A Heavy Traffic Fluid Limit For Large Capacity Retrial Networks

Glen K. Takahara*
Department of Math. & Stats.
Queen's University
Kingston, ON, Canada

Joanna S. Meadows†
Bank of Montréal
Emfysis
Toronto, ON, Canada

Abstract — We consider a model for a broadband connection-oriented network that models connection admission control, routing, topology, bandwidth, multiple call classes, and connection retrials. A fluid limit is proved in a limiting regime in which link capacities and loads grow together to infinity. The limit is a generalization of a similar result proved in [6], which did not incorporate retrials.

1 Introduction

Intelligent evolution of modern communications networks towards future high speed integrated services broadband networks requires an understanding of several interacting system elements; among these are network topology, bandwidth, routing and connection admission protocols, multiple call classes, and traffic behaviour. Typically, these elements are studied individually, and often by simulation. However, simulation techniques can quickly become inefficient as the model complexity increases, which usually occurs before two distinct system elements can be modeled jointly to study their interaction effects. There is a need for tractable stochastic models on a macro scale that can provide insights into joint, qualitative, effects of several network elements.

In this paper we generalize the model and results in [6] to incorporate connection retrials. In section 2 we describe the retrial model and the limiting regime we consider for our fluid limit. In section 3 we prove the main result - a functional law of large numbers for the sequence of vector processes defined in the limiting regime.

2 System Model

We consider a retrial network consisting of a set $\mathcal{J} = \{1, \ldots, J\}$ of links, with link j of capacity C_j, and a set $\mathcal{R} = \{1, \ldots, R\}$ of routes. Call types are distinguished from one another by the route on which they are requesting service; as a result, we consider the $J \times R$ matrix A with entries $A_{jr} \in \mathbb{Z}_+$ corresponding to the number of circuits from link j required by a call on route r.

Calls arrive to route r from an external source as a Poisson process of rate κ_r. A call accepted on route r holds A_{jr} circuits from link j for an exponential holding time of mean μ_r^{-1}. An external route r call rejected from service may, with probability $(1-\alpha_r)$, elect to enter retrial population r where it will remain for an exponentially distributed holding time of mean ρ_r^{-1} before retrying. If a route r call from the retrial population is again rejected from service it may, with probability $(1-\gamma_r)$, elect to return to the retrial population, at which time the retrial process recommences. All arrival streams and holding times are assumed mutually independent.

Let $\mathbf{n}(t) = (n_r(t))_{r\in\mathcal{R}}$ and $\mathbf{a}(t) = (a_r(t))_{r\in\mathcal{R}}$ be the call- and retrial-profiles of the network at time t; specifically, $n_r(t)$ is the number of route r calls in progress at time t, and $a_r(t)$ is the number of route r calls in retrial at time t. Let $\mathbf{m}(t) = (m_j(t))_{j\in\mathcal{J}}$ where $m_j(t) = C_j - \sum_r A_{jr}n_r(t)$, the number of free circuits on link j at time t.

We now consider some rules by which calls of type r are accepted or rejected. Let $\mathcal{A}_r \subset \mathbb{Z}_+^J$ be the acceptance set for type r calls, such that a type r call connection request at time t is accepted if and only if $\mathbf{m}(t) \in \mathcal{A}_r$. The acceptence sets can be used to model any call admission control protocol whose acceptence rule is determined only by the network call profile at the time of the connection request. For example, with a complete sharing policy, we have $\mathcal{A}_r = \{\mathbf{m} : m_j \geq A_{jr} \text{ for all } j \in \mathcal{J}\}$. With a trunk reservation policy, we have $\mathcal{A}_r = \{\mathbf{m} : m_j \geq A_{jr} + s_{jr} \text{ for all } j \in \mathcal{J}\}$, with the trunk reservation parameters given by $s_{jr} \geq 0$. We may also consider *alternative routing* which can be employed in conjunction with trunk reservation. Under this acceptance policy, a call is given more than one chance to be routed; that is, if a type r call is blocked due to an overloaded link on its route of choice, say link j, it may immediately attempt an alternate route which bypasses link j. In this case, we can view the model as follows. Suppose a type r call will be routed *directly* if $\mathbf{m} \in \mathcal{A}_r^1$ (where $\mathcal{A}_r^1$ is the acceptance set for type r calls that are accepted onto their route of choice) and *alternately* if $\mathbf{m} \notin \mathcal{A}_r^1$ but $\mathbf{m} \in \mathcal{A}_r^2$ (where $\mathcal{A}_r^2$ is some acceptance set for type r calls that are accepted onto an alternate route when their first-choice route is not available, and $\mathcal{A}_r^2 \subset (\mathcal{A}_r^1)^c$), then the stochastic process giving the state of this network is identical to one in

