Design of Sample Adaptive Product Quantizers for Noisy Channels*

Zahir Raza[†], *Fady Alajaji*[‡] and *Tamás Linder*[‡]

[†]T-Mobile
12920 South-East 38th Street
Bellevue, WA 98006, USA

[‡]Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6, Canada

Abstract

Channel optimized vector quantization (COVQ) has proven to be an effective joint source-channel coding technique that makes the underlying quantizer robust to channel noise. Unfortunately, COVQ retains the high encoding complexity of the standard vector quantizer (VQ) for medium to high quantization dimensions and moderate to good channel conditions. A technique called sample adaptive product quantization (SAPQ) was recently introduced by Kim and Shroff to reduce the complexity of the VQ while achieving comparable distortions. In this work, we generalize the design of SAPQ for the case of memoryless noisy channels by optimizing the quantizer with respect to both source and channel statistics. Numerical results demonstrate that the channel optimized SAPQ (COSAPQ) achieves comparable performance to the COVQ (within 0.2 dB), while maintaining considerably lower encoding complexity (up to half of that of COVQ) and storage requirements. Robustness of the COSAPQ system against channel mismatch is also examined.

Index Terms: Channel-optimized quantization, joint source-channel coding, structurally constrained vector quantization, encoding/storage complexity.

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I Introduction

In recent works [7, 8], Kim and Shroff introduced a constrained vector quantizer structure, which they called the sample adaptive product quantizer (SAPQ), that achieves a comparable performance to the unconstrained vector quantizer (VQ) [6] while maintaining a lower encoding complexity.[1] However, like most data compression schemes that solely remove source redundancy, the compressed source tends to be sensitive to channel noise. The traditional approach uses tandem source-channel coding to achieve reliable transmission of information by separately designing the source and channel codes. It is however known that when there are delay and complexity constraints, it is more advantageous to employ jointly designed or coordinated source and channel codes (e.g., [1, 3, 4, 5, 9, 11, 12, 14]). A VQ-based joint source-channel coding system that exploits both source and channel statistics is called a channel optimized vector quantizer (COVQ). COVQ has received considerable attention due to its improvement in performance over VQ in the presence of channel noise (e.g., [4, 5]). However, COVQ incurs relatively high encoding complexity particularly when the channel conditions are moderate or good. In this work, we study the design of SAPQ for noisy memoryless channels, or channel optimized SAPQ (COSAPQ), in order to find a less complex alternative to COVQ.

The rest of this letter is organized as follows. We introduce our COSAPQ system in Section II, and we establish its necessary conditions for optimal encoding and decoding in Section III. We briefly discuss the system encoding complexity and storage requirements in Section IV. In Section V, we evaluate the performance and complexity of the COSAPQ system and compare it with the COVQ and other systems. We also present performance results under mismatch channel conditions. Finally, we provide conclusions in Section VI.

II COSAPQ System Description

The general structure of the proposed COSAPQ system, which is a generalization of SAPQ [7, 8] for noisy channels, is depicted in Fig. 1. As in the case of COVQ, the objective of this system is to convey a random source described by a probability density function (p.d.f.) over a binary

[1] For previous related work on adaptive quantization, see [2, 10] and the references in [7, 8].

symmetric channel (BSC) with bit error rate (BER) $\epsilon \in (0,1)$, and reproduce it at the receiver with the aim of minimizing the overall expected mean square error distortion.

For every source vector to be quantized, a (k,m,N,η) COSAPQ employs a codebook from a previously designed set of 2^η km-dimensional codebooks $\{\mathbf{C}_j\}_{j=1}^{2^\eta}$, available at both encoder and decoder. Each of the 2^η codebooks is that of a product quantizer (PQ) (e.g., see [6, 7]), and the choice of the particular codebook used for encoding depends on the source vector. Hence when transmitting the indexes representing the codevectors or reconstruction vectors, the encoder must also transmit an overhead index that indicates the codebook used for that source vector. Thus the codebook of a (k,m,N,η) COSAPQ is a union of the 2^η product quantizer codebooks. As for the case of SAPQ (for noiseless channels) in [7, 8], we distinguish between two types of COSAPQ: COm-SAPQ and CO1-SAPQ.

