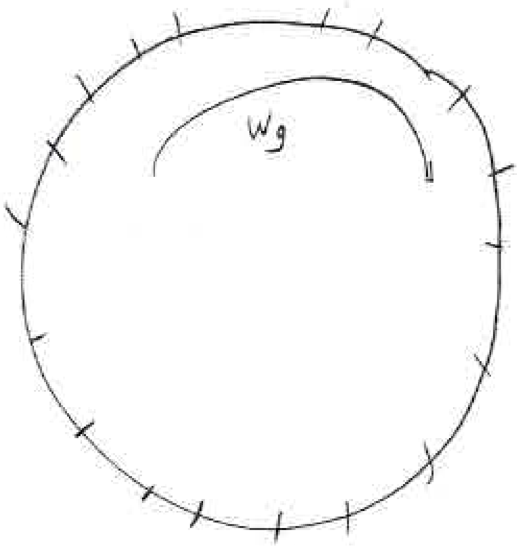
CURRENTS, LECTURE - 4, Ilya Kapovich

Recall:

$\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$ marking.

$g \in F_N$, $g \neq 1$, $g \rightsquigarrow w_g$ - immersed circuit in Γ



W_g

Given $v \in \Omega(\Gamma)$ - finite reduced edge path

$$\langle v, \eta_g \rangle_\alpha = \langle v, w_g \rangle_\alpha$$

There is a map

$$\langle\ ,\ \rangle : cv_N \times Curr(F_N) \longrightarrow \mathbb{R}_{\geq 0}$$

$T \in cv_N$, $\Gamma := T/F_N$, For any $\mu \in Curr(F_N)$

$\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$

$L : E\Gamma \longrightarrow \mathbb{R}_{>0}$

$L(e) = L(\bar{e})$

$$\langle T, \mu \rangle = \sum_{e \in E^+\Gamma} \langle e, \mu \rangle_\alpha \cdot L(e) = \frac{1}{2} \sum_{e \in E^+\Gamma} \langle e, \mu \rangle_\alpha \cdot L(e).$$

Prop:

1. $\forall g \in F_N,\ g \neq 1$

   $\forall T \in cv_N$

$$\langle T, \eta_g \rangle = \|g\|_T$$

2. $\langle\ ,\ \rangle : cv_N \times Curr(F_N) \to \mathbb{R}_{\geq 0}$ continuous

3. $\langle T, c_1\mu_1 + c_2\mu_2 \rangle = c_1 \langle T, \mu_1 \rangle + c_2 \langle T, \mu_2 \rangle$ where $c_1, c_2 \geq 0$.

4. $\langle cT, \mu \rangle = c \langle T, \mu \rangle$

5. $\forall T \in cv_N$, $\forall \mu \in Curr(F_N)$, $\forall \varphi \in Out(F_N)$

$$\langle T\varphi, \mu \rangle = \langle T, \varphi\mu \rangle$$

i.e $\langle \varphi T, \varphi\mu \rangle = \langle T, \mu \rangle$

$\varphi \in Out(F_N)$

$T \in \overline{cv_N}$

$\varphi T := T\varphi^{-1}$

To prove (5) it suffices to show this for counting currents by linearity and denseness of rational currents

$\mu = \eta_g$, $g \in F_N$, $g \neq 1$

$$\langle T\varphi, \eta_g \rangle = \|g\|_{T\varphi} = \|\varphi(g)\|_T = \langle T, \eta_{\varphi(g)} \rangle$$

$$= \langle T, \varphi\eta_g \rangle \quad \square$$

Projectivization

$X = Curr(F_N) - \{0\}$

$\sim$: equivalence relation on X.

$\mu_1, \mu_2 \in X$, $\mu_1 \sim \mu_2$ iff $\exists\, c > 0$ s.t $\mu_2 = c\mu_1$

$\mathbb{P}Curr(F_N) := (Curr(F_N) - \{0\}) / \sim$ with the quotient topology.

If $\mu \neq 0$, $\mu \in Curr(F_N)$

$\mathbb{P}curr(F_N) \ni [\mu]$ is the $\sim$-equivalance class of μ.

For $\varphi \in Out(F_N)$, $c > 0$, $\mu \in Curr(F_N)$

then $\varphi(c\mu) = c\varphi(\mu)$

$\Rightarrow$ $Out(F_N)$ naturally acts on $\mathbb{P}Curr(F_N)$

$\varphi[\mu] := [\varphi\mu]$

<u>Prop:</u> $\mathbb{P}Curr(F_N)$ is compact

<u>Idea of Proof:</u>

$\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ marking $L(e) = 1$ $\forall e \in E\Gamma$

$T = \tilde{\Gamma}$

$$Q = \{\mu \in Curr(F_N) \mid \langle T, \mu \rangle = 1\}$$

$$p : Curr(F_N) - \{0\} \longrightarrow \mathbb{P}Curr(F_N)$$

$$\mu \longmapsto [\mu]$$

$p_{|Q} : Q \to \mathbb{P}Curr(F_N)$ is a homeomorphism

$v \in \Omega(\Gamma)$

$$\langle v, \mu \rangle_\alpha = \sum_{e \in q_+(v)} \langle ve, \mu \rangle_\alpha$$

$\mu \in Q$

$$\langle T, \mu \rangle = \sum_{e \in E^+\Gamma} \langle e, \mu \rangle_\alpha = 1.$$

$v = e_1 \dots e_m$

$$\langle e_1, \mu \rangle_\alpha \geqslant \langle e_1 e_2, \mu \rangle_\alpha \geqslant \dots \geqslant \langle v, \mu \rangle_\alpha$$

Why is α compact?

