On the Structure of Entropy-Constrained Scalar Quantizers

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Abstract — **New results concerning the structure and existence of optimal entropy-constrained scalar quantizers (ECSQs) are presented. One main result shows that for continuous sources and a wide class of distortion measures, any finite-level ECSQ can be replaced by an ECSQ with *interval* cells which has the same entropy and equal or less distortion. Furthermore, the existence of an optimal ECSQ for an arbitrary entropy constraint is shown under rather general conditions.**

I. Introduction and Preliminaries

An important consequence of the nearest neighbor condition of *fixed-rate* quantization is that an MSE optimal fixed-rate quantizer can be assumed to be *regular* (a quantizer is called regular if its cells are convex and its code points lie inside the associated cells). The cells of a regular scalar quantizer are intervals, and the cells of a regular vector quantizer with a finite number of code points are convex polytopes. In this sense, the structure of MSE optimal fixed-rate quantizers is well understood. As well, the nearest neighbor condition makes it relatively easy to prove the existence of optimal (minimum distortion) fixed-rate quantizers.

A modified version of the nearest neighbor condition [1, 2] provides a necessary condition for the optimality of an entropy-constrained quantizer that achieves the lower convex hull of the operational distortion-rate function. However, this condition cannot be applied to an arbitrary optimal entropy-constrained quantizer since the operational distortion-rate function may be nonconvex (even for continuous sources [3]). This fact considerably complicates existence and structural problems concerning optimal entropy-constrained quantization. One fundamental question is: Under what conditions (if any) is it sufficient to consider only regular quantizers when searching for an optimal entropy-constrained quantizer? A simple example shows that for discrete distributions regularity can preclude optimality; there exists a discrete scalar source distribution and an interval of entropy constraints for which no quantizer with interval cells is MSE optimal. In this work, among other things, we show that such a pathological example cannot exist when the source distribution is continuous.

An N-level ($N=\infty$ is allowed) *scalar quantizer* Q is characterized by its code points $\{c_i;\, i=1,\dots,N\}$ and partition cells $\{S_i;\, i=1,\dots,N\}$ via the rule $Q(x)=c_i$ iff $x\in S_i$. The distortion of Q in quantizing a real random variable X is measured by the expectation $D(Q)=E[d(X,Q(X))]$, where $d(\cdot,\cdot)$ is a nonnegative distortion measure. Throughout we assume that $d(x,y)=\rho(|x-y|)$, where $\rho:[0,\infty)\to[0,\infty)$ is convex and nondecreasing. The *entropy-constrained rate* of Q is the entropy $H(Q)=H(Q(X))$ of the discrete random variable $Q(X)$. A quantizer Q is called an *optimal* ECSQ for an entropy constraint R if $H(Q)\le R$ and $D(Q)=D_h(R)$, where $D_h(R)$ is the operational distortion-rate function $D_h(R)=\inf\{D(Q): H(Q)\le R\}$.

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II. Regularizing ECSQs

The first result shows that if the distribution of X has no discrete component, then an arbitrary finite-level ECSQ can be "regularized" so that the resulting regular ECSQ has equal or better performance.

Theorem 1 *If X has a nonatomic distribution P_X, then for any finite-point quantizer Q with cells $\{S_1,\dots,S_N\}$ there exists a regular quantizer $\widehat{Q}$ with interval cells $\{\widehat{S}_1,\dots,\widehat{S}_N\}$ such that $P_X(S_i)=P_X(\widehat{S}_i)$, $i=1,\dots,N$, and $D(\widehat{Q})\le D(Q)$.*

Regularizing an arbitrary infinite-level scalar quantizer is more involved, and we need to relax somewhat the strict condition of regularity. A scalar quantizer Q is called *almost regular* if there exists an $S\subset\mathbb{R}$ with $P_X(S)=0$ such that Q is defined on $\mathbb{R}\setminus S$, and every cell of Q is an interval containing the associated code point.

Theorem 2 *If X has a nonatomic distribution, then for any infinite-point quantizer Q there exists an almost regular quantizer $\widehat{Q}$ with the same cell probabilities such that $D(\widehat{Q})\le D(Q)$.*

III. Existence of optimal ECSQs

Theorem 2 makes it possible to show the existence of an optimal ECSQ for any source with a nonatomic distribution.

Theorem 3 *If X has a nonatomic distribution, then for any $R\ge 0$ there exists an almost regular ECSQ that is optimal for the entropy constraint R.*

The following result resolves the problem of the regularity of optimal ECSQs in the special, but important case of the squared error distortion measure and piecewise monotone and piecewise continuous source densities.

Theorem 4 *Let X be a random variable with a piecewise monotone and piecewise continuous probability density function. Assume that $d(x,y)=(x-y)^2$. Then for any entropy constraint $R\ge 0$ there is an optimal ECSQ which is regular.*

Note that all densities commonly used in source modeling are covered by Theorem 4, including the generalized Gaussian, Cauchy, and beta densities.

References

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