# $Out(F_3)$ INDEX REALIZATION

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Abstract. By proving precisely which singularity index lists arise from the pair of invariant foliations for a pseudo-Anosov surface homeomorphism, Masur and Smillie [MS93] determined a Teichmüller flow invariant stratification of the space of quadratic differentials. In this paper we determine an analog to the theorem for $Out(F_3)$. That is, we determine which index lists permitted by the [GJLL98] index sum inequality are achieved by ageometric fully irreducible outer automorphisms of the rank-3 free group.

## 1. Introduction

We let $Out(F_r)$ denote the outer automorphism group of the rank-$r$ free group. In this paper we prove realization results for an outer automorphism invariant dependent only on the conjugacy class (within $Out(F_r)$) of the outer automorphism, namely the "index list." This work is motivated both by the important role index lists have played in mapping class group theory and by the role they are already playing in studying the dynamics of the groups $Out(F_r)$.

The outer automorphism groups have been studied for many years. More recent developments have encouraged and enabled rapid analysis of deep relationships between the mapping class groups and the groups $Out(F_r)$. For a compact surface $\Sigma$, the *mapping class group* $\mathcal{MCG}(\Sigma)$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma$. The relationship between the mapping class groups and $Out(F_r)$ is particularly visible in rank 2, where there are even isomorphisms $Out(F_2) \cong Out(\pi_1(\Sigma_{1,1})) \cong \mathcal{MCG}(\Sigma_{1,1})$ for the one-holed torus $\Sigma_{1,1}$. It can be noted that even in higher ranks, many outer automorphisms are still induced by homeomorphisms of compact surfaces with boundary. For future reference, such outer automorphisms are called *geometric*.

While not necessary for following the content of this paper, we first briefly explain indices in the mapping class group setting to orient the reader more familiar with surface theory. The index list is an important invariant of a "pseudo-Anosov" mapping class. Pseudo-Anosovs are the most common mapping class group elements (see for example [Mah11]) and are characterized by having a representative leaving invariant a pair of transverse measured singular minimal foliations. In [MS93] Masur and Smillie determined precisely which singularity index lists, permitted by the Poincaré-Hopf index formula, come from these invariant foliations of pseudo-Anosovs. The stratification they give of the space of quadratic differentials is not only invariant under the Teichmüller flow, but has been extensively studied in papers such as [KZ03], [Lan04], [Lan05], [EMR12], and [Zor10].

The index list for a pseudo-Anosov can identically be viewed in terms of its invariant foliation or in terms of its dual $\mathbb{R}$-tree. In fact, the singularities of the invariant foliation, lifted to the universal cover, are in one-to-one correspondence with the branchpoints of the dual $\mathbb{R}$-tree. In the respective settings, the index list has an entry of $1 - \frac{k}{2}$ obtained by counting the number $k$ of prongs at the singularity or the valence of the branch point. Alternatively, one can ascertain the index list from singularities of the expanding invariant lamination (obtained as the limit of any simple closed curve

The author was supported by the ARCHIMEDE Labex (ANR-11-LABX- 0033) and the A*MIDEX project (ANR-11-IDEX-0001-02) funded by the "Investissements dAvenir" French government program managed by the ANR..

under repeated application of the pseudo-Anosov) or from the invariant train track. Much of this theory can be found in [FdPDdm91].

A "fully irreducible" (iwip) outer automorphism is the most commonly used analogue to a pseudo-Anosov mapping class. An element $\phi \in Out(F_r)$ is *fully irreducible* if no positive power $\phi^k$ fixes the conjugacy class of a proper free factor of $F_r$. The index theory for automorphisms of free groups dates back to the work of Cooper [Coo87], Dyer and Scott [DPS75], Gersten [Ger87], and later Bestvina and Handel [BH92] in understanding the fixed point sets for an automorphism. Gersten [Ger87] proved the Scott conjecture [DPS75] that rank of the fixed subgroup $Fix(\Phi) = \{g \in F_r \mid \Phi(g) = g\}$ for a $\Phi \in Aut(F_r)$ is finite. In [BH92], Bestvina and Handel proved the strengthened Scott conjecture, stating that $Fix(\Phi)$, for $\Phi \in Aut(F_r)$, has rank $\leq r$. Gaboriau, Jaeger, Levitt, and Lustig [GJLL98], utilizing the index theory for $\mathbb{R}$-trees already developed in [GL95], introduced an index theory for automorphisms of free groups in order to prove (as a positive answer to a conjecture of Cooper [Coo87], pg. 455) that, in fact, $rk(Fix(\Phi)) + \frac{a(\Phi)}{2}$. Here $a(\Phi)$ denotes the number of attracting fixed points for the action of $\Phi$ on $\partial F_r$. [GJLL98] also provides an inequality bounding the index sum for a nongeometic fully irreducible, as described below.

