NORMALIZERS AND CENTRALIZERS OF CYCLIC SUBGROUPS GENERATED BY LONE AXIS FULLY IRREDUCIBLE OUTER AUTOMORPHISMS

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Abstract. We let φ be an ageometric fully irreducible outer automorphism so that its Handel-Mosher [HM11] axis bundle consists of a single unique axis (as in [MP13]). We show that the centralizer $Cen(\langle\varphi\rangle)$ of the cyclic subgroup generated by φ equals the stabilizer $\mathrm{Stab}(\Lambda^+_\varphi)$ of the attracting lamination Λ^+_φ and is isomorphic to $\mathbb{Z}$. We further show, via an analogous result about the commensurator, that the normalizer $N(\langle\varphi\rangle)$ of $\langle\varphi\rangle$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.

1. Introduction

It is well known [McC94] that, given a pseudo-Anosov mapping class φ, the centralizer $Cen(\langle\varphi\rangle)$ and normalizer $N(\langle\varphi\rangle)$ of the cyclic subgroup $\langle\varphi\rangle$ are virtually cyclic. In fact, this property characterizes pseudo-Anosov mapping classes.[1]

We recall some of the history for this problem for the outer automorphism groups $\mathrm{Out}(F_r)$. In [BFH97], Bestvina, Feighn, and Handel constructed for a fully irreducible outer automorphism $\varphi \in \mathrm{Out}(F_r)$ the attracting lamination Λ^+_φ. They proved that the stabilizer $\mathrm{Stab}(\Lambda^+_\varphi)$ of Λ^+_φ in $\mathrm{Out}(F_r)$ is virtually cyclic. The centralizer $Cen(\langle\varphi\rangle)$ of φ in $\mathrm{Out}(F_r)$ is a subgroup of $\mathrm{Stab}(\Lambda^+_\varphi)$, see Lemma 2.21. Moreover, the normalizer $N(\langle\varphi\rangle)$ of $\langle\varphi\rangle$ in $\mathrm{Out}(F_r)$ has a subgroup of index at most 2 which is contained in $\mathrm{Stab}(\Lambda^+_\varphi)$. With this relationship in mind, the result of [BFH97] can be reinterpreted as saying that the groups $\mathrm{Stab}(\Lambda^+_\varphi), Cen(\langle\varphi\rangle), N(\langle\varphi\rangle)$ are each virtually cyclic.

Using attracting trees instead of laminations, Kapovich and Lustig were able to reprove and strengthen this result as follows. Given the dilitation homomorphism $\sigma\colon \mathrm{Stab}(T^+_\varphi) \to \mathbb{R}_{>0}$ (see Equation 2), and denoting its kernel P_T, they proved [KL11, Thereom 4.4] that $\mathrm{Stab}(T^+_\varphi) = P_T \rtimes \mathbb{Z}$ and P_T is finite. (In fact their theorem applies more generally to stabilizers of very small F_r-trees where $\mathrm{Stab}(T^+_\varphi)$ is infinite).

This article is concerned with identifying the centralizer and normalizer of $\langle\varphi\rangle$ when φ is an ageometric lone axis fully irreducible outer automorphism, as defined in Subsection 2.6. The term "lone axis" is connected with the axis bundle defined by Handel and Mosher [HM11]. The axis bundle is an analogue of the axis of a pseudo Anosov, but in general contains many fold lines.

We start with the following theorem.

Theorem A. Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a single unique axis, then $Cen(\langle\varphi\rangle) = \mathrm{Stab}(\Lambda^+_\varphi) \cong \mathbb{Z}$.

To motivate this theorem consider the faithful and discrete action of $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ on the upper half-plane model of $\mathbb{H}^2$ via Möbius transformations (see Remark 3.5 for more details regarding this example). Let $A \in \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ act hyperbolically on $\mathbb{H}^2$ with fixed points $\lambda, \frac{1}{\lambda} \in \mathbb{R}$. If $C \in$

[1] The following argument for this fact is given by Sisto in
`http://mathoverflow.net/questions/82889/centralizers-of-non-iwip-elements-of-outf-n?rq=1`
If the centralizer $Cen(\langle\varphi\rangle)$ is not virtually cyclic, then $\langle\varphi\rangle$ has infinite index in $Cen(\langle\varphi\rangle)$ and hence φ is not a Morse element of the mapping class group. Thus, by [Beh06] or alternatively by [DMS10, Theorem 1.5], φ is not a pseudo-Anosov mapping class.

$\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ fixes the ordered pair $(\lambda, \frac{1}{\lambda})$, then C preserves the hyperbolic geodesic between these two points. Consider the homomorphism $\sigma\colon \mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}((\lambda, \frac{1}{\lambda})) \to (\mathbb{R}, +)$ given by the signed hyperbolic translation length. If $C \in \mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}((\lambda, \frac{1}{\lambda}))$ is not the identity, then C cannot fix any other point on this geodesic. Therefore the kernel of σ is trivial. An easy argument (such as in Corollary 3.3, for example) implies that $\mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}((\lambda, \frac{1}{\lambda})) = Cen_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{A} \rangle) \cong \mathbb{Z}$.

Returning to the group $\mathrm{Out}(F_r)$ and the case where φ is a lone axis ageometric fully irreducible outer automorphism, our main task was to prove that the kernel of the analogous homomorphism $\rho\colon \mathrm{Stab}(\Lambda_\varphi^+) \to (\mathbb{R}, +)$, defined in Lemma 4.3, is trivial. This is achieved by appealing to the theorem of Mosher-Pfaff [MP13] characterizing these outer automorphisms. Proposition 4.6 then shows that the kernel of ρ is trivial.

Our next result involves the commensurator of $\langle\varphi\rangle$ (see Definition 2.18), denoted $Comm(\langle\varphi\rangle)$. Recall that $N(\langle\varphi\rangle) \leq Comm(\langle\varphi\rangle)$.

Theorem B. Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a single unique axis, then either

(1) $Comm(\langle\varphi\rangle) \cong \mathbb{Z}$ and $Comm(\langle\varphi\rangle) = N(\langle\varphi\rangle) = Cen(\langle\varphi\rangle)$ or
(2) $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ and $Comm(\langle\varphi\rangle) = N(\langle\varphi\rangle)$.

In particular, $N(\langle\varphi\rangle) \cong \mathbb{Z}$ or $N(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Further, in the case where $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we have that φ^{-1} is also an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_{\varphi^{-1}}$ consists of a single unique axis.

Example 4.1 reveals the necessity of the "lone axis" condition. It is a consequence of [Pfa13] that ageometric lone axis fully irreducible outer automorphisms exist in each rank and it is proved in [KP15] that this situation is generic along a specific "train track directed" random walk, but understanding what properties transfer to inverses of outer automorphisms is much more elusive. Theorem B gives a condition which guarantees that φ^{-1} also admits a lone axis. However, we do not know if the latter case in fact occurs, prompting the following question.

Question 1.1. *Does there exist some ageometric lone axis fully irreducible outer automorphism such that $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ (i.e $N(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$)?*

We pose two further questions.

Question 1.2.

