A lower bound on the probability of a finite union of events*

H. Kuai, *F. Alajaji* and *G. Takahara*

Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6, Canada
Email: fady@polya.mast.queensu.ca

Abstract

A new lower bound on the probability $P(A_1 \cup \cdots \cup A_N)$ is established in terms of only the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s. This bound is shown to be always at least as good as two similar lower bounds: one by de Caen (1997) and the other by Dawson and Sankoff (1967). Numerical examples for the computation of this inequality are also provided.

Keywords: Probability of a union, lower bound, linear programming.

1 Main results

Consider a finite family of events $A_1, A_2, \ldots, A_N$ in a finite[1] probability space (Ω, P), where N is a fixed positive integer. For each $x \in \Omega$, let $p(x) \stackrel{\triangle}{=} P(\{x\})$, and let the degree of x – denoted by $deg(x)$ – be the number of A_i's that contain x. Define

$$B_i(k) \stackrel{\triangle}{=} \{x \in A_i : \deg(x) = k\}$$

and

$$a_i(k) \stackrel{\triangle}{=} P(B_i(k)),$$

where $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$. We obtain the following lemma.

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[1]For a general probability space, the problem can be directly reduced to the finite case since there are only finitely many Boolean atoms specified by the A_i's [2].

Lemma 1

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{k=1}^{N} \frac{a_i(k)}{k}.$$

Proof: We know from [2] that

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{x\in A_i} \frac{p(x)}{deg(x)}.$$

But

$$\begin{aligned}
\sum_{x\in A_i} \frac{p(x)}{deg(x)} &= \sum_{k=1}^{N} \sum_{x\in A_i : deg(x)=k} \frac{p(x)}{deg(x)} \\
&= \sum_{k=1}^{N} \sum_{x\in A_i : deg(x)=k} \frac{p(x)}{k} \\
&= \sum_{k=1}^{N} \frac{1}{k} \sum_{x\in B_i(k)} p(x) = \sum_{k=1}^{N} \frac{a_i(k)}{k}.
\end{aligned}$$

This completes the proof. □

This brings us to our main result.

Theorem 1

$$\begin{aligned}
P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \Bigg(& \frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} \\
+ & \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \Bigg), \qquad (1)
\end{aligned}$$

where

$$\theta_i \triangleq \frac{\beta_i}{\alpha_i} - \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor,$$

$$\alpha_i \triangleq \sum_{k=1}^{N} a_i(k) = P(A_i),$$

and

$$\beta_i \triangleq \sum_{k=1}^{N} (k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j).$$

Proof of Theorem 1: From Lemma 1, we can write

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{k=1}^{N}\frac{a_i(k)}{k} = \sum_{i=1}^{N} V_i,$$

where

$$V_i \triangleq \sum_{k=1}^{N}\frac{a_i(k)}{k}.$$

To obtain a lower bound on $P\left(\bigcup_{i=1}^{N} A_i\right)$, we proceed by finding (for each i) the minimum of the linear expression

$$V_i = \sum_{k=1}^{N}\frac{a_i(k)}{k} \tag{2}$$

subject to the constraints:

$$a_i(k) \geq 0, \quad k = 1, \ldots, N, \tag{3}$$

$$\sum_{k=1}^{N} a_i(k) = P(A_i) \triangleq \alpha_i, \tag{4}$$

and

$$\sum_{k=1}^{N}(k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j) \triangleq \beta_i. \tag{5}$$

This constrained minimization problem is solved using the same methodology as proposed in [1].

Step 1: For $r \geq 2$, solving (4) for $a_i(r-1)$ gives:

$$a_i(r-1) = \alpha_i - \sum_{k:k\neq r-1} a_i(k).$$

Substituting the above expression of $a_i(r-1)$ in (5) yields:

$$(r-2)\left[\alpha_i - \sum_{k:k\neq r-1} a_i(k)\right] + \sum_{k:k\neq r-1}(k-1)a_i(k) = \beta_i$$

or

$$\sum_{k:k\neq r-1}[k-(r-1)]a_i(k) = \beta_i - (r-2)\alpha_i.$$

Dividing by r, we get:

$$\frac{1}{r}\sum_{k:k\neq r-1}[k-(r-1)]a_i(k)=\frac{1}{r}[\beta_i-(r-2)\alpha_i] \tag{6}$$