*Supported by NSERC grant ORG-92-J-1171
†Supported by the Dept. of Math. & Stats., Queen's University

which there are calls of types r_1 and r_2 with independent Poisson arrival processes of rate κ_r and acceptance sets $\mathcal{A}_r^1$ and $\mathcal{A}_r^2$ respectively.

We represent this model as a solution of the following system:

$$\begin{aligned}
n_r(t) &= n_r(0) + N_{ar}\left(\int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}(\kappa_r + a_r(s)\rho_r)\,ds\right)\\
&- N_{dr}\left(\int_0^t n_r(s)\mu_r\,ds\right),\\
a_r(t) &= a_r(0) + N_{ar}\left(\int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r^c\}}(1-\alpha_r)\kappa_r\,ds\right)\\
&- N_{dr}\left(\int_0^t \left[I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}a_r(s)\rho_r\right.\right.\\
&\qquad \left.\left. + I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r^c\}}\gamma_r a_r(s)\rho_r\right]ds\right),
\end{aligned}$$

where N_{ar} and N_{dr} are unit Poisson processes. We see that the second term on the right-hand side of the equation for $n_r(t)$ gives the number of type r calls accepted into service during the time interval $[0,t]$ and the third term gives the number of departures from service in $[0,t]$. We also see that the second term on the right-hand side of the equation for $a_r(t)$ gives the number of calls entering retrial population r over $[0,t]$, and the third term gives the number of departures from retrial pool r in $[0,t]$. Letting $\tilde{N}_{ar}(u) = N_{ar}(u) - u$ and $\tilde{N}_{dr}(u) = N_{dr}(u) - u$:

$$\begin{aligned}
n_r(t) &= n_r(0) + \tilde{N}_{ar}\left(\int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}(\kappa_r + a_r(s)\rho_r)ds\right)\\
&- \tilde{N}_{dr}\left(\int_0^t n_r(s)\mu_r ds\right)\\
&+ \int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}(\kappa_r + a_r(s)\rho_r)ds) - \int_0^t n_r(s)\mu_r ds \quad (1)\\
a_r(t) &= a_r(0) + \tilde{N}_{ar}\left(\int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r^c\}}(1-\alpha_r)\kappa_r\,ds\right)\\
&- \tilde{N}_{dr}\left(\int_0^t \left[I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}a_r(s)\rho_r\right.\right.\\
&\qquad \left.\left. + I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r^c\}}\gamma_r a_r(s)\rho_r\right]ds\right)\\
&- \int_0^t I_{\{\mathbf{m}(s^-)\in\mathcal{A}_r\}}\big((1-\alpha_r)\kappa_r + (1-\gamma_r)\rho_r a_r(s)\big)\,ds\\
&+ \int_0^t (1-\alpha_r)\kappa_r - \gamma_r\rho_r a_r(s)\,ds. \quad (2)
\end{aligned}$$

We consider a limiting regime with large capacities and offered traffics. We consider a sequence of networks indexed by N and suppose that as $N\to\infty$,

$$\frac{\kappa(N)}{N} \to \kappa \qquad \frac{\mathbf{C}(N)}{N} \to \mathbf{C} \tag{3}$$

where $\kappa(N)$ is the vector of external arrival rates in the N^{th} network and $\mathbf{C} = (C_j)_{j\in\mathcal{J}}$. We hold the matrix A and the acceptance sets $\mathcal{A}_r$ constant. This limit was previously studied by Kelly in [8] for a fixed routing network in equilibrium for complete sharing and without retrials. By incorporating retrials, we seek to generalize the results of [6], which generalized Kelly's results not only by incorporating general acceptance sets, but by considering the network when started away from equilibrium and proving a functional law of large numbers. Accordingly we let $\mathbf{x}^N(t) = \mathbf{n}^N(t)/N$ and $\mathbf{y}^N(t) = \mathbf{a}^N(t)/N$.

