol{n}_t = (N_t^1, ..., N_t^L)^T$, and ^{T} denotes transposition. $\boldsymbol{H}$ is the $L \times K$ path gains matrix with elements $\{H_{ji}\}$. We assume that the input, noise, and fading processes are all independent.

Let $\mathbf{c} = (c_1, ..., c_\tau)^T$ be a vector of τ consecutive constellation points and $\mathbf{S} = (\mathbf{s}_1, ..., \mathbf{s}_{T_w})$ be the space-time code corresponding to it, where T_w is the codeword length. In the case of STOB codes, we have $T_w = g\tau$, where g is the coding gain and $\mathbf{S}\mathbf{S}^\dagger = g\|\mathbf{c}\|^2 \boldsymbol{I}_K$, where $\boldsymbol{I}_K$ is the $K \times K$ identity matrix and † represents complex conjugate transposition. As an example, for the code $\mathcal{G}^3$ in [14], $T_w = 8, \tau = 4$, and $g = 2$, and for Alamouti's code [2], $g = 1$ and $T_w = \tau = 2$.

It can be verified that the PEP conditioned on the path gains is given by

$$P(c_i \to c_j | \boldsymbol{H}) = Q\left(\delta_{ij} \sqrt{Y}\right), \tag{2}$$

where $c_i \to c_j$ denotes the event that c_j has a larger metric than c_i when c_i is sent (c_i and c_j are a pair of symbols input to the STOB encoder), $\delta_{ij} = \sqrt{\frac{g\gamma_s}{2K}} |c_i - c_j|$, and

$$Y = \sum_{l=1}^{L} Y_l = \sum_{k=1}^{K} \sum_{l=1}^{L} |H_{lk}|^2 \tag{3}$$

is the sum of the squared magnitudes of all path gains.[1] Using the moment generating function of Gaussian random variables, it can be verified that the probability density function of Y is

$$f_Y(y) = \frac{1}{(n-1)!} y^{n-1} e^{-y}, \qquad y > 0, \tag{4}$$

where $n = KL$. As noticed in [3], the PEP of diversity reception systems with maximum ratio combining has a form similar to that of (2) with the exception that δ_{ij} is given by $\sqrt{\gamma_s/2}|c_i - c_j|$. Therefore, the

[1]We assume here that all path gains contribute equally in Y. This assumption is valid for most orthogonal codes used in the literature, including Alamouti's $\mathcal{G}^2$ code and the $\mathcal{G}^3$ and $\mathcal{G}^4$ codes of [14].

conditional PEP for those systems is the same as (2). MRC systems were previously analyzed in, for example, [8, 12], whose results can be used to obtain the expected value of (2) as

$$P(c_i \to c_j) = \frac{1}{2}\left(1 - \frac{\delta_{ij}}{\sqrt{\delta_{ij}^2 + 2}} \sum_{k=0}^{KL-1} \binom{2k}{k} \frac{1}{(2(\delta_{ij}^2 + 2))^k}\right). \tag{5}$$

3 Error Rate Bounds for Space-Time Orthogonal Block Codes

In this section, we show how the algorithms in [6] for two tight Bonferroni-type upper and lower bounds can be used for space-time orthogonal block codes.

3.1 The Bonferroni-Type Lower and Upper Bounds

For a positive integer M, let $A_1, ..., A_M$ be events in an arbitrary probability space. A stepwise algorithm is given in [6] to find a lower bound for the probability of the union of the A_i. The bound is given by

$$P\left(\bigcup_{i=1}^{M} A_i\right) \geq \max_{\mathcal{I}} \left\{ \sum_{i \in \mathcal{I}} P(A_i) - \sum_{\substack{i,j \in \mathcal{I} \\ i<j}} P(A_i \cap A_j) \right\}, \tag{6}$$

where $\mathcal{I} \subseteq \{1, 2, ..., M\}$. Also, a greedy algorithm is described in [6] which finds an upper bound for the probability of the union of A_i, which is given by