Each codebook $\mathbf{C}_j$ of COm-SAPQ is a product of m codebooks: $\mathbf{C}_j = \mathrm{C}_{1,j} \times \cdots \times \mathrm{C}_{m,j}$ where $\mathrm{C}_{s,j}$ is a collection of N k-dimensional codevectors $\{\underline{c}_l^{[s,j]}\}_{l=1}^N$.[2] As shown in Fig. 1, the COm-SAPQ encoder consists of 2^η vector functions called product encoders $\{\mathrm{PE}_j\}_{j=1}^{2^\eta}$. Copies of the source vector $\underline{\mathbf{x}} = (\underline{\mathrm{x}}_1, \cdots, \underline{\mathrm{x}}_m) \in \mathbb{R}^{km}$ are encoded by each PE_j producing index vector $\underline{I}_j$: $\mathrm{PE}_j(\underline{\mathbf{x}}) = \underline{I}_j \in J_N^m = \{1, \cdots, N\}^m$. Each index vector in the set $\{\underline{I}_j\}_{j=1}^{2^\eta}$ has an associated distortion, and the index vector $\underline{I}$ with minimum distortion is transmitted over the channel along with index j^*, representing PE_{j^*} that produced $\underline{I}$. Due to channel noise, potentially corrupted versions of $\underline{I}$ and j, $\underline{L} = (l_1, \ldots, l_m) \in J_N^m$ and $j' \in J_{2^\eta} = \{1, \cdots, 2^\eta\}$, are received at the decoder. The decoder decodes $\underline{L}$ using decoding function $G_{j'}$ from the set $\{\mathrm{G}_j\}_{j=1}^{2^\eta}$ as follows: $\mathrm{G}_{j'}(\underline{L}) = (g_{1,j'}(l_1), g_{2,j'}(l_2), \cdots, g_{m,j'}(l_m)) = (\underline{c}_{l_1}^{[1,j']}, \ldots, \underline{c}_{l_m}^{[m,j']}) = \underline{\mathbf{c}}_{\underline{L}}^{[j']}$ where $\underline{c}_l^{[s,j']} \in \mathrm{C}_{s,j'}$ for $l \in J_N$, and $\underline{\mathbf{c}}_{\underline{L}}^{[j']} \in \mathbf{C}_{j'}$.

The CO1-SAPQ is a simplified version of COm-SAPQ in that each codebook $\mathbf{C}_j$ is an m-fold product of the same codebook: $\mathbf{C}_j = \mathrm{C}_j \times \cdots \times \mathrm{C}_j$. The rate of both the (k,m,N,η) CO1-SAPQ and the (k,m,N,η) COm-SAPQ is given by

$$R = \frac{\log_2 N}{k} + \frac{\eta}{km} \quad \text{bits/source sample.}$$

[2] In the original definition of m-SAPQ [7] the codebooks $\mathrm{C}_{s,j}$ are allowed to have different sizes n_s'. Our formulation introduces the restriction that $n_s' = N$ for all $s = 1, \ldots, m$.

III Optimality Conditions for System Design

We next establish the necessary encoding and decoding conditions for optimality that can be used in an iterative design algorithm (as the generalized Lloyd-Max algorithm in COVQ) that attempts to minimize the overall distortion. Let $\mathbf{S}_{\underline{Z}}^{[j]}$ be the encoding region for index vector $\underline{Z}$ and index j: $\mathbf{S}_{\underline{Z}}^{[j]} = \{\underline{\mathbf{x}} : \text{encoder}(\underline{\mathbf{x}}) = (\underline{Z}, j)\}$. Then the average mean squared end-to-end distortion can be written as follows

$$\mathrm{D}_{COm\text{-}SAPQ} = \sum_{j=1}^{2^\eta} \sum_{\underline{Z} \in J_N^m} \int_{\mathbf{S}_{\underline{Z}}^{[j]}} \sum_{j'=1}^{2^\eta} \sum_{\underline{L} \in J_N^m} P(j'|j) P(\underline{L}|\underline{Z}) \sum_{s=1}^{m} \| u_s(\underline{\mathbf{x}}) - \underline{c}_{v_s(\underline{L})}^{[s,j']} \|^2 p(\underline{\mathbf{x}}) d\underline{\mathbf{x}}, \tag{1}$$

where $p(\underline{\mathbf{x}})$ is the source p.d.f. with $\underline{\mathbf{x}} \in \mathbb{R}^{km}$, $P(a|b)$ is the channel transition probability of receiving index a given that index b was sent (which can be easily expressed in terms of the BSC BER ϵ assuming that each index is sent using a natural binary assignment), $\mathrm{v}_s(\underline{I})$ = the s^{th} index component of $\underline{I}$ ($\mathrm{v}_s : J_N^m \to J_N$) and $\mathrm{u}_s(\underline{\mathbf{x}}) = \underline{\mathbf{x}}_s$; $s = 1, \ldots, m$ ($\mathrm{u}_s : \mathbb{R}^{km} \to \mathbb{R}^k$).