As with a pseudo-Anosov acting on Teichmüller space, a fully irreducible acts with north-south dynamics [LL03] on the natural compactification of Culler-Vogtmann Outer space [CV86]. Both the attracting and repelling points for the action are $\mathbb{R}$-trees, denoted respectively $T_{\phi}^{+}$ and $T_{\phi}^{-}$. The repelling tree is an extension, to nongeometric fully irreducibles, of the dual tree to the invariant foliation for a pseudo-Anosov. As with a pseudo-Anosov, the index list for a fully irreducible, as defined in [HM11], has an entry of $1 - \frac{k}{2}$ obtained by counting the valence $k$ of the branch point. The index list can again also be computed from the expanding lamination of [BFH97]. For a fully irreducible the lamination can be obtained by applying an automorphism in the class repeatedly to any generator, then taking the closure. The description of the index list we use here (explained in Section 2) uses the "train track representative" proved to exist for a fully irreducible in [BH92].

While the $Out(F_r)$ groups resemble mapping class groups, there is added depth to the $Out(F_r)$. A particularly good example of this arises when trying to generalize the Masur-Smillie pseudo-Anosov index theorem to nongeometric fully irreducibles. One facet of this depth is expanded upon in [Pfa12b], [Pfa13a], and [Pfa13b], where we show that, unlike with pseudo-Anosovs, where the ideal Whitehead graph can be determined by the singularity index list, the ideal Whitehead graph actually gives a finer invariant of a fully irreducible giving, in particular, more detailed behavior of the lamination at a singularity. In this paper we focus on the fact that, instead of being restricted by an index sum equality, such as the Poincaré-Hopf index equality, the index sum for a fully irreducible is only restricted by an inequality. Gaboriau, Jäeger, Levitt, and Lustig proved in [GJLL98] that each fully irreducible $\phi \in Out(F_r)$ satisfies that index sum inequality $0 > i(\phi) \geq 1 - r$. Here we revise their index definition to be invariant under taking powers and to have the sign consistent with the mapping class group case. If one takes an adequately high power (the "rotationless power" of [FH11], see Subsection 2.2), the definitions differ only by a sign change. This is simply because we compute the index using periodic points, instead of fixed points, on the boundary of $T^{+}$.

Index lists of geometric fully irreducibles are understood by the Masur-Smillie index theorem. Complexity of the nongeometric case prompted Handel and Mosher to ask ([HM11] Question 6):

**Question 1.1.** *Which index types, satisfying $0 > i(\phi) \geq 1 - r$, are achieved by nongeometric, fully irreducible $\phi \in Out(F_r)$?*

We answered the rank 3 case with:

**Theorem 1.2.** *Each of the six possible index lists, $\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$, $\{-\frac{1}{2}, -1\}$, $\{-\frac{3}{2}\}$, $\{-\frac{1}{2}, -\frac{1}{2}\}$, $\{-1\}$, and $\{-\frac{1}{2}\}$, satisfying $0 > i(\phi) > 1 - r$ are realized by fully irreducible $\phi \in Out(F_3)$. In fact, they are realized by ageometric fully irreducible $\phi \in Out(F_3)$.*

One may notice that we restrict to looking at outer automorphisms for which the right-hand inequality is strict. This is because we focus on ageometric outer automorphisms, as defined in [GJLL98]. While ageometrics are believed generic, there does exists a second class of nongeometric outer automorphisms, namely the parageometrics (which could be classified as nongeometric fully irreducible outer automorphisms with geometric attracting tree). These have been studied in papers such as [HM07] and [Gui05], where in fact they show that the inverse of a parageometric is ageometric. It can additionally be noted that Bestvina and Feighn give in [BF94] a nice description of the distinction between ageometrics, parageometrics, and geometrics.

It is proved in [GL95] that a fully irreducible has geometric attracting tree precisely if the index sum satisfies $i(\phi) = 1 - r$. Thus, like geometrics, parageometrics have index sum $i(\phi) = 1 - r$. It would be interesting to understand whether index lists satisfying $i(\phi) = 1 - r$, but not realized by geometrics, are in fact realized by parageometric outer automorphisms.

While this paper is the first to focus on index list realization, the index theory for free group outer automorphisms has in some directions already been extensively developed. In fact, there are three types of $Out(F_r)$ index invariants in the literature, those of [GL95], [GJLL98], and [CH]. The index of $\phi$, as defined and studied in [GJLL98], is equal to the geometric index of $T_{\phi}^{+}$, as established by Gaboriau-Levitt [GL95] for more general $R$-trees. [CH12] provides a relationship between the index of [CH] and the geometric index, as well as uses the index to relate different properties of the attracting and repelling tree for a fully irreducible. There are also even index realization results of a different nature. For example, [JL09] gives examples of automorphisms with the maximal number of fixed points on $\partial F_r$, as dictated by a related inequality in [GJLL98]. Focusing on an $Out(F_r)$-version of the Masur-Smillie theorem, we restrict attention to fully irreducibles and the [GJLL98] index inequality given above.

