Ideal Whitehead Graphs in $Out(F_r)$ II: The Complete Graph in Each Rank

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Abstract

We show how to construct, for each $r \geq 3$, an ageometric, fully irreducible $\phi \in Out(F_r)$ whose ideal Whitehead graph is the complete graph on $2r - 1$ vertices.

This paper is the second in a series of three where we show that precisely eighteen of the twenty-one connected, simplicial, five-vertex graphs are ideal Whitehead graphs of fully irreducible $\phi \in Out(F_3)$. The result is a first step to an $Out(F_r)$ version of the Masur-Smillie theorem proving precisely which index lists arise from singular measured foliations for pseudo-Anosov mapping classes.

In this paper we additionally give a method for finding periodic Nielsen paths and prove a criterion for identifying representatives of ageometric, fully irreducible $\phi \in Out(F_r)$

1 Introduction

In [MS93] Masur and Smillie list precisely which singularity index lists arise from the pair of invariant foliations for a pseudo-Anosov mapping class. The index lists were significant in their stratification of the space of quadratic differentials into strata invariant under the Teichmuller flow. Several papers studying the stratification include [KZ03], [Lan04], [Lan05], and [Zor10]. While $Out(F_r)$ index theory has been developed in papers such as [GJLL98], [GL95], [CH], and [CH12], this is the first on index realization.

Our search for ideal Whitehead graphs arising from fully irreducible free group outer automorphisms is motivated by our goal of determining the $Out(F_r)$-version of the Masur-Smillie theorem. An ideal Whitehead graph (see Section 2) is a finer invariant than a singularity index list and encodes information about the attracting lamination for a fully irreducible outer automorphism.

We construct, for each $r \geq 3$, an ageometric, fully irreducible $\phi \in Out(F_r)$ whose ideal Whitehead graph is the complete $(2r - 1)$-vertex graph. We consequently prove:

Theorem. *Let $\mathcal{C}_r$ denote the complete $(2r-1)$-vertex graph. For each $r \geq 3$, there exists an ageometric, fully irreducible $\phi \in Out(F_r)$ such that $\mathcal{C}_r$ is the ideal Whitehead graph $\mathcal{IW}(\phi)$ for ϕ.*

That the $(2r - 1)$-vertex complete graph occurs as an ideal Whitehead graph in each rank is both nonobvious and significant. First, the complete graph has no cut vertices. This phenomena holds significance for other versions of Whitehead graphs (see, for example, [Cas10], [MM10], [Sta99]). Additionally, by [HM11], cut vertices in an ideal Whitehead graph have implications about periodic Nielsen paths. Second, the $(2r - 1)$-vertex complete graph is a connected graph yielding the index sum $\frac{3}{2} - r$. This sum is as close as possible to that of $1 - r$, achieved by *geometrics* (fully irreducibles induced by pseudo Anosov surface homeomorphisms), without being achieved by a geometric outer automorphism. As in [Pfa12b], we denote the set of connected $(2r - 1)$-vertex simplicial graphs by $\mathcal{PI}_{(r;(\frac{3}{2}-r))}$. Third, the existence of such complicated ideal Whitehead graphs highlights significant depth within $Out(F_r)$ theory,

beyond that of mapping class groups, as the ideal Whitehead graph of any pseudo-Anosov is simply a disjoint union of circles.

To show that our examples indeed represent fully irreducibles, we prove a folk lemma, Proposition 4.1, the "Full Irreducibility Criterion (FIC)." [Kap12] gives another criterion inspired by our FIC.

Proposition. 4.1 **(The Full Irreducibility Criterion)** *Let g be a train track representing an outer automorphism $\phi \in Out(F_r)$ such that*

(I) g has no periodic Nielsen paths,

(II) the transition matrix for g is Perron-Frobenius, and

(III) all local Whitehead graphs $\mathcal{LW}(x; g)$ for g are connected.

Then ϕ is fully irreducible.

We address three issues when applying the criterion to a train track representative $g \colon \Gamma \to \Gamma$. First, we must verify that g has no "periodic Nielsen paths." Recall [BH92] that, for a train track map $g \colon \Gamma \to \Gamma$, a nontrivial path ρ in Γ is called a *periodic Nielsen Path (pNp)* if, for some k, $g^k(\rho) \simeq \rho$ rel endpoints. We ensure our particular representatives are pNp-free using a "Nielsen path prevention sequence," see Lemma 5.5. Proposition 5.2 provides a method for identifying pNp's of train track maps *ideally decomposed* in the sense of [Pfa12b] (a method of a different nature can be found in [Tur94]). This procedure can also be used to prove that an ideally decomposed representative has no pNp's. Second, we need that the local Whitehead graph $\mathcal{LW}(x; g)$ at each vertex x of Γ is connected, a condition satisfied in our case by the ideal Whitehead graph being connected. Third, the FIC includes a condition on the transition matrix (as defined in [BH92]) for g, satisfied when some g^n maps each edge of Γ over each other edge of Γ. We use a "switch sequence" to ensure this property is satisfied (Lemmas 3.11 and 3.13).

Finally, to ensure our representatives actually have the correct ideal Whitehead graphs, the representatives are constructed using paths in the lamination train track structures of [Pfa12b]. The paths correspond to Dehn twist automosphisms $x \mapsto xw$ and construct the lamination, as do the Dehn twists in [Pen88]. In [CP10] Clay and Pettet also use Dehn twists to construct fully irreducibles, but focus on subgroups generated by powers of two Dehn twists for two filling cyclic splittings. Further construction methods for fully irreducible outer automorphisms can be found in [KL10] and [Ham09].

In [Pfa12b] we gave, for each $r \geq 3$, examples of connected, simplicial $(2r-1)$-vertex graphs that are not the ideal Whitehead graph of any fully irreducible outer automorphism $\phi \in Out(F_r)$. In [Pfa13] we will finish our proof that precisely eighteen of the twenty-one connected, simplicial, five-vertex graphs are ideal Whitehead graphs of fully irreducible $\phi \in Out(F_3)$. The results of this paper are used for proving the theorem in [Pfa13], but also make progress in a second direction, as we prove existence of the complete graph in each rank instead of focusing exclusively on rank-three, as we will in [Pfa13].

Acknowledgements

2   Preliminary definitions and notation

We continue with the introduction's notation. Additionally, unless otherwise stated, we assume throughout this document that outer automorphism representatives are train track

(tt) representatives in the sense of [BH92].

For a rank r free group F_r, $\mathcal{FI}_r$ will denote the set of the fully irreducible elements of $Out(F_r)$.

Directions and turns

In general we use definitions from [BH92] and [BFH00] for discussing train track maps. We remind the reader of additional definitions and notation given in [Pfa12b]. Here Γ will be a rose and $g\colon \Gamma \to \Gamma$ will represent $\phi \in Out(F_r)$. $\mathcal{E}^+(\Gamma) := \{E_1, \ldots, E_n\} = \{e_1, e_1, \ldots, e_{2n-1}, e_{2n}\}$ will be the edge set of Γ with a prescribed orientation. $\mathcal{E}(\Gamma) := \{E_1, \overline{E_1}, \ldots, E_n, \overline{E_n}\}$, where $\overline{E_i}$ denotes E_i oppositely oriented. If an edge indexing $\{E_1, \ldots, E_n\}$ (thus indexing $\{e_1, e_1, \ldots, e_{2n-1}, e_{2n}\}$) is prescribed, Γ is called *edge-indexed*. $\mathcal{D}(v)$ or $\mathcal{D}(\Gamma)$ will denote the set of directions at the vertex v. For each $e \in \mathcal{E}(\Gamma)$, $D_0(e)$ will denote the initial direction of e. Also, $D_0\gamma := D_0(e_1)$ for any path $\gamma = e_1 \ldots e_k$ in Γ. Dg will denote the direction map g induces. We call $d \in \mathcal{D}(\Gamma)$ *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* if $k = 1$.

$\mathcal{T}(v)$ will denote the set of turns at v and $D^t g$ the induced turn map. Sometimes we abusively write $\{\overline{e_i}, e_j\}$ for $\{D_0(\overline{e_i}), D_0(e_j)\}$. For a path $\gamma = e_1 e_2 \ldots e_{k-1} e_k$ in Γ, we say γ *traverses* $\{\overline{e_i}, e_{i+1}\}$ for each $1 \leq i < k$. Recall that a turn is called *illegal* for g if $Dg^k(d_i) = Dg^k(d_j)$ for some k.

Transition matrices and irreducibility.

The *transition matrix* for a topological representative g is the square matrix where the ij^{th} entry is the number of times $g(E_j)$ traverses E_i in either direction. A matrix $A = [a_{ij}]$ is *irreducible* if each entry $a_{ij} \geq 0$ and if, for each i and j, there exists a $k > 0$ so that the ij^{th} entry of A^k is strictly positive. The matrix is *Perron-Frobenius (PF)* if each sufficiently high k works for all index pairs $\{i, j\}$. [BH92]

The Full Irreducibility Criterion will require that the transition matrix for a representative be PF. It will be relevant that any power of a PF matrix is both PF and irreducible. Additionally, a topological representative is irreducible if and only if its transition matrix is irreducible [BH92].

Periodic Nielsen paths and ageometric outer automorphisms.

Recall [BH92], for $g\colon \Gamma \to \Gamma$, a nontrivial path ρ in Γ is called a *periodic Nielsen path (pNp)* if, for some k, $g^k(\rho) \simeq \rho$ rel endpoints. For $k = 1$, ρ is called a *Nielsen path (Np)*. ρ is called an *indivisible Nielsen path (iNp)* if it cannot be written as a nontrivial concatenation $\rho = \rho_1 \cdot \rho_2$ of Np's ρ_1 and ρ_2. If ρ is an iNp for an irreducible train track map g, then (Lemma 3.4, [BF94]) there exist unique, nontrivial, legal paths α, β, and τ in Γ so that $\rho = \bar{\alpha}\beta$, $g(\alpha) = \tau\alpha$, and $g(\beta) = \tau\beta$. In [BF94], immersed paths $\alpha_1, \ldots, \alpha_k \in \Gamma$ are said to form an *orbit of periodic Nielsen paths* if $g^k(\alpha_i) \simeq \alpha_{i+1 \bmod k}$ rel endpoints, for all $1 \leq i \leq k$. The orbit is called *indivisible* when α_1 is not a concatenation of subpaths belonging to orbits of pNps. Each α_i in an indivisible orbit is called an *indivisible periodic Nielsen path (ipNp)*.

Recall [GJLL98], that an outer automorphism is *ageometric* whose stable representative, in the sense of [BH92], has no pNp's. We denote by $\mathcal{AFI}_r$ the subset of $\mathcal{FI}_r$ consisting of the ageometric elements.

Ideal Whitehead graphs and lamination train track (ltt) structures.

We remind the reader of an [HM11] ideal Whitehead graph definition and [Pfa12b] lamination train track structure definition. See [Pfa12a] and [HM11] for descriptions of ideal Whitehead graph alternative definitions and outer automorphism invariance. Note that, while we use a representative to construct it, the ideal Whitehead graph does not depend on the choice of representative for an outer automorphism.

Let Γ be a marked graph, $g\colon \Gamma \to \Gamma$ a tt representive of $\phi \in Out(F_r)$, and $v \in \Gamma$ a *singularity* (the endpoint of an ipNp or a vertex with at least three periodic directions). The *local Whitehead graph* $\mathcal{LW}(g; v)$ for g at v has:
(1) a vertex for each direction $d \in \mathcal{D}(v)$ and
(2) edges connecting vertices for $d_1, d_2 \in \mathcal{D}(v)$ when $\{d_1, d_2\}$ is taken by some $g^k(e)$, with $e \in \mathcal{E}(\Gamma)$.
The *local stable Whitehead graph* $\mathcal{SW}(g; v)$ is the subgraph obtained by restricting precisely to vertices

with periodic direction labels. For a rose Γ with vertex v, we denote the single local stable Whitehead graph $\mathcal{SW}(g;v)$ by $\mathcal{SW}(g)$ and the single local Whitehead graph $\mathcal{LW}(g;v)$ by $\mathcal{LW}(g)$.

For a pNp-free g, the *ideal Whitehead graph* $\mathcal{IW}(\phi)$ *of* ϕ is isomorphic to $\bigsqcup_{\text{singularities } v \in \Gamma} \mathcal{SW}(g;v)$. In particular, when Γ is a rose, $\mathcal{IW}(\phi) \cong \mathcal{SW}(g)$.

Let g be a pNp-free tt map on a marked rose Γ with vertex v. Recall from [Pfa12b] the definition of the *lamination train track (ltt) structure* $G(g)$ for g: The *colored local Whitehead graph* $\mathcal{CW}(g)$ *at* v is $\mathcal{LW}(g)$ with the subgraph $\mathcal{SW}(g)$ colored purple and $\mathcal{LW}(g) - \mathcal{SW}(g)$ red (nonperiodic direction vertices are red). Let $\Gamma_N = \Gamma - N(v)$, where $N(v)$ is a contractible neighborhood of v. For each $E_i \in \mathcal{E}^+$, add vertices $D_0(E_i)$ and $\overline{D_0(E_i)}$ at the corresponding boundary points of the partial edge $E_i - (N(v) \cap E_i)$. $G(g)$ is formed from $\Gamma_N \bigsqcup \mathcal{CW}(g)$ by identifying the vertex d_i in Γ_N with the vertex d_i in $\mathcal{CW}(g)$. Nonperiodic directions vertices are red, edges of Γ_N are black, and periodic vertices are purple.

$G(g)$ is given a *smooth structure* via a partition of the edges into the set of black edges $\mathcal{E}_b$ and the set of colored edges $\mathcal{E}_c$. A *smooth path* will mean a path alternating between colored and black edges.

We refer the reader to [Pfa12a] or [Pfa12b] for a thorough presentation of abstract lamination train track structures. We summarize just several definitions here.