Optimal Encoding: Given that codebooks $\{\mathbf{C}_j\}_{j=1}^{2^\eta}$ are fixed, the optimal encoding of a source sample $\underline{\mathbf{x}}$ into $(\underline{I}, j^*)$ is achieved via two optimization steps: one to minimize the distortion over all index vectors $\underline{Z} \in J_N^m$, and the other over all indexes $j \in J_{2^\eta}$. The first optimization over all $\underline{Z} \in J_N^m$ is obtained via the PEs

$$\underline{I}_j = \mathrm{PE}_j(\underline{\mathbf{x}}) = \arg\min_{\underline{Z} \in J_N^m} \sum_{j'=1}^{2^\eta} \sum_{\underline{L} \in J_N^m} P(j'|j) P(\underline{L}|\underline{Z}) \sum_{s=1}^{m} \| u_s(\underline{\mathbf{x}}) - \underline{c}_{v_s(\underline{L})}^{[s,j']} \|^2. \tag{2}$$

When the source $\underline{\mathbf{x}}$ is encoded by PE_j into $\underline{I}_j$, the associated distortion is given by

$$D_j(\underline{\mathbf{x}}) = \min_{\underline{Z} \in J_N^m} \sum_{j'=1}^{2^\eta} \sum_{\underline{L} \in J_N^m} P(j'|j) P(\underline{L}|\underline{Z}) \sum_{s=1}^{m} \| u_s(\underline{\mathbf{x}}) - \underline{c}_{v_s(\underline{L})}^{[s,j']} \|^2. \tag{3}$$

The optimum index j^* then chooses the index $\underline{I}$ with smallest associated distortion, resulting in the following optimal encoding region

$$\mathbf{S}_{\underline{I}}^{[j^*]} = \left\{ \underline{\mathbf{x}} : j^* = \arg\min_{j \in J_{2^\eta}} D_j(\underline{\mathbf{x}}) \quad \text{and } \underline{I} = \underline{I}_{j^*} = PE_{j^*}(\underline{\mathbf{x}}) \right\}.$$

Encoding simplifications can be obtained using a similar approach as in [4, Section IV]. If we define

$$\underline{y}_{s,j}(\gamma) = \sum_{j'=1}^{2^\eta} \sum_{l=1}^{N} P(j'|j) P(l|\gamma) \underline{c}_l^{[s,j']} \quad \text{and} \quad \alpha_{s,j}(\gamma) = \sum_{j'=1}^{2^\eta} \sum_{l=1}^{N} P(j'|j) P(l|\gamma) \| \underline{c}_l^{[s,j']} \|^2, \tag{4}$$

then the product encoder operation (2) can be written in the following simpler expression

$$\mathrm{PE}_j(\underline{\mathbf{x}}) = \underline{I}_j = \arg\min_{\underline{I}\in J_N^m} \sum_{s=1}^{m} \alpha_{s,j}(v_s(\underline{I})) - 2\left\langle u_s(\underline{\mathbf{x}}), \underline{y}_{s,j}(v_s(\underline{I}))\right\rangle, \tag{5}$$

where $\langle \underline{\mathrm{x}}, \underline{y}\rangle$ is the standard inner product in $\mathbb{R}^k$. In other words, the encoding of $\underline{\mathbf{x}}$ by a COm-SAPQ requires the prior calculation of $2^\eta mN$ k-dimensional vectors $\underline{y}_{\gamma,j}^{[s]}$ and $2^\eta mN$ scalars $\alpha_{\gamma,j}^{[s]}$.