$$P\left(\bigcup_{i=1}^{M} A_i\right) \leq \sum_{i=1}^{M} P(A_i) - \max_{T_0 \in \mathcal{T}} \sum_{(i,j) \in T_0} P(A_i \cap A_j), \tag{7}$$

where $\mathcal{T}$ is the set of all spanning trees of the M indices, i.e., the trees whose set of nodes is the set of indices $\{1, ..., M\}$. The greedy algorithm in [6] is as follows. First, each pair of nodes (i, j) is connected via edges of weight $P(A_i \cap A_j)$ to form a fully connected graph. To form the tree T_0, the algorithm starts from the edge with the largest weight (the (i, j) with the largest $P(A_i \cap A_j)$). Then, at each step, the edge with the largest weight is added to T_0, subject to the constraint that there is no cycle in T_0. This step is executed until all of the nodes are in T_0.

3.2 The Symbol Error Rate

For a constellation of size $M = 2^p$, where p is a positive integer, and an i.i.d. uniform bit stream, the SER is given by

$$\text{SER} = \sum_{u=1}^{M} P(\epsilon|c_u)P(c_u) = \frac{1}{M}\sum_{u=1}^{M} P_u\left(\bigcup_{i\neq u}\epsilon_{iu}\right), \tag{8}$$

where $P(\cdot|c_u) \triangleq P_u(\cdot)$ is the conditional probability of error given that c_u was sent, and ϵ_{iu} indicates the event that c_i has a larger metric than c_u. Note that $P_u(\epsilon_{iu})$ is the PEP and is given in (5). In order to find lower and upper bounds on the probability of each union in (8) via (6) and (7), respectively, we need to find the probability of the intersection of ϵ_{iu} and ϵ_{ju}, the 2-D PEP of symbols c_i and c_j with c_u. This can be done by averaging [6, Equation (7)], i.e.,

$$P_u(\epsilon_{iu}\cap\epsilon_{ju}) = \mathrm{E}_Y\left\{\Psi\left(\rho_{iju}, \delta_{iu}\sqrt{Y}, \delta_{ju}\sqrt{Y}\right)\right\}, \tag{9}$$

where $\rho_{iju} = \frac{\langle c_i - c_u, c_j - c_u\rangle}{d_{iu}d_{ju}}$, $\langle x, y\rangle = \Re\{x\}\Re\{y\} + \Im\{x\}\Im\{y\}$ ($\Re\{\cdot\}$ and $\Im\{\cdot\}$ are the real and imaginary parts, respectively), $d_{iu} = |c_i - c_u|$, Y is defined in (3), and

$$\Psi(\rho, \phi_i(y), \phi_j(y)) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{\phi_i(y)}^{\infty}\int_{\phi_j(y)}^{\infty}\exp\left(-\frac{\tau^2 - 2\rho\tau\lambda + \lambda^2}{2(1-\rho^2)}\right)d\tau d\lambda, \tag{10}$$

with $\phi_i(y) = \delta_{iu}\sqrt{y}$. As $\phi_i(y)$ and $\phi_j(y)$ are non-negative, we can use the result of [10] to write (10) as

$$\Psi\left(\rho, \delta_{iu}\sqrt{y}, \delta_{ju}\sqrt{y}\right) = \frac{1}{2\pi}\int_0^{\varphi\left(\frac{d_{iu}}{d_{ju}},\rho\right)}\exp\left(\frac{-\delta_{iu}^2}{2\sin^2\theta}y\right)d\theta + \frac{1}{2\pi}\int_0^{\varphi\left(\frac{d_{ju}}{d_{iu}},\rho\right)}\exp\left(\frac{-\delta_{ju}^2}{2\sin^2\theta}y\right)d\theta, \tag{11}$$

where $\varphi(x,\rho) = \tan^{-1}\left(x\sqrt{1-\rho^2}/(1-\rho x)\right)$, and $\tan^{-1}(x)$ is defined here as $\pi - \tan^{-1}(-x)$ for negative x. Using the pdf of Y in (4), we find the expected value of each of the integrals in (11) as follows.