(1) Can one give a concrete description of $Cen(\langle\varphi\rangle)$ and $N(\langle\varphi\rangle)$ when φ is not an ageometric lone axis fully irreducible outer automorphism?
(2) Does there exist a reducible outer automorphism with a virtually cyclic centralizer and normalizer?[2]

In the more general context of determining the centralizer of the cyclic subgroup generated by an element $\varphi \in \mathrm{Out}(F_r)$ we mention the following additional results. Using the machinery of completely split relative train track maps, Feighn and Handel [FH09] present an algorithm that virtually determines the weak centralizer of $\langle\varphi\rangle$, i.e. all elements that commute with some power of φ. When φ is a Dehn twist, Rodenhausen and Wade [RW15] give an algorithm determining a presentation of a finite index subgroup of $Cen(\langle\varphi\rangle)$. They use this to compute a presentation of the centralizer of a Whitehead generator.

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[2] A positive answer to this question is outlined in the *mathoverflow* conversation of the first footnote.

2. Preliminary definitions and notation

To keep this section at a reasonable length, we will provide only references for the definitions that are better known.

2.1. **Train track maps, Nielsen paths, and principal vertices.** Irreducible elements of $\mathrm{Out}(F_r)$ are defined in [BH92] and fully irreducible outer automorphisms are those such that each of their powers is irreducible. Every irreducible outer automorphism can be represented by a special kind of graph map called a train track map, as defined in [BH92]. In particular, we will require that vertices map to vertices. Moreover, we can also choose these maps so that they are defined on graphs with no valence-1 or valence-2 vertices (from the proof of [BH92] Theorem 1.7). We refer the reader to [BH92] for the definitions of *directions*, *periodic directions*, *fixed directions*, *legal paths*, *Nielsen paths* (denoted *NP*) and *periodic Nielsen paths* (*PNP*).

Definition 2.1 (Principal points)**.** Given a train track map $g\colon \Gamma \to \Gamma$, following [HM11] we call a point *principal* that is either the endpoint of a PNP or is a periodic vertex with ≥ 3 periodic directions. Thus, in the absence of PNPs, a point is principal if and only if it is a periodic vertex with ≥ 3 periodic directions

2.2. **Outer Space CV_r and the attracting tree T_+^{φ} for a fully irreducible $\varphi \in \mathbf{Out}(F_r)$.** Let CV_r denote the Culler-Vogtmann Outer Space in rank r, as defined in [CV86], with the asymmetric Lipschitz metric, as defined in [AKB12]. The group $\mathrm{Out}(F_r)$ acts naturally on CV_r on the right by homeomorphisms. An element $\varphi \in CV_r$ sends a point $X = (\Gamma, m, \ell) \in CV_r$ to the point $X \cdot \varphi = (\Gamma, m \circ \Phi, \ell)$ where Φ is a lift in $\mathrm{Aut}(F_r)$ of φ. Let $\overline{CV_r}$ denote the compactification of CV_r, as defined in [CL95, BF12]. The action of $\mathrm{Out}(F_r)$ on CV_r extends to an action on $\overline{CV_r}$ by homeomorphisms.

Definition 2.2 (Attracting tree T_+^{φ})**.** Let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible outer automorphism. Then φ acts on $\overline{CV_r}$ with North-South dynamics (see [LL03]). We denote by T_+^{φ} the attracting fixed point of this action and by T_-^{φ} the repelling fixed point of this action.

2.3. **The attracting lamination Λ_φ for a fully irreducible outer automorphism.** We give a concrete description of Λ_φ^+ using a particular train track representative $g\colon \Gamma \to \Gamma$. This is the original definition appearing in [BFH97]. Note that apriori it is not clear that it does not depend on the train track representative.

Definition 2.3 (Iterating neighborhoods)**.** Let $g\colon \Gamma \to \Gamma$ be an affine irreducible train track map so that, in particular, there has been an identification of each edge e of Γ with an open interval of its length $\ell(e)$ determined by the Perron-Frobenius eigenvector. Let $\lambda = \lambda(\varphi)$ be its stretch factor and assume $\lambda > 1$. Let x be a periodic point which is not a vertex (such points are dense in each edge). Let $\varepsilon > 0$ be sufficiently small so that the ε-neighborhood of x, denoted U, is contained in the interior of an edge. There exists an $N > 0$ such that x is fixed, $U \subset g^N(U)$, and Dg^N fixes the directions at x. We choose an isometry $\ell\colon (-\varepsilon, \varepsilon) \to U$ and extend it to the unique locally isometric immersion $\ell\colon \mathbb{R} \to \Gamma$ so that $\ell(\lambda^N t) = g^N(\ell(t))$. We then say that ℓ is *obtained by iterating a neighborhood of x*.

Definition 2.4 (Leaf segments, equivalent isometric immersions)**.** We call isometric immersions $\gamma_1\colon [a,b] \to \Gamma$, $\gamma_2\colon [c,d] \to \Gamma$ *equivalent* when there exists an isometry $h\colon [a,b] \to [c,d]$ so that $\gamma_1 = \gamma_2 \circ h$. Let $\ell\colon \mathbb{R} \to \Gamma$ be an isometric immersion. *A leaf segment* of ℓ is the equivalence class of the restriction to a finite interval of $\mathbb{R}$. Two isometric immersions ℓ, ℓ' are equivalent if each leaf segment ℓ is a leaf segment of ℓ' and vice versa.

Definition 2.5 (The realization in Γ of the attracting lamination $\Lambda^+_\varphi(\Gamma)$)**.** The *attracting lamination realized in* Γ, denoted $\Lambda^+_\varphi(\Gamma)$, is the equivalence class of a line ℓ obtained by iterating a periodic point in Γ (as in Definition 2.3). An element of $\Lambda^+_\varphi(\Gamma)$ is called a leaf. Notice that $\Lambda^+_\varphi(\Gamma)$ can be realized as an F_r-invariant set of bi-infinite geodesics in $\widetilde{\Gamma}$, the universal cover of Γ. We shall denote this set by $\Lambda^+_\varphi(\widetilde{\Gamma})$

The marking of Γ induces an identification of $\partial\Gamma$ with ∂F_r. The attracting lamination Λ^+_φ is the image of $\Lambda^+_\varphi(\widetilde{\Gamma})$ under this identification. In [BFH97] it is proved that this set is independent of the choice of g.

Definition 2.6 (The action of $\mathrm{Out}(F_r)$ on the set of laminations $\Lambda^\pm_\varphi$)**.** Let $\psi \in \mathrm{Out}(F_r)$, then by [BFH97, Lemma 3.5],

$$\psi \cdot (\Lambda^+_\varphi, \Lambda^-_\varphi) = (\Lambda^+_{\psi\varphi\psi^{-1}}, \Lambda^-_{\psi\varphi\psi^{-1}}). \tag{1}$$

2.4. **Whitehead graphs.** The following definitions are in [HM11] and [MP13].

Definition 2.7 (Stable Whitehead graphs and local Whitehead graphs)**.** Let $g\colon \Gamma \to \Gamma$ be a train track map. The *local Whitehead graph* $LW(v;\Gamma)$ at a point $v \in \Gamma$ has a vertex for each direction at v and an edge connecting the vertices corresponding to the pair of directions $\{d_1, d_2\}$ if the turn $\{d_1, d_2\}$ is taken by an image of an edge. The *stable Whitehead graph* $SW(v;\Gamma)$ at a principal point v is then the subgraph of $LW(v;\Gamma)$ obtained by restricting to the periodic direction vertices.

