CURRENTS – LECTURE -1- – 06/24/2013

Outline: & Motivation

$\mathcal{C}(F_N) = \{[g] \mid g \in F_N\}$ → conjugacy classes

$F_N \curvearrowright T$

There will be a function

$$\langle \; , \; \rangle : \mathcal{C}(F_N) \times \overline{cv}_N \longrightarrow \mathbb{R}_{\geq 0}$$

$$([g], T) \longmapsto \|g\|_T$$

For $g \neq 1$

$[g] \rightsquigarrow \eta_g \in Curr(F_N)$

counting current (will be defined)

finite dimensional (very good)

$$\langle \; , \; \rangle : Curr(F_N) \times \overline{cv}_N \longrightarrow \mathbb{R}_{\geq 0}$$

infinite dimen. (not good)

$$\langle \eta_g , T \rangle = \|g\|_T$$

$Out(F_N)$ acts on both of the spaces above (in a nice way)

Preliminaries:

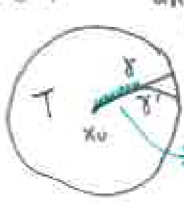
Boundary of F_N

∂F_N

In general:

T - any $\mathbb{R}$-tree, ∂T is described as follows

$x_0 \in T$ - any basepoint



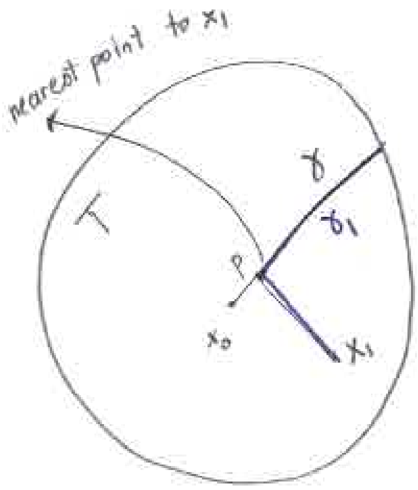
$\partial_{x_0} T = \{\gamma \mid \gamma : [0,\infty) \to T$ isometric embedding, $\gamma(0) = x_0\}$

$C_{x_0}(\gamma, \gamma')$ - gromov product

For $\gamma, \gamma' \in \partial_{x_0} T$ ②

Put $d_{x_0}(\gamma,\gamma') = \dfrac{1}{2^{c_{x_0}(\gamma,\gamma')}}$ defines a metric.

What happens if we change the basepoint?



The path starting at x_1 going to p and then following γ is a geodesic path. call it γ_1

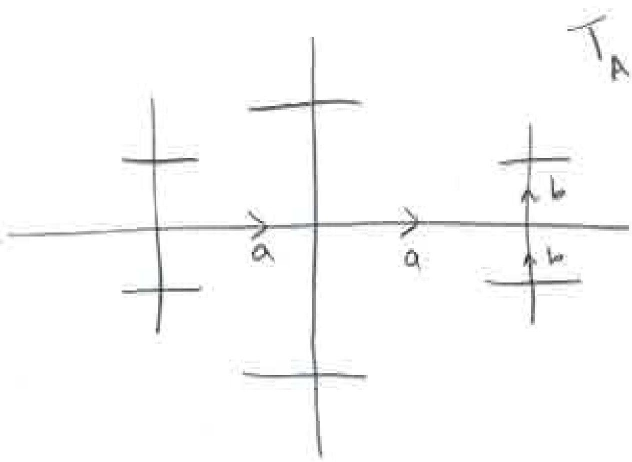
$$j_{x_0, x_1} : \partial_{x_0} T \rightarrowtail\!\!\!\!\twoheadrightarrow \partial_{x_1} T$$
$$\gamma \longmapsto \gamma_1$$

Bi-Lipshitz equivalence. So one can ignore the base-point.

$\boxed{\text{Back to } \partial F_N}$

$F_N = F(a_1, \cdots, a_N)$, $F(a,b)$ $N = 2$

T_A - Cayley graph of F_N with respect to a free basis A.

