Tight Error Bounds for Non-Uniform Signaling over AWGN Channels*

Hongyan Kuai, *Fady Alajaji* and *Glen Takahara*

Abstract

We consider a Bonferroni-type lower bound due to Kounias on the probability of a finite union. The bound is expressed in terms of only the individual and pairwise event probabilities; however, it suffers from requiring an exponentially complex search for its direct implementation. We address this problem by presenting a practical algorithm for its evaluation. This bound is applied together with two other bounds, a recent lower bound (the KAT bound) and a greedy algorithm implementation of an upper bound due to Hunter, to examine the symbol error (P_s) and bit error (P_b) probabilities of an uncoded communication system used in conjunction with M-ary PSK/QAM modulations and maximum a posteriori (MAP) decoding over additive white Gaussian noise (AWGN) channels. It is shown that the bounds – which can be efficiently computed – provide an excellent estimate of the error probabilities over the entire range of the signal-to-noise ratio E_b/N_0. The new algorithmic bound and the greedy bound are particularly impressive as they agree with the simulation results even during very severe channel conditions.

Index terms – Probability of a union, lower bound, upper bound, bit error rate, symbol error rate, PSK and QAM modulations, MAP and ML decoding, AWGN channels.

---

* This work was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada. The authors are with Mathematics and Engineering, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6, Canada. Parts of this paper were presented at the Eighth Communication Theory Mini-Conference, ICC'99, Vancouver, June 1999.

1 Introduction

We consider a Bonferroni-type [12] lower bound on the probability $P(A_1 \cup \cdots \cup A_N)$ of a finite union of N events. This bound – which is due to Kounias [16] – is expressed in terms of only the individual ($P(A_i)$'s) and pairwise ($P(A_i \cap A_j)$'s) event probabilities. It is obtained as an optimal bound over a family of bounds which is indexed by the set of all subsets of $\{1, 2, \cdots, N\}$; thus a direct search to determine the optimal bound requires a search over $2^N - 1$ non-empty subsets, which is of exponential complexity. We have found in experiments in which various lower bounds were compared for small values of N that the above bound was almost always the tightest. Thus it is of interest to have a practical implementation of this bound, or near optimal instance of it.

In this paper, we provide a novel stepwise algorithm which implements a suboptimal version of the Kounias lower bound; the complexity of this algorithm is empirically observed (in our applications) to be usually linear in N and at worst $O(N^2)$. It is worth mentioning that this proposed algorithmic bound, along with a recent lower bound established by the authors in [19] (the KAT bound) and a greedy algorithm [1] implementation of an upper bound due to Hunter [14], provide new and computationally efficient techniques for the error analysis of a variety of communication systems which has been traditionally handled using the union upper bound and variations of it.

One important communication system for which exact error probabilities are not available is when a non-uniformly distributed Bernoulli source is transmitted via coherent M-ary Phase-Shift Keying (PSK) and Quadrature Amplitude Modulation (QAM) signaling with Gray code bit mapping over additive white Gaussian noise (AWGN) channels. We next investigate the application of the two algorithmic bounds (the stepwise algorithm lower bound and the greedy algorithm implementation of the Hunter upper bound) and the KAT lower bound to the probabilities of symbol error P_s and bit error P_b for this system. The justification for the non-uniformity assumption of the source stems from the fact that in many practical image and speech compression techniques, after some transformation, the transform coefficients are turned into bit streams (binary source). Due to the suboptimality of the compression scheme, the bit stream often exhibits a certain amount of redundancy [3, 6, 13, 29]. This embedded residual redundancy can be characterized by modeling the bitstream as an i.i.d. non-uniform (Bernoulli) process or as a Markov process [2, 3, 6, 17, 18].

When a non-uniform M-ary signal s_u is transmitted over an AWGN channel, maximum a posteriori (MAP) decoding – which is optimal in the sense of minimizing the symbol error probability – is utilized at the receiver. Moreover, the more non-uniform the source is, the better is the system performance under the MAP decoding criterion ([2, 6, 13], see also Section 4). Under the MAP decoding rule, the symbol error probability can be written as

$$P_s = \sum_{u=1}^{M} P(\epsilon|s_u)P(s_u),$$

where $P(\epsilon|s_u)$ is the conditional probability of error given that s_u was sent, and $P(s_u)$ is the probability of transmitting signal s_u. For most signal sets, an exact calculation of the right hand side of the above equation is very difficult due to the complexity and non-symmetrical structure of the decision regions. For this reason, upper and lower bounds on P_s are of importance. Furthermore, calculation of the bit error probability P_b is more complex than computing P_s by an order of magnitude. In this work, the above three bounds are employed to estimate P_s and P_b. The numerical computations of the bounds are shown to be excellent as both the stepwise lower bound and the greedy upper bound often coincide and agree with the simulation results even when the channel is very noisy.

In previous related work [7], Craig provided an elegant method for computing P_s for two-dimensional M-ary signaling over AWGN channels. Although he implemented his technique exclusively for uniformly distributed (equally likely) signaling used with maximum likelihood (ML) decoding, he briefly considered the problem of non-uniform signaling (with MAP decoding) and gave an expression for P_s (cf. equations (13) and (14) in [7]). However, the expression (13) in [7] is not completely correct, as it assumes that each signal is inside its decision region which is not necessarily true for non-uniform signals with MAP decoding. Moreover, this expression is a sum of single integrals, the number and limits of which depend explicitly on the geometry of each decision region. In the case of uniform signaling, these regions are straightforward to determine and are symmetrically shaped. But for non-uniform signaling, these regions are non-symmetric and are shaped differently for each signal. More specifically, each decision region is a (possibly infinite) polygon with up to $M-1$ sides determined by the intersection of $M-1$ half-planes, where the equation of each half-plane boundary is a function of the apriori signal probabilities and the channel noise variance. We have not encountered any efficient and systematic methods for numerically evaluating this expression in the literature. In this paper, we employ a different approach which does not require any knowledge of the geometry of the decision regions. With this method, we are also able to efficiently estimate P_b for non-uniform signaling.

In [4], [26] and [10], Craig's method was extensively generalized to provide a unified framework for the determination of P_s for various uniform modulation schemes over generalized fading channels. In other work, the computation of P_b for uniform M-ary PSK and QAM signaling over AWGN channels was investigated in [21, 15, 20]. In an approach similar in spirit to our paper, Séguin [23] employed an inequality by de Caen's [9] to derive a lower bound on the probability of codeword error of uniform M-ary signals derived from binary linear codes and sent over AWGN channels. We note that de Caen's bound is weaker than the KAT lower bound [19]; thus the results in [23] can be tightened using the bounds presented in this paper.

The rest of this paper is organized as follows. In Section 2, the three general bounds on the probability of a finite union of events are described in detail. In Section 3, the problem of estimating the symbol and bit error probabilities of non-uniform M-ary signals transmitted via PSK/QAM modulation over AWGN channels is investigated. Numerical results and discussions are presented in Section 4. Finally, conclusions are stated in Section 5.

2 Bounds on the Probability of a Union

Consider a finite family of events $A_1, A_2, \ldots, A_N$ in an arbitrary probability space (Ω, P), where N is a fixed positive integer. In this section, we present a practical algorithmic lower bound to $P(\bigcup_{i=1}^N A_i)$ and briefly describe an algorithmic implementation of an upper bound due to Hunter [14] and a recent lower bound established by the authors in [19].