**Acknowledgements.** The author would like to thank her thesis advisor Lee Mosher for his continued discussions and support. She would like to thank Thierry Coulbois for his invaluable computer program and assistance. She would like to thank Ilya Kapovich for helpful discussions and advice. And, finally, she would like to thank Martin Lustig for his interest in her work.

## 2. Definitions and Background

### 2.1. **Train track representatives.**

Let $R_r$ denote the $r$-petaled rose (graph with one vertex and $r$ edges) together with an identification $\pi(R_r) \cong F_r$. We call a connected 1-dimensional CW-complex $\Gamma$ such that each vertex has valence at least two, together with a homotopy equivalence (*marking*) $R_r \to \Gamma$, a *marked graph*. Each $\phi \in Out(F_r)$ is represented by a homotopy equivalence $g\colon \Gamma \to \Gamma$ of a marked graph, where $\phi = g_*$. When $g$ additionally sends vertices to vertices and satisfies that $g^k$ is locally injective on edge interiors for each $k > 0$, we say $g$ is a *train track (tt) representative* for $\phi$, and a *train track (tt) map*. In [BH92], Bestvina and Handel prove that a fully irreducible outer automorphism always has a train track representative.

Many of the definitions and notation for discussing train track representatives were established in [BH92] and [BFH00]. We remind the reader here of a few that are relevant.

Let $g\colon \Gamma \to \Gamma$ be a tt map. For each $x \in \Gamma$, we let $\mathcal{D}(x)$ denote the set of *directions* at $x$, i.e. germs of initial segments of edges emanating from $x$. For an edge $e \in \mathcal{E}(\Gamma)$, we let $D_0(e)$ denote the initial direction of $e$. For a path $\gamma = e_1 \dots e_k$, define $D_0\gamma = D_0(e_1)$. We denote the map of directions induced by $g$ by $Dg$, i.e. $Dg(d) = D_0(g(e))$ for $d = D_0(e)$. A direction $d$ is *periodic* if $Dg^k(d) = d$ for some $k > 0$.

We call an unordered pair of directions $\{d_i, d_j\}$ a *turn*. It is an *illegal turn* for $g$ if $Dg^k(d_i) = Dg^k(d_j)$ for some $k$ and *legal* otherwise. Considering the directions of an illegal turn equivalent, one can define an equivalence relation on the set of directions at a vertex. Each equivalence class is called a *gate*. For a path $\gamma = e_1 e_2 \dots e_{k-1} e_k$ in $\Gamma$, we say $\gamma$ *takes* $\{\overline{e_i}, e_{i+1}\}$ for each $1 \leq i < k$. For both edges and paths we more generally use the "overline" to denote a reversal of orientation.

We call a tt map *reducible* if it has an invariant subgraph with a noncontractible component and is otherwise called *irreducible*. An outer automorphism $\phi$ is *fully irreducible* if every representative of every power is irreducible.

The *transition matrix* for a tt map $g$ is the square matrix such that, for each $i$ and $j$, the $ij^{th}$ entry is the number of times $g(E_j)$ contains $E_i$ with either orientation. A transition matrix $A = [a_{ij}]$ is *Perron-Frobenius (PF)* if there exists an $N$ such that, for all $k \geq N$, $A^k$ is strictly positive (see for example [BH92]).

We call a tt map $g$ *expanding* if for each edge $e$ in $\Gamma$ we have that $|g^n(e)| \to \infty$ as $n \to \infty$, where for a path $\gamma$ we use $|\gamma|$ to denote the number of edges $\gamma$ traverses (with multiplicity).

In this paper we only deal with expanding irreducible train track maps and hence will give definitions, etc, in this context, even when it is not strictly necessary.

2.2. **Periodic Nielsen paths and rotationless powers.** Let $g \colon \Gamma \to \Gamma$ be an expanding irreducible tt map. A *periodic Nielsen path (PNP)* for $g$ is a nontrivial path $\rho$ in $\Gamma$ such that, for some $k$, we have $g^k(\rho) \simeq \rho$ rel endpoints. $\rho$ is called an *indivisible periodic Nielsen path (iPNP)* if it cannot be written as a nontrivial concatenation $\rho = \rho_1 \cdot \rho_2$, where $\rho_1$ and $\rho_2$ are nontrivial PNPs. An (indivisible) periodic Nielsen path is just called an (indivisible) Nielsen path when its period is 1. The notation reflects this.

As it is used in Section 3, we remark that iPNPs have a specific structure, described in [BH92] Lemma 3.4:

**Proposition 2.1.** *Let $g \colon \Gamma \to \Gamma$ be an expanding irreducible tt map. Then every iPNP $\rho$ in $\Gamma$ has the form $\rho = \overline{\rho_1}\rho_2$, where $\rho_1$ and $\rho_2$ are nondegenerate legal paths sharing their initial vertex $v \in \Gamma$ and such that the turn at $v$ between $\rho_1$ and $\rho_2$ is an illegal nondegenerate turn for $g$.*

The notion of a rotationless tt map is first defined in [FH11] Definition 3.18. However, because we only deal with expanding irreducible tt maps, we instead use the version of the definition and results surrounding it given in [HM11].[1]

Given an expanding irreducible tt map $g \colon \Gamma \to \Gamma$, we call a periodic vertex $v$ of $\Gamma$ *principal* that is either the endpoint of an iPNP or that has at least 3 periodic directions. (In order to include all endpoints of iPNPs under this definition, one typically adds valence-2 vertices at the endpoints of iPNPs, by [FH11] Lemma 2.12 there are only finitely many such points.) An expanding irreducible tt map $g \colon \Gamma \to \Gamma$ such that each principal vertex is fixed and each principal direction of each principal vertex is fixed is called *rotationless*.

