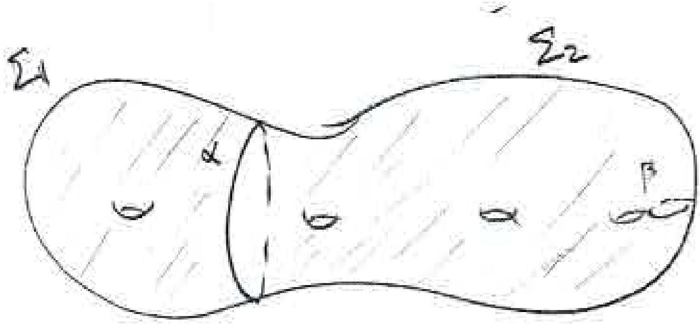
Currents - Lecture 6 - Ilya Kapovich

S - closed, hyperbolic surface



$\mathcal{C}(S)$ - curve graph

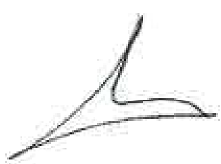
$V = \{[\alpha] \mid \alpha$ is an essential simple closed curves on $S\}$

E

, $i(\alpha, \beta) = 0$

$\mathrm{Mod}(S) \curvearrowright \mathcal{C}(S)$

- $\mathcal{C}(S)$ is connected
- $\operatorname{diam} \mathcal{C}(S) = \infty$
- $\mathcal{C}(S)$ is Gromov-hyperbolic, i.e. ($\exists\, \delta > 0$: all geodesic triangles are δ-thin)



$[\alpha] \in V\mathcal{C}(S)$

$\swarrow$ $[g_\alpha]$, $g_\alpha \in \pi_1(S)$

$\downarrow$ complementary subsurface

$\searrow$ T_α splitting of fundamental group

$\pi_1(S) = \pi_1(\Sigma_1) *_{\langle g_\alpha \rangle} \pi_1(\Sigma_2)$

Free Factor Graph

$F_N, \quad N \geq 3$

$FF(F_N) = FF_N = \mathcal{F}_N$ - free factor graph

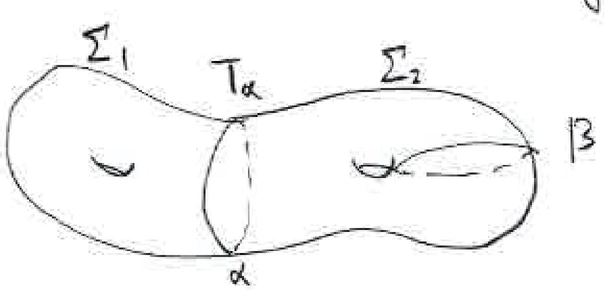
$$V = \{ [A] \mid \underbrace{1 \neq A \subsetneq F_N}_{\text{proper free factor}} \}$$

E: $[A]$ ——— $[B]$

$\exists\, A \in [A],\ B \in [B] : A \subseteq B$

$Out(F_N) \curvearrowright FF_N$

- FF_N - connected
- $\operatorname{diam}(FF_N) = \infty$
- FF_N is Gromov-Hyperbolic (Bestvina-Feighn)



$i(\beta, \alpha) = 0 \quad \longleftrightarrow \quad \|\beta\|_{T_\alpha} = 0.$

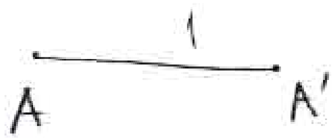
$\mathcal{Y}_N$ = bipartite graph

- two types of vertices:

a) $[a]$, $a \in F_N$ - primitive

b) $T \in FS(F_N)$: T/F_N has a single edge.

F_N



$A * A'$

F_N

$\{a_1, \ldots, a_{N-1}, t\}$ - a free basis of F_N

$F_N = \langle A, t \mid t^{-1} \cdot 1 \cdot t = 1 \rangle$

$A = F(a_1, \ldots, a_{N-1})$

Edges in $\mathcal{Y}_N$

$\|a\|_T = 0$

$\exists$ an $Out(F_N)$ equivariant quasi-isometry

$\gamma: FF_N \longrightarrow \mathcal{Y}_N$

Define I_N - intersection graph:

- bipartite graph with vertices

a) $[T]$, $T \in \overline{cv_N}$

b) $[\mu]$ $\in \mathbb{P}Curr(F_N)$

Edges:

$[T]$ ——— $[\mu]$

$\langle T, \mu \rangle = 0$

$Out(F_N) \curvearrowright I_N$ on the left.

$\varphi T := T\varphi^{-1}$

$\langle T, \mu \rangle = \langle \varphi T, \varphi\mu \rangle$

- Facts about I_N

- I_N is disconnected
  e.g if $T \in cv_N$, $[T]$ - isolated vertex
- If φ - hyperbolic iwip.

$[T_-]$ ——— $[\mu_+]$ isolated edge.

$$\mathcal{Y}_N \underset{\substack{\downarrow \\ \text{Out}(F_N)\text{ invariant} \\ \text{connected subgraph}}}{\subseteq} \mathcal{I}_N$$

$$T \overset{}{\text{———}} [a] \quad \rightsquigarrow \quad [T] \text{———} [\eta_a]$$

<u>Lemma:</u>

Suppose $[T_n] \text{——} [\mu_n]$ is a sequence of edges

s.t $[T_n] \xrightarrow{n\to\infty} [T_\infty]$

$[\mu_n] \xrightarrow{n\to\infty} [\mu_\infty]$

then $[T_\infty] \text{——} [\mu_\infty]$ is again an edge of $\mathcal{I}_N$.