- $\forall \mu \in Q$, $\forall e \in E\Gamma$

$$0 \leqslant \langle e, \mu \rangle_\alpha \leqslant 1$$

$$\sum_{e \in E^+\Gamma} \langle e, \mu \rangle_\alpha = 1$$

- $\forall v \in \Omega(\Gamma)$

$$0 \leqslant \langle v, \mu \rangle_\alpha \leqslant 1$$

$(\mu_n)_1^\infty$, $\mu_n \in Q$

$E^+\Gamma = \{e_1, \ldots, e_k\}$

$$(\langle e_1, \mu_n\rangle, \ldots, \langle e_k, \mu_n\rangle) \in \underbrace{[0,1]\times[0,1]\times \cdots \times [0,1]}_{k\text{-times}}$$

$\langle e_1, \mu_n\rangle \xrightarrow[\text{up to sub}]{n\to\infty} \theta(e_1) \in [0,1]$

$\langle e_k, \mu_n\rangle \longrightarrow \theta(e_k) \in [0,1]$

k-times

compact

Pass to a further subsequence of μ_n so that $\forall\, v = ee' \in \Omega(\Gamma$

$$\langle ee', \mu_n\rangle \longrightarrow \theta(ee') \in [0,1]$$

Repeat this process to obtain sequential compactness for Q and so on.

Thm: (Kapovich–Lustig)

$\langle\ ,\ \rangle : cv_N \times Curr(F_N) \to \mathbb{R}_{\geq 0}$ admits a unique continuous extension

$$\langle\ ,\ \rangle : \overline{cv_N} \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

has the same list of properties as before

$\forall T \in \overline{cv}_N \,, \forall g \in F_N$, $g \neq 1$

- $\langle T, \eta_g \rangle = \|g\|_T$
- $\langle T\varphi, \mu \rangle = \langle T, \varphi\mu \rangle$

<u>Q:</u> If $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$ when do we have
$\langle T, \mu \rangle = 0$.

$F_N = A * B$, $A \neq 1$, $B \neq 1$

T – Bass-Serre Tree, $T \in \overline{cv}_N$.

Let $a \in A$, $a \neq 1$: then

$\langle T, \eta_a \rangle = \|a\|_T = 0$

<u>Thm</u> (Kapovich - Lustig)

Let $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$. Then $\langle T, \mu \rangle = 0$
$\iff supp(\mu) \subseteq L(T)$

$L(T) \subset \partial^2 F$

closed, F_N invariant, flip invariant

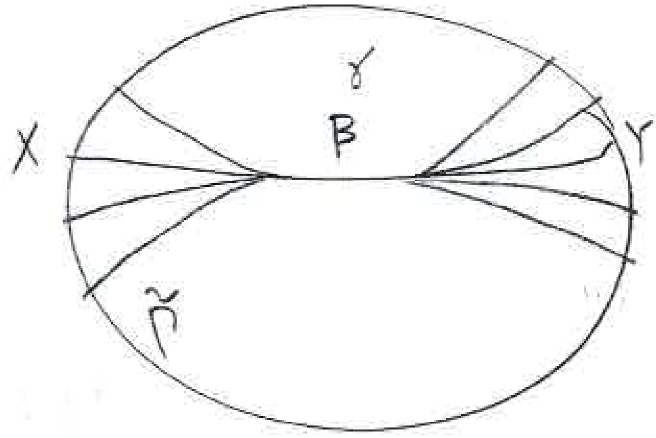
$\mu \in Curr(F_N) \longrightarrow$ measure on $\partial^2 F_N$

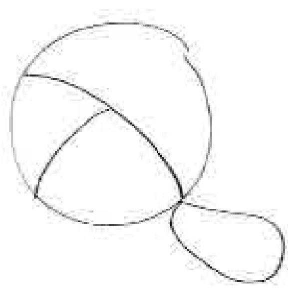
$$supp(\mu) = \underbrace{\partial^2 F_N - \bigcup_{\substack{U \text{ open} \\ \text{and } \mu(U)=0}} U}$$

closed, F_N-inv, flip invariant subset of $\partial^2 F_N$.

<u>Proposition</u>: Let $\alpha : F_N \simeq \pi_1(\Gamma)$ a marking. Let $\mu \in Curr(F_N)$. Let $(X, Y) \in \partial^2 F_N$. Let γ be the geodesic from X to Y in $\tilde{\Gamma}$. Then $(X,Y) \in supp(\mu) \iff$ for any finite subpath β of γ with label $v = p(\beta)$ we have

$$\langle v, \mu \rangle_\alpha > 0.$$





- End of Lecture 4 -