In terms of $\big(\mathbf{x}^N(\cdot), \mathbf{y}^N(\cdot)\big)$ and (3), equations (1) and (2) become

$$\begin{aligned}
x_r^N(t) &= x_r^N(0) + \int_0^t I_{\{\mathbf{m}^N(s^-)\in\mathcal{A}_r\}}\Big(\frac{\kappa_r(N)}{N} + y_r^N(s)\rho_r\Big)\,ds\\
&- \int_0^t x_r^N(s)\mu_r\,ds + o_p(1) \qquad (4)\\
y_r^N(t) &= y_r^N(0) + \int_0^t \Big((1-\alpha_r)\frac{\kappa_r(N)}{N} - \gamma_r y_r^N(s)\rho_r\Big)\,ds\\
&+ \int_0^t I_{\{\mathbf{m}^N(s^-)\in\mathcal{A}_r\}}\big[(\gamma_r - 1)y_r^N(s)\rho_r\\
&\qquad - (1-\alpha_r)\frac{\kappa_r(N)}{N}\big]\,ds + o_p(1), \qquad (5)
\end{aligned}$$

where, after introducing a bound on $\mathbf{y}^N(\cdot)$, the $o_p(1)$ terms are square-integrable martingales. From Doob's inequality, their supremum over any bounded time interval converges in probability to zero.

Now consider the third term on the RHS of (4) and the second term on the RHS of (5). Let Ω denote the space of all sample paths of $\big(\mathbf{m}^N(\cdot), \mathbf{x}^N(\cdot), \mathbf{y}^N(\cdot)\big)$, for $N\geq 1$. Then, as a function of t, each of these terms satisfies a Lipschitz condition uniformly in $\omega\in\Omega$ and $N\geq 1$. Furthermore, each of these terms is a sequence of function-valued random variables where the sample points are continuous functions. Let $C_E[0,\infty)$ be the space of all continuous functions from $[0,\infty)$ to a metric space (E,d). Then for each i we have a sequence of $C_E[0,\infty)$-valued random variables. Referring to Hunt and Kurtz[6](Section 1), given that the sequences are uniformly Lipschitz over $\omega\in\Omega$ and $N\geq 1$, they are relatively compact for convergence in distribution in the topology of uniform convergence on bounded time intervals. Our primary problem is determining the asymptotic behaviour of the second term on the RHS of (4) and the third term on the RHS of (5).

3 Functional law of large numbers

Since call acceptance at time t is determined from the value of $\mathbf{m}^N(t)$, we consider the Markov process $\big(\mathbf{x}^N(\cdot), \mathbf{y}^N(\cdot), \mathbf{m}^N(\cdot)\big)$, $\mathbf{m}^N(t)$ being defined on $\mathbb{Z}_+^J$. Following the method of Hunt and Kurtz in [6], we endow $\mathbb{Z}_+^J$ with the metric and σ-algebra induced by the Euclidean metric and Borel σ-algebra on $\mathbb{R}^J$ and the one-to-one map from $\mathbb{Z}_+^J$ to $\mathbb{R}^J$

$$\begin{aligned}
i : E &\hookrightarrow [0,1]^J\\
(x_1,\ldots,x_J) &\mapsto \big((x_1+1)^{-1},\ldots,(x_J+1)^{-1}\big),
\end{aligned}$$

where $E = (\mathbb{Z}_+^\Delta)^J$ and $\mathbb{Z}_+^\Delta = \mathbb{Z}_+ \cup \{\infty\}$. Under this mapping, E is a compact metric space. Furthermore, E is complete and separable.