2.1 Stepwise Algorithm Lower Bound

In this subsection, we present an algorithmic lower bound on the probability of a finite union. The bound is a suboptimal instance of the following bound by Kounias [16], which is generally infeasible to determine exactly:

$$P(\bigcup_{i=1}^{N} A_i) \geq \max_{\mathcal{J}} \Big\{ \sum_{i \in \mathcal{J}} P(A_i) - \sum_{\substack{i,j \in \mathcal{J} \\ i<j}} P(A_i \bigcap A_j) \Big\}, \qquad (1)$$

where $\mathcal{J} \subseteq \mathcal{S} \stackrel{\triangle}{=} \{1, 2, \ldots, N\}$. We note that the above maximization requires a search over $2^N - 1$ subsets; this results in a search complexity that increases exponentially with N. We significantly reduce this complexity by proposing a suboptimal algorithm whose search complexity is empirically observed (in our applications) to be usually linear in N and at worst $O(N^2)$. It employs a stepwise searching technique which is commonly used in statistics to search for the optimal subset of predictors in regression problems involving a large number of candidates (see, for example, [28]). In the statistical applications, the objective function is usually based on a goodness of fit measure, such as the Akaike Information Criterion (AIC) [28]. In our problem, the objective is to maximize the term in braces on the RHS of (1).

Let $M(\mathcal{J})$ denote the term in braces on the RHS of (1) for subset $\mathcal{J} \subseteq \mathcal{S}$. The basic idea of the algorithm is to (iteratively) construct a subset $\mathcal{J}$ such that both $M(\mathcal{J} \cup \{i\})$ and $M(\mathcal{J} \backslash \{j\})$ are not greater than $M(\mathcal{J})$ for any index $i \notin \mathcal{J}$ or index $j \in \mathcal{J}$. Specifically, the algorithm is as follows.

Stepwise Algorithm:

- *Step 0*: Set $\mathcal{J}_1 = \emptyset$ and $\mathcal{J}_2 = \mathcal{S}$. Set $M(\mathcal{J}_1) = 0$ and compute $M(\mathcal{J}_2)$.
- *Step 1*:
  - *1a*: Augment $\mathcal{J}_1$ by adding to it (if possible) the index $i \in \mathcal{S} \backslash \mathcal{J}_1$ such that the term in braces on the RHS of (1) is maximized and $M(\mathcal{J}_1 \bigcup \{i\}) > M(\mathcal{J}_1)$.
  - *1b*: Shrink $\mathcal{J}_2$ by deleting from it (if possible) the index $j \in \mathcal{J}_2$ such that the term in braces on the RHS of (1) is maximized and $M(\mathcal{J}_2 \backslash \{j\}) > M(\mathcal{J}_2)$.
- *Step 2*: Repeat Step 1 until the term in braces on the RHS of (1) can no longer be improved in either Steps 1a or 1b.
- *Step 3*:
  - *3a*: Shrink $\mathcal{J}_1$ by deleting (if possible) the index $j \in \mathcal{J}_1$ such that the term in braces on the RHS of (1) is maximized and $M(\mathcal{J}_1 \backslash \{j\}) > M(\mathcal{J}_1)$.
  - *3b*: Augment $\mathcal{J}_2$ by adding to it (if possible) the index $i \in \mathcal{S} \backslash \mathcal{J}_2$ such that the term in braces on the RHS of (1) is maximized and $M(\mathcal{J}_2 \bigcup \{i\}) > M(\mathcal{J}_2)$.
- *Step 4*: Repeat Step 3 until the term in braces on the RHS of (1) can no longer be improved in either Steps 3a or 3b.

- *Step 5*: Repeat Steps 1–4 until the metric M can no longer be improved. Output the stepwise lower bound as $\max\{M(\mathcal{J}_1), M(\mathcal{J}_2)\}$.

The above algorithm actually implements the stepwise procedure twice, once starting with the empty set and once starting with the set of all indices $\mathcal{S}$, and chooses the better of the two results. In our applications we observed that, starting with either set ($\emptyset$ or $\mathcal{S}$), the algorithm goes through a process of successively deleting or adding indices from the current subset at most once, resulting in a linear search complexity.

2.2 Greedy Algorithm Upper Bound

We herein describe an algorithmic upper bound on the probability of a union. As in the case for the lower bound, this upper bound is expressed in terms of the individual and pairwise error probabilities. We first present a family of upper bounds due to Hunter.

Theorem 1 ([14, 12]) Let $A_1, A_2, \cdots, A_N$ be N sets, where $N \geq 3$. Then

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \sum_{(i,j)\in T_0} P(A_i \bigcap A_j), \tag{2}$$

where T_0 is any tree spanning the N indices of the sets $A_1, A_2, \cdots, A_N$ and (i,j) is an edge in T_0.

Since T_0 is arbitrary, the tightest bound in this family is obtained as follows.

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \max_{T_0\in\mathcal{T}} \sum_{(i,j)\in T_0} P(A_i \bigcap A_j), \tag{3}$$

where $\mathcal{T}$ is the set of all spanning trees of the N indices. This set $\mathcal{T}$ has N^{N-2} elements [27]; thus a direct search to compute (3) has an exponential complexity. However, the problem of finding the optimal tree can be formulated as that of determining the maximal spanning tree of a completely connected weighted graph G with nodes corresponding to the event indices and edge weights corresponding to the pairwise event probabilities. Viewed in this way, the Greedy Algorithm [27, 1] (also known as Kruskal's algorithm) for obtaining a minimal spanning tree for a weighted graph can be applied to construct the optimal spanning tree T_0 for (3). The algorithm is described as follows.

Greedy Algorithm: Consider a (fully) connected graph G with N vertices and $\binom{N}{2}$ edges (i, j) of weights $P(A_i \cap A_j)$. Construct a set of edges T_0 as follows.

- *Step 0*: $T_0 = \emptyset$.
- *Step 1*: Add to T_0 the edge with maximum weight.
- *Step 2*: From the remaining edges, add to T_0 the edge with maximum weight subject to the constraint that T_0 remain cycle-free.
- *Step 3*: Repeat Step 2 until T_0 contains $N - 1$ edges.

The complexity of this algorithm is theoretically at worst $O(N^3)$ [1]. However, as for the stepwise bound, we observed in our applications a search complexity that was usually linear. We also note that there exists an improved implementation of this algorithm which has theoretically at worst $O(N^2)$ search complexity [1].

2.3 KAT lower bound

In [19], Kuai, Alajaji and Takahara prove the following bound.

Theorem 2 ([19]) *KAT Lower Bound*

$$\begin{aligned} P\left(\bigcup_{i=1}^{N} A_i\right) \;\geq\; & \sum_{i=1}^{N} \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i) P(A_i)} \right. \\ & \left. + \; \frac{(1-\theta_i) P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right), \end{aligned} \tag{4}$$

where

$$\theta_i \triangleq \frac{\beta_i}{\alpha_i} - \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor,$$

$$\alpha_i \triangleq P(A_i),$$

and

$$\beta_i \triangleq \sum_{j:j \neq i} P(A_i \cap A_j).$$

In [19], it was also demonstrated that de Caen's lower bound [9] and the Dawson-Sankoff lower bound [8] are special cases of the above lower bound. When $\theta_i = 0$ for all i, the KAT lower bound reduces to de Caen's bound; when the ratio $\frac{\beta_i}{\alpha_i}$ is constant, it reduces to Dawson-Sankoff's bound. Furthermore, it is shown in [19] that the KAT bound is always sharper than these two bounds.