$$\begin{aligned} E_Y\left[\int_0^{\varphi} e^{-a(\theta)Y}d\theta\right] &= \frac{1}{(n-1)!}\int_0^{\infty} y^{n-1}e^{-y}\int_0^{\varphi} e^{-a(\theta)y}d\theta dy \\ &= \frac{1}{(n-1)!}\int_0^{\varphi}\int_0^{\infty} y^{n-1}e^{-(1+a(\theta))y}dyd\theta \qquad (12) \\ &= \int_0^{\varphi}\left(\frac{1}{1+a(\theta)}\right)^n d\theta, \qquad (13) \end{aligned}$$

where $a(\theta)$ is a given non-negative function of θ. The step from (12) to (13) follows (for example) by writing the inner integral in (12) in terms of the $(n-1)^{\text{st}}$ derivative of the Laplace transform of the unit step function. Using (13) with $a(\theta) = \delta^2/2\sin^2\theta$ and [12, Eq. 5A.35], we have

$$\begin{aligned} I(\varphi,\delta) &\triangleq \frac{1}{2\pi}\int_0^{\varphi}\left(\frac{2\sin^2\theta}{\delta^2+2\sin^2\theta}\right)^n d\theta \\ &= \frac{\varphi}{2\pi} - \frac{\beta}{2\pi}\frac{\delta}{\sqrt{2+\delta^2}}\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{(2(2+\delta^2))^k} \\ &\quad -\frac{\delta}{\pi\sqrt{2+\delta^2}}\sum_{k=0}^{n-1}\sum_{m=0}^{k-1}\binom{2k}{m}\frac{(-1)^{m+k}}{(2(2+\delta^2))^k}\frac{\sin[2\beta(k-m)]}{2(k-m)}, \end{aligned} \tag{14}$$

where

$$\beta = \frac{1}{2}\tan^{-1}\frac{A}{B} + \frac{\pi}{2}\left[1 - \left(\frac{1+\operatorname{sgn} B}{2}\right)\operatorname{sgn} A\right],$$

with $A = \delta\sqrt{2+\delta^2}\sin 2\varphi$, $B = (1+\delta^2)\cos 2\varphi - 1$, and $\operatorname{sgn} x \triangleq |x|/x$ if $x \neq 0$ and 0 otherwise. Therefore, from (9), (11), and (14), we obtain the following expression for the 2-D PEP

$$P_u(\epsilon_{iu}\cap\epsilon_{ju}) = I\left(\varphi\left(\frac{\delta_{iu}}{\delta_{ju}},\rho_{iju}\right),\delta_{iu}\right) + I\left(\varphi\left(\frac{\delta_{ju}}{\delta_{iu}},\rho_{iju}\right),\delta_{ju}\right). \tag{15}$$

3.3 The Bit Error Rate

The same algorithm can be used as outlined in [6] to estimate the BER. We have

$$\text{BER} = \frac{1}{M}\sum_{u=1}^{M} P_b(u),$$

where $P_b(u)$ is the bit error probability when c_u is sent and is given by

$$P_b(u) = \frac{1}{\log_2 M}\sum_{j=1}^{M} H(j,u)P(\hat{c}=c_j|c=c_u) = \frac{1}{\log_2 M}\sum_{j=1}^{M} H(j,u)\left(1 - P_u\left(\bigcup_{i\neq j}\epsilon_{ij}\right)\right), \tag{16}$$

where $H(j,u)$ is the Hamming distance between the bit assignments of c_j and c_u, $P(\hat{c}=c_j|c=c_u)$ is the probability that c_j is decoded given that c_u is sent, and ϵ_{ij} is the event that c_i has a larger metric than c_j. From the above, it is clear that finding upper and lower bounds on the BER requires evaluating lower and upper bounds on the probability of the union in (16). These bounds, in turn, require the computation of

