Fourier and Hankel Bandlimited Functions

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Abstract

The relationship between the sampling theorem associated with the Hankel transform and the sampling theorem associated with the Fourier transform is examined further. It is shown that a Fourier bandlimited function is also Hankel bandlimited under fairly general conditions.

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1 Introduction

It is known [9] that, for $\nu \geq -\frac{1}{2}$, the Bessel function $J_\nu(z)$ has infinitely many positive zeros, which we denote $\{z_{1,\nu}, z_{2,\nu}, z_{3,\nu}, \dots\}$, and that the functions $\sqrt{x}J_\nu(z_{k,\nu}x)$ form a complete orthogonal set in $L^2(0,1)$. We call the function f "Hankel bandlimited" to the band $[0,1]$ if there is a function F such that

$$f(t) = \int_0^1 J_\nu(tx)F(x)x\,dx. \tag{1}$$

Kramer's sampling theorem [5, 3, 4, 10] shows that, if f is Hankel bandlimited and if $\sqrt{x}F \in L^2(0,1)$, then

$$f(t) = \sum_{k=1}^{\infty} f(z_{k,\nu})S_k(t), \tag{2}$$

where

$$S_k(t) := \frac{2z_{k,\nu}J_\nu(t)}{[(z_{k,\nu})^2 - t^2]J_{\nu+1}(z_{k,\nu})}.$$

The spacing between the sample points $z_{k,\nu}$ is asymptotically equal to π.

For some other theoretical results related to this series, see Genuit and Schöttler [2], Rawn [6], and Higgins [3], and for some numerical experiments using a version of the series, see Stark and Sarna [7].

Some time ago, the author [1] raised the question of the relationship between the ordinary Whittaker-Shannon-Kotelnikov (WSK) sampling theorem [10] and this sampling theorem (and also others derived from Kramer's theorem). Using the Poisson integral representation [9, p.48]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{izy}(1-y^2)^{\nu-\frac{1}{2}}\, dy, \tag{3}$$

it was shown [1] that any f given by (1) is also bandlimited in the Fourier sense. More precisely, substitution of (3) in (1) and a little manipulation shows that

$$f(t) = t^\nu \int_{-1}^{1} H(\omega) e^{i\omega t}\, d\omega, \tag{4}$$

where

$$H(\omega) := \frac{1}{2^\nu \Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_{|\omega|}^{1} (x^2-\omega^2)^{\nu-\frac{1}{2}} x^{-\nu+1} F(x)\, dx. \tag{5}$$

The representation (3) is valid for $\nu > -\frac{1}{2}$. The calculation leading from (1) to (4) and (5) is easily justified using Fubini's theorem for $\nu > -\frac{1}{2}$ when $\sqrt{x}F(x) \in L^2(0,1)$, or even under the less restrictive condition that $\sqrt{x}F(x) \in L^1(0,1)$.

Genuit and Schöttler [2] have shown that H as defined by (5) is in $L^2(0,1)$ if $\sqrt{x}F \in L^2(0,1)$. From this there follows the WSK sampling expansion

$$t^{-\nu} f(t) = \sum_{n=-\infty}^{\infty} (n\pi)^{-\nu} f(n\pi) \frac{\sin(t-n\pi)}{t-n\pi},$$

with the same asymptotic spacing of sample points. (From (1), or (4) and (5), it follows that $t^{-\nu}f(t)$ is an even entire function of t and so is well-defined for all values of t.)

For $\nu = -\frac{1}{2}$, the relation

$$J_{-\frac{1}{2}}(tx) = \sqrt{\frac{2}{\pi t x}} \cos tx$$

shows that

$$f(t) = t^{-1/2} \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{x} F(x) \cos xt\, dx,$$

so that f has a WSK sampling expansion with sample point spacing equal to π in this case too, provided that $\sqrt{x}F \in L^2(0,1)$. Indeed, in this case, the Fourier and Hankel integrals are really the same.