We will use that rotationless powers always exist (and are in fact bounded by the rank of the free group). To understand this, one needs from [FH11] Proposition 3.29 that $\phi \in Out(F_r)$ is rotationless if one (hence all) of its expanding irreducible tt representatives is rotationless. The following is [FH11] Corollary 4.43:

**Proposition 2.2.** *For each $r \geq 2$, there exists an $R(r) \in \mathbb{N}$ such that $\phi^{R(r)}$ is rotationless for each $\phi \in Out(F_r)$.*

2.3. **Local Whitehead graphs, ideal Whitehead graphs, and index lists.** We assume in this subsection that $g \colon \Gamma \to \Gamma$ is an expanding irreducible tt representative of $\phi \in Out(F_r)$. We also assume that $g$ is PNP-free.

[1]The definitions in [HM11] are also adjusted from those in [FH11] to resolve an omission that meant that endpoints of closed periodic Nielsen paths were not considered principal.

The following definitions are from [HM11]. One can reference [Pfa12a] Section 2.9 for more extensive explanations of the definitions and their invariance (notice that the index list being an outer automorphism invariant follows from the ideal Whitehead graph being an outer automorphism invariant, since it is computed from the ideal Whitehead graph). It is notable that, while we use a tt representative here to define the ideal Whitehead graph and index list for a fully irreducible outer automorphism, both the ideal Whitehead graph and index list are invariants of the outer automorphism (again by [Pfa12a]). In fact, they are invariants of the conjugacy class within $Out(F_r)$ of the outer automorphism.

Let $g\colon \Gamma \to \Gamma$ be a tt representative of $\phi \in Out(F_r)$ and $v$ a vertex of $\Gamma$. The *local Whitehead graph* $\mathcal{LW}(g;v)$ for $g$ at $v$ has a vertex for each direction at $v$ and an edge connecting the vertices for $d_i$ and $d_j$ when there exists an edge $e$ of $\Gamma$ and $k > 0$ so that $g^k(e)$ takes the turn $\{d_i, d_j\}$. Restricting to periodic directions, one obtains a subgraph called the *local stable Whitehead graph* $\mathcal{SW}(g;v)$. Still assuming $g$ has no PNPs, the *ideal Whitehead graph* $\mathcal{IW}(\phi)$ *of* $\phi$ is then isomorphic to the disjoint union $\bigsqcup \mathcal{SW}(g;v)$ taken over all principal vertices.

**Remark 2.3.** *A useful observation is that, if $\mathcal{LW}(g;v)$ is connected, then so is $\mathcal{SW}(g;v)$. Hence, if $g$ is PNP-free and all local Whitehead graphs are connected, then the connected components of the ideal Whitehead graph are in one-to-one correspondence with the principal vertices.*

Let $\phi$ be a nongeometric fully irreducible outer automorphism and let $C_1, \dots, C_l$ be the connected components of the ideal Whitehead graph $\mathcal{IW}(\phi)$. For each $j$, let $k_j$ denote the number of vertices of $C_j$. The index list for $\phi$ can be defined as

$$\{i_1, \dots, i_j, \dots, i_l\} = \{1 - \frac{k_1}{2}, \dots, 1 - \frac{k_j}{2}, \dots, 1 - \frac{k_l}{2}\},$$

where we only include nonzero entries. The index sum is then $i(\phi) = \sum i_j$.

2.4. **Full irreducibility criterion.** In order to show that our maps represent fully irreducible outer automorphisms, we use the "Full Irreducibility Criterion (FIC)" proved in [Pfa13a] (Lemma 4.1):

**Lemma.** (*The Full Irreducibility Criterion (FIC)*) Let $g\colon \Gamma \to \Gamma$ be a PNP-free, irreducible train track representative of $\phi \in Out(F_r)$. Suppose that the transition matrix for $g$ is Perron-Frobenius and that all the local Whitehead graphs are connected. Then $\phi$ is fully irreducible.

## 3. Verification Procedures

In Theorem 4.1 we used a computer program [Cou14] to verify that each example is indeed a PNP-free tt representative of the correct rank. We include here a procedure for finding by hand all PNPs of a train track map. This procedure is not too different from that given in [Pfa13a] and is that applied in [HM11] Example 3.4.

We also include here procedures for computing by hand local Whitehead graphs and ideal Whitehead graphs. In the proof of Theorem 4.1 we only use the procedures for PNP-free train track maps.