The map g induces a continuous simplicial map $Dg\colon LW(g,v) \to LW(g,g(v))$. When g is rotationless and v a principal vertex, Dg acts as the identity on $SW(g,v)$, when viewed as a subgraph of $LW(g,v)$, and hence gives an induced surjection $Dg\colon LW(g,v) \to SW(g,v)$. We recall that for a train track representative of a fully irreducible outer automorphism the local Whitehead graph at each vertex is connected. Hence

Lemma 2.8. *If $g\colon \Gamma \to \Gamma$ is a train track map representing a fully irreducible outer automorphism φ and $v \in \Gamma$ is a principal vertex, then $SW(g,v)$ is connected.*

Lemma 2.9. *Let $g\colon \Gamma \to \Gamma$ be a rotationless PNP-free train track representative of an ageometric fully irreducible $\varphi \in \mathrm{Out}(F_r)$. Let $\widetilde{\Gamma}$ be the universal cover of Γ and $\widetilde{v} \in \widetilde{\Gamma}$ a vertex that projects to a principal vertex $v \in \Gamma$. Then there exist two leaves ℓ_1, ℓ_2 of the lamination $\Lambda^+_\varphi(\widetilde{\Gamma})$ so that $\ell_1 \cup \ell_2$ is a tripod whose vertex is $\widetilde{v}$.*

Proof. Since v is a principal vertex and there are no PNPs, $SW(g,v)$ will have ≥ 3 vertices. Since $SW(g,v)$ is connected, one of these vertices d_1 will belong to at least 2 edges ϵ_1, ϵ_2. Let d_2, d_3 be the directions corresponding to the other vertices of these edges. Since g is rotationless, periodic directions are in fact fixed directions. We may lift g to a map $\widetilde{g}\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ that fixes $\widetilde{v}$. Iterating the lifts of the edges that correspond to d_1, d_2, d_3 will give us three eigenrays R_1, R_2, R_3 initiating at $\widetilde{v}$. The 2 edges ϵ_1, ϵ_2 correspond to 2 leaves ℓ_1 and ℓ_2 of $\Lambda^+_\varphi(\widetilde{\Gamma})$ [HM11]. We have $\ell_1 \cup \ell_2 = R_1 \cup R_2 \cup R_3$. Hence, as desired, $\ell_1 \cup \ell_2$ is a tripod whose vertex is $\widetilde{v}$. $\square$

2.5. **Axis bundles.** Three equivalent definitions of the axis bundle $\mathcal{A}_\varphi$ for a nongeometric fully irreducible $\varphi \in \mathrm{Out}(F_r)$ are given in [HM11]. We include only the definition that we use.

Definition 2.10 (Fold lines)**.** A *fold line* in CV_r is a continuous, injective, proper function $\mathbb{R} \to CV_r$ defined by

1. a continuous 1-parameter family of marked graphs $t \to \Gamma_t$ and
2. a family of homotopy equivalences $h_{ts}\colon \Gamma_s \to \Gamma_t$ defined for $s \leq t \in \mathbb{R}$, each marking-preserving, satisfying:

***Train track property*:** h_{ts} is a local isometry on each edge for all $s \leq t \in \mathbb{R}$.

***Semiflow property*:** $h_{ut} \circ h_{ts} = h_{us}$ for all $s \leq t \leq u \in \mathbb{R}$ and $h_{ss} \colon \Gamma_s \to \Gamma_s$ is the identity for all $s \in \mathbb{R}$.

Definition 2.11 (Axis Bundle)**.** $\mathcal{A}_\varphi$ is the union of the images of all fold lines $\mathcal{F} \colon \mathbb{R} \to CV_r$ such that $\mathcal{F}$(t) converges in $\overline{CV_r}$ to T_-^φ as $t \to -\infty$ and to T_+^φ as $t \to +\infty$.

Definition 2.12 (Axes)**.** We call the fold lines in Definition 2.11 the *axes* of the axis bundle.

2.6. **Lone Axis Fully Irreducibles Outer Automorphisms.**

Definition 2.13 (Lone axis fully irreducibles)**.** A fully irreducible $\varphi \in \mathrm{Out}(F_r)$ will be called a *lone axis fully irreducible outer automorphism* if $\mathcal{A}_\varphi$ consists of a single unique axis.

[MP13, Theorem 3.9] gives necessary and sufficient conditions on an ageometric fully irreducible outer automorphism $\varphi \in \mathrm{Out}(F_r)$ to ensure that $\mathcal{A}_\varphi$ consists of a single unique axis. It is also proved there that, under these conditions, the axis will be the periodic fold line for a (in fact any) train track representative of φ. In particular, as is always true for axis bundles, $\mathcal{A}_\varphi$ contains each point in Outer Space on which there exists an affine train track representative of a power of φ.

Remark 2.14. It will be important for our purposes that no train track representative of an ageometric lone axis fully irreducible φ has a periodic Nielsen path. This follows from [MP13, Lemma 4.4], as it shows that each train track representative of each power of φ is stable, hence (in the case of an ageometric fully irreducible outer automorphism) has no Nielsen paths.

The following proposition is a direct consequence of [MP13, Corollary 3.8].

Proposition 2.15 ([MP13])**.** *Let φ be an ageometric lone axis fully irreducible outer automorphism, then there exists a train track representative $g \colon \Gamma \to \Gamma$ of some power φ^R of φ so that all vertices of Γ are principal, and fixed, and all but one direction is fixed.*

2.7. **The stabilizer Stab(Λ_φ^+) of the lamination.**

Definition 2.16 (Stab(Λ_φ^+))**.** Given a fully irreducible $\varphi \in \mathrm{Out}(F_r)$, we let $\mathrm{Stab}(\Lambda_\varphi^+)$ denote the subgroup of $\mathrm{Out}(F_r)$ fixing Λ_φ^+ setwise, i.e. sending leaves of Λ_φ^+ to leaves of Λ_φ^+.

In [BFH97], Bestvina, Feighn, and Handel define a homomorphism (related to the expansion factor)

$$\sigma \colon \mathrm{Stab}(\Lambda_\varphi^+) \to (\mathbb{R}_{>0}, \cdot) \tag{2}$$

that they use to prove the following theorem ([BFH97, Theorem 2.14]):

Theorem 2.17 ([BFH97, Theorem 2.14] or [KL11, Theorem 4.4])**.** *For each fully irreducible $\varphi \in \mathrm{Out}(F_r)$, we have that $\mathrm{Stab}(\Lambda_\varphi^+)$ is virtually cyclic.*

2.8. **Commensurators.**

Definition 2.18 (Commensurator $Comm(\langle\varphi\rangle)$)**.** Given a group G and subgroup $H \leq G$, the *commensurator* or *virtual normalizer* of H in G is defined as

$$Comm_G(H) := \{g \in G \mid [H : H \cap g^{-1}Hg] < \infty \text{ and } [g^{-1}Hg : H \cap g^{-1}Hg] < \infty\}.$$

Convention 2.19 ($\langle\varphi\rangle$, $Cen(\langle\varphi\rangle)$, $N(\langle\varphi\rangle)$)**.** Given an element $\varphi \in \mathrm{Out}(F_r)$, we let $\langle\varphi\rangle$ denote the cyclic subgroup generated by φ, we let $Cen(\langle\varphi\rangle)$ denote its centralizer in $\mathrm{Out}(F_r)$, and we let $N(\langle\varphi\rangle)$ denote its normalizer in $\mathrm{Out}(F_r)$.

