Currents - Lecture 5 - Ilya Kapovich

Recall:

$$\langle\ ,\ \rangle : \overline{cv}_N \times Curr(F_N) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\langle T\varphi, \mu \rangle = \langle T, \varphi\mu \rangle$$

$$\langle \varphi T, \varphi\mu \rangle = \langle T, \mu \rangle$$

$g \in F_N$, $g \neq 1$

$$\langle T, \eta_g \rangle = \|g\|_T$$

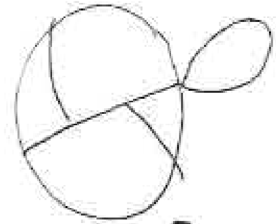
For $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$

$$\langle T, \mu \rangle = 0 \iff supp(\mu) \subseteq L(T) \subseteq \partial^2 F_N$$

$\varphi \in Out(F_N)$, φ - iwip

$\exists$ a train-track representative f of φ.

$f : \Gamma \longrightarrow \Gamma$, $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$



$E\Gamma = E_+ \sqcup E_-$, $E_+ = \{e_1, \dots, e_k\}$

$M(f)$ $k \times k$ matrix

m_{ij} = # of times $e_i, \overline{e_i}$ occurs in $f(e_j)$

Remark:

- $M(f^n) = M(f)$ $\forall n \geq 1$
- $\exists\, n_0 \geq 1$ s.t. $\forall\, i,j \in \{1, \dots, k\}$
  $[M(f^{n_0})]_{ij} > 0$

PF-Theory $\Rightarrow$

- The spectral radius λ of $M(f)$, $\lambda(f) > 1$ and $\lambda(f)$ is an eigenvalue of $M(f)$ with multiplicity 1.

- $\exists$ $V_L = (x_1, \dots, x_k) \in \mathbb{R}^k$

$$V_R = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \mathbb{R}^k$$

s.t. $x_i, y_j > 0$ and

$$M(f)\, V_R = \lambda(f)\, V_R$$

$$V_L\, M(f) = \lambda(f)\, V_L$$

Can get a geodesic current μ_+ from $f : \Gamma \to \Gamma$

$f : \Gamma \to \Gamma$, $\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$.

Pick $e \in E\Gamma$

Let $v \in \Omega(\Gamma)$, $\langle v, f^n(e) \rangle = \#$ of occurances of v in $f^n(e)$.

$\forall\, v \in \Omega(\Gamma)$, put

③

$$\circledast \quad \langle v, \mu_+ \rangle_\alpha := \lim_{n\to\infty} \frac{\langle v, f^n(e) \rangle}{|f^n(e)|}$$

$$\langle v, \mu_+ \rangle = \sum_{e' \in q_{L_+}(v)} \langle ve', \mu_+ \rangle = \sum_{e'' \in q_{L_-}(v)} \langle e''v, \mu_+ \rangle .$$

$\circledast$ does define a current $\mu_+ \in Curr(F_N)$.

Moreover,

$$\varphi \mu_+ = \underset{\underset{\lambda(f)}{\parallel}}{\lambda_+} \mu_+ \qquad \lambda_+ > 1$$

Thus φ fixes $[\mu_+]$ in $\mathbb{P}Curr(F_N)$

$[\mu_+]$ is called the attracting current of φ.

Use φ^{-1}

$f' : \Gamma' \to \Gamma'$

$\downarrow$

train-track rep. of φ^{-1}

$\xrightarrow{\text{Apply this procedure}}$

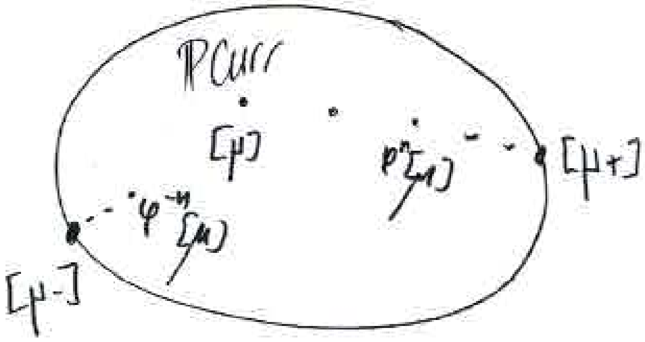
$[\mu_-] \in \mathbb{P}Curr(F_N)$

$\varphi^{-1} \mu_- = \lambda_- \mu_-$

$\varphi \mu_- = \dfrac{1}{\lambda_-} \mu_-$

$\lambda_- > 1$

Usually $\lambda_+ \neq \lambda_-$.



$\mathbb{P}Curr(F_N)$

φ-iwip, hyperbolic

$\Downarrow$

There are no $n>0$, $g \in F_N$, $g \neq 1$ s.t $\varphi^n[g] = [g]$.

φ has exactly two fixed points on $\mathbb{P}Curr$.