Recall that a *train track (tt) graph* is a finite graph G satisfying:

tt1: G has no valence-1 vertices;

tt2: each edge of G has 2 distinct vertices (single edges are never loops); and

tt3: the edge set of G is partitioned into two subsets, $\mathcal{E}_b$ (the "black" edges) and $\mathcal{E}_c$ (the "colored" edges), such that each vertex is incident to at least one $E_b \in \mathcal{E}_b$ and at least one $E_c \in \mathcal{E}_c$.

A *lamination train track (ltt) structure* G is a pair-labeled colored train track graph (black edges will be included, but not considered colored) satisfying:

ltt1: Vertices are either purple or red.

ltt2: Edges are of 3 types ($\mathcal{E}_b$ comprises the black edges and $\mathcal{E}_c$ comprises the red and purple edges):

(Black Edges): A single black edge connects each pair of (edge-pair)-labeled vertices. There are no other black edges. In particular, each vertex is contained in a unique black edge.

(Red Edges): A colored edge is red if and only if at least one of its endpoint vertices is red.

(Purple Edges): A colored edge is purple if and only if both endpoint vertices are purple.

ltt3: No pair of vertices is connected by two distinct colored edges.

We denote the purple subgraph of G (from $\mathcal{SW}(g)$) by $\mathcal{PI}(G)$ and, if $\mathcal{G} \cong \mathcal{PI}(G)$, say G *is an ltt structure for* $\mathcal{G}$. An $(r; (\frac{3}{2} - r))$ *ltt structure* is an ltt structure G for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ such that:

ltt(*)4: G has precisely 2r-1 purple vertices, a unique red vertex, and a unique red edge.

We consider ltt structures *equivalent* that differ by an ornamentation-preserving homeomorphism and refer the reader to the Standard Notation and Terminology 2.2 of [Pfa12b]. In particular, in abstract and nonabstract ltt structures, $[d_i, d_j]$ is the edge connecting a vertex pair $\{d_i, d_j\}$, $[e_i]$ denotes the black edge $[d_i, \overline{d_i}]$ for $e_i \in \mathcal{E}(\Gamma)$, and $\mathcal{C}(G)$ denotes the colored subgraph (from $\mathcal{LW}(g)$). Purple vertices are *periodic* and red vertices *nonperiodic*. G is *admissible* if birecurrent as a train track structure (i.e has a locally smoothly embedded line traversing each edge infinitely many times as $\mathbb{R} \to \infty$ and as $\mathbb{R} \to -\infty$).

For an $(r; (\frac{3}{2} - r))$ ltt structure G for $\mathcal{G}$, additionally:

1. d^u labels the unique red vertex and is called the *unachieved direction*.

2. $e^R = [t^R]$ denotes the unique red edge, $\overline{d^a}$ labels its purple vertex, thus $t^R = \{d^u, \overline{d^a}\}$ ($e^R = [d^u, \overline{d^a}]$).

3. $\overline{d^a}$ is contained in a unique black edge, which we call the *twice-achieved edge*.

4. d^a will label the other twice-achieved edge vertex and be called the *twice-achieved direction*.

5. If G has a subscript, the subscript carries over to all relevant notation. For example, in G_k, d_k^u will label the red vertex and e_k^R the red edge.

4

We call a $2r$-element set of the form $\{x_1, \overline{x_1}, \ldots, x_r, \overline{x_r}\}$, with elements paired into *edge pairs* $\{x_i, \overline{x_i}\}$, a *rank-r edge pair labeling set* (we write $\overline{\overline{x_i}} = x_i$). We call a graph with vertices labeled by an edge pair labeling set a *pair-labeled* graph, and an *indexed pair-labeled* graph if an indexing is prescribed. A $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ is *(index) pair-labeled* whose vertices are labeled by a $2r - 1$ element subset of the rank-r (indexed) edge pair labeling set. An ltt structure, index pair-labeled as a graph, is an *indexed pair-labeled* ltt structure if the vertices of the black edges are indexed by edge pairs. Index pair-labeled ltt structures are *equivalent* that are equivalent as ltt structures via an equivalence preserving the indexing of the vertex labeling set. By rank-r index pair-labeling an $(r;(\frac{3}{2}-r))$ ltt structure G and edge-indexing the edges of an r-petaled rose Γ, one creates an identification of the vertices in G with $\mathcal{D}(v)$, where v is the vertex of Γ. With this identification, we say G is *based* at Γ. In such a case we may use the notation $\{d_1, d_2, \ldots, d_{2r-1}, d_{2r}\}$ for the vertex labels. Additionally, $[e_i]$ denotes $[D_0(e_i), D_0(\overline{e_i})] = [d_i, \overline{d_i}]$ for each edge $e_i \in \mathcal{E}(\Gamma)$. We call a permutation of the indices $1 \leq i \leq 2r$ combined with a permutation of the elements of each pair $\{x_i, \overline{x_i}\}$ an *edge pair (EP) permutation*. Edge-indexed graphs will be considered *edge pair permutation (EPP) isomorphic* if there is an EP permutation making the labelings identical.

Ideal decompositions.

M. Feighn and M. Handel defined rotationless train track representatives and outer automorphisms in [FH11]. Recall [HM11]: Let a $\phi \in \mathcal{AFI}_r$ be such that $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$, then ϕ is *rotationless* if and only if the vertices of $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ are fixed by the action of ϕ.

The following is Proposition 3.3 of [Pfa12b]:

Proposition. [Pfa12b] Let $\phi \in \mathcal{AFI}_r$ with $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$. There exists a pNp-free tt map on the rose representing a rotationless power $\psi = \phi^R$ and decomposing as $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$, such that:

(I) the index set $\{1, \ldots, n\}$ is viewed as the set $\mathbf{Z}/n\mathbf{Z}$ with its natural cyclic ordering;

(II) each Γ_k is an edge-indexed rose with an indexing $\{e_{(k,1)}, e_{(k,2)}, \ldots, e_{(k,2r-1)}, e_{(k,2r)}\}$ where:

(a) one can edge-index Γ with $\mathcal{E}(\Gamma) = \{e_1, e_2, \ldots, e_{2r-1}, e_{2r}\}$ such that, for each t with $1 \leq t \leq 2r$, $g(e_t) = e_{i_1} \ldots e_{i_s}$ where $(g_n \circ \cdots \circ g_1)(e_{0,t}) = e_{n,i_1} \ldots e_{n,i_s}$;

(b) for some i_k, j_k with $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$,

$$g_k(e_{k-1,t}) := \begin{cases} e_{k,t}e_{k,j_k} & \text{for } t = i_k \\ e_{k,t} & \text{for all } e_{k-1,t} \neq e_{k-1,j_k}^{\pm 1}; \text{ and} \end{cases}$$

(c) for each $e_t \in \mathcal{E}(\Gamma)$ such that $t \neq j_n$, we have $Dh(d_t) = d_t$, where $d_t = D_0(e_t)$.

Recall that tt maps satisfying (I)-(II) are called *ideally decomposable ($\mathcal{ID}$)* with an *ideal decomposition ($\mathcal{ID}$)*. An $\mathcal{ID}$ g such that $\phi \in \mathcal{AFI}_r$ and $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ has *type* $(r;(\frac{3}{2}-r))$. In [Pfa12b] we proved for a $(r;(\frac{3}{2}-r))$ tt map $g : \Gamma \to \Gamma$, that $G(g)$ is an $(r;(\frac{3}{2}-r))$ ltt structure with base Γ.

Again we denote e_{k-1,j_k} by e_{k-1}^{pu}, denote e_{k,j_k} by e_k^u, denote e_{k,i_k} by e_k^a, and denote $e_{k-1,i_{k-1}}$ by e_{k-1}^{pa}. Also, $\mathcal{D}_k := \mathcal{D}(\Gamma_k)$, $\mathcal{E}_k := \mathcal{E}(\Gamma_k)$, and $G_k := G(f_k)$ where $f_k := g_k \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_{k+1} : \Gamma_k \to \Gamma_k$. And

$$g_{k,i} := \begin{cases} g_k \circ \cdots \circ g_i : \Gamma_{i-1} \to \Gamma_k \text{ if } k > i \text{ and} \\ g_k \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_i \text{ if } k < i \end{cases}.$$

It is proved in [Pfa12b] that $D_0(e_k^u) = d_k^u$, $D_0(e_k^a) = d_k^a$, $D_0(e_{k-1}^{pu}) = d_{k-1}^{pu}$, and $D_0(e_{k-1}^{pa}) = d_{k-1}^{pa}$. As described in [Pfa12b], for any k, l, we have a *direction map* $Dg_{k,l}$, an *induced map of turns* $Dg_{k,l}^t$, and an *induced map of ltt structures* $Dg_{k,l}^T : G_{l-1} \mapsto G_k$. $Dg_{k,l}^C$ denotes the restriction to $\mathcal{C}(G_{l-1})$ of $Dg_{k,l}^T$.

Extensions and switches.

By a proper full fold we mean the identification of a (proper) partial edge with a full edge. A *triple* (g_k, G_{k-1}, G_k) is an ordered set of three objects where $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of roses and, for $i = k - 1, k$, G_i is an ltt structure with base Γ_i. Recall [Pfa12b], in an $\mathcal{ID}$ of a $(r; (\frac{3}{2} - r))$ representative, each (g_k, G_{k-1}, G_k) satisfies the "admissible map properties" $\mathcal{AM}$I-VII of [Pfa12b] and is either a "switch" or "extension."

A *generating triple (gt)* is a triple (g_k, G_{k-1}, G_k) where

(gtI) $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of edge-indexed roses defined by

    a. $g_k(e_{k-1,j_k}) = e_{k,i_k} e_{k,j_k}$ where $d_k^a = D_0(e_{k,i_k})$, $d_k^u = D_0(e_{k,j_k})$, and $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$ and

    b. $g_k(e_{k-1,t}) = e_{k,t}$ for all $e_{k-1,t} \neq (e_{k,j_k})^{\pm 1}$;

(gtII) G_i is an indexed pair-labeled $(r; (\frac{3}{2} - r))$ ltt structure with base Γ_i for $i = k - 1, k$; and

(gtIII) The induced map of based ltt structures $D^T(g_k) : G_{k-1} \to G_k$ exists and, in particular, restricts to an isomorphism from $\mathcal{PI}(G_{k-1})$ to $\mathcal{PI}(G_k)$.

G_{k-1} is the *source ltt structure* and G_k the *destination ltt structure*. If both are admissible, the triple is *admissible*. We sometimes write $g_k : e_{k-1}^{pu} \mapsto e_k^a e_k^u$ for g_k, write d_{k-1}^{pu} for d_{k-1,j_k}, and write e_{k-1}^{pa} for e_{k-1,i_k}. If G_k and G_{k-1} are index pair-labeled $(r; (\frac{3}{2} - r))$ ltt structures for $\mathcal{G}$, then (g_k, G_{k-1}, G_k) will be a *generating triple for $\mathcal{G}$*.

The *switch* determined by a purple edge $[d_k^a, d_{(k,l)}]$ in G_k is the gt (g_k, G_{k-1}, G_k) for $\mathcal{G}$ satisfying:

(swI): $D^T(g_k)$ restricts to an isomorphism from $\mathcal{PI}(G_{k-1})$ to $\mathcal{PI}(G_k)$ defined by $d_{k-1}^{pu} \mapsto d_k^a = d_{k,i_k}$ $(d_{k-1,t} \mapsto d_{k,t}$ for $d_{k-1,t} \neq d_{k-1}^{pu})$ and extended linearly over edges.

(swII): $d_{k-1}^{pa} = d_{k-1}^u$.
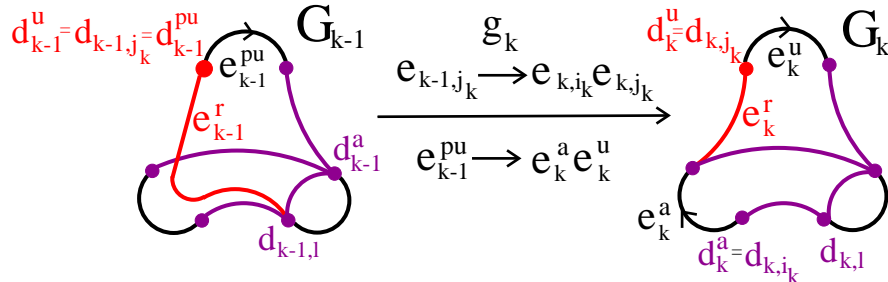
(swIII): $\overline{d_{k-1}^a} = d_{k-1,l}$.
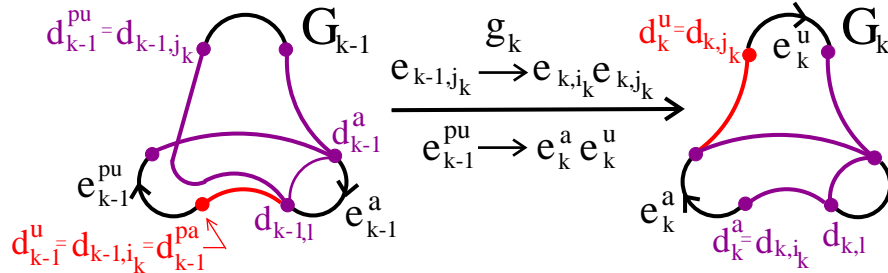
The *extension determined by* $[d_k^a, d_{k,l}]$, is the gt (g_k, G_{k-1}, G_k) for $\mathcal{G}$ satisfying:

(extI): The restriction of $D^T(g_k)$ to $\mathcal{PI}(G_{k-1})$ is defined by sending, for each j, the vertex labeled $d_{k-1,j}$ to the vertex labeled $d_{k,j}$ and extending linearly over edges.

