Sampling-Type Theorems in More than One Dimension

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SUMMARY

The object of this note is to suggest that there might be more than one appropriate generalization of the sampling theorem to the multidimensional case. In one dimension, bandlimited functions are determined by their values at isolated points and are reconstructed by interpolating between these values. In n dimensions, there are more possibilities. One might again find a set of points such that function values at these points determine the function completely. One might also find other lower-dimensional manifolds, curves or surfaces for example, with the property that function values on these manifolds determine the function completely. This possibility will be illustrated in two and three dimensions. The series expansions to be considered here are derived from expansions in the eigenfunctions of the Laplace operator and may not be new in themselves. Rather it is the point of view that they might be considered as multidimensional extensions of the sampling theorem which is to be examined here.

In dimension two, we exhibit a way of representing a function that is bandlimited to a disc (i.e. its Fourier transform vanishes outside the disc) by a sequence of measurements on a countable set of circles. In dimension three, we give two examples. For a function bandlimited to a solid cylinder, the function is determined by measurements on a countable set of circles. For a function bandlimited to a ball, the function is determined by measurements

on a countable set of spheres. In all the examples here, the number of measurements which are required per unit area or per unit volume to reconstruct the function is equal to the Nyquist density. This density is seen to be related to the distribution of eigenvalues of certain boundary value problems for the Laplace operator. For the purpose of storing images, for example, these results would seem to be as applicable as uses of the Whittaker-Shannon-Kotelnikov (WSK) sampling theorem. The proofs of these results are very much like Kramer's proof of his generalized sampling theorem and the reconstruction functions are quite similar to those that arise in use of Kramer's sampling theorem.