Two-Way Gaussian Networks With a Jammer and Decentralized Control

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***Abstract*—We consider the existence and structure of (zero-sum game) Nash equilibria for a two-way network in the presence of an intelligent jammer capable of tapping the channel signals in both directions. We assume that the source and channel noise signals are all Gaussian random variables, where the source signals are independent of each other, while the noise signals are arbitrarily correlated. We show that for fixed jammer power constraints, a Nash equilibrium exists with respect to the system wide mean square error, and equilibrium jamming policies are always Gaussian. We derive the equilibrium policies in closed form under various system parameters. Finally for two system scenarios, we analytically determine the optimal power allocation levels the jammer can deploy in each channel link, when allowed to operate under an overall power constraint.**

***Index Terms*—Decentralized signaling and estimation, game theory, information theory, source-channel coding, two-way Gaussian networks.**

I. Introduction

DECENTRALIZED control and estimation problems involve decision makers who aim for a common goal, but who only have local information. Networked control systems are those involving multiple decision makers connected over communication channels. In this paper, we consider a setup where two decision makers wish to encode information to each other over a shared medium.

The question of how a communication system performs in the worst case scenario is one of vital importance, for if one can guarantee a suitable operation in the least favorable case, then the system will operate at least in all other cases as well. In determining the worst case faced by a system, it is useful to personify the noise as an intelligent "jammer," actively working to steer the system toward a deleterious behaviour. Furthermore, in many scenarios, there may indeed be a malicious agent who intends to suppress communication and control in a decentralized system. These setups motivate the use of game theoretic methods in communications and networked control applications.

One of the earliest works to consider such a problem is by Başar [1], where the transmission of a Gaussian source over a one-way additive Gaussian noise channel is investigated in the presence of an intelligent jammer. Başar establishes complete solutions to a zero-sum formulation, proving the optimality of linear/affine/Gaussian policies under various setups and assumptions. Further relevant studies include [2]–[4].

The problem stated above focuses on one-way (or point-to-point) communications. However, modern communication systems are increasingly decentralized and multiterminal for better utilization of limited channel resources. The simplest networked system is the two-way channel (TWC) first introduced by Shannon in [5].

In this paper, we consider the case of a Gaussian two-way source-channel system with an intelligent jammer. In such a system, each user transmits and receives signals simultaneously (i.e., the system is in full duplex mode). This allows each encoder, when sending a codeword vector, to interactively adapt the current input to its own message and all previously received signals, hence rendering it more resilient to channel noise. The reader is referred to [5]–[15] and the references therein for coding theorems and channel capacity results for two-way channels. In particular, it is shown in [15] that zero-delay linear (scalar) coding and decoding achieve the Shannon theoretical mean square error (MSE) distortion limit for the two-way Gaussian system with independent sources and are hence optimal. We herein focus on the same scalar two-way Gaussian system in the presence of a jammer capable of accessing the channel's signals in both directions. Since all the system's respective (Gaussian) random variables are temporally uncorrelated and identically distributed, our setting is indeed a one shot MSE optimization problem.

The identification of optimal linear/affine/Gaussian policies for decentralized systems involving Gaussian variables under quadratic criteria [such as in linear quadratic Gaussian (LQG)] is a recurring problem in stochastic networked control and estimation theory. These certainly include the classical problem of communicating a scalar Gaussian source over a Gaussian channel [17]–[20], where linear encoding policies are optimal, which also extends to the vector case under certain conditions [21]–[28]. For nonclassical decentralized stochastic control problems, Witsenhausen's counterexample [29] shows that

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optimal policies for LQG systems may be nonlinear and this suboptimality also extends to various decentralized LQG problems as reviewed in [16, Ch. 11] and [30].

For game-theoretic formulations, somewhat surprisingly, optimality and linearity again coincide for a large class of setups: in Witsenhausen's counterexample, if the first encoder is viewed as a maximizer and the decoder is a minimizer, then affine policies may be optimal [31], [32], where similar to the related results in the literature, the *ordered interchangeability property of saddle points in zero-sum games [37, Corollary 2.1 and Property 4.1, p. 177]* is a crucial tool in the analysis.

For a setup similar to the work presented in [1], but with the game being played only between an encoder and a jammer (with the decoder being an impartial Bayesian decision maker), it is shown in [33] that the worst additive channel noise is Gaussian and the optimal encoder is linear. This result may be viewed as a Stackelberg extension of the Nash equilibrium setup given in [1] (for a detailed discussion on the distinction between Nash versus Stackelberg equilibria in signaling games, see [14], [34], and [35]), where the receiver is a follower and the encoder/jammer pair is a leader.

In view of the aforementioned discussion, this paper provides further conditions in which affine and Gaussian policies may constitute equilibria for such decentralized quadratic optimization problems. In particular, we show that for a two-way Gaussian scalar networked system with an intelligent jammer, an essentially unique zero-sum Nash equilibrium exists and the equilibrium policies are affine/Gaussian. We derive the closed form of the equilibrium policies under various system parameters. Thus, this paper provides a two-way (and thus a decentralized) generalization, in the sense that there exists a team of encoders/decoders against a single jammer, of the findings of [1] where a single encoder/decoder pair is present against a jammer. We also point out that a problem in which a team of agents plays against another player can lead to subtleties with regard to existence of equilibria (even for finite games) that do not arise when a single agent plays against another one, as demonstrated in [36]. Such a setup is precisely what is studied in this paper. Finally, we note that the work presented in [7] and [13] consider aspects related to our jamming problem, though in a quite different channel coding context.

The nature of two-way channels adds significant complexity to the problem. The correlation between the noise signals of each channel direction requires special analysis in different situations, and ultimately results in different jamming policies depending on the noise correlation and variance values. Finding the equilibrium jamming policy amounts to solving for the fixed point of a best response function involving multiple variables. Furthermore, we investigate the optimal power allocation the jammer can employ for each channel direction under a given overall budget; this problem has no counterpart in the one-way setup of [1].

The rest of this paper is organized as follows. In Section II, we formulate the problem. In Section III-A, we examine the common setup for the two-way channel noise variables and derive full closed form solutions for the equilibrium policies. In Section III-B, we analyze using a slightly different approach a special "degenerate" case under which the noise variables coincide in each channel direction. We compare the results to previous work and provide a qualitative analysis in Section III-C. In Section III-D, we investigate the optimal power allocation levels when the jammer is allowed to choose its power constraints subject to an overall budget. Two examples are shown in Section IV and concluding remarks are drawn in Section V. All proofs are presented in the Appendix section.

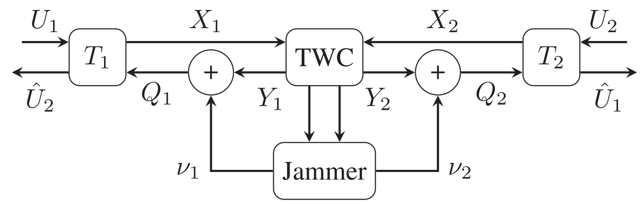

Fig. 1. System diagram of a two-way source-channel system with an intelligent jammer.

II. Problem Setup

Consider two terminals (decision makers) T_1 and T_2 attempting to concurrently exchange Gaussian independent source signals U_1 and U_2, respectively, where $U_i \sim \mathcal{N}(0,1)$ has zero mean and unit variance for $i = 1, 2$, across a two-way additive Gaussian noise channel as depicted in Fig. 1. More specifically, each terminal T_i observes signal U_i and uses transmitter policy $\gamma_i : \mathbb{R} \to \mathbb{R}$ to generate signal X_i subject to the power constraint

$$E[(\gamma_i(U_i))^2] \leq c_i, i = 1, 2. \tag{1}$$

The two-way channel inputs are X_1, X_2 and its outputs are

$$Y_i = X_1 + X_2 + Z_i, \quad i = 1, 2 \tag{2}$$

where Z_1 and Z_2 are Gaussian random variables [which are independent of (U_1, U_2)] with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} E[Z_1^2] & E[Z_1 Z_2] \\ E[Z_2 Z_1] & E[Z_2^2] \end{pmatrix} = \begin{pmatrix} \zeta_1 & \zeta_{1,2} \\ \zeta_{1,2} & \zeta_2 \end{pmatrix} \tag{3}$$

where $\zeta_{1,2}$ takes values in $[-(\zeta_1\zeta_2)^{\frac{1}{2}}, (\zeta_1\zeta_2)^{\frac{1}{2}}]$.