Thus it is known that Hankel bandlimited functions are Fourier bandlimited for $\nu \geq -\frac{1}{2}$. In Sec. 3 we examine the reverse question: Are Fourier bandlimited functions also Hankel bandlimited? Given f defined by (4), is there a function F such that (1) holds? As part of a somewhat differently oriented investigation, Rawn [6] answered this question in the affirmative for $\nu = 0$ (see Rawn's eq.(8)–(11)) by reducing (5) to an Abel integral equation for the unknown function F.[1] We show that, under reasonable smoothness assumptions on H, it is possible for all $\nu > -\frac{1}{2}$ to find F satisfying (5) and thus (1). Our results are slightly more general than Rawn's for $\nu = 0$. In Sec. 4 it is shown by examples that $H \in L^2(0,1)$ does not imply that $\sqrt{x}F \in L^2(0,1)$. Consequently, the question of the existence of a sampling series (2) is not settled even when F is found.

2 Conversion to an Abel integral equation

We now regard (5) as an integral equation for the unknown function F, given the function H. Clearly, for a solution to exist, we must have $H(\omega) = H(-\omega)$. This means that $|\omega|$ may be replaced by ω and attention may be restricted to the interval $[0,1]$. Next, let $G(x) = x^{-\nu}F(x)$ and make the substitutions $z = 1 - \omega^2$ and $y = 1 - x^2$ in (5). Then

$$H(\sqrt{1-z}) = \frac{1}{2^{\nu+1}\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^z (z-y)^{\nu-\frac{1}{2}} G(\sqrt{1-y})\, dy.$$

Defining

$$h(z) := H(\sqrt{1-z}), \quad g(y) := \frac{G(\sqrt{1-y})}{2^{\nu+1}\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}, \quad \alpha := \frac{1}{2} - \nu, \tag{6}$$

we have

$$h(z) = \int_0^z \frac{g(y)}{(z-y)^\alpha}\, dy. \tag{7}$$

When $-\frac{1}{2} < \nu < \frac{1}{2}$, or $0 < \alpha < 1$, this is an Abel integral equation. Tonelli [8] has shown that, for $h \in L^1(0,1)$ and $0 < \alpha < 1$, a necessary and sufficient condition for the existence of a unique solution of (7) is that the function

$$q(y) := \int_0^y \frac{h(z)}{(y-z)^{1-\alpha}}\, dz$$

be absolutely continuous on $[0,1]$ and satisfy $q(0) = 0$. Under these conditions, the solution of (7) is

$$g(y) = \frac{\sin \pi\alpha}{\pi} q'(y),$$

as can be deduced by a simple Laplace transform calculation.

[1] Rawn [6] also claimed to have found F for $\nu = 2K$, where K is any integer, but there appears to be an error beginning with the equation on p.555, left column, ninth line from the bottom, which relates $H_n(w)$ and $h_{n,1}$.

3 When Fourier bandlimited functions are Hankel bandlimited

The cases $-\frac{1}{2} < \nu < \frac{1}{2}$ and $\nu = \frac{1}{2}$ will be dealt with first. Other cases will be dealt with by differentiating often enough to bring the appropriate parameter into this range.

3.1 The case $-\frac{1}{2} < \nu < \frac{1}{2}$

For $-\frac{1}{2} < \nu < \frac{1}{2}$, we can apply the results of Sec. 2 directly after making the changes of variables used there. Define the function

$$\phi(x) := 2\int_x^1 \frac{y\, H(y)}{(y^2 - x^2)^{\nu+\frac{1}{2}}} dy. \tag{8}$$

If H is absolutely continuous on $[0,1]$, then integration by parts shows that

$$(\frac{1}{2} - \nu)\phi(x) = H(1)\left(1 - x^2\right)^{\frac{1}{2}-\nu} - \int_x^1 \left(y^2 - x^2\right)^{\frac{1}{2}-\ \nu} H'(y)\, dy. \tag{9}$$

Note that for this H, ϕ is absolutely continuous on $[0,1]$ and that $\phi(1) = 0$. (If H is absolutely continuous except for a finite number of finite jumps, a similar expression for ϕ can be derived, with extra terms of the form $j(a)\left(a^2 - x^2\right)^{\frac{1}{2}-\nu}$ where $j(a)$ is the jump of H at a.)