(extII): $d_{k-1}^u = d_{k-1}^{pu}$, i.e. $d_{k-1}^{pu} = d_{k-1,j_k}$ labels the single red vertex in G_{k-1}.

(extIII): $\overline{d_{k-1}^a} = d_{k-1,l}$.

3 Compositions of extensions and switches

Compositions of a sequence of extensions or a sequence of switches play an important role in our proofs.

Definition 3.1. A *preadmissible composition* $(g_{i-k}, \ldots, g_i, G_{i-k-1}, \ldots, G_i)$ for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ is a sequence of proper full folds of (edge-pair)-indexed roses,

$$\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i,$$

with *associated sequence of* $(r;(\frac{3}{2}-r))$ *ltt structures* for $\mathcal{G}$,

$$G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i,$$

where, for each $i - k - 1 \leq j < i$, (g_{j+1}, G_j, G_{j+1}) is an extension or switch for $\mathcal{G}$.

The Definition 3.1 notation will be standard. A composition is *admissible* if each G_j is. We call $g_{i,i-k}$ the *associated automorphism*, G_{i-k-1} the *source ltt structure*, and G_k the *destination ltt structure*.

To ensure $\mathcal{IW}(g) \cong C_r$ in Theorem 6.2, we use "building block" compositions of extensions: If each (g_j, G_{j-1}, G_j) with $i - k < j \leq i$ is an admissible extension and $(g_{i-k}, G_{i-k-1}, G_{i-k})$ is an admissible switch, then we call $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ an *admissible construction composition* for $\mathcal{G}$. We call $g_{i,i-k}$ a *construction automorphism*. Leaving out the switch, gives a *purified construction automorphism* $g_p = g_i \circ \cdots \circ g_{i-k+1}$ and *purified construction composition* $(g_{i-k+1}, \ldots, g_i; G_{i-k}, \ldots, G_i)$.

A construction automorphism always has the form of a Dehn twist automorphism $e_{i-k-1}^{pu} \mapsto w e_{i-k}^u$, where $w = e_{i-k}^a \ldots e_i^a$. One can view the composition as twisting the edge corresponding to e_{i-k-1}^{pu} around the path corresponding to w in the destination ltt structure. In the next section we describe these paths and prove (Proposition 3.6) they "construct" a smooth path in their destination ltt structure.

3.1 Construction Paths

Corresponding to a construction composition is a path in its destination ltt structure. A key property of such a path (Lemma 3.6) holds when the construction composition is part of the ideal decomposition of a type $(r;(\frac{3}{2}-r))$ representative g: the image of the path's purple edges live in $G(g)$.

We abuse notation throughout this section by dropping indices. While not necessary, it may aid in visualization of the properties and procedures, as well as reduce potential confusion over indices.

Lemma 3.2. *Let* $(g_1, \ldots, g_n, G_0, \ldots, G_n)$ *be an* $\mathcal{ID}$ *for a* $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ *and* $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ *a construction composition. Then*

$$[d_i^u, \overline{d_i^a}, d_i^a, d_i, d_{i-1}^a, d_{i-1}, \ldots, d_{i-k+1}^a, d_{i-k+1}, d_{i-k}^a] = [d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, \ldots, \overline{d_{i-k}^a}, d_{i-k}^a]$$

is a smooth path in the ltt structure G_i.

Proof. We proceed by induction for decreasing s. Proof by induction is valid, as the proof does not rely on G_{i-k-1} (the only thing distinguishing $(g_{i-k}, G_{i-k-1}, G_{i-k})$ as a switch). For the base case note that $e_i^R = [d_i^u, \overline{d_i^a}]$. So $[d_i^u, \overline{d_i^a}, d_i^a]$ is a path in G_i and smooth, as it alternates between colored and black edges ($[d_i^u, \overline{d_i^a}]$ is colored and $[\overline{d_i^a}, d_i^a]$ is black). For the induction assume, for $i > s > i - k$, $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}, \ldots, d_{s+1}^a, \overline{d_s^a}, d_s^a]$ is a smooth path in G_i (ending with the black edge $[\overline{d_s^a}, d_s^a]$).

By [Pfa12b] Corollary 5.6b, $e_{s-1}^R = [d_{s-1}^u, \overline{d_{s-1}^a}]$. By [Pfa12b] Lemma 5.7, $D^C g_s([d_{s-1}^u, \overline{d_{s-1}^a}])$ $= [d_s^a, \overline{d_{s-1}^a}]$ is a purple edge in G_s. Since purple edges are always mapped to themselves by extensions (in the sense that D^C preserves the second index of their vertex labels) and $D^C g_s([d_s^u, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is in $\mathcal{PI}(G_s)$, $D^C g_{n,s}(\{d_{s-1}^u, \overline{d_{s-1}^a}\}) = D^C g_{n,s+1}(D^C g_s([d_{s-1}^u, \overline{d_{s-1}^a}])) = D^C g_{n,s+1}([d_s^a, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is in

$\mathcal{PI}(G_i)$. Thus, including the purple edge $[d_s^a, \overline{d_{s-1}^a}]$ in the smooth path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \ldots, d_{s+1}^a, \overline{d_s^a}, d_s^a]$ gives the smooth path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, \ldots, d_{s+1}^a, \overline{d_s^a}, d_s^a, \overline{d_{s-1}^a}]$. (It is smooth, as we added a colored edge to a path with edges alternating between colored and black, ending with black). By including the black edge $[\overline{d_{s-1}^a}, d_{s-1}^a]$ we get the construction path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \ldots, d_s^a, \overline{d_{s-1}^a}, d_{s-1}^a]$. (Also smooth, as we added a black edge to a path with edges alternating between colored and black, ending colored). This concludes the inductive step, hence proof. $\qquad\square$

The path of Lemma 3.2 (depicted in Example 3.5) is called the *construction path* for $(g_{i-k}, \ldots, g_i;$ $G_{i-k-1}, \ldots, G_i)$ and denoted $\gamma_{g_{i,i-k}}$. One obtains it by traversing the red edge $[d_i^u, \overline{d_i^a}]$ from the red vertex d_i^u to the vertex $\overline{d_i^a}$, the black edge $[\overline{d_i^a}, d_i^a]$ from $\overline{d_i^a}$ to d_i^a, the extension determining purple edge $[d_i^a, d_i] = [d_i^a, \overline{d_{i-1}^a}]$ from d_i^a to $d_i = \overline{d_{i-1}^a}$, the black edge $[\overline{d_{i-1}^a}, d_{i-1}^a]$ from $\overline{d_{i-1}^a}$ to d_{i-1}^a, the extension determining purple edge $[d_{i-1}^a, d_{i-1}] = [d_{i-1}^a, \overline{d_{i-2}^a}]$ from d_{i-1}^a to $d_{i-1} = \overline{d_{i-2}^a}$, the black edge $[\overline{d_{i-2}^a}, d_{i-2}^a]$ from $\overline{d_{i-2}^a}$ to d_{i-2}^a, continuing as such through the purple edges determining each g_j (inserting black edges between), and finally traversing $[d_{i-k+1}^a, d_{i-k+1}] = [d_{i-k+1}^a, \overline{d_{i-k}^a}]$ and then $[\overline{d_{i-k}^a}, d_{i-k}^a]$ from $\overline{d_{i-k}^a}$ to d_{i-k}^a.

Let G be an admissible $(r; (\frac{3}{2} - r))$ ltt structure with the standard notation. The *construction subgraph* G_C is constructed from G via the following procedure:

1. Remove the interior of the black edge $[e^u]$, the purple vertex $\overline{d^u}$, and the interior of any purple edges containing the vertex $\overline{d^u}$. Call the graph with these edges and vertices removed G^1.
2. Given G^{j-1}, recursively define G^j: Let $\{\alpha_{j-1,i}\}$ be the set of vertices in G^{j-1} not contained in any colored edge of G^{j-1}. G^j is obtained from G^{j-1} by removing all black edges containing a vertex $\alpha_{j-1,i} \in \{\alpha_{j-1,i}\}$, as well as the interior of each purple edge containing a vertex $\overline{\alpha_{j-1,i}}$.
3. $G_C = \bigcap_j G^j$.

A construction path actually always lives in the construction subgraph of its destination ltt structure.

Example 3.3. To find the construction subgraph G_C for the ltt structure G on the left (1), we remove the interior of the black edge $[\bar{a}, a]$ to obtain the middle graph (2), then remove a and the interior of all purple edges containing a to obtain G_C (graph (3) depicted on the right).
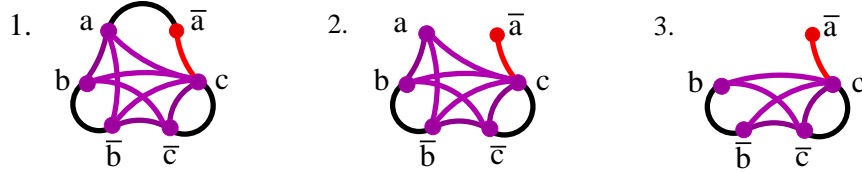
The following lemma gives some conditions under which a path in an admissible (edge-pair)-indexed $(r; (\frac{3}{2} - r))$ ltt structure G is guaranteed to be the construction path for a construction composition with destination ltt structure G. It also explains how to find such a construction composition.

Lemma 3.4. *Let G be an admissible $(r; (\frac{3}{2} - r))$ ltt structure and consider a smooth path*

$$\gamma = [d^u, \overline{x_1}, x_1, \overline{x_2}, x_2, \ldots, x_{k+1}, \overline{x_{k+1}}]$$

in G_C starting with e^R (oriented from d^u to $\overline{d^a}$) and ending with the black edge $[x_{k+1}, \overline{x_{k+1}}]$.
Edge-index r-petaled roses $\Gamma_{i-k-1}, \ldots, \Gamma_i$ and define the homotopy equivalences

$$\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$$

by $g_l : e_{l-1,s} \mapsto e_{l,t_l} e_{l,s}$, where $D_0(e_{l,t_l}) = \overline{x_{i-l+1}}$, and $g_l(e_{l-1,j}) = e_{l,j}$ for $e_{l-1,j} \neq e_{l-1,s}^{\pm 1}$.
Define the ltt structures (with respective bases Γ_j) G_t, for $i - k - 1 \leq t \leq i$, by having:
1. each $\mathcal{PI}(G_l)$ isomorphic to $\mathcal{PI}(G_i)$ via an isomorphism preserving the vertex label second indices,

2. the second index of the label on the single red vertex in each G_l be "s" (the same as in G_i), and

3. the single red edge in G_l be $[d_{l,s}, \overline{d_{l,t_l}}]$.

If each G_j is an admissible $(r; (\frac{3}{2}-r))$ ltt structure for $\mathcal{G}$ with base Γ_j, then $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ is a purified construction composition with construction path γ. For each $i - k + 1 \le l \le i$, the triple (g_l, G_{l-1}, G_l) is the extension determined by $[\overline{x_{i-l+1}}, x_{i-l+2}]$.

Proof. It suffices to show: A. each (g_l, G_{l-1}, G_l) is the extension determined by $[\overline{x_{i-l+1}}, x_{i-l+2}]$ (so that $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ is indeed a construction composition) and B. the corresponding construction path is $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \ldots, d_{i-k+1}^a, \overline{d_{i-k}^a}, d_{i-k}^a]$.

(extI) holds by our requiring each G_j be an $(r; (\frac{3}{2} - r))$ ltt structure with rose base graph. The G_l are $(r; (\frac{3}{2} - r))$ ltt structures for $\mathcal{PI}(G)$ by (1)-(3) in the lemma statement. This, with how we defined our notation, implies (gtIII) and (extI). The second index of the red vertex label is the same in each G_l as in G_i, giving (ext II). To see (extIII) holds by (1), note that $[\overline{x_{i-l+1}}, x_{i-l+2}]$ is in $\mathcal{PI}(G_l)$ (it is in G and $\mathcal{PI}(G) \cong \mathcal{PI}(G_l)$) and would be the determining edge for the extension. (A) is proved.

The construction path is $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \ldots, d_{i-k+1}^a, \overline{d_{i-k}^a}, d_{i-k}^a]$ by Lemma 3.2, proving (B). $\qquad\square$

It is proved in [Pfa12a] that $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ is in fact the unique construction composition with γ as its construction path. We call $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$, together with its sequence of ltt structures $G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i$, as in the lemma, the construction composition *determined by* the path $\gamma = [d^u, \overline{x_1}, x_1, \overline{x_2}, x_2, \ldots, x_{k+1}, \overline{x_{k+1}}]$.