Optimal Decoding: Given that the $2^\eta N^m$ encoding regions $\{\mathbf{S}_{\underline{I}}^{[j^*]}\}$ are fixed, we can determine the optimal centroids via standard minimum mean squared estimation based on the distortion expression in (1). This results in the following centroids

$$\underline{c}_l^{[s,j']} = \frac{\sum_{j^*=1}^{2^\eta}\sum_{i=1}^{N} P(j'|j^*)P(l|i)\int_{S_i^{[s,j^*]}} u_s(\underline{\mathbf{x}})p(\underline{\mathbf{x}})d\underline{\mathbf{x}}}{\sum_{j^*=1}^{2^\eta}\sum_{i=1}^{N} P(j'|j^*)P(l|i)\int_{S_i^{[s,j^*]}} p(\underline{\mathbf{x}})d\underline{\mathbf{x}}}, \quad \text{where} \quad S_i^{[s,j^*]} = \bigcup_{\underline{I}:v_s(\underline{I})=i} \mathbf{S}_{\underline{I}}^{[j^*]}. \tag{6}$$

IV Encoding Complexity and Storage Requirement

We next briefly discuss the encoding complexity and the storage requirement of a (k,m,N,η) COm-SAPQ system.

Encoding Complexity: As in [7, 8, 13], the encoding complexity is defined as the number of multiplications needed to encode one source sample. It is important to point out that unlike conventional VQ systems (designed for noiseless channels), the encoding complexity of VQ schemes for noisy channels (such as COVQ and COSAPQ) is strongly dependent on the design channel BER. Indeed, when the design BER is very high, the encoding complexity of such channel optimized schemes can be considerably low since a large number of the encoding cells become empty (e.g., see [5]). Thus we evaluate the COSAPQ encoding complexity in terms of the number of non-empty encoding cells. More specifically, the encoding complexity of a (k,m,N,η) COm-SAPQ is given by

$$\sum_{j=1}^{2^\eta}\sum_{s=1}^{m}\frac{kN_{sj}}{km} = \sum_{j=1}^{2^\eta}\sum_{s=1}^{m}\frac{N_{sj}}{m}, \tag{7}$$

where N_{sj} denotes the number of non-empty encoding cells in codebook $\mathrm{C}_{s,j}$, which consists of N k-dimensional codevectors. Similarly, the encoding complexity of a (k,m,N,η) CO1-SAPQ is given by

$$\sum_{j=1}^{2^\eta}\frac{kmN_j}{km} = \sum_{j=1}^{2^\eta} N_j, \tag{8}$$

where N_j denotes the number of non-empty encoding cells in codebook C_j.

Storage Requirement: The storage requirement is defined as the total number of scalars needed to implement the quantizer operations in both encoding and decoding stages [7, 8, 13]. At the encoder, we need to store the sets $\{\underline{y}^{[s]}_{\gamma,j}\}$ and $\{\alpha^{[s]}_{\gamma,j}\}$ given in (4). We need $k2^{\eta}mN$ scalars for $\{\underline{y}^{[s]}_{\gamma,j}\}$ and $2^{\eta}mN$ scalars for $\{\alpha^{[s]}_{\gamma,j}\}$. Hence, at the encoder we require $k2^{\eta}mN + 2^{\eta}mN$ scalars in total. At the decoder, we need to store the entire COSAPQ codebook which requires $k2^{\eta}mN$ scalars. Therefore, the total storage requirement for a (k,m,N,η) COm-SAPQ is given by

$$2k2^{\eta}mN + 2^{\eta}mN. \tag{9}$$

For a (k,m,N,η) CO1-SAPQ, the total storage requirement is analogously given by

$$2k2^{\eta}N + 2^{\eta}N. \tag{10}$$

V Numerical Results

We next evaluate the performance in terms of signal-to-distortion ratio (SDR), encoding complexity (via (7) and (8)) and storage requirement (via (9) and (10)) of our COSAPQ system for the compression and transmission of two (unit variance) sources: a memoryless Gaussian source and a Gauss-Markov source with correlation coefficient 0.9. Comparisons with the following competing quantization systems are also provided: COVQ and channel optimized product quantizer (COPQ).[3] For a (k,N) COVQ, the encoding complexity is given by the number of non-empty encoding cells N^*, and the storage requirement is given by $2kN + N$. The encoding complexity of a (k,m,N) COPQ is given by $(1/m)\sum_{i=1}^{m} N_i$, where N_i is the number of non-empty encoding cells in the ith codebook, while its storage requirement is given by $2kmN + mN$.