$$\begin{aligned} P_u(\epsilon_{ij}) &= E_Y\left\{Q\left(\frac{\delta_{iu}^2-\delta_{ju}^2}{\delta_{ij}}\sqrt{Y}\right)\right\} \\ &= \frac{1}{2}\left(1-\sqrt{\frac{\delta_{iu}^2-\delta_{ju}^2}{(\delta_{iu}^2-\delta_{ju}^2)^2-2\delta_{ij}^2}}\right)\sum_{k=0}^{KL-1}\binom{2k}{k}\left(\frac{\delta_{ij}^2}{2((\delta_{iu}^2-\delta_{ju}^2)^2+2\delta_{ij}^2)}\right)^k, \end{aligned}$$

as well as

$$
\begin{aligned}
P_u(\epsilon_{ij} \cap \epsilon_{kj}) &= E_Y\left\{\Psi\left(\rho_{ikj}, \frac{\delta_{iu}^2-\delta_{ju}^2}{\delta_{ij}}\sqrt{Y}, \frac{\delta_{ku}^2-\delta_{ju}^2}{\delta_{kj}}\sqrt{Y}\right)\right\} \\
&= I\left(\varphi\left(\frac{\delta_{kj}(\delta_{iu}^2-\delta_{ju}^2)}{\delta_{ij}(\delta_{ku}^2-\delta_{ju}^2)}, \rho_{ikj}\right), \frac{\delta_{iu}^2-\delta_{ju}^2}{\delta_{ij}}\right) \\
&\quad + I\left(\varphi\left(\frac{\delta_{ij}(\delta_{ku}^2-\delta_{ju}^2)}{\delta_{kj}(\delta_{iu}^2-\delta_{ju}^2)}, \rho_{ikj}\right), \frac{\delta_{ku}^2-\delta_{ju}^2}{\delta_{kj}}\right),
\end{aligned}
\tag{17}
$$

where $I(\cdot,\cdot)$ is defined in (14) and if the second and the third arguments of $\Psi(\cdot,\cdot,\cdot)$ are non-negative. If at least one of the two arguments are negative, the $\Psi(\cdot,\cdot,\cdot)$ function can be written as sum of a number of $\Psi(\cdot,\cdot,\cdot)$ functions with non-negative arguments as follows

$$
\Psi(\rho,\alpha,\beta) = \begin{cases}
\Psi(\rho,\alpha,0)+\Psi(-\rho,\alpha,0)-\Psi(-\rho,\alpha,-\beta) & \alpha \geq 0, \beta < 0, \\
\Psi(\rho,0,\beta)+\Psi(-\rho,0,\beta)-\Psi(-\rho,-\alpha,\beta) & \alpha < 0, \beta \geq 0, \\
1-\Psi(\rho,0,-\beta)-\Psi(-\rho,0,-\beta) & \\
-\Psi(\rho,-\alpha,0)-\Psi(-\rho,-\alpha,0)+\Psi(\rho,-\alpha,-\beta) & \alpha < 0, \beta < 0.
\end{cases}
$$

In this case, we can also obtain an expression for the 2-D PEP as shown in (17).

4 Simulation Results

For this part, the length of the input bit-sequence is max(100000, 100 $\times$ BER^{-1}). The computation time of the algorithmic Bonferroni bounds is negligible for our code that is written in C and run on a SUN Ultra 60 machine. The Chernoff and the union bounds are also given for the SER.

The SER and BER versus CSNR curves are presented in Figures 1 through 4 for various K, L, space-time codes, and M-ary PSK and QAM constellations with Gray or quasi-Gray mappings. Figures 1 and 2 show the performance of Alamouti's $\mathcal{G}^2$ code [2], while the performance of the codes $\mathcal{G}^3$ and $\mathcal{G}^4$ of [14] are presented in Figures 3 and 4, respectively. The lower and upper Bonferroni bounds based on the 2-D PEP formulas for both SER and BER curves are very tight and can hardly be distinguished from each other and from the performance curve obtained via simulation. It can be seen that as the constellation size grows, the union bound becomes less reliable and the Chernoff bound gets farther from the other curves, while the lower and upper bounds based on the exact 2-D PEP remain very tight. Figure 3 shows our

BER upper and lower bounds as well as simulation results for a system with three transmit and $L = 1, 2,$ and 4 receive antennas for a wide range of CSNRs. The bounds are tight even at negative CSNR values and high diversity orders. Notice that the Star 8-QAM constellation of Figure 4 $\big($with signal points at $\frac{1}{2\sqrt{10}}\{1, 3, i, 3i, -1, -3, -i, -3i\}$, where $i = \sqrt{-1}$ and a corresponding quasi-Gray bit mapping given by $\{000, 100, 001, 101, 011, 111, 010, 110\}\big)$ is not a regular square QAM nor 8-PSK, but the bounds are still very tight.