Example 3.5. In the following ltt structure, G_i, the numbered edges give a construction path determined by the construction automorphism $a \mapsto ab\overline{c}\overline{c}bbcb$ (the automorphism fixes all other edges).
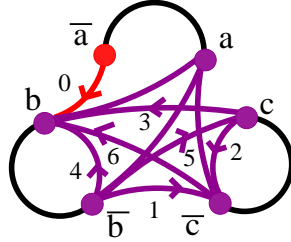
We retrieve each ltt structure G_{i-k} in the construction composition by moving the red edge of G_i to be attached to the terminal vertex of edge k in the construction path. If the red vertex of G_j is d_s and the red edge is $[d_s, d_t]$, then g_j is defined by $e_s \mapsto \bar{e}_t e_s$. We show the construction composition, leaving out the source ltt structure G_{i-7} of the switch to highlight that it does not affect the construction path.
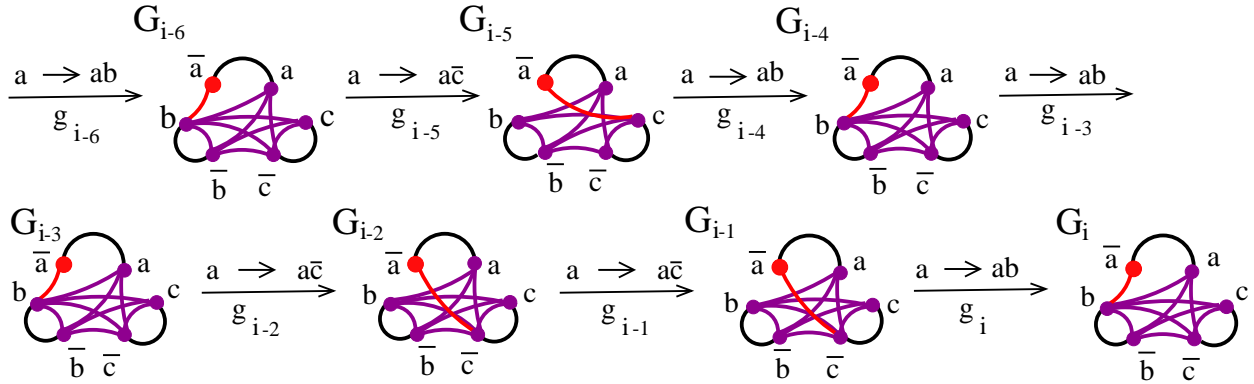
Lemma 3.6 is fundamental to our construction techniques. It says that construction compositions, in fact, "build" in the ideal Whitehead graph the images of the purple edges of the construction path:

Lemma 3.6. *Let g be an $\mathcal{ID}$ type $(r; (\frac{3}{2} - r))$ representative of $\phi \in Out(F_r)$ with $\mathcal{IW}(\phi) = \mathcal{G}$. Suppose g decomposes as $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$, with the sequence of ltt structures for $\mathcal{G}$:*

$$G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i.$$

If $g' = g_n \circ \cdots \circ g_{k+1}$ is a construction composition, then $\mathcal{G}$ contains as a subgraph the purple edges in the construction path for g'.

Proof. We proceed by induction for decreasing k. Proof by induction is valid here since nothing in the proof will rely on G_k (the only thing distinguishing (g_k, G_k, G_{k+1}) as a switch instead of an extension).

For the base case consider $g_n \circ g_{n-1}$. By [Pfa12b] Corollary 5.6b G_{n-1} has red edge $[d_{n-1}^u, \overline{d_{n-1}^a}]$. We know g_n is defined by g_n: $e_{n-1}^{pu} \mapsto e_n^a e_n^u$ and $g_n(e_{n-1,l}) = e_{n,l}$ for all $e_{n-1,l} \neq (e_{n-1}^{pu})^{\pm 1}$. Thus, since $d_{n-1}^{pu} = d_{n-1}^u \neq \overline{d_{n-1}^a}$, we know that $Dg_n(\overline{d_{n-1}^a}) = \overline{d_{n-1}^a}$. So $D^C g_n([d_{n-1}^u, \overline{d_{n-1}^a}]) = D^C g_n([d_{n-1}^{pu}, \overline{d_{n-1}^a}]) = [d_n^a, \overline{d_{n-1}^a}]$ and, since $D^C g_n(\mathcal{C}(G_{n-1})) \subset \mathcal{PI}(G_n)$, $[d_n^a, \overline{d_{n-1}^a}]$ is in $\mathcal{PI}(G_n)$. The base case is proved.

For the inductive step assume, for $n > s > k+1$, G_n contains the purple edges of $\gamma_{g_n,s}$. Again by [Pfa12b] Corollary 5.6b, $e_{s-1}^R = [d_{s-1}^u, \overline{d_{s-1}^a}]$. As above, $D^C g_s([d_{s-1}^u, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is in $\mathcal{PI}(G_s)$. Since extensions map purple edges to themselves and $D^C g_s([d_s^a, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$, $D^C g_{n,s}([d_{s-1}^u, \overline{d_{s-1}^a}]) = D^C g_{n,s+1}(D^t g_s([d_{s-1}^u, \overline{d_{s-1}^a}])) = D^C g_{n,s+1}([d_s^a, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$, proving the inductive step. $\square$

3.2 Switch Paths

We use "switch paths" to find switch sequences. Here switch sequences play two primary roles: ensuring our ideal decomposition actually gives a loop in $\mathcal{ID}(\mathcal{G})$ and ensuring our transition matrix is PF.

We continue with the notational abuse of the previous section (primarily ignoring second indices).

Definition 3.7. (See Example 3.12) An admissible *switch sequence* for a $(r; (\frac{3}{2} - r))$ graph $\mathcal{G}$ is an admissible composition $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ for $\mathcal{G}$ such that

(ss1) each (g_j, G_{j-1}, G_j) with $i - k \leq j \leq i$ is a switch and

(ss2) $d_{n+1}^a = d_n^u \neq d_l^u = d_{l+1}^a$ and $\overline{d_l^a} \neq d_n^u = d_{n+1}^a$ for all $i \geq n > l \geq i - k$.

We call the associated automorphism $g_{i,i-k} = g_i \circ \cdots \circ g_{i-k}$ a *switch sequence automorphism*.

Remark 3.8. (ss2) is not implied by (ss1) and is necessary for a switch path to indeed be a path. Certain statements in the Lemma 3.11 proof below (showing that the switch path for a switch sequence is a smooth path in the destination ltt structure) would be incorrect without (ss2).

Definition 3.9. Let $(g_j, \ldots, g_k; G_{j-1}, \ldots, G_k)$ be an admissible switch sequence. Its *switch path* is a path in the destination ltt structure G_k traversing the red edge $[d_k^u, \overline{d_k^a}]$ from its red vertex d_k^u to $\overline{d_k^a}$, the black edge $[\overline{d_k^a}, d_k^a]$ from $\overline{d_k^a}$ to d_k^a, what is the red edge $[d_{k-1}^u, \overline{d_{k-1}^a}] = [d_k^a, \overline{d_{k-1}^a}]$ in G_{k-1} (purple edge in G_k) from $d_k^a = d_{k-1}^u$ to $\overline{d_{k-1}^a}$, the black edge $[\overline{d_{k-1}^a}, d_{k-1}^a]$ from $\overline{d_{k-1}^a}$ to d_{k-1}^a, continues as such through the red edges for the G_i with $j \leq i \leq k$ (inserting black edges between), and ends by traversing the black edge $[\overline{d_{j+1}^a}, d_{j+1}^a]$ from $\overline{d_{j+1}^a}$ to d_{j+1}^a, what is the red edge $[d_j^u, \overline{d_j^a}] = [d_{j+1}^a, \overline{d_j^a}]$ in G_j (purple edge in G_k), and then the black edge $[\overline{d_j^a}, d_j^a]$ from $\overline{d_j^a}$ to d_j^a. In other words, a switch path alternates between the red edges (oriented from d_j^u to $\overline{d_j^a}$) for the G_j (for descending j) and the black edges between.

Remark 3.10. We clarify here some ways in which switch paths and construction paths differ:

1. The purple edges in the construction path for a construction composition $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ are purple in each G_l with $i - l \leq l < i$, for a switch path. They are red edges in the structure G_l they

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are created in and then will not exist at all in the structures G_m with $m < l$. The change of color (and disappearance) of red edges is the reason for (ss2) in the switch sequence definition.

2. Unlike constructions paths, switch paths do not give subpaths of lamination leaves.

The following lemma proves that switch paths are indeed smooth paths in destination LTT structures. It is important to note that this only holds when (ss1) and (ss2) hold.

Lemma 3.11. *Let $(g_1, \ldots, g_n, G_0, \ldots, G_n)$ be an $\mathcal{ID}$ for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ and $(g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)$ a switch sequence. Then the associated switch path forms a smooth path in the ltt structure G_k.*

*Proof.* The red edge in G_k is $[d_k^u, \overline{d_k^a}]$. We are left to show (by induction) that:
(1) For each $1 \le l < k$, $[d_l^u, \overline{d_l^a}] = [d_{l+1}^a, \overline{d_l^a}]$ is a purple edge of G_k and
(2) alternating the purple edges $[d_{l+1}^a, \overline{d_l^a}]$ with the black edges $[\overline{d_l^a}, d_l^a]$ gives a smooth path in G_k.

We prove the base case. By the switch properties, e_{k-1}^R is $[d_{k-1}^u, \overline{d_{k-1}^a}] = [d_k^a, \overline{d_{k-1}^a}]$. Since $d_k^a \ne d_k^u$ and $\overline{d_{k-1}^a} \ne d_k^u$ (by the switch sequence definition), $D^t g_k(\{d_k^a, \overline{d_{k-1}^a}\}) = \{d_k^a, \overline{d_{k-1}^a}\}$. So $[d_k^a, \overline{d_{k-1}^a}]$, is a purple edge in G_k. Since $e_k^R = [d_k^u, \overline{d_k^a}]$, by including the black edge $[\overline{d_k^a}, d_k^a]$, we have a path $[d_k^u, \overline{d_k^a}, d_k^a, \overline{d_{k-1}^a}]$ in G_k (smooth, as it alternates between colored and black edges). The base case is proved.

We prove the inductive step. By the inductive hypothesis we assume the sequence of switches for $g_k, \ldots, g_{k-i}$ gives us a smooth path $[d_k^u, \ldots, \overline{d_{k-i}^a}]$ in G_k ending with a purple edge with "free" vertex $\overline{d_{k-i-1}^a}$. We know $e_{k-i-1}^R = [d_{k-i-1}^u, \overline{d_{k-i-1}^a}] = [d_{k-i}^a, \overline{d_{k-i-1}^a}]$. As long as $d_l^u \ne d_{k-i}^a$ and $d_l^u \ne \overline{d_{k-i-1}^a}$ for $k-i \le l \le k$ (holding by the switch sequence definition), $D^t g_{k,k-i}(\{d_{(k-i-1)}^u, \overline{d_{(k-i-1)}^a}\}) = D^t g_{k,k-i}(\{d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}\}) = \{d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}\}$. So, $[d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}]$ is a purple edge in G_k.

Since $[d_k^u, \ldots, \overline{d_{k-i}^a}]$ is a smooth path in G_k ending with a black edge, $[d_k^u, \ldots, \overline{d_{k-i}^a}, d_{k-i}^a, \overline{d_{k-i-1}^a}]$ is also a smooth path in G_k, as $[\overline{d_{k-i}^a}, d_{k-i}^a]$ is a black edge in G_k and $[d_{k-i}^a, \overline{d_{k-i-1}^a}]$ a purple edge in G_k. □

Example 3.12. In the ltt structure G_i of Example 3.5 we number the colored edges of a switch path:
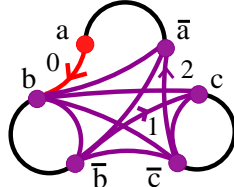
The switch sequence constructed from the switch path is:
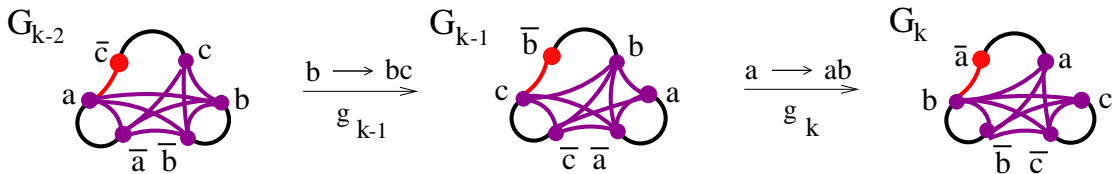
The red edge e_k^r in G_k is (0), the red edge e_{k-1}^r in G_{k-1} is (1), and the red edge e_{k-2}^r in G_{k-2} is (2).

The following lemma explains how, by inserting construction compositions into a well-chosen switch sequence, one can ensure their transition matrix is PF (for purposes of applying the FIC).

Lemma 3.13. *Suppose g decomposes as $g_{i'_m, i_m} \circ \cdots \circ g_{i'_1, i_1}$ where:*

1. *Each $g_{i'_k, i_k}$ is a construction composition whose pure construction composition starts and ends with the same ltt structure.*
2. *For each edge pair $\{d_i, \overline{d_i}\}$, either d_i or $\overline{d_i}$ is the red unachieved direction vertex for some G_{i_k-1}.*

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Then the transition matrix for g is PF.

Proof. It suffices to show each $e^u_{i_k}$ is in the image of each $e^u_{i_j}$. In fact, it suffices to show each $e^u_{i_{k-1}}$ is in the image of each $e^u_{i_k}$. Note $g_{i_k}(e^{pu}_{(i_k-1)}) = e^a_{i_k}e^u_{i_k}$. Since $(g_{i_k}, G_{(i_k-1)}, G_{i_k})$ is a switch, $e^{pa}_{(i_k-1)} = e^u_{(i_k-1)}$. Since $g_{i'_{(k-1)}, i_{(k-1)}}$ is a construction composition whose pure construction composition starts and ends with the same ltt structure, $e^u_{(i_{k}-1)} = e^u_{i_{(k-1)}}$. So $e^{pa}_{(i_k-1)} = e^u_{i_{(k-1)}}$ and $e^u_{i_k}$ maps over $e^u_{i_{k-1}}$. ($e^u_{i_k}$ has the same second index as $e^{pu}_{i_k}$ and $e^u_{i_{k-1}}$ has the same second index as $e^a_{i_k}$. Use Lemma 5.3 of [Pfa12b].) $\quad\square$

4 Full Irreducibility Criterion

We prove a "Folk Lemma" giving a criterion, the "Full Irreducibility Criterion (FIC)," for an irreducible tt representative to be fully irreducible. Our original approach involved the "Weak Attraction Theorem," several notions of train tracks, laminations, and the basin of attraction for a lamination. However, Michael Handel graciously provided a method to complete it, making much of our initial work unnecessary. Our proof here uses Handel's recommendation. [K12] gives a criterion based on ours.