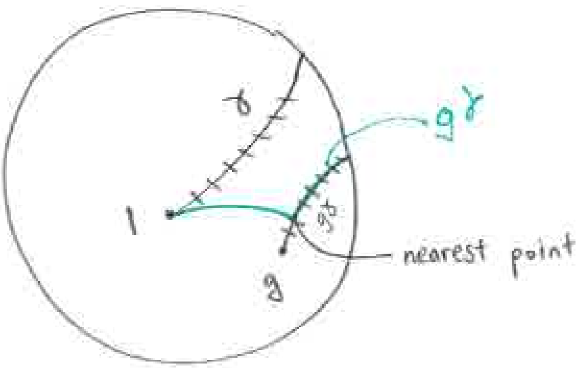


$$\partial T_A := \partial_1 T_A \approx \text{Cantor Set}$$

Now,

$$\partial F_N := \partial T_A \quad \text{(as top. spaces)}.$$

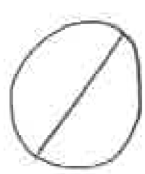
The left action of F_N on F_N extends to an action on ∂F_N by homeomorphisms.



What happens if we pick another basis or marking?

If $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$, Γ - finite, connected graph without val. 1 vertices.

Ex: For $N=2$,



There is a natural map:

$$\tilde{\alpha} : F_N \longrightarrow \tilde{\Gamma}, \quad y_0 \in \tilde{\Gamma}$$

$$g \longmapsto g y_0$$

an F_N-equivariant quasi-isometry

· Important Fact: $\tilde{\alpha}$ extends to a canonical F_N-equivariant (4)

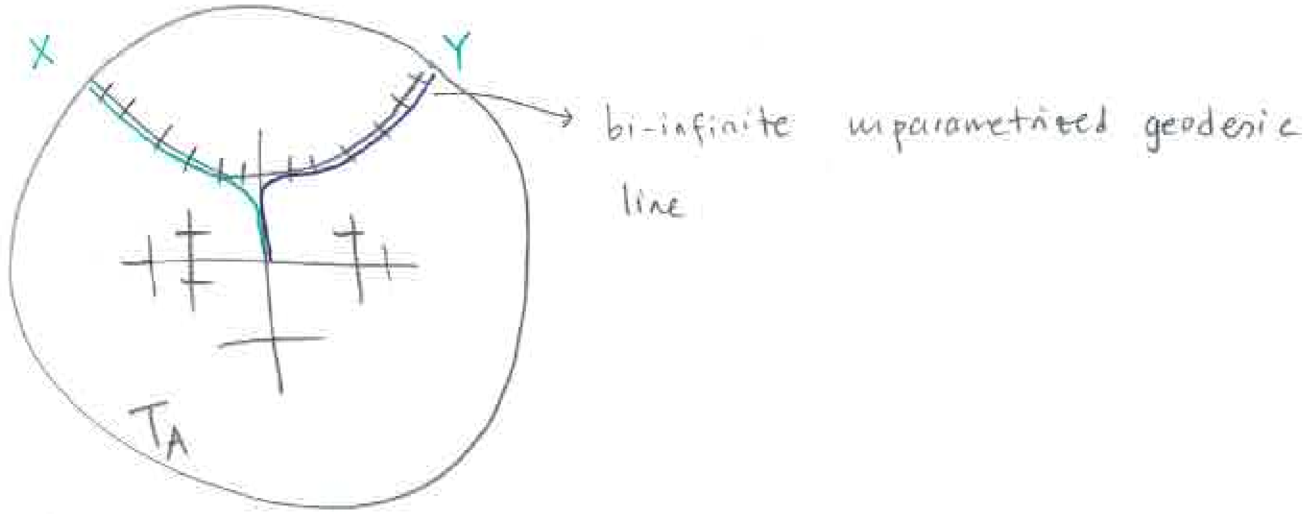
homeomorphism

$$\partial\tilde{\alpha} : \partial F_N \to \partial\tilde{\Gamma}$$

So ∂F_N "=" $\partial\tilde{\Gamma}$

—— ∘ ——

$F_N = F(a_1, -, a_N)$; $A = \{a_1, --, a_N\}$ a basis.