An upper bound on the BER can be computed by upper bounding $P(\hat{c} = c_j | c = c_u)$ in (16) by the PEP to get

$$\text{BER} \leq \frac{1}{M \log_2 M} \sum_u \sum_j H(u, j) P(c_u \to c_j),$$

where $H(u, j)$ is the Hamming distance between the bits corresponding to c_u and c_j. This bound is also plotted in all figures. It is clearly observed that our upper bound is much tighter, particularly for larger constellations.

As Gray mapping is used in the above systems (except for Figure 4), it is expected that at high enough CSNRs the BER of PSK-modulated systems converges to [8] $\text{BER} \approx \frac{1}{p}$ SER, where 2^p is the constellation size. The curves labeled with "SER/p" in Figure 1 show this approximation. We also observe that as the constellation size shrinks or the number of the receive antennas grows, this estimate becomes tighter. Therefore, at high enough CSNR values, it would be enough to find only one of the SER bounds to obtain a good estimate of system SER and BER. For example, for 16-PSK signaling and at a CSNR of 15 dB, the coinciding SER lower and upper bounds are equal to 0.195682. The above approximation yields BER $\approx$ 0.048921, while the lower and upper bounds on the BER coincide and equal 0.051838. It is also important to note that although we have only presented the BER results for the Gray or quasi-Gray signal mappings, our bounds also apply to any other mapping and show the same behavior.

5 Conclusions

We derive very tight Bonferroni-type lower and upper bounds on the symbol and bit error rates for space-time orthogonal block codes with arbitrary signal constellations. The lower and upper bounds derived for these codes have very close values (up to six identical significant digits) even at low channel signal-to-noise ratios. Several constellation types and sizes, space-time codes, and diversity orders were considered in the numerical results. Simulations support the tightness and accuracy of the formulas derived.

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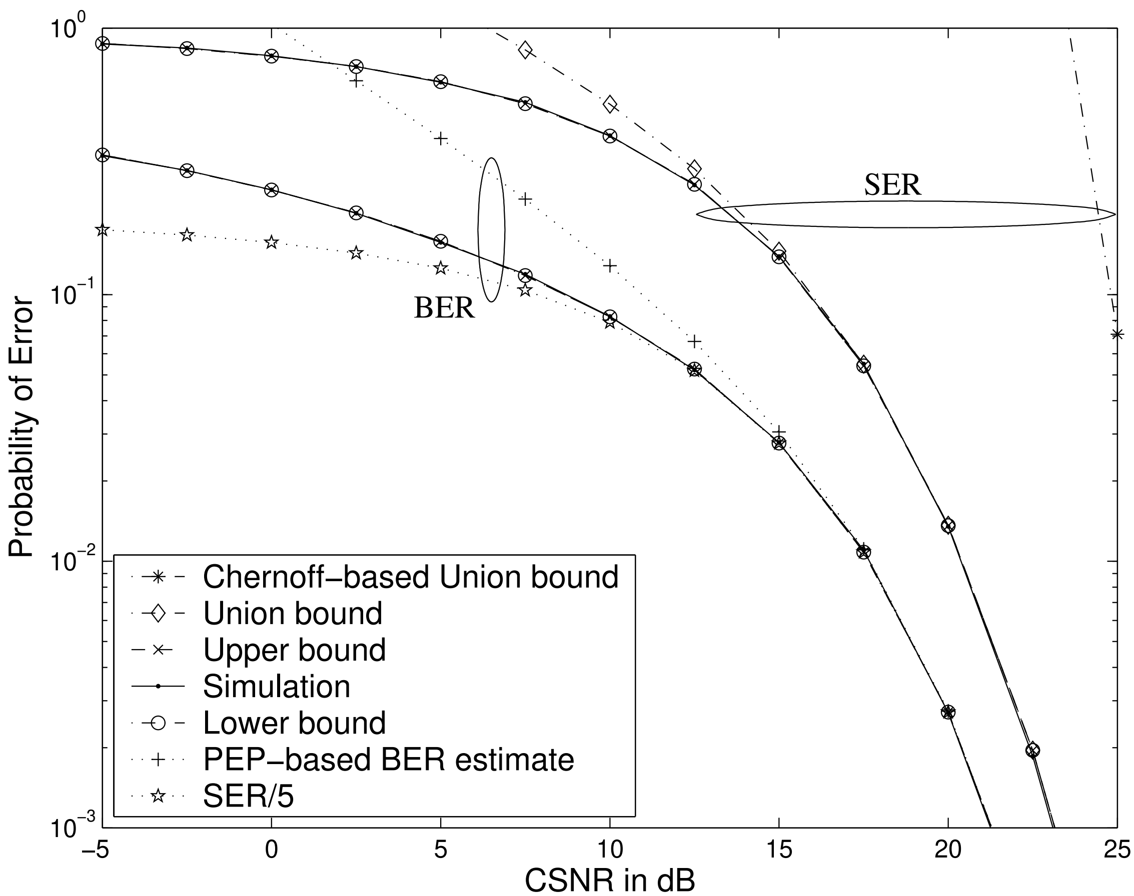