Remark 2.20. $N_G(H) \leq Comm_G(H)$.

Lemma 2.21. *Let $\varphi \in Out(F_r)$ be fully irreducible. Then:*

(1) $Comm(\langle\varphi\rangle) = Stab(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}) = Stab(\{T^+_\varphi, T^-_\varphi\})$.

And, in particular, each element $\psi \in N(\langle\varphi\rangle)$ *fixes the unordered pair* $\{T^+_\varphi, T^-_\varphi\}$ *and the unordered pair* $\{\Lambda^+_\varphi, \Lambda^-_\varphi\}$.

(2) *Each element* $\psi \in Cen(\langle\varphi\rangle)$ *fixes the ordered pair* $(T^+_\varphi, T^-_\varphi)$ *and the ordered pair* $(\Lambda^+_\varphi, \Lambda^-_\varphi)$. *In particular,* $Cen(\langle\varphi\rangle) < \mathrm{Stab}(\Lambda^+_\varphi)$.

Proof. (1) By the proof of Corollary 5.8 in [KL10],

$$Comm(\langle\varphi\rangle) \leq Stab(\{T^+_\varphi, T^-_\varphi\}).$$

Thus, by [BFH97, Lemma 3.5],

$$Comm(\langle\varphi\rangle) \leq Stab(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}).$$

Notice that, since $\langle\varphi\rangle$ is cyclic,

$$Comm(\langle\varphi\rangle) := \{\psi \in Out(F_r) \mid \exists\, m, n \in \mathbb{Z} \text{ so that } \psi\varphi^n\psi^{-1} = \varphi^m\}. \tag{3}$$

Now suppose $\psi \in Stab(\{\Lambda^+_\varphi, \Lambda^-_\varphi\})$. Then, by Equation 1, we know $\psi\varphi\psi^{-1} = \varphi^n$ for some $n \in \mathbb{Z}$. So $\psi \in Comm(\langle\varphi\rangle)$.

Since $N(\langle\varphi\rangle) \leq Comm(\langle\varphi\rangle)$, the last statement follows also.

(2) Let $\psi \in Cen(\langle\varphi\rangle)$ then by Equation 1 we have $\psi \cdot (\Lambda^+_\varphi, \Lambda^-_\varphi) = (\Lambda^+_\varphi, \Lambda^-_\varphi)$. That ψ fixes the ordered pair $(T^+_\varphi, T^-_\varphi)$ now follows from [BFH97, Lemma 3.5] or [KL10, Corollary 5.8]. □

3. The normalizer of a fully irreducible outer automorphism

Proposition 3.1. *Let* $\varphi \in \mathrm{Out}(F_r)$ *be fully irreducible. Then there exists some* $k \in \mathbb{N}$ *such that* $\mathrm{Stab}(\Lambda^+_\varphi)$ *is a subgroup of index* ≤ 2 *in* $N(\langle\varphi^k\rangle)$.

Proof. If $\nu \in \mathrm{Stab}(\Lambda^+_\varphi)$ define $\psi = \nu\varphi\nu^{-1}$. Then ψ is a fully irreducible element of $\mathrm{Stab}(\Lambda^+_\varphi)$. Therefore, ψ is exponentially growing and, by [BFH97, Corollary 2.13] or [KL11, Proposition 3.14], $\sigma(\psi) > 1$, where σ is the map from Equation 2. By [BFH97, Corollary 2.15] or [KL11, Theorem 4.4], φ and ψ have common nonzero powers, i.e. there exist integers k and m so that $\psi^k = \varphi^m$ and hence $\nu \circ \varphi^k \circ \nu^{-1} = \varphi^m$.

We denote by $\omega\colon \mathrm{Out}(F_r) \to \mathrm{Out}(F_r)$ the isomorphism defined by conjugation by ν, i.e.

$$\omega(\theta) = \nu \circ \theta \circ \nu^{-1}.$$

Note that $\omega(\mathrm{Stab}(\Lambda^+_\varphi)) = \mathrm{Stab}(\Lambda^+_\varphi)$. Since $\mathrm{Stab}(\Lambda^+_\varphi)$ is virtually cyclic (Theorem 2.17) and $\varphi \in \mathrm{Stab}(\Lambda^+_\varphi)$, we have that $\langle\varphi\rangle$ is a finite index subgroup in $\mathrm{Stab}(\Lambda^+_\varphi)$, let n be its index. Then the index of $\langle\varphi^k\rangle$ in $\mathrm{Stab}(\Lambda^+_\varphi)$ is $|k|n$. The index of $\langle\varphi^m\rangle$ in $\mathrm{Stab}(\Lambda^+_\varphi)$ is $|m|n$. On the other hand, $\langle\varphi^m\rangle = \omega(\langle\varphi^k\rangle)$ has index $|k|n$ in $\omega(\mathrm{Stab}(\Lambda^+_\varphi)) = \mathrm{Stab}(\Lambda^+_\varphi)$. Hence $|k| = |m|$. This proves that $\nu \in N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)$ and hence $\mathrm{Stab}(\Lambda^+_\varphi) \leq N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)$.

Finally, by Lemma 2.21(1), we have $\left[N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)\right]^2 \leq \mathrm{Stab}(\Lambda^+_{\varphi^k}) = \mathrm{Stab}(\Lambda^+_\varphi) \leq N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)$. Moreover, the index of $\left[N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)\right]^2$ in $N_{\mathrm{Out}(F_r)}(\langle\varphi^k\rangle)$ is 2. This proves the proposition. □

Corollary 3.2. *If* $\nu \in \mathrm{Stab}(\Lambda^+_\varphi)$, *then* ν *fixes the ordered pair* $(T^+_\varphi, T^-_\varphi)$.

Proof. By Proposition 3.1, we have $\nu \in N(\langle\varphi^k\rangle)$ for some k. Hence, by Lemma 2.21(1), we have $\nu(\{T^+_\varphi, T^-_\varphi\}) = \{T^+_\varphi, T^-_\varphi\}$. Since $\nu(\Lambda^+_\varphi) = \Lambda^+_\varphi$ we get that $\nu(T^-_\varphi) = T^-_\varphi$. Thus, ν fixes the ordered pair $(T^+_\varphi, T^-_\varphi)$. □

Lemma 3.3. *If* $\varphi \in \mathrm{Out}(F_r)$ *is fully irreducible and* $\mathrm{Stab}(\Lambda^+_\varphi)$ *is an infinite cyclic group, then* $Cen(\langle\varphi\rangle) = \mathrm{Stab}(\Lambda^+_\varphi)$.

Proof. Let ψ be the generator of $\mathrm{Stab}(\Lambda_\varphi^+)$ then $\varphi = \psi^m$ for some $m \in \mathbb{Z}$. Thus, $\psi \in Cen(\langle\varphi\rangle)$ and $\mathrm{Stab}(\Lambda_\varphi^+) \leq Cen(\langle\varphi\rangle)$, but $Cen(\langle\varphi\rangle) \leq \mathrm{Stab}(\Lambda_\varphi^+)$ by Lemma 2.21(2). □

Proposition 3.4. *There exists a number K so that for each $j \in \mathbb{Z}$ we have*

$$N(\langle\varphi^j\rangle) < N(\langle\varphi^K\rangle).$$

Proof. Define $N_s^m := N(\langle\varphi^m\rangle) \cap \mathrm{Stab}(\Lambda_\varphi^+)$. Then $\left[N(\langle\varphi^m\rangle)\right]^2 < N_s^m$, so N_s^m is a subgroup of index at most 2 in $N(\langle\varphi^m\rangle)$.