We furthermore assume the existence of a third party, the jammer. The latter taps the channel in both directions and captures signals Y_1 and Y_2; in return, it sends an adversarial signal ν_i to each terminal T_i using the jamming policy $\nu_i = \beta_i(y_1, y_2)$, $i = 1, 2$, where each β_i is in general a random mapping. Let $\mathcal{M}$ denote the set of the pairs (μ_1, μ_2), where μ_i is the probability measure associated with jamming signal ν_i under the power constraint

$$E[\nu_i^2] \leq k_i, \quad i = 1, 2. \tag{4}$$

Terminal T_i then receives signal $Q_i = Y_i + \nu_i$, which it uses together with side information X_i (i.e., its own signal sent to Terminal T_j) to reconstruct U_j via $\hat{U}_j$ under decoding policy $\delta_i : \mathbb{R} \times \mathbb{R} :\to \mathbb{R}, i \neq j, i, j = 1, 2$. We will refer to the channel $T_j \to T_i$ as channel i throughout this paper.

The average MSE of the system is denoted by $R(\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2)$ and calculated as

$$\frac{1}{2}\sum_{i=1}^{2}\left(\int_{-\infty}^{\infty} E[(\hat{U}_i - U_i)^2|\nu_i]d\mu_i(\nu_i)\right). \tag{5}$$

Let Γ_t^i, Γ_r^i be the set of admissible (as specified earlier) transmitter and receiver policies for terminal T_i, $i = 1, 2$. Naturally, the objective of terminals T_1 and T_2 is to choose their encoding/decoding policies so that the system MSE is minimized, while the jammer aims at designing its policies in order to maximize MSE.

Definition 2.1: A policy tuple $(\gamma_1^*, \gamma_2^*, \delta_1^*, \delta_2^*, \mu_1^*, \mu_2^*)$ is a Nash equilibrium if

$$\begin{aligned} R(\gamma_1^*, \gamma_2^*, \delta_1^*, \delta_2^*, \mu_1, \mu_2) &\le R(\gamma_1^*, \gamma_2^*, \delta_1^*, \delta_2^*, \mu_1^*, \mu_2^*) \\ &\le R(\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1^*, \mu_2^*) \end{aligned} \tag{6}$$

$\forall \gamma_i \in \Gamma_t^i, \delta_i \in \Gamma_r^i, i = 1, 2, (\mu_1, \mu_2) \in \mathcal{M}$.

We will separately consider a special case of this problem, which can be viewed as a *degenerate case*: the case when the same noise variable affects both the channel directions, i.e., $Z_1 = Z_2$ (almost surely). This yields that $Y_1 = Y_2$ and the jammer sees two identical signals. We first consider the *nondegenerate* case, where the elements of the channel noise covariance matrix Σ are not all identical.

For certain power regions, the problem becomes uninteresting in the sense that the jammer can employ a linear policy $\beta_i(y_1, y_2) = a_{i,i}y_i + a_{i,j}y_j$ that fully cancels the signal at the terminal. If $a_{i,i} + a_{i,j} = -1$, then we have

$$\nu_i = \beta_i(y_1, y_2) = a_{i,i}y_i + a_{i,j}y_j \tag{7}$$

$$= (a_{i,i} + a_{i,j})(X_1 + X_2) + a_{i,i}Z_i + a_{i,j}Z_j \tag{8}$$

where $i \neq j$. Then, we have

$$\begin{aligned} Q_i &= Y_i + \nu_i \\ &= (1 + a_{i,i} + a_{i,j})(X_1 + X_2) + (1 + a_{i,i})Z_i + a_{i,j}Z_j \\ &= (1 + a_{i,i})Z_i + a_{i,j}Z_j. \end{aligned}$$

Thus, the signal Q_i received at the terminal is pure noise and the MSE is maximized at 1, irrespective of the transmitter and receiver policies. We next determine the minimum power level k_i, denoted by $\tilde{k}_i$, that admits a linear jamming policy with $a_{i,i} + a_{i,j} = -1$. To this end, for a linear jamming policy we have

$$E[\nu_i^2] = a_{i,i}^2(C + \zeta_i) + 2a_{i,i}a_{i,j}(C + \zeta_{1,2}) + a_{i,j}^2(C + \zeta_j)$$

where $C = c_1 + c_2$. Setting $a_{i,j} = -1 - a_{i,i}$ and $E[\nu_i^2] = k_i$, we have

$$a_{i,i}^2(\zeta_i + \zeta_j - 2\zeta_{1,2}) + 2a_{i,i}(\zeta_j - \zeta_{1,2}) + (C + \zeta_j - k_i) = 0.$$

The lowest k_i value for which this equation (which is quadratic in $a_{i,i}$) admits a real solution is when the discriminant is equal to zero. Hence

$$\tilde{k}_i = C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}, \quad i = 1, 2 \tag{9}$$

TABLE I

NOTATION REFERENCE TABLE

Notation	Description
U_i	Signal at terminal T_i to be transmitted to T_j
$\hat{U}_i$	Reconstructed signal at terminal T_j, $j \neq i$
γ_i	Transmission policy at terminal T_i
X_i	$X_i = \gamma_i(U_i)$
c_i	Power constraint for X_i
Z_i	Channel noise
ζ_i	Variance of Z_i
$\zeta_{1,2}$	Covariance of Z_1, Z_2
Σ	Covariance matrix for Z_1, Z_2
Y_i	$Y_i = X_1 + X_2 + Z_i$
C	$C = c_1 + c_2$
ω	$\omega = C(\zeta_1 + \zeta_2) + \zeta_1\zeta_2 - \zeta_{1,2}^2$
β_i	Jammer policy for channel i
ν_i	$\nu_i = \beta_i(y_1, y_2)$
k_i	Power constraint for ν_i
Q_i	$Y_i + \nu_i$
R_i^1	$\left\{k_i \ge C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1+\zeta_2-2\zeta_{1,2}}\right\}$
R_i^2	$\left\{k_i < C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1+\zeta_2-2\zeta_{1,2}}\right\}$
$a_{i,i}, a_{i,j}, b_i$	Coefficients for equilibrium jamming policy, see Table II
$\tilde{Q}_i$	$\tilde{Q}_i = Q_i - (1 + a_{i,i} + a_{i,j})\sqrt{c_i}U_i$
α_i	$\frac{\sqrt{c_j}(1+a_{i,i}+a_{i,j})}{(1+a_{i,i}+a_{i,j})^2c_j+(1+a_{i,i})^2\zeta_i+(a_{i,j})^2\zeta_j+b_i}$
K	$K = k_1 + k_2$
$f(k_1, k_2)$	Returns equilibrium MSE
$\rho_{i,i}$	See (33)
$\rho_{i,j}$	See (34)
π_i	$\pi_i(y_1, y_2) = \frac{\alpha_i}{\omega}(\rho_{i,i}y_i + \rho_{i,j}y_j)$

with corresponding jamming coefficients

$$a_{i,i} = -\frac{\zeta_j - \zeta_{1,2}}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}, \quad a_{i,j} = -\frac{\zeta_i - \zeta_{1,2}}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}. \tag{10}$$

Thus, for each $i = 1, 2$, we divide our analysis into the regions

$$R_1^i = \left\{k_i \ge C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}\right\} \tag{11}$$

$$R_2^i = \left\{k_i < C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}\right\}. \tag{12}$$

In region R_1^i, there is no interesting problem to consider for channel i as the jammer can simply employ the policy in (10) regardless of the encoding/decoding policy. Furthermore, the region R_1^1 does not affect the region R_2^2; hence, there can be a nontrivial problem to consider along channel 2, even if the problem along channel 1 is trivial or vice versa.

Remark 2.1: Note that we do not assume linear/affine policies in region R_2^i *a priori*. In conducting our analysis, we consider any admissible policy and linear/affine policies arise naturally as the optimal policies.

Remark 2.2: Note that we chose the two-way channel to be symmetric in X_1, X_2, but this is not required. The problem can be formulated for a general channel of the form $Y_i = e_{i,1}X_1 + e_{i,2}X_2 + Z_i$. However, this will significantly complicate the already elaborate expressions making the problem unnecessarily tedious.

We conclude this section by referring to Table I, which provides a summary of all the main variables used in this paper.

TABLE II
JAMMER COEFFICIENTS FOR DIFFERENT RELATIONSHIPS BETWEEN THE NOISE VARIANCE AND COVARIANCE

	$a^*_{i,i}$	$a^*_{i,j}$	b^*_i
$\zeta_i > \zeta_{1,2}$	$\frac{-k_i\omega^{\frac{1}{2}} + (C+\zeta_{1,2})(k_i(C+\zeta_i-k_i))^{\frac{1}{2}}}{(C+\zeta_i)\omega^{\frac{1}{2}}}$	$-\left(\frac{k_i(C+\zeta_i-k_i)}{\omega}\right)^{\frac{1}{2}}$	0
$\zeta_i < \zeta_{1,2}$	$\frac{-k_i\omega^{\frac{1}{2}} - (C+\zeta_{1,2})(k_i(C+\zeta_i-k_i))^{\frac{1}{2}}}{(C+\zeta_i)\omega^{\frac{1}{2}}}$	$\left(\frac{k_i(C+\zeta_i-k_i)}{\omega}\right)^{\frac{1}{2}}$	0
$\zeta_i = \zeta_{1,2}$	$-\left(\frac{k_i}{C+\zeta_i}\right)$	0	$k_i\left(1-\frac{k_i}{C+\zeta_i}\right)$

III. MAIN RESULTS

A. Equilibrium Policies in the Nondegenerate Case

We consider a system where $(k_1, k_2) \in R^1_2 \times R^2_2$. If either k_i were to belong to the R^i_1 region, then the jammer can just send ν_i according to the policy described in (10), and the transmitter and receiver policies are irrelevant. Define

$$\omega = C(\zeta_1 + \zeta_2 - 2\zeta_{1,2}) + \zeta_1\zeta_2 - \zeta^2_{1,2}. \tag{13}$$

We then have the following theorem.