Figure 1: Results for 32-PSK signaling with $K = 2$, $L = 4$, and $\mathcal{G}^2$ STOB code (5 bits/s/Hz).

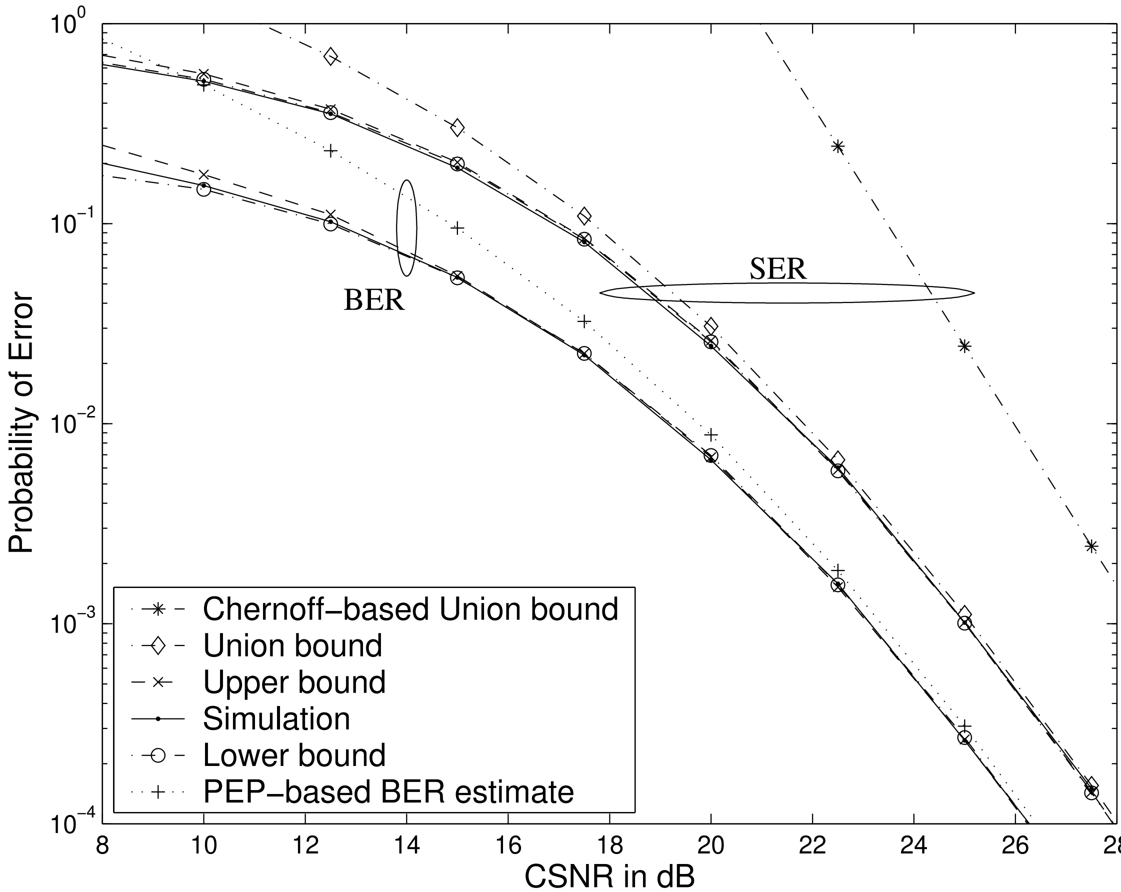

Figure 2: Results for 64-QAM signaling with $K = 2$, $L = 