Any such geod. line determines a pair of points X, Y

$(X,Y) \in \partial F_N \times \partial F_N$, $X \neq Y$

$\partial^2 F_N := \partial F_N \times \partial F_N$ - diagonal

$= \{ (X,Y) \mid X, Y \in \partial F_N, X = Y \}$

equip with the subspace topology from $\partial F_N \times \partial F_N$.

There is a natural F_N action on $\partial^2 F_N$ (by homeomorphisms)

$$g(X,Y) := (gX, gY)$$

<u>Defn:</u> A $\boxed{\text{geodesic current}}$ on F_N is a measure μ on $\partial^2 F_N$ such that

1. μ is Borel
2. μ is F_N-invariant:

$$\mu(S) = \mu(gS) \quad \text{for any } g \in F_N, \ S\text{-Borel subset of } \partial^2 F_N$$

3. μ is locally finite, i.e

For all compact subset $K \subseteq \partial^2 F_N$

$$\mu(K) < \infty$$

4. μ is τ-invariant:

$$\mu(S) = \mu(\tau(S))$$

Where τ is the flip map

$$\tau : \partial^2 F_N \to \partial^2 F_N$$
$$(X,Y) \to (Y,X).$$

Currents defined by conjugacy classes.

Fix a free basis, $F_N = F(a_1, \ldots, a_N)$

$g \in F_N$, $g \neq 1$, where $g \neq h^n$ for $n \geq 2$.

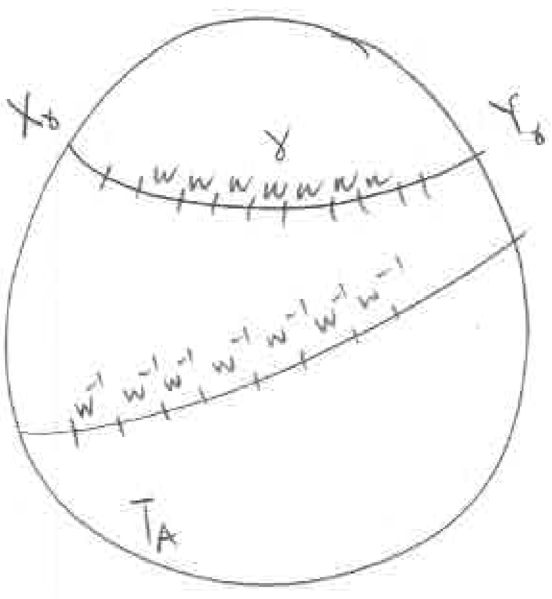
$g = v(a_1, \ldots, a_N) \to$ freely reduced

$v \equiv u w u^{-1}$

$w \to$ cyclically reduced

Ex: $F_2 = F(a, b)$

$$g = \underbrace{ab}_{u}\,\underbrace{aab^{-1}a}_{w}\,\underbrace{b^{-1}a^{-1}}_{u^{-1}}$$



$\Lambda_w = \{ \gamma \mid \gamma$ is a bi-infinite geod. line in T_A labelled by $\ldots wwww \ldots$ or $\ldots w^{-1}w^{-1}w^{-1} \ldots \}$

$\Lambda_w \ni \gamma \longmapsto (X_\gamma, Y_\gamma) \in \partial^2 F_N$

One can think of the delta measure corresponding to (X_γ, Y_γ):

$$\delta_{(X_\gamma, Y_\gamma)}(S) = \begin{cases} 1 & \text{if } (X_\gamma, Y_\gamma) \in S \\ 0 & \text{otherwise} \end{cases}$$

where $S \subset \partial^2 F$.

$$\eta_g := \sum_{\gamma \in \Lambda_g} \delta_{(X_\gamma, Y_\gamma)}$$

Fact: η_g is a geodesic current on F_N.