Let k be as in the Proposition 3.1. Then $\mathrm{Stab}(\Lambda_\varphi^+) = N_s^k$. Therefore, if it so happens that for each $j \in \mathbb{Z}$ we have that $N(\langle\varphi^j\rangle) = N_s^j < \mathrm{Stab}(\Lambda_\varphi^+)$, then $N(\langle\varphi^j\rangle) < N(\langle\varphi^k\rangle)$ for all j.

Thus, we assume that there exists a number m so that $N(\langle\varphi^m\rangle) = \langle N_s^m, \psi\rangle$ for some $\psi \notin \mathrm{Stab}(\Lambda_\varphi^+)$. Let $K = km$. Since $N(\langle\varphi^K\rangle) \geq N(\langle\varphi^k\rangle), N(\langle\varphi^m\rangle)$, we have that $\psi \in N(\langle\varphi^K\rangle)$ and $\mathrm{Stab}(\Lambda_\varphi^+) < N(\langle\varphi^K\rangle)$. Then, since N_s^K is properly contained in $\langle\mathrm{Stab}(\Lambda_\varphi^+), \psi\rangle < N(\langle\varphi^K\rangle)$ and has at most index 2 in $N(\langle\varphi^K\rangle)$, we have that $N(\langle\varphi^K\rangle) = \langle\mathrm{Stab}(\Lambda_\varphi^+), \psi\rangle$.

We show that the same K works for an arbitrary $j \in \mathbb{Z}$. If $N(\langle\varphi^j\rangle) < \mathrm{Stab}(\Lambda_\varphi^+) < N(\langle\varphi^K\rangle)$ then we are done. So assume that there exists some $\theta \notin \mathrm{Stab}(\Lambda_\varphi^+)$ such that $N(\langle\varphi^j\rangle) = \langle N_s^j, \theta\rangle$. As in the previous paragraph, we have $\langle\mathrm{Stab}(\Lambda_\varphi^+), \psi\rangle, \langle\mathrm{Stab}(\Lambda_\varphi^+), \theta\rangle < N(\langle\varphi^{jK}\rangle)$ and $\mathrm{Stab}(\Lambda_\varphi^+)$ has index 2 in each of the groups $\langle\mathrm{Stab}(\Lambda_\varphi^+), \psi\rangle, \langle\mathrm{Stab}(\Lambda_\varphi^+), \theta\rangle, N(\langle\varphi^{jK}\rangle)$. This implies that the three groups are equal and, in particular, $N(\langle\varphi^K\rangle) = \langle\mathrm{Stab}(\Lambda_\varphi^+), \psi\rangle = \langle\mathrm{Stab}(\Lambda_\varphi^+), \theta\rangle = N(\langle\varphi^j\rangle)$. □

Example 3.5. We work out an example where the number K of Proposition 3.4 is > 1. To see this, we recall that $\mathrm{Out}(F_2) \cong \mathrm{GL}(2,\mathbb{Z})$ via the abelianization map. Thus, it suffices to carry out the computations in $\mathrm{GL}(2,\mathbb{Z})$. In fact we will work with $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$. Letting I denote the identity matrix, $\mathrm{SL}(2,\mathbb{Z}) \cong \mathbb{P}\mathrm{GL}(2,\mathbb{Z}) \times \langle -I\rangle$ and $\mathrm{GL}(2,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2$, where the $\mathbb{Z}_2$ subgroup is generated by $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (this follows by considering the determinant homomorphism).

By standard facts about Möbius transformations, each matrix A in $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ acts either:

- elliptically - fixing a point inside $\mathbb{H}^2$ but with no fixed point on $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, or
- parabolically - with a single fixed point on $\mathbb{R} \cup \{\infty\}$, or
- hyperbolically - with precisely two fixed points on $\mathbb{R} \cup \{\infty\}$.

Let $A \in \mathrm{GL}(2,\mathbb{Z})$ be a matrix such that its projectivization $\bar{A} \in \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ acts hyperbolically on $\mathbb{H}^2$ and fixes the points $\lambda, \frac{1}{\lambda} \in \mathbb{R}$. We have seen in the introduction that the stablizer in $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ of the ordered pair $(\lambda, \frac{1}{\lambda})$, denoted $\mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, \frac{1}{\lambda})$, is isomorphic to $\mathbb{Z}$ and is equal to $Cen_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle\bar{A}\rangle)$. Let $C \in \mathrm{SL}(2,\mathbb{Z})$ be such that $\bar{C}$ is the generator of $Cen_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle\bar{A}\rangle)$. Note that $-I$ is in the center of $\mathrm{SL}(2,\mathbb{Z})$, thus $Cen_{\mathrm{SL}(2,\mathbb{Z})}(\langle A\rangle) = \langle C, -I\rangle$. Moreover, S only commutes with diagonal matrices. Thus, if $A \in \mathrm{GL}(2,\mathbb{Z})$ is not diagonal, $Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = Cen_{\mathrm{SL}(2,\mathbb{Z})}(\langle A\rangle) = \langle C, -I\rangle \cong \mathbb{Z} \times \mathbb{Z}_2$. This gives us an example where the centralizer is not isomorphic to $\mathbb{Z}$ (in contrast to the conclusion of our theorem).

Moreover, note that for all $m \in \mathbb{Z}$, we have $Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^m\rangle) = Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle)$ since both A and A^m fix the same points on $\partial\mathbb{H}^2$.

Consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}), \qquad B = A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \qquad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have $Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$. A direct calculation shows that $\overline{A}$ is not a power of any matrix in $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$. Therefore, $\mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, \frac{1}{\lambda}) = \langle\overline{A}\rangle$. Hence, $Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = \langle A, -I\rangle$. One can check directly that $P \in N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$ and $P \notin N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle)$. This gives us an example where $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$ strictly contains $Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$. Moreover, we see that $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) \neq N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle)$. Thus $K \neq 1$.

We can in fact compute $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle)$ and $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$. We have

$$N_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{B}\rangle) > \langle \mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, \tfrac{1}{\lambda}), \overline{P}\rangle.$$

The subgroup $\mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, \frac{1}{\lambda})$ of $N_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{B}\rangle)$ has index ≤ 2 (since for each $\psi \in N_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{B}\rangle)$, we have that ψ preserves $\{\lambda, \frac{1}{\lambda}\}$). Hence $N_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{B}\rangle) = \langle \mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, \frac{1}{\lambda}), \overline{P}\rangle$. The image of $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle)$ under the homomorphism $\mathrm{GL}(2,\mathbb{Z}) \to \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ is $N_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{B}\rangle)$. Therefore, $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle) = \langle A, -I, P\rangle$. Moreover, we have $\langle A, -I\rangle < N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) < N_{\mathrm{GL}(2,\mathbb{Z})}(\langle B\rangle) = \langle A, -I, P\rangle$, we have $P^2 = -I$, and we have $P \notin N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle)$. Thus, $N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = \langle A, -I\rangle$. In conclusion,

$$N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = Cen_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle) = \langle A, -I\rangle \cong \mathbb{Z} \times \mathbb{Z}_2 \text{ and}$$
$$N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle) = \langle A, P, -I\rangle \cong (\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2.$$

4. Proof of Main Theorems

Before we prove the main theorems, we give an example revealing the necessity of the "lone axis" condition in the main theorems.