Rather than considering the process $\mathbf{m}^N(\cdot)$, we consider the random measure processes $\nu_{x_r}^N(\cdot)$ and $\nu_{y_r}^N(\cdot)$ on $[0,\infty)\times E$ defined by, for each $r \in \mathcal{R}$,

$$\begin{aligned}
\nu_{x_r}^N([0,t]\times\Gamma) &= \int_0^t I_{\{\mathbf{m}^N(s-)\in\Gamma\}}\big(\frac{\kappa_r(N)}{N} + \rho_r y_r^N(s)\big)\,ds,\\
\nu_{y_r}^N([0,t]\times\Gamma) &= \int_0^t I_{\{\mathbf{m}^N(s-)\in\Gamma\}}\big((1-\alpha_r)\frac{\kappa_r(N)}{N}\\
&\qquad + (1-\gamma_r)\rho_r y_r^N(s)\big)\,ds,
\end{aligned}$$

So, letting Ω be the space of all sample paths for $\big(\mathbf{x}^N(\cdot),\mathbf{y}^N(\cdot),\mathbf{m}^N(\cdot)\big)$, then for a given N, t, r and for each $\omega\in\Omega$, $\nu_{x_r}^N$ and $\nu_{y_r}^N$ are finite measures on $\big([0,t]\times E, \mathcal{B}([0,t])\times\mathcal{B}(E)\big)$.

In terms of $\nu_{x_r}^N(\cdot)$ and $\nu_{y_r}^N(\cdot)$, (4) and (5) become

$$\begin{aligned}
x_r^N(t) &= x_r^N(0) + \nu_{x_r}^N([0,t]\times\mathcal{A}_r) - \int_0^t x_r^N(s)\mu_r\,ds, &(6)\\
y_r^N(t) &= y_r^N(0) - \nu_{y_r}^N([0,t]\times\mathcal{A}_r)\\
&\quad + \int_0^t (1-\alpha_r)\frac{\kappa_r(N)}{N} - \gamma_r\rho_r y_r^N(s)\,ds. &(7)
\end{aligned}$$

In [10], the following lemma is shown:

Lemma 3.1 *Suppose $\big(\mathbf{x}^N(0),\mathbf{y}^N(0)\big) \Rightarrow \big(\mathbf{x}(0),\mathbf{y}(0)\big)$. Then letting $\mathcal{L}_0(E)$ denote the space of all σ-finite measures on $([0,\infty)\times E)^{2R}$ and letting $\nu^N = (\nu_{u_r}^N)_{u=x,y,r\in\mathcal{R}}$, the sequence $\{\big(\mathbf{x}^N(\cdot),\mathbf{y}^N(\cdot),\nu^N(\cdot)\big)\}$ is relatively compact in $D_{\mathbb{R}^{2R}}[0,\infty)\times\mathcal{L}_0(E)$, and the limit $\{\big(\mathbf{x}(\cdot),\mathbf{y}(\cdot),\nu(\cdot)\big)\}$ of any convergent subsequence satisfies*

$$\begin{aligned}
x_r(t) &= x_r(0) + \nu_{x_r}([0,t]\times\mathcal{A}_r) - \int_0^t x_r(s)\mu_r ds &(8)\\
y_r(t) &= y_r(0) - \nu_{y_r}([0,t]\times\mathcal{A}_r)\\
&\quad + \int_0^t (1-\alpha_r)\kappa_r - \gamma_r y_r(s)\rho_r ds &(9)
\end{aligned}$$

We now turn our attention to determining ν_{x_r} and ν_{y_r}. In order to do this, we consider the Markov process $\mathbf{m}_{\mathbf{x},\mathbf{y}}(\cdot)$ on E with transition rates

$$\mathbf{m}_{\mathbf{x},\mathbf{y}} \to \begin{cases} \mathbf{m}_{\mathbf{x},\mathbf{y}} - \mathbf{A}_r & \text{at rate } (\kappa_r + y_r\rho_r)I_{\mathcal{A}_r} \\ \mathbf{m}_{\mathbf{x},\mathbf{y}} + \mathbf{A}_r & \text{at rate } \mu_r x_r. \end{cases} \qquad (10)$$