For an arbitrary element $g \in F_N$, $g \neq 1$

write $g = h^n$ $n \geq 1$ where h is not a proper power.

Define

$$\eta_g := n \cdot \eta_h$$

Exercise: Show that if $g = g_1 g' g_1^{-1}$ in F_N, then

$$\eta_g = \eta_{g'}$$

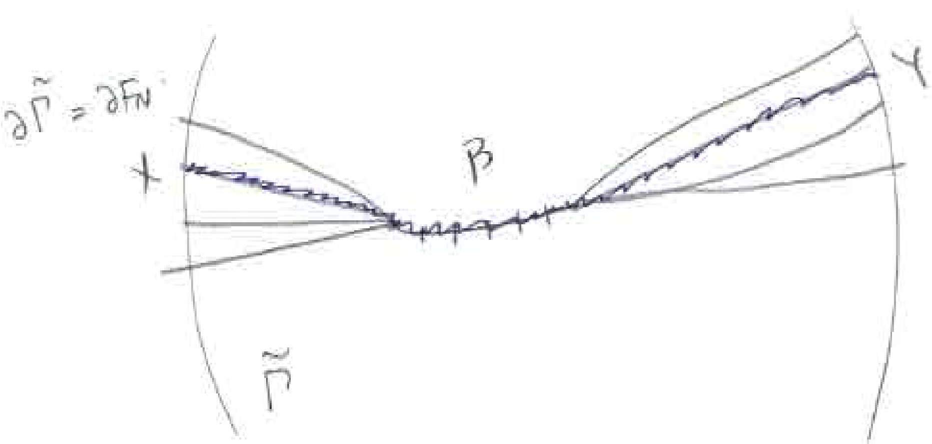
i.e η_g depends only on the conjugacy class of g.

Notation:

$$Curr(F_N) = \{\mu \mid \mu \text{ is a geodesic current on } F_N\}.$$

Choose a marking $\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$

$$\partial F_N \text{ "="} \ \partial \tilde{\Gamma}$$



β - finite-reduced edge-path in $\tilde{\Gamma}$ (non-degenerate)

$$Cyl_\alpha(\beta) = \{ (X,Y) \in \partial^2 F_N \mid \text{geodesic from } X \text{ to } Y \text{ in } \tilde{\Gamma} \text{ passes thru } \beta \}$$

$\downarrow$

Cylinder set

FACTS:

① $\forall$ finite geodesic β in $\tilde{\Gamma}$

$Cyl_\alpha(\beta)$ is compact-open.

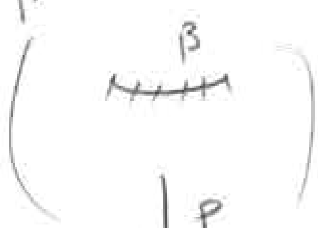
② $\{ Cyl_\alpha(\beta) \mid \beta \}$ is a basis for the topology on $\partial^2 F_N$.

— ∘ —

$p : \tilde{\Gamma} \longrightarrow \Gamma$

covering map

$\tilde{\Gamma}$



$\downarrow p$

$p(\beta)$ - a finite reduced edge path in Γ

$p(\beta)$ is called the label of β.

