Mathematical Problems in Error Calculations for Interference

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Abstract— **Several treatments of interference problems are examined with a view to determining whether doubtful mathematical assumptions about the existence of probability density functions affect the conclusions. Most derivations can be recast so that the existence of a density function is not necessary for the conclusions to be valid.**

Keywords—**Interference, probability of error.**

I. Introduction

In THE LAST decade, a number of authors [1], [2], [3], [6], [8], [9], [10], [11] have presented methods for approximating or bounding the probability of error in digital communications systems which are affected by interference and additive noise. Some of these authors assume in their derivations that the interference Z, or $Z + N$, where N is a noise variable, has a probability density function. However, it has been known for some time that this assumption is often not valid [5], [7], [11].

This concern applies particularly to the case where the interference is modelled as a random variable Z where

$$Z = \sum_{k=1}^{\infty} \beta_k X_k, \tag{1}$$

$\{\beta_k\}$ are known constants and $\{X_k\}$ are independent identically distributed random variables which take a finite number of values. It will be assumed that

$$\sum_{k=1}^{\infty} (\beta_k)^2 < \infty. \tag{2}$$

When (2) is true, Kolmogorov's three-series theorem [4,p.317] shows that the series in (1) converges with probability one to a random variable. If β_k satisfies the stronger condition

$$\sum_{k=1}^{\infty} |\beta_k| < \infty, \tag{3}$$

then Z is a bounded random variable.

If there are infinitely many non-zero coefficients in (1) and if the coefficients β_k asymptotically decay exponentially, say $\beta_k = O(r^k)$, then it is possible for the probability distribution function of Z to be continuous but not absolutely continuous. In this situation there is no density function, neither as an ordinary function nor as a combination of delta functions [2], [5], [7], [11]. If the terms in the series (1) are sampled values of the output of a lumped parameter filter, this continuous distribution function with no density function occurs for bandwidths that would quite normally occur in communications receivers [11].

These considerations raise questions about the validity of the conclusions in the papers by Beaulieu [1], Helstrom [6], and Metzger [8], whose method is also adapted for use in [9]. It will be shown here that there is no real difficulty with Helstrom's [6] derivation, and that the work of Beaulieu [1] and Metzger [8] can be recast in terms of distribution functions in such a way that their conclusions are unchanged. Carlisle *et al.* [3] deal with a finite series (1) to which these comments do not apply while [2], [10], [11] explicitly take account of the difficulties mentioned.

This note is not concerned with the relative efficiencies of the methods of calculation in these papers. The only concern is with the use of possibly non-existent density functions in the derivations.

II. Comments on Derivations

In any receiver which makes decisions according to whether a received signal falls above or below a threshold, the error probability depends on values of the cumulative distribution function of $Y = Z + N$ where Z is the interference variable above and N is noise which is assumed to be independent of Z. If we write F_Y, F_Z, and F_N for the distribution functions of Y, Z, and N respectively, then

$$F_Y(y) = \int_{-\infty}^{\infty} F_N(y-z) F_Z\{dz\}. \tag{4}$$

A. Helstrom's calculation

Helstrom [6] has developed a method of calculating error probabilities using numerical quadrature of a Laplace inversion integral. At one point he uses the probability density function $p_Y(y)$ of Y. As shown by Feller [4, p.146, Theorem 4] F_Y is absolutely continuous and thus has a density function whenever N has a density function. This seems to deal with any cases of practical interest. Thus the question of the existence of a density function is not a difficulty for this method.

B. Metzger's calculation

Metzger [8] developed a numerical algorithm for approximately calculating the probability density function of interference and then used it to calculate error probabilities. His initial discussion assumes that (3) holds and that X_k takes the values ± 1 with probabilities $1/2$.The algorithm

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splits the sum in (1) into two parts,

$$Z = \sum_{k=1}^{M} \beta_k X_k + Z_{M+1}. \tag{5}$$

The approximation consists in assigning a uniform distribution to Z_{M+1} on its interval of possible values $(-S_M, S_M)$, where

$$S_M = \sum_{k=M+1}^{\infty} |\beta_k|.$$

Starting with this, a recursive calculation leads to an approximating density function $f_Z^{(M)}(z)$ for Z. (There is a further approximation in [8] to smooth the function, but this is not relevant to this discussion.)