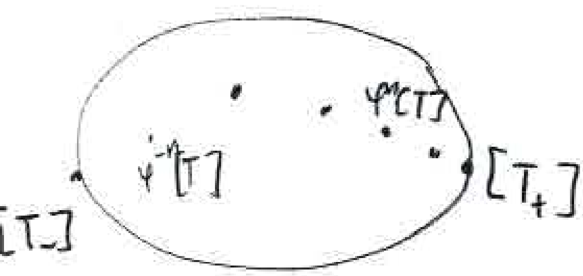
$\forall [\mu] \in \mathbb{P}Curr(F_N)$ s.t $[\mu] \neq [\mu_-], [\mu_+]$ Then

$$\lim_{n\to\infty} \varphi^n[\mu] = [\mu_+]$$

(Reiner Martin)

$$\lim_{n\to\infty} \varphi^{-n}[\mu] = [\mu_-].$$

Similarly for $\overline{CV_N} = \mathbb{P}\overline{cv_N}$, φ-iwip



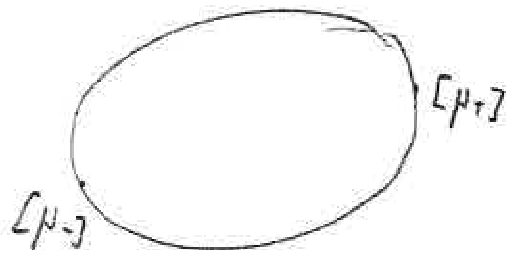
Thm: Let $\varphi \in Out(F_N)$ be a hyp. iwip then

1. $\langle T_-, \mu_+ \rangle = 0$, $\langle T_+, \mu_- \rangle = 0$

2. If $\mu \neq 0$, $\langle T_-, \mu \rangle = 0 \Rightarrow [\mu] = [\mu_+]$

3 If $T \in \overline{cv_N}$ is such that $\langle T, \mu_+ \rangle = 0$ then $[T] = [T_-]$.

Reason:



Take $\underset{\neq 1}{g} \in F_N$

$\varphi^n[\eta_g] \longrightarrow [\mu_+]$. In fact we have

$$\lim_{n\to\infty} \frac{\varphi^n(\eta_g)}{\lambda^n} = c \cdot \mu_+ \quad \text{for some } c>0.$$

$T_0 \in \overline{cv_N}$ $\qquad [T_0]\varphi^{-n} \longrightarrow [T_-]$

$$\frac{T_0\varphi^{-n}}{\lambda_-^{\,n}} \xrightarrow[n\to\infty]{} T_-$$

cont.

$$\langle T_-, \mu_+ \rangle = \lim_{n\to\infty} \left\langle \frac{T_0\varphi^{-n}}{\lambda_-^{\,n}}, \frac{\varphi^n\eta_g}{\lambda_+^{\,n}} \right\rangle$$

$$= \lim_{n\to\infty} \frac{1}{(\lambda_-\lambda_+)^n} \langle T_0\varphi^{-n}, \varphi^n\eta_g \rangle$$

$$= \lim_{n\to\infty} \frac{1}{(\lambda_- \lambda_+)^n} \langle T_0, \varphi^{-n}\varphi^{n}\eta_g \rangle$$

$$= \lim_{n\to\infty} \frac{\|g\|_{T_0}}{(\lambda_- \lambda_+)^n} = 0.$$

Ex: Let $\varphi \in Out(F_N)$ a hyperbolic iwip. Let T_+, T_- be attracting and repelling tree of φ.

Let A be a free basis of F_N.

$\Rightarrow \exists C > 1 : \forall g \in F_N$

$$\frac{1}{C}\|g\|_A \leq \|g\|_{T_+} + \|g\|_{T_-} \leq C\|g\|_A$$

Proof: Define $J : \mathbb{P}Curr(F_N) \longrightarrow \mathbb{R}_{\geq 0}$

$$J([\mu]) := \frac{\langle T_+, \mu\rangle + \langle T_-, \mu\rangle}{\langle T_A, \mu\rangle}$$

Remarks:

- $\langle T_A, \mu\rangle \neq 0$, since $T_A \in \overline{cv}_N$ & $\mu \neq 0$.
- well-defined continuous function.

Note: T_A - Cayley tree corres. to basis A.

$\tau([\mu]) > 0$, since otherwise μ has to be scalar multiples of both μ_-, and μ_+. 7

$\mathbb{P}Curr(F_N)$ is compact.

$\therefore$ τ achieves a max. and min on $\mathbb{P}Curr(F_N)$

$0 < C_2 \leq \tau[\mu] \leq C_1 < \infty.$

$\forall\, g \in F_N$, $g \neq 1$

$$C_2 \leq \frac{\langle T_+, \eta_g \rangle + \langle T_-, \eta_g \rangle}{\langle T_A, \eta_g \rangle} \leq C_1$$

$$C_2 \cdot \|g\|_{T_A} \leq \|g\|_{T_+} + \|g\|_{T_-} \leq C_1 \|g\|_{T_A}$$