Step 2: Solving (5) for $a_i(r)$ gives:

$$a_i(r)=\frac{1}{r-1}\left[\beta_i-\sum_{k:k\neq r}(k-1)a_i(k)\right].$$

Substituting the expression for $a_i(r)$ in (4) yields:

$$\frac{1}{r-1}\left[\beta_i-\sum_{k:k\neq r}(k-1)a_i(k)\right]+\sum_{k:k\neq r}a_i(k)=\alpha_i$$

or

$$\frac{1}{r-1}\sum_{k=1}^{N}(r-k)a_i(k)=\alpha_i-\frac{\beta_i}{r-1}. \tag{7}$$

Step 3: Solving (6) for $a_i(r)$ and solving (7) for $a_i(r-1)$ respectively yield:

$$\frac{a_i(r)}{r}=\frac{\beta_i}{r}-\frac{(r-2)\alpha_i}{r}-\sum_{k:k\neq r}\frac{k-(r-1)}{r}a_i(k)$$

and

$$\frac{a_i(r-1)}{r-1}=\alpha_i-\frac{\beta_i}{r-1}-\sum_{k:k\neq r-1}\frac{r-k}{r-1}a_i(k).$$

Substituting the above two expressions in (2) yields:

$$V_i-\frac{\beta_i}{r}+\frac{r-2}{r}\alpha_i+\sum_{k:k\neq r}\frac{k-(r-1)}{r}a_i(k)-\alpha_i+\frac{\beta_i}{r-1}+\sum_{k:k\neq r-1}\frac{r-k}{r-1}a_i(k)$$

$$=\sum_{k:k\neq r-1,r}\frac{a_i(k)}{k},$$

or

$$V_i-\frac{2}{r}\alpha_i+\frac{1}{r(r-1)}\beta_i=\sum_{k=1}^{N}\frac{(r-k)(r-k-1)}{r(r-1)}\frac{a_i(k)}{k}\geq 0,$$

where $r \geq 2$.

Step 4: Define

$$f_i(r) \stackrel{\triangle}{=} \frac{2}{r}\alpha_i - \frac{\beta_i}{r(r-1)}. \tag{8}$$

We thus get that

$$V_i \geq f_i(r) \tag{9}$$

where

$$r \geq 2.$$

We want to maximize $f_i(r)$ over $r \geq 2$ in order to render (9) as tight as possible. Setting

$$\begin{cases} f_i(r) - f_i(r-1) \geq 0 \\ f_i(r) - f_i(r+1) \geq 0 \end{cases},$$

we get

$$1 + \frac{\beta_i}{\alpha_i} \leq r \leq 2 + \frac{\beta_i}{\alpha_i}.$$

Since r is an integer, we obtain

$$1 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor \leq r \leq 2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor.$$

Let $r_1' \stackrel{\triangle}{=} 1 + \lfloor \frac{\beta_i}{\alpha_i} \rfloor$, $r_2' \stackrel{\triangle}{=} 2 + \lfloor \frac{\beta_i}{\alpha_i} \rfloor$ and $\theta_i = \frac{\beta_i}{\alpha_i} - \lfloor \frac{\beta_i}{\alpha_i} \rfloor$. So

$$\begin{aligned} f_i(r_1') &= \frac{(1+\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} - \frac{\theta_i {\alpha_i}^2}{\beta_i - \theta_i \alpha_i}, \\ f_i(r_2') &= \frac{\theta_i {\alpha_i}^2}{\beta_i + (2-\theta_i)\alpha_i} + \frac{(1-\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i}. \end{aligned}$$

If r_1' is valid – i.e., if $r_1' \geq 2$ – it is easy to prove that $f_i(r_1') \leq f_i(r_2')$. This is verified as follows:

$$\begin{aligned} f_i(r_2') - f_i(r_1') &= \frac{\theta_i {\alpha_i}^2}{\beta_i + (2-\theta_i)\alpha_i} + \frac{(1-\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} \\ &\quad - \frac{(1+\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} + \frac{\theta_i {\alpha_i}^2}{\beta_i - \theta_i \alpha_i} \\ &= \frac{2\theta_i(\alpha_i)^4}{[\beta_i + (2-\theta_i)\alpha_i][\beta_i + (1-\theta_i)\alpha_i][\beta_i - \theta_i\alpha_i]} \\ &\geq 0. \end{aligned}$$

Substituting $f_i(r_2')$ into (9) and summing over i yields (1). $\square$

2 Comparison with de Caen's bound

In a recent work [2], de Caen also presented a lower bound on $P(\cup_{i=1}^N A_i)$ in terms of the $P(A_i)$'s and the $P(A_i \cap A_j)$'s.