3 Non-Uniform Signaling over AWGN Channels

We apply the bounds to estimate the symbol error probability (P_s) and the bit error probability (P_b) of non-uniform M-PSK or M-QAM modulated additive white Gaussian noise (AWGN) channels. The problem formulation is as follows.

We consider a non-uniform independent and identically distributed (i.i.d.) binary source $\{X_i\}$, with distribution $P\{X = 0\} = p$, that is transmitted via M-PSK or M-QAM modulation over an AWGN channel with single-sided power spectral density N_0. The source stream is grouped in blocks of $\log_2 M$ bits which are each subsequently mapped to a modulation signal for transmission over the channel (we assume that M is a power of 2). At the receiver, optimal maximum a posteriori decoding (MAP) is performed in estimating the transmitted M-ary signal. More specifically, if one of M signals $s_1, s_2, \ldots, s_M$ is sent, then the MAP decoder declares that s_k was sent if, for $i = 1, 2, \ldots, M$ and $i \neq k$, the MAP metric of s_k is bigger than the metric of s_i; i.e.,

$$P\left(s_k|r\right) \geq P\left(s_i|r\right),$$

where $r = s_u + n$ is the received signal and n is Gaussian distributed noise with zero-mean and covariance matrix $\frac{N_0}{2} I_2$, where I_2 is the 2×2 identity matrix.

3.1 Average Symbol Error Rate

The probability of symbol decoding error P_s is

$$P_s = \sum_{u=1}^{M} P(\epsilon|s_u)P(s_u) = \sum_{u=1}^{M} P\Big(\bigcup_{i \neq u} \epsilon_{iu}\Big)P(s_u), \tag{5}$$

where $P(\epsilon|s_u)$ is the conditional probability of error given that s_u was sent, and ϵ_{iu} represents the event that s_i has a higher MAP metric than s_u given that s_u was sent. Therefore,

$$P\left(\epsilon_{iu}\right) = Pr\left\{p(r|s_i)p(s_i) \geq p(r|s_u)p(s_u)\right\} \tag{6}$$

and

$$P\left(\epsilon_{iu} \cap \epsilon_{ju}\right) = Pr\{p(r|s_i)p(s_i) \geq p(r|s_u)p(s_u),\ p(r|s_j)p(s_j) \geq p(r|s_u)p(s_u)\}.$$

The right hand side of equation (6) can be easily reduced to

$$Pr\{N_0 \ln p(s_i) - ||n + s_u - s_i||^2 \geq N_0 \ln p(s_u) - ||n||^2\}$$

which is

$$Pr\left\{\frac{\sqrt{2} < n, s_i - s_u >}{\sqrt{N_0}||s_i - s_u||} \geq \frac{||s_i - s_u||}{\sqrt{2N_0}} + \frac{N_0 ln\frac{p(s_u)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_u||}\right\}.$$

Notice that

$$X_i = \frac{\sqrt{2}}{\sqrt{N_0}||s_i - s_u||} < n, s_i - s_u >$$

is Gaussian with 0-mean and unit variance. Then

$$P\left(\epsilon_{iu}\right) = Q\left(\frac{d_{iu}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_i\right)}{2d_{iu}}\right),$$

and similarly

$$P\left(\epsilon_{iu} \cap \epsilon_{ju}\right) = \Psi\left(\rho_{iju}, \frac{d_{iu}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_i\right)}{2d_{iu}}, \frac{d_{ju}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_j\right)}{2d_{ju}}\right), \tag{7}$$

where

$$d_{iu} = ||s_i - s_u||,$$

$$\rho_{iju} = \frac{< s_i - s_u, s_j - s_u >}{||s_i - s_u|| \cdot ||s_j - s_u||},$$

$$Q\left(x\right) = \frac{1}{\sqrt{2\pi}} \int_x^\infty exp\left(-y^2/2\right) dy,$$

$$\Psi\left(\rho_{iju}, a, b\right) = \frac{1}{2\pi\sqrt{1-\rho_{iju}^2}} \int_a^\infty \int_b^\infty e^{-\frac{\left(x^2 - 2\rho_{iju}xy + y^2\right)}{2\left(1-\rho_{iju}^2\right)}} dx\ dy,$$

where $||\cdot||$ is the Euclidean norm and $< \cdot, \cdot >$ denotes the usual dot product.

If we apply the algorithms in Sections 2.1 and 2.2 and Theorem 2 to $P(\bigcup_{i\neq u} \epsilon_{iu})$ and substitute in (5), we obtain two lower bounds and one upper bound on P_s in terms of $P(\epsilon_{iu})$, $P(\epsilon_{iu} \cap \epsilon_{ju})$ and $P(s_u)$.

3.2 Average Bit Error Rate

In many cases, the bit error probability is a more useful performance measure. Under the MAP decoding criterion, the bit error probability P_b can be written as

$$P_b = \sum_{u=1}^{M} P_b(u) P(s_u),$$

where

$$
\begin{aligned}
P_b(u) &= \frac{1}{\log_2 M} E(\texttt{\# of bit errors}|s_u \texttt{ is sent}) \\
&= \frac{1}{\log_2 M} \sum_{j=1}^{M} d(c_j, c_u) A_{j/u}, \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
A_{j/u} &= P(s_j \texttt{ is decoded}|s_u \texttt{ is sent}) \\
&= 1 - P\Big(\bigcup_{i\neq j} \{P(s_i|r) \geq P(s_j|r)|s_u \texttt{ is sent}\}\Big) \\
&= 1 - P\Big(\bigcup_{i\neq j} \epsilon_{iju}\Big), \qquad (9)
\end{aligned}
$$

where $u = 1, \ldots, M$, c_j and c_u are the bit assignments for signals s_j and s_u, respectively, $d(c_j, c_u)$ is the Hamming distance between c_j and c_u, and ϵ_{iju} represents the event that symbol s_i has a higher metric than symbol s_j given that symbol s_u was sent. As in the case for the symbol error probability, $P(\epsilon_{iju})$ and $P(\epsilon_{iju} \cap \epsilon_{kju})$ can be expressed in terms of the $Q(\cdot)$ and $\Psi(\cdot)$ functions, respectively. More precisely, we obtain that

$$
\begin{aligned}
P(\epsilon_{iju}) &= Pr\{p(r|s_i)p(s_i) \geq p(r|s_j)p(s_j)|s_u \texttt{ is sent}\} \\
&= Pr\{N_0 \ln p(s_i) - ||n + s_u - s_i||^2 \geq N_0 \ln p(s_j) - ||n + s_u - s_j||^2\} \\
&= Pr\left\{\frac{\sqrt{2} < n, s_i - s_j >}{\sqrt{N_0}||s_i - s_j||} \geq \frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||} \right. \\
&\qquad \left. - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}\right\} \\
&= Q\left(\frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}\right), \qquad (10)
\end{aligned}
$$

and

$$
P(\epsilon_{iju} \cap \epsilon_{kju}) = Pr\{p(r|s_i)p(s_i) \geq p(r|s_j)p(s_j),\ p(r|s_k)p(s_k) \geq p(r|s_j)p(s_j)\}
$$

$$= \Psi\left(\rho_{ikj}, \frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}, \right.$$
$$\left. \frac{N_0 \ln \frac{p(s_j)}{p(s_k)}}{\sqrt{2N_0}||s_k - s_j||} + \frac{||s_k - s_u||^2}{\sqrt{2N_0}||s_k - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_k - s_j||}\right), \tag{11}$$

where

$$\rho_{ikj} = \frac{< s_i - s_j, s_k - s_j >}{||s_i - s_j|| \cdot ||s_k - s_j||}.$$

Finally, applying the algorithms in Sections 2.1 and 2.2 and Theorem 2 to $P(\bigcup_{i \neq j} \epsilon_{iju})$ in (9) yields two upper bounds and one lower bound on the bit error probability P_b.