2$, and $\mathcal{G}^2$ STOB code (6 bits/s/Hz).

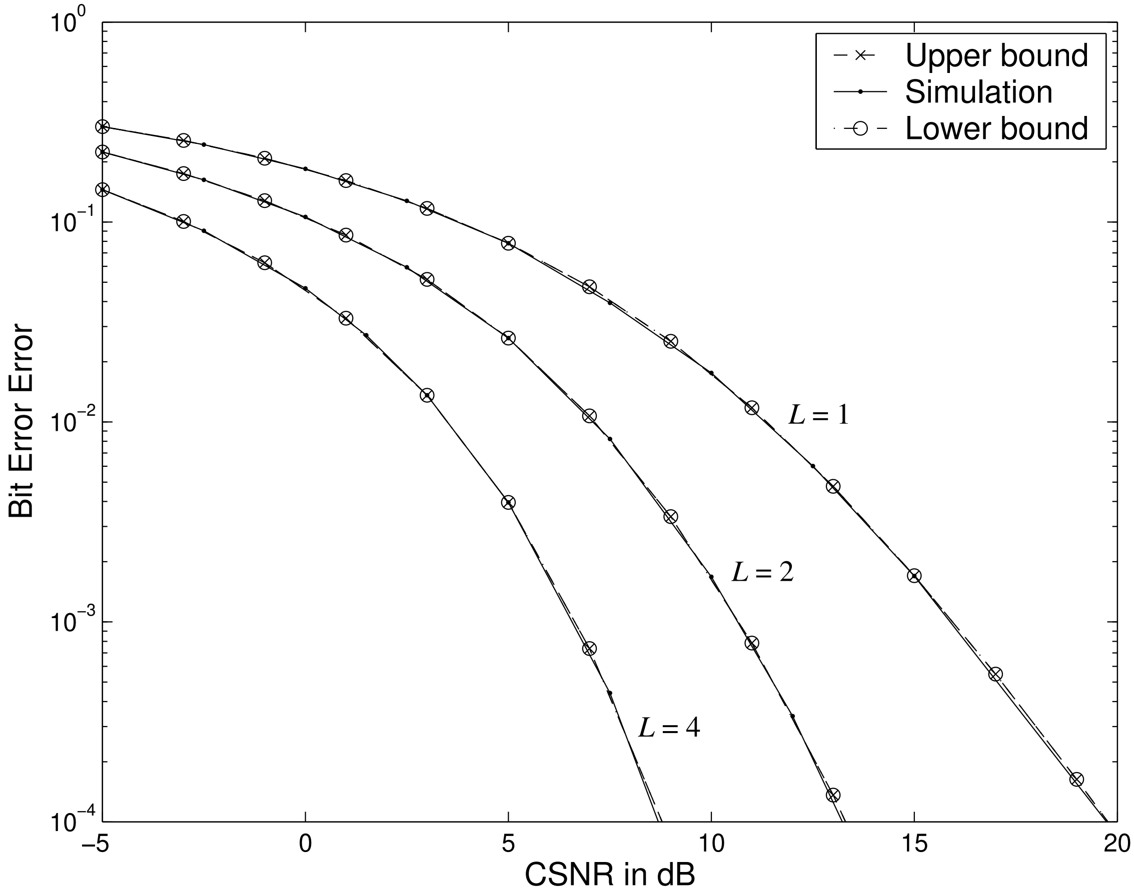

Figure 3: Results for 8-PSK signaling with $K = 3$, $L = 1, 2$, and 4, and $\mathcal{G}^3$ STOB code (1.5 bits/s/Hz).