Theorem 3.1: Fix $(k_1, k_2) \in R^1_2 \times R^2_2$.

1) There exist four saddle-point solutions $(\gamma^*_1, \gamma^*_2, \delta^*_1, \delta^*_2, \mu^*_1, \mu^*_2)$ depending on whether the transmitter uses $\sqrt{c_i}$ or $-\sqrt{c_i}$. Assuming both transmitters use the positive amplification, the equilibrium policies for the system are given by

$$\gamma^*_i(u_i) = \sqrt{c_i}u_i \tag{14}$$

$$\delta^*_i(q_i, u_i) = \alpha_i(q_i - (1 + a^*_{i,i} + a^*_{i,j})\sqrt{c_i}u_i) \tag{15}$$

$$\beta^*_i(y_1, y_2) = a^*_{i,i}y_i + a^*_{i,j}y_j + \eta_i \tag{16}$$

where $\eta_i \sim \mathcal{N}(0, b^*_i)$ is a Gaussian signal with zero mean and variance b^*_i that is independent of the system signals

$$\alpha_i = \frac{\sqrt{c_j}(1 + a^*_{i,i} + a^*_{i,j})}{G}$$
$$G = (1 + a^*_{i,i} + a^*_{i,j})^2 c_j + (1 + a^*_{i,i})^2\zeta_i + (a^*_{i,j})^2\zeta_j + (1 + a^*_{i,i})(a^*_{i,j})\zeta_{1,2} + b^*_i \tag{17}$$

and the coefficients $a^*_{i,i}, a^*_{i,j}, b^*_i$ are detailed in Table II, depending on the relationship between ζ_i and $\zeta_{1,2}$.

2) Furthermore, the zero-sum Nash equilibria are essentially unique up to the changes of the signs of the encoding/decoding coefficients.

Proof: The proof is presented in Appendix A. ■

B. Equilibrium Policies in the Degenerate Case

We next consider the degenerate case of having identical noise signals in both the channel directions ($Z_1 = Z_2$). In this case, Y_1 and Y_2 are the same signal and hence the jammer only has access to one unique signal. To address this scenario, we consider the modified problem where the jamming signal is given by $\nu_i = \beta_i(y_i)$, that is the jammer makes no use of y_j. Again, β_i is in general a random mapping.

We must redefine our regions of interest since the game may become uninteresting in the sense that the signal is fully cancelled at the receiver when $\beta_i(y_i) = -y_i$; the lowest power constraint, which admits this policy is $k_i = C + \zeta_i$. Therefore, defining the regions

$$\tilde{R}^i_1 = \{k_i \geq C + \zeta_i\} \tag{18}$$

$$\tilde{R}^i_2 = \{k_i < C + \zeta_i\} \tag{19}$$

we obtain the following result.

Theorem 3.2: Fix $(k_1, k_2) \in \tilde{R}^1_2 \times \tilde{R}^2_2$ and assume that the jammer only has access to signal y_i to jam channel i. Then, there exist four saddle-point solutions $(\gamma^*_1, \gamma^*_2, \delta^*_1, \delta^*_2, \mu^*_1, \mu^*_2)$ depending on whether the transmitters use $\pm\sqrt{c_i}$. If the transmitters use positive amplification, the equilibrium policies are

$$\gamma^*_i(u_i) = \sqrt{c_i}u_i \tag{20}$$

$$\beta^*_i(y_i) = a^*_i y_i + \eta_i \tag{21}$$

$$\delta^*_i(q_i, u_i) = \frac{\sqrt{c_j}}{c_j + \zeta_i - a^*_i c_i}(q_i - (1 + a^*_i)(\sqrt{c_i}u_i)) \tag{22}$$

$$a^*_i = -\frac{k_i}{C + \zeta_i} \qquad \eta_i \sim \mathcal{N}\left(0, \left(1 - \frac{k_i}{C + \zeta_i}\right)k_i\right). \tag{23}$$

Furthermore, when the jammer is given access to only signal y_i to jam channel i, the zero-sum Nash equilibria are essentially unique up to change in sign of the encoding/decoding coefficients.

Proof: The proof follows a similar approach as the one for the nondegenerate case and is presented in Appendix B. ■

Remark 3.1: Given access to only one signal, the jammer's optimal policy consists of sending $\beta^*(y) = -(\frac{k_i}{C+\zeta_i})y + \eta_i$. However, in the degenerate case since the jammer has access to the same signal ($Y_1 = Y_2$), any policy of the form $\beta_i(y_1, y_2) = a_{i,1}y_1 + a_{i,2}y_2 + \eta_i$, which satisfies $a_{i,1} + a_{i,2} = -(\frac{k_i}{C+\zeta_i})$ will indeed be an equilibrium policy. We can then see that there is actually an infinite number of equilibrium jamming policies, which produce the same output signal ν_i.

C. Discussion

Let us consider a special case with the parameters $c_2 = 0, \zeta_1 \to \infty, \zeta_{1,2} = 0$. These values correspond to shutting down the channel in the direction $T_2 \to T_1$, reducing the system to a one-way system going from T_1 to T_2, which falls under the analysis considered in [1]. Note that the notation used in each paper is different, but each variable in [1] has a counterpart in this paper. Staying consistent with our notation, we next show that our results agree with the results of [1]. Under our analysis, the boundary between the R_1^2 and R_2^2 regions now becomes

$$C + \left.\frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}\right|_{\zeta_1\to\infty, \zeta_{1,2}=0, c_2=0} = c_1 + \zeta_2$$

which is identical to the definition of the $R2$ region in [1] under the following notational equivalences $k^2 = k_2, c^2 = c_1, \xi_1 = \zeta_2, \sigma = 0$, where the left-hand side (LHS) terms in each identity are from [1]. The results in [1] state that for a k_2 (i.e., k^2 in [1]) value in the R_2^2 (i.e., $R2$ in [1]) region, the equilibrium jamming policy is given by

$$\nu_2 = -\left(\frac{k_2}{c_1 + \zeta_2}\right) y_2 + \eta_2$$

where η_2 is given by (23) using $i = 2$ and $C = c_1$, while our results state that the policy will be of the form $\beta_2^*(y_1, y_2) = a_{2,1}y_1 + a_{2,2}y_2$ where the values for $a_{2,1}, a_{2,2}$ are specified in the first row of Table II. These two results may at first not seem to agree: one is a combination of a negative feedback term and Gaussian noise, while the other is a linear combination of the two received signals. However, setting $\zeta_1 \to \infty$ yields

$$\begin{aligned}\lim_{\zeta_1\to\infty} a_{2,2} &= \lim_{\zeta_1\to\infty}\left(\frac{-k_2\omega^{\frac{1}{2}} + (c_1)(k_2(c_1 + \zeta_2 - k_2))^{\frac{1}{2}}}{(c_1 + \zeta_2)\omega^{\frac{1}{2}}}\right)\\ &= -\frac{k_2}{c_1 + \zeta_2}\end{aligned}$$

and transforms signal Y_1 into pure noise. Thus, in the jamming policy, the term $a_{2,1}Y_1$ acts as a zero mean Gaussian random variable (which is independent of the other system signals) with variance given by

$$\begin{aligned}\lim_{\zeta_1\to\infty} E[(a_{2,1}Y_1)^2] &= \lim_{\zeta_1\to\infty}\left(\frac{k_2(c_1 + \zeta_2 - k_2)}{c_1(\zeta_2 + \zeta_1) + \zeta_1\zeta_2}\right)(c_1 + \zeta_1)\\ &= k_2\left(1 - \frac{k_2}{c_1 + \zeta_2}\right).\end{aligned}$$

Therefore, we conclude that the two results indeed coincide.