— ∘ —

③ $\forall g \in F_N$, $g\, Cyl(\beta) = Cyl(g\beta)$

— End of Lecture - 1 -

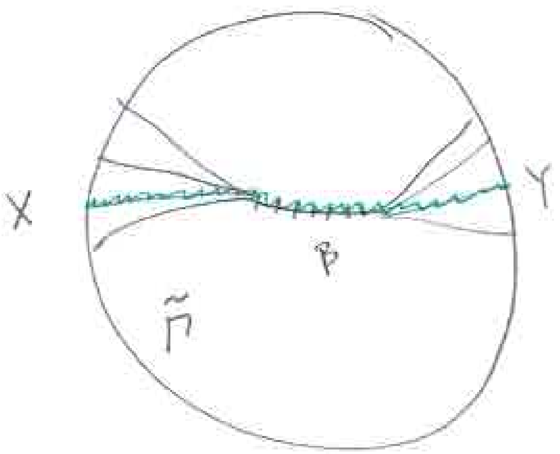
<u>Recall:</u> $\partial^2 F_N = \{ (X,Y) \mid X, Y \in \partial F_N, \ X \neq Y \}$

A geodesic current on F_N is a measure μ on $\partial^2 F_N$ s.t.

- μ is Borel, loc. finite
- μ is F_N-invariant
- μ is flip-invariant

$\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$ - marking

$\partial F_N \approx \partial \tilde{\Gamma}$



$Cyl_\alpha(\beta) := \{ (X,Y) \in \partial^2 F_N$ the geodesic in $\tilde{\Gamma}$ from X to Y passes thru $\beta \}$

$v = p(\beta)$ - edge-path in Γ

Properties of Cylinders

1. $\forall \beta$. $Cyl_\alpha(\beta) \subseteq \partial^2 F_N$ is both open and compact.
2. $\{ Cyl_\alpha(\beta) \mid \beta \}$ is a basis of open sets for the topology on $\partial^2 F_N$.

(2

③ $\forall g \in F_N$, $\forall \beta$

$$g\, Cyl_\alpha(\beta) = Cyl_\alpha(g\beta)$$

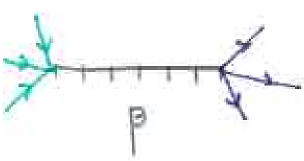
④ For any compact $C \subseteq \partial^2 F_N$, $\exists \beta_1, \dots, \beta_n$ such that

$$C \subseteq Cyl(\beta_1) \cup \dots \cup Cyl(\beta_n)$$

Notation: Δ - a locally finite graph

$\Omega(\Delta) = \{\beta \mid \beta$ is a finite non-degenerate reduced edge-path in $\Delta\}$

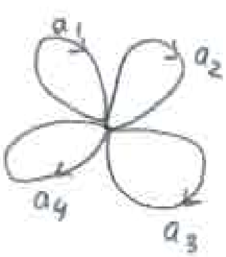
If $\beta \in \Omega(\Delta)$



$$q_+(\beta) := \{e \in E\Delta \mid \beta e \in \Omega(\Delta)\}$$

$$q_-(\beta) := \{e' \in E\Delta \mid e'\beta \in \Omega(\Delta)\}$$

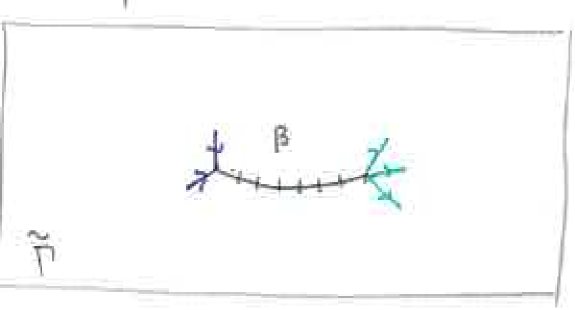
Ex:

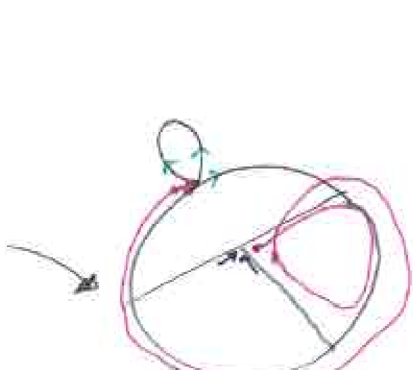
$\Delta =$ 

$F_4 = F(a_1, a_2, a_3, a_4)$

$\gamma = a_1 a_2^{-1} a_1 a_3$ - reduced edge path in Δ.