Theorem 1 *Let $-\frac{1}{2} < \nu < \frac{1}{2}$ and let H be an even function in $L^1(-1,1)$. Let H be such that the function ϕ defined by* (8) *is absolutely continuous on* $[0,1]$ *and satisfies* $\phi(1) = 0$. *Define*

$$F(x) := -\frac{2^\nu\sqrt{\pi}}{\Gamma(\frac{1}{2} - \nu)} x^{\nu-1}\phi'(x). \tag{10}$$

If f is defined by (4) *then F and f satisfy* (1).

Proof: Tonelli's theorem, which is outlined in Sec. 2, shows that F satisfies (5). Substitution of the expression on the right of (5) for $H(\omega)$ in (4) and a reversal of the steps leading from (1) to (4) shows that (1) is satisfied. The absolute continuity of ϕ ensures that $x^{1-\nu}F(x) \in L^1(0,1)$. This, in turn, allows the change in order of integration that is needed in order to pass from (4) to (1) via (3). □

If ϕ is given by (9), then F has the form

$$F(x) = \frac{2^{\nu+1}\sqrt{\pi}\, x^\nu}{\Gamma(\frac{1}{2} - \nu)} \left[\frac{H(1)}{(1 - x^2)^{\nu+\frac{1}{2}}} - \int_x^1 \frac{H'(u)\, du}{(u^2 - x^2)^{\nu+\frac{1}{2}}}\right]. \tag{11}$$

For $\nu = 0$, this is equivalent to Rawn's [6] expression. An example is given in Sec. 4 for which (11) is inapplicable while (10) can be used.

3.2 The case $\nu = \frac{1}{2}$

If $\nu = \frac{1}{2}$ and $\omega > 0$ (5) takes the form

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\omega}^{1} \sqrt{x} F(x)\, dx.$$

Differentiation gives F directly.

Theorem 2 *Let H be even and differentiable on $[-1,1]$ and let $H(1) = 0$. Let*

$$F(x) := -\sqrt{\frac{2\pi}{x}} H'(x). \tag{12}$$

If f is defined by (4) for $\nu = \frac{1}{2}$ then F and f satisfy (1) for $\nu = \frac{1}{2}$.

The verification of this theorem follows easily from substitution of the expression

$$J_{\frac{1}{2}}(tx) = \sqrt{\frac{2}{\pi t x}} \sin tx$$

in (1) and an integration by parts.

3.3 The case $\nu > \frac{1}{2}$

For $\nu > \frac{1}{2}$ we differentiate (5) often enough to bring the exponent of $(x^2 - \omega^2)$ within the range $(-\frac{1}{2}, \frac{1}{2}]$. Let k be the positive integer that satisfies $k - \frac{1}{2} < \nu \le k + \frac{1}{2}$. In order to determine suitable conditions to impose on H, we examine (5). For $\nu > \frac{1}{2}$,

$$\left(x^2 - \omega^2\right)^{\nu - \frac{1}{2}} x^{\frac{1}{2} - \nu} \le \left(1 - \omega^2\right)^{\nu - \frac{1}{2}} \omega^{\frac{1}{2} - \nu} \qquad \text{for } \omega \le x \le 1.$$

Therefore,

$$\left| \int_{\omega}^{1} (x^2 - \omega^2)^{\nu - \frac{1}{2}} x^{-\nu + 1} F(x)\, dx \right| \le \left(1 - \omega^2\right)^{\nu - \frac{1}{2}} \omega^{\frac{1}{2} - \nu} \int_{0}^{1} \sqrt{x}\, |F(x)|\, dx.$$

Whenever the integral on the right is finite and $\nu > \frac{1}{2}$, this shows that $H(1) = 0$. In fact, a similar argument shows that when $\sqrt{x}F(x) \in L^1(0,1)$, $H(\omega)$ has $k - 1$ continuous derivatives, all of which vanish at $\omega = 1$ and a k-th derivative, which is not necessarily continuous.