The optimality conditions derived in Section III were used in a generalized Lloyd-Max algorithm to design the COm-SAPQ and CO1-SAPQ codes. Details of the algorithm are available in [13]. For the COm-SAPQ system, the algorithm was initialized by a COPQ codebook with the same design parameters, while the CO1-SAPQ system was initialized with a COVQ codebook. We employed

[3] A (k,m,N) COPQ is a (k,m,N) PQ optimized for noisy channels; its overall codebook is a product of m codebooks, each consisting of N k-dimensional codevectors. Refer to [13] for a detailed description of this system and its design algorithm.

200,000 training samples for the codes design and 200,000 test samples for the simulations. In all cases, the simulation and training results were in agreement.

Table 1 shows the performance (both SDR, encoding complexity and storage) of COVQ, COPQ, COm-SAPQ and CO1-SAPQ at various rates R and design channel BERs ϵ_d for memoryless Gaussian sources. We remark that COm-SAPQ performs within 0.2 dB of COVQ of the same rate, while enjoying an encoding complexity that is 23 to 50 % smaller than that of COVQ (note that for $R = 1$, the encoding complexity reduction is by 50 % for all design BERs) and a storage requirement that is lower by a factor of 40 to 44 %. Furthermore, it is observed in [13] that for Gauss-Markov sources, a gain of up to 0.7 dB over COVQ can be attained by a CO1-SAPQ with the same complexity while keeping a lower storage requirement (by 40 to 67 %).[4] This gain is due to the fact that the CO1-SAPQ can employ a higher dimension km while keeping the same encoding complexity as COVQ. Table 2 compares channel optimized quantizers with quantizers designed for noiseless channels. The LBGVQ system is the same as in [5] that employs simulated annealing for index assignment. We note that COSAPQ offers significant gains over SAPQ particularly at high BERs.

So far, we assumed that the quantizers' encoder and decoder are perfectly matched to the channel conditions; however, in many practical situations, the channel characteristics may be time varying. We next test the robustness of both COSAPQ and COVQ systems when they are designed for a fixed channel BER ϵ_d while the actual or true BER is $\epsilon_a \neq \epsilon_d$. Mismatch simulation results are presented in Fig. 2 for $R = 2.0$, $\epsilon_d = 0.05$ and memoryless Gaussian sources. We observe that COm-SAPQ performs within 0.3 dB from COVQ for $\epsilon_a \leq 0.01$; as ϵ_a increases, COm-SAPQ is as robust as COVQ. For Gauss-Markov sources, gains of up to 0.2 dB were attained by CO1-SAPQ over COVQ under mismatch BER conditions [13]. [5]

VI Conclusion

We introduced COSAPQ systems (COm-SAPQ and CO1-SAPQ) for the efficient compression and reliable transmission of Gaussian sources over BSCs. Numerical results show that for memoryless Gaussian sources, performance within 0.2 dB of that of COVQ can be attained by COm-SAPQ

[4] Compare, for example, (2,64) COVQ with (1,4,4,4) CO1-SAPQ in Table 2 for low values of BERs.

[5] Additional mismatch results for various rates and design BERs are available in [13].

of the same rate with up to half the encoding complexity and lower storage requirements. For Gauss-Markov sources, CO1-SAPQ with dimension km outperforms COVQ of dimension k by up to 0.7 dB for the same rate and encoding complexity, while having lower storage requirement. Finally, COSAPQ was shown to be as robust as COVQ under mismatched channel conditions.

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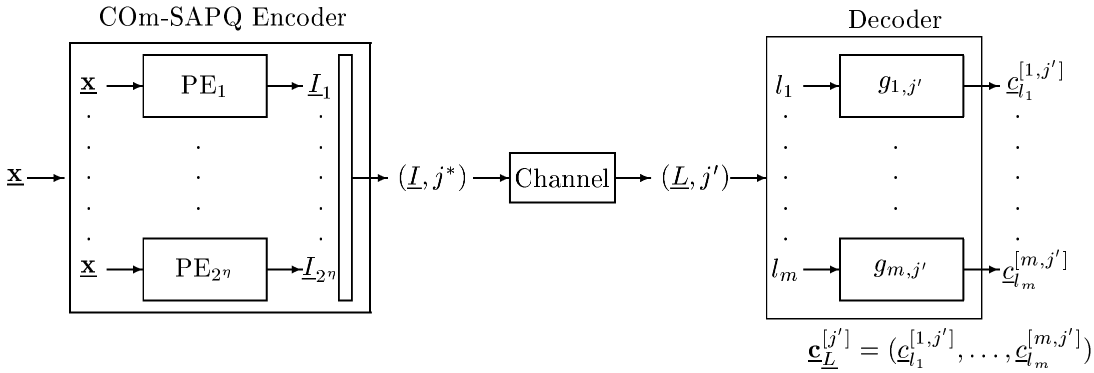