If the input to the receiver is $R = \pm A + Z + N$, where N is Gaussian noise which is independent of the signal and the interference, and $A > 0$ is the signal amplitude, the decision is $+1$ if $R > 0$ and -1 if $R < 0$. The probability of error is $p_e = Pr\{Y \leq -A\} = F_Y(-A)$, where F_Y is given by (4). When the approximate density function $f_Z^{(M)}(z)$ is used, p_e is replaced by $p_e^{(M)} = F_Y^{(M)}(-A)$, where

$$F_Y^{(M)}(y) = \int_{-\infty}^{\infty} F_N(y-z) F_Z^{(M)}\{dz\}$$

and

$$F_Z^{(M)}(u) = \int_{-\infty}^{u} f_Z^{(M)}(z)\, dz.$$

From (5), and the fact that the distribution of Z_{M+1} is approximated by the uniform distribution, the characteristic function corresponding to $F_Z^{(M)}(u)$ is

$$\phi_Z^{(M)}(t) = \left(\prod_{k=1}^{M} \cos \beta_k t\right) \left(\frac{\sin S_M t}{S_M t}\right),$$

from which it follows easily (using the fact that $S_M \to 0$ as $M \to \infty$) that $\phi_Z^{(M)}(t) \;\to\; \phi_Z(t)$ as $M \to \infty$, where $\phi_Z(t)$ is the characteristic function of the random variable Z. Now $\phi_Z(t)$ is a characteristic function and hence is continuous. The continuity theorem [4, p.508] for characteristic functions shows that $F_Z^{(M)}(u) \to F_Z(u)$. From this it follows by the dominated convergence theorem [4, p. 111] that $F_Y^{(M)}(y) \to F_Y(y)$ and finally that $p_e^{(M)} \to p_e$ as $M \to \infty$. That is, for large enough M, the error probability calculated from the approximation is very close to the exact error probability.

In a later section, Metzger [8] replaces the assumption (3) by the weaker assumption (2). The approximation used in this case is to assume that Z_{M+1} has a normal distribution with variance equal to

$$\sigma_M^2 = \sum_{k=M+1}^{\infty} (\beta_k)^2.$$

Since this variance approaches zero as $M \to \infty$, we can repeat the above argument, replacing $(\sin S_M t)/S_M t$ by $e^{-\sigma_M^2 t^2/2}$ and again conclude that, for large enough M, the error probability calculated from the approximation is very close to the exact error probability.

Thus, at least for the purpose of calculating error probability, the possibly invalid assumption that Z has a probability density function does not affect the validity of the final calculation.

C. Beaulieu's calculation

Beaulieu [1] approximates $F_N(y-z)$ by a Fourier series in the integral

$$\int_{-\infty}^{\infty} F_N(y-z) f_Z(z)\, dz$$

and then integrates term by term, where $f_Z(z)$ is the presumed density function of Z. However, if $f_Z(z)$ exists, this integral is equivalent to (4). The only place where $f_Z(z)$ appears is in the integrals of the terms of the series. These integrals are then interpreted as values of the characteristic function of Z. The characteristic function is then calculated as a product using (1), without reference to $f_Z(z)$. The resulting series is just what would be obtained from the use of (4). There are some other approximations in the paper to improve the efficiency of computation, but none of these depends on the existence of $f_Z(z)$. Once again, the assumption that Z has a density function can be omitted without affecting the calculated values of error probability.

III. Conclusion

In the three cases studied, the assumption that Z has a density function can be dropped without changing the final calculation of error probability.

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