The proof we give uses the relative train tracks (rtts) of [BH92] and the completely split relative train tracks (CTs) of [FH11]. If $\phi \in Out(F_r)$ is rotationless and $\mathcal{C}$ is a nested sequence of ϕ-invariant free factor systems, then ϕ is represented by a CT and filtration realizing $\mathcal{C}$ ([FH11], Theorem 4.29). We use from the CT definition (CT5): for a fixed stratum H_t with unique edge E_t, either E_t is a loop or each end of E_t is in Γ_{t-1}.

Our definition of the attracting lamination for an outer automorphism will be as in [BFH00]. A complete summary of relevant definitions can be found in [Pfa12a]. $\mathcal{L}(\phi)$ will denote the set of attracting laminations for ϕ. By [BFH00], for a $\phi \in Out(F_r)$, there exists a correspondence between $\mathcal{L}(\phi)$ and the EG-strata of a rtt representative $g\colon \Gamma \to \Gamma$ of ϕ: For each EG stratum H_t, there exists a unique attracting lamination Λ_t with H_t as the highest stratum crossed by the realization $\lambda \subset \Gamma$ of a Λ_t-generic line. $\Lambda(\phi)$ will denote the unique attracting lamination for an irreducible ϕ.

Free factor support is defined in [BFH00] (Corollary 2.6.5). The relevant information for our FIC proof is: if a lamination is carried by a proper free factor, then its support is a proper free factor.

Proposition 4.1. *(The Full Irreducibility Criterion) Let g be a train track representive of an outer automorphism $\phi \in Out(F_r)$ such that*

(I) *g has no periodic Nielsen paths,*

(II) *the transition matrix for g is Perron-Frobenius, and*

(III) *all local Whitehead graphs $LW(x; g)$ for g are connected.*

Then ϕ is fully irreducible.

Proof. Suppose $g\colon \Gamma \to \Gamma$ is as in the statement. Since g has a PF transition matrix, as an rtt, it has precisely one stratum and that stratum is EG. Hence, it has precisely one attracting lamination [BFH00]. Since the number of attracting laminations for a tt representative is independent of the representative choice, any ϕ representative would also have precisely one attracting lamination.

Suppose, for contradiction's sake, ϕ were not fully irreducible. Then some ϕ^k would be reducible. If necessary, take an even higher power R so that ϕ^R is also rotationless (this does not change reducibility). Note, since $\mathcal{L}(\phi)$ is ϕ-invariant, any ϕ^R representative also has precisely one attracting lamination.

Since ϕ^R is reducible (and rotationless), there exists a CT representative $h : \Gamma' \to \Gamma'$ of ϕ^R with more than one stratum ([FH11], Theorem 4.29). Since ϕ^R has precisely one attracting lamination, h has precisely one EG-stratum H_t. Each stratum H_i, other than H_t and any zero strata, would be an

NEG-stratum consisting of a single edge E_i ([FH11], Lemma 4.22). We consider separately the cases where $t = 1$ and $t > 1$.

Since the transition submatrix for any zero stratum is zero (hence every edge of the stratum is mapped to a lower filtration element by h), H_1 could not be a zero stratum. Thus, if $t > 1$, then H_1 is NEG and must consist of a single edge E_1. Since H_1 is bottom-most, it would be fixed, as there are no lower strata for its edge to be mapped into. According to (CT5), E_1 would then be an invariant loop. This would imply ϕ^R has a rank-1 invariant free factor. However, g (hence g^R) was pNp-free. So ϕ^R could not have a rank-1 invariant free factor. We have reached a contradiction for $t > 1$.

Assume $t = 1$. Then $\Lambda(\phi^R)(= \Lambda(\phi))$ is carried by a proper free factor. Proposition 2.4 of [BFH97] says, if a finitely generated subgroup $A \subset F_r$ carries Λ_ϕ, then A has finite index in F_r. The necessary conditions for this proposition are actually only: 1. the transition matrix of g is irreducible and 2. each $LW(g; v)$ is connected. (Up to the contradiction in the proof of Proposition 2.4 of [BFH97], the only properties used in the proof are that the support is finitely generated, proper, and carries the lamination. The contradiction uses Lemma 2.1 in [BFH97], which shows (1) and (2) carry over to lifts of g to finite-sheeted covering spaces, using no properties other than properties (1) and (2).) Assumptions (1) and (2) are assumptions in this lemma's hypotheses and $\Lambda(\phi)$ is still the attracting lamination for g. So we can apply the proposition to contradict $\Lambda(\phi)$ having proper free factor support: Applying the proposition, since proper free factors have infinite index, the support must be the whole group. This contradicts that the EG-stratum is H_1 and that there must be more than one stratum.

We have thus shown that we cannot have more than one stratum with $t = 1$ or $t > 1$. So all powers of ϕ must be irreducible and ϕ is fully irreducible, as desired. $\qquad\square$

Lemma 4.2. *Suppose $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ and $(g_1, \ldots, g_k; G_0, \ldots, G_k)$, $g = g_k \circ g_1$, is an $\mathcal{ID}$, satisfying:*

1. *$\mathcal{PI}(G(g)) \cong \mathcal{G}$. More precisely, $\mathcal{G} \cong \mathcal{SW}(g; v)$.*
2. *And for each $1 \leq i, j \leq q$, there exists some $k \geq 1$ such that $g^k(E_j)$ contains either E_i or $\bar{E}_i$.*
3. *And g has no periodic Nielsen paths.*

Then g is a tt representative of a $\phi \in \mathcal{AFI}_r$ such that $\mathcal{IW}(\phi) = \mathcal{G}$.

Proof. By the FIC, we only need to show g is a tt map, the transition matrix of g is PF, and $\mathcal{IW}(\phi) = \mathcal{G}$. (2) implies that the transition matrix is PF. g is a tt map by [Pfa12b] Lemma 5.3a. Since g has no pNp's, $\mathcal{IW}(g) = \mathcal{SW}(g)$. By the definition of $G(g)$, we know $\mathcal{PI}(G(g)) = \mathcal{SW}(v; g)$. $\qquad\square$

Remark 4.3. In the language of [Pfa12b], the conditions of Lemma 4.2 could be stated as

$$L(g_1, \ldots, g_k; G_0, G_1 \ldots, G_{k-1}, G_k) = E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$$

being a loop in $\mathcal{ID}(\mathcal{G})$ satisfying (1)-(3) and inducing a map fixing all periodic directions.
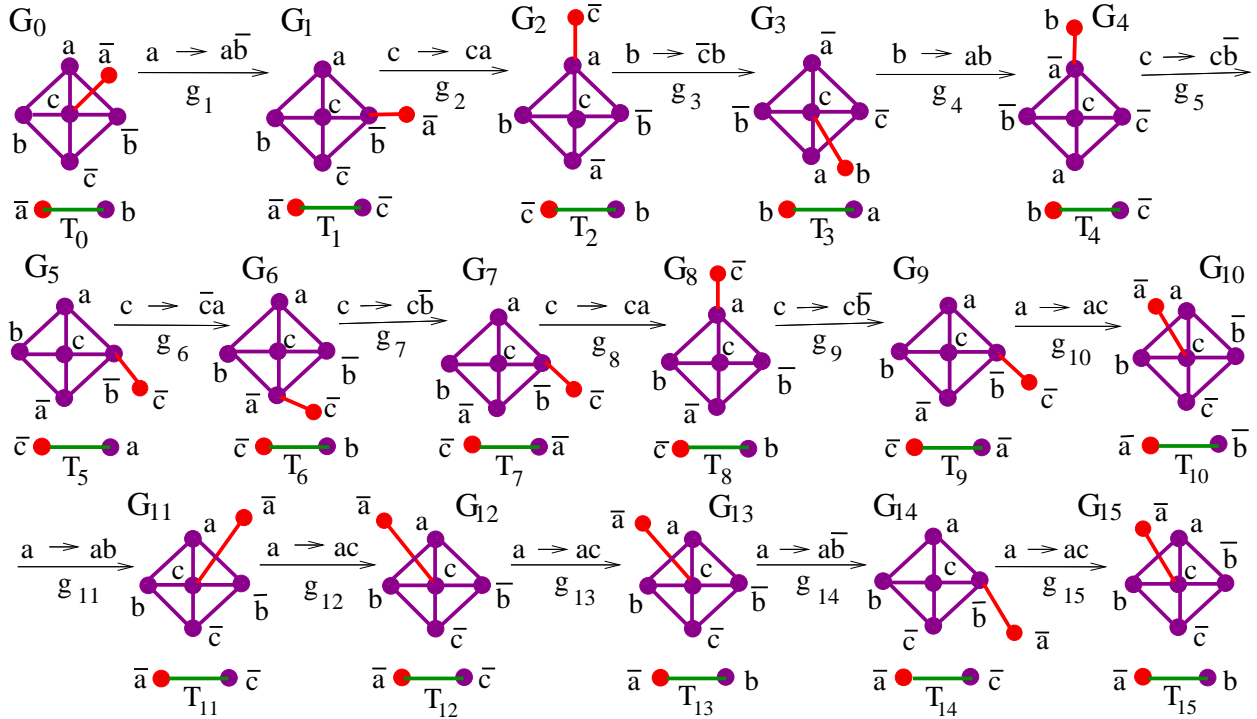
5 Nielsen path identification and prevention

We give (Proposition 5.2) a method for finding all ipNp's, thus pNp's, for a tt map $g \colon \Gamma \to \Gamma$ ideally decomposed as $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ where $(g_0, \ldots, g_n; G_0, \ldots, G_n)$ is an admissible composition. Note that, for each k, we know that $T_k = \{d_k^{pu}, d_k^{pa}\}$ is the unique illegal turn for $f_k = g_k \circ \cdots \circ g_{k+1} \colon \Gamma_k \to \Gamma_k$ and $\mathcal{AMII}$ guarantees each f_k is also a tt map.

As a warm-up, Example 5.1 demonstrates the procedure's application to an ideally decomposed representative. We then explain the procedure's steps and how we used them in Example 5.1. This section concludes with a proof of the procedure's validity.

Example 5.1. We apply the procedure to show the following ideally decomposed tt map has no pNp's.

For simplicity, the ltt structures $G_i = G(f_i)$ are shown without black edges $[e_j]$ connecting the vertex pairs $\{d_j, \bar{d}_j\}$. Underneath each ltt structure G_i we included the illegal turn T_i for the generator g_i. We often abuse notation by writing e for $D_0(e)$ where $e \in \{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$.

$G_0 \quad \xrightarrow[g_1]{a \to a\bar{b}} \quad G_1 \quad \xrightarrow[g_2]{c \to ca} \quad G_2 \quad \xrightarrow[g_3]{b \to \bar{c}b} \quad G_3 \quad \xrightarrow[g_4]{b \to ab} \quad G_4 \quad \xrightarrow[g_5]{c \to c\bar{b}}$

$T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4$

$G_5 \quad \xrightarrow[g_6]{c \to \bar{c}a} \quad G_6 \quad \xrightarrow[g_7]{c \to c\bar{b}} \quad G_7 \quad \xrightarrow[g_8]{c \to ca} \quad G_8 \quad \xrightarrow[g_9]{c \to c\bar{b}} \quad G_9 \quad \xrightarrow[g_{10}]{a \to ac} \quad G_{10}$

$T_5 \quad T_6 \quad T_7 \quad T_8 \quad T_9 \quad T_{10}$

$G_{11} \quad \xrightarrow[g_{11}]{a \to ab} \quad \xrightarrow[g_{12}]{a \to ac} \quad G_{12} \quad \xrightarrow[g_{13}]{a \to ac} \quad G_{13} \quad \xrightarrow[g_{14}]{a \to a\bar{b}} \quad G_{14} \quad \xrightarrow[g_{15}]{a \to ac} \quad G_{15}$

$T_{11} \quad T_{12} \quad T_{13} \quad T_{14} \quad T_{15}$

Procedure Applied:

Since $\{\bar{a}, b\}$ is the only illegal turn for g, any ipNp would contain it as its unique illegal turn. We are thus trying to find an ipNp $\overline{\rho_1}\rho_2$ where $\rho_1 = \bar{a}e_2e_3\dots$ and $\rho_2 = be_2'e_3'\dots$ are legal paths and all $e_i, e_i' \in \mathcal{E}(\Gamma)$, except that the final edge in either ρ_1 or ρ_2 may be partial.

Since $g_1(b) = b$ is the initial subpath of $g_1(\bar{a}) = b\bar{a}$, proper cancelation would force ρ_2 to contain another edge e_2' after b (Proposition 5.2 I). Since $\bar{a}$ labels the red vertex in G_1 (i.e. $D_0(\bar{a})$ is the unachieved direction d_1^u), the edge e_2' would be such that $D_0(e_2')$ is a preimage under Dg_1 of $D_0(\bar{c})$ (proper cancelation requires $Dg_{2,1}(e_2') = Dg_2(\bar{a}) = D_0(\bar{c})$ but, since $Dg_1(e_2')$ cannot be the unachieved direction $\bar{a}$, and the only other preimage under Dg_2 of the twice-achieved direction $\bar{c}$ is the other direction in the illegal turn $T_1 = \{\bar{a}, \bar{c}\}$, namely the twice-achieved direction $\bar{c}$, we must have $Dg_1(e_2') = \bar{c}$). The only preimage under Dg_1 of $\bar{c}$ is $\bar{c}$. So $e_2' = \bar{c}$. Thus, so far, we have $\rho_1 = \bar{a}\dots$ and $\rho_2 = b\bar{c}\dots$

Since $g_{2,1}(\bar{a}) = g_2(b)\bar{a}$ is the initial subpath of $g_{2,1}(b\bar{c}) = g_2(b)\bar{a}c$, we know ρ_1 would contain another edge e_2 after $\bar{a}$ (Proposition 5.2 III). Since $\bar{c}$ labels the red unachieved direction vertex in G_2, we know $D_0(e_2)$ cannot be a preimage under $Dg_{2,1}$ of $\bar{c}$. So our best hope is $Dg_{3,1}(e_2) = Dg_3(\bar{c})$. This can only happen if $Dg_{2,1}(e_2)$ is a preimage under Dg_3 of the other direction in the illegal turn $T_2 = \{\bar{c}, b\}$, namely the twice-achieved direction b. There are two such preimages under $Dg_{2,1}$ (the two direction in the illegal turn $T_0 = \{\bar{a}, b\}$): Case 1 will be where $e_2 = \bar{a}$ and Case 2 where $e_2 = b$. We analyze each of these case.