In general, when faced with a Gaussian system, be it one-way or two-way, there are certain traits that appear in the jamming policies. There is an R_1-type region where the jammer has too much power and can fully cancel the transmitted signal before it reaches the receiver, making the game trivial. When the jammer cannot fully cancel the signal, its equilibrium policies are either linear or affine by combining a linear policy with Gaussian noise. For a two-way channel, the choice of linear or affine in the jamming policy is determined by the covariance matrix of the noise variables, Σ. If $\zeta_i \neq \zeta_{1,2}$, then a linear policy is used. If $\zeta_i = \zeta_{1,2}$ (in either the degenerate or nondegenerate case), then an affine policy is used. The reasoning for this involves an in depth discussion of the best response function used in the proof of Theorem 3.1. In essence, if the function admits a fixed point then the policy is linear, and if it does not admit a fixed point then an affine policy is used.

D. Jamming Power Allocation

Theorems 3.1 and 3.2 fully describe the problem when the jamming power levels k_1 and k_2 are fixed. Let us consider a modified problem. Assume that the covariance matrix Σ is fixed, and fix the transmitter power levels c_1, c_2. Maintain the constraint (4), but now allow

$$k_1 + k_2 \leq K \tag{24}$$

the jammer then has an overall budget that it can allocate to either channel as it sees fit.

From our earlier results in Section III, we know that for some choice of k_1, k_2 there exists a unique (up to change in sign of the encoding policy) Nash equilibrium. One can produce a function $f(k_1, k_2)$ that outputs the equilibrium MSE for these power constraints. If we call channel i as the channel from terminal T_j to T_i, we can see that the jamming signal v_j plays no role in the jamming of channel i. Then, we have

$$f(k_1, k_2) = \frac{1}{2}\left(g_1(k_1) + g_2(k_2)\right) \tag{25}$$

where g_i is the distortion on channel i. Note that if $k_i \in R_1^i$, then $g_i(k_i) = 1$. Furthermore, for $k_i \in R_2^i$ we can easily calculate g_i as

$$g_i(k_i) = E[(\alpha_i\tilde{Q}_i - U_j)^2] \tag{26}$$

where $\tilde{Q}_i = Q_i - (1 + a_{i,i} + a_{i,j})X_i$, that is the received signal with X_i removed. (See Appendix A for more discussion on $\tilde{Q}_i$.)

We can then formulate a constrained optimization problem (OP-1)

$$\begin{aligned}\max f(k_1, k_2) \text{ s.t.} \qquad h_1(k_1, k_2) &= k_1 \geq 0\\ h_2(k_1, k_2) &= k_2 \geq 0\\ h_3(k_1, k_2) &= k_1 + k_2 - K \leq 0.\end{aligned}$$

Note that as long as $K > 0$, all points are regular. Let $\kappa = (k_1, k_2)$ and $\xi \in \mathbb{R}^3$.

Observe that an optimal solution exists due to the continuity of f in k_1, k_2 and the compactness of the set of feasible allocations. By [38, Th. 1, p. 249], it follows that a variational stationarity condition through the introduction of a Lagrange multiplier would need to be satisfied at an optimal solution. The Karush–Kuhn–Tucker (KKT) conditions then become

$$\nabla L(\kappa, \xi) = \begin{pmatrix}\frac{\partial}{\partial k_1}(f(\kappa)) + \xi_1 + \xi_3\\ \frac{\partial}{\partial k_2}(f(\kappa)) + \xi_2 + \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\xi_1 k_1 = 0 \quad \xi_2 k_2 = 0 \quad \xi_3(k_1 + k_2 - K) = 0.$$

Therefore, we know that some optimal solution exists, and we have a necessary condition provided by the KKT conditions. The general form of our proofs will be going about finding the unique pair (k_1, k_2), which satisfies the KKT conditions, and hence must be the optimal solution.

Note that two cases in particular are of interest, the degenerate case and the uncorrelated case. In both of these cases, g_1 and g_2 are the same function since both channels use the same jamming policy from Table II. This introduces a symmetry to the problem, which results in an analytical solution for the KKT conditions.

Theorem 3.3: In the degenerate case of $Z_1 = Z_2$ (where we have $\zeta_1 = \zeta_2 = \zeta$), the jammer's optimal power allocation is as follows. For $i, j = 1, 2$ with $i \neq j$, let

$$K_i^* = \frac{(C+\zeta)\left((c_j)^{\frac{1}{2}}(c_i+\zeta) - (c_i)^{\frac{1}{2}}(c_j+\zeta)\right)}{(c_j)^{\frac{3}{2}}} \tag{27}$$

$$\hat{k}_i = \frac{(c_j)^{\frac{3}{2}}(K_i^* + K)}{(c_i)^{\frac{3}{2}} + (c_j)^{\frac{3}{2}}}. \tag{28}$$

If $K < |\min_{i=1,2} K_i^*|$, then the allocation is given by

$$(k_1, k_2) = \begin{cases} (K, 0) & \text{if } c_1(c_2+\zeta)^2 \leq c_2(c_1+\zeta)^2 \\ (0, K) & \text{if } c_1(c_2+\zeta)^2 > c_2(c_1+\zeta)^2 \end{cases}.$$

If $K \geq |\min_{i=1,2} K_i^*|$, the jammer allocates according to

$$(k_1, k_2) = \begin{cases} \left(\min(\hat{k}_1, C+\zeta), K - k_1\right) \\ \text{if } c_1(c_2+\zeta)^2 \leq c_2(c_1+\zeta)^2 \\ \left(K - k_2, \min(\hat{k}_2, C+\zeta_2)\right) \\ \text{if } c_1(c_2+\zeta)^2 > c_2(c_1+\zeta)^2 \end{cases}.$$

Proof: The proof is given in Appendix C. ■

Theorem 3.4: For the uncorrelated noise case ($\zeta_{1,2} = 0$), the jammer allocates as follows.

If $K < 2\left(C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}\right)$, the optimal allocation is the solution to the equation

$$\begin{aligned} &c_2(C+\zeta_1)^2 \frac{(-2\zeta_1\omega^{\frac{1}{2}})k_1 + (\omega - \zeta_1^2)x_1 + \zeta_1(\omega)^{\frac{1}{2}}(C+\zeta_1)}{x_1(\lambda_{1,4}k_1 + 2\lambda_{1,5}x_1 + \lambda_{1,6})^2} \\ &= c_1(C+\zeta_2)^2 \\ &\times \frac{(-2\zeta_2\omega^{\frac{1}{2}})k_2 + (\omega - \zeta_2^2)x_2 + \zeta_2(\omega)^{\frac{1}{2}}(C+\zeta_2)}{x_2(\lambda_{2,4}k_2 + 2\lambda_{2,5}x_2 + \lambda_6)^2} \end{aligned} \tag{29}$$

$$x_i = (k_i(C + \zeta_i - k_i))^{\frac{1}{2}}$$

$$\lambda_{i,4} = \zeta_i C^2 + \zeta_j(C+\zeta_i)^2 - \zeta_i\omega + c_j(\zeta_i^2 - \omega)$$

$$\lambda_{i,5} = \zeta_i c_i(\omega)^{\frac{1}{2}} \quad \lambda_{i,6} = (\zeta_i + c_j)(C+\zeta_i)\omega.$$

If $K \geq 2\left(C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}\right)$, the optimal allocation is $k_i = C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}$, $i = 1, 2$ so that both channels are in the R_1^i region and there is no signal reaching either terminal.

Proof: The proof is given in Appendix D. ■

Furthermore, the power allocation, which is the optimal solution to OP-1 is actually itself a Nash equilibrium, in that both the jammer and transmitter/receiver are best responding.

Theorem 3.5: Fix a correlation matrix Σ, transmitter power constraints c_1, c_2, and some overall jamming power budget K. Assume that there exists a power allocation $\kappa^* = (k_1^*, k_2^*)$ that solves OP-1. Let $\gamma^* = (\gamma_1^*, \gamma_2^*)$, $\delta^* = (\delta_1^*, \delta_2^*)$, and $\beta^* = (\beta_1^*, \beta_2^*)$ be the transmission, receiving, and jamming policies determined by Theorem 3.1 (or Theorem 3.2 if the system is degenerate) for this set of power constraints (c_1, c_2, k_1^*, k_2^*). Then, these policies form a Nash equilibrium overall transmission policies $\gamma = (\gamma_1, \gamma_2)$, which satisfy $E[\gamma_i(U_i)^2] \leq c_i$ and all jamming policies $\beta = (\beta_1, \beta_2)$, which satisfy $E(\beta_1(Y_1, Y_2)^2) + E(\beta_2(Y_1, Y_2)^2) \leq K$

Proof: The proof is given in Appendix E. ■

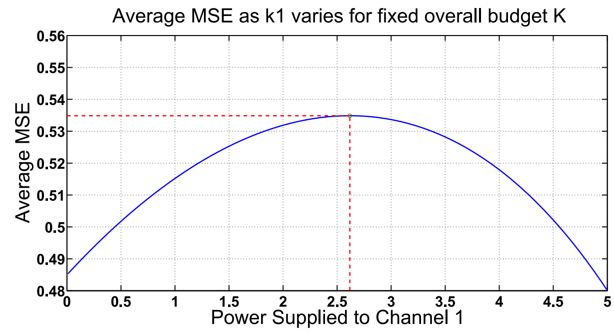

Fig. 2. System wide MSE for Example 1, as we vary k_1 and let $k_2 = K - k_1$.