Example 4.1. We show that there exists an ageometric fully irreducible outer automorphism φ such that $Cen(\langle\varphi\rangle) \ncong \mathbb{Z}$, and moreover $Cen(\langle\varphi\rangle) \ncong \mathbb{Z} \times \mathbb{Z}_2$ (as in $\mathrm{Out}(F_2)$, whose center is $\mathbb{Z}_2$). Consider $F_3 = \langle a, b, c\rangle$. Let R_3 be the 3 petaled rose and define

$$\Psi : a \to b \to c \to ab.$$

It is straight-forward (see [Pfa13, Proposition 4.1]) to check that this map represents an ageometric fully irreducible outer automorphism. Denote by Δ the 3-fold cover corresponding to the subgroup

$$\langle b, c, a^3, abA, acA, a^2bA^2, a^2cA^2\rangle.$$

We claim that Ψ^{13} lifts to Δ. Indeed, let A be the transition matrix of Ψ, then

$$A^{13} = \begin{pmatrix} 7 & 9 & 12 \\ 12 & 16 & 21 \\ 9 & 12 & 16 \end{pmatrix}.$$

In particular both $\Psi(b)$ and $\Psi(c)$ cross a a multiple of three times. Thus Ψ^{13} lifts to Δ. Denote the vertices of Δ by v_1, v_2, v_3. We denote by $g\colon \Delta \to \Delta$ the lift of Ψ^{13} that sends v_1 to itself. Let $T\colon \Delta \to \Delta$ denote the deck transformation sending v_1 to v_2. The action of T on $H_1(\Delta, \mathbb{Z})$ is nontrivial, so T does not represent an inner automorphism. Moreover, we claim that $g \circ T = T \circ g$. First note that both of the maps $g \circ T, T \circ g$ are lifts of Ψ^{13}. Moreover, since a appears in $\Psi^{13}(a)$ 7 times (see the matrix A^{13}) then $g(v_2) = v_2$. Therefore,

$$g \circ T(v_1) = g(v_2) = v_2 = T(v_1) = T \circ g(v_1).$$

Therefore, $g \circ T = T \circ g$. Let $\varphi \in \mathrm{Out}(F_7)$ be the outer automorphism represented by g, and θ the outer automorphism represented by T. An elementary computation shows that g is an irreducible train track map and that each local Whitehead graph is connected. Moreover, a PNP for g would descend to a PNP for Ψ. Since Ψ contains no such paths, then there are no PNPs for g. Thus the outer automorphism φ is ageometric fully irreducible (see [Pfa13, Proposition 4.1]). In conclusion, θ is an order-3 element in $Cen_{\mathrm{Out}(F_7)}(\langle\varphi\rangle)$ in contrast to the conclusion of our theorem for a lone axis ageometric fully irreducible outer automorphism.

Lemma 4.2. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism. If $\psi \in \mathrm{Out}(F_r)$ is an outer automorphism fixing the pair $(T_\varphi^+, T_\varphi^-)$, then ψ fixes $\mathcal{A}_\varphi$ as a set, and also preserves its orientation.*

Proof. $\mathcal{A}_\varphi$ consists precisely of all fold lines $\mathcal{F}\colon \mathbb{R} \to CV_r$ such that $\mathcal{F}(t)$ converges in $\overline{CV_r}$ to T_-^φ as $t \to -\infty$ and to T_+^φ as $t \to +\infty$. Further, since $\varphi \in \mathrm{Out}(F_r)$ is a lone axis fully irreducible outer automorphism, there is only one such fold line. Hence, since ψ fixes $(T_\varphi^+, T_\varphi^-)$, it suffices to show that the image of the single fold line $\mathcal{A}_\varphi$ under ψ is a fold line. Indeed given the fold line $t \to \Gamma_t$ with the semi-flow family $\{h_{ts}\}$, the new fold line is just $t \to \Gamma_t \cdot \psi$ with the same family of homotopy equivalences $\{h_{ts}\}$. Hence the properties of Definition 2.10 still hold. ☐

Recall that $\mathcal{A}_\varphi$ is a directed geodesic and suppose that the map $t \to \Gamma_t$ is a parametrization of $\mathcal{A}_\varphi$ according to arc-length with respect to the Lipschitz metric, i.e.

$$d(\Gamma_t, \Gamma_{t'}) = t' - t \qquad \text{for } t' > t. \tag{4}$$

Lemma 4.3. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism and $\psi \in N(\langle\varphi\rangle)$, then there exists a number $\rho(\psi) \in \mathbb{R}$ so that for all $t \in \mathbb{R}$, we have $\psi(\Gamma_t) = \Gamma_{\rho(\psi)+t}$.*

Proof. By Lemma 4.2, $\psi(\mathcal{A}_\varphi) = \mathcal{A}_\varphi$ and ψ preserves the direction of the fold line. Therefore, there exists a strictly monotonically increasing surjective function $f\colon \mathbb{R} \to \mathbb{R}$ so that $\psi(\Gamma_t) = \Gamma_{f(t)}$. Moreover, since ψ is an isometry with respect to the Lipschitz metric, for $t < t'$, since $f(t) < f(t')$, Equation (4) implies

$$f(t') - f(t) = d(\Gamma_{f(t)}, \Gamma_{f(t')}) = d(\psi(\Gamma_t), \psi(\Gamma_{t'})) = d(\Gamma_t, \Gamma_{t'}) = t' - t.$$

Hence $f(t') = f(t) + t' - t$. This implies that for all $s \in \mathbb{R}$, $f(s) = f(0) + s$. Define $\rho(\psi) = f(0)$, then

$$\psi(\Gamma_t) = \Gamma_{f(t)} = \Gamma_{f(0)+t} = \Gamma_{\rho(\psi)+t}.$$

☐

Lemma 4.4. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism, then the map $\rho\colon \mathrm{Stab}(\Lambda_\varphi^+) \to (\mathbb{R}, +)$ is a homomorphism.*

Proof. For each $t \in \mathbb{R}$,

$$\Gamma_t = \psi^{-1}\psi(\Gamma_t) = \psi^{-1}(\Gamma_{\rho(\psi)+t}) = \Gamma_{\rho(\psi^{-1})+\rho(\psi)+t}.$$

Thus, $t = \rho(\psi^{-1}) + \rho(\psi) + t$, i.e. $\rho(\psi^{-1}) = -\rho(\psi)$. Moreover, let $\psi, \nu \in \mathrm{Stab}(\Lambda_\varphi^+)$, then

$$\Gamma_{\rho(\psi\circ\nu)+t} = \psi \circ \nu(\Gamma_t) = \psi(\nu(\Gamma_t)) = \psi(\Gamma_{\rho(\nu)+t}) = \Gamma_{\rho(\psi)+\rho(\nu)+t}.$$

Thus, $\rho(\psi \circ \nu) = \rho(\psi) + \rho(\nu)$. We therefore obtain that ρ is a homomorphism. ☐

Since $\mathrm{Stab}(\Lambda_\varphi^+)$ is virtually cyclic and $\rho(\varphi) \neq 0$, the image of $\mathrm{Stab}(\Lambda_\varphi^+)$ under ρ is infinite cyclic. Thus it gives rise to a surjective homomorphism

$$\tau\colon \mathrm{Stab}(\Lambda_\varphi^+) \to \mathbb{Z} \tag{5}$$

with finite kernel. Note that the kernel consists precisely of those elements of $\mathrm{Out}(F_r)$ that, when acting on CV_r, fix the axis $\mathcal{A}_\varphi$ pointwise. We show in Corollary 4.7 that $ker(\tau) = id$.