For Case 1, suppose $e_2 = \bar{a}$. Since $g_{3,1}(b\bar{c}) = g_{3,2}(b)g_3(\bar{a})\bar{c}$ is the initial subpath of $g_{3,1}(\bar{a}\bar{a}) = g_{3,2}(b)g_3(\bar{a})\bar{c}b\bar{a}$, we know ρ_2 would contain another edge e_3' after $\bar{c}$. Since b labels the red unachieved direction vertex in G_3, we would need $D_0(e_3')$ to be a preimage of a under Dg_3 (since $T_3 = \{a, b\}$, this follows as above). The only such preimage is a. So $e_3' = a$, giving $\rho_1 = \bar{a}\bar{a}\dots$ and $\rho_2 = b\bar{c}a\dots$.

Note that $g_{4,1}(b\bar{c}a) = g_{4,2}(b)g_{4,3}(\bar{a})g_4(\bar{c})ab\bar{a}c$ and $g_{4,1}(\bar{a}\bar{a}) = g_{4,2}(b)g_{4,3}(\bar{a})g_4(\bar{c})ab\bar{a}$. So, since $\{\bar{b}, b\} \neq T_4$ (and $b \neq \bar{b}$), we could not have $\rho_1 = \bar{a}\bar{a}$ and $\rho_2 = b\bar{c}a$ (Proposition 5.2 IIc). Case 1 could not occur.

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For Case 2, suppose $e_2 = b$. Since $g_{3,1}(b\bar{c}) = g_{3,2}(b)g_3(\bar{a})\bar{c}$ is the initial subpath of $g_{3,1}(\bar{a}b) = g_{3,2}(b)g_3(\bar{a})\bar{c}b$, we know ρ_2 would contain another edge e_3' after $\bar{c}$. Since b labels the red unachieved direction vertex in G_3, we can follow the logic above and see that e_3' would be a, giving $\rho_1 = \bar{a}b\ldots$ and $\rho_2 = b\bar{c}a\ldots$. Since $g_{4,1}(b\bar{c}a) = g_{4,2}(b)g_{4,3}(\bar{a})g_4(\bar{c})a\bar{b}\bar{a}c$ and $g_{4,1}(\bar{a}b) = g_{4,2}(b)g_{4,3}(\bar{a})g_4(\bar{c})ab$, cancelation leaves $\{\bar{b}, b\}$. As above, we could not have $\rho_1 = \bar{a}b\ldots$ and $\rho_2 = b\bar{c}a\ldots$.

This rules out all possibilities for $\overline{\rho_1}\rho_2$ and so g has no ipNp's, thus no pNp's, as desired.

Explanation of Procedure Applied:

Let $(g_0, \ldots, g_n; G_0, \ldots, G_n)$ be an $\mathcal{ID}$ admissible composition with the standard notation. We give the general procedure for finding any ipNp $\rho = \overline{\rho_1}\rho_2$ for $g = g_0 \circ \cdots \circ g_n$, where $\rho_1 = e_1 \ldots e_m$; $\rho_2 = e_1' \ldots e_{m'}'$; $e_1, \ldots, e_m, e_1', \ldots, e_{m'}' \in \mathcal{E}(\Gamma)$; and $\{D_0(e_1), D_0(e_1')\} = \{d_1, d_1'\}$ is the unique illegal turn of ρ. We let $\rho_{1,k} = e_1 \ldots e_k$ and $\rho_{2,l} = e_1' \ldots e_l'$ throughout the procedure. After each step is explained in italics, we explain its use in Example 5.1.

(I) *Apply g_1, g_2, etc, to e_1 and e_1' until $Dg_{j,1}(e_1') = Dg_{j,1}(e_1)$. Either $g_{j,1}(e_1)$ is the initial subpath of $g_{j,1}(e_1')$ or vice versa. Without generality loss (or adjust notation) assume $g_{j,1}(e_1')$ is the subpath of $g_{j,1}(e_1)$, so $g_{j,1}(e_1) = g_{j,1}(e_1')t_2 \ldots$, for some edge t_2. Then ρ_2 must contain another edge e_2'.*

$g_1(b) = b$ was the initial subpath of $g_1(\bar{a}) = b\bar{a}$. This implied ρ_2 had an edge after b.

(II) *Continue composing generators g_i until either:*

(a) *one obtains a $g^{p'}$ with $g^{p'}(\rho_{2,k}) = \tau'e_1' \ldots$ and $g^{p'}(\rho_{1,s}) = \tau'e_1 \ldots$ for a legal path τ' (proceed to V),*

IIa did not occur in Example 5.1. When it occurs, it makes an ipNp promising. Va may identify an ipNp, if it exists, as does IVc. However, IVc involves "trimming," since it involves the case where the path's initial and final edges are only partial edges. IVb and IVd direct one to possibly find an ipNp by continuing to add edges.

(b) *$g_{j,l}(\rho_{2,k})$ is a subpath of $g_{j,l}(\rho_{1,s})$ or vice versa (proceed to III),*

IIb occurs in both Case 1 and Case 2 of Example 5.1. In Case 1, IIIa is used to obtain $\rho_{1,2} = \bar{a}\bar{a}$ and $\rho_{2,3} = b\bar{c}a$. In Case 2, IIIa yields $\rho_{1,2} = \bar{a}b$ and $\rho_{2,3} = b\bar{c}a$.

(c) *or some $g_{l,1} \circ g^{p'}(\rho_{2,k}) = \tau'\gamma_{2,k}$ and $g_{l,1} \circ g^{p'}(\rho_{1,s}) = \tau'\gamma_{1,s}$ where $T_l \neq \{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\}$. In this case there cannot be an ipNp with $\overline{\rho_{2,k}}\rho_{1,s}$ as a subpath. Proceed to VI.*

In both Case 1 and Case 2, after applying IIIa to obtain $\rho_{1,2}$ and $\rho_{2,3}$, IIc occurs with $\{D_0(\bar{b}), D_0(b)\} \neq T_4$. In both cases, $\tau = g_{4,2}(b)g_{4,3}(\bar{a})g_4(\bar{c})a$. In Case 1, $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\} = \{D_0(\bar{b}\bar{a}c), D_0(b\bar{a})\} = \{D_0(\bar{b}), D_0(b)\}$. In Case 2, $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\} = \{D_0(\bar{b}\bar{a}c), D_0(b)\} = \{D_0(\bar{b}), D_0(b)\}$.

(III) *Suppose $g_{j,1}(\rho_{1,k}) = g_{j,1}(\rho_{2,s})t_{s+1} \ldots$ (or switch e_i for e_i', ρ_1 for ρ_2, etc). ρ_2 must have another edge e_{s+1}' after $\rho_{2,s}$. There are two cases to consider:*

Since $\rho_{1,1} = b$, $\rho_{2,1} = \bar{a}$, and $g_{2,1}(\bar{a}) = g_2(b)\bar{a}$ is the initial subpath of $g_{2,1}(b\bar{c}) = g_2(b)\bar{a}c$, we assumed $g_{2,1}(\rho_{1,1}) = g_{2,1}(\rho_{2,1})t_2 \ldots$. In particular, $t_{s+1} = \bar{a}$.

(a) *If $D_0(t_{s+1}) = d_j^u$, then the different possibilities for e_{s+1}' are determined by the directions d_{s+1}' such that $T_j = \{Dg_{j,1}(d_{s+1}'), D_0(t_{s+1})\}$ where $D_0(e_{s+1}') = d_{s+1}'$.*

$D_0(\bar{a})$ labeled the red vertex in G_1, implying $D_0(\bar{a}) = d_1^u$, i.e. $D_0(t_{s+1}) = d_j^u$. $T_1 = \{\bar{a}, \bar{c}\}$ implied $Dg_1(d_2') = \bar{c}$. Since the only preimage of $\bar{c}$ under Dg_1 was $\bar{c}$, we needed $d_2' = D_0(\bar{c})$, $e_2' = \bar{c}$, and $\rho_{2,2} = b\bar{c}$. We hit this case again when determining possibilities for e_2.

(b) *If $D_0(t_{s+1}) \neq d_j^u$, the possibilities for e_{s+1}' are all edges e_{s+1}' such that $Dg_{j,1}(D_0(e_{s+1}')) = D_0(t_{s+1})$.*

IIIb did not occur in Example 5.1. If $D_0(\bar{a}) \neq d_1^u$, we would have looked for edges e_2' with $Dg_1(d_2') = \bar{a}$. (Otherwise we could not have had $Dg_{2,1}(d_2') = \bar{a}$.)

Since ρ_2 must be legal, ignore choices for d_{s+1}' where $T_0 = \{\overline{d_s'}, d_{s+1}'\}$ is the illegal turn for g. Each remaining d_{s+1}' in (a) or (b) gives another prospective ipNp one must apply the steps to.

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Only $e_2' = \bar{c}$ satisfied $Dg_1(e_2') = \bar{c}$. However, both $\bar{a}$ and b (referred to as Case 1 and Case 2) gave prospective directions analyzed in finding e_2. For Case 1 and Case 2 we separately continued through the steps.

(IV) *For each $p' \geq 1$ with $g^{p'}(\rho_{2,m}) = \tau'e_1\ldots$ and $g^{p'}(\rho_{1,n}) = \tau'e_1'\ldots$ for some legal path τ' (and appropriate m and n), check if $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ or vice versa.*

    (a) *If, for some $p' \geq 1$, $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\rho_{1,n}}\rho_{2,m}$, then $\overline{\rho_{1,n}}\rho_{2,m}$ is the only possible ipNp for g.*

    (b) *For each $p' \geq 1$ such that $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ (containment proper), proceed to V.*

    (c) *If $\overline{\rho_{1,n}}\rho_{2,m} \subset g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ (containment proper): The final occurrence of e_n in the copy of $\rho_{1,n}$ in $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ must be from $g^{p'}(e_n)$ and the final occurrence of e_m' in the copy of $\rho_{2,m}$ in $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ must be from $g^{p'}(e_m')$. Thus, e_n and e_m' have fixed points. Replace $\overline{\rho_{1,n}}\rho_{2,m}$ with $\overline{\rho_{1,n}'}\rho_{2,m}'$, where $\overline{\rho_{1,n}'}\rho_{2,m}'$ is $\overline{\rho_{1,n}}\rho_{2,m}$, but with e_n and e_m' replaced by partial edges ending at the fixed points. Repeat until some $\overline{\rho_{1,n}'}\rho_{2,m}'$ is an ipNp.*

    (d) *If we do not have $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}'$ or vice versa for any $1 \leq p' \leq b$, there is only one possibility for $\overline{\rho_{2,m}}\rho_{1,n}$ to be a subpath of an ipNp. It is when $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\gamma_{1,n}}\gamma_{2,m}$ and either $\gamma_{1,n} \subset \rho_{1,n}$ and $\rho_{2,m} \subset \gamma_{2,m}$ or $\gamma_{2,m} \subset \rho_{2,m}$ and $\rho_{1,n} \subset \gamma_{1,n}$. In this case, apply V to the side too short. Otherwise, there cannot be an ipNp with $\overline{\rho_{2,m}}\rho_{1,n}$ as a subpath (proceed to VI).*

(VI) *Assume $g^{p'}(\bar{\rho}_{1,n}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ (containment proper) and, without generality loss, $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})t_{m+1} \subset \overline{\rho_{1,n}}\rho_{2,m}$ for some t_{m+1}. For each e_i such that $Dg^{p'}(D_0(e_i)) = D_0(t_{m+1})$ and $\{D_0(\overline{e_{i-1}}), D_0(e_i)\} \neq \{D_0(e_1), D_0(e_1')\}$ (the illegal turn for g), return to V with $\rho_{2,m+1}$, where $e_{m+1} = e_i$.*

(VI) *Rule out the other possible subpaths that arose via this procedure (by different choices of d_i, as in III or V). If there are no other possible subpaths, there are no ipNp's (thus no pNp's) for g.*

    In discovering options for e_2 (after analyzing Case 1), VI sent us back to Case 2.

Proposition 5.2. *Let $(g_0,\ldots,g_n; G_0,\ldots,G_n)$ be an $\mathcal{ID}$ admissible composition with notation as above. Then the procedure described in steps (I)-(VI) determines all ipNp's for $g = g_0 \circ \cdots \circ g_n$.*

We use the following lemma in the proposition proof (see [Pfa12a] for proofs).

Lemma 5.3. *[Pfa12a]*
a. *Subpaths of legal paths are legal.*
b. *For train tracks, images of legal paths and turns are legal.*

We also remind the reader:
1. Since d_k^u will be one vertex of the illegal turn T_k of G_k, T_k cannot also be a purple edge in G_k.
2. d_k^u must be a vertex of the red edge $[t_k^R]$ of G_k.
3. For each k, T_k is not the red edge in G_k, so is not represented by any edge in G_k.

Proof. (of Proposition) We begin with an argument used repeatedly. Since $\rho = \overline{\rho_1}\rho_2$ is an ipNp, ρ_1 and ρ_2 are both legal. Since subpaths of legal paths are legal and since the $g_{k,1}$ images of legal paths are legal, $g_{k,1}(e_1 \ldots e_l)$ and $g_{k,1}(e_1' \ldots e_{l'}')$ are legal paths for each $1 \leq k \leq n$, $1 \leq l \leq m$, and $1 \leq l' \leq n'$.