Pf:

$$[T_n] \longrightarrow [T_\infty], \quad [\mu_n] \to [\mu_\infty]$$

$\Rightarrow \exists\, c_n, c_n' > 0$ s.t

$$c_n T_n \longrightarrow T_\infty$$

$$c_n' \mu_n \longrightarrow \mu_\infty$$

$$0 = \langle T_n, \mu_n \rangle = c_n c_n' \langle T_n, \mu_n \rangle = \langle c_n T_n, c_n' \mu_n' \rangle$$

$$\longrightarrow \underset{\substack{\shortparallel \\ 0}}{\langle T_\infty, \mu_\infty \rangle}$$

$$\mathcal{Y}_N \underset{\text{subgraph}}{\subseteq} I_N$$ ⑥

Prop: Let φ be a hyp. iwip. Let $a \in F_N$ - primitive then

$$d_{I_N}([\eta_a], \varphi^n[\eta_a]) \xrightarrow{n \to \infty} \infty$$

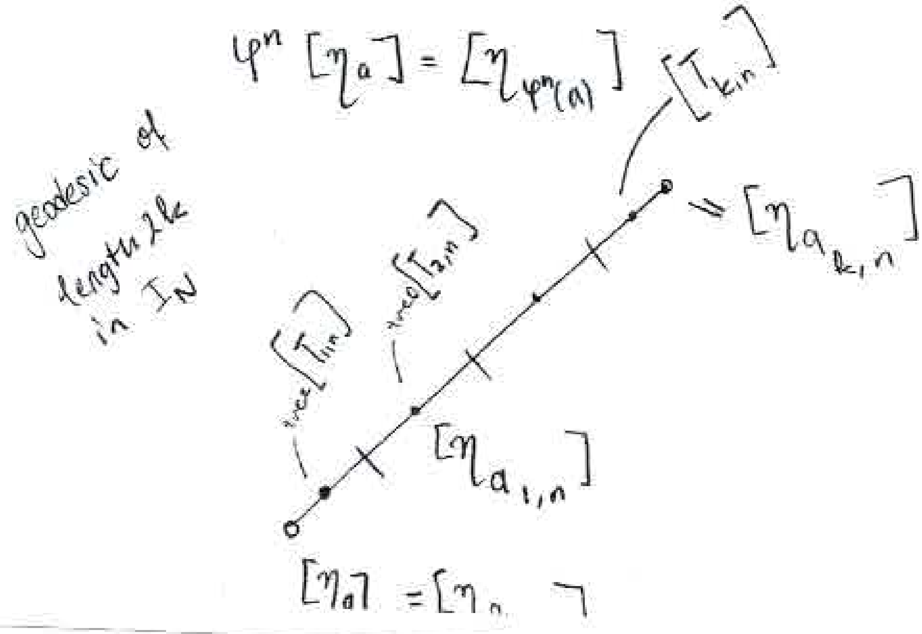
Proof. Suppose not; $\Rightarrow \exists n_i \to \infty$, $\exists C > 0$ s.t

$$d_{I_N}([\eta_a], \varphi^{n_i}[\eta_a]) \leq C \quad \forall i$$

Can assume that $\exists n_i \to \infty$; $\exists k > 0$:

$$\forall i \quad d_{I_N}([\eta_a], \varphi^{n_i}[\eta_a]) = 2k$$

Drop i for simplicity



$\mathbb{P}CV_N$, $\mathbb{P}Curr(F_N)$ – compact

So $[\eta_{0,n}] \longrightarrow [\eta_0]$ up to passing to a subsequence

$[\eta_{1,n}] \longrightarrow [\eta_1]$

$[\eta_{k,n}] \longrightarrow [\eta_k]$

Similarly,

$[T_{1,n}] \longrightarrow [T_1]$

$\vdots$

$[T_{k,n}] \longrightarrow [T_k]$

We know $\varphi^n[\eta_a] = [\eta_{a_{k,n}}] \xrightarrow[n\to\infty]{} [\mu_+]$

& $[\eta_{a_{0,n}}] = [\eta_a] \xrightarrow[n\to\infty]{} [\eta_a]$

There is an edge b/w $[\eta_k]$ & $[T_k]$

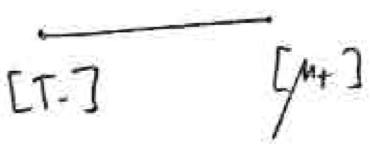
" " " " " $[T_k]$ & $[\eta_{k-1}]$

$\vdots$

$[\eta_a]$

This gives a path of length $2k$ in I_N from $[\mu_a]$ to $[\mu_+]$.

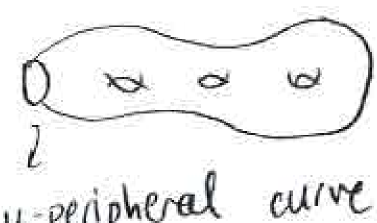
This contradicts the fact that the connected component of $[\mu_+]$ in I_N is



<u>Remarks:</u>

If φ is non-hyperbolic iwip

$\Rightarrow$ φ is induced by pA on a compact surface with 1 boundary component.



$\varphi([\eta_u]) = [\eta_u]$

$N \geq 3$

- Bestvina-Feighn: For all $\varphi \in Out(F_N)$
  - if φ is an iwip $\Rightarrow$ then φ acts as a hyperbolic isometry (hence w/ unbounded orbits) on I_N and FF_N.

- if φ is not an iwip $\Rightarrow \varphi$ acts with bounded orbits on $\mathcal{Y}_N$ and FF_N.

FF_N & $\mathcal{Y}_N$ are quasi-isometric!

$$\begin{array}{ccc} \mathcal{Y}_N & \longrightarrow & FF_N \\ [a] & \longrightarrow & [\langle a \rangle] \end{array}$$ is a q.i.

- End of lectures -

Note-taker: Caglar Uyanik