Define

$$H_0(\omega) := H(\omega), \quad H_n(\omega) := -\omega^{-1} H'_{n-1}(\omega) \text{ for } n = 1, 2, \ldots, k. \tag{13}$$

Then differentiation of (5) gives

$$H_k(\omega) = \frac{2^{k-\nu}}{\Gamma(\nu + \frac{1}{2} - k)\sqrt{\pi}} \int_{\omega}^{1} \frac{x^{k-\nu+1}}{(x^2 - \omega^2)^{k - \nu + \frac{1}{2}}} \left(x^{-k} F(x)\right) dx. \tag{14}$$

This is the same as (5) with H replaced by H_k, ν replaced by $\nu - k$, and $F(x)$ replaced by $x^{-k}F(x)$. Thus we can apply the results of Section 3.1. To do this it is convenient to define

$$\phi_k(x) := 2\int_x^1 \frac{y\,H_k(y)}{(y^2-x^2)^{\nu-k+\frac{1}{2}}}dy. \tag{15}$$

Theorem 3 *Let $\nu > \frac{1}{2}$ and let $\nu \neq n + \frac{1}{2}$ for any integer n. Let k be the integer satisfying $k - \frac{1}{2} < \nu < k + \frac{1}{2}$. Let H be even and k times differentiable on $[-1,1]$. Let H and its first $k-1$ derivatives be continuous on $[-1,1]$. Let $H(1) = H'(1) = \cdots = H^{(k-1)}(1) = 0$. Let H_k be such that the function ϕ_k defined by (15) is absolutely continuous on $[0,1]$ and satisfies $\phi_k(1) = 0$. Define*

$$F(x) := -\frac{2^{\nu-k}\sqrt{\pi}}{\Gamma(\frac{1}{2}-\nu+k)}x^{\nu-1}\phi_k'(x). \tag{16}$$

If f is defined by (4) then F and f satisfy (1).

Proof: As in the the proof of Theorem 1, F satisfies (14). Repeated integration of (14), using (13) and the vanishing of H and its first $k-1$ derivatives shows that (5) is satisfied. As in the proof of Theorem 1, this shows that F and f satisfy (1). □

Theorem 4 *Let $\nu = k + \frac{1}{2}$. Let H be even and $k+1$ times differentiable on $[-1,1]$. Let H and its first k derivatives be continuous and let $H^{(k+1)}$ be integrable on $[-1,1]$. Let $H(1) = H'(1) = \cdots = H^{(k)}(1) = 0$. Let H_k be given by (13) and define*

$$F(x) := -\sqrt{2\pi}x^{k-\frac{1}{2}}H_k'(x). \tag{17}$$

If f is defined by (4) then F and f satisfy (1).

The proof follows from substitution of this expression for F in (1), repeated integrations by parts, and use of the differentiation formula

$$\frac{d}{du}\left[u^\mu J_\mu(tu)\right] = tu^\mu J_{\mu-1}(tu).$$

4 Integrability conditions

While the theorems of Sec. 3 yield a function F satisfying (1) for each H satisfying suitable conditions, the theorems do not establish that $\sqrt{x}F \in L^2(0,1)$. The square integrability is a sufficient condition for Kramer's [5] derivation of the sampling theorem, since his proof uses the Cauchy-Schwarz inequality. Thus, before using F as derived from Sec. 3 in a sampling theorem, it may be necessary to check whether $\sqrt{x}F \in L^2(0,1)$. It should be noted that, while $\sqrt{x}F \in L^2(0,1)$ is a sufficient condition for (2) to hold, it is not known to be necessary. For the case of the WKS sampling theorem, many conditions other than $H \in L^2(-1,1)$ are known. See, for example, Higgins [4, Theorems 6.13, 6.16, 6.21].