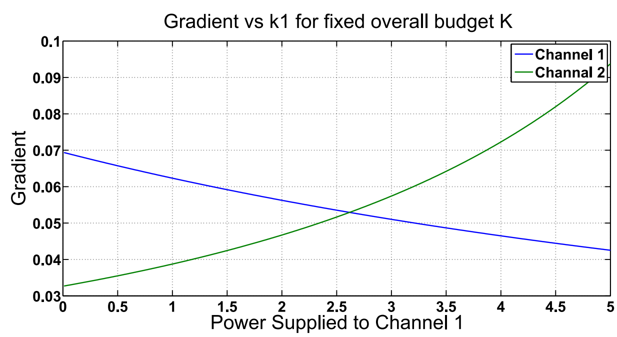

Fig. 3. Gradients of the MSE along channels 1 and 2, as we vary k_1 and let $k_2 = K - K_1$. Gradients intersect at peak system wide power allocation.

IV. Examples

A. Example 1

Consider a system with parameters $c_1 = 3, c_2 = 5, K = 5, \zeta_1 = \zeta_2 = \zeta_{1,2} = 1$. Since this choice of parameters yield the degenerate case, we use Theorem 3.3 to determine the power allocation and Theorem 3.2 to find the policy structures. We have that $K_1^* = -1.1656$ and $K_2^* = 2.5081$. We also have that $K > |\min_{i=1,2} K_i^*|$; hence the jammer will employ a power splitting policy between the two channels. Furthermore, the following inequalities hold: $c_1(c_2+\zeta)^2 > c_2(c_1+\zeta)^2$ and $\hat{k}_2 = 2.3823 < C + \zeta$. Thus, the power allocation is given by $(k_1, k_2) = (2.6177, 2.3823)$. This can be seen in Fig. 2, which displays the system wide MSE for different values of k_1. The peak occurs at $k_1 = 2.6177$. In Fig. 3, we plot the gradients of the MSE along each channel as we vary k_1. The power allocation, which causes the gradients to intersect is exactly the one resulting in the highest system wide MSE, as inferred in the proof of Theorem 3.3 found in Appendix C.

With the determined power allocation, we obtain via Theorem 3.2 the following transmitter, receiver, and jammer

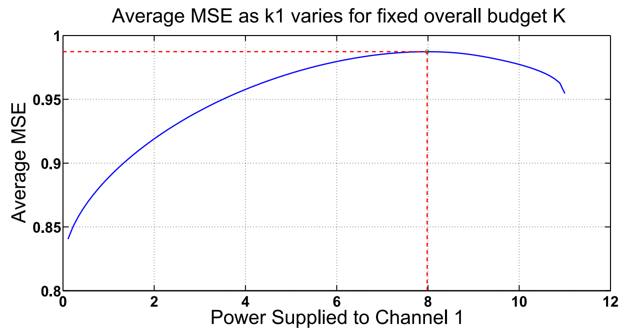

Fig. 4. System wide MSE for Example 2, as we vary k_1 and let $k_2 = K - k_1$.

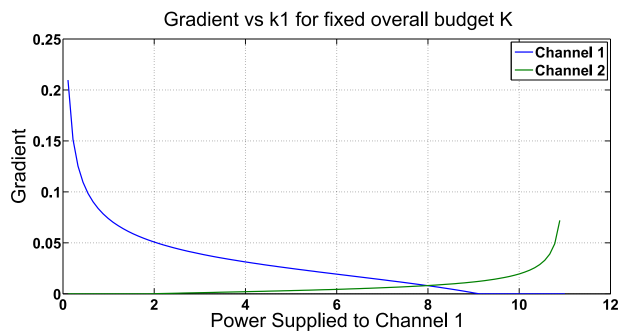

Fig. 5. Gradients of the MSE along channels 1 and 2, as we vary k_1 and let $k_2 = K - K_1$. Gradients intersect at peak system wide power allocation.

policies:

$$\begin{aligned}\gamma_1(u_1) &= \sqrt{3}u_1 \quad \delta_1(q_1, u_1)\\ &= 0.325(q_1 - (0.71)(\sqrt{3}u_1))\\ \beta_1(y_1, y_2) &= a_{1,1}y_1 + a_{1,2}y_2 + \eta_1\\ a_{1,1} + a_{1,2} &= -(0.29) \quad \eta_1 \sim N(0, 1.856)\\ \gamma_2(u_2) &= \sqrt{5}u_2 \quad \delta_2(q_2, u_2)\\ &= 0.325(q_2 - (0.735)(\sqrt{5}u_2))\\ \beta_2(y_1, y_2) &= a_{2,1}y_1 + a_{2,2}y_2 + \eta_2\\ a_{2,1} + a_{2,2} &= -0.265 \quad \eta_2 \sim N(0, 1.752).\end{aligned}$$

B. Example 2

Consider now an uncorrelated system with parameters $c_1 = 1, c_2 = 4, K = 11, \zeta_1 = 7, \zeta_2 = 10, \zeta_{1,2} = 0$. This system is solved using Theorem 3.4. We have that $K < 2(C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2})$; thus the optimal jammer power allocation is also to split power between the two channels in order to solve (29). While this equation is difficult to solve directly, using a bisection method the allocation is approximately $(k_1, k_2) = (7.98, 3.12)$. This allocation is seen in Fig. 4 where the system wide MSE peaks, it also corresponds to the allocation, which equates the gradients along the two channels seen in Fig. 5. We then calculate the transmitter, receiver, and jammer policies using Theorem 3.1.

Since the correlation is zero, on both channels the jammer will use the policy outlined in the first row of Table II. We have

$$\begin{aligned}\gamma_1(u_1) = u_1 \quad \delta_1(q_1, u_1) &= 0.0346(q_1 - (0.0694)u_1)\\ \beta_1(y_1, y_2) &= -0.4757y_1 - 0.4549y_2\\ \gamma_2(u_2) = 2u_2 \quad \delta_2(q_2, u_2) &= 0.0431(q_2 - 0.9536u_2)\\ \beta_2(y_1, y_2) &= -0.4830y_1 - 0.0402y_2.\end{aligned}$$

V. Concluding Remarks

The results established in this paper provide a full set of solutions for the two-way communication system presented in Fig. 1, with independent Gaussian sources and arbitrarily correlated Gaussian noise signals. The results in many ways provide a natural extension of [1] to a two-way system, maintaining the existence of a Nash equilibrium and the optimality of linear/affine policies. In the special degenerate case, there is actually an infinite set of equilibrium jamming policies due to signals Y_1 and Y_2 being identical. There are a number of intricacies for the two-way system that make the analysis more complicated than the one-way case. The correlation between the noise signals now plays a pertinent role in determining the jamming policy.

We note that the assumption in this paper that the encoders have access to Gaussian data that are not dependent is essential, for the counterexamples presented in both [16, Ch. 11] and [30] imply that when noisy side information is available at the decoder, the optimal encoder may not be linear through a Gaussian channel, which would precisely be the setup in the context of this paper.

Extensions of this paper include examining correlation between the sources, which would alter how terminal T_i decides to decode with side information X_i, as well as considering non-Gaussian noise and source variables. Another worthwhile future direction is to investigate the vector (finite-time horizon) setup of the problem in the sense that the two-way channel is time correlated and each user utilizes the channel multiple times (via a block encoding operation) to convey its source to the other user, hence necessitating it to interactively adapt its codeword to the previously received signals.

Appendix A
Proof of Theorem 3.1

We proceed in two steps by separately validating the right-hand side (RHS) and the LHS inequalities of (6) under the stated policies in (14)–(16). We then examine the uniqueness of the equilibrium.

A. RHS Inequality

Assume that the jammer is using an affine policy of the form (16), not necessarily with the assumed equilibrium values $(a^*_{i,i}, a^*_{i,j}, b^*_i)$, but any general values $(a_{i,i}, a_{i,j}, b_i)$. We can view the two-way system as two separate one-way channels with each channel going from transmitter $\gamma_j(\cdot)$ to receiver $\delta_i(\cdot)$ as displayed in Fig. 6. In the figure, $\tilde{\gamma}_j(u_j) = (1 + a_{i,i} +$

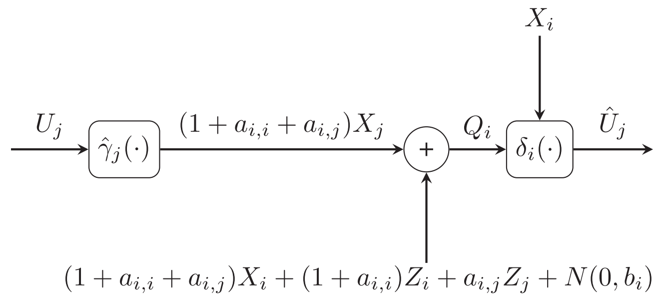

Fig. 6. Equivalent one-way Gaussian test system in the direction $T_j \to T_i$ for nondegenerate case.