We leave verification of the validity of all of the procedures to the reader.

3.1. **Finding periodic Nielsen paths.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible tt map and $\{T_1, \dots, T_n\}$ the set of illegal turns for $g$. The following procedure will identify if there exists an iPNP $\rho = \overline{\rho_1}\rho_2$ for $g$, where $\rho_1 = e_1 \dots e_m$ and $\rho_2 = e'_1 \dots e'_{m'}$ are edge paths (with possibly $e_m$ and $e'_{m'}$ being partial edges) and with illegal turn $T_i = \{D_0(e_1), D_0(e'_1)\} = \{d_1, d'_1\}$. Since PNPs can be decomposed into iPNP, as such, one can find all PNPs for $g$. Throughout the procedure, we use the notation $\rho_{1,k} = e_1 \dots e_k$ and $\rho_{2,l} = e'_1 \dots e'_l$ for respectively the length-$k$ and length-$l$ initial segments of a proposed $\rho_1$ and $\rho_2$.

We suppose one has followed the procedure to obtain legal paths $\rho_{1,k} = e_1 \dots e_k$ and $\rho_{2,l} = e'_1 \dots e'_l$ and explain the next step of the procedure in each of the possible cases.

**(Case A)** Suppose that either $g^j(\rho_{1,k})$ is the initial subpath of $g^j(\rho_{2,l})$ or vice versa.

Without generality loss (or by adjusting notation) we can assume $g^j(\rho_{1,k})$ is the initial subpath of $g^j(\rho_{2,l})$, so $g^j(\rho_{2,l}) = g^j(\rho_{1,k})\sigma$, for some legal path $\sigma$. Then if $\rho_{1,k}$ were the initial subpath of some $\rho_1$ and $\rho_{2,l}$ were the initial subpath of some $\rho_2$ so that $\rho = \overline{\rho_1}\rho_2$ were a PNP for $g$, we would need for $\rho_1$ to contain another edge $e_{k+1}$. With each choice of $e_{k+1}$ such that $\rho_{1,k}e_{k+1}$ is legal and $Dg^{j+P}(e_{k+1}) = Dg^P(\sigma)$ for some $P$, one must continue to compose with $g$ until following the procedure either leads to a PNP or shows the choice does not lead to a PNP. (Notice that to check if $Dg^{j+P}(e_{k+1}) = Dg^P(\sigma)$ for some $P$ it is enough to check that either $Dg^{j+1}(e_{k+1}) = Dg(\sigma)$ or $\{Dg^{j+1}(e_{k+1}), Dg(\sigma)\}$ is an illegal turn).

**(Case B)** Suppose that $g^j(\rho_{1,k}) = \gamma\alpha'_1$ and $g^j(\rho_{2,l}) = \gamma\alpha'_2$ where $\{D_0(\alpha_1), D_0(\alpha_2)\}$ is a legal turn.

Then $\rho_{1,k}$ and $\rho_{2,l}$ could not yield an iPNP containing $T_i$ and one must start the procedure over for each possible edge addition arising in (A).

**(Case C)** Suppose that $g^j(\rho_{1,k}) = \gamma\alpha'_1$ and $g^j(\rho_{2,l}) = \gamma\alpha'_2$ where $\{D_0(\alpha_1), D_0(\alpha_2)\}$ is an illegal turn. And either

(i) $\rho_{1,k}$ is an initial subpath of $\alpha'_1$ and $\rho_{2,l}$ is an initial subpath of $\alpha'_2$

(ii) or (i) does not hold.

In the case of (i), there exists an iPNP from a fixed point of $e_k$ to a fixed point in $e'_l$. In the case of (ii), one must continue composing with $g$ until either they land in the case of (A), (B), or (i) or they reach a rotationless power, in which case (ii) would indicate there is no iNP.

**Remark 3.1.** *One can note that this procedure is finite, as there are only finitely many illegal turns and a bound on the length of an iNP (as described in* [BH92] *Corollary 3.5 to be a consequence of the "bounded cancellation lemma"). Since we never reach a case where we have to take advantage of the bound of the length of an iNP, we refer the interested reader to* [BH92] *for its explicit computation.*

*While the procedure in theory also could require computation of the rotationless power for a given rank, in practice it also is not used in our applications of the procedure and its computation can be somewhat involved. Hence, we refer the reader to* [FH11] *Corollary 4.43.*

3.2. **Computing index lists.** Suppose $g\colon \Gamma \to \Gamma$ is a train track representative for an ageometric fully irreducible. And supposed $\Gamma$ has periodic vertices $v_1, \dots, v_k$. For each $1 \le i \le k$, let $n_i$ denote the number of gates at the vertex $v_i$. Define an equivalence relation on the set of all periodic points by: $x_i \sim x_j$ if there exists a PNP running from $x_i$ to $x_j$. Call an equivalence class a *Nielsen class*. For a Nielsen class $N_i = \{x_1, \dots, x_n\}$, let $g_i$ denote the number of gates at $x_i$. Now let

$$n_i = (\sum g_i) - \#\{\text{iNPs } \rho \text{ such that both endpoints of } \rho \text{ are in } N_i\}.$$

The index list is then

$$\{1 - \frac{n_1}{2}, \dots, 1 - \frac{n_t}{2}\},$$

where we only include nonzero entries.