Figure 1: Block diagram of a (k,m,N,η) COm-SAPQ system, where $\underline{\mathbf{x}} = (\underline{\mathrm{x}}_1, \ldots, \underline{\mathrm{x}}_m) \in \mathbb{R}^{km}$, $\underline{I} = (i_1, \ldots, i_m) \in J_N^m$, $\underline{L} = (l_1, \ldots, l_m) \in J_N^m$, and $j^*, j' \in J_{2^\eta}$.

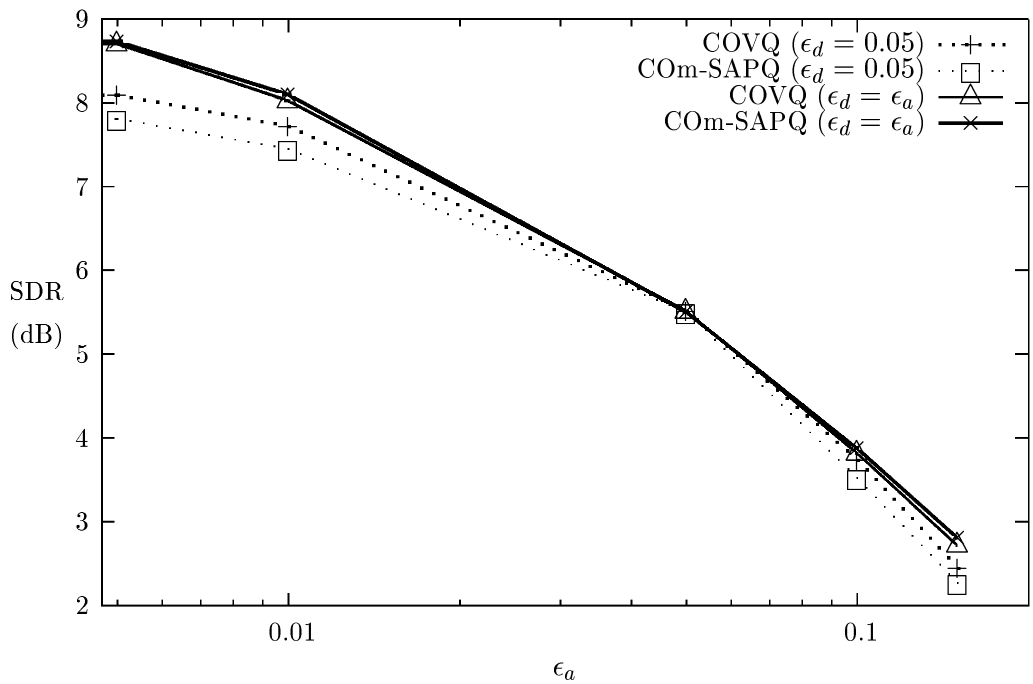

Figure 2: SDR performance in dB of (2,16) COVQ and (1,2,2,2) COm-SAPQ at rate $R = 2.0$ and BSC design BER $\epsilon_d = 0.05$, using 200,000 memoryless Gaussian testing samples and a simulated BSC with actual BER ϵ_a.