Since $\{D_0(e_1), D_0(e_1')\}$ is illegal, $Dg_{j,1}(e_1') = Dg_{j,1}(e_1)$, for some j. We show either $g_{j,1}(e_1)$ is the initial subpath of $g_{l,1}(e_1')$ or vice versa. Let $d_1 = D_0(e_1)$ and $d_1' = D_0(e_1')$. Since $\{D_0(e_1), D_0(e_1')\}$ is illegal for g and $T_0 = \{d_0^{pu}, d_0^{pa}\}$ is g's only illegal turn, $\{d_1, d_1'\} = \{d_0^{pu}, d_0^{pa}\}$. Without generality loss suppose $d_1 = d_0^{pu}$ (and $d_1' = d_0^{pa}$). Since $g_1(e_0^{pu}) = e_1^a e_1^u$, we have $g_1(e_1) = g_1(e_0^{pu}) = e_1^a e_1^u$ and $g_1(e_1') = g_1(e_0^{pa}) = e_1^a$. So $g_1(e_1')$ is the initial subpath of $g_1(e_1)$. Since $g_{j,2}$ is an automorphism, $g_{j,2}(g_1(e_1')) = g_{j,2}(e_1^a)$ is the initial subpath of $g_{j,2}(g_1(e_1)) = g_{j,2}(e_1^a e_1^u) = g_{j,2}(e_1^a)g_{j,2}(e_1^u)$, as desired. Left to show for I is that ρ_2 must contain

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a second edge e_2'. Suppose it did not. Then tightening would cancel all of $g_{j,1}(\rho_2)$ with an initial subpath of $g_{j,1}(\rho_1)$, so would cancel all of $g_{j,1}(\rho_2)$ with an initial subpath of $g_{j,1}(\rho_1)$. Thus, $(g_{j,1})_\#(\rho) = (g_{j,1})_\#(\overline{\rho_1}\rho_2)$ would be a subpath of $g_{j,1}(\rho_2)$, hence would be legal, as would $g_\#^p(\rho) = (g^{p-1} \circ g_{n,j+1})_\#((g_{j,1})_\#(\rho))$ for all p. This contradicts that some $g_\#^p(\rho) = \rho$, which has an illegal turn.

We next show applying the g_i results in IIa, IIb, or IIc. For each l, k, and s, $g_{l,1}(\rho_{2,k})$ and $g_{l,1}(\rho_{1,s})$ are both legal. Write $g_{l,1}(\rho_{2,k}) = \tau'\gamma_{2,k}$ and $g_{l,1}(\rho_{1,s}) = \tau'\gamma_{1,s}$ where $D_0(\gamma_{2,k}) \neq D_0(\gamma_{1,s})$. If $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\}$ is legal, we are in IIc. If it is illegal and we are not in IIb, we can continue to apply the g_i until we are in IIb, are in IIc, or reach a g^p. Since the only illegal turn for any g^p is $\{d_1, d_1'\}$, we would by necessity be in IIa.

We show for IIc: if $g_{l,1} \circ g^{p'}(\rho_{2,k}) = \tau'\gamma_{2,k}$ and $g_{l,1} \circ g^{p'}(\rho_{1,s}) = \tau'\gamma_{1,s}$ where $\{D_0(\gamma_{1,s}), D_0(\gamma_{2,k})\}$ is legal for g_l, then adding edges to $\rho_{2,k}$ and $\rho_{1,s}$ cannot give an ipNp for g. Now $g_{l,1} \circ g^{p'}(\rho_2) = \tau'\gamma_2$ and $g_{l,1} \circ g^{p'}(\rho_1) = \tau'\gamma_1$ where $\gamma_{2,k}$ is an initial subpath of γ_2 (both legal) and $\gamma_{1,s}$ is an initial subpath of γ_1 (both legal). Since $\{D_0(\gamma_1), D_0(\gamma_2)\} = \{D_0(\gamma_{1,s}), D_0(\gamma_{2,k})\}$ is legal, $(g_{l,1} \circ g^{p'})_\#(\rho) = (g_{l,1} \circ g^{p'})_\#(\overline{\rho_1}\rho_2) = \overline{\gamma}_1\gamma_2$, which is a legal path. Let p be such that $g_\#^p(\rho) = \rho$. (Without generality loss assume $p > p'$, by replacing p by an integer multiple of p if necessary). Then, $g_\#^p(\rho) = ((g^{p-p'-1} \circ g_{n,l+1}) \circ (g_{l,1} \circ g^{p'}))_\#(\rho) = (g^{p-p'-1} \circ g_{n,l+1})_\#((g_{l,1} \circ g^{p'})_\#(\rho)) = (g^{p-p'-1} \circ g_{n,l})_\#(\overline{\gamma_1}\gamma_2) = (g^{p-p'-1} \circ g_{n,l})(\overline{\gamma_1}\gamma_2)$, since $\overline{\gamma_1}\gamma_2$ is a legal path. So $g_\#^p(\rho)$ is legal, as images under admissible compositions of legal paths are legal. This contradicts that $g_\#^p(\rho) = \rho$ is not legal. We have verified all in II needing verification.

We prove the claims of IIIa and IIIb: $g_{j,1}(\rho_{1,k}) = g_{j,1}(\rho_{2,s})t_{s+1} \dots$ implies $(g_{j,1})_\#(\overline{\rho_{1,k}}\rho_{2,s}) = \dots \overline{t_{s+1}} = \overline{\gamma}$ for a legal path γ (legal, since a subpath of a legal path). Since e_{s+1}' is a legal path, $g_{j,1}(e_{s+1}')$ is also. Thus, $(g_{j,1})_\#(\overline{\rho_{1,k}}\rho_{2,s}) = (\overline{\gamma}g_{j,1}(e_{s+1}'))_\#$, which is $\overline{\gamma}g_{j,1}(e_{s+1}')$ unless $Dg_{j,1}(d_{s+1}') = D_0(t_{s+1})$. It is then legal (causing a contradiction) unless $\{D_0(\gamma), D_0(g_{j,1}(e_{s+1}'))\}$ is illegal, i.e. $T_j = \{Dg_{j,1}(D_0(e_{s+1}')), D_0(t_{s+1})\}$.

Suppose, as in IIIa, that $D_0(t_{s+1}) = d_j^u$. Then, $D_0(t_{s+1})$ is not in the image of Dg_j, thus is not in the image of $Dg_{j,1} = D(g_j \circ g_{j-1,1})$. So $Dg_{j,1}(d_{s+1}') \neq D_0(t_{s+1})$. This implies $(\overline{\gamma}g_{j,1}(e_{s+1}'))_\# = \overline{\gamma}g_{j,1}(e_{s+1}')$, which will be a legal path unless $T_j = \{Dg_{j,1}(d_{s+1}'), D_0(t_{s+1})\}$. However, if $(g_{j,1})_\#(\overline{\rho_{1,k}}\rho_{2,s}) = \overline{\gamma}g_{j,1}(e_{s+1}')$, then $(g_{j,1})_\#(\rho) = (g_{j,1})_\#(\overline{\rho_1}\rho_2) = (g_{j,1})_\#(\overline{e_m} \dots \overline{e_{k+1}})\overline{\gamma}g_{j,1}(e_{s+1}')(g_{j,1})_\#(e_{s+2}' \dots e_m') = g_{j,1}(\overline{e_m} \dots \overline{e_{k+1}})\overline{\gamma}g_{j,1}(e_{s+1}')g_{j,1}(e_{s+2}' \dots e_m')$, since ρ_1 and ρ_2 are legal and edge images are legal. But $g_{j,1}(\overline{e_m} \dots \overline{e_{k+1}})\overline{\gamma}$ is a subpath of $g_{j,1}(\overline{\rho_1})$, so is legal, and $g_{j,1}(e_{s+1}')g_{j,1}(e_{s+2}' \dots e_m')$ is a subpath of $g_{j,1}(\rho_2)$, so is legal, and we still have $\overline{\gamma}g_{j,1}(e_{s+1}')$ is legal, together making $(g_{j,1})_\#(\rho)$ legal. This contradicts that some $g_\#^p(\rho) = (g^{p-1} \circ g_{j,n+1})_\#(g_{j,1})_\#(\rho))$ must be ρ, which has an illegal turn. So, $T_j = \{Dg_{j,1}(d_{s+1}'), D_0(t_{s+1})\}$, as desired.

Suppose, as in IIIb, $D_0(t_{s+1}) \neq d_j^u$. For contradiction's sake suppose $Dg_{j,1}(d_{s+1}') \neq D_0(t_{s+1})$, where $D_0(e_{s+1}') = d_{s+1}'$. Again $(\overline{\gamma}g_{j,1}(e_{s+1}'))_\# = \overline{\gamma}g_{j,1}(e_{s+1}')$. Also, $Dg_{j,1}(d_{s+1}')$ cannot be d_j^u (as $d_j^u \notin \mathcal{IM}(Dg_j)$) and $D_0(t_{s+1}) \neq d_j^u$, implying $T_j \neq \{Dg_{j,1}(d_{s+1}'), D_0(t_{s+1})\}$. Then, since $Dg_{j,1}(d_{s+1}') \neq D_0(t_{s+1})$, that $\overline{\gamma}g_{j,1}(e_{s+1}')$ is legal, causes a contradiction as above. So $Dg_{j,1}(d_{s+1}') = D_0(t_{s+1})$, as desired. Finally observe that choices for e_{s+1}' with $T_0 = \{D_0(e_s'), D_0(e_{s+1}')\}$ must be thrown out, as ρ_2 must be legal.

As in IV, suppose $g_{l,1} \circ g^{p'}(\rho_{2,m}) = \tau'e_1' \dots$ and $g_{l,1} \circ g^{p'}(\rho_{1,n}) = \tau'e_1 \dots$ for a legal path τ' (and appropriate m and n). It is clear these are the only options. Also, Va is true by definition, Vb refers us to a later step, and IVc is fairly straight-forward. So we focus on IVd. We first need that there is only possibility for $\overline{\rho_{1,n}}\rho_{2,m}$ to be a subpath of an ipNp. Suppose, for no power p', is $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\rho_{1,n}}\rho_{2,m}$ or $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ or $\overline{\rho_{1,n}}\rho_{2,m} \subset g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ or $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\gamma_{1,n}}\gamma_{2,m}$ where either $\gamma_{1,n} \subset \rho_{1,n}$ and $\rho_{2,m} \subset \gamma_{2,m}$ or $\gamma_{2,m} \subset \rho_{2,m}$ and $\rho_{1,n} \subset \gamma_{1,n}$. For contradiction's sake, suppose some $\overline{\rho_{1,n+k}}\rho_{2,m+l}$ containing $\overline{\rho_{1,n}}\rho_{2,m}$ is a period-p ipNp. Since $\overline{\rho_{1,n+k}}\rho_{2,m+l}$ is an ipNp, $\rho_{1,n+k}$ and $\rho_{2,m+l}$ are both legal (as are $\rho_{1,n}$ and $\rho_{2,m}$). Thus, $g_\#^p(\overline{\rho_{1,n+k}}\rho_{2,m+l}) = g^p(\overline{e_{n+k}'} \dots \overline{e_{n+1}})g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m})g^p(e_{m+1} \dots e_{m+l})$. So $g_\#^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset g_\#^p(\overline{\rho_{1,n+k}}\rho_{2,m+l}) = \overline{\rho_{1,n+k}}\rho_{2,m+l}$. We are in a scenario we said could not occur, a contradiction.

There is nothing to prove in V since the conditions for IV still hold. □

Definition 5.4. An admissible composition $(g_1, \ldots, g_k; G_0, \ldots, G_k)$ will be a *Nielsen path prevention sequence* if no $\mathcal{ID}$ map $(g'_1, \ldots, g'_{k+m}; G'_0, \ldots, G'_{k+m})$, with $(g_i; G_{i-1}, G_i) \sim (g'_i; G'_{i-1}, G'_i)$ for each $1 \leq i \leq k$, can have a pNp.

Lemma 5.5. $G_0 \xrightarrow{g_1: z \mapsto xz} G_1 \xrightarrow{g_2: w \mapsto zw} G_2 \xrightarrow{g_3: y \mapsto y\bar{w}} G_3 \xrightarrow{g_4: y \mapsto y\bar{x}} G_4 \xrightarrow{g_5: y \mapsto y\bar{w}} G_5 \xrightarrow{g_6: y \mapsto y\bar{x}} G_6$ *is a Nielsen path prevention sequence where each $\mathcal{PI}(G_i)$ is the complete $(2r-1)$-vertex graph and, listed red vertex first: $e_0^R = [x, y]$, $e_1^R = [z, \bar{x}]$, $e_2^R = [w, \bar{z}]$, $e_3^R = [\bar{y}, \bar{w}]$, $e_4^R = [\bar{y}, \bar{x}]$, $e_5^R = [\bar{y}, \bar{w}]$, $e_6^R = [\bar{y}, \bar{x}]$.*

Proof. Suppose ρ were an ipNp for g. Then ρ would have to contain the illegal turn $\{x, z\}$ and could be written $\overline{\rho_1}\rho_2$ where $\rho_1 = xe_2e_3 \ldots$ and $\rho_2 = ze'_2e'_3 \ldots$ are legal. Now, $g_1(x) = x$ and $g_1(z) = xz$. So ρ_1 would contain an additional edge e_2. Also, $dg_1(e_2) = w$. So $e_2 = w$. Since $g_{2,1}(xw) = xzw$ and $g_{2,1}(z) = xz$, we know ρ_2 would contain an additional edge e'_2 with $dg_{2,1}(e'_2) = \bar{y}$. The only option is $e'_2 = \bar{y}$. Since $g_{3,1}(xw) = xzw$ and $g_{3,1}(z\bar{y}) = xzw\bar{y}$, also ρ_1 would contain an additional edge e_3 with $dg_{3,1}(e_3) = x$. So either $e_3 = x$ or $e_3 = z$. Suppose $e_3 = x$. Since $g_{4,1}(xwx) = xzwx$ and $g_{4,1}(z\bar{y}) = xzwx\bar{y}$, we know ρ_1 would contain an edge e_4 with $dg_{4,1}(e_4) = w$. So $e_4 = \bar{y}$. Note that $g_{5,1}(xwx\bar{y}) = xzwxxw\bar{y}$ and $g_{5,1}(z\bar{y}) = xzwxw\bar{y}$ both start with $xzwx$ and $\{x, w\}$ is not the illegal turn for g_6. So we must return to the case of $e_3 = z$. Since $g_{4,1}(xwz) = xzwxz$ and $g_{4,1}(z\bar{y}) = xzwx\bar{y}$ and $\{\bar{y}, z\}$ is not the illegal turn for g_5, we have exhausted all possibilities for ρ. Thus, g was not an ipNp. □

6   Achieving all $(2r-1)$-vertex complete graphs

Denote by $\mathcal{C}_r$ the complete $(2r-1)$-vertex graph. We construct (Theorem 6.2), for each $r \geq 3$, a $\phi \in \mathcal{AFI}_r$, such that $\mathcal{C}_r \cong \mathcal{IW}(\phi)$. First we establish notation and prove a lemma.