$a_{i,j})\gamma_j(u_j)$, as the jammer is amplifying the signal X_j by sending back a scaled version of Y_i, Y_j to the channel. Then, the power constraint on the transmitter is $E[\tilde{\gamma}_i(u_i)^2] \le c_i(1 + a_{i,i} + a_{i,j})^2$. Note that the receiver has side information about the signal X_i since terminal T_i is composed of both the transmitter γ_i and receiver δ_i (see Fig. 1). This setup has a classical solution building on the data-processing inequality of information theory: the transmitter will amplify the signal as much as possible and the receiver will use the minimal MSE estimator $\delta_i^*(q_i, u_i) = E[U_j | Q_i = q_i, U_i = u_i]$ (see [16, Ch. 11], or [17]–[20]). Thus, $\tilde{\gamma}_i^*(u_i) = \sqrt{c_i}(1 + a_{i,i} + a_{i,j})u_i$, which yields that $\gamma_i^*(u_i) = \sqrt{c_i}u_i$. Note that $\gamma_i^*(u_i) = -\sqrt{c_i}u_i$ is also a valid solution; hence there are four possible equilibrium policies as stated in the theorem. We determine the best receiver policy when both transmitters use positive amplification (the policies for the other equilibria can be found by simply changing the sign of $\sqrt{c_i}$ in the transmitter and receiver functions). Writing $Q_i = \tilde{Q}_i + (1 + a_{i,i} + a_{i,j})\sqrt{c_i}U_i$ and noting that U_i is independent of $\tilde{Q}_i$ and U_j, it can be shown that the receiver policy satisfies

$$\delta_i^*(q_i, u_i) = \frac{E[U_j \tilde{Q}_i]}{E[\tilde{Q}_i^2]}\tilde{q}_i = \alpha_i \tilde{q}_i$$

where α_i is given in (17). This verifies the RHS inequality of (6).

B. LHS Inequality

Assuming the policies in (14) and (15) at each terminal, again for some general $(a_{i,i}, a_{i,j}, b_i)$ and not the assumed equilibrium values, the maximization problem faced by the jammer is given by

$$\max_{(\mu_1,\mu_2)\in\mathcal{M}} \frac{1}{2}\left(\sum_{i=1}^{2}\int_{-\infty}^{\infty} E\left[\left(\alpha_i\tilde{Q}_i - U_j\right)^2 \middle| \nu_i\right] d\mu_i(\nu_i)\right). \tag{30}$$

Setting

$$\tilde{Q}_i = W_i + \nu_i \tag{31}$$

$$W_i = X_j + Z_i - (a_{i,i} + a_{i,j})X_i \tag{32}$$

we expand the ith MSE term in the abovementioned sum as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} E\left[\left(\alpha_i\tilde{Q}_i - U_j\right)^2 \middle| \nu_i\right] d\mu_i(\nu_i)\\
&= E_{Y_1,Y_2}\left[\int_{-\infty}^{\infty} E\left[\left(\alpha_i\tilde{Q}_i - U_j\right)^2 \middle| \nu_i, y_1, y_2\right] d\mu_i(\nu_i|y_1,y_2)\right]\\
&= \alpha_i^2(\zeta_i + (a_{i,i} + a_{i,j})^2 c_i) + (\alpha_i\sqrt{c_j} - 1)^2 +\\
&E_{Y_1,Y_2}\left[\int_{-\infty}^{\infty} \alpha_i^2\nu_i^2 + 2\nu_i E[\alpha_i^2 W_i - \alpha_i U_j | y_1, y_2]\right.\\
&\qquad \left. \times\, d\mu_i(\nu_i|y_1,y_2)\right].
\end{aligned}$$

It can be shown that

$$E[\alpha_i^2 W_i - \alpha_i U_j | y_1, y_2] = \pi_i(y_1, y_2)$$

where

$$\begin{aligned}
\rho_{i,i} &= \alpha_i[c_j(\zeta_i + \zeta_j - 2\zeta_{1,2}) + c_i(\zeta_i - \zeta_{1,2} - (\zeta_j - \zeta_{1,2})\\
&\qquad \times (a_{i,i} + a_{i,j})) + \zeta_1\zeta_2 - \zeta_{1,2}^2] - (\zeta_j - \zeta_{1,2})\sqrt{c_j}
\end{aligned} \tag{33}$$

$$\rho_{i,j} = -\alpha_i(\zeta_i - \zeta_{1,2})c_i(1 + a_{i,i} + a_{i,j}) - (\zeta_i - \zeta_{1,2})\sqrt{c_j} \tag{34}$$

$$\pi_i(y_1, y_2) = \frac{\alpha_i}{\omega}\left(\rho_{i,i}y_i + \rho_{i,j}y_j\right). \tag{35}$$

Ignoring the terms, which are independent of ν_i, we can focus on determining

$$J = \max_{\mu_1,\mu_2}\sum_{i=1}^{2} E[\alpha_i^2\nu_i^2] + 2E[\nu_i\pi_i(Y_1, Y_2)]. \tag{36}$$

Applying the Cauchy–Schwarz inequality and the power constraint (4), we have

$$J \le \sum_{i=1}^{2}\alpha_i^2 k_i + 2(k_i)^{\frac{1}{2}} E[\pi_i(Y_1, Y_2)^2]^{\frac{1}{2}}. \tag{37}$$

From here we break our analysis into two parts.

Case 1: $\zeta_i \ne \zeta_{1,2}$

We have that $E[\pi_i(Y_1, Y_2)^2]^{\frac{1}{2}} \ne 0$; thus we can uniquely achieve the upper bound with the linear jammer policy

$$\begin{aligned}
\nu_i = \beta_i^*(y_1, y_2) &= \frac{(k_i)^{\frac{1}{2}}}{E[\pi_i(Y_1, Y_2)^2]^{\frac{1}{2}}}\pi_i(y_1, y_2)\\
&= \frac{(k_i)^{\frac{1}{2}}\left(p_{i,i}y_i + p_{i,j}y_j\right)}{(C(p_{i,i} + p_{i,j})^2 + p_{i,i}^2\zeta_i + p_{i,j}^2\zeta_j + 2\rho_{i,i}\rho_{i,j}\zeta_{1,2})^{\frac{1}{2}}}.
\end{aligned} \tag{38}$$

Therefore, the best response jamming policy to transmission and receiving policies (14) and (15) is linear, yet these policies were constructed when faced with an affine (linear if $b_i = 0$) jamming policy. Setting

$$\lambda_i = \left(\frac{k_i}{E[\pi_i(Y_1, Y_2)^2]}\right)^{\frac{1}{2}} \tag{39}$$

for each linear jamming policy with coefficients $(a_{i,i}, a_{i,j})$, we use (38) to define the best response mapping as follows:

$$T\begin{pmatrix} a_{i,i} \\ a_{i,j} \end{pmatrix} \mapsto \begin{pmatrix} (\lambda_i)\rho_{i,i} \\ (\lambda_i)\rho_{i,j} \end{pmatrix}. \tag{40}$$

The equilibrium policy then has coefficients $(a^*_{i,i}, a^*_{i,j})$, which are the fixed point of this mapping and satisfy (4) for a given k_i. However, directly finding this fixed point is computationally difficult. We instead consider an equivalent set of conditions, which can be proved via a geometric approach.

Proposition A.1: A sufficient and necessary set of conditions for equilibria are

$$a_{i,i}\rho_{i,j} = a_{i,j}\rho_{i,i} \tag{41}$$

$$E[(a_{i,i}Y_i + a_{i,j}Y_j)^2] = k_i \tag{42}$$

$$\frac{a_{i,i}}{|a_{i,i}|} = \frac{\rho_{i,i}}{|\rho_{i,i}|} \tag{43}$$

$$\frac{a_{i,j}}{|a_{i,j}|} = \frac{\rho_{i,j}}{|\rho_{i,j}|}. \tag{44}$$

Proof: Assume that coefficients $a_{i,i}$ and $a_{i,j}$ are equilibrium policies. Then, via being a fixed point of T, we have that $a_{i,i} = (\lambda_i)\rho_{i,i}$ and $a_{i,j} = (\lambda_i)\rho_{i,j}$, thus

$$a_{i,i}\rho_{i,j} = \lambda_i\rho_{i,i}\rho_{i,j} = a_{i,j}\rho_{i,i}$$

and hence condition (41) is necessary. Also

$$E[(a_{i,i}Y_i + a_{i,j}Y_j)^2] = \lambda_i^2 E[\pi_i(Y_1, Y_2)^2] = k_i$$

and thus condition (42) is also necessary. Furthermore, we have

$$\frac{a_{i,i}}{|a_{i,i}|} = \left(\frac{\lambda_i}{|\lambda_i|}\right)\frac{\rho_{i,i}}{|\rho_{i,i}|} = \frac{\rho_{i,i}}{|\rho_{i,i}|}$$

since $\lambda_i > 0$; thus condition (43) is necessary as well. A similar argument shows that condition (44) is necessary. This concludes one direction of the proof.