4 Numerical Results and Discussion

In this section, we apply the results of Sections 2 and 3 to present bounds for the probability of symbol error P_s and bit error P_b for non-uniform signals with 8-PSK, 16-PSK, 32-PSK and 16-QAM modulation (using Gray mapping) over AWGN channels. We consider $p = 0.5$ and 0.9 for the probability that an input source bit is zero. The results for symbol and bit error are displayed in terms of the SNR E_b/N_0, where E_b is the energy per information bit. The SNR ranges considered are all from SNR=-4 to SNR=h, with h ranging from 4 to 14, so the focus is on noisy channels for which the usual union bound performs poorly. For evaluation of the bounds, we also provide simulation results for $p = 0.9$. We provide the union upper bound in the plots for SER. We do not show the KAT-based upper bound in the BER plots, as its performance in relation to the stepwise bound is similar to its performance in the SER plots.

We first consider the case $p = 0.5$ for which MAP estimation is equivalent to maximum likelihood (ML) estimation. We note that for $p = 0.5$ with M-PSK or M-QAM modulation, exact methods are available for P_s (eg, [22]) and P_b ([21, 15, 20]), so that bounds are unnecessary. However, for M-PSK modulation (M = 8, 16, 32) we have found the lower and upper bounds for both symbol error (P_s) and bit error (P_b) derived in Sections 3.1 and 3.2, based on the stepwise and greedy bounds from Sections 2.1 and 2.2, to coincide, and agree with the exact values for P_s and P_b. Thus, either of these bounds may simply be viewed as alternative methods for computing the exact symbol or bit error rates in these cases. For the 16-QAM case, we plot the symbol and bit errors in Figures 1 and 2. Though not exact for this case, the stepwise and greedy-based bounds are excellent over the entire range of SNR values. We also show in these figures the simulation results for a highly non-uniform source with $p = 0.9$. These curves, when compared with the plots for $p = 0.5$, illustrate the fact that the system performance is considerably improved via the use of MAP decoding.

As mentioned, exact methods are available in the case of uniform signals. The use of these bounds provide equally precise probabilities in a practical sense. Their added value is that the same code can be used to estimate performance for non-uniform signals, which are considered in the following.

For the case of non-uniform signaling, we have found no published results for efficiently calculating P_s or P_b exactly; thus bounds are very useful. In Figures 3 to 6, the bounds for the symbol and bit error rates are plotted for $p = 0.9$. In all cases, the stepwise and greedy bounds provide excellent results, being very close to exact over the entire range of SNR values. The union upper bound for SER, as expected, does not perform well at lower SNR, particularly for the 32-PSK case in Figure 5.

The KAT lower bound for the SER provides very good results, though not as good as the stepwise lower bound for SER, and the accuracy seems to degrade as M increases in the case of M-PSK modulation. It should be remembered, however, that the KAT lower bound is a single formula which, given the single and pairwise event error probabilities, does not involve a search over a large class of bounds, and so may be more practical for applications which involve a very large number of events in the union of interest; for example, in analyzing code or bit error rates in complex coded systems (cf. [23]).

For the applications considered in this paper, all bounds were practical to compute. For the SER curves, for a given value of SNR, and including the time to calculate all the single and pairwise event probabilities, the combined computing time to calculate all bounds (KAT, stepwise, greedy, and union) was less than a second for 8-PSK and 1 or 2 seconds for 32-PSK, on a computer with a 266 MHz MMX Pentium chip running Linux. For the P_b plots (described below), the corresponding times for just the KAT upper bound (computed but not shown in this paper) were less than a second even for 32-PSK, while the times for the stepwise or greedy bounds were about 1 second for 8-PSK and up to 2 minutes for 32-PSK.

We next mention the algorithms used to compute the Q and ψ functions of Sections 3.1 and 3.2. For $Q(\cdot)$, we adopted the algorithm developed by N. Beaulieu [5], which takes only a few lines of code and which we have found to be very efficient and accurate. For the $\Psi(\cdot)$ function, we adopted an algorithm for computing Bivariate Normal tail probabilities written in Fortran by Donnelly [11] (see also [24]). We note that naive computation of the double integral in the $\Psi(\cdot)$ function by simple numerical integration rules results in a very inefficient algorithm that takes over a day of computing to produce, for example, the values for the SER curves for 16-PSK in Figure 4. For nonnegative lower limits, Simon and Divsalar[25] (see also [26]) provide an alternate representation of the Ψ function as the sum of two single integrals over finite ranges. For uniform signaling, the lower limits in (7) or (11) are both nonnegative, but for non-uniform signaling, one or both of these limits can be negative, in which case the Ψ function can be expressed as a linear combination of four instances of the Ψ function with nonnegative lower limits. However, with the application of Simpson's composite numerical integration rule, this method proved much less efficient for a given accuracy as compared with Donnelly's algorithm, which is based on rational function approximations. Thus the latter method was adopted.

Additional results for the computation of the SER and BER bounds when $p = 0.7$ were obtained. The accuracy of the stepwise and greedy bounds were again shown to be excellent. However, we do not include them for the sake of brevity.

5 Conclusion

A stepwise algorithmic lower bound on the probability of a finite union, which is expressed in terms of only the individual and pairwise intersection event probabilities, was presented. This bound is a practical implementation of a bound that is otherwise exponentially complex to evaluate.

This bound along with two other bounds, a recent lower bound due to the authors (the KAT bound) and a greedy algorithm implementation of an upper bound due to Hunter, were applied for the estimation of the symbol and bit error probabilities of non-uniform M-PSK/QAM signaling over very noisy AWGN communication channels used in conjunction with MAP decoding. The stepwise and greedy bounds provided excellent and often exact estimates of P_s and P_b for all examples considered. The KAT bound was also shown to provide very good bounds for most cases considered. The main advantage of the KAT bound is its computational simplicity, which can be an advantage when one wishes to calculate the probability of the union of a very large number of events, or the probability of a small or moderate union many times.

It should be noted that the bounds we have presented are very general and can be applied to any problem involving calculation of the probability of events which can be expressed as a finite union of events whose individual and pairwise intersection probabilities can be efficiently computed. Future work may include the application of these bounds to Rayleigh fading channels and the analysis of channel coded communication systems with non-uniform inputs using MAP decoding.

