# Constellation Mappings For Two-Dimensional Non-Uniform Signaling

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*Abstract* — **In this work we investigate the design of constellation mappings for the transmission of non-uniform memoryless sources over AWGN channels via $M$-ary modulation schemes. We show that constellation mappings which minimize the average symbol energy and, given this, maximize the decoding probability of the most likely signals, can yield SER and BER performance that is better than Gray encoding maps. We also find that for highly non-uniform sources, 16-QAM can perform better than 2-QAM, in terms of *both* throughput and BER.**

## I. Introduction

For equally likely signals, Gray mapping in two-dimensional signaling is generally accepted as optimal for minimizing bit error rate (BER). However, many data sources generate non-uniformly distributed symbols, often with memory (e.g. image or speech signals). Thus, they contain a substantial amount of (natural or residual) redundancy which, after transmission over a noisy channel, can be appropriately exploited by a maximum-a-posteriori (MAP) detector to improve the overall error resilience of the communication system [1].

In this work we propose criteria for constructing mappings from a set of signals to points of a two-dimensional constellation. We show that for non-uniform sources Gray mapping is not necessarily optimal for minimizing BER or symbol error rate (SER). We illustrate this in the context of an uncoded communication system with QAM modulated, non-uniform signals sent over an AWGN channel, and decoded using MAP decoding. We also illustrate that, when using MAP decoding for highly non-uniform signals, the BER performance of 16-QAM can be better than that of 2-QAM, even though 16-QAM has four times higher throughput.

## II. Constellation Mappings for MAP Decoding

We propose the following criteria (listed in order of priority) for constructing mappings from a set of $M$ non-uniformly distributed symbols to the points of a two-dimensional constellation: (i) minimize the average energy per symbol for the $M$ given symbol probabilities, and (ii) successively minimize the conditional symbol decoding error probabilities, going from the most likely to the least likely symbol. The following determines the mapping which satisfies criterion (i), up to permutations within sets of symbols with the same energy: given $M$ symbol probabilities $\{p_i\}_{i=1}^{M}$ with energies $E_1 \leq \ldots \leq E_M$, any permutation $\pi$ of $\{1, 2, \ldots, M\}$ which satisfies $p_{\pi(1)} \geq \ldots \geq p_{\pi(M)}$ minimizes $\sum_{i=1}^{M} E_i p_{\pi(i)}$.

Subject to criterion (i), we next consider criterion (ii). Let $s_1, \ldots, s_M$ denote the signals listed from most likely to least likely. We propose a simple heuristic for successively minimizing the conditional probabilities $P(\text{Symbol Error}|s_i \text{ sent})$. Starting with symbol $s_1$, and subject to not violating criterion (i), choose neighbours of $s_1$ to be least likely signals to maximize the area of the decoding region of signal $s_1$. Continue to allocate signals in this way until there are no signals left to allocate.

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## III. Numerical Results

We consider a Bernoulli($p$) source sent over an AWGN channel with 16-QAM modulation and MAP decoding. BER calculations were done using the upper and lower bounds in [2], which coincide with each other when plotted. Fig. 1 shows a 16-QAM constellation with a mapping $M_1$. For $p > 0.5$, the mapping $M_1$ minimizes the average symbol energy (criterion (i)) and, subject to this, for any noise variance $N_0/2$, the mapping $M_1$ also maximizes the conditional probability that symbol 0000 (the most likely symbol) is decoded, given that 0000 is sent. This is due to the fact that symbol 0000 has the least likely neighbours, subject to criterion 1; thus the decision region for 0000 is maximized. The remaining symbols are placed in the constellation to successively maximize the decoding regions of 0001, 0100, and 0010, in that order.

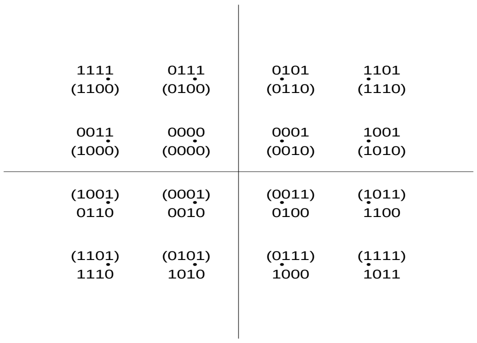


Figure 1: Mappings $M_1$ and Gray (in parentheses).

Under the mapping $M_1$, 16-QAM modulation with $p = 0.9$ and MAP decoding performs better than the usual Gray mapping, gaining roughly 1 dB and 0.75 dB in $E_b/N_0$ (at error rates between $10^{-5}$ and $10^{-2}$) for SER and BER, respectively. We also note that 16-QAM with the mapping $M_1$ achieves around 1 dB gain over 2-QAM for $p = 0.9$ and the same BER. This leads us to the interesting observation that while the conventional wisdom for equally likely signals is that there is a tradeoff between throughput and BER, with non-uniform signals there need not be such a tradeoff. Indeed, in this example 16-QAM achieves *both* four times the throughput and better BER performance than 2-QAM when $p = 0.9$.

## References

[1] F. Alajaji, N. Phamdo and T. Fuja, "Channel codes that exploit the residual redundancy in CELP-encoded speech," *IEEE Trans. Speech and Audio Processing*, pp. 325-336, Sept. 1996.

[2] H. Kuai, F. Alajaji and G. Takahara, "Performance bounds of non-uniform signaling over AWGN channels," *Proc. CTMC'99*, Vancouver, pp. 96-100, June 1999.