Lemma 2 (de Caen [2]) Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \sum_{i=1}^N \frac{P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j)}. \tag{10}$$

We next demonstrate that our new bound is *always* at least as good as de Caen's bound. More specifically, we prove the following.

Lemma 3 Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$\begin{aligned}\sum_{i=1}^N \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j) + (1-\theta_i)P(A_i)} \right. &+ \left. \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j) - \theta_i P(A_i)}\right) \\ &\geq \sum_{i=1}^N \frac{P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j)},\end{aligned}$$

where

$$\theta_i \stackrel{\triangle}{=} \frac{\beta_i}{\alpha_i} - \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor.$$

In order to prove Lemma 3, we need the following fact.

Lemma 4 Suppose $a > 0$, $b \geq 0$, and $0 \leq x \leq 1$, then

$$\frac{xa^2}{b+(2-x)a} + \frac{(1-x)a^2}{b+(1-x)a} \geq \frac{a^2}{b+a}.$$

Proof: Let

$$f(x) = \frac{a^2 x}{b+(2-x)a} + \frac{a^2(1-x)}{b+(1-x)a}.$$

- For $b = 0$,
$$f(x) = \frac{a^2 x}{(2-x)a} + a \geq \frac{a^2}{b+a} = a.$$
We are done.

- For $b > 0$, $f(x)$ is continuous for all $x \in [0,1]$.
$$f'(x) = \frac{a^2 b + 2a^3}{[b+(2-x)a]^2} - \frac{a^2 b}{[b+(1-x)a]^2}.$$
Let $x_0 \in [0,1]$ such that $f'(x_0) = 0$. Then we get a unique solution
$$x_0 = \frac{2a + b - \sqrt{2ab + b^2}}{2a} \in [0,1],$$
and
$$\begin{aligned} f(x_0) &= \frac{x_0 a^2}{b + (2-x_0)a} + \frac{(1-x_0)a^2}{b+(1-x_0)a} \\ &= 2a + 2b - 2\sqrt{2ab+b^2}. \end{aligned}$$
It is easy to prove that
$$2a + 2b - 2\sqrt{2ab+b^2} > \frac{a^2}{b+a}.$$
Therefore
$$\begin{aligned} \min_{x\in[0,1]} f(x) &= \min\{f(0), f(1), f(x_0)\} \\ &= \min\left\{\frac{a^2}{b+a}, 2a+2b-2\sqrt{2ab+b^2}\right\} = \frac{a^2}{a+b}; \end{aligned}$$
thus,
$$\frac{xa^2}{b+(2-x)a} + \frac{(1-x)a^2}{b+(1-x)a} \geq \frac{a^2}{b+a}$$
for all $x \in [0,1]$. □

Proof of Lemma 3: Letting

$$a = P(A_i),\ b = \sum_{j:j\neq i} P(A_i \cap A_j),\ x = \theta_i = \frac{b}{a} - \left\lfloor \frac{b}{a} \right\rfloor$$

in Lemma 4 gives

$$\frac{\theta_i P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + (2-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + (1-\theta_i)P(A_i)}$$

$$\geq \frac{P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + P(A_i)} = \frac{P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j)}.$$

Therefore, (1) is always stronger than (10); i.e.,

$$\sum_{i=1}^N \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j) + (1-\theta_i)P(A_i)} \quad + \quad \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^N P(A_i \cap A_j) - \theta_i P(A_i)} \right)$$

$$\geq \sum_{i=1}^N \frac{P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + P(A_i)}.$$

Note: de Caen's bound is tight (i.e., (10) is an equality) if and only if the degrees $deg(x)$ are constant on each A_i [2]. Since (1) is stronger than (10), we conclude that the above condition is only a sufficient (but *not* necessary, cf. Example 1 in Section 4) condition for the tightness of (1). □

Observation 1 If $\theta_i = 0 \quad \forall i$, then our bound reduces to de Caen's lower bound. This leads us to also conclude that de Caen's bound is a special case of our bound.

3 Comparison with the Dawson-Sankoff bound

We next prove that our bound is also *always* at least as good as the Dawson-Sankoff bound [1, 3].