We next show the reverse direction, by proving that any pair of coefficients $(a_{i,i}, a_{i,j})$ satisfying conditions (41)–(44), forms a fixed point of T; therefore $a_{i,i}$ and $a_{i,j}$ are equilibrium policies. By conditions (43) and (44) we have that $(a_{i,i}, a_{i,j})$ and $T(a_{i,i}, a_{i,j})$ must exist in the same quadrant on the Cartesian plane. Combining this fact with (41) yields that they both have the same polar angle. Finally, condition (42) means that they both live on the same ellipse. Therefore, points $(a_{i,i}, a_{i,j})$ and $T(a_{i,i}, a_{i,j})$ live on the same ellipse, in the same quadrant, and have the same polar angle; thus they form an identical point and hence the conditions are sufficient. ■

Lemma A.1: The ellipse described by

$$(C+\zeta_i)(a_{i,i}+0.5)^2 + 2a_{i,j}(a_{i,i}+0.5)(C+\zeta_{1,2}) + (C+\zeta_2)a_{i,j}^2 = \frac{C+\zeta_i}{4} \tag{45}$$

satisfies condition (41).

Lemma A.2: The ellipse defined by

$$a_{i,i}^2(C+\zeta_i) + 2a_{i,i}a_{i,j}(C+\zeta_{1,2}) + a_{i,j}^2(C+\zeta_j) = k_i$$

satisfies condition (42).

Intersecting these two ellipses gives two possible solutions, one in each half plane. We can then use the remaining conditions (43) and (44) to find the only sufficient and necessary solution, which yields the first two rows of Table II.

Case 2: $\zeta_i = \zeta_{1,2}$

If $\zeta_i = \zeta_{1,2}$, then $\rho_{i,j} = 0$ for any affine jamming policy. Thus, $E[\pi_i(Y_1, Y_2)^2] \neq 0$ is not guaranteed. Revisiting (36), if there were a choice of $a_{i,i}, a_{i,j}, b_i$, which forced $\rho_{i,i} = 0$, then $J = \max_{\mu_1,\mu_2}\sum_{i=1}^2 \alpha_i^2 E[\nu_i^2] \le \alpha_i^2 k_i$, which is clearly maximized by a policy with variance k_i. Therefore, the equilibrium policy when $\zeta_i = \zeta_{1,2}$ must force $\rho_{i,i} = 0$ and achieve variance k_i, which is the policy given in the third row of Table II.

C. Uniqueness of the Equilibrium Policies

Let the presented equilibrium be referred to as

$$\Lambda^* = (\gamma_1^*, \gamma_2^*, \beta_1^*, \beta_2^*, \delta_1^*, \delta_2^*)$$

and denote any other solution by

$$\Lambda = (\gamma_1, \gamma_2, \beta_1, \beta_2, \delta_1, \delta_2).$$

If Λ is also a Nash equilibrium, then the *ordered interchangeability* property of saddle points in zero-sum games [37, Corollary 2.1, Property 4.1, p. 177] would require that both

$$(\gamma_1^*, \gamma_2^*, \beta_1, \beta_2, \delta_1^*, \delta_2^*) \tag{46}$$

$$(\gamma_1, \gamma_2, \beta_1^*, \beta_2^*, \delta_1, \delta_2) \tag{47}$$

are also Nash equilibria (note that the fact that there are multiple encoders as a team does not violate the *rectangularity* condition of the policies, that is the policy sets do not depend on the realized policies of the opposing player, the jammer).

Considering the first candidate solution (46), when faced with γ_i^*, δ_i^*, an optimal jamming solution must achieve the maximum of (36). When $\zeta_i \neq \zeta_{1,2}$, the upper bound of the Cauchy–Schwarz inequality can only be achieved when ν_i is linearly dependent on $\pi_i(Y_1, Y_2)$ and has variance k_i, of which there are following two possibilities:

$$\pm\left(\frac{k_i}{E[\pi_i(Y_1, Y_2)^2]}\right)^{\frac{1}{2}}\pi_i(y_1, y_2).$$

However, the negative option would result in a negative second term in (36), lowering the overall result. Therefore, β_i^* is the only policy to achieve the upper bound in (36).

When $\zeta_i = \zeta_{1,2}$, there is only one policy that can force $\rho_{i,i} = 0$ and achieve variance k_i. Any other policy has $E[(\pi_i(y_1, y_2)^2] \neq 0$, forcing the equilibrium policy to be a scaled version of $\pi_i(y_1, y_2)$, which achieves variance k_i. Since $\rho_{i,j} = 0$, there are following two possibilities:

$$\beta_i(y_1, y_2) = \pm\left(\frac{k_i}{C+\zeta_i}\right)^{\frac{1}{2}} y_i.$$

The best response mapping (40) simplifies to

$$T\begin{pmatrix} a_{i,i} \\ a_{i,j} \end{pmatrix} = \begin{pmatrix} -\left(\frac{k_i}{C+\zeta_i}\right)^{\frac{1}{2}}\frac{a_{i,i}}{|a_{i,i}|} \\ 0 \end{pmatrix}. \tag{48}$$

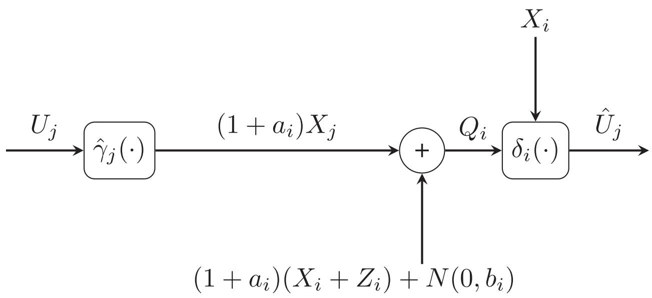

Fig. 7. Equivalent one-way Gaussian test system in the direction $T_j \to T_i$ for degenerate case.

We see that the best response flips the sign of $a_{i,i}$; therefore, each of these possible policies is the best response of the other and neither is a fixed point of T.

For the second candidate solution (47), as mentioned in the theorem, the transmitter can choose $\pm\sqrt{c_i}$ and the receiver can follow suit and both of these policies are optimal with respect to the jamming policy. Also, the jamming policy only depends on c_i and c_j, it is irrelevant to the jammer if the transmitter used positive or negative amplification. Therefore, there exist four Nash equilibria, which have the same jamming policy, and have transmission and receiving policies, which are identical up to the absolute value of $\sqrt{c_i}$. Hence, the equilibrium is essentially unique.

Appendix B
Proof of Theorem 3.2

A. RHS Inequality

Assuming the jammer uses a policy of the form seen in (21), we can do a similar analysis as before and see that the TWC systems have two one-way channels— each going from transmitter $\gamma_j(\cdot)$ to receiver $\delta_i(\cdot)$. We view the channel as a Gaussian test channel displayed in Fig. 7.

Then, following the same analysis on this channel we can show that the transmitter will amplify the signal with either $\pm\sqrt{c_i}$, and the receiver will compute the Bayes estimator with side information of u_i

$$\delta_i(q_i, u_i) = E[u_j|q_i, u_i] = \frac{\sqrt{c_j}}{(1+a_i)(c_j+\zeta_i)+\frac{b_i}{1+a_i}}\tilde{q}_i.$$

B. LHS Inequality

Assume that the transmitter and receiver are using their equilibrium policies. Let $\alpha_i = \frac{\sqrt{c_j}}{(1+a_i)(c_j+\zeta_i)+\frac{b_i}{1+a_i}}$. As before, we consider

$$\int_{-\infty}^{\infty} E\left[(\alpha_i\tilde{q}_i - u_j)^2\middle|\nu_i\right] d\mu_i(\nu_i)$$
$$= E_{y_i}\Bigg[\int_{-\infty}^{\infty}(\alpha_i^2\nu_i^2 + 2\alpha_i\nu_i E[\alpha_i(X_j+Z_i-a_iX_i)$$
$$-U_j|\,y_i])d\mu_i(\nu_i|y_i)\Bigg] + \alpha_i^2(c_j+\zeta_i+a_i^2c_i) + -2\alpha_i\sqrt{c_j}+1.$$

TABLE III
Possible Constraints That Yield a Solution to the KKT Conditions

ξ_1	ξ_2	ξ_3	Conditions
0	0	$\neq 0$	$k_1>0, k_2>0, k_1+k_2=K$
$\neq 0$	0	$\neq 0$	$k_1=0, k_2>0, k_1+k_2=K$
0	$\neq 0$	$\neq 0$	$k_1>0, k_2=0, k_1+k_2=K$

Setting

$$\pi_i(y_i) = \left(\frac{\alpha_i c_j + z_i + c_i a_i - \sqrt{c_j}}{C+z_i}\right) y_i = \rho_i y_i. \tag{49}$$

We can then arrive at an analogous expression (36). Now, similar to abovementioned Case 2, we have that (a_i^*, b_i^*) force $\rho_i = 0$ and achieve the power constraint k_i, therefore, jamming policy (21) satisfies the LHS inequality of (6). We can then use the ordered interchangeability property to show that the solution is essentially unique, that is, the transmitter can use positive or negative amplification, which produces four possible solutions.