References

[1] R. K. Ahuja, T. L. Magnanti and J. B. Orlin, *Network Flows,* Prentice Hall, 1993.

[2] F. Alajaji, N. Phamdo, N. Farvardin and T. Fuja, "Detection of Binary Markov Sources over Channels with Additive Markov Noise," *IEEE Trans. Inform. Theory*, Vol. 42, No. 1, pp. 230–239, January. 1996.

[3] F. Alajaji, N. Phamdo and T. Fuja, "Channel Codes That Exploit the Residual Redundancy in CELP-Encoded Speech," *IEEE Trans. Speech and Audio Processing*, Vol. 4, No. 5, pp. 325-336, September 1996.

[4] M.-S. Alouini and A. Goldsmith, "A Unified Approach for Calculating Error Rates of Linearly Modulated Signals over Generalized Fading Channels," *IEEE Trans. Communications*, Vol. 47, No. 9, pp. 1324-1334, September 1999.

[5] N. C. Beaulieu, "A Simple Series for Personal Computer Computation of the Error Function (.)," *IEEE Trans. Communications*, Vol. 37, No. 9, pp. 989-991, September 1989.

[6] P. Burlina and F. Alajaji, "An Error Resilient Scheme for Image Transmission Over Noisy Channels with Memory," *IEEE Trans. Image Processing*, Vol. 7, No. 4, pp. 593-600, April 1998.

[7] J. W. Craig, "A New Simple and Exact Result for Calculating the Probability of Error for Two-Dimensional Signal Constellations," *Proc. IEEE Military Commun. Conf.*, pp. 571-575, McLean, VA, October 1991.

[8] D. A. Dawson and D. Sankoff, "An Inequality for Probabilities," *Proc. Amer. Math. Soc.*, Vol. 18, pp. 504-507, 1967.

[9] D. de Caen, "A Lower Bound on the Probability of a Union," *Discrete Mathematics*, Vol. 169, pp. 217-220, 1997.

[10] X. Dong, N. C. Beaulieu and P. H. Wittke, "Error Probabilities of Two-Dimensional M-ary Signaling in Fading," *IEEE Trans. Communications*, Vol. 47, No. 3, pp. 352-355, March 1999.

[11] T. G. Donnelly, "Bivariate Normal Distribution," Algorithm 462, *Communications ACM*, Vol. 16, p. 636, 1973.

[12] J. Galambos and I. Simonelli, *Bonferroni-Type Inequalities with Applications*, Springer, 1996.

[13] J. Hagenauer, "Joint Source and Channel Coding for Broadcast Applications," *Audio and Video Digital Radio Broadcasting Systems and Techniques,* Elsevier, 1994.

[14] D. Hunter, "An Upper Bound for the Probability of a Union," *Journal of Applied Probability,* Vol. 13, pp. 597-603, 1976.

[15] M. I. Irshid and I. S. Salous, "Bit Error Probability for Coherent M-ary PSK Systems," *IEEE Trans. Communications*, Vol. 39, No. 3, pp. 349-352, March 1991.

[16] E. G. Kounias, "Bounds on the Probability of a Union, with Applications," *Annals of Mathematical Statistics,* Vol. 39, No. 6, pp. 2154-2158, 1968.

[17] J. Kroll and N. Phamdo, "Analysis and Design of Trellis Codes Optimized for a Binary Symmetric Markov Source with Maximum A Posteriori Detection," *IEEE Trans. Inform. Theory*, Vol. 44, No. 6, pp. 2977–2987, November 1998.

[18] J. Kroll and N. Phamdo, "Source-Channel Optimized Trellis Codes for Bitonal Image Transmission over AWGN Channels," *IEEE Trans. Image Processing,* Vol. 8, No. 7, pp. 899-912, July 1999.

[19] H. Kuai, F. Alajaji and G. Takahara, "A Lower Bound on the Probability of a Finite Union of Events," *Discrete Mathematics*, Vol. 215, pp. 147-158, March 2000.

[20] P. J. Lee, "Computation of Bit Error Rate of Coherent M-ary PSK with Gray Code Bit Mapping," *IEEE Trans. Communications*, Vol. 34, No. 5, pp. 488-491, May 1986.

[21] J. Lu, K. B. Letaief, J. C.-I. Chuang and M. L. Liou, "M-PSK and M-QAM BER Computation Using Signal-Space Concepts," *IEEE Trans. Communications*, Vol. 47, No. 2, pp. 181-184, February 1999.

[22] J. G. Proakis, *Digital Communications,* McGraw-Hill, New York, 1995.

[23] G. E. Séguin, "A Lower Bound on the Error Probability for Signals in White Gaussian Noise," *IEEE Trans. Inform. Theory*, Vol. 44, No. 7, pp. 3168-3175, November 1998.

[24] M. J. Schervish, "Multivariate Normal Probabilities with Error Bound," *Applied Statistics,*, Algorithm AS 195, Vol. 33, No. 1, pp 81-94, 1984.

[25] M. K. Simon and D. Divsalar, "Some New Twists to Problems Involving the Gaussian Probability Integral," *IEEE Trans. Communications,* Vol. 46, No. 2, pp. 200-210, February 1998.

[26] M. K. Simon and M. S. Alouini, "A Unified Approach to the Performance Analysis of Digital Communication over Generalized Fading Channels," *Proceedings of the IEEE,* Vol. 86, No. 9, pp. 1860-1877, September 1998.

[27] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics,* Cambridge, 1992.

[28] W. N. Venables and B. D. Ripley, *Modern Applied Statistics with S-Plus, 2nd Ed.*, Springer, New York, 1997.

[29] W. Xu, J. Hagenauer and J. Hollmann, "Joint Source-Channel Decoding Using the Residual Redundancy in Compressed Images," *Proc. International Conference on Communications*, Dallas, TX, June 1996.