$q_+(\gamma) = \{a_1^{\pm 1}, a_2^{\pm 1}, a_3, a_4^{\pm 1}\}$





Γ

5. $\forall \beta \in \Omega(\tilde{\Gamma})$

$$Cyl(\beta) = \bigsqcup_{e \in q_+(\beta)} Cyl(\beta e) = \bigsqcup_{e' \in q_-(\beta)} Cyl(e'\beta)$$

Notation:

Let $\mu \in Curr(F_N)$ be a geodesic current. $v \in \Omega(\Gamma)$

$\langle v, \mu \rangle_\alpha := \mu(Cyl_\alpha(\beta))$ where β is any lift of v to $\beta \in \Omega(\tilde{\Gamma})$.

This is well-defined:

If $\beta \in \Omega(\tilde{\Gamma})$, $g \in F_N$, $\mu \in Curr(F_N)$, then

$$\mu(Cyl(\beta)) = \mu(g\,Cyl(\beta)) = \mu(Cyl(g\beta))$$

Prop: Let $\mu \in Curr(F_N)$, $\alpha : F_N \xrightarrow{\cong} \pi_1(\Gamma)$ marking

Then

1. $\forall v \in \Omega(\Gamma)$, $\langle v, \mu \rangle_\alpha \geq 0$ and finite.

2. $\forall v \in \Omega(\Gamma)$

$$\langle v, \mu \rangle_\alpha = \sum_{e \in q_+(v)} \langle ve, \mu \rangle_\alpha = \sum_{e' \in q_-(v)} \langle e'v, \mu \rangle_\alpha$$

3. $\forall v \in \Omega(\Gamma)$

$$\langle v, \mu \rangle_\alpha = \langle v^{-1}, \mu \rangle_\alpha$$

$v = e_1 e_2 \cdots e_n$

$v^{-1} = \bar{e}_n \bar{e}_{n-1} \cdots \bar{e}_1$

④ If $\mu_1, \mu_2 \in Curr(F_N)$; $\forall v \in \Omega(\Gamma)$, $\langle v, \mu_1\rangle_\alpha = \langle v, \mu_2\rangle_\alpha$

then $\mu_1 = \mu_2$

<u>Prop:</u> Let $(x_v)_{v\in\Omega(\Gamma)}$ is a family of numbers such that

① $x_v \geq 0$

② $\forall v$, $x_v = \sum_{e\in q_1(v)} x_{ve} = \sum_{e'\in q_{-1}(v)} x_{e'v}$

③ $x_v = x_{v^{-1}}$, $\forall v \in \Omega(\Gamma)$

then there exist a unique $\mu \in Curr(F_N)$ such that

$\forall v \in \Omega(\Gamma)$

$\langle v, \mu\rangle_\alpha = x_v$

key word : "Kolmogorov measure extension thm"

- TOPOLOGY

$Curr(F_N)$ is endowed with weak* topology:

$\lim_{n\to\infty} \mu_n = \mu \iff \forall f \in C_0(\partial^2 F_N)$

$$\int f d\mu_n \xrightarrow[n\to\infty]{} \int f d\mu$$

Prop: Let $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ - marking

Let $\mu_n \in \mathrm{Curr}(F_N)$, $\mu \in \mathrm{Curr}(F_N)$

then $\mu_n \xrightarrow[n\to\infty]{} \mu \iff \forall v \in \Omega(\Gamma)$ $\langle v, \mu_n \rangle_\alpha \xrightarrow[n\to\infty]{} \langle v, \mu \rangle_\alpha$

Prop: Let $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ be a marking. Let $\mu \in \mathrm{Curr}(F_N)$

For $\varepsilon > 0$, $k \geq 1$, $(k \in \mathbb{Z})$

Put $U_\alpha(\mu, \varepsilon, k) = \{ \mu' \in \mathrm{Curr}(F_N) \mid \forall v \in \Omega(\Gamma)$ with $|v| \leq k$
$|\langle v, \mu \rangle_\alpha - \langle v, \mu' \rangle_\alpha| < \varepsilon \}$

Then $\{ U_\alpha(\mu, \varepsilon, k) \mid 1 > \varepsilon > 0, k \in \mathbb{N} \}$

is a basis of neighborhoods for μ in $\mathrm{Curr}(F_N)$.