Lemma 5 (Dawson-Sankoff [1]) Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \frac{\theta S_1^2}{(2-\theta)S_1 + 2S_2} + \frac{(1-\theta)S_1^2}{(1-\theta)S_1 + 2S_2}, \tag{11}$$

where

$$S_1 \triangleq \sum_{i=1}^{N} P(A_i),$$

$$S_2 \triangleq \sum_{i=1}^{N} \sum_{j=1}^{i-1} P(A_i \cap A_j),$$

and

$$\theta \triangleq \frac{2S_2}{S_1} - \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

Lemma 6 Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then (1) is always sharper than (11); i.e.,

$$\sum_{i=1}^{N} \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right) \geq \frac{\theta S_1^2}{(2-\theta)S_1 + 2S_2} + \frac{(1-\theta)S_1^2}{(1-\theta)S_1 + 2S_2}.$$

Proof of Lemma 6: From the proof of Theorem 1, we know that

$$f_i\left(2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor\right) \geq f_i(r), \quad \forall r \geq 2,$$

where the function $f_i(\cdot)$ is described in (8). In particular, we have that

$$f_i\left(2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor\right) \geq f_i\left(2 + \left\lfloor \frac{\beta}{S_1} \right\rfloor\right),$$

where

$$\beta \triangleq \sum_{i=1}^{N} \sum_{j:j \neq i} P(A_i \cap A_j) = \sum_{i=1}^{N} \beta_i,$$

and

$$S_1 \triangleq \sum_{i=1}^{N} \alpha_i.$$

It can be easily verified that $\beta = 2S_2$, where S_2 is defined in Lemma 5.

Noting that $\sum_i f_i \left(2 + \lfloor \frac{\beta_i}{\alpha_i} \rfloor\right)$ yields our bound (the right-hand side of (1)), and letting $s = 2 + \lfloor \frac{\beta}{S_1} \rfloor$ we get

$$\begin{aligned}
\sum_{i=1}^{N} f_i \left(2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor\right) &\geq \sum_{i=1}^{N} f_i \left(2 + \left\lfloor \frac{\beta}{S_1} \right\rfloor\right) \\
&= \frac{2}{s} \sum_{i=1}^{N} \alpha_i - \frac{1}{s(s-1)} \sum_{i=1}^{N} \beta_i \\
&= \frac{2S_1}{s} - \frac{1}{s(s-1)} \beta \\
&= \frac{2S_1}{s} - \frac{2S_2}{s(s-1)}. \qquad (12)
\end{aligned}$$

The proof is completed by observing that the right-hand side of (12) is indeed equal to the Dawson-Sankoff bound given in (11). □

Observation 2 If $\frac{\beta_i}{\alpha_i} = C \ \ \forall i$, where C is a constant, then $\theta_i = \theta \ \forall i$ and our lower bound reduces to the Dawson-Sankoff lower bound. Thus, Dawson-Sankoff's lower bound is a special case of our bound.

4 Numerical examples

Example 1 We first give an example in which our proposed bound is tight. Let $3|n$ (n is a multiple of 3) and

$$A_i = \begin{cases} \{\frac{3i-1}{2}, \frac{3i+1}{2}\}, & \text{if } i \text{ is odd,} \\ \{\frac{3i}{2} - 1, \frac{3i}{2}\}, & \text{if } i \text{ is even,} \end{cases}$$

where $1 \leq i \leq \frac{2n}{3}$. Then $A_i \cap A_j \neq \phi$ if and only if $\lceil \frac{i}{2} \rceil = \lceil \frac{j}{2} \rceil$. If the points are uniformly distributed with probability $\frac{1}{n}$, then

$$P(A_i) = \frac{2}{n},$$

$$\sum_{j:j\neq i} P(A_i \cap A_j) = \sum_{j\neq i:\lceil \frac{i}{2} \rceil = \lceil \frac{j}{2} \rceil} P(A_i \cap A_j) = \frac{1}{n},$$

and

$$\theta_i = \frac{1}{2}.$$

Clearly

$$P\left(\bigcup_{i=1}^{\frac{2n}{3}} A_i\right) = 1.$$

(1) gives

$$\sum_{i=1}^{\frac{2n}{3}} \left(\frac{\frac{1}{2}(\frac{2}{n})^2}{\frac{3}{n} + \frac{1}{2}\frac{2}{n}} + \frac{\frac{1}{2}(\frac{2}{n})^2}{\frac{3}{n} - \frac{1}{2}\frac{2}{n}} \right) = \sum_{i=1}^{\frac{2n}{3}} \frac{3}{2n} = 1.$$

However (10) gives

$$\sum_{i=1}^{\frac{2n}{3}} \frac{(\frac{2}{n})^2}{\frac{3}{n}} = \sum_{i=1}^{\frac{2n}{3}} \frac{4}{3n} = \frac{8}{9}.$$

Thus, in this case, (1) is stronger than (10).