Notice that there are only finitely many nonzero entries, as there are only finitely many iNPs and a periodic point $x_i$ that is not a vertex and not the endpoint of an iNP will have $1 - \frac{x_i}{2} = 0$. Additionally notice that one does not need to find all periodic points to make this computation, but only needs to consider Nielsen classes that contain a vertex or endpoint of a PNP.

3.3. **Computation of local Whitehead graphs.** Now let $g\colon \Gamma \to \Gamma$ be any expanding irreducible tt map. Recall that the local Whitehead graph $\mathcal{LW}(g; v)$ has a vertex for each direction at $v$ and an edge connecting the vertices for $d_i$ and $d_j$ if there is some edge $e$ of $\Gamma$ and some $k > 0$ so that $g^k(e)$ takes the turn $\{d_i, d_j\}$. We explain a finite procedure for computing all such $\{d_i, d_j\}$. We denote by $T$ this list of turns taken by some $g^k(e)$.

Enumerate the edges $e_i$ of $\Gamma$. For each $e_i$, find the list of turns traversed by $g(e_i)$. Let

$$\mathcal{T} = \{\{d_{i_1}, d_{j_1}\}, \dots, \{d_{i_m}, d_{j_m}\}\}$$

be the union of these lists. That is, each $\{d_{i_b}, d_{j_b}\} \in \mathcal{T}$ is a turn taken by some $g(e_i)$. We now construct the list of turns $T$ as follows: $T$ first off includes all elements of $\mathcal{T}$, but it will also include all $Dg^k(\{d_{i_b}, d_{j_b}\})$ where $k > 0$. To ensure this:

Start with $\{d_{i_1}, d_{j_1}\}$. Add $Dg(\{d_{i_1}, d_{j_1}\})$, $Dg^2(\{d_{i_1}, d_{j_1}\})$, etc, to $\mathcal{D}$ until reaching some $Dg^N(\{d_{i_1}, d_{j_1}\})$ already in $\mathcal{D}$. Now do the same for $\{d_{i_2}, d_{j_2}\}$, for $\{d_{i_3}, d_{j_3}\}$, etc.

Notice that, not only is this set $T$ finite, but it will contain fewer than $mR$ elements, where $R$ is the minimal rotationless power (see Subsection 2.2). So the procedure is finite.

**Example 3.2.** *We consider the train track map on the rose:*

$$g = \begin{cases} a \mapsto cab \\ b \mapsto ca \\ c \mapsto acab \end{cases}$$

*We will show how to verify that this is a tt representative of an ageometric fully irreducible with single-entry index list $\{-1\}$.*

*Since the automorphism is positive, it is easily verified to be a tt map. It is irreducible since the iterated image of each edge contains each other edge (notice that, since it is a tt map, this also implies that the transition matrix is Perron-Frobenius). It is expanding since the image of each edge has length $\geq 2$.*

*The direction map $Dg$ sends:*
$a \mapsto c \mapsto a \mapsto \dots$
$b \mapsto c \mapsto \dots$
$c \mapsto a \mapsto \dots$
$\bar{a} \mapsto \bar{b} \mapsto \bar{a} \mapsto \dots$
$\bar{b} \mapsto \bar{a} \mapsto \dots$
$\bar{c} \mapsto \bar{b} \mapsto \dots$
*From this one can see that the periodic directions are $a$, $c$, $\bar{a}$, and $\bar{b}$ and the gates are $\{a, b\}$, $\{c\}$, $\{\bar{a}, \bar{c}\}$, and $\{\bar{b}\}$. Since there are four gates at the single vertex, the index list has a single entry $1 - \frac{4}{2} = -1$, provided that the map is PNP-free and represents a fully irreducible. (Without PNPs, there is precisely one ideal Whitehead graph component for each local stable Whitehead graph, of which there is only one here since there is only one vertex.)*

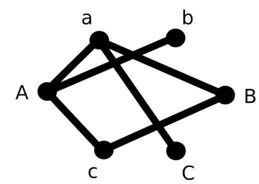
*The turns taken by $g(a)$ are $\{\bar{c}, a\}$ and $\{\bar{a}, b\}$. The turns taken by $g(b)$ are $\{\bar{c}, a\}$.The turns taken by $g(c)$ are $\{\bar{a}, c\}$, $\{\bar{c}, a\}$, and $\{\bar{a}, b\}$. Thus,*

$$\mathcal{T} = \{\{\bar{c}, a\}, \{\bar{a}, b\}, \{\bar{a}, c\}\}.$$

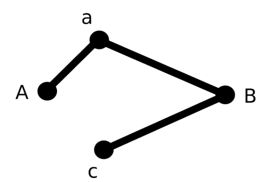
$\{\bar{c}, a\} \mapsto \{\bar{b}, c\} \mapsto \{\bar{a}, a\} \mapsto \{\bar{b}, c\}$, *which is already in $T$.*
$\{\bar{a}, b\} \mapsto \{\bar{b}, c\}$, *which is already in $T$.*
$\{\bar{a}, a\} \mapsto \{\bar{b}, a\} \mapsto \{\bar{a}, c\}$, *which is already in $T$.*
*So*

$$T = \{\{\bar{c}, a\}, \{\bar{a}, b\}, \{\bar{a}, c\}, \{\bar{b}, c\}, \{\bar{a}, a\}, \{\bar{b}, a\}\}.$$