—∘—

Let $\Phi \in \mathrm{Aut}(F_N)$

$\Rightarrow$ $\Phi : F_N \to F_N$ is quasi-isometry

$\Rightarrow$ Φ induces a homeomorphism $\Phi : \partial F_N \xrightarrow{\sim} \partial F_N$. Also, get a homeo. $\Phi : \partial^2 F_N \to \partial^2 F_N$

$$\Phi(X, Y) := (\Phi(X), \Phi(Y))$$

- OUT(F_N) action

Prop -
Defn: Let $\Phi \in \mathrm{Aut}(F_N)$, $\mu \in \mathrm{Curr}(F_N)$. Define a measure $\Phi\mu$ on $\partial^2 F$ as follows: $\forall$ Borel $S \subseteq \partial^2 F_N$

$$(\Phi\mu)(S) := \mu(\Phi^{-1}(S))$$

Then $\Phi\mu \in \mathrm{Curr}(F_N)$

This defines a left action of $Aut(F_N)$ on $Curr(F_N)$ by continuous linear transformations.

<u>Prop:</u> $Inn(F_N)$ acts trivially on $Curr(F_N)$.

<u>Reason:</u> Pick $u \in F_N$, consider $\Phi \in Aut(F_N)$ given by

$$\Phi(w) = uwu^{-1} \quad \forall w \in F_N.$$

Then $\forall X \in \partial F_N$

$$\Phi(X) = uX$$

<u>Cor:</u> The action of $Aut(F_N)$ on $Curr(F_N)$ factors thru $Out(F_N)$:

Given $\varphi \in Out(F_N)$, $\mu \in Curr(F_N)$, pick any representative $\Phi \in Aut(F_N)$ of φ

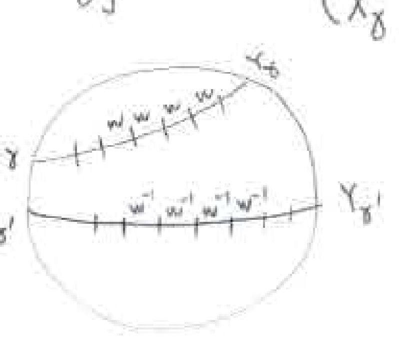
then $\varphi\mu := \Phi\mu$

<u>Prop:</u> Let $g \in F_N$, $g \neq 1$. Assume g is not a proper power. Let $\varphi \in Out(F_N)$ then

$$\varphi \cdot \eta_g = \eta_{\varphi(g)}.$$

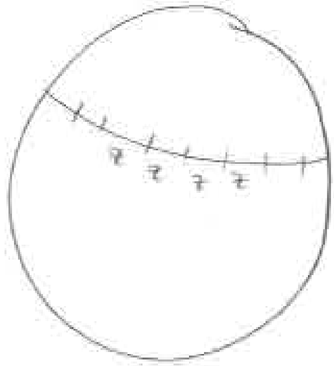
<u>Reason:</u>

$$\eta_g = \sum \delta_{(X_\gamma, Y_\gamma)}, \quad (X_\gamma, Y_\gamma) \in \Lambda_g \qquad g \rightsquigarrow w \text{ - cyc. red.}$$



$A = \{a_1, \dots, a_N\}$

$$(\Phi \eta_g)(s) := \eta_g(\Phi^{-1}(s))$$



$\Phi(g) \rightsquigarrow z$

$\downarrow$

cyl. red

- End of Lecture 2 -