for fixed $\mathbf{x}$ and $\mathbf{y}$. As discussed in [6], the interplay between $\mathbf{m}_{\mathbf{x},\mathbf{y}}(\cdot)$ and the process $(\mathbf{x}(\cdot),\mathbf{y}(\cdot))$ is striking. As we shall see in theorem 3.1, in the limit the process $\mathbf{m}(\cdot)$ and the process $(\mathbf{x}(\cdot),\mathbf{y}(\cdot))$ experience a complete separation of time scale. Informally, at a given instant, $(\mathbf{x},\mathbf{y})$ is fixed, and this determines the evolution of $\mathbf{m}(\cdot)$ in that instant to be that of the Markov process $\mathbf{m}_{\mathbf{x},\mathbf{y}}(\cdot)$, whose stationary distribution in turn determines the evolution of $(\mathbf{x}(\cdot),\mathbf{y}(\cdot))$ in the next instant.

We now restrict our attention to convergent subsequences to deduce properties of the limit process. We further assume that $\{\big(\mathbf{x}^N(\cdot),\mathbf{y}^N(\cdot),\nu^N(\cdot)\big)\} \to \{\big(\mathbf{x}(\cdot),\mathbf{y}(\cdot),\nu(\cdot)\big)\}$ almost surely along any convergent subsequence (using Skorokhod's representation theorem, [4](Theorem 3.1.8)).

Lemma 3.2 *Let $\{\big(\mathbf{x}^N(\cdot),\mathbf{y}^N(\cdot),\nu^N(\cdot)\big)\}$ be a convergent subsequence where the limit process is $\{\big(\mathbf{x}(\cdot),\mathbf{y}(\cdot),\nu(\cdot)\big)\}$. Then for all $\Gamma_1\in\mathcal{B}([0,\infty))$, $\Gamma_2\in\mathcal{B}(E)$, we may write*

$$\begin{aligned}
\nu_{x_r}(\Gamma_1\times\Gamma_2) &= \int_{\Gamma_1}\pi_s^x(\Gamma_2)\big(\kappa_r + y_r(s)\rho_r\big)\,ds &(11)\\
\nu_{y_r}(\Gamma_1\times\Gamma_2) &= \int_{\Gamma_1}\pi_s^y(\Gamma_2)\big((1-\alpha_r)\kappa_r\\
&\quad + (1-\gamma_r)y_r(s)\rho_r\big)\,ds &(12)
\end{aligned}$$

where π_s^x, π_s^y are probability measures on E.

Proof. See [9](Lemma 1.4) and [10].

Before proceeding to the main theorem, we note that the measures π_s^x and π_s^y are independent of r since the dependence on Γ_2 in $\nu_{x_r}(\Gamma_1\times\Gamma_2)$ and $\nu_{y_r}(\Gamma_1\times\Gamma_2)$ is only through the process $\mathbf{m}(\cdot)$. We are now in a position to state the main result of this section.

Theorem 3.1 *Suppose $\big(\mathbf{x}^N(0),\mathbf{y}^N(0)\big) \Rightarrow \big(\mathbf{x}(0),\mathbf{y}(0)\big)$ as $N\to\infty$. Then $\{\big(\mathbf{x}^N(\cdot),\mathbf{y}^N(\cdot),\nu^N(\cdot)\big)\}$ is relatively compact and any convergent subsequence has a continuous limit $\big(\mathbf{x}(\cdot),\mathbf{y}(\cdot)\big)$ which satisfies*

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{x}(0)\\
&\quad + \int_0^t\{\sum_r\big[(\kappa_r + y_r(s)\rho_r)\pi_s(\mathcal{A}_r) - x_r(s)\mu_r\big]\mathbf{e}_r\}\,ds,\\
\mathbf{y}(t) &= \mathbf{y}(0)\\
&\quad + \int_0^t\{\sum_r\big[((\gamma_r-1)y_r(s)\rho_r - (1-\alpha_r)\kappa_r)\pi_s(\mathcal{A}_r)\\
&\qquad + ((1-\alpha_r)\kappa_r - \gamma_r y_r(s)\rho_r)\big]\mathbf{e}_r\}\,ds,
\end{aligned}$$

where $\pi_s = \pi_s^x = \pi_s^y$ are as in lemma 3.2 and $\pi_s(\mathcal{A}_r) = \pi_{\{\mathbf{x}(s),\mathbf{y}(s)\}}(\mathcal{A}_r)$ for some stationary distribution $\pi_{\mathbf{x}(s),\mathbf{y}(s)}$ of the Markov process (10) satisfying $\pi_{\{\mathbf{x}(t),\mathbf{y}(t)\}}\{\mathbf{m}: m_j=\infty\} = 1$ if $\sum_r A_{jr}x_r(t) < C_j$.