Proposition 4.5. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism and let $\psi \in \mathrm{Stab}(\Lambda_\varphi^+)$ be an outer automorphism that fixes $\mathcal{A}_\varphi$ pointwise. Let $f\colon \Gamma \to \Gamma$ be an affine train track representative of some power φ^R of φ such that all vertices of Γ are principal (guaranteed by Proposition 2.15) and let $h\colon \Gamma \to \Gamma$ be any isometry representing ψ. Then h permutes the f-fixed directions and hence fixes the (unique) nonfixed direction.*

Proof. ψ fixes the points Γ and $\Gamma\varphi$. Thus there exist isometries $h\colon \Gamma \to \Gamma$ and $h'\colon \Gamma\varphi \to \Gamma\varphi$ that represent an automorphism Ψ in the outer automorphism class of ψ, i.e. the following diagrams commute

$$\begin{tikzcd} R_r \arrow[r, "\Psi"] \arrow[d, "m"] & R_r \arrow[d, "m"] \\ \Gamma \arrow[r, "h"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} R_r \arrow[r, "\Psi"] \arrow[d, "f\circ m"] & R_r \arrow[d, "f\circ m"] \\ \Gamma \arrow[r, "h'"] & \Gamma \end{tikzcd}$$

Therefore, the following diagram commutes up to homotopy

$$\begin{tikzcd} \Gamma \arrow[r, "h"] \arrow[d, "f"] & \Gamma \arrow[d, "f"] \\ \Gamma \arrow[r, "h'"] & \Gamma \end{tikzcd}$$

We will show that this diagram commutes and in fact that $h' = h$. Let $H\colon \Gamma \times I \to \Gamma$ be the homotopy so that $H(x,0) = f \circ h(x)$ and $H(x,1) = h' \circ f(x)$. Choose a lift $\widetilde{f}$ of f and a lift $\widetilde{h}$ of h to $\widetilde{\Gamma}$. Note that $\widetilde{f}\circ\widetilde{h}$ is a lift of $f\circ h$. Let $\widetilde{H}$ be a lift of H that starts with the lift $\widetilde{f}\circ\widetilde{h}$. Then $\widetilde{H}(x,1)$ is a lift of $h'\circ f$, which we denote by $\widetilde{h'\circ f}$. This in turn determines a lift $\widetilde{h}'$ of h' so that $\widetilde{h'\circ f} = \widetilde{h}'\circ\widetilde{f}$. There exists a constant M so that for all $x \in \widetilde{\Gamma}$, we have $d(\widetilde{f}\circ\widetilde{h}(x), \widetilde{h}'\circ\widetilde{f}(x)) \leq M$, hence $\widetilde{f}\circ\widetilde{h}(x)$ and $\widetilde{h}'\circ\widetilde{f}(x)$ induce the same homeomorphism on $\partial\widetilde{\Gamma}$. Let $v \in \widetilde{\Gamma}$ be any vertex. By Lemma 2.9 there exist leaves ℓ_1, ℓ_2 of $\Lambda_+(\widetilde{\Gamma})$ that form a tripod whose vertex is v. Then $\widetilde{f}\circ\widetilde{h}(\ell_1), \widetilde{f}\circ\widetilde{h}(\ell_2), \widetilde{f}\circ\widetilde{h}(\ell_3)$ are embedded lines forming a tripod, as are $\widetilde{h}'\circ\widetilde{f}(\ell_1), \widetilde{h}'\circ\widetilde{f}(\ell_2), \widetilde{h}'\circ\widetilde{f}(\ell_3)$. Moreover, the ends of the two tripods coincide. Thus, $\widetilde{f}\circ\widetilde{h}(v) = \widetilde{h}'\circ\widetilde{f}(v)$. Since v was arbitrary and the maps are linear, we have $\widetilde{f}\circ\widetilde{h} = \widetilde{h}'\circ\widetilde{f}$ and $f\circ h = h'\circ f$.

We now show that $h' = h$. Let e_1 be the oriented edge representing the nonfixed direction of Df. For all $i \neq 1$, $Df(e_i) = e_i$. Let k be such that $h(e_k) = e_1$. We have $Dh'\circ Df = Df \circ Dh$. Thus for $i \neq 1, k$ we have $Dh'(e_i) = Dh(e_i)$. Since h and h' are isometries, this implies that $h'(e_i) = h(e_i)$ for $i \neq 1, k$. If $k = 1$ then h and h' agree on all but one oriented edge and therefore coincide, so we assume $k \neq 1$. If $e_1 \neq \bar{e}_k$ then $h(\bar{e}_i) = h'(\bar{e}_i)$ for both $i = 1$ and $i = k$, hence $h' = h$. Therefore we may assume that $\bar{e}_1 = e_k$. We have $h(e_k) = e_1$, hence $h(e_1) = e_k$. So $h'(\{e_1, e_k\}) = \{e_1, e_k\}$, hence we assume $h'(e_k) = e_k$ and $h'(e_1) = e_1$. Notice that the edge of e_1 must be a loop, since h and h' coincide on all other edges. Further, the orientation of the loop is preserved by h' and flipped by h. Now let $j \neq 1$ be so that $Df(e_1) = e_j$ and let u be an edge path so that $f(e_1) = e_j u e_1$. Thus, $f(e_k) = \overline{f(e_1)} = e_k \bar{u} \bar{e}_j$. We have

$$e_k \bar{u} \bar{e}_j = f(e_k) = f(h(e_1)) = h'(f(e_1)) = h'(e_j)h'(u)h'(e_1).$$

Thus $h'(e_j) = e_k$, so $j = k$. Hence $Df(e_1) = e_k = Df(e_k)$. So the unique illegal turn of f is $\{e_1, \bar{e}_1\}$. But this is impossible since f is a homotopy equivalence and must fold to the identity. Thus, $h = h'$ and so, since we have from the previous paragraph that $f \circ h = h' \circ f$, we know that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "h"] \arrow[d, "f"] & \Gamma \arrow[d, "f"] \\ \Gamma \arrow[r, "h"] & \Gamma \end{tikzcd}$$

Let e be an edge so that the direction defined by e is fixed by Df. We have $Dh(e) = Dh(Df(e)) = Df(Dh(e))$, therefore $Dh(e)$ is also a fixed direction. Thus $h(e)$ defines a fixed direction, hence the f-fixed directions are permuted by h. $\square$

Proposition 4.6. *Under the conditions of Proposition 4.5, h is the identity on Γ.*

Proof. Let e be the oriented edge of Γ representing the unique direction that is not f-fixed (or f-periodic). By Proposition 4.5, we know that $h(e) = e$. Let p be an f-periodic point in the interior of e. We can switch to a power of f fixing p. Let $\ell \in \Lambda^+_\varphi(\Gamma)$ be the leaf of the lamination obtained by iterating a neighborhood of p (see Definition 2.3). Denote by $\widetilde{\Gamma}$ the universal cover of Γ and let $\widetilde{p}$ be a lift of p and $\widetilde{e}$ and $\widetilde{\ell}$ be the corresponding lifts of e and ℓ. Let $\widetilde{h}$ and $\widetilde{f}$ be the respective lifts of h and f fixing the point $\widetilde{p}$. The lift $\widetilde{f}$ fixes $\widetilde{\ell}$, since this leaf is generated by $\widetilde{f}$-iterating a neighborhood of $\widetilde{p}$ contained in $\widetilde{e}$.