One fairly simple sufficient set of conditions for $\sqrt{x}F \in L^2(0,1)$, which does not seem unduly restrictive for applications, is given for $-\frac{1}{2} < \nu < \frac{1}{2}$ in the next theorem. Higher values of ν can be dealt with as in Sec. 3.3.

Theorem 5 *Let $-\frac{1}{2} < \nu < \frac{1}{2}$. Let H be continuous and have a bounded derivative on $[-1,1]$. Let $H(\pm 1) = 0$. Then $\sqrt{x}F(x)$ is bounded on $[0,1]$.*

Proof. From (11), the condition $H(1) = 0$, and the boundedness of $|H'|$,

$$|F(x)| \le K x^\nu \int_x^1 (u^2 - x^2)^{-\nu-\frac{1}{2}} du,$$

where K is a constant. Now $(u+x)^{-\nu-\frac{1}{2}} \le (2x)^{-\nu-\frac{1}{2}}$. Thus

$$\sqrt{x}|F(x)| \le 2^{-\nu-\frac{1}{2}} K \int_x^1 (u-x)^{-\nu-\frac{1}{2}} du = \frac{2^{\frac{1}{2}-\nu}K}{1-2\nu}(1-x)^{\frac{1}{2}-\nu},$$

so that $\sqrt{x}|F(x)|$ is bounded on $[0,1]$. $\square$

To illustrate the difficulty that may arise, consider the case $H(\omega) = 1$ for $-1 \le \omega \le 1$ and zero otherwise. From (11),

$$F(x) = \frac{2^{\nu+1}\sqrt{\pi}\, x^\nu}{\Gamma(\frac{1}{2}-\nu)\,(1-x^2)^{\nu+\frac{1}{2}}} \qquad \text{for } -\frac{1}{2} < \nu < \frac{1}{2}.$$

Substitution in (1) yields

$$f(t) = 2\, t^{\nu-1} \sin t,$$

in agreement with (4). However, it is easily checked that $\int_0^1 x|F(x)|^2\, dx$ (and, correspondingly, $\int_0^\infty t|f(t)|^2\, dt$) converges for $-\frac{1}{2} < \nu < 0$ and diverges for $0 \le \nu < \frac{1}{2}$. Consequently, for this example, the Kramer sampling theorem (2) is established for $-\frac{1}{2} < \nu < 0$, but not for $0 \le \nu < \frac{1}{2}$.

Another, more complicated, example is provided by

$$H(\omega) = \left(1-\omega^2\right)^{\mu-\frac{1}{2}}, \qquad -1 \le \omega \le 1, \qquad \mu > -\frac{1}{2}.$$

From (4),

$$f(t) = 2^\mu \sqrt{\pi}\,\Gamma(\mu + \frac{1}{2})\, t^{\nu-\mu} J_\mu(t).$$

For $\mu < \frac{1}{2}$, (11) cannot be used because $H(1)$ is undefined. However, by (8),

$$\phi(x) = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2}-\nu)}{\Gamma(\mu-\nu+1)}\left(1-x^2\right)^{\mu-\nu}.$$

The conditions of Theorem 1 are satisfied if $-\frac{1}{2} < \nu < \mu$ and we find that

$$F(x) = 2^{\nu+1}\sqrt{\pi}\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu-\nu)}x^{\nu}\left(1-x^2\right)^{\mu-\nu-1}.$$

Under the conditions $-\frac{1}{2} < \nu < \mu$, $H \in L^1(0,1)$ and $\sqrt{x}F \in L^1(0,1)$, but $H \notin L^2(0,1)$ unless $\mu > 0$ and $\sqrt{x}F \notin L^2(0,1)$ unless $\mu - \nu > \frac{1}{2}$.

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