*Thus, the single local Whitehead graph (which is connected) is:*




*From this, by restricting to the periodic directions, one can ascertain the single local stable Whitehead graph (yielding the ideal Whitehead graph):*




*We now show how to apply the procedure to verify that there are no PNPs. The two illegal turns are $\{a, b\}$ and $\{\bar{a}, \bar{c}\}$. We verify that there is no iPNP containing $\{a, b\}$ and leave the verification for $\{\bar{a}, \bar{c}\}$ to the reader.*

$a \mapsto cab$

$b \mapsto ca$

*Thus we are in the case of (A) with $e_1 = b$, and there must be another edge $e_2$ after $b$. $e_2$ must satisfy that either $Dg(e_2) = a$ or $Dg(e_2) = b$. The only such possibility is $e_2 = c$. So $\rho_{1,2} = bc$. We apply $g$ twice because after the first application, cancellation ends in the illegal turn $\{\bar{a}, \bar{c}\}$, but not in the manner of (C)(i):*

$a \mapsto cab \mapsto acabcabca$

$bc \mapsto caacab \mapsto acabcabcabacabcabca$

*We are now in the case of (A) and must add another edge $e_2'$ after $e_1' = a$. $e_2'$ must satisfy that either $Dg^2(e_2) = a$ or $Dg^2(e_2) = b$. So we must check $e_2' = a$ and $e_2' = b$. The cancellation of $g^3(aa)$ and $g^3(bc)$ ends with $\{\bar{b}, \bar{c}\}$, which is a legal turn. And the same is true for the cancellation of $g^3(ab)$ and $g^3(bc)$. So we are done.*

*Notice that we have shown that $g$ is an irreducible PNP-free tt map with Perron-Frobenius transition matrix and a unique connected local Whitehead graph. Hence, $g$ represents an ageometric fully irreducible outer automorphism. The index list $\{-1\}$ could additionally have been computed from the ideal Whitehead graph because the single component has 4 vertices.*

## 4. Main Theorem

**Theorem 4.1.** *Each of the six possible index lists, $\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$, $\{-\frac{1}{2}, -1\}$, $\{-\frac{3}{2}\}$, $\{-\frac{1}{2}, -\frac{1}{2}\}$, $\{-1\}$, and $\{-\frac{1}{2}\}$, satisfying $0 > i(\phi) > 1 - r$ are realized by fully irreducible $\phi \in Out(F_3)$. In fact, they are realized by ageometric fully irreducible $\phi \in Out(F_3)$.*

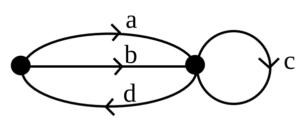
*Proof.* For each index list we give an explicit example. We used a computer program [Cou14] to verify that each example is indeed a tt map of the correct rank and additionally has no PNPs. We apply the FIC to show that the example is indeed a fully irreducible outer automorphism. To verify that a given representative has PF transition matrix, since our representatives are tt maps, it suffices to prove that a sufficiently high power maps each edge over each other edge. We compute the local Whitehead graphs to show that they are connected. (There should be precisely one connected graph for each vertex.) Having no PNPs, having PF transition matrix, and having connected local Whitehead graphs, our representatives are fully irreducible by the FIC. Having no PNPs additionally implies they in fact represent an ageometric fully irreducible. Restricting the local Whitehead graphs to their periodic direction vertices (and the edges between them) gives the local stable Whitehead graphs. Since there are no PNPs, these local stable Whitehead graphs are precisely the components of the ideal Whitehead graph. As explained in Section 2, one then computes the index list by including for each component $C$ of the ideal Whitehead graph an entry $1 - \frac{k}{2}$, where $k$ is the number of vertices of $C$.

INDEX LIST $\{-\frac{3}{2}\}$:

A plethora of examples with this index list can be found in [Pfa13b].

INDEX LIST $\{-\frac{1}{2}, -1\}$:

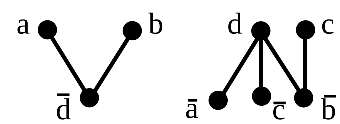
The representative on the graph



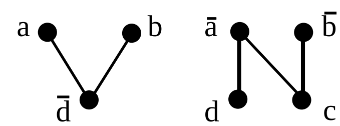

is:

$$g = \begin{cases} a \mapsto adbcdbdbda \\ b \mapsto bdadb \\ c \mapsto cdbdbdadbdbdadbdbdbdadbdbdadbcdbdbdadbdbdadb \\ d \mapsto dbd \end{cases}$$

As you can see from the below figure, the local Whitehead graphs are connected.