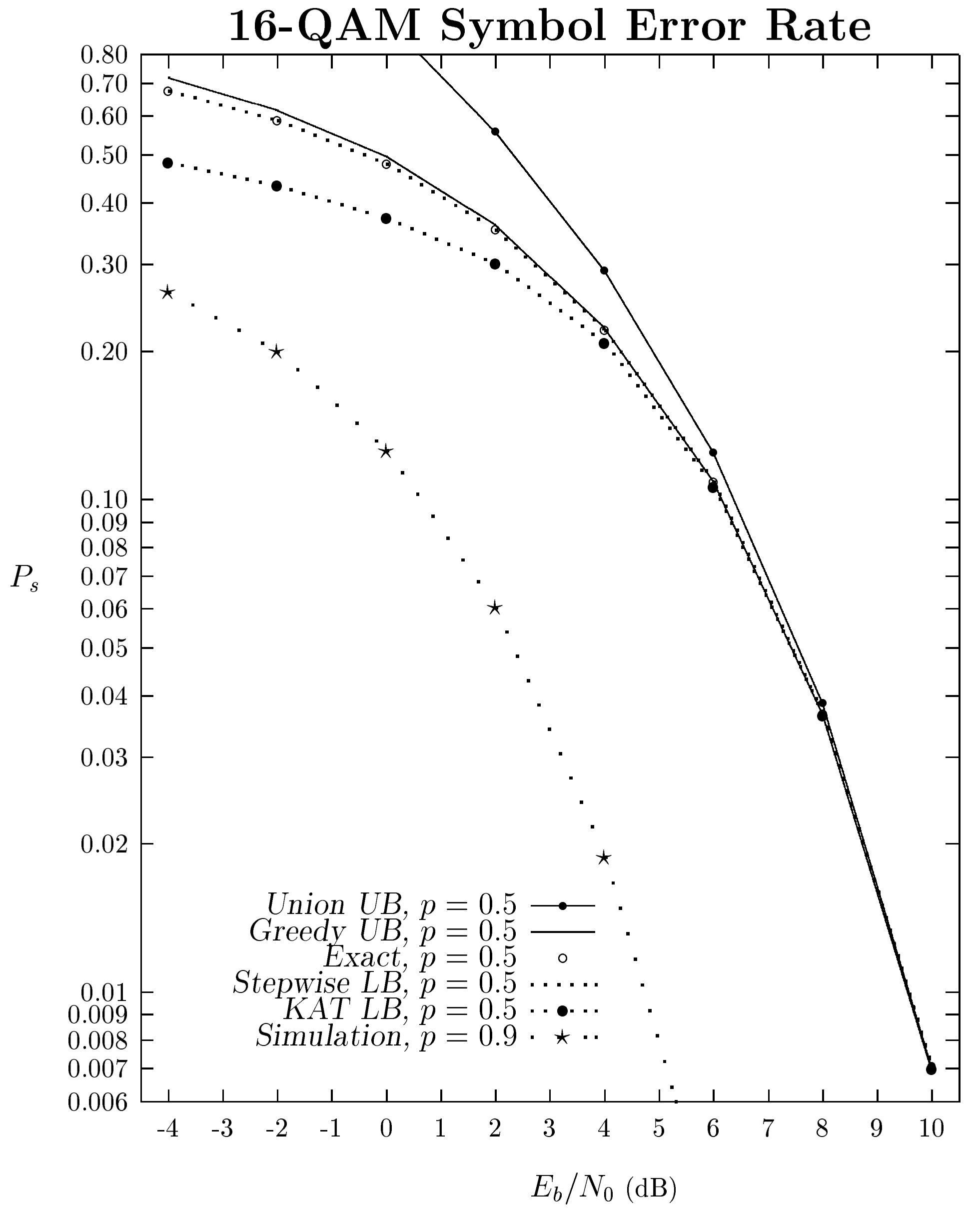

Figure 1: Symbol Error Rate P_s for 16-QAM.

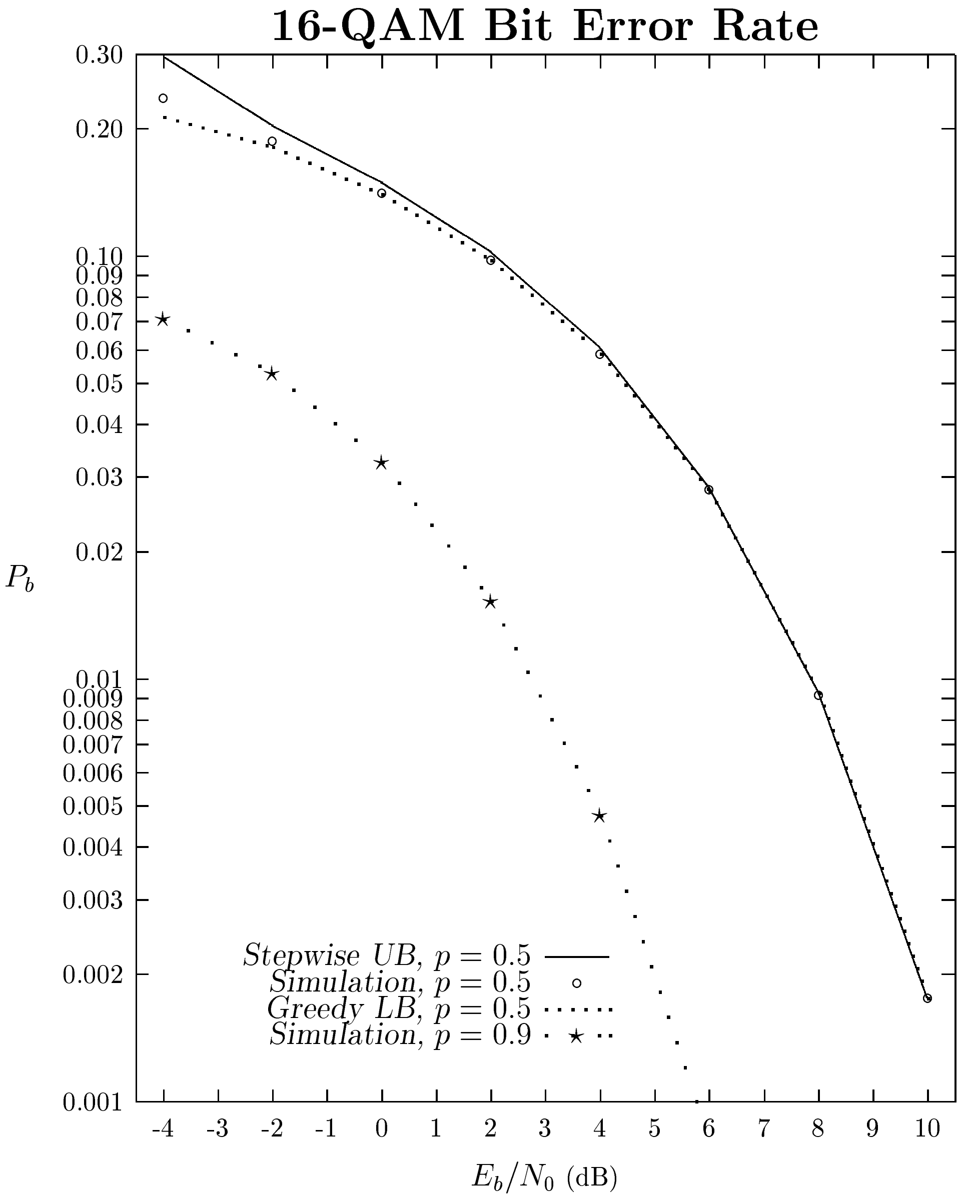

Figure 2: Bit Error Rate P_b for 16-QAM.