R	km	Quantizer	ϵ_d 0.000	0.005	0.010	0.050	0.100	0.150	storage
3.0	2	COVQ	15.23	12.19	11.07	7.35	5.11	3.78	320
		$k=2,\ N=64$	(64)	(64)	(57)	(30)	(27)	(25)	
	2	COPQ	14.57	12.00	10.47	5.62	4.65	3.48	48
		$k=1,\ N=8$	(8)	(8)	(8)	(6)	(6)	(6)	
	2	COm-SAPQ	15.07	12.38	11.04	7.20	5.14	3.75	192
		$k=1,\ N=2,\ \eta=4$	(32)	(32)	(32)	(23)	(16)	(14)	
	2	CO1-SAPQ	13.59	11.54	10.29	6.48	4.35	3.13	96
		$k=1,\ N=2,\ \eta=4$	(32)	(32)	(32)	(22)	(18)	(12)	
	4	CO1-SAPQ	15.19	12.41	10.92	6.61	4.47	3.18	192
		$k=1,\ N=4,\ \eta=4$	(64)	(64)	(52)	(30)	(26)	(17)	
2.0	2	COVQ	9.64	8.71	8.02	5.52	3.82	2.71	80
		$k=2,\ N=16$	(16)	(16)	(16)	(13)	(13)	(13)	
	2	COPQ	9.27	8.50	7.86	4.85	3.04	1.99	24
		$k=1,\ N=4$	(4)	(4)	(4)	(4)	(4)	(4)	
	2	COm-SAPQ	9.51	8.71	8.10	5.48	3.86	2.79	48
		$k=1,\ N=2,\ \eta=2$	(8)	(8)	(8)	(8)	(8)	(8)	
	2	CO1-SAPQ	8.72	8.04	7.50	5.16	3.61	2.50	24
		$k=1,\ N=2,\ \eta=2$	(8)	(8)	(8)	(8)	(8)	(8)	
	3	CO1-SAPQ	8.92	8.16	7.58	5.06	3.45	2.45	48
		$k=1,\ N=2,\ \eta=3$	(16)	(16)	(16)	(16)	(16)	(16)	
1.0	4	COVQ	4.66	4.44	4.24	3.14	2.26	1.61	144
		$k=4,\ N=16$	(16)	(16)	(16)	(16)	(16)	(16)	
	4	COPQ	4.38	4.16	3.96	2.72	1.75	1.14	40
		$k=2,\ N=4$	(4)	(4)	(4)	(4)	(4)	(4)	
	4	COm-SAPQ	4.47	4.28	4.09	3.13	2.26	1.61	80
		$k=2,\ N=2,\ \eta=2$	(8)	(8)	(8)	(8)	(8)	(8)	
	4	CO1-SAPQ	4.41	4.15	3.95	2.81	1.96	1.44	40
		$k=2,\ N=2,\ \eta=2$	(8)	(8)	(8)	(8)	(8)	(8)	
	6	CO1-SAPQ	4.53	4.25	4.01	2.73	1.95	1.39	80
		$k=2,\ N=2,\ \eta=3$	(16)	(16)	(16)	(16)	(16)	(16)	

Table 1: SDR performance in dB, encoding complexity (in parentheses) and storage requirement of COVQ, COPQ, COm-SAPQ, and CO1-SAPQ with rate R and dimension km. All systems were designed for a BSC with BER ϵ_d using 200,000 memoryless Gaussian training samples.

Quantizer	ϵ_a					
	0.000	0.005	0.010	0.050	0.100	0.150
LBGVQ (+ sim. annl.) $k = 2, N = 64$	19.01	13.63	11.07	5.29	2.37	0.71
PQ $k = 1, m = 2, N = 8$	14.64	12.01	10.39	5.01	2.25	0.57
m-SAPQ $k = 1, m = 2, N = 2, \eta = 4$	16.65	12.82	10.74	4.85	2.09	0.48
1-SAPQ $k = 1, m = 4, N = 4, \eta = 4$	19.71	13.42	10.98	4.55	1.69	0.05
COVQ $k = 2, N = 64$	19.01	14.62	13.62	9.48	6.85	5.20
COPQ $k = 1, m = 2, N = 8$	14.60	12.02	10.50	5.57	4.60	3.45
CO-m-SAPQ $k = 1, m = 2, N = 2, \eta = 4$	16.64	14.30	12.75	8.45	6.10	4.66
CO-1-SAPQ $k = 1, m = 4, N = 4, \eta = 4$	19.71	15.27	13.94	9.52	6.91	5.21

Table 2: SDR performance in dB of COVQ, COPQ, COm-SAPQ and CO1-SAPQ versus LBGVQ (with simulated annealing), PQ, m-SAPQ and 1-SAPQ with rate $R = 3.0$, using 200,000 Gauss-Markov testing samples (with correlation coefficient 0.9) and a simulated BSC with actual BER ϵ_a (here, the design BER ϵ_d is matched to ϵ_a; i.e., $\epsilon_d = \epsilon_a$).