For each $r \geq 3$ and $1 \leq j \leq 2r - 1$, color $\mathcal{C}_r$ purple and let $\mathcal{G}_j$ be the graph formed from $\mathcal{C}_r$ by:
1. index edge-pair labeling $\mathcal{C}_r$'s vertices with $\{x_1, x_2, \ldots, x_{2r-1}\}$, writing X_i for x_{2i-1} and $\overline{X_i}$ for x_{2i};
2. adding a black edge $[x_i, \overline{x_i}]$ for each $1 \leq i \leq 2r - 1$ (leaving vertices x_i and $\overline{x_i}$ purple); and
3. adding a single red vertex x_{2r} and single red edge $[x_{2r}, x_j]$ (x_j remains purple).

Note that, for $j = 2r - 1$, $\mathcal{G}_j$ contains as a subgraph the graph (we denote $\mathcal{G}_{1,\ldots,2r-2}$) consisting of the (purple) complete graph on the vertices $x_1, \ldots, x_{2r-2}$, together with the black edges $[X_l, \overline{X_l}]$ for $1 \leq j \leq r - 1$. Hence, for each $1 \leq i, k \leq 2r - 2$ where $i \neq k$, $\mathcal{G}_j$ also contains:

1. the locally smoothly embedded loop $L_{i,k} := [x_i, \overline{x_k}, x_k, \overline{x_k}, x_k, \overline{x_i}, x_i, \overline{x_i}, x_i, x_k, \overline{x_k}, \overline{x_i}, x_i]$, starting with the purple edge $[x_i, \overline{x_k}]$ and ending with the black edge $[\overline{x_i}, x_i]$,
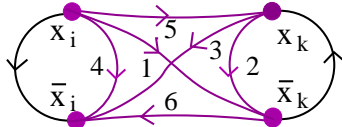
2. and the smooth path $T_{i,k} := [x_i, \overline{x_k}, x_k]$, traversing the purple edge $[x_i, \overline{x_k}]$, then black edge $[\overline{x_k}, x_k]$.

Lemma 6.1. *For each $1 \leq j \leq 2r - 2$, $\mathcal{G}_j$ is an admissible $(r; (\frac{3}{2} - r))$ ltt structure.*

Proof. The only step not following immediately from the definition and construction, is that $\mathcal{G}_j$ is birecurrent, thus admissible. To prove birecurrency we construct a locally smoothly embedded loop γ traversing each edge of $\mathcal{G}_j$. Repeating γ^j gives the locally smoothly embedded (periodic) line proving birecurrency. Since birecurrency is EPP-invariant, we assume $j = 2r - 2$.

Let γ_1 be the following smooth path traversing all edges of $\mathcal{G}_{1,\ldots,2r-2}$:

$$L_{1,3} * L_{1,5} * \cdots * L_{1,2r-3} * T_{1,3} * L_{3,5} * \cdots * L_{3,2r-3} * T_{3,5} * L_{5,7} * \cdots * L_{5,2r-3} * \cdots * T_{2l-1,2l+1}*$$

$$L_{2l+1,2l+3} * \cdots * L_{2l+1,2r-3} * \cdots * T_{2r-7,2r-5} * L_{2r-5,2r-3} * T_{2r-5,1}.$$

Remaining are the purple edges $[x_{2r-1}, x_i]$ for each $1 \le i \le 2r-2$, the red edge $[x_{2r}, x_j]$, and black edge $[x_{2r-1}, x_{2r}]$. $\gamma^j = \gamma_1 * \gamma_2$, where γ_2 is a concatenation of the loops $[x_1, x_{2r}, x_{2r-1}, \overline{x_i}, x_i, x_2, x_1]$
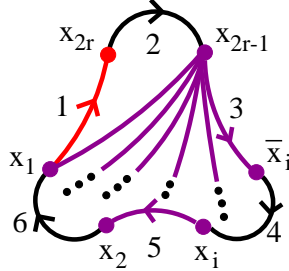
for $3 \le i \le 2r - 2$, together with the path $[x_1, x_{2r-1}, x_{2r}, x_1, x_{2r-3}, x_{2r-2}, x_2, x_1]$ to return to x_1:

$$\gamma_2 = [x_1, x_{2r}, x_{2r-1}, \overline{x_3}, x_3, x_2, x_1] * \cdots * [x_1, x_{2r}, x_{2r-1}, \overline{x_i}, x_i, x_2, x_1] * \cdots$$

$$\cdots * [x_1, x_{2r}, x_{2r-1}, \overline{x_{2r-2}}, x_{2r-2}, x_2, x_1] * [x_1, x_{2r}, x_{2r-1}, x_2, x_1] * [x_1, x_{2r-1}, x_{2r}, x_1, x_{2r-3}, x_{2r-2}, x_2, x_1].$$

$\square$

Theorem 6.2. *Let $\mathcal{C}_r$ denote the complete $(2r-1)$-vertex graph. For each $r \ge 3$, there exists an ageometric, fully irreducible $\phi \in Out(F_r)$ such that $\mathcal{C}_r$ is the ideal Whitehead graph $\mathcal{IW}(\phi)$ for ϕ.*

Proof. We first give the representative g with $\mathcal{IW}(g) \cong \mathcal{C}_3$, as this case is slightly different (ideal decomposition depicted below):

$$g = \begin{cases} a \mapsto aba\bar{b}aac\bar{b}aba\bar{b}aacbaba\bar{b}aacaba\bar{b}aac\bar{b}a \\ b \mapsto baba\bar{b}aac\bar{a}b\bar{c}\bar{a}\bar{a}b\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}ab\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}ab\bar{a}\bar{b}\bar{a}b \\ c \mapsto aba\bar{b}aac \end{cases}$$
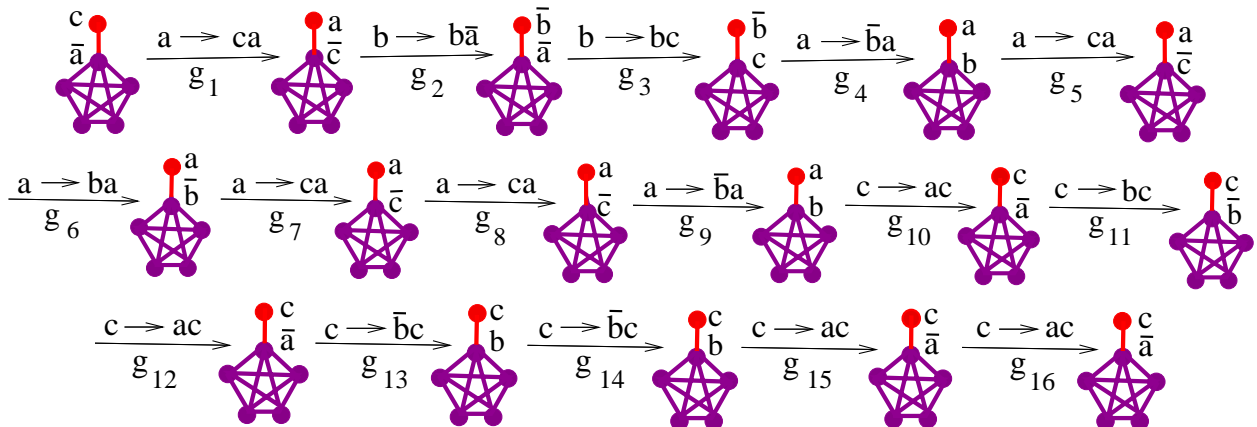
Suppose $r > 3$. Let γ_I be the construction loop in $\mathcal{G}_1$ starting with purple edge $[x_2, \overline{x_3}] = [x_2, x_4]$ and ending with purple edge $[x_{2r-2}, \overline{x_2}] = [x_{2r-2}, x_1]$ and black edge $[\overline{x_2}, x_2] = [x_1, x_2]$:

$$L_{2,3} * L_{2,5} * \cdots * L_{2,2r-3} * T_{2,4} * L_{4,5} * L_{4,7} * \cdots * L_{4,2r-3} * T_{4,6} * L_{6,7} * L_{6,9} * \cdots * L_{6,2r-3} * T_{6,8} * \cdots$$

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$$\cdots * T_{2l,2l+2} * L_{2l+2,2l+3} * T_{2l+2,2l+4} * \cdots * L_{2l+2,2r-3} * \cdots * T_{2r-6,2r-4} * L_{2r-4,2r-3} * T_{2r-4,2}.$$

Lemma 6.1 implies, with a suitable switch, γ_I induces an admissible construction composition g_{γ_I}. By Lemma 3.6, the only purple edges left to construct are the $[x_{2r-1}, x_i]$ for each $1 \leq i \leq 2r - 2$.

The initial ltt structure for the pure construction composition has red edge (red vertex first) $[x_{2r}, x_1]$. The switch for g_{γ_I} will be determined by the purple edge $[x_2, x_3]$ and has initial ltt structure with red vertex x_2 and red edge $[x_2, x_3]$. In this ltt structure, the preimages of the purple edge left are $[x_{2r-1}, x_i]$, for each $1 \leq i \leq 2r - 2$ with $i \neq 2$, and $[x_{2r-1}, x_{2r}]$. We ensure inclusion of all these edges except $[x_1, x_{2r-1}]$ by the construction loop $\gamma_{II} = T_{4,2r-1} * L_{2r-1,3} * L_{2r-1,5} * \cdots * L_{2r-1,2r-3} * T_{2r-1,4}$. The red edge for the pure construction composition is (red vertex first) $[x_2, x_3]$.

The next switch is determined by the purple edge $[x_4, x_5]$ and has initial ltt structure with red edge $[x_4, x_5]$ and red vertex x_4. In this ltt structure, the preimage of the purple edge $[x_1, x_{2r-1}]$ left still looks like $[x_1, x_{2r-1}]$. We construct it using the construction loop $[x_6, x_2, x_1, x_{2r-1}, x_{2r}, x_5, x_6]$. The associated pure construction composition has initial ltt structure with red vertex x_4, and red edge $[x_4, x_5]$.

We use a sequence of switches to ensure that g has PF transition matrix (see Lemma 3.13):

$$G_6 \xrightarrow{g_7: x_{2r} \mapsto x_2 x_{2r}} G_7 \xrightarrow{g_8: x_{2r-2} \mapsto x_{2r} x_{2r-2}} G_8 \xrightarrow{g_9: x_{2r-4} \mapsto x_{2r-2} x_{2r-4}} \cdots$$

$$\xrightarrow{g_{r+3}: x_8 \mapsto x_{10} x_8} G_{r+3} \xrightarrow{g_{r+4}: x_6 \mapsto x_8 x_6} G_{r+4} \xrightarrow{g_{r+5}: x_4 \mapsto x_6 x_4} G_{r+5}$$

where, listed red vertex first: $e_6^R = [x_2, x_{2r-1}]$, $e_7^R = [x_{2r}, x_1]$, $e_8^R = [x_{2r-2}, x_{2r-1}]$, $e_9^R = [x_{2r-4}, x_{2r-3}]$, $e_{r+2}^R = [x_{10}, x_{11}]$, $e_{r+3}^R = [x_8, x_9]$, $e_{r+4}^R = [x_6, x_7]$, and $e_{r+5}^R = [x_4, x_5]$.

We end with the Nielsen path prevention sequence (see Lemma 5.5)

$$G_0 \xrightarrow{g_1: x_5 \mapsto x_{2r} x_5} G_1 \xrightarrow{g_2: x_3 \mapsto x_5 x_3} G_2 \xrightarrow{g_3: x_2 \mapsto x_3 x_2} G_3 \xrightarrow{g_4: x_2 \mapsto x_{2r} x_2} G_4 \xrightarrow{g_5: x_2 \mapsto x_3 x_2} G_5 \xrightarrow{g_6: x_2 \mapsto x_{2r} x_2} G_6$$

where, listed red vertex first: $e_0^R = [x_{2r}, x_1]$, $e_1^R = [x_5, x_{2r-1}]$, $e_2^R = [x_3, x_6]$, $e_3^R = [x_2, x_4]$, $e_4^R = [x_2, x_{2r-1}]$, $e_5^R = [x_2, x_4]$, $e_6^R = [x_2, x_{2r-1}]$.

Take an adequately high power so that all periodic directions are fixed.

By Lemma 4.2, it suffices that the constructed representatives are (1) pNp-free, (2) have PF transition matrices, and (3) have the appropriate ideal Whitehead graphs. Lemma 5.5 implies (1), Lemma 3.13 implies (2), and Lemma 3.6 implies (3). □

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