Proof. What remains to be shown is that $\pi_t^x = \pi_t^y$, and that both are equal to $\pi_{\{\mathbf{x}(t),\mathbf{y}(t)\}}$. We follow the proof of [6], theorem 3, generalizing to incorporate retrials. For any bounded and continuous function $f(\cdot)$ on E, we have

$$\begin{aligned}
f(\mathbf{m}^N(t)) &= f(\mathbf{m}^N(0)) + \int_0^t\sum_r(f(\mathbf{m}^N(s-)+\mathcal{A}_r)\\
&\quad - f(\mathbf{m}^N(s-)))dN_{dr}\left(\int_0^s N\mu_r(N)x_r^N(u)du\right)\\
&\quad + \int_0^t\sum_r(f(\mathbf{m}^N(s-)-\mathcal{A}_r) - f(\mathbf{m}^N(s-)))I_{\mathbf{m}^N(s-)\in\mathcal{A}_r}\\
&\qquad dN_{ar}\left(\int_0^s(\kappa_r(N) + Ny_r(u)\rho_r)du\right).
\end{aligned}$$

Since $\tilde{N}_{dr}(\int_0^t N\mu_r(N)x_r^N(u)du)$ and $\tilde{N}_{ar}(\int_0^t(\kappa_r(N) + Ny_r(u)\rho_r)du)$ are martingales, it follows that

$$M_f^N(t) = f(\mathbf{m}^N(t)) - f(\mathbf{m}^N(0)) -$$
$$\int_0^t \sum_r (f(\mathbf{m}^N(s-) + \mathcal{A}_r) - f(\mathbf{m}^N(s-)))N\mu_r(N)x_r^N(s)ds$$
$$- \int_0^t \sum_r (f(\mathbf{m}^N(s-) - \mathcal{A}_r) - f(\mathbf{m}^N(s-)))$$
$$I_{\mathbf{m}^N(s-)\in\mathcal{A}_r}(\kappa_r(N) + Ny_r(s)\rho_r)ds$$

is an $\{\mathcal{F}_t^{\mathbf{x}^N,\mathbf{y}^N}\}$-martingale, where $\{\mathcal{F}_t^{\mathbf{x}^N,\mathbf{y}^N}\} = \sigma(\mathbf{x}^N(u), \mathbf{y}^N(u) : u \le t)$. Dividing M_f^N by N to get $\tilde{M}_f^N = N^{-1}M_f^N$, we have that $E[(\tilde{M}_f^N(t))^2] \to 0$, so that from Doob's inequality $\tilde{M}_f^N \Rightarrow 0$. Since $N^{-1}(f(\mathbf{m}^N(t)) - f(\mathbf{m}^N(0)))$ also converges to zero, the remaining terms of $\tilde{M}_f^N$ converge weakly to zero. These remaining terms we may write in the form

$$\int_{[0,t]\times E} \sum_r \Big\{ I_{z\in\mathcal{A}_r}[f(z-\mathcal{A}_r) - f(z)]$$
$$+ \ (N^{-1}\kappa_r(N) + y_r^N(u)\rho_r)^{-1}\mu_r(N)x_r^N(u)$$
$$[f(z+\mathcal{A}_r) - f(z)]\Big\} \nu_{x_r}^N(du \times dz),$$

which, by the continuous mapping theorem, converges weakly to

$$\int_{[0,t]\times E} \sum_r \Big\{ I_{z\in\mathcal{A}_r}[f(z-\mathcal{A}_r) - f(z)] \ +$$
$$(\kappa_r + y_r(u)\rho_r)^{-1}\mu_r x_r(u)[f(z+\mathcal{A}_r) - f(z)]\Big\} \nu_{x_r}(du \times dz)$$
$$= \int_0^t \int_E \sum_r \Big\{ (\kappa_r + y_r(u)\rho_r)I_{z\in\mathcal{A}_r}[f(z-\mathcal{A}_r) - f(z)]$$
$$+ \mu_r x_r(u)[f(z+\mathcal{A}_r) - f(z)]\Big\} \pi_u^x(dz)du$$