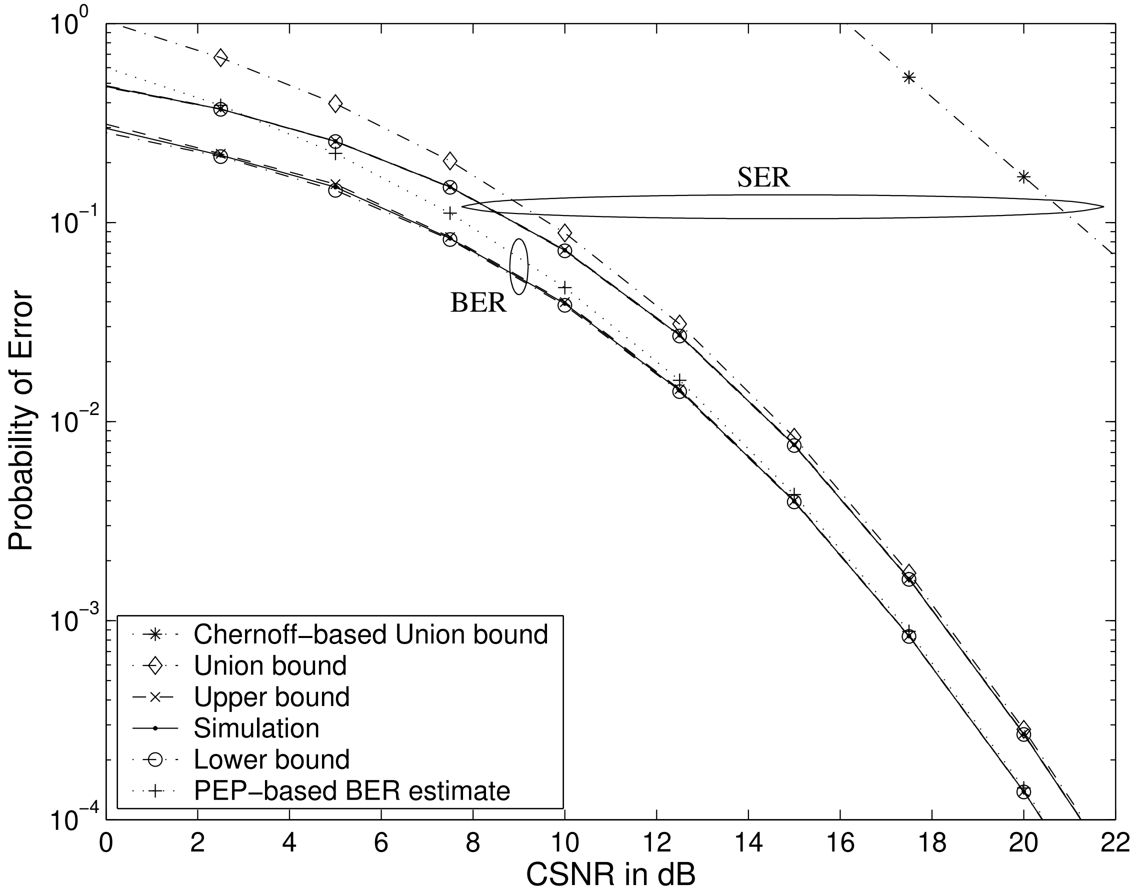

Figure 4: Results for Star 8-QAM signaling and quasi-Gray mapping with $K = 4$, $L = 1$, and $\mathcal{G}^4$ STOB code (1.5 bits/s/Hz).