Appendix C
Proof of Theorem 3.3

For the degenerate case, we have that

$$\frac{\partial}{\partial k_i} f(\kappa) = \frac{c_j(C+\zeta_i)^2}{2((c_j+\zeta_i)^2 + c_i(c_j+k_i+\zeta_i))^2} \tag{50}$$

we now go about solving the KKT conditions. Recall an optimal solution that exists due to continuity and compactness, and if we can find a unique pair of (k_1, k_2), which satisfies the KKT conditions, it must be the optimal policy. Since the constraints are not tight, we will by process of elimination determine which of the constraints should be treated as tight or not. As we can see, the derivative is always positive, hence adding more power to a channel increases the MSE and the constraint $k_1 + k_2 \leq K$ should be tight for an optimal solution. It is not possible for all the three constraints to be tight simultaneously, hence we really have three options to consider as outlined in Table III.

When $\xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0$, the KKT condition becomes

$$\nabla L(\kappa, \xi) = \begin{pmatrix} \frac{\partial}{\partial k_1}(f(\kappa)) + \xi_3 \\ \frac{\partial}{\partial k_2}(f(\kappa)) + \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the only way that a solution could exist is if

$$\frac{\partial}{\partial k_1}(f(\kappa)) = \frac{\partial}{\partial k_2}(f(\kappa)). \tag{51}$$

That is, if we could find a power level that makes the partial derivatives equal.

If $\xi_i \neq 0, \xi_j = 0$, in this case we force one of the $k_i = 0$, which means $k_j = K$. Then, the Lagrangian condition becomes

$$\nabla L(\kappa, \xi) = \begin{pmatrix} \frac{\partial}{\partial k_i}(f(\kappa)) + \xi_3 \\ \frac{\partial}{\partial k_j}(f(\kappa)) + \xi_j + \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We must then have

$$\xi_3 = -\frac{\partial}{\partial k_i} f(\kappa) \qquad \xi_j = -\left(\frac{\partial}{\partial k_j} f(\kappa) - \frac{\partial}{\partial k_i} f(\kappa)\right).$$

However, ξ_j must be nonnegative since it is a greater than or equal constraint in a maximization problem.

On an asymmetrical channel $c_1 \neq c_2$, the derivatives in (50) evaluated at $k_i = 0$ are not equal, one is larger. Moreover, as one adds more power, the derivative decreases. Therefore, for small power levels the jammer cannot make the derivatives equal and supplies all power to the channel with the higher derivative yielding a unique solution to the KKT conditions of either $(K, 0)$ or $(0, K)$, depending on the system parameters. For higher power levels, the jammer can make the derivatives equal and supplies the appropriate power level to each channel to achieve equality (51). The specific values to achieve these allocations are those provided in the theorem.

Appendix D
Proof of Theorem 3.4

For the uncorrelated case, we have that

$$\frac{\partial}{\partial k_i} f(\kappa) = c_j \omega (C + \zeta_i)^2 \times \left(\frac{(-2\zeta_i \omega^{\frac{1}{2}})k_i + (\omega - \zeta_i^2)x_i + \zeta_i(\omega)^{\frac{1}{2}}(C + \zeta_i)}{x_i(\lambda_{i,4}k_i + 2\lambda_{i,5}x_i + \lambda_{i,6})^2} \right) \tag{52}$$

where the various notations are previously described in the theorem. We see that evaluated at $k_i = 0$ the derivative is ∞, and evaluated at $k_i = K$ it is some finite positive value. Therefore, for any power budget K we can find an allocation (k_1, k_2) such that the derivatives are equal, which is exactly the condition for optimality in the theorem.

Appendix E
Proof of Theorem 3.5

Let $\kappa = (k_1^*, k_2^*)$ be the power allocation that maximizes $f(k_1, k_2)$. Let $\gamma = (\gamma_1, \gamma_2), \delta = (\delta_1, \delta_2)$, and $\beta = (\beta_1, \beta_2)$ be the equilibrium policies determined by Theorem 3.1 (or Theorem 3.2 if the system is degenerate) for this set of power constraints. Clearly, if we fix the jamming policy β then γ and δ satisfy the RHS of inequality (6); therefore we focus on the LHS of the inequality.

Assume that the transmitter and receiver employ policies γ and δ designed with respect to (c_1, c_2, k_1^*, k_2^*). Clearly, if the jammer chooses the allocation $(k_1, k_2) = (k_1^*, k_2^*)$, any policy, which is not β will have a lower MSE, since β is the Nash equilibrium policy for this power allocation as proven previously.

Assume that faced with the fixed policies γ and δ, the jammer tries to find another allocation $(k_1, k_2) \neq (k_1^*, k_2^*)$ that achieves a higher MSE. Following the LHS analysis of Appendix A (or Appendix B), by (37) for a fixed set of transmission and receiving policies γ and δ, if the jammer has a power allocation (k_1, k_2) then the average MSE is upper bounded by

$$\frac{1}{2}\left(\sum_{i=1}^{2} \alpha_i^2 k_i + \sqrt{k_i} E\left[(\pi_i(y_1, y_2))^2\right]^{\frac{1}{2}}\right). \tag{53}$$

Let us now consider the degenerate and uncorrelated cases separately. For the degenerate case, γ and δ will force $\pi_i(y_1, y_2) = 0$ and therefore the jammer chooses the allocation that maximizes

$$\alpha_1^2 k_1 + \alpha_2^2 k_2. \tag{54}$$

However, for the degenerate case we have

$$a_i = -\left(\frac{k_i^*}{C + \zeta_i}\right) \qquad b_i = k_i^*\left(1 - \frac{k_i^*}{C + \zeta_i}\right)$$

$$\alpha_i = \frac{\sqrt{c_i}}{(1 + a_i)(c_j + \zeta_i) + \frac{b_i}{1 + a_i}} = \frac{\sqrt{c_i}(C + \zeta_i)}{k_i^* c_i + (C + \zeta_i)(c_j + \zeta_i)}.$$

Thus, α_i^2 is exactly the expression we see for $\frac{\partial}{\partial k_i} f(\kappa)$ in (50) and (k_1^*, k_2^*) satisfy the KKT conditions. Therefore, when $(k_1^*, k_2^*) = (K, 0), \alpha_1 > \alpha_2$, so the optimal solution to (54) is again $(k_1, k_2) = (K, 0)$. Similarly, when $(k_1^*, k_2^*) = (0, K), \alpha_2 > \alpha_1$ and the optimal solution to (54) is $(k_1, k_2) = (0, K)$. Finally, if $\alpha_1 = \alpha_2$ then any allocation is the same under (54), hence $(k_1, k_2) = (k_1^*, k_2^*)$ is not outperformed by any other policy. Therefore, when we design γ and δ with respect to (c_1, c_2, k_1^*, k_2^*), we see that $(k_1, k_2) = (k_1^*, k_2^*)$ maximizes (54) and therefore satisfies the LHS inequality.

For the uncorrelated case, now $E[(\pi_i(y_1, y_2)^2]^{\frac{1}{2}} \neq 0$ and (53) is maximized by the choice of k_1, k_2 such that

$$\alpha_1^2 + \frac{1}{2}\left(\frac{E[(\pi_1(y_1, y_2)^2]}{k_1}\right)^{\frac{1}{2}} = \alpha_2^2 + \frac{1}{2}\left(\frac{E[(\pi_2(y_1, y_2)^2]}{k_2}\right)^{\frac{1}{2}}.$$

One can see that the LHS and RHS of this equation are symmetrical, and it can be shown that expression is the same as $\frac{\partial}{\partial k_i} f(\kappa)$ given in (52). Therefore, if we design γ and δ with respect to (c_1, c_2, k_1^*, k_2^*), we see that $(k_1, k_2) = (k_1^*, k_2^*)$ maximizes (53) and hence satisfies the LHS inequality.

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