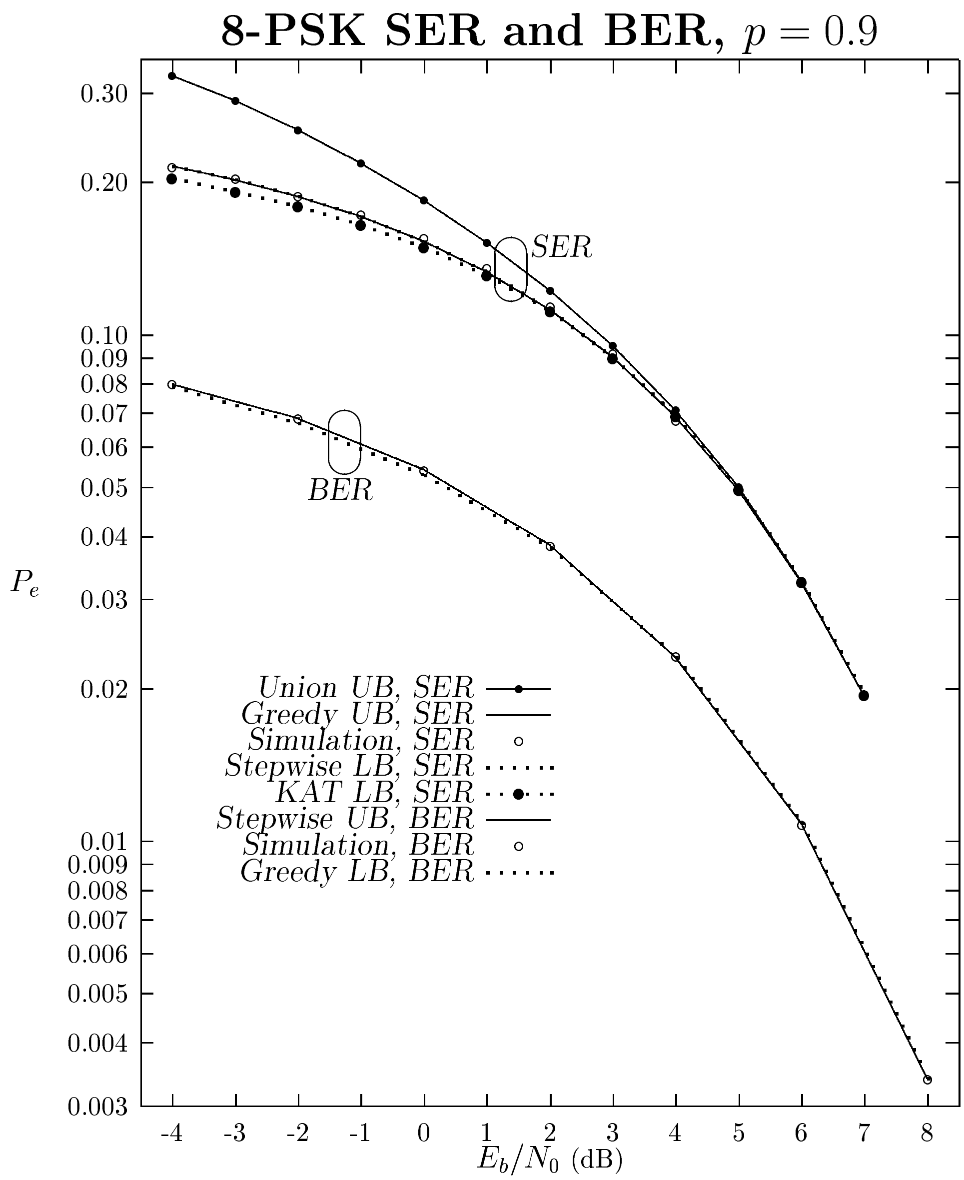

Figure 3: Symbol Error Rate P_s and Bit Error Rate P_b for 8-PSK, $p = 0.9$.

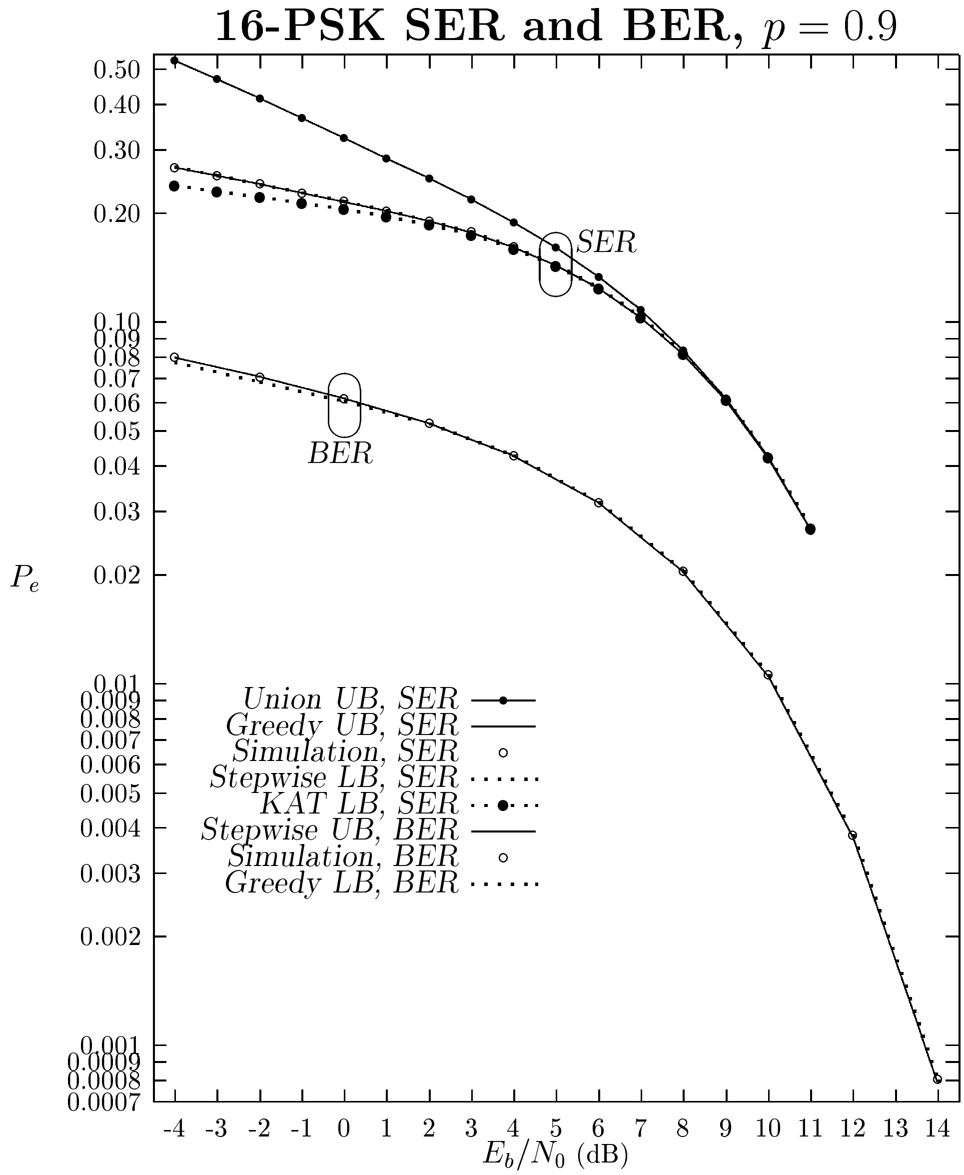

Figure 4: Symbol Error Rate P_s and Bit Error Rate P_b for 16-PSK, $p = 0.9$.