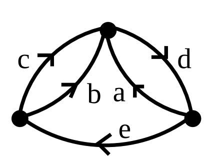



Restricting to periodic directions, since there are no periodic Nielsen paths, this gives the ideal Whitehead graph, from which the index list is computed to be $\{-\frac{1}{2}, -1\}$:




INDEX LIST $\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$:

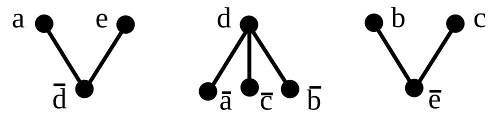
The representative on the graph



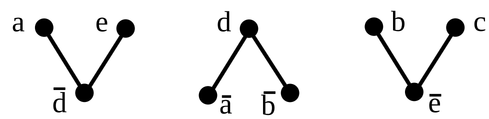

is:

$$g = \begin{cases} a \mapsto adecdadebda \\ b \mapsto bdadadecdadebdadadebdadebdadadecdadebdadadeb \\ c \mapsto cdadebdadadebdadebdadadebdadadecdadebdadadebdadebdadadecdadebdadadebdadebdadadecdadebda \\ d \mapsto dadebdad \\ e \mapsto ebdadade \end{cases}$$

As you can see from the below figure, the local Whitehead graphs are connected.



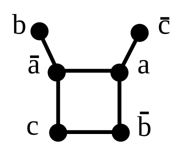
Restricting to periodic directions, since there are no periodic Nielsen paths, this gives the ideal Whitehead graph, from which the index list is computed to be $\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$:



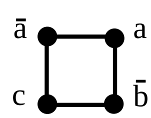
INDEX LIST $\{-1\}$: The representative on the rose is:

$$g=\begin{cases} a\mapsto cab \\ b\mapsto ca \\ c\mapsto acab \end{cases}$$

As you can see from the below figure, the single local Whitehead graph is connected.

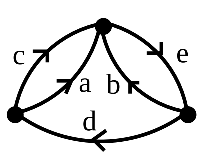


Restricting to periodic directions, since there are no periodic Nielsen paths, this gives the ideal Whitehead graph, from which the index list is computed to be $\{-1\}$:



INDEX LIST $\{-\frac{1}{2},-\frac{1}{2}\}$:

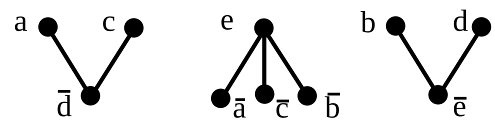
The representative on the graph



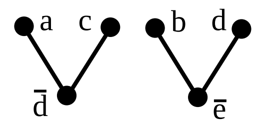
is:

$$g=\begin{cases} a\mapsto aebedcebedcebebedcebebeda \\ b\mapsto beda \\ c\mapsto cebebeda \\ d\mapsto dcebebed \\ e\mapsto ebedcebe \end{cases}$$

As you can see from the below figure, the local Whitehead graphs are connected.

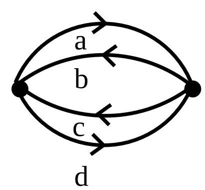



Restricting to periodic directions, since there are no periodic Nielsen paths, this gives the ideal Whitehead graph, from which the index list is computed to be $\{-\frac{1}{2}, -\frac{1}{2}\}$:




INDEX LIST $\{-\frac{1}{2}\}$:

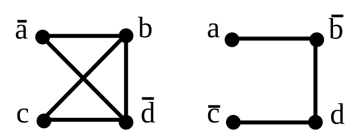
The representative on the graph



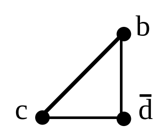

is:

$$g = \begin{cases} a \mapsto \bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c} \\ b \mapsto \bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b} \\ c \mapsto cdbabdcdbdbdcdbdbdcdbdcdbabdcdbabdcdbdbdcdbdbdcdbdcdbabdcdbabdcdbd \\ \bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}cdbabdcdbdbdcdbdbdcdbdcdbabdcdbabdcdbdbdcdbdbdcdbdcdbabdcdbabdcdbd \\ \bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b} \\ d \mapsto \bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{a}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b}\bar{d}\bar{c}\bar{d}\bar{b} \end{cases}$$

As you can see from the below figure, the local Whitehead graphs are connected.




Restricting to periodic directions, since there are no periodic Nielsen paths, this gives the ideal Whitehead graph, from which the index list is computed to be $\{-\frac{1}{2}\}$:




It can be noted that this example was in fact produced using the random automorphism generating capacity of [Cou14]. The sage worksheet can be found at `http://www.math.uni-bielefeld.de/~cpfaff/SageWorksheet.pdf`

$\square$

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