Example 2 We next consider several systems and compare our bound to the de Caen and Dawson-Sankoff bounds. The different systems are described in Tables 1-4. The lower bounds for each system are computed below.

System	$P(\cup_i A_i)$	de Caen (10)	Dawson (11)	New Bound (1)
I	0.7890	0.7087	0.7007	0.7247
II	0.6740	0.6154	0.6150	0.6227
III	0.7890	0.7048	0.6933	0.7222
IV	0.9689	0.8759	0.8881	0.8911

It can be clearly observed from the above table that the new bound is sharper than the de Caen and the Dawson-Sankoff bounds.

References

[1] D. A. Dawson and D. Sankoff, An inequality for probabilities, Proc. Amer. Math. Soc., 18 (1967) 504-507.

[2] D. de Caen, A lower bound on the probability of a union, Discrete Mathematics, 169 (1997) 217-220.

[3] J. Galambos and I. Simonelli, Bonferroni-type Inequalities with Applications (Springer, 1996).

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	×		×		×	
x_1	0.022		×		×		×
x_2	0.023	×		×		×	
x_3	0.033		×				
x_4	0.034	×				×	×
x_5	0.044		×	×		×	
x_6	0.045		×			×	×
x_7	0.055		×	×	×		×
x_8	0.056	×		×			
x_9	0.066				×	×	
x_{10}	0.067		×		×	×	
x_{11}	0.077		×		×		
x_{12}	0.078	×			×		×
x_{13}	0.088		×				
x_{14}	0.089	×		×		×	×

Table 1: Description of System I with $N = 6$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.023	$\times$		$\times$		$\times$	
x_1	0.034		$\times$		$\times$		
x_2	0.045	$\times$		$\times$		$\times$	
x_3	0.056		$\times$				
x_4	0.067	$\times$				$\times$	$\times$
x_5	0.078		$\times$	$\times$		$\times$	
x_6	0.067		$\times$			$\times$	$\times$
x_7	0.056			$\times$	$\times$		$\times$
x_8	0.045	$\times$		$\times$			
x_9	0.038				$\times$	$\times$	
x_{10}	0.011		$\times$		$\times$	$\times$	
x_{11}	0.022		$\times$				
x_{12}	0.033	$\times$			$\times$		$\times$
x_{13}	0.044		$\times$				
x_{14}	0.055	$\times$		$\times$		$\times$	$\times$

Table 2: Description of System II with $N = 6$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	$\times$		$\times$		$\times$	
x_1	0.022		$\times$		$\times$		
x_2	0.023	$\times$		$\times$		$\times$	
x_3	0.033		$\times$				
x_4	0.034	$\times$				$\times$	$\times$
x_5	0.044		$\times$	$\times$		$\times$	
x_6	0.045		$\times$			$\times$	$\times$
x_7	0.055			$\times$	$\times$		$\times$
x_8	0.056	$\times$		$\times$			
x_9	0.066				$\times$	$\times$	
x_{10}	0.067		$\times$		$\times$	$\times$	
x_{11}	0.077		$\times$				
x_{12}	0.078	$\times$			$\times$		$\times$
x_{13}	0.088		$\times$				
x_{14}	0.089	$\times$		$\times$		$\times$	$\times$

Table 3: Description of System III with $N = 6$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6	A_7
x_0	0.0329			$\times$				
x_1	0.1076	$\times$	$\times$	$\times$				$\times$
x_2	0.0599					$\times$		
x_3	0.1108			$\times$		$\times$		
x_4	0.0420		$\times$					
x_5	0.0055		$\times$	$\times$				$\times$
x_6	0.0508					$\times$	$\times$	$\times$
x_7	0.1142	$\times$				$\times$		
x_8	0.0480						$\times$	$\times$
x_9	0.0235						$\times$	$\times$
x_{10}	0.0676	$\times$	$\times$					$\times$
x_{11}	0.0295		$\times$		$\times$			
x_{12}	0.0441	$\times$		$\times$			$\times$	
x_{13}	0.1265	$\times$			$\times$		$\times$	
x_{14}	0.1058				$\times$	$\times$		$\times$

Table 4: Description of System IV with $N = 7$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.