But this is zero for almost all t, and therefore

$$\int_E \sum_r \Big\{ (\kappa_r + y_r(t)\rho_r)I_{z\in\mathcal{A}_r}[f(z-\mathcal{A}_r) - f(z)]$$
$$+ \mu_r x_r(t)[f(z+\mathcal{A}_r) - f(z)]\Big\} \pi_t^x(dz) = 0$$

for almost all t. By [4], proposition 9.2, π_t^x is invariant for $\mathbf{m}_{\mathbf{x}(t),\mathbf{y}(t)}$. To see that $\pi_t^y = \pi_t^x$, we note that

$$\int_{\Gamma_1} I_{\{\mathbf{m}^N(s-)\in\Gamma_2\}} \Big(\frac{c_1\kappa_r(N)}{N} + c_2\rho_r y_r^N(s)\Big)\, ds$$
$$\Longrightarrow \quad \int_{\Gamma_1} \pi_s^x(\Gamma_2)(c_1\kappa_r + c_2\rho_r y_r(s))ds,$$

for any positive constants c_1 and c_2, since $\kappa_r(N) > 0$ and $\rho_r > 0$ are arbitrary. Setting $c_1 = 1 - \alpha_r$ and $c_2 = 1 - \gamma_r$ gives $\pi_t^y = \pi_t^x$.

The question of the uniqueness of the driving invariant distribution π_t in theorem 3.1 remains to be fully addressed. In general it is not unique, as discussed in [7]. Another question of practical importance is the implementation of the fluid limit for deriving blocking probabilities, for even a simple system. We refer the reader to [1] for an analysis of a single link model with multiple call classes, incorporating trunk reservation into the model, and to [10], where connection retrials are also incorporated into the model. Both these references also develop approximations for blocking probabilities based on the asymptotic analyses which perform well even for smaller capacity systems. Further work is required to obtain practical implementations of the fluid limit of theorem 3.1 for more general networks.

References

[1] N. G. Bean, R. J. Gibbens, S. Zachary, Asymptotic analysis of single resource loss systems in heavy traffic, with applications to integrated networks. *Adv. Appl. Prob.* **27** (1995), 273-292.

[2] C. Clos, An aspect of the dialing behaviour of subscribers and its effect on the trunk plant. *Bell System Technical Journal* **27** (1948), 424-445.

[3] J.W. Cohen, Basic problems of telephone traffic theory and the influence of repeated calls. *Philips Telecom. Review* **18** (1957), 49-100.

[4] S.N. Ethier, T.G. Kurtz, *Markov Processes: Characterization and Convergence* Wiley, New York (1986).

[5] G.I. Falin, A survey of retrial queues. *Queueing Systems* **7** (1990), 127-168.

[6] P.J. Hunt, T.G. Kurtz, Large loss networks. *Stoch. Proc. Appl.* **53** (1994), 363-378.

[7] P.J. Hunt, Pathological behaviour in loss networks. *J. Appl. Prob* **32** (1995), 519-533.

[8] F.P. Kelly, Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob* **18** (1986), 473-505.

[9] T.G. Kurtz, Averaging for martingale problems and stochastic approximation, in: Applied Stochastic Analysis Proc. US-French Workshop, *Lecture Notes in Control and Information Sciences* **177** Springer-Berlin (1992), 186-209.

[10] J.S. Meadows and G.K. Takahara, Large retrial networks in heavy traffic, Preprint (1998).

[11] G.K. Takahara, Fixed point approximations for retrial networks. *Prob. Eng. Inf. Sci.* **10** (1996), 243-259.