We first claim $\widetilde{f}$ fixes only one leaf of $\widetilde{\Lambda}^+_\varphi(\Gamma)$. Indeed, if $\widetilde{\ell}'$ is another such leaf, both ends of $\widetilde{\ell}'$ are $\widetilde{f}$-attracting, so there exists an $\widetilde{f}$-fixed point $\widetilde{q} \in \widetilde{\ell}'$ [3]. If $\widetilde{q} \neq \widetilde{p}$, then the segment between them is an NP, contradicting the fact that f has no PNPs (see Remark 2.14). Thus $\widetilde{q} = \widetilde{p}$. The intersection $\widetilde{\ell}' \cap \widetilde{\ell}$ contains $\widetilde{p}$ but since $\widetilde{p}$ is not a branch point, it must also contain $\widetilde{e}$, i.e. the edge containing $\widetilde{p}$. But since $\widetilde{\ell}$ and $\widetilde{\ell}'$ are both $\widetilde{f}$-fixed they must both contain $\widetilde{f}^k(e)$ for each k. Thus $\widetilde{\ell} = \widetilde{\ell}'$.

We now claim that $\widetilde{h}(\widetilde{\ell}) = \widetilde{\ell}$. By the previous paragraph, it suffices to show that $\widetilde{f}(\widetilde{h}(\widetilde{\ell})) = \widetilde{h}(\widetilde{\ell})$. We have $\widetilde{h}(\widetilde{\ell}) = \widetilde{h} \circ \widetilde{f}(\widetilde{\ell}) = \widetilde{f} \circ \widetilde{h}(\widetilde{\ell}) = \widetilde{f}(\widetilde{h}(\widetilde{\ell}))$, and our claim is proved.

Recall from before that $\widetilde{h}(\widetilde{e}) = \widetilde{e}$. Since $\widetilde{h}$ is an isometry, it restricts to the identity on $\widetilde{\ell}$. Projecting to Γ, since ℓ covers all of Γ, we get that h equals the identity on Γ. ☐

Corollary 4.7. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a single unique axis, then $Ker(\tau) = \{id\}$.*

Recall the surjective homomorphism τ from Equation 5.

Theorem A. Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a single unique axis, then $Cen(\langle\varphi\rangle) = \mathrm{Stab}(\Lambda^+_\varphi) \cong \mathbb{Z}$.

Proof. We showed in Corollary 4.7 that $Ker(\tau) = id$. It then follows from Equation 5 that $\mathrm{Stab}(\Lambda^+_\varphi) \cong \mathbb{Z}$. ☐

Theorem B. Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a single unique axis, then either

(1) $Comm(\langle\varphi\rangle) \cong \mathbb{Z}$ and $Comm(\langle\varphi\rangle) = Cen(\langle\varphi\rangle)$ or
(2) $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ and $Comm(\langle\varphi\rangle) = N(\langle\varphi\rangle)$.

In particular, $N(\langle\varphi\rangle) \cong \mathbb{Z}$ or $N(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Further, in the case where $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we have that φ^{-1} is also an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_{\varphi^{-1}}$ consists of a single unique axis.

Proof. Let $C_s := \mathrm{Stab}(\Lambda^+_\varphi) \cap Comm(\langle\varphi\rangle)$. By Lemma 2.21, C_s is a subgroup in $Comm(\langle\varphi\rangle)$ of index ≤ 2. Thus, either $Comm(\langle\varphi\rangle) = C_s \cong \mathbb{Z}$ or there is a short exact sequence

$$1 \to C_s \to Comm(\langle\varphi\rangle) \to \mathbb{Z}_2 \to 1. \tag{6}$$

We assume we are in the latter case, i.e. $Comm(\langle\varphi\rangle) \neq C_s$, since the other case is already part of the theorem. There are two homomorphisms $\mathbb{Z}_2 \to \mathrm{Aut}(C_s)$. We call the one whose image is the identity in $\mathrm{Aut}(C_s)$ the trivial action and we call the one mapping the identity in $\mathbb{Z}_2$ to the automorphism in $\mathrm{Aut}(C_s)$ taking a generator to its inverse the nontrivial action. First suppose $\mathbb{Z}_2$ acts trivially. Let $\psi \in Comm(\langle\varphi\rangle)$ be any outer automorphism mapping to $1 \in \mathbb{Z}_2$, then $\psi \notin C_s$ and $\psi\varphi\psi^{-1} = \varphi$ (because the action is trivial). Thus $\psi \in Cen(\langle\varphi\rangle) < C_s$, which is a contradiction. If $\mathbb{Z}_2$ acts nontrivially, then $H^2(\mathbb{Z}_2, \mathbb{Z}) \cong \{0\}$ classifies the possible group extensions in the short exact sequence (6) (see [Ben91, Proposition 3.7.3]). Hence, the only possible extension is $Comm(\langle\varphi\rangle) \cong C_s \rtimes \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

[3] There exists an interval on the leaf whose $\widetilde{f}^n$ image contains this interval.

$Comm(\langle\varphi\rangle) \geq Cen(\langle\varphi\rangle)$. Suppose $Comm(\langle\varphi\rangle) \cong \mathbb{Z}$. Given any $\eta \in Comm(\langle\varphi\rangle)$, since $\varphi \in Comm(\langle\varphi\rangle)$ and $Comm(\langle\varphi\rangle)$ is an abelian group, η commutes with φ. So $Comm(\langle\varphi\rangle) = Cen(\langle\varphi\rangle)$.

Now suppose $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, and recall $Comm(\langle\varphi\rangle) \geq N(\langle\varphi\rangle)$. As in the first paragraph of the proof, the identity $\psi \in \mathbb{Z}_2$ acts by conjugation on $C_s \cong \mathbb{Z}$ sending each element of $\mathbb{Z}$ to its inverse. Since $\varphi \in C_s$, we have $\psi\varphi\psi^{-1} = \varphi^{-1}$. Hence $\psi \in N(\langle\varphi\rangle)$ also and $Comm(\langle\varphi\rangle) \cong N(\langle\varphi\rangle)$.

We now prove the last part of the theorem. If $Comm(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, then it contains an element ψ mapping to the nonzero element in $\mathbb{Z}_2$ (as before) so that $\psi\varphi\psi^{-1} = \varphi^{-1}$. In other words, φ^{-1} is in the conjugacy class of φ. Hence, it has the same index list and ideal Whitehead graph as φ (and is also ageometric fully irreducible). In particular, φ^{-1} satisfies the conditions to be a lone axis fully irreducible outer automorphism [MP13, Theorem 4.6] □

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