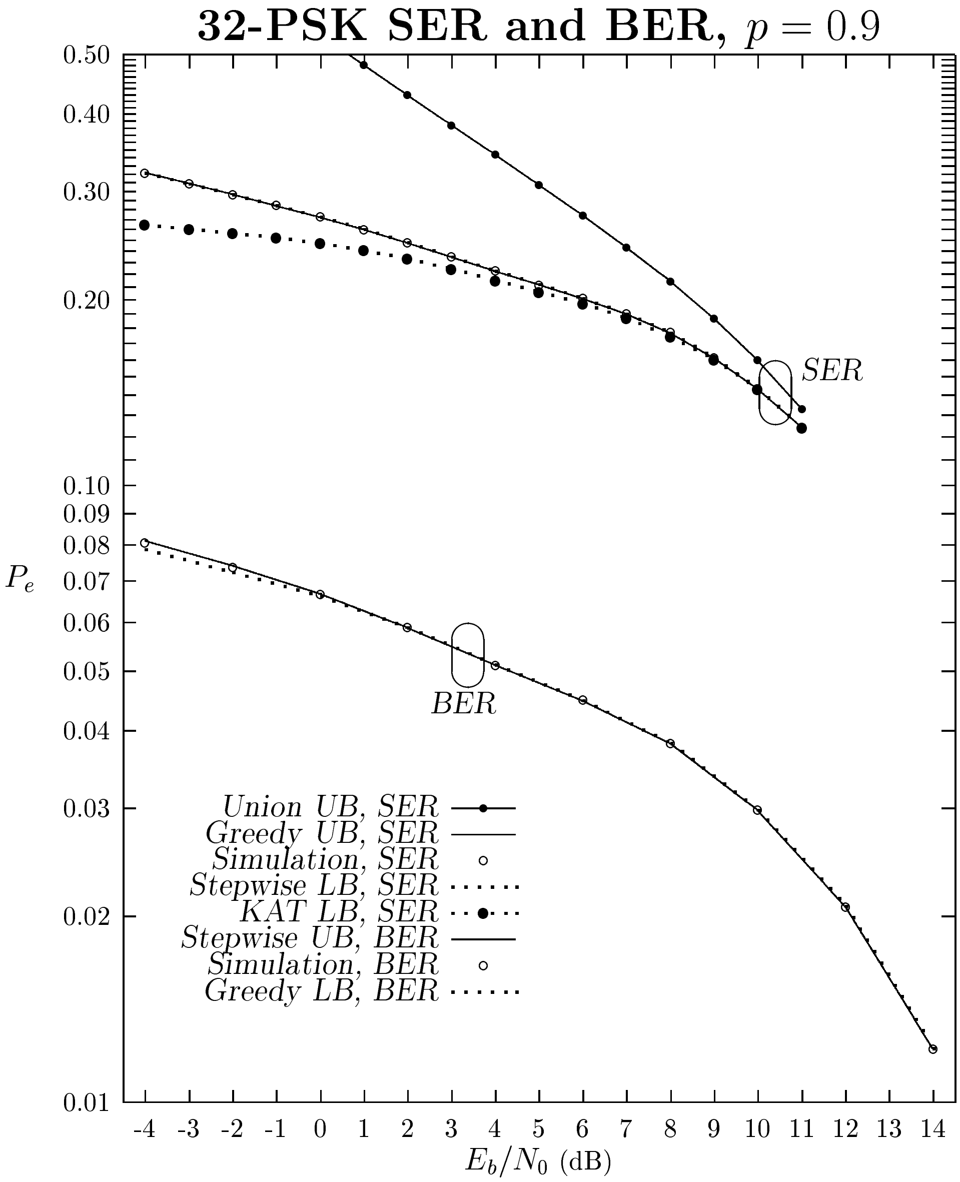

Figure 5: Symbol Error Rate P_s and Bit Error Rate P_b for 32-PSK, $p = 0.9$.

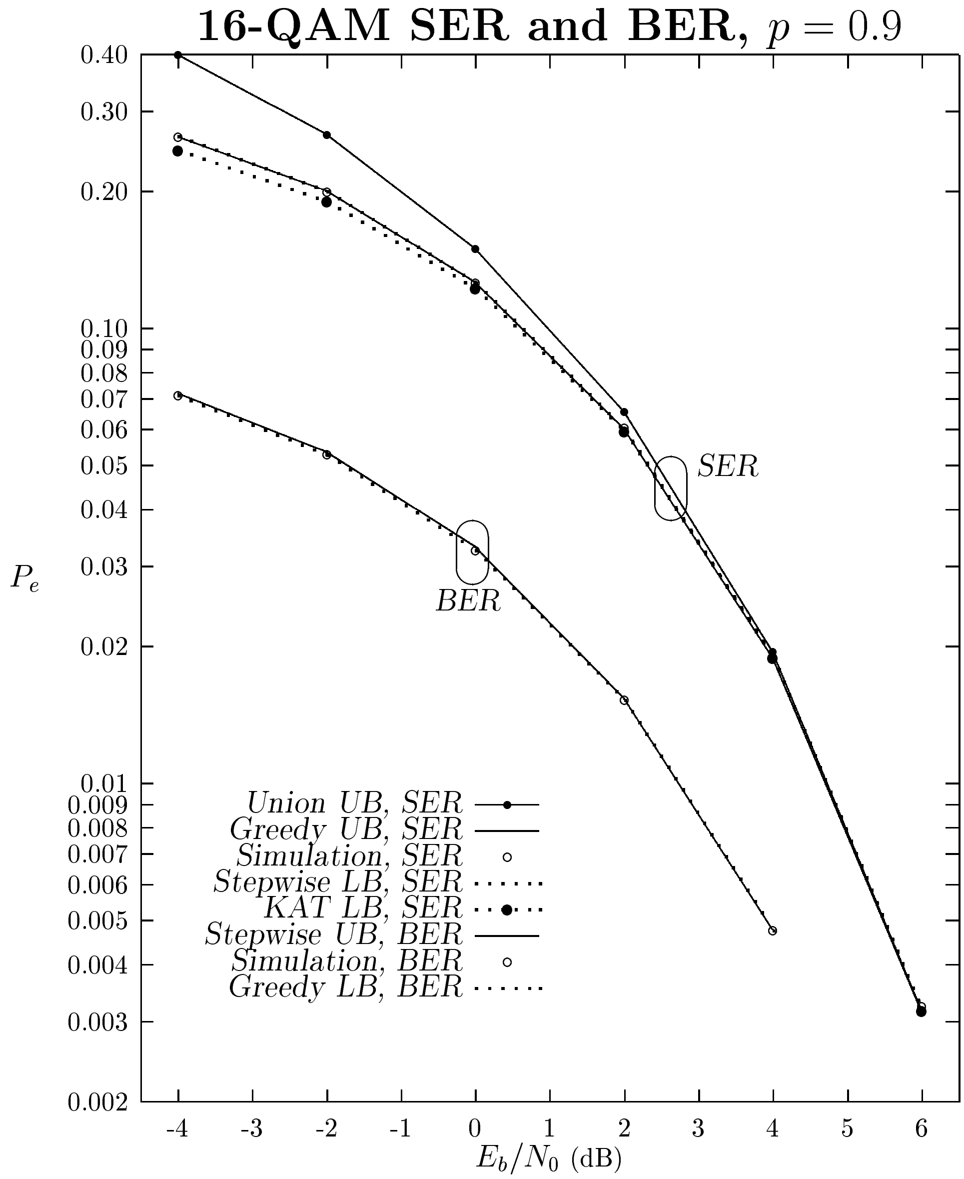

Figure 6: Symbol Error Rate P_s and Bit Error Rate P_b for